Metric embeddings and geometric inequalities

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Introduction

The topic of this course is geometric inequalities with applications to metric embeddings; and we are actually going to do things more general than metric embeddings: Metric geometry. Today I will roughly explain what I want to cover and hopefully start proving a first major theorem. The strategy for this course is to teach novel work. Sometimes topics will be covered in textbooks, but a lot of these things will be a few weeks old. There are also some things which may not have been even written yet. I want to give you a taste for what's going on in the field. Notes from the course I taught last spring are also available.

One of my main guidelines in choosing topics will be topics that have many accessible open questions. I will mention open questions as we go along. I'm going to really choose topics that have proofs which are entirely self-contained. I'm trying to assume nothing. My intention is to make it completely clear and there should be no steps you don't understand.

Now this is a huge area. I will explain some of the background today. I'm going to use proofs of some major theorems as excuses to see the lemmas that go into the proofs. Some of the theorems are very famous and major, and you'er going to see some improvements, but along the way, we will see some lemmas which are **immensely powerful**. So we will always be proving a concrete theorem. But actually somewhere along the way, there are lemmas which have wide applicability to many many areas. These are excuses to discuss methods, though the theorems are important.

The course can go in many directions: If some of you have some interests, we can always change the direction of the course, so express your interests as we go along.

Chapter 1

The Ribe Program

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1 The Ribe Program

The main motivation for most of what we will discuss is called the Ribe Program, which is a research program many hundreds of papers large. We will see some snapshots of it, and it all comes from a theorem from 1975, **Ribe's rigidity theorem** 1.1.9, which we will state now and prove later in a modern way. This theorem was Martin Ribe's dissertation, which started a whole direction of mathematics, but after he wrote his dissertation he left mathematics. He's apparently a government official in Sweden. The theorem is in the context of Banach spaces; a relation between their linear structure and their structure as metric spaces. Now for some terminology.

Definition 1.1.1: Banach space.
A Banach space is a complete, normed vector space. Therefore, a Banach space is equipped with a metric which defines vector length and distances between vectors. It is complete, so every Cauchy sequence of converges to a limit defined inside the space.

Definition 1.1.2: Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. We say that X is (crudely) **finitely representable** in Y if there exists some constant $K > 0$ such that for every finite-dimensional linear subspace $F \subseteq X$, there is a linear operator $S : F \to Y$ such that for every $x \in F$,

$$\|x\|_X \leq \|Sx\|_Y \leq K \|x\|_X .$$

Note K is decided once and for all, before the subspace F is chosen.

(Some authors use "finitely representable" to mean that this is true for any $K = 1 + \varepsilon$. We will not follow this terminology.)

Finite representability is important because it allows us to conclude that X has the same finite dimensional linear properties (**local properties**) as Y. That is, it preserves any invariant involves finitely many vectors, their lengths, etc.

Let's introduce some local properties like type. To motivate the definition, consider the triangle inequality, which says

$$\|y_1 + \cdots + y_n\|_Y \le \|y_1\|_Y + \cdots + \|y_n\|_Y .$$

In what sense can we improve the triangle inequality? In L^1 this is the best you can say. In many spaces there are ways to improve it if you think about it correctly.

For any choice $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$,

$$\left\| \sum_{i=1}^n \varepsilon_i y_i \right\|_Y \le \sum_{i=1}^n \|y_i\|_Y .$$

Definition 1.1.3: df:type Say that X has **type** p if there exists $T > 0$ such that for every $n, y_1, \ldots, y_n \in Y$,

$$\mathop{\mathbb{E}}_{\varepsilon \in \{\pm 1\}^n} \left\| \sum_{i=1}^n \varepsilon_i y_i \right\|_Y \le T \left[\sum_{i=1}^n \|y_j\|_Y^p \right]^{\frac{1}{p}} .$$

The L^p norm is always at most the L^1 norm; if the lengths are spread out, this is asymptotically much better. Say Y has **nontrivial type** if $p > 1$.

For example, $L_p(\mu)$ has type $\min(p, 2)$.

Later we'll see a version of "type" for metric spaces. How far is the triangle inequality from being an equality is a common theme in many questions. In the case of normed spaces, this controls a lot of the geometry. Proving a result for $p > 1$ is hugely important.

Proposition 1.1.4: pr:finrep-type If X is finitely representable and Y has type p then also X has type p.

Proof. Let $x_1, \ldots, x_n \in X$. Let $F = \operatorname{span}\{x_1, \ldots, x_n\}$. Finite representability gives me $S : F \to Y$. Let $y_i = Sx_i$. What can we say about $\sum \varepsilon_i y_i$?

$$\begin{aligned}
\mathop{\mathbb{E}}_{\varepsilon} \left\| \sum_{i=1}^n \varepsilon_i y_i \right\|_Y &= \mathop{\mathbb{E}}_{\varepsilon} \left\| S(\sum_{i=1}^n \varepsilon_i x_i) \right\|_Y \\
&\ge \mathop{\mathbb{E}}_{\varepsilon} \left\| \sum_{i=1}^n \varepsilon_i X_i \right\|_X \\
\mathop{\mathbb{E}}_{\varepsilon} \left\| \sum_{i=1}^n \varepsilon_i y_i \right\|_Y &\le T \left(\sum_{i=1}^n \|Sx_i\|^p \right)^{\frac{1}{p}} \\
&\le TK \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} .
\end{aligned}$$

Putting these two inequalities together gives the result. □

Theorem 1.1.5 (Kahane's inequality)**.** *For any normed space Y and $q \ge 1$, for all n, $y_1, \ldots, y_n \in Y$,*

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i y_i\right\| \gtrsim_q \left(\mathbb{E}\left[\left\|\sum_{i=1}^n \varepsilon_i y_i\right\|_Y^q\right]\right)^{\frac{1}{q}}.$$

Here $\gtrsim_q$ means "up to a constant"; subscripts say what the constant depends on. The constant here does not depend on the norm Y.

Kahane's Theorem tells us that the LHS of Definition 1.1.3 can be replaced by any norm, if we change $\le$ to $\lesssim$. We get that having type p is equivalent to

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i y_i\right\|_Y^p \lesssim T^p \sum_{i=1}^n \|y_i\|_Y^p.$$

Recall the **parallelogram identity** in a Hilbert space H:

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i y_i\right\|^2 = \sum_{i=1}^n \|y_i\|_H^2.$$

A different way to understand the inequality in the definition of "type" is: how far is a given norm from being an Euclidean norm? The **Jordan-von Neumann Theorem** says that if parallelogram identity holds then it's a Euclidean space. What happes if we turn it in an inequality?

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i y_i\right\|_H^2 \begin{matrix}\ge\\ \le\end{matrix} T \sum_{i=1}^n \|y_i\|_H^2.$$

Either inequality *still* characterizes a Euclidean space.

What happens if we add constants or change the power? We recover the definition for type and cotype (which has the inequality going the other way):

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i y_i\right\|_H^q \begin{matrix}\gtrsim\\ \lesssim\end{matrix} \sum_{i=1}^n \|y_i\|_H^q.$$

Definition 1.1.6: Say it has **cotype** q if

$$\sum_{i=1}^n \|y_i\|_Y^q \lesssim C^q \mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i y_i\right\|_Y^q$$

R. C. James invented the local theory of Banach spaces, the study of geometry that involves properties involving finitely many vectors ($\forall x_1, \ldots, x_n, P(x_1, \ldots, x_n)$ holds). As a counterexample, reflexivity cannot be characterized using finitely many vectors (this is a theorem).

Ribe discovered link between metric and linear spaces.

First, terminology.

Definition 1.1.7: Two Banach spaces are **uniformly homeomorphic** if there exists $f : X \to Y$ that is 1-1 and onto and f, f^{-1} are uniformly continuous.

Without the word "uniformly", if you think of the spaces as topological spaces, all of them are equivalent. Things become interesting when you quantify! "Uniformly" means you're controlling the quantity.

Theorem 1.1.8 (Kadec)**.** *Any two infinite-dimensional separable Banach spaces are homeomorphic.*

This is a amazing fact: these spaces are all topologically equivalent to Hilbert spaces!

Over time people people found more examples of Banach spaces that are homeomorphic but not uniformly homeomorphic. Ribe's rigidity theorem clarified a big chunk of what was happening.

