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| Analysis | Math Notes • Study Guide |
|  | Real Analysis |
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| 1 | Ordered Fields |
| 1-1 | Ordered sets and fields Let S be an ordered set and let . is bounded below, above if there exists (called a lower or upper bound) such that for all , respectively.  If is a lower bound such that any is not a lower bound for E, then is the **greatest lower bound (supremum)** of E, denoted by . The supremum is unique when it exists. Similarly, if is an upper bound such that any is not an upper bound for E, then is the **least upper bound (infimum)** of E, denoted by .  S has the **least upper bound property** if whenever is nonempty and bounded above, exists in S. This is equivalent to the greatest lower bound property.  An **ordered field** is a field that is an ordered set satisfying:   1. If then . 2. If then .   An ordered field is **Archimedean** if for all with , there exists such that . and are both Archimedean. |
| 1-2 | Construction of the Reals 1: Dedekind Cuts There exists a unique ordered field (the **real numbers**)with the least upper bound property; it contains as a subfield.  *Pf.*   1. The real numbers are associated with subsets (called cuts) satisfying:    1. .    2. If and , then . (If contains , it contains all numbers less than .)    3. If then for some . (No matter which we choose, we can always find in larger than it.) 2. We say if . 3. has the LUB property. 4. Let . Verify the axioms for addition. The inverse of is (some rational number smaller than is not in ). 5. Show that . 6. For positive , let . 7. Complete the definition by defining multiplication involving negative elements. Verify the remaining axioms. 8. Each rational number is associated with . Check that with this embedding, the rational numbers are an ordered subfield.   In the set , every nonempty subset has a infimum and supremum.[[1]](#footnote-1) |

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| 2 | Metric Spaces |
| 2-1 | Metric Spaces A set X with a real-valued function (a metric) on pairs of points in X is a **metric space** if:   1. with equality iff . 2. (Triangle inequality)   *Ex.*   * Discrete space: For any set X, define the metric * **N-dimensional Euclidean space** , with distance defined as * :    + To prove this is a metric, use Hölder’s Inequality…   + …to derive Minkowski’s Inequality: * : Continuous functions defined on * : * : Infinite sequences with : * : Bounded infinite sequences |
| 2-2 | Definitions  |  |  |  | | --- | --- | --- | | Term | Definition in metric space X | Definition in topology X | | neighborhood | For , the -**neighborhood** of a point is the set | A **neighborhood** of is an open set containing . | | contact point | is a **contact point** of if every neighborhood of contains a point of . | | | limit point | is a **limit point** of if every neighborhood of contains a point of besides . is the set of limit points of E. | | | isolated point | If but is not a limit point of E, then p is an **isolated point**. (Contact points = limit points isolated points.) | | | closed | E is **closed** if every limit point of E is in E. | E is **closed** if X-E is open. | | closure | The **closure** of E is the set of contact points of E. | The **closure** of X is the intersection of all closed sets contained in E. | | interior point | is an **interior point** if there is a neighborhood N of such that . The interior of N, denoted by , is the set of interior points (or union of open sets contained in E). | | | open | E is **open** if every point of E is an interior point of E. | A **topology** on a set X is a collection of subsets, called **open sets** satisfying:   1. The union of an arbitrary collection of sets in is in . 2. The intersection of a finite number of sets in is in . | | perfect | E is **perfect** if E is closed and every point of E is a limit point of E. | | | bounded | E is **bounded** if there exists and so that for all . | N/A | | dense | E is **dense** in X if every point of X is a contact point of E, i.e. . | |   Closed sets satisfy the following:   1. are closed. 2. Arbitrary intersections of closed sets are closed. 3. Finite unions of closed sets are closed.   A metric space is a topology- the definitions in topology hold in a metric space. (Note that neighborhoods in metric spaces are more strictly defined.)  If p is a limit point of E, then every neighborhood of p contains infinitely many points of E. Thus a finite point set has no limit points.  %On closure:   1. is closed. 2. iff E is closed. 3. for every closed set F with .   Let E be a nonempty subset of that is bounded above. If E is closed, then .  Subspace topology: Let . E is open relative to Y iff for some open set U in X. |
