Geometry of Manifolds

Lectures delivered by Tobias Colding
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Introduction

Tobias Colding taught a course (18.965) on Geometry of Manifolds at MIT in Fall 2012. These are my "live-TEXed" notes from the course. The template is borrowed from Akhil Mathew.

Please email corrections to `holden1@mit.edu`.

Lecture 1
Thu. 9/6/12

Today Bill Minicozzi (2-347) is filling in for Toby Colding.

We will follow the textbook Riemannian Geometry by Do Carmo. You have to spend a lot of time on basics about manifolds, tensors, etc. and prerequisites like differential topology before you get to the interesting topics in geometry. Do Carmo gets to the interesting topics much faster than other books.

Today we give a quick overview of Riemannian geometry, and then introduce the basic definitions (manifolds, tangent spaces, etc.) that we'll need throughout the course. You will see how these definitions generalize concepts you are already familiar with from calculus.

§1 What is Riemannian geometry?

On Euclidean space we can do calculus; we can measure distances, angles, volumes, etc.

However, we want to do all that geometry on more general spaces, called **Riemannian manifolds**.

First, we'll have to rigorously define what those spaces are. What is a manifold? We need to generalize the basic notions of calculus in the manifold setting: what is a derivative? A derivative is basically a linear approximation, because the tangent line is the best linear approximation. We'll define the notion of a **tangent space** for a manifold.

Once we have a manifold, we can define have functions, curves and (sub)surfaces on the manifolds, and objects called **tensors**. The idea of tensors generalizes the idea of vector fields, which are 1-tensors. We can differentiate tensors; for instance, the covariant derivative of two-tensor is three-tensor.

Next, we'll see that a **Riemannian metric** allows us to calculuate distance and angles. A **geodesic** is the shortest path connecting two points, or more generally, paths that are *locally* shortest. For instance, the equator of a sphere is a geodesic: any connected part of the diameter that doesn't include antipodal points gives the shortest path between two points. We can view our spaces as metric spaces and do some geometry. We have *comparison theorems*, where we use the geometry of the space to get information about the metric. For instance, in the Bonnie-Meyer theorem, we use the curvature of a space to learn about its metric.

Later in the course, we will cover topic such as Cartan-Hadamard manifolds, harmonic maps, and minimal surfaces.

§2 Manifolds

We want to do calculus on more general spaces, called manifolds. In particular, we care about (smooth) differential manifolds. Before we give a formal definition, we first develop some intuition through examples.

2.1 Examples

The following are all manifolds.

- $\mathbb{R}^n$: n-dimensional Euclidean space

$$\mathbb{R}^n := \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}.$$

- S^n: the unit n-sphere

$$S^n := \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}.$$

  Here the Euclidean norm is defined by $|x|^2 := \sum_{i=1}^n x_i^2$. Note this is an example of a "submanifold" of $\mathbb{R}^{n+1}$.

Note that Euclidean geometry descends to geometry on any submanifold. A theorem of Nash says any abstract manifold can be embedded (at least locally) in Euclidean space. This means it is sufficient to learn about geometry of submanifolds of Euclidean space.

However, just as linear algebra is often simpler with "linear transformations" than with matrices, we will see that geometry is often simpler when we think of manifolds in the abstract.

Here are some more examples.

- T^n: n-torus $\mathbb{R}^n/\mathbb{Z}^n$. This means that we are modding out $\mathbb{R}^n$ by the equivalence relation $\sim$ where $x \sim (x+z)$ for every tuple $z = (z_1, \ldots, z_n)$ with $z_i \in \mathbb{Z}$. Note any small piece of T^n looks like $\mathbb{R}^n$ because don't see the wraparound.

  This local property means we can calculate derivatives of a function defined on T^n the same way we calculate derivatives of a function on $\mathbb{R}^n$.

- $\mathbb{R}P^n$: real projective n-space, the space of lines through 0 in $\mathbb{R}^{n+1}$. Note $\mathbb{R}P^n$ is closely related to the S^n, as follows. Each line through origin cuts sphere in 2 points, so we can think of $\mathbb{R}P^n$ as S^n modded out by the antipodal map $p \mapsto -p$.

These are all differential manifolds, but we don't get a *geometry* on them until we get a Riemannian metric (something we'll develop later in the course).