Theorem 1.1.9 (Rigidity Theorem, Martin Ribe (1975))**.** thm:ribe *Suppose that X, Y are uniformly homeomorphic Banach spaces. Then X is finitely representable in Y and Y is finitely representable in X.*

For example, for L^p and L^q, for $p \neq q$ it's always that case that one is not finitely representable in the other, and hence by Ribe's Theorem, L^p, L^q are not uniformly homeomorphic. (When I write L_p, I mean $L_p(\mathbb{R})$.)

Theorem 1.1.10. *For every $p \geq 1, p \neq 2$, L_p and ℓ_p are finitely representable in each other, yet not uniformly homeomorphic.*

(Here ℓ_p is the sequence space.)

Exercise 1.1.11: Prove the first part of this theorem: L_p is finitely representable in ℓ_p.

Hint: approximate using step functions. You'll need to remember some measure theory.

When $p = 2$, L_p, ℓ_p are separable and isometric.

The theorem in various cases was proved by:

1. $p = 1$: Enflo
2. $1 < p < 2$: Bourgain
3. $p > 2$: Gorelik, applying the Brouwer fixed point theorem (topology)

Every linear property of a Banach signs which is local (type, cotype, etc.; involving summing, powers, etc.) is preserved under a general nonlinear deformation.

After Ribe's rigidity theorem, people wondered: can we reformulate the local theory of Banach spaces without mentioning anything about the linear structure? Ribe's rigidity theorem is more of an existence statement, we can't see anything about an explicit dictionary which maps statements about linear sums into statements about metric spaces. So people started to wonder whether we could reformulate the local theory of Banach spaces, but only looks at distances between pairs instead of summing things up. Local theory is one of the hugest subjects in analysis. If you could actually find a dictionary which takes one linear theorem at a time, and restate it with only distances, there is a huge potential here! Because the definition of type only involves distances between points, we can talk about a metric space's type or cotype. So maybe we can use the intution given by linear arguments, and then state things for metric spaces which often for very different reasons remain true from the linear domain. And then now maybe you can apply these arguments to graphs, or groups! We could be able to prove things about the much more general metric spaces. Thus, we end up applying theorems on linear spaces in situations with *a priori nothing* to do with linear spaces. This is massively powerful.

There are very crucial entries that are missing in the dictionary. We don't even now how to define many of the properties! This program has many interesting proofs. Some of the most interesting conjectures are how to define things!

Corollary 1.1.12. cor:uh-type *If X, Y are uniformly homeomorphic and if one of them is of type p, then the other does.*

This follows from Ribe's Theorem 1.1.9 and Proposition 1.1.4. Can we prove something like this theorem without using Ribe's Theorem 1.1.9? We want to reformulate the definition of type using only the distance, so this becomes self-evident.

Enflo had an amazing idea. Suppose X is a Banach space, $x_1, \dots, x_n \in X$. The type p inequality says

$$\text{eq:type-p}\,\mathbb{E}\left[\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^p\right] \lesssim_X \sum_{i=1}^n \|x_i\|^p . \tag{1.1}$$

Let's rewrite this in a silly way. Define $f : \{\pm 1\}^n \to X$ by

$$f(\varepsilon_1, \dots, \varepsilon_n) = \sum_{i=1}^n \varepsilon_i x_i.$$

Write $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. Multiplying by 2^n, we can write the inequality (1.1) as

$$\text{eq:type-gen}\,\mathbb{E}\left[\|f(\varepsilon) - f(-\varepsilon)\|^p\right] \lesssim_X \sum_{i=1}^n \mathbb{E}\left[\|f(\varepsilon) - f(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n)\|^p\right]. \tag{1.2}$$

This inequality just involves distances between points $f(\varepsilon)$, so it is the reformulation we seek.

Definition 1.1.13: df:enflo A metric space (X, d_X) has **Enflo type** p if there exists $T > 0$ such that for every n and every $f : \{\pm 1\}^n \to X$,

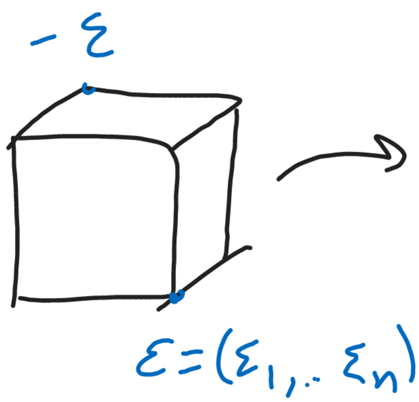
$$\mathbb{E}[d_X(f(\varepsilon), f(-\varepsilon))^p] \le T^p \sum_{i=1}^n \mathbb{E}[d_X(f(\varepsilon), f(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n))^p].$$

This is bold. It wasn't true before for a general function! The discrete cube (Boolean hypercube) $\{\pm 1\}^n$ is all the ϵ vectors, of which there are 2^n. Our function just assigns 2^n

points arbitrarily. No structure whatsoever. As they are indexed this way, you see nothing. But you're free to label them by vertices of the cube however you want. But there are many labelings! In (1.2), the points had to be vertices of a cube, but in Definition 1.1.13, they are arbitrary. The moment you choose the labelings, you impose a cube structure between the points. Some of them are diagonals of the cube, some of them are edges. ϵ and $-\epsilon$ are antipodal points. But it's not really a diagonal. They are points on a manifold, and are a function of how you decided to label them. What this sum says is that the sum over all diagonals, the length of the diagonals to the power p is less than the sum over edges to the p^{th} powers (these are the points where one ϵ_i is different). Thus we can see

$$\sum \text{diag}^p \lesssim_X \sum \text{edge}^p.$$

The total pth power of lengths of diagonals is up to a constant, at most the same thing over all edges.



This is a vast generalization of type; we don't even know a Banach space satisfies this. The following is one of my favorite conjectures.

Conjecture 1.1.14 (Enflo)**.** *If a Banach space has type p then it also has Enflo type p.*

This has been open for 40 years. We will prove the following.

Theorem 1.1.15 (Bourgain-Milman-Wolfson, Pisier)**.** *If X is a Banach space of type $p > 1$ then X also has type $p - \varepsilon$ for every $\varepsilon > 0$.*

If you know the type inequality for parallelograms, you get it for arbitrary sets of points, up to ε. Basically, you're getting arbitrarily close to p instead of getting the exact result. We also know that the conjecture stated before is true for a lot of specific Banach spaces, though we do not yet have the general result. For instance, this is true for the L_4 norm. Index functions by vertices; some pairs are edges, some are diagonals; then the L^4 norm of the diagonals is at most that of the edges.

How do you go from knowing this for a linear function to deducing this for an arbitrary function?

Once you do this, you have a new entry in the hidden Ribe dictionary. If X and Y are uniformly homeomorphic Banach spaces and Y has Enflo type p, then so is X. The minute you throw away the linear structure, Corollary 1.1.12 becomes easy. It requires a tiny argument. Now you can take a completely arbitrary function $f : \{\pm 1\}^n \to X$. There exists a homeomorphism $\psi : X \to Y$ such that ψ, ψ^{-1} are uniformly continuous. Now we want to deduce that the same inequality in Y gives the same inequality in X.

Proposition 1.1.16: pr:uh-enflo If X, Y are uniformly homeomorphic Banach spaces and Y has Enflo type p, then so does X.

This is an example of making the abstract Ribe theorem explicit.

Lemma 1.1.17 (Corson-Klee). *lem:corson-klee* *If X, Y are Banach spaces and $\psi : X \to Y$ are uniformly continuous, then for every $a > 0$ there exists $L(a)$ such that*

$$\|x_1 - x_2\|_X \ge a \implies \|\psi(x_1) - \psi(x_2)\| \le L \|x_1 - x_2\| .$$

Proof sketch of 1.1.16 given Lemma 1.1.17. By definition of uniformly homeomorphic, there exists a homeomorphism $\psi : X \to Y$ such that ψ, ψ^{-1} are uniformly continuous. Lemma 1.1.17 tells us that ψ perserves distance up to a constant. Dividing so that the smallest nonzero distance you see is at least 1, we get the same inequality in the image and the preimage. □

Proof details. Let ε^i denote ε with the ith coordinate flipped. We need to prove

$$\mathbb{E}(d_X(f(\varepsilon), f(-\varepsilon))^p) \le T_f^p \sum_{i=1}^{n} \mathbb{E}(d_X(f(\varepsilon), f(\varepsilon^i))^p)$$

Without loss of generality, by scaling f we may assume that all the points $f(\varepsilon)$ are distance at least 1 apart. (X is a Banach space, so distance scales linearly; this doesn't affect whether the inequality holds.)