| 2-3 | Separability A metric space is **separable** if it contains a countable[[2]](#footnote-2) dense subset.  *Ex.* is separable since is dense in . is not separable.  Any subset of a separable space is separable.  A **base** for a topology on X is a collection of subsets, called base elements, of X such that   1. For each , there is at least one base element containing . 2. If for some , then for some .   In the topology generated by , a subset U is open if for each , there is a base element so that . In particular, each base element is open.  Two other equivalent formulations:   * is the collection of all unions of elements of . * is a collection of open sets of X such that for each open set U of X and each , there is an element such that .   Ex. -neighborhoods  A metric space is separable iff it has a countable base.   * SeparableCountable base: Let P be a countably dense subset and take . * Countable baseSeparable: (true for any topology) Choose a point in each base element. |
| 2-4 | Compact Sets An **open cover** of a set E in a topology X is a collection of open subsets such that . A subset is **compact** if every open cover of K contains a finite subcover. K is **sequentially (or countably) compact** if every infinite subset of K has a limit point in K.  On subsets:   * Suppose . Then K is compact relative to X if it is compact relative to Y. In other words, compactness is an intrinsic property. * Compact subsets are closed. * Closed subsets of compact sets are compact.   Theorems on compact sets:   * K is compact iff K is sequentially compact.   + : Let E be infinite subset. If no limit points, each has a neighborhood containing at most 1 point of E- neighborhoods form a cover with no finite subcover.   + :     - Sequentially compactSeparable: Given , choose any , take to be away from all . This must stop. Let range over ; putting together ’s gives countable dense subset.     - SeparableCountable base     - Countable baseEvery cover has at most countable subcover: For a base and a subcover, associate each element contained in some with . is a finite subcover.     - Sequentially compactNested nonempty sets have nonempty intersection: Take . If finite then done; else it has limit point, which is in the intersection.     - Take countable subcover; let . If for any , there is by above. Then , contradiction. * If is a family of compact subsets of X such that the intersection of every finite subcollection is nonempty, then . * (Nested Intervals Theorem) If is a nested sequence of intervals (), then . If the length of the intervals goes to 0, then the intersection consists of a single point. * If is a nested sequence of k-cells (closed boxes in ), then . * Every k-cell is compact.   + *Pf.* Suppose there’s an open cover without a finite subcover. Find a nested sequence of k-cells whose dimensions go to 0, such that the cells can’t be covered by a finite subcollection of . Some point x is in . It’s in an open set in which is contained in for n large enough, contradiction.   Heine-Borel Theorem: For a subset , the following properties are equivalent:   1. E is compact (every cover has a finite subcover). 2. E is closed and bounded.   *Cor.* Every bounded infinite subset of has a limit point in . |
| 2-5 | Perfect Sets Any nonempty perfect set in is uncountable. Thus every interval is uncountable.  The Cantor set: Let . Once is defined, write it as a disjoint union of intervals in the form , and replace each with to form . The **Cantor set** is . C is a (uncountable) perfect, compact set containing no segment. The Cantor set consists of all numbers whose ternary expansion consists only of the digits 0 and 2 (an infinite string of 2s being allowed). |
| 2-6 | Connected Sets Two subsets of a metric space are **separated** if . A subset E is **disconnected** if it is a union of two nonempty separated sets, and **connected** otherwise.  Equivalent condition (see below): E is disconnected if there exist disjoint nonempty open so that .  The union of sets in is connected if every distinct pair of sets in are not separated.  For , the union of all connected subsets containing x is the **connected component** of X containing x. The connected components form a partition of E, and they are all closed sets.  If X is a metric space with finitely many components, then the components are both closed and open (clopen). Conversely, any clopen set is a union of components of X. In particular, if X is connected, the only clopen sets are X and .  In a **totally disconnected set**, all connected components are point sets.  *Ex.* and the Cantor set C |