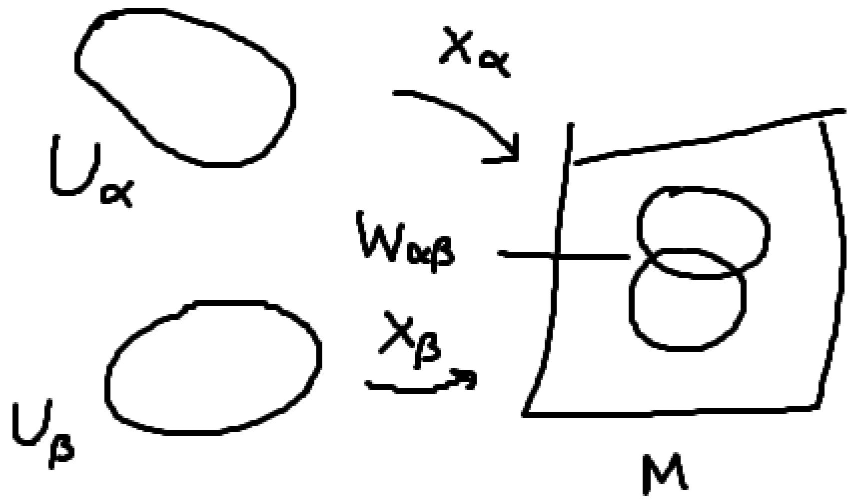
We also need a notion of a tangent vector. We'll give a formal definition of a manifold, then go back to talk about tangent spaces on manifolds.

2.2 Formal definition

Definition 1: A (smooth) n-dimensional **manifold** M is...

1. a set, denoted M, equipped with
2. a family of open sets $U_\alpha \subseteq \mathbb{R}^n$ and injective maps $x_\alpha : U_\alpha \hookrightarrow M$ (together called a **chart**) such that
   - (The open sets cover the manifold) $\bigcup_\alpha x_\alpha(U_\alpha) = M$.

- (Overlap properties) Set $W_{\alpha\beta} := x_\alpha(U_\alpha) \cap x_\beta(U_\beta)$ for each α and β. Suppose $W_{\alpha\beta} \neq \phi$. Then
  (a) $x_\alpha^{-1}(W_{\alpha\beta})$ is open,
  (b) $x_\beta^{-1} \circ x_\alpha$ is C^∞, i.e. infinitely differentiable
  (c) (*) This family is maximal with respect to A and B.



Let's analyze this definition. The open sets U_α tell us that locally, each point is parameterized by an open set in Euclidean space. We saw this in each of the examples. (For S^n, you can "flatten" any local part of the sphere.) Note that M inherits a topology by deeming that each $x_\alpha(U_\alpha)$ be a homeomorphism onto an open set of M.

The technical overlap properties force the x_α to be nice maps. (b) is why we call the manifold "smooth." We can loosen, tighten, or change the condition, for instance,

- A real analytic manifold is where $x_\beta^{-1} \circ x_\alpha$ are all real analytic.
- A complex manifold is one where we replace $\mathbb{R}$ with $\mathbb{C}$ and C^n by holomorphic.
- A PL manifold is where $x_\beta^{-1} \circ x_\alpha$ are all piecewise linear.

Without condition (c), we would have a lot of manifolds. Suppose we have $(M, \{x_\alpha\}, \{U_\alpha\})$ satisfying all the conditions except (c). For each U_α, we can take a subset $V \subseteq U_\alpha$ and restrict x_α to V. This is still a good parameterization. Adding V and $x_\alpha|_V$, we get a new manifold.

Thus (c) gives uniqueness: two manifolds that should be the same *are* the same.

An alternative approach is as follows: call something satisfying just (a) and (b) quasi-manifolds. Define an equivalence relation: two manifolds are the same if you can refine the two families of mappings $\{(U_\alpha, x_\alpha)\}$ to be the same family. Modding out quasi-manifolds by this equivalence relation gives a manifold.

Note n has to be constant: A sphere with two 1-dimensional antlers is not a manifold.

2.3 Reconciling definition with example

Let's show that $\mathbb{R}P^n$ is a manifold. Define **homogeneous coordinates** as follows: Consider $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \text{at least one } x_i \neq 0\}$, and mod out by the equivalence relation

$$(x_1, \ldots, x_{n+1}) \sim \lambda(x_1, \ldots, x_{n+1})$$

where $\lambda \in \mathbb{R}\backslash\{0\}$. Let the $[x_1, \ldots, x_{n+1}]$ denote the equivalence class of $(x_1, \ldots, x_{n+1})$; it is called homogeneous coordinates.