Let $\psi : X \to Y$ be such that ψ, ψ^{-1} are uniform homeomorphisms. Because ψ^{-1} is uniformly homeomorphic, there is C such that $d_Y(y_1, y_2) \le 1$ implies $d_X(\psi^{-1}(y_1), \psi^{-1}(y_2)) < C$. WLOG, by scaling f we may assume that all the points $f(\varepsilon)$ are $\max(1, C)$ apart, so that the points $\psi \circ f(\varepsilon)$ are at least 1 apart.

We know that for any $g : \{\pm 1\}^n \to Y$ that

$$\mathbb{E}(d_X(g(\varepsilon), g(-\varepsilon))^p) \le T_g^p \sum_{i=1}^{n} \mathbb{E}(d_X(g(\varepsilon), g(\varepsilon^i))^p).$$

We apply this to $g = \psi \circ f$,

$$\begin{tikzcd}
& X \arrow[dd, "\psi"] \\
\{\pm 1\}^n \arrow[ur, "f"] \arrow[dr, "g=\psi\circ f"'] & \\
& Y
\end{tikzcd}$$

to get

$$\begin{aligned}
\mathbb{E}(d_X(f(\varepsilon), f(-\varepsilon))^p) &= \mathbb{E}(d_X(\psi^{-1}\circ g(\varepsilon), \psi^{-1}\circ g(-\varepsilon))^p)\\
&\le L_{\psi^{-1}}(1)\mathbb{E}(d_Y(g(\varepsilon), g(-\varepsilon))^p)\\
&\le L_{\psi^{-1}}(1)T_g^p\sum_{i=1}^n \mathbb{E}(d_Y(g(\varepsilon), g(\varepsilon^i)))\\
&\le L_{\psi^{-1}}(1)L_\psi(1)T_g^p\sum_{i=1}^n \mathbb{E}(d_X(g(\varepsilon), g(\varepsilon^i))^p)
\end{aligned}$$

as needed. □

The parallelogram inequality for exponent 1 instead of 2 follows from using the triangle inequality on all possible paths for all paths of diagonals. Type $p>1$ is a strengthening of the triangle inequality. For which metric spaces does it hold?

What's an example of a metric space where the inequality doesn't hold with $p>1$? The cube itself (with L^1 distance).

$$n^p \not\lesssim n.$$

I will prove to you that this is the only obstruction: given a metric space that doesn't contain bi-Lipschitz embeddings of arbitrary large cubes, the inequality holds.

We know an alternative inequality involving distance equivalent to type; I can prove it. It is, however, not a satisfactory solution to the Ribe program. There are other situations where we have complete success.

We will prove some things, then switch gears, slow down and discuss Grothendieck's inequality and applications. They will come up in the nonlinear theory later.

2 Bourgain's Theorem implies Ribe's Theorem

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We will use the Corson-Klee Lemma 1.1.17.

Proof of Lemma 1.1.17. Suppose $x, y \in X$, $\|x-y\| \ge a$. Break up the line segment from x, y into intervals of length a; let $x = x_0, x_1, \dots, x_k = y$ be the endpoints of those intervals, with

$$\|x_{i+1} - x_i\| \le a.$$

The **modulus of continuity** is defined as

$$W_f(t) = \sup\{\|f(u) - f(v)\| : u, v \in X, \|u - v\| \le t\}.$$

Uniform continuity says $\lim_{t\to 0} W_f(t) = 0$. The number of intervals is

$$k \le \frac{\|x-y\|}{a} + 1 \le \frac{2\|x-y\|}{a}.$$

Then

$$\begin{aligned}\|f(x) - f(y)\| &\le \sum_{i=1}^{k} \|f(x_i) - f(x_{i-1})\| \\ &\le KW_f(a) \le \frac{2W_f(a)}{a} \|x - y\|,\end{aligned}$$

so we can let $L(a) = \frac{2W_f(a)}{a}$. □

2.1 Bourgain's discretization theorem

There are 3 known proofs of Ribe's Theorem.

1. Ribe's original proof, 1987.

2. HK, 1990, a genuinely different proof.

3. Bourgain's proof, a Fourier analytic proof which gives a quantitative version. This is the version we'll prove.

Bourgain uses the Discretization Theorem 1.2.4. There is an amazing open problem in this context.

Saying δ is big says there is a not-too-fine net, which is enough. Therefore we are interested in lower bounds on δ.

Definition 1.2.1: Discretization modulus.
Let X be a finite-dimensional normed space $\dim(X) = n < \infty$. Let target space Y be an arbitrary Banach space. Consider the unit ball B_X in X. Take an δ-net (the distance between points is at most δ) $\mathcal{N}_\delta$ in B_X (a maximal δ-separated subset). Suppose we can embed $\mathcal{N}_\delta$ into Y via $f : \mathcal{N}_\delta \to Y$. Suppose we know in Y that

$$\|x - y\| \le \|f(x) - f(y)\| \le D \|x - y\|.$$

for all $x, y \in N_\delta$. (We say that $\mathcal{N}_\delta$ embeds with distortion D into Y.)

You can prove using a nice compactness argument that if this holds for δ is small enough, then the entire space X embeds into Y with rough the same distortion. Bourgain's discretization theorem 1.2.4 says that you can choose $\delta = \delta_n$ to be independent of the geometry of X and Y such that if you give a δ-approximation of the unit-ball in the n-dimensional norm, you succeed in embedding the whole space.

I often use this theorem in this way: I use continuous methods to show embedding X into Y requires big distortion; immediately I get an example with a finite object. Let us now make the notion of distortion more precise.

Definition 1.2.2: Distortion.
Suppose $(X, d_X), (Y, d_Y)$ are metric spaces $D \geq 1$. We say that X embeds into Y with **distortion** D if there exists $f : X \to Y$ and $s > 0$ such that for all $x, y \in X$,

$$Sd_X(x, y) \leq d_Y(f(x), f(y)) \leq DSd_X(x, y).$$

The infimum over those $D \geq 1$ such that X embeds into Y with distortion is denoted $C_Y(X)$. This is a measure of how far X is being from a subgeometry of Y.

Definition 1.2.3: Let be a n-dimensional normed space and Y be any Banach space, $\varepsilon \in (0, 1)$. Let $\delta_{X \hookrightarrow Y}(\varepsilon)$ be the supremum over all those $\delta > 0$ such that for every δ-net $\mathcal{N}_\delta$ in B_X,

$$C_Y(\mathcal{N}_\delta) \geq (1 - \varepsilon)C_Y(X).$$

Here $B_X := \{x \in X : \|x\| \leq 1\}$.

In other words, the distortion of the δ-net is not much larger than the distortion of the whole space. That is, the discrete δ-ball encodes almost all information about the space when it comes to embedding into Y: If you got $C_Y(\mathcal{N}_\delta)$ to be small, then the distortion of the entire object is not much larger.

Theorem 1.2.4 (Bourgain's discretization theorem)**.** *For every n, $\varepsilon \in (0, 1)$, for every X, Y, $\dim X = n$,*

$$\delta_{X \hookrightarrow Y}(\varepsilon) \geq e^{-\left(\frac{n}{\varepsilon}\right)^{Cn}}.$$

*Thus there is a delta which is just dependent on the dimension such that in any n-dim norm space if you look at the unit ball it encodes all the information of embedding X into **anything else**. It's only a function of the dimension, not of any of the relevant geometry. Moreover for $\delta = e^{-(2n)^{Cn}}$, we have $C_Y(X) \leq 2C_Y(\mathcal{N}_\delta)$ via a linear operator.*

The theorem says that if you look at a δ-net in the unit ball, it encodes all the information about X when it comes to embedding into everything else. The amount you have to discretize is just a function of the dimension, and not of any of the other relevant geometry.

Remark 1.2.5: The proof is via a linear operator. All the inequality says is that you can find a *function* with the given distortion. The proof will actually give a *linear operator*.