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| 3 | Sequences and Series |
| 3-1 | Sequences and Convergence Let be a **sequence** of points in a metric space X. The sequence **converges** to a point if for every there exists such that for every , . Else it **diverges**.   * converges to p if every open set containing p contains for all but finitely many . (This is the definition of convergence in a topological space.) * If converges then it converges to a unique , denoted by . * If converges then it is bounded. * If , is a contact point of E iff there exists a sequence such that .   A **Cauchy sequence** is a sequence such that for every , there is an integer so that for all . In other words, letting and defining , .  Cauchy Criterion: Every convergent sequence is Cauchy.  A sequence is **monotonically increasing, decreasing** if , , respectively. A monotonically increasing, decreasing sequence is convergent iff it is bounded above, below, respectively.  Basic properties (for ): Suppose .   * , * Squeeze Theorem: If and then .   The same properties and definitions hold if are replace with functions defined on reals, letting the variable range over the reals.  A **subsequence** of is in the form , where are positive integers. The limits of subsequences are called subsequential limits.   * If is a sequence in a compact metric space X, then some subsequence of converges to a point of X. In particular, every bounded subsequence of contains a convergent subsequence. * The subsequential limits for a closed subset.   Césaro-Stolz Lemma: Let be two sequences of real numbers and suppose either of the following holds.   1. and is decreasing for sufficiently large . 2. and is increasing for sufficiently large .   Then , provided the latter limit exists. |
| 3-2 | Lim inf and Lim sup Given a sequence , let . Let and . Define  Properties:   * Any sequence in has a monotonic subsequence converging to , (allowing ). * Let S be the set of subsequential limit points (including ). Then |
| 3-3 | Construction of the Reals 2: Cauchy Sequences  1. Identify the real numbers with equivalence classes of Cauchy sequences of rational numbers. Two sequences are equivalent if . Each rational number is associated with its constant sequence. 2. Define addition and multiplication as termwise addition and multiplication, and show it is well-defined. 3. For the multiplicative inverse, take the reciprocal of all terms, except those that are 0. (Sequence is eventually nonzero.) 4. Structure of ordered field: A real number is positive (greater than 0) if the sequence is eventually positive. if is positive. Check the order axioms. 5. is Archimedean: We can find so the terms of a given positive are eventually at least . 6. is dense in : follows from construction. 7. has the LUB property: Construct real sequences of upper and lower bounds of so that (you can make it halve each time). approach the same real number, the sup. 8. is complete: funness. |
| 3-4 | Completion In a **complete** metric space, every convergent sequence is Cauchy.   * Every compact metric space is complete. * Any Euclidean space is complete.   Each metric space has a completion :   1. The elements of are equivalence classes of Cauchy sequences in . Two sequences are equivalent if . Each is associated with a constant sequence. 2. Define distance by . 3. is complete: funness. 4. is dense in and if is complete. |
| 3-5 | Infinite Series The partial sums of are . Define  The sum converges if this limit exists; else it diverges. Infinite products are defined similarly.  Convergence/ Divergence Tests:   * Divergence Theorem: If converges then . * A series of nonnegative terms converges iff its partial sums form a bounded sequence. * Basic Comparison Test: Suppose for all .   + (Convergence) If converges then so does .   + (Divergence) If diverges then so does . * Limit Comparison Test: Let be eventually positive sequences. If is finite and nonzero, then and either both converge or both diverge. * Ratio Test: Let be a sequence of nonzero terms.   + If then converges.   + If then diverges. * Root Test: Let .   + If then converges.   + If then diverges. If for infinitely many distinct values of then diverges. * Cauchy’s Condensation Criterion: Suppose is nonincreasing. converges iff converges. * P-Test: converges iff . * Absolute Convergence: If converges then converges. It is said to **converge absolutely**. * Alternating Series (Leibniz) Test: If for all and , then converges. If the series converges but does not converge absolutely, it converges **conditionally**.   + Alternating Series Approximation Theorem: Suppose satisfies the conditions above. Then the th partial sum approximates the infinite series with an error of at most : * Kummer’s Test: Let be positive sequences. Suppose diverges and let . Then converges if and diverges if for all .   For products:  Coriolis Test: If and converge, then so does . (Pf. Take ln and use Taylor expansion.)  Let be a bijection. Then is a rearrangement of .   * If is absolutely convergent to then any rearrangement is also absolutely convergent to . * If is conditionally convergent then for every , there exists a rearrangement that is conditionally convergent to .[[3]](#footnote-3)   + Pf. Break up into a positive and negative sequence. Add terms from the positive sequence until sum overshoots B, add terms from the negative sequence until sum below B, and repeat. |
| 3-6 | Power Series A **power series** is in the form . There exists (possibly ), called the **radius of convergence**, so that   1. converges for all complex . 2. diverges for all complex .   A **Laurent series** is in the form . |

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| 4 | Limits and Continuity |
| 4-1 | Limits Let X and Y be metric spaces, , and be a limit point of . Let be a function.  The **limit** of at is if for every there exists such that for every with we have .  Note that does not matter (it need not exist).  Equivalently, for any sequence such that and , we have . (This allows basic properties of sequences to carry over as below.)  Infinite limits: (Definitions with are similar.)   * if for every there exists such that whenever . * if for every there exists such that whenever .   The limit is unique if it is defined, and satisfies the following:  Suppose .   * , * Squeeze Theorem: If for all in a neighborhood of except possibly at , and , then . |
| 4-2 | Continuity Let . is **continuous** at if for every there exists such that for every such that , we have . is continuous iff either is an isolated point of E or .  Equivalently, for every sequence converging to , converges to .  is continuous if is continuous at every point .  If and are continuous then is continuous.  (One topological definition) Let be any function between topological spaces. is continuous iff for any open set , is open.  Pf.   1. : For every open set U, can find neighborhood of in U. By continuity some neighborhood of is in . 2. : Take U equal to neighborhood of of radius . is open and contains ; some neighborhood of is in .   is continuous for every closed set , is closed in . (Use .  A **homeomorphism** is a continuous bijective function such that is continuous.  Basic properties:   * If are continuous , then (if ) are continuous. * Let . is continuous in iff are continuous. * is continuous.   *Ex.* Any polynomial is continuous. Any rational function is continuous except at points where the denominator is 0. |
| 4-3 | Compactness and Uniform Continuity Let be a continuous map of metric spaces, where is compact. Then is also compact.  *Pf.* For an open cover of , take the inverse of each subset. They’re open since is continuous; choose a finite subcover and take the image.  *Cor.* Any continuous map from compact is bounded.  *Cor.* (Weierstrass) A continuous function attains its maximum and minimum.  If X is compact and is continuous and bijective, then is also continuous.  *Pf.* Take open . is closed and hence compact (since X is compact). Then (since is bijective) is compact. Hence is open.  A function is **uniformly continuous** if for every there exists (independent of ) such that for every with .  Heine-Cantor Theorem: A continuous function on a compact metric space is uniformly continuous.  *Pf.* Suppose else. For every , we can find so but . By compactness, some subsequence converges; then converges to the same point . They get arbitrarily close to but , contradicting continuity at .  If is continuous and is connected then is also connected.  *Pf.* If are open sets whose union is , then their inverses under would be open sets whose union is .  Intermediate Value Theorem:   1. Let be continuous on a connected metric space. If then there exists so that . 2. If is continuous, then has the **intermediate value property**: If then there exists so that .   X is **pathwise connected** if for any there exists a continuous function so that . Any pathwise connected set is connected.  *Pf.* If X is a disjoint union of nonempty open sets, take , let connect them. Take of ; we get that is disconnected, contradiction. (Topo)  Counterexample to converse: Topologist’s sine curve is connected but not pathwise connected. |
