1 Rational Approximations

Theorem 4.1: Let α and $Q \geq 1$ be real numbers. There exist integers a and Q such that

$$1 \leq q \leq Q, \qquad \gcd(a, q) = 1$$

and

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{(\lfloor Q \rfloor + 1)q} < \frac{1}{qQ}. \tag{1}$$

Proof. Let $N = \lfloor Q \rfloor$.

We want $q\alpha$ to be close to an integer a, because then $|q\alpha - a|$ would be small, and $\left|\alpha - \frac{a}{q}\right|$ would be small. Specifically, to make the last quantity be less than $\frac{1}{(N+1)q}$, we would like $|q\alpha - a| \leq \frac{1}{N+1}$. Thus we want to prove the following:

Claim: $\{q\alpha\} \in \left[0, \frac{1}{N+1}\right] \cup \left[\frac{N}{N+1}, 1\right)$ for some integer q such that $1 \leq q \leq N$.

Proof of claim. Suppose no such q exists; then the $q\alpha, 1 \leq q \leq N$ fall into the $N-1$ intervals

$$\left[\frac{1}{N+1}, \frac{2}{N+1}\right), \left[\frac{2}{N+1}, \frac{3}{N+1}\right), \ldots, \left[\frac{N-1}{N+1}, \frac{N}{N+1}\right).$$

By the Box Principle, two of the $q\alpha$ fall in the same interval, say $q_1\alpha$ and $q_2\alpha$. Then $\{|q_1 - q_2|\alpha\} \in \left[0, \frac{1}{N+1}\right) \cup \left(\frac{N}{N+1}, 1\right)$, a contradiction.

Now that we have q as in the claim, we let a be the closest integer to $q\alpha$. If q, a are relatively prime, then we are done; else divide by their gcd and (1) would still hold. □

2 Difference operators

Definition: Let f be a function. Define the difference operators:

$$(\Delta_d f)(x) = f(x+d) - f(x)$$
$$(\Delta f)(x) = f(x+1) - f(x)$$

Define the iterated difference operators:

$$\Delta_{d_l,\ldots,d_1} = \Delta_{d_l} \circ \cdots \circ \Delta_{d_1}$$
$$\Delta^{(l)} = \Delta_{1,\ldots,1}$$

The difference operator Δ is the discrete version of a derivative; their properties are somewhat similar to derivatives. The study of differences and sums (the discrete version of integrals) is called "finite calculus"; a few results of note are included in the exercises.

Lemma 4.1: Let $l \geq 1$. Then

$$\Delta^l f(x) = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} f(x+j).$$

Proof. Let E denote the shift operator, that sends a function $f(x)$ to $g(x) := f(x)+1$. Let I denote the identity operator. Then $\Delta = E - I$. Note that E, I commute, and E is a linear operator so composition distributes over addition. Hence the Binomial Theorem applies to $(E-I)^l$, and we have

$$\begin{aligned}\Delta^l f(x) &= (E-I)^l f(x) \\ &= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} E^j f(x) \\ &= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} f(x+j)\end{aligned}$$

□

Lemma 4.2: Let $k \geq 1$ and $1 \leq l \leq k$. Then

$$\begin{aligned}\Delta_{d_1,\ldots,d_l}(x^k) &= \sum_{\substack{j_1+\cdots+j_l+j=k \\ j\geq 0, j_i \geq 1}} \binom{k}{j, j_1, \ldots, j_l} d_1^{j_1} \cdots d_l^{j_l} x^j \\ &= d_1 \cdots d_l p_{k-l}(x)\end{aligned}$$

where $p_{k-l}(x)$ is an polynomial of degree $k-l$ and leading coefficient $k^{\underline{l}} = k(k-1)\cdots(k-l+1)$. If the d_i are all integers then $p_{k-l}(x)$ has integer coefficients.

If $l > k$, then

$$\Delta_{d_1,\ldots,d_l}(x^k) = 0.$$

Proof. An uninspiring induction, which can be found in the book. □

The following expresses a linear function as an iterated difference of x^k. This will be useful in Section 4.3 (Easier Waring's Problem), because this lemma says that every number in a certain arithmetic progression can be written the sum or difference of a fixed number of kth powers.