The best known upper bound is

$$\delta_{X \hookrightarrow Y}\left(\frac{1}{2}\right) \lesssim \frac{1}{n}.$$

The latest progress was 1987, there isn't a better bound yet. You have a month to think about it before you get corrupted by Bourgain's proof.

There is a better bound when the target space is a L^p space.

Theorem 1.2.6 (Gladi, Naor, Shechtman)**.** *For every $p \geq 1$, if $\dim X = n$,*

$$\delta_{X \hookrightarrow L_p}(\varepsilon) \gtrsim \frac{\varepsilon^2}{n^{\frac{5}{2}}}$$

(We still don't know what the right power is.) The case $p = 1$ is important for applications. There are larger classes for spaces where we can write down axioms for where this holds. There are crazy Banach spaces which don't belong to this class, so we're not done. We need more tools to show this: Lipschitz extension theorems, etc.

2.2 Bourgain's Theorem implies Ribe's Theorem

With the "moreover," Bourgain's theorem implies Ribe's Theorem 1.1.9.

Proof of Ribe's Theorem 1.1.9 from Bourgain's Theorem 1.2.4. Let X, Y be Banach spaces that are uniformly homeomorphic. By Corson-Klee 1.1.17, there exists $f : X \to Y$ such that

$$\|x - y\| \geq 1 \implies \|x - y\| \leq \|f(x) - f(y)\| \leq K \|x - y\| .$$

(Apply the Corson-Klee lemma for both f and the inverse.)

In particular, if $R > 1$ and $\mathcal{N}$ is a 1-net in

$$RB_X = \{x \in X : \|x\| \leq R\} ,$$

then $C_Y(\mathcal{N}) \leq K$. Equivalently, for every $\delta > 0$ every δ-net in B_X satisfies $C_Y(\mathcal{N}) \leq K$. If $F \subseteq X$ is a finite dimension subspace and $\delta = e^{-(2\dim F)^{C \dim F}}$, then by the "moreover" part of Bourgain's Theorem 1.2.4, there exists a linear operator $T : F \to Y$ such that

$$\|x - y\| \leq \|Tx - Ty\| \leq 2K \|x - y\|$$

for all $x, y \in F$. This means that X is finitely representable. □

The motivation for this program comes in passing from continuous to discrete. The theory has many applications, e.g. to computer science whcih cares about finite things. I would like an improvement in Bourgain's Theorem 1.2.4.

First we'll prove a theorem that has nothing to do with Ribe's Theorem. There are lemmas we will be using later. It's an easier theorem. It looks unrelated to metric theory, but the lemmas are relevant.

Chapter 2

Restricted invertibility principle

1 Restricted invertibility principle

1.1 The first restricted invertibility principles

We take basic facts in linear algebra and make things quantitative. This is the lesson of the course: when you make things quantitative, new mathematics appears.

Proposition 2.1.1: If $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear operator, then there exists a linear subspace $V \subseteq \mathbb{R}^n$ with $\dim(V) = \operatorname{rank}(A)$ such that $A : V \to A(V)$ is invertible.

What's the quantitative question we want to ask about this? Invertibility just says that an inverse exists. Can we find a large subspace where not only is A invertible, but the inverse has small norm?

We insist that the subspace is a coordinate subspace. Let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{R}^m$, $e_j = (0, \ldots, \underbrace{1}_{j}, 0, \ldots)$. The goal is to find a "large" subset $\sigma \subseteq \{1, \ldots, m\}$ such that A is invertible on $\mathbb{R}^\sigma$ where

$$\mathbb{R}^\sigma := \{(x_1, \ldots, x_n) \in \mathbb{R}^m : x_i = 0 \text{ if } i \notin \sigma\}$$

and the norm of $A^{-1} : A(\mathbb{R}^\sigma) \to \mathbb{R}^\sigma$ is small.

A priori this seems a crazy thing to do; take a small multiple of the identity. But we can find conditions that allow us to achieve this goal.

Theorem 2.1.2 (Bourgain-Tzafriri restricted invertibility principle, 1987)**.** thm:btrip *Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator such that*

$$\|Ae_j\|_2 = 1$$

for every $j \in \{1, \ldots, m\}$. Then there exist $\sigma \subseteq \{1, \ldots, m\}$ such that

1. *$|\sigma| \geq \frac{cm}{\|A\|^2}$, where $\|A\|$ is the operator norm of A.*

2. A is invertible on $\mathbb{R}^\sigma$ and the norm of $A^{-1} : A(\mathbb{R}^\sigma) \to \mathbb{R}^\sigma$ is at most C' (i.e., $\|AJ_\sigma\|_{S^\infty} \le C'$, to use the notation introduced below).

Here c, C' are universal constants.

Suppose the biggest eigenvalue is at most 100. Then you can always find a coordinate subset of proportional size such that on this subset, A is invertible and the inverse has norm bounded by a universal constant.

All of the proofs use something very amazing.

This proof is from 3 weeks ago. This has been reproved many times. I'll state a theorem that gives better bound than the entire history.

This was extended to rectangular matrices. (The extension is nontrivial.)

Given $V \subseteq \mathbb{R}^m$ a linear subspace with $\dim V = k$ and $A : V \to \mathbb{R}^m$ a linear operator, the singular values of A

$$s_1(A) \ge s_2(A) \ge \cdots \ge s_k(A)$$

are the eigenvalues of $(A^*A)^{\frac{1}{2}}$. We can decompose

$$A = UDV$$

where D is a matrix with $s_i(A)$'s on the diagonal, and U, V are unitary.

Definition 2.1.3: For $p \ge 1$ the **Schatten-von Neumann p-norm** of A is

$$\begin{aligned}
\|A\|_{S_p} &:= \left(\sum_{i=1}^k s_i(A)^p\right)^{\frac{1}{p}} \\
&= (\mathrm{Tr}((A^*A)^{\frac{p}{2}}))^{\frac{1}{p}} \\
&= (\mathrm{Tr}((AA^*)^{\frac{p}{2}}))^{\frac{1}{p}}
\end{aligned}$$

The cases $p = \infty, 2$ give the operator and Frobenius norm,

$$\begin{aligned}
\|A\|_{S_\infty} &= \text{operator norm} \\
\|A\|_{S_2} &= \sqrt{\mathrm{Tr}(A^*A)} = \left(\sum a_{ij}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Exercise 2.1.4: $\|\cdot\|_{S_p}$ is a norm on $\mathcal{M}_{n\times m}(\mathbb{R})$. You have to prove that given A, B,

$$(\mathrm{Tr}([(A+B)^*(A+B)]^{\frac{p}{2}}))^{\frac{1}{p}} \le (\mathrm{Tr}((A^*A)^{\frac{p}{2}}))^{\frac{1}{p}} + (\mathrm{Tr}((B^*B)^{\frac{p}{2}}))^{\frac{1}{p}}.$$

This requires an idea. Note if A, B commute this is trivial. Apparently von Neumann wrote a paper called "Metric Spaces" in the 1930s in which he just proves this inequality and doesn't know what to do with it, so it got forgotten for a while until the 1950s, when Schatten wrote books on applications. When I was a student in grad school, I was taking a class on random matrices. There was two weeks break, I was certain that it was trivial because the professor had not said it was not, and it completely ruined my break though I came up with a different proof of it. It's short, but not trivial: It's not typical linear algebra!. This is like another triangle inequality, which we may need later on.

Spielman and Srivastava have a beautiful theorem.

Definition 2.1.5: Stable rank.
Let $A : \mathbb{R}^m \to \mathbb{R}^n$. The **stable rank** is defined as

$$\operatorname{srank}(A) = \left(\frac{\|A\|_{S_2}}{\|A\|_{S_\infty}}\right)^2.$$

The numerator is the sum of squares of the singular values, and the denominator is the maximal value. Large stable rank means that many singular values are nonzero, and in fact large on average. Many people wanted to get the size of the subset in the Restricted Invertibility Principle to be close to the stable rank.