| 4-4 | Discontinuities One-sided limits   * if for every there exists such that for every with we have . * if for every there exists such that for every with we have .   Discontinuity of the first kind:   1. Jump discontinuity: 2. Removable discontinuity: The function can be redefined at to make it continuous.   Discontinuity of the second kind: Everything else.  Ex.   1. Dirichlet’s function 2. Riemann’s function is continuous at irrational points, and has removable discontinuities at rational points. ( is the denominator of x in lowest terms) 3. has a discontinuity of the second kind at .   Monotonic functions  Increasing:  Strictly increasing:  Decreasing:  Strictly decreasing:  Monotonic: Increasing or decreasing   * If is increasing then and exist for all and  Moreover, for all , . Reverse inequalities for decreasing. * The only discontinuities of a monotonous function are jump discontinuities. * The set of discontinuities points of a monotonic function are at most countable. *Pf.* For each discontinuity point , associate it with a rational number in .   Given an at most countable set , there exists a monotonic function such that has discontinuities exactly at S.  A: Take any positive convergent series; define . |

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| 5 | Differentiation |
| 5-1 | Derivative The **derivative** of at is  Left and right-sided derivatives are defined with left and right-sided limits. If exists is **differentiable**.  Generalizes to vector-valued functions.  If is differentiable at then is continuous at .  *Pf.* Multiply derivative by .  Rules:   1. (Pf. add and subtract .) 2. (Prove for and use product rule.)   Tells us how to differentiate polynomials and rational functions.  A **local maximum (minimum)** of is a point such that there exists such that () for all with .  If is defined on and has a local maximum or minimum at , and exists, then .  Mean Value Theorem: If are continuous real functions on which are differentiable in , then there exists a point such that  In particular, there exists a point at which  When this is called Rolle’s Theorem.  *Pf.* Let . Then , need to find so that . Take the point where attains maximum or minimum.  is increasing if , constant if , and decreasing if .  If is differentiable on then satisfies the Intermediate Value Theorem, and cannot have any simple discontinuities.  L’Hospital’s Rule:  Suppose are real and differentiable in , for all , and one of the following holds:  Then  if the latter limit is defined. (This rule may need to be used multiple times, and is true for .)  Higher derivatives:  denotes the set of functions (or ) with continuous th derivatives.  denotes the set of functions with derivatives of all orders.  For vector-valued functions: Suppose is continuous and differentiable in . There exists so that  *Pf.* Project onto line connecting with . |
| 5-2 | Taylor and Power Series Power series (and Laurent series) are continuous in the open ball of convergence.  *Pf.* If where is the radius of convergence, then is uniformly continuous on .  Factor out from each term in and use Triangle Inequality and Root Test.  If has derivatives of all orders at , the **Taylor series** of around is  Taylor’s Theorem: Suppose is a real function on , n is a positive integer, is continuous on , exists for every . Let be distinct points of , and let  Then there exists a point between and such that  *Pf.* Let , so . Let . Then . Need so that . for and . By induction and the Mean Value Theorem, there exists such that .  *Remarks:* For this is the Mean Value Theorem. Useful when there is a convenient upper bound for on . |

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| 6 | Riemann Integration |