Defining the open sets and maps:

Define sets $V_i = \{[x_1, \ldots, x_{n+1}] : x_i \neq 0\}$. It's clear that $\bigcup V_i = \mathbb{R}P^n$. Because we need to get $n+1$ coordinates out of n coordinates, we define maps $x_i : \mathbb{R}^n \to V_i$ by

$$x_i(y_1, \ldots, y_n) = [y_1, \ldots, \underbrace{1}_{i}, \ldots, y_n].$$

For instance, for $\mathbb{R}P^3$, we have

$$\begin{aligned} x_1(y_1, y_2) &= [1, y_1, y_2] \\ x_2(y_1, y_2) &= [y_1, 1, y_2] \\ x_3(y_1, y_2) &= [y_1, y_2, 1] \end{aligned}$$

Note these maps are all bijective. They are onto because any element of $[x_1, \ldots, x_n] \in V_i$ has $x_i \neq 0$, and $[x_1, \ldots, x_n] = \left[\frac{x_1}{x_i}, \ldots, \frac{x_i}{x_i} = 1, \ldots, \frac{x_n}{x_i}\right]$.

Verifying overlap properties:

(a) We have
$$W_{12} = x_1(\mathbb{R}^2) \cap x_2(\mathbb{R}^2) = V_1 \cap V_2 = \{[z_1, z_2, z_3] : z_1 z_2 \neq 0\}.$$
Consider $x_1^{-1}(W_{12})$. We have $[z_1, z_2, z_3] \sim \left[1, \frac{z_2}{z_1}, \frac{z_3}{z_1}\right]$, so
$$x_1^{-1} W_{12} = \{(y_1, y_2) : y_1 \neq 0\}.$$
which is open.

(b) Now consider $x_2^{-1} \circ x_1 : W_{12} \to W_{21}$. We have for $y_1 \neq 0$ that
$$(y_1, y_2) \xrightarrow{x_1} [1, y_1, y_2] = \left[\frac{1}{y_1}, 1, \frac{y_2}{y_1}\right] \xrightarrow{x_2^{-1}} \left(\frac{1}{y_1}, \frac{y_2}{y_1}\right).$$
This is rational, so smooth.

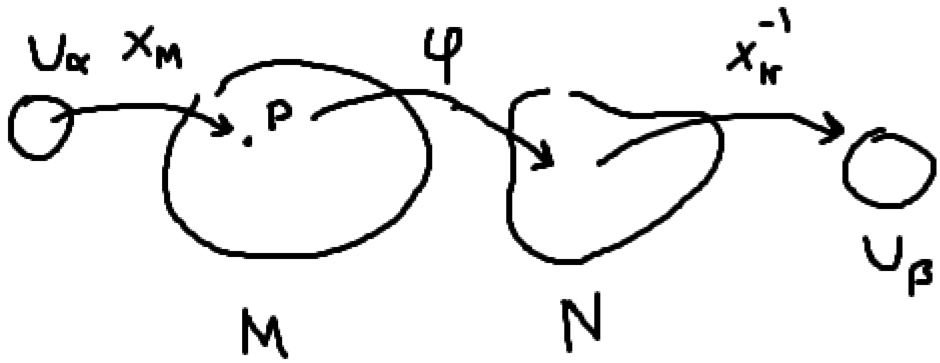
(c) To satisfy condition (c), we take a maximal family of (V_α, x_α) satisfying (a) and (b) and containing all the (V_i, x_i). (I.e. take all intersections among all the (V_i, x_i) and add them in; now take all subsects, not take intersections again, *ad infinitum.*)

Because the calculations are straightforward, this is the first and last time we're going to check something is a manifold.

2.4 Maps between manifolds

Definition 2: Let M and N be smooth manifolds. We say that $\varphi : M \to N$ is smooth at $p \in M$ if $x_N^{-1} \circ \varphi \circ x_M$ is smooth at $x_M^{-1}(p)$.

Here, x_M is any x_α such that $p \in x_\alpha(U_\alpha)$, and x_N is any x_β such that $\varphi(p) \in x_\beta(U_\beta)$.