Theorem 2.1.6 (Spielman-Srivastava)**.** thm:ss *For every linear operator $A : \mathbb{R}^m \to \mathbb{R}^n$, $\varepsilon \in (0,1)$, there exists $\sigma \subseteq \{1, \ldots, m\}$ with $|\sigma| \ge (1-\varepsilon) srank(A)$ such that*

$$\left\|(AJ_\sigma)^{-1}\right\|_{S_\infty} \lesssim \frac{\sqrt{m}}{\varepsilon \|A\|_{S_2}}.$$

Here, J_σ is the identity function restricted to $\mathbb{R}^\sigma$, $J : \mathbb{R}^\sigma \hookrightarrow \mathbb{R}^m$.

This is stronger than Bourgain-Tzafriri. In Bourgain-Tzafriri the columns were unit vectors.

Proof of Theorem 2.1.2 from Theorem 2.1.6. Let A be as in Theorem 2.1.2. Then $\|A\|_{S_2} = \sqrt{\operatorname{Tr}(A^*A)} = \sqrt{m}$ and $\operatorname{srank}(A) = \frac{m}{\|A\|_{S_\infty}^2}$. We obtain the existence of

$$|\sigma| \ge (1-\varepsilon)\frac{m}{\|A\|_{S_\infty}^2}$$

with $\|(AJ_\sigma)^{-1}\|_{S_\infty} \lesssim \frac{\sqrt{m}}{=}\frac{1}{\varepsilon}$. □

This is a sharp dependence on ε.

The proof introduces algebraic rather than analytic methods; it was eye-opening. Marcus even got sets bigger than the stable rank and looked at $pf\|A\|_{S_2}\|A\|_{S_4}{}^2$, which is much stronger.

1.2 A general restricted invertibility principle

I'll show a theorem that implies all these intermediate theorems. We use (classical) analysis and geometry instead of algebra. What matters is not the ratio of the norms, but the tail of the distribution of $s_1(A)^2, \ldots, s_m(A)^2$.

Theorem 2.1.7. thm:gen-srank *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator. If $k < \operatorname{rank}(A)$ then there exist $\sigma \subseteq \{1, \ldots, m\}$ with $|\sigma| = k$ such that*

$$\left\|(AJ_\sigma)^{-1}\right\|_{S_\infty} \lesssim \min_{k<r\le \operatorname{rank}(A)} \sqrt{\frac{mr}{(r-k)\sum_{i=r}^m s_i(A)^2}}.$$

You have to optimize over r. You can get the ratio of L_p norms from the tail bounds. This implies all the known theorems in restricted invertibility. The subset can be as big as you want up to the rank, and we have sharp control in the entire range. This theorem generalizes Spielman-Srivasta (Theorem 2.1.6), which had generalized Bourgain-Tzafriri (Theorem 2.1.2). 2-8-16

Now we will go backwards a bit, and talk about a less general result. After Theorem 2.1.6, a subsequent theorem gave the same theorem but instead of the stable rank, used something better.

Theorem 2.1.8 (Marcus, Spielman, Srivastava)**.** thm:mss4 *If*

$$k < \frac{1}{4}\left(\frac{\|A\|_{S_2}}{\|A\|_{S_4}}\right)^4,$$

there exists $\sigma \subseteq \{1, \dots, m\}$, $|\sigma| = k$ such that

$$\left\|(AJ_\sigma)^{-1}\right\|_{S_\infty} \lesssim \frac{\sqrt{m}}{\|A\|_{S_2}}.$$

A lot of these quotients of norms started popping up in people's results. The correct generalization is the following notion.

Definition 2.1.9: For $p > 2$, define the **stable pth rank** by

$$\operatorname{srank}_p(A) = \left(\frac{\|A\|_{S_2}}{\|A\|_{S_p}}\right)^{\frac{2p}{p-2}}.$$

Exercise 2.1.10: Show that if $p \geq q > 2$, then

$$\operatorname{srank}_p(A) \leq \operatorname{srank}_q(A).$$

(Hint: Use Hölder's inequality.)

Now we would like to prove how Theorem 2.1.7 generalizes the previously listed results:

Proof of Generalizability of Theorem 2.1.7. Using Hölder's inequality with $\frac{p}{2}$,

$$\begin{aligned}
\|A\|_{S_2}^2 &= \sum_{j=1}^m s_j(A)^2 \\
&= \sum_{j=1}^{r-1} s_j(A)^2 + \sum_{j=r}^m s_j(A)^2 \\
&\leq (r-1)^{1-\frac{2}{p}} \left(\sum_{j=1}^{r-1} s_j(A)^p\right)^{\frac{2}{p}} + \sum_{j=r}^m s_j(A)^2
\end{aligned}$$

$$\le (r-1)^{1-\frac{2}{p}} \|A\|_{S_p}^2 + \sum_{j=r}^m s_j(A)^2$$

$$\sum_{j=r}^m s_j(A)^2 \ge \|A\|_{S_2}^2 \left(1 - (r-1)^{-\frac{2}{p}} \frac{\|A\|_{S_p}^2}{\|A\|_{S_2}^2}\right)$$
$$= \|A\|_{S_2}^2 \left(1 - \left(\frac{r-1}{\text{srank}_p(A)}\right)^{1-\frac{2}{p}}\right)$$

Now we can plug the previous calculation into Theorem 2.1.7 to demonstrate the way the new theorem generalizes the previous results:

$$\left\|(AJ_\sigma)^{-1}\right\| \lesssim \min_{k+1\le r\le \text{rank}(A)} \sqrt{\frac{mr}{(r-k)\|A\|_{S_2}^2 \left(1 - \left(\frac{r-1}{\text{srank}_p(A)}\right)^{1-\frac{2}{p}}\right)}}$$
$$= \frac{\sqrt{m}}{\|A\|_\infty} \min_{k+1\le r\le \text{rank}(A)} \sqrt{\frac{r}{(r-k)\left(1 - \left(\frac{r-1}{\text{srank}_p(A)}\right)^{1-\frac{2}{p}}\right)}}$$

This equation implies all the earlier theorems. □

To optimize, fix the stable rank, differentiate in r, and set to 0. All theorems in the literature follow from this theorem; in particular, we get all the bounds we got before. There was nothing special about the number 4 in Theorem 2.1.8; this is about the distribution of the eigenvalues.

1.3 Ky Fan maximum principle

We'll be doing linear algebra. It's mostly mechanical, except we'll need this lemma.

Lemma 2.1.11 (Ky Fan maximum principle)**.** *Suppose that $P : \mathbb{R}^m \to \mathbb{R}^m$ is a rank k orthogonal projection. Then*

$$Tr(A^*AP) \le \sum_{i=1}^k s_i(A)^2.$$

Proof. This proof isn't complete. I will fix it next time.

We will prove that if $B : \mathbb{R}^m \to \mathbb{R}^m$ is positive semidefinite, then

$$\text{Tr}(BP) \le \sum_{i=1}^k s_i(B).$$

To get the lemma, set $B = A^*A$.

Apply arbitrarily small perturbation s_i so that

$$s_1(B) > s_2(B) > \cdots > s_m(B).$$

Let $v_1, \dots, v_m$ be an orthonormal basis such that $Bv_i = s_i(B)v_i$. Let $u_1, \dots, u_k$ be an orthonormal basis of $P\mathbb{R}^m$ ordered so that

$$\langle Bu_1, u\rangle \ge \langle Bu_2, u_2\rangle \ge \cdots \ge \langle Bu_k, u_k\rangle .$$

We calculate

$$\begin{aligned}\mathrm{Tr}(BP) &= \sum_{i=1}^{k} \langle BPu_i, u_i\rangle \\ &= \sum_{i=1}^{k} \langle Bu_i, u_i\rangle .\end{aligned}$$

We will prove by induction on i that

$$\langle Bu_i, u_i\rangle \le s_i(B).$$

For $i = 1$,

$$s_1(B) = \|B\|_{S_\infty},$$

and (when $\|u_1\| = 1$)

$$\langle Bu_1, u_1\rangle \le \|B\|_{S_\infty} = s_1(B).$$

Suppose we proved the claim for $j-1$. If $?\langle Bu_i, u_i\rangle \le s_j(B)$ $\langle Bu_{j-1}, u_{j-1}\rangle \le s_j(B)$ then we're done because $\langle Bu_j, u_j\rangle \le \langle Bu_{j-1}, u_{j-1}\rangle \le s_j(B)$. So we may assume that $\langle Bu_{j-1}, u_{j-1}\rangle > s_j(B)$.