| 6-1 | Riemann-Stiltjes Integral A partition of is a finite collection of points . Define .  Let be bounded on , , .   1. The lower integral sum is . The lower integral is 2. The upper integral sum is . The upper integral is   (Exist when bounded.)  is **Riemann integrable** on if the lower and upper integral sums are equal. Then  (If the integral is 0.)  More general context: Let be a monotonically increasing function on , and let . Define sums and integrals () similarly with replaced by . The integral is called the **Riemann-Stieltjes integral**. The set of Riemann-Stieltjes integrable functions with respect to is denoted .  Useful in probability- random variables. If is the distribution function, is the expected value of .  The integral of a vector (or complex) valued function is taken componentwise (real and imaginary parts separately). |
|  | Integrability A refinement of a partition obtained from adding a set of division points. Any two partitions have a common refinement.  Preliminary Results:   * If is a refinement of then * . * iff for every there exists a partition P such that (\*) .   + If (\*) holds, then it holds for any refinement of P.   + If (\*) holds for and then .   + If (\*) holds and then .   Main Results:   * Let be continuous on . Then is Riemann-Stieltjes integrable for any . Proof:   + is uniformly continuous. Given choose , take partition so that all intervals are shorter than . Then (\*) holds (choosing depending on small enough). * If is monotonic and is continuous then . Proof:   + By IVT and continuity of , we can choose a partition so that . Choose large enough to make (\*) hold. * If is bounded with only finitely many discontinuity points and is continuous at all these points, then .   + Take very small intervals around discontinuity points: Surround the discontinuity points by nonoverlapping intervals where changes by less than , where . Delete the intervals (and endpoints if they are discontinuity points). is uniformly continuous on the resulting set (union of closed intervals), take a partition with intervals of length at most , including . Upper bound depends on and (for the intervals ), and can be made small. (Sum consists of two parts.)   + Counterexample when not continuous: . does not exist. * A set has **measure 0** if for all there exists a countable collection of open intervals such that and . is Riemann integrable on iff the set of discontinuity points of has measure 0.   + Let be intervals of total length at most covering discontinuity points. Let V be the union of “bad balls,” those where .   + Lemma: There exists such that if , , , then , i.e. any small interval with large variation must be contained in V. Proof: Else take a sequence that violate the lemma, for . By sequential compactness, take a simultaneously convergent subseqence; it must be in since is closed, but must also be in .   + Break Riemann sum into two parts: the intervals in V (bounded by ) and others (bounded by ).   + Any subset of a set of measure 0 has measure 0.   + A closed interval doesn’t have measure 0. (Use compactness) Sets of measure 0 can’t contain an interval.   + Any countable set (ex. ) has measure 0. (Choose sequence of lengths to make sum converge to arbitrarily small number.)   + Cantor set has measure 0.   + A countable union of sets of measure 0 has measure 0.   + Baire Category Theorem: is not a countable union of nowhere dense sets. (A set S is nowhere dense if does not contain an interval.) I.e. intervals are of the second category.   + Nowhere dense set of measure >0: Like Cantor set but remove intervals whose sum of lengths is <1. * Let on , , and let be a continuous function. Then on .   + is uniformly continuous, so can choose so . Choose so . Let . Divide indices into 2 classes.     - : Bound this part of this sum by .     - : Bound this part by . |
| 6-3 | Properties  1. Linearity: . 2. . 3. If then . 4. . 5. For .   More integrability: (Use composition theorem to prove.)   1. If then . 2. If then and .   *Ex.* Let . iff is right continuous at 0. Then .  Let be a convergent nonnegative series. Let . For any continuous, .  Assume is Riemann integrable on and is bounded on . Then iff , and if so, . “.”  *Pf.* By the Mean Value Theorem, there exists such that . The upper (lower) sums of the two integrals can be made arbitrarily close; the upper and lower integrals are equal (use refinement).  Change of variable: Let be a strictly increasing continuous function. If on then and .  *Pf.* A partition of induces a partition of .  *Cor.* If is differentiable and in , then . (If is strictly monotone then .)  Integration and differentiation.  Fundamental Theorem of Calculus: Let on . For define . Then is continuous on and it is continuous at , then is differentiable at and .  *Pf.* is Lipschitz with constant so F is uniformly continuous. Using continuity of , choose from ; using integral bounds the difference is at most . Take limit.  Integration by Parts: If are differentiable on , then  *Pf.* is integrable.  Assuming the integrals are defined, for (or ), . (Use Cauchy-Schwarz.) |