Note the choice of $x_M = x_\alpha$ and $x_N = x_\beta$ doesn't matter, because the transition condition will give that it is true for any choice.

Some particularly important smooth maps are those with domain or target inside $\mathbb{R}$:

- Smooth functions on M, i.e. smooth maps $M \to \mathbb{R}$. This set is denoted by $\mathcal{D}$.
- Curves, maps from an interval $I \subseteq \mathbb{R} \to M$.

§3 Tangent vectors

There are two approaches to defining the derivative of a function on a manifold.

- The computational approach is to give it in terms of coordinates. We have to define how it transforms when we change coordinates.
- We can define it in a more abstract way, invariant under choice of charts. This is Do Carmo's approach and the approach we'll take. This automatically forces what the derivative has to be when we *do* express it in coordinates.

We'll first look at derivatives/tangent vectors in $\mathbb{R}^n$, and then generalize to manifolds.

3.1 Tangent vectors in $\mathbb{R}^n$

Definition 3: Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, so that $f \circ \alpha$ is a smooth function $(-\varepsilon, \varepsilon) \to \mathbb{R}$. Define the **directional derivative** or **tangent vector** in the direction of α of f to be

$$(f \circ \alpha)'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha(0))\alpha_i'(0) = \langle \nabla f, \alpha'(0) \rangle.$$

(The equality is by the chain rule.) Thus, we can think of the tangent vector as a function sending α to the linear map $f \mapsto \langle \nabla f, \alpha'(0) \rangle$.

Note the derivative depends only on $\alpha'(0)$. It didn't matter what the curve was; we could have covered all possibilities with curves that are straight lines, $\alpha(t) = p + tq$.

Definition 4: A **derivation** D of an $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear function from A to $\mathbb{R}$ that satisfies the Leibniz rule

$$D(fg) = (Df)g + f(Dg), \qquad f, g \in A.$$

Denote the space of derivations by $\mathrm{Der}(A)$.

Proposition 5: Let $C^\infty(x_1, \ldots, x_n)$ be the set of C^∞ functions on $x_1, \ldots, x_n$. The map $v \mapsto (f \mapsto \langle \nabla f, v \rangle)$ is a vector space isomorphism from $\mathbb{R}^n$ to $\mathrm{Der}(C^\infty(x_1, \ldots, x_n))$.

(Proof of surjectivity is omitted.) Think of v as $\alpha'(0)$, so the map is

$$f \mapsto \langle \nabla f, \alpha'(0) \rangle.$$

Now why did we define the directional derivative in terms of α instead of $v = \alpha'(0)$? Because we want something that *doesn't depend on coordinates*. Associated to α we get a *linear map* $f \mapsto \langle \nabla f, \alpha'(0) \rangle$ that we can define without coordinates. This is why we define the tangent vector as a linear map on a space of functions.

> We define the tangent vector to be a linear map on a space of functions, so that it does not depend on coordinates.

This will be important in the general manifold setting.

3.2 Tangent vectors in general

Our viewpoint naturally generalizes to manifolds.

Definition 6: Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a smooth curve. Then define the tangent vector as the linear map $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by

$$\alpha'(0)f = (f \circ \alpha)'(0)$$

The **tangent space** to M at p is

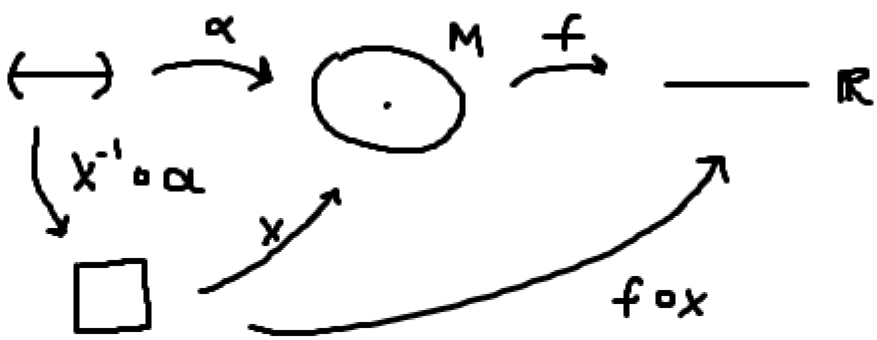
$$T_pM = \{\text{All tangent vectors to curves through } p\}$$

Note that $T_p \cong \mathbb{R}^n$. We will explain why.