Lemma 4.3: For $k \geq 2$,

$$\Delta_{d_{k-1},\ldots,d_1}(x^k) = d_1 \cdots d_{k-1} k! \left(x + \frac{d_1 + \cdots + d_{k-1}}{2} \right).$$

Proof. Plug in $l = k-1$ into Lemma 4.2. □

Lemma 4.4: If the leading term of $f(x)$ is αx^k and $1 \leq l \leq k$, then the leading term of $\Delta_{d_l,\ldots,d_1} f(x)$ is

$$d_1 \cdots d_l k^{\underline{l}} = d_1 \ldots d_l k(k-1)\cdots(k-l+1)\alpha x^{k-l}.$$

In particular, the leading term when $l = k-1$ is

$$d_1 \ldots, d_{k-1} k! \alpha x.$$

For $l > k$, $\Delta_{d_l,\ldots,d_1} f(x) = 0$.

Proof. Just plug into Lemma 4.2 and note the summand with highest power of x is that with $j_1, \ldots, j_l = 1$. $\square$

The following gives an estimate of how large an iterated difference can get:

Lemma 4.5: Let $1 \leq l \leq k$. If $d_1, \ldots, d_l, x \in [-P, P]$, then

$$\Delta_{d_1,\ldots,d_l} x^k \leq CP^k$$

for some constant depending only on k.

Proof. Using Lemma 4.2 and the Multinomial Theorem

$$\begin{aligned}
\Delta_{d_1,\ldots,d_l}(x^k) &\leq \sum_{\substack{j_1+\cdots+j_l+j=k \\ j\geq 0, j_i \geq 1}} \binom{k}{j, j_1, \ldots, j_l} |d_1^{j_1} \cdots d_l^{j_l}| P^{j_1+\cdots+j_l+j} \\
&\leq \sum_{\substack{j_1+\cdots+j_l+j=k \\ j, j_i \geq 0}} \binom{k}{j, j_1, \ldots, j_l} |d_1|^{j_1} \cdots |d_l|^{j_l} P^k \\
&\leq (P + |d_1| + \cdots + |d_l|)^k \leq (k+1)^k P^k
\end{aligned}$$

$\square$

3 Exercises

1. Let $\alpha_1, \ldots, \alpha_n$ be real numbers. Show that for any $\varepsilon > 0$ there exists nonzero positive p and integers m_i so that

$$|p\alpha_i - m_i| < \varepsilon$$

for all i. Given ε and n, how small can you choose p? (Hint: Use the Box Principle and a similar argument to Theorem 4.1.)

2. Define the *falling power* for $k \in \mathbb{N}_0$ by

$$x^{\underline{k}} = x(x-1)\cdots(x-k+1).$$

Show that $\Delta(x^{\underline{k}}) = kx^{\underline{k-1}}$, and that $\Delta(c^x) = (c-1)c^x$.

3. (Indefinite sum) Define $\sum f(x)\,\delta x$ to be a function $F(x)$ such that $\Delta F(x) = f(x)$. Compute $\sum x^{\underline{k}}\,\delta x$ and $\sum c^x\,\delta x$.

4. (Fundamental Theorem of Finite Calculus) Define F as in Problem 3. Define

$$\sum_a^b f(x)\,\delta x = \sum_{x=a}^{b-1} f(x).$$

Prove that

$$\sum_a^b f(x)\,\delta x = F(b) - F(a).$$

5. (Product Rule) Let E be the shift operator. Prove that

$$\Delta(uv) = u\Delta v + (Ev)\Delta u.$$

6. (Summation by Parts) Prove that

$$\sum u\Delta v\,\delta x = uv - \sum Ev\Delta u\,\delta x.$$

   (Hint: Copy the proof for integrals, but use the product rule above.) Use this to evaluate $\sum_{x=1}^{n} xc^x$.

7. Let χ be a function $\mathbb{N} \to \mathbb{C}$ with period L and such that $\chi(1) + \cdots + \chi(L) = 0$. Let $\{a_n\}_{n=1}^{\infty}$ be a positive decreasing sequence. Prove that the sequence of partial sums

$$\sum_{j=1}^{n} \chi(j)a_j$$

   stays bounded. (Hint: Use summation by parts.)

References

[1] Nathanson, M.: "Additive Number Theory: The Classical Bases," Springer, NY, 1996, p. 97-102.