|  | Rectifiable Curves A **curve** in is a continuous function . If is injective it is an **arc**; if it is closed.  Let when . Define  The curve is rectifiable if is finite.  *Ex.* Nonrectifiable curve- Koch snowflake.  If is continuous on , then is rectifiable, and .  *Pf.* By FTC, . Summing, is rectifiable.  Using uniform continuity of , take a partition with distances less than ; bound the error by . (There are 2 parts to the error, sum of. |

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| 7 | Sequences of Functions |
| 7-1 | Uniform Convergence A sequence of functions converges to () if for all . In general, pointwise convergence does not preserve limits (continuity), derivatives, or integrals. Convergence for series of functions is defined similarly.  **converges uniformly** to (X a complete metric space) if for every there exists so that for every and , .  Cauchy Criterion for Uniform Convergence: is uniformly convergent iff for all there exists such that for every .  *Pf.* Choose for and use triangle inequality.  Weierstrass M-Test for Uniform Convergence:  Suppose pointwise, and let . Then uniformly iff as .  *Cor.* Let , . If is convergent then converges uniformly.  *Pf.* Use Cauchy criterion on difference of partial sums.  Suppose converges uniformly to . Let be a limit point of (subset of metric space). Then  *Pf.* Let . Choose and so that   1. for all (use uniform continuity). 2. . 3. for .   Then .  *Cor.* If is continuous and uniformly on , then is continuous on .  Suppose is a continuous real-valued functions on a compact set , , and pointwise. Then uniformly.  *Pf.* Consider . As closed subsets of K they are compact. By monotonicity, . Since , one of the sets, and all subsequent sets, are empty.  Let be the space of bounded continuous functions. For define and . Then with this metric iff uniformly on X.  is complete because if is Cauchy, then it is uniformly convergent. Hence it is continuous (and bounded).  Note continuity is not important. The space of continuous functions on compact is a complete metric space.  Integration: Suppose is a monotonically increasing on , , and uniformly. Then and .  *Pf.* Let . Then ; both sides go to ; integrate.  uniformly does not imply .  Suppose are differentiable, converges uniformly, and converges for some . Then is uniformly continuous to some function , and .  *Pf.* Choose so that for , and . Use MVT for on to get difference at most . Using Triangle Inequality and Cauchy criterion, uniformly, continuous.  Let . Then is uniformly convergent to when so by exchanging limits .  Everywhere continuous but nowhere differentiable function:  Weierstrass:  Or: Let have period 2. . is nowhere differentiable because of increasing oscillations, but continuous by the M-Test (oscillations on smaller scale). In the difference quotient choose (direction so that don’t hit cusps→ linear at this scale scale); becomes a finite sum but the difference quotient increases as increases ( is large; the smaller ones don’t cancel out).  Holder ½  (Differentiable functions have Baire category 2 in continuous functions.)  Let X be compact (so continuous functions are bounded) and be the space of continuous functions with the metric . is complete: is complete so a Cauchy sequence converges uniformly. Since are continuous, their limit is continuous.  Heine-Borel fails: Take to be a function with a spike of height 1 at and 0 elsewhere. is closed but the functions are all distance 1 from each other. |
| 7-2 | Equicontinuity A family of functions is **equicontinuous** if for every there exists such that for all so that . For a finite collection, this is equivalent to all elements being uniformly continuous.  is **uniformly bounded** if there exists so that for all and .  Suppose X is compact and is uniformly convergent in . Then is equicontinuous. (Holds if X is not compact but are uniformly continuous.  *Pf.* uniformly Cauchy. Choose for and then choose for .  Arzela-Ascolli Theorem: If is compact and is a pointwise bounded equicontinuous sequence in then has a uniformly convergent subsequence.  (Separable X implies existence of pointwise convergent subsequence.)  