To actually perform computations involving tangent vectors, we need to work on the actual charts, so the maps $(-\varepsilon, \varepsilon) \xrightarrow{\alpha} M \xrightarrow{f} \mathbb{R}$ are unsatisfactory. So given a point p, let $x : U \to M$ be a parameterization around p. Then we can work on the chart U, because we have the following commutative diagram (for small enough ε)

$$18965-1-cd \quad \begin{tikzcd} (-\varepsilon,\varepsilon) \arrow[r, "\alpha"] \arrow[dr, "x^{-1}\circ\alpha"'] & M \arrow[r, "f"] & \mathbb{R} \\ & U \arrow[u, "x"] \arrow[ur, "f\circ x"'] & \end{tikzcd} \tag{1}$$

Note $(-\varepsilon, \varepsilon) \xrightarrow{x^{-1}\circ\alpha} U \xrightarrow{f\circ x} \mathbb{R}$ are maps staying in $\mathbb{R}$, so we can do multivariable calculus with them.



Write

$$\begin{gathered} x^{-1} \circ \alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)) \\ f \circ x(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \text{ as shorthand} \\ (f \circ \alpha)(t) = (f \circ x) \circ (x^{-1} \circ \alpha)(t). \end{gathered}$$

The chain rule gives

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \alpha_i'(0).$$

(By abuse of notation $\frac{\partial f}{\partial x_i} = \frac{\partial f \circ x}{\partial x_i}$ and $\alpha_i'(0)$ is the ith component of $(x^{-1} \circ \alpha)'(0)$.) Think of this as the directional derivative "in coordinates," just like we can express a linear transformation as a matrix once we have coordinates.

Now if we had a different $(V, x') \neq (U, x)$ with $p \in x'(V)$, we can add V to our commutative diagram (1) We can then compute the directional derivative in the coordinates of the chart V instead of U, and we can see how the derivative changes from U to V using the chain rule on $y^{-1} \circ x$.

Thus we see that the directional derivative is an invariant notion—we don't need coordinates to define it, but once we do have coordinates, we can calculate it in terms of coordinates, and we know exactly how this expression changes when we change coordinates.

One thing to note is that if the $\alpha'(0)$ are all 0, then no matter *what* coordinates we choose, all the α_i' are still 0. But if $\alpha'(0)$ is nonzero, then we can mix things up any way we like.

Remarks:

1. $\alpha'(0)$ depends only on $\alpha'(0)$ in a coordinate system.
2. $T_p(M)$ is a n-dimensional vector space with a natural basis

$$\frac{\partial}{\partial x_i} := \text{tangent vector to curve where we only vary } x_i.$$

(More precisely, we are considering the curve $\alpha(t) = x(x^{-1}(p) + tx_i)$.) We have $\alpha_i'(0) = \delta_{ij}$.

§4 Differentials

Definition 7: A smooth map $\varphi : M \to N$ induces linear maps

$$d\varphi_p : T_pM \to T_{\varphi(p)}N$$

by taking the curve α to curve $\varphi \circ \alpha$:

$$d\varphi_p(\alpha'(0)) := (\varphi \circ \alpha)'(0).$$

The map $d\varphi_p$ is called the **differential** of φ at p.

Again this does not depend on the choice of α, only on $\alpha'(0)$.

Any smooth map φ gives rise to a differential map on the tangent spaces.

Once we have the differential, we can talk about immersions and embeddings.

Definition 8: We say that $\varphi : M \to N$ is a diffeomorphism if it is

1. smooth
2. bijective, and
3. φ^{-1} is smooth.

Definition 9: We say that $\varphi : M \to N$ is a **local diffeomorphism** at $p \in M$ if there exists an open set U containing p such that $\varphi_i : U \to \varphi(U)$ is a diffeomorphism

Definition 10: $\varphi : M \to N$ is an **immersion** if $d\varphi_p$ is *injective* at each p.

We have the following theorem

Theorem 11 (Inverse function theorem for manifolds)**:** If $d\varphi_p$ is bijective, then φ is local diffeomorphism to p.

This tells us that if the linearization of φ, i.e. $d\varphi_p$, is a bijection at p, then φ is actually a diffeomorphism at p.

Proof. Appeal to Euclidean Inverse Function Theorem (see Analysis on Manifolds, by Munkres) and compose with charts at either end. $\square$