Write the u's in the basis of v's:

$$u_i = \sum_{l=1}^{m} c_{il} v_l.$$

The fact that the u_i's are orthonormal means that the c_i's are probability vectors,

$$\sum_{l=1}^{m} c_{il}^2 = 1.$$

We have

$$\langle Bu_i, u_i\rangle = \sum_{l=1}^{m} m c_{il}^2 s_l(B).$$

If $c_{il}^2 > 0$ for any $l \ge j$. □

1.4 Finding big subsets

We'll present 4 lemmas for finding big subsets with certain properties. We'll put them together at the end.

Theorem 2.1.12 (Little Grothendieck inequality). thm:lgi *Fix $k, m, n \in \mathbb{N}$. Suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear operator. Then for every $x_1, \ldots, x_k \in \mathbb{R}^m$,*

$$\sum_{r=1}^{k} \|Tx_r\|_2^2 \le \frac{\pi}{2} \|T\|_{\ell_\infty^m \to \ell_2^n}^2 \sum_{r=1}^{k} x_{ri}^2$$

for some $i \in \{1, \ldots, m\}$ where $x_r = (x_{r1}, \ldots, x_{rm})$.

Later we will show $\frac{\pi}{2}$ is sharp.

If we had only 1 vector, what does this say?

$$\|Tx_1\|_2 \le \sqrt{\frac{\pi}{2}} \|T\|_{\ell_\infty^m \to \ell_2^n} \|x_1\|_\infty$$

We know the inequality is true for $k = 1$ with constant 1, by definition of the operator norm. The theorem is true for arbitrary many vectors, losing an universal constant $(\frac{\pi}{2})$. After we see the proof, the example where $\frac{\pi}{2}$ is attained will be natural.

We give Grothendieck's original proof.

The key claim is the following.

Claim 2.1.13. clm:lgi

$$\text{eq:lgi1} \sum_{j=1}^{m} \left(\sum_{r=1}^{k} (T^*Tx_r)_j^2 \right)^{\frac{1}{2}} \le \sqrt{\frac{\pi}{2}} \|T\|_{\ell_\infty^m \to \ell_2^n} \left(\sum_{r=1}^{k} \|Tx_r\|^2 \right)^{\frac{1}{2}}. \tag{2.1}$$

Proof of Theorem 2.1.12. Assuming the claim, we prove the theorem.

$$\begin{aligned}
\sum_{r=1}^{k} \|Tx_r\|_2^2 &= \sum_{r=1}^{k} \langle Tx_r, Tx_r \rangle \\
&= \sum_{r=1}^{k} \langle x_r, T^*Tx_r \rangle \\
&= \sum_{r=1}^{k} \sum_{j=1}^{m} x_{rj} (T^*Tx_r)_j \\
&\le \sum_{j=1}^{m} \left(\sum_{r=1}^{k} x_{rj}^2 \right)^{\frac{1}{2}} \left(\sum_{r=1}^{k} (T^*Tx_r)_j^2 \right)^{\frac{1}{2}} && \text{by Cauchy-Schwarz} \\
&\le \left(\max_{1 \le j \le m} \left(\sum_{r=1}^{k} x_{rj}^2 \right)^{\frac{1}{2}} \right) \left(\sum_{j=1}^{m} \sum_{r=1}^{k} (T^*Tx_r)_j^2 \right)^{\frac{1}{2}} \\
&\le \max_{1 \le j \le m} \left(\sum_{i=1}^{k} x_{ij}^2 \right)^{\frac{1}{2}} \sqrt{\frac{\pi}{2}} \|T\|_{\ell_\infty^m \to \ell_2^n} \left(\sum_{r=1}^{k} \|Tx_r\|_2^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$\sum_{r=1}^{k} \|Tx_r\|_2^2 \le \frac{\pi}{2} \|T\|_{\ell_\infty^m \to \ell_2^n}^2 \max_j \sum_{r=1}^{k} x_{ij}^2.$$

We bounded by a square root of the multiple of the same term, a bootstrapping argument. In the last step, divide and square. □

Proof of Claim 2.1.13. Let $g_1, \ldots, g_k$ be iid standard Gaussian random variables. For every fixed $j \in \{1, \ldots, m\}$,

$$\sum_{r=1}^{k} g_r (T^* T x_r)_j.$$

This is a Gaussian random variable with mean 0 and variance $\sum_{r=1}^{k} (T^* T x_r)_j^2$. Taking the expectation,[1]

$$\mathbb{E}\left|\sum_{r=1}^{k} g_r (T^* T x_r)_j\right| = \left(\sum_{r=1}^{k} (T^* T x_r)_j^2\right)^{\frac{1}{2}} \sqrt{\frac{2}{\pi}}.$$

Sum these over j:

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{m}\left|T^*(\sum_{r=1}^{k} g_r T x_r)_j\right|\right] &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{m}\left(\sum_{r=1}^{k}(T^*Tx_r)_j^2\right)^{\frac{1}{2}} \\ \sum_{j=1}^{m}\left(\sum_{r=1}^{k}(T^*Tx_r)_j^2\right)^{\frac{1}{2}} &= \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\sum_{j=1}^{m}\left|T^*\sum_{r=1}^{k} g_r (T x_r)_j\right|\right]. \text{eq:lgi2} \qquad (2.2)\end{aligned}$$

Define a random sign vector $z \in \{\pm 1\}^m$ by

$$z_j = \operatorname{sign}\left(\left(T^* \sum_{r=1}^{k} g_r T x_r\right)_j\right)$$

Then

$$\begin{aligned}\sum_{j=1}^{m}\left|(T^*\sum_{r=1}^{k} g T x_r)_j\right| &= \left\langle z, T^*\sum_{r=1}^{k} g_r T x_r\right\rangle \\ &= \left\langle Tz, \sum_{r=1}^{k} g_r T x_r\right\rangle \\ &\le \|Tz\|_2 \left\|\sum_{r=1}^{k} g_r T x_r\right\|_2 \\ &\le \|T\|_{\ell_\infty^m \to \ell_2^n}\left\|\sum_{r=1}^{k} g_r T x_r\right\|_2\end{aligned}$$

This is a pointwise inequality. Taking expectations and using Cauchy-Schwarz,

$$\text{eq:lgi3}\mathbb{E}\left[\sum_{j=1}^{m}\left|\left(T^*\sum_{r=1}^{k} g_r T x_r\right)_j\right|\right] \le \|T\|_{\ell_\infty^m \to \ell_2^n}\left(\mathbb{E}\left\|\sum_{r=1}^{k} g_r T x_r\right\|_2^2\right)^{\frac{1}{2}}. \qquad (2.3)$$

[1] $\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}|x|e^{-\frac{x^2}{2}} = -2\sqrt{\frac{1}{2\pi}}[e^{-\frac{x^2}{2}}]_0^\infty = \sqrt{\frac{2}{\pi}}$

What is the second moment? Expand:

$$\text{eq:lgi4}\mathbb{E}\left\|\sum_{r=1}^{k} g_i T x_r\right\|_2^2 = \mathbb{E}\left[\sum_{ij} g_i g_j \langle Tx_i, Tx_j\rangle\right] = \sum_{r=1}^{k} \|Tx_r\|_2^2 . \tag{2.4}$$

Chaining together (2.2), (2.3), (2.4) gives the result. □

Why use the Gaussians? The identity characterizes the Gaussians using rotation invariance. Using other random variables gives other constants that are not sharp.

There will be lots of geometric lemmas:

- A fact about restricting matrices.
- Another geometric argument to give a different method for selecting subsets.
- A combinatorial lemma for selecting subsets.

Finally we'll put them together in a crazy induction.

From this proof you can reverse engineer vectors that make the inequality sharp. You need to come up with T and the points.

Example 2.1.14: Let $g_1, g_2, \ldots, g_k$ be iid Gaussians on the probability space (Ω, P). Let $T : L_\infty(\Omega, P) \to \ell_2^k$ be

$$Tf = (\mathbb{E}[fg_1], \ldots, \mathbb{E}[fg_k]).$$

Let $x_r \in L_\infty(\Omega, P)$,

$$x_r = \frac{g_r}{\left(\sum_{i=1}^k g_i^2\right)^{\frac{1}{2}}}.$$

2-10-16

We were in the process of proving three or four subset selection principles, which we will somehow use to prove the RIP.