A closed and bounded equicontinuous family of functions C(X) is compact.  *Pf.*   1. Pointwise bounded implies uniformly bounded: Choose for equicontinuity for , the -neighborhoods form an open cover; take a finite subcover and take . 2. Take a countable dense subset . is bounded so has a convergent subsequence . Given , take a convergent subsequence . is row i. Take the diagonal ; by -argument, converges. 3. For , choose for equicontinuity for for . covers ; take a finite subcover . For each take so for . Take . For this , compare to to show . 4. A closed, bounded, and equicontinuous family in is sequentially compact so it is compact.   Application:  Show the existence of the solution to a differential equation. Solution to is the minimizer of given by . Restricting to a compact set of , if is continuous there must be a minimum. |
| 7-3 | Approximation Theorems An algebra of functions is a set of functions closed under addition, multiplication, and scalar multiplication. is self-adjoint if implies .  The **uniform closure** of is the set of limits of uniform convergent sequences in ; i.e. the closure of in the uniform metric. If is its own uniform closure, then is uniformly closed.  Weierstrass Approximation Theorem: Let be a compact interval in , and let be continuous. Then there exists a sequence of polynomials such that on . I.e. the uniform closure of the set of polynomials on is .  *Pf.* WLOG and . Set .   1. Choose so that . . 2. converges to 0 uniformly on for . The polynomials “squish” to 0 and become higher at 0.   Let . Let (a convolution). Let . Pick so that . Then . Split into  *Cor*. There exists a sequence of polynomials such that and uniformly on .  Stone-Weierstrass Theorem: Let be a compact metric space, and ( is a subalgebra of the set of continuous function from to ). Suppose that …   * Separates points: for every there exist such that . * Does not vanish at any point: there does not exist such that for all .   Then is dense in ; i.e. the uniform closure of is the set of all continuous functions.  If is a self-adjoint algebra of complex functions that separates points and does not vanish at any point, then is dense in .  *Pf.*   1. For every there exists such that , . 2. . Let . Take so that on . Then on . 3. . . 4. For , there exists such that and . From (1) take so that , . There exists an open set so that . Take a finite subcover ; take . 5. For each there exists containing so that . Take a finite subcover and let . 6. For complex: Use .   *Corollary*: Functions can be uniformly approximated by trigonometric polynomials (linear combinations of ). Any complex continuous function on the unit circle can be uniformly approximated by Laurent polynomials. |

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| 8 | Power Series |
| 8-1 | Analytic Functions A function on is **analytic** if it is representable as the sum of a convergent power series .  A power series with radius of convergence is uniformly convergent on for all and  By induction, ; i.e. all derivatives exist.  *Pf.* converges uniformly by the Root Test.  An analytic function is determined completely by all its derivatives at 0, in particular by values of in for any . We can define an **analytic continuation**.  Suppose is a convergent series. Let . Then .  *Pf.* .  *Cor.* Suppose , . Then .  This is true if A or B converges absolutely. Else, let , , . Then for ; take .  Inversion of order of sums: If converges, then .  *Pf.* Take ; let . uniformly on so we can exchange double sums.  Taylor’s Theorem: Suppose that converges for . If then can be expanded into a power series about which converges for , and .  *Pf.* . Change order of sum (legal since gives absolute convergence by applying Binomial Theorem backwards): converges since . The series must be its Taylor series.  Suppose and converge for , and let . If is not discrete in (i.e. has a limit point in ) then . Let be the set of all limit points of in and . is closed so is open. However is open: Expanding near , we get either as for some in a neighborhood of , or in a neighborhood of is internal point of .  Since is both open and closed, either or discrete. |

1. The definitions of limit, etc. extend to numbers in is we let the neighborhoods of be all sets of the form , and similarly for . [↑](#footnote-ref-1)
2. Here countable means finite or having same cardinality as . [↑](#footnote-ref-2)
3. For complex number sequences, the set of possible sums is a point, a line, or the whole plane. (This is difficult.) [↑](#footnote-ref-3)