Now I owe you a proof (just ask me for the linear algebra proof) - I'll show you an analytic proof.

We proved the little Grothendieck inequality (Theorem **??**), which is part of an amazing area of mathematics with many applications. It's little, but it's also very useful. Just to remind you, we had an linear operator $T : \mathbb{R}^m \to \mathbb{R}^n$. Then for every $x_1, \cdots, x_k \in \mathbb{R}^m$, we get a bounded operator. If you look at the sum of the Euclidean lengths $\left(\sum_{i=1}^k \|Tx\|_2^2\right)^{1/2} \le \sqrt{\pi/2}\|T\|_{l_\infty^m \to l_2^n} \cdot \max_{1\le j\le m}\left(\sum_{i=1}^k x_{rj}^2\right)^{1/2}$. This is really the way Grothendieck did it, but the proof we saw is really the original proof, re-organized. For completeness, we'll show the fact that this inequality is sharp (cannot be improved).

Corollary 2.1.15. *$\sqrt{\pi/2}$ is the best constant in Theorem* **??**.

Proof. Define $g_1, \cdots, g_k$ be i.i.d standard Gaussians, defined on probability space (Ω, P). We define $T : L_\infty(\Omega, \mathbb{P}) \to \mathbb{R}^k$. Then $Tf = (\mathbb{E}(fg_1), \mathbb{E}(fg_2), \cdots, \mathbb{E}(fg_k))$. Choose $X_r = \frac{g_r}{\left(\sum_i^k g_i^2\right)^{1/2}}$. This is nothing more than a vector on the k-dimensional unit sphere. So it's a bounded function. We also note that x_r is a function on the measure space Ω. We can also write

$$\sum_{r=1}^k x_r(\omega)^2 = \sum_{r=1}^k \frac{g_r(\omega)^2}{\sum_{i=1}^r g_i(\omega)^2} = 1$$

We can use the Central Limit Theorem to make things precise: $g_r \approx \frac{\epsilon_{r_1} + \cdots + \epsilon_{r_N}}{\sqrt{N}}$ as $N \to \infty$. So all these statements will be asymptotically true. Where does the family of random variables $\{g_r\}$ live in Ω? Well $\Omega = \{\pm 1\}^{NK}$. So $L_\infty(\Omega) = l_\infty^{2^{NK}}$, which is some huge dimension, but it's still finite. So ω will really be a coordinate in Ω.

Now we show two things; nothing more than computations.

1. $\|T\|_{L_\infty(\Omega, \mathbf{P}) \to l_2^k} = \sqrt{2/\pi}$,
2. We also show $\sum_{r=1}^k \|Tx_r\|_2^2 \to^{k \to \infty} 1$.

First we tackle the first case. We have

$$\begin{aligned}
\|T\|_{\infty \to 1} &= \sup_{\|f\|_\infty \le 1} \left(\sum_{r=1}^k \mathbb{E}\left[fg_r\right]^2 \right)^{1/2} \\
&= \sup_{\|f\|_\infty \le 1} \sup_{\sum_{r=1}^k \alpha_r^2 = 1} \sum_{r=1} \alpha_r \mathbb{E}\left[fg_r\right] \\
&= \sup_{\sum_{r=1}^k} \sup_{\|f\|_\infty \le 1} \mathbb{E}\left[f \sum_{i=1}^k \alpha_r g_r \right] \\
&= \sup_{\sum_{r=1}^k} \mathbb{E}\left| \sum_{r=1}^k \alpha_r g_r \right| = \mathbb{E}|g_1| = \sqrt{\frac{2}{\pi}}
\end{aligned} \tag{2.5}$$

as we claimed. Now we tackle the second computation:

$$\begin{aligned}
\sum_{r=1}^k \|Tx_r\|_2^2 &= \sum_{r=1}^k \left(\mathbb{E}\left[\frac{g_r^2}{\left(\sum_{i=1}^k g_i^2\right)^{1/2}} \right] \right)^2 \\
&= K \left(\mathbb{E}\left[\frac{g_1^2}{\left(\sum_{i=1}^k g_i^2\right)^{1/2}} \right] \right) \\
&= K \left(\frac{1}{K} \mathbb{E}\left[\sum_{r=1}^k \frac{g_r^2}{\left(\sum g_i^2\right)^{1/2}} \right] \right)^2 \\
&= \frac{1}{K} \left(\mathbb{E}\left[\left(\sum_{i=1}^k g_i^2 \right)^{1/2} \right] \right)^2
\end{aligned} \tag{2.6}$$

and you can use Stirling to finish. This is just a χ^2-distribution.

In this case $\mathbb{E}\frac{g_1 g_2}{\left(\sum_i g_i^2\right)^{1/2}} = \mathbb{E}\frac{g_1(-g_2)}{\left(\sum g_i^2\right)^{1/2}}$. Also note that if $(g_1, \cdots, g_k) \in \mathbb{R}^k$ is a standard Gaussian, then $\frac{(g_1,\cdots,g_k)}{\left(\sum_{i=1}^k g_i^2\right)^{1/2}}$ and $\left(\sum_{i=1}^k g_i^2)^{1/2}\right)$ are independent. In other words, the length and angle are independent: This is just polar coordinates, you can check this. □

Now, how does this relate to the Restricted Invertibility Problem?

Theorem 2.1.16. *Pietsch Domination Theorem.* thm:pdt
Fix $m, n \in \mathbb{N}$ and $M > 0$. Suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear operator such that for every $x_1, \cdots, x_k \in \mathbb{R}^m$ have

$$\left(\sum_{r=1}^k \|Tx_r\|_2^2\right)^{1/2} \le M max_{1\le j\le m}\left(\sum_{r=1}^k x_{rj}^2\right)^{1/2}$$

Then there exist $\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{R}^m$ with $\mu_1 \ge 0$ and $\sum_{i=1}^m = 1$ such that for every $x \in \mathbb{R}^m$

$$\|Tx\|_2 \le M\left(\sum_{i=1}^M \mu_i x_i^2\right)^{1/2}$$

It's really an iff: The latter is a stronger statement than the former, and in fact they are equivalent. You can come out with a probability measure, a way to weight the coordinates, such that the norm of T as an operator as a standard norm from l_∞ to l_2, is bounded by M.

Proof. Define $K \subseteq \mathbb{R}^m$ with

$$K = \left\{y \in \mathbb{R}^m : y_i = \sum_{r=1}^k \|Tx_r\|_2^2 - M^2\sum_{r=1}^m x_{ri}^2 \text{ for some } k, x_1, \cdots, x_k \in \mathbb{R}^m\right\}$$

Basically we cleverly select a convex set. Every n-tuple of vectors in $\mathbb{R}^m$ gives you a new vector in $\mathbb{R}^m$. Let's check that K is convex. We have to check if two vectors $y, z \in K$ have all points on the line between them in K. $y \in K$ means that

$$y_i = \sum_{r=1}^k \|Tx_r\|_2^2 - M^2\sum_{r=1}^m x_{ri}^2$$

$$z_i = \sum_{r=1}^l \|Tw_r\|_2^2 - M^2\sum_{r=1}^l w_{ri}^2$$

for all i. So what can you say about the average $\frac{y_i+z_i}{2}$? It comes from $\left(\frac{x_1}{\sqrt{2}}, \cdots, \frac{x_k}{\sqrt{2}}, \frac{w_1}{\sqrt{2}}, \cdots, \frac{w_l}{\sqrt{2}}\right)$. So trivially by design this is a convex set.

Now, the assumption of the theorem says that

$$\left(\sum_{r=1}^k \|Tx_r\|_2^2\right)^{1/2} \le M\max_{1\le j\le m}\left(\sum_{r=1}^k x_{rj}^2\right)^{1/2}$$

which implies

$$\|Tx_r\|_2^2 - M^2 \max_{1\le j\le m} \sum_{r=1}^m x_{rj}^2 \le 0$$

which implies $K \cap (0,\infty)^m = \emptyset$. By the hyperplane separation theorem (for two disjoint convex sets in $\mathbb{R}^m$ with at least one compact, there is a hyperplane between them), there exists $0 \ne \mu = (\mu_1, \cdots, \mu_m) \in \mathbb{R}^m$. We have

$$\langle \mu, y\rangle \le \langle \mu, z\rangle$$

for all $y \in K$ and $z \in (0,\infty)^m$. By renormalizing, $\sum_{i=1}^m = 1$. Moreover μ cannot have any strictly negative coordinate: Otherwise you could take z to have arbitrarily large value at a strictly negative coordinate with zeros everywhere else, implying $\langle u, z\rangle$ is no longer bounded from below, a contradiction. Therefore, μ is a probability vector and $\langle \mu, z\rangle$ can be arbitrarily small. So for every $y \in K$, $\sum_{i=1}^m \mu_i y_i \le 0$. Then $y_i = \|Tx\|_2^2 - M^2 x_i \in K$, and if you write this out, $\|Tx\|_2^2 - M^2 \sum_{i=1}^n \mu_i y_i \le 0$, which is exactly what we wanted. □

Lemma 2.1.17. lem:projbound *$m, n \in \mathbb{N}$, $\epsilon \in (0,1)$, $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear operator. Then $\exists \sigma \subset \{1, \cdots, m\}$ with $|\sigma| \ge (1-\epsilon)m$ such that*

$$\|Proj_{\mathbb{R}^\sigma} T\|_{S_\infty} \le \sqrt{\frac{\pi}{2\epsilon m}} \|T\|_{l_2^n \to l_1^m}$$

We will find ways to restrict a matrix to a big big submatrix, but we won't be able to control its operator norm, but we will be able to control the norm from $l_2^n \to l_1^m$. So then you go to a further subset, which this becomes an operator norm on, which is an improvement which Grothendieck gave us. This is the first very useful tool to start finding big sub matrices.

Proof. We have $T : l_2^n \to l_1^m$, $T^* : l_\infty^m \to l_2^n$. Now some abstract nonsense gives us that for Banach spaces, the norm of an operator and its adjoint are equal, i.e. $\|T\|_{l_2^n \to l_1^m} = \|T^*\|_{l_\infty^m \to l_2^n}$. This statement follows from Hahn-Banach theorem (come see me if you haven't seen this before, I'll tell you what book to read). From the Little Grothendieck inequality (Theorem 2.1.12), T^* satisfies the assumption of the Pietsch domination theorem with $M = \sqrt{\frac{\pi}{2}} \|T\|_{l_2^n \to l_1^m}$ (we're applying it to T^*). So we have probability vector $(\mu_1, \cdots, \mu_m)$ such that for every $y \in \mathbb{R}^m$

$$\|T^* y\|_2 = M \left(\sum_{i=1}^m \mu_i y_i^2 \right)^{1/2}$$

with $M = \sqrt{\frac{\pi}{2}} \|T\|_{l_2^n \to l_1^m}$. Define $\sigma = \left\{ i \in \{1, \cdots, m\} : \mu_i \le \frac{1}{m\epsilon} \right\}$, then $|\sigma| \ge (1-\epsilon)m$ by Markov's inequality. We can also see this by writing

$$1 = \sum \mu_i = \sum_{i\in\sigma} \mu_i + \sum_{i \notin \sigma} \mu_i > \sum_{i \in \sigma} \mu_i + \frac{m - |\sigma|}{m\epsilon}$$

which follows since μ_j for $j \notin \sigma$ has $\mu_j > \frac{1}{m\epsilon}$. Continuing,

$$\frac{m\epsilon - m + |\sigma|}{m\epsilon} \ge \sum_{i\in\sigma} \mu_i$$

$$|\sigma| \geq (m\epsilon)\sum_{i\in\sigma}\mu_i + m(1-\epsilon)$$

Then, since $(m\epsilon)\sum_{i\in\sigma}\mu_i \geq 0$ since μ is a probability distribution, we have

$$|\sigma| \geq m(1-\epsilon)$$

Now take $x \in \mathbb{R}^n$ and choose $y \in \mathbb{R}^m$ with $\|y\|_2 = 1$. Then

$$\langle y, \text{Proj}_{\mathbb{R}^\sigma} Tx\rangle^2 = \|\text{Proj}_{\mathbb{R}^\sigma} Tx\|_2^2 \leq \langle T^*\text{Proj}_{\mathbb{R}^\sigma} y, x\rangle^2 \leq \|T^*\text{Proj}_{\mathbb{R}^\sigma} y\|_2^2 \cdot \|x\|_2^2$$

$$\leq \frac{\pi}{2}\|T\|_{l_2^n\to l_1^m}\left(\sum_{i\in\sigma}\mu_i y_i^2\right)\|x\|_2^2 \leq \frac{\pi}{2}\|T\|_{l_2^n\to l_1^m}^2 \frac{1}{m\epsilon}\|x\|_2^2$$

by Cauchy-Schwarz. Then, taking square roots gives the desired result. □

In the previous proof, we used a lot of duality to get an interesting subset.

Remark 2.1.18: In Lemma 2.1.17, I think that either the constant $\pi/2$ is sharp (no subset are bigger; it could come from the Gaussians), or there is a different constant here. If the constant is 1, I think you can optimize the previous argument and get the constant to be arbitrarily close to 1, which would have some nice applications: In other words, getting $\sqrt{\frac{\pi}{2\epsilon m}}$ as close to 1 as possible would be good. I didn't check before class, but you might want to check if you can carry out this argument using the Gaussian argument we made for the sharpness of $\frac{\pi}{2}$ in Grothendieck's inequality (Theorem **??**). It's also possible that there is a different universal constant.

Now we will give another lemma which is very easy and which we will use a lot.

Lemma 2.1.19. *Sauer-Shelah.* lem:saushel
Take integers $m, n \in \mathbb{N}$ and suppose that we have a large set $\Omega \subseteq \{\pm 1\}^n$ with

$$|\Omega| > \sum_{k=0}^{m-1}\binom{n}{k}$$

Then $\exists \sigma \subseteq \{1, \cdots, n\}$ such that with $|\sigma| = m$, if you project onto $\mathbb{R}^\sigma$ the set of vectors, you get the entire cube: $\text{Proj}_{\mathbb{R}^\sigma}(\Omega) = \{\pm 1\}^\sigma$. For every $\epsilon \in \{\pm 1\}^\sigma$, there are signs $\delta = (\delta_1, \cdots, \delta_n) \in \Omega$ such that $\delta_j = \epsilon_j$ for $j \in \sigma$.

Note that Lemma 2.1.19 is used in the proof of the Fundamental Theorem of Statistical Learning Theory.

Proof. We want to prove by induction on n. First denote the shattering set

$$\text{sh}(\Omega) = \{\sigma \subseteq \{1, \cdots, n\} : \text{Proj}_{\mathbb{R}^\sigma}\Omega = \{\pm 1\}^\sigma\}$$

The claim is that the number of sets shattered by a given set is $|\text{sh}(\Omega)| \geq |\Omega|$. The empty set case is trivial. What happens when $n = 1$? $\Omega \subset \{-1, 1\}$, and thus the set is shattered. Assume that our claim holds for n, and now set $\Omega \subseteq \{\pm 1\}^{n+1} = \{\pm 1\}^n \times \{\pm 1\}$. Define

$$\Omega_+ = \{\omega \in \{\pm 1\}^n : (\omega, 1) \in \Omega\}$$

$$\Omega_- = \{\omega \in \{\pm 1\}^n : (\omega, -1) \in \Omega\}$$

Then, letting $\tilde{\Omega}_+ = \{(\omega, 1) \in \{\pm 1\}^{n+1} : \omega \in \Omega_+\}$ and $\tilde{\Omega}_-$ similarly, we have $|\Omega| = |\tilde{\Omega}_+| + |\tilde{\Omega}_-| = |\Omega_+| + |\Omega_-|$. By our inductive step, we have $\text{sh}(\Omega_+) \geq |\Omega_+|$ and $\text{sh}(\Omega_-) \geq |\Omega_-|$. Note that any subset that shatters Ω_+ also shatters Ω, and likewise for Ω_-. Note that if a set Ω' shatters both of them, we are allowed to add on an extra coordinate to get $\Omega' \times \{\pm 1\}$ which shatters Ω. Therefore,

$$\text{sh}(\Omega_+) \cup \text{sh}(\Omega_-) \cup \{\sigma \cup \{n+1\} : \sigma \in \text{sh}(\Omega_+) \cap \text{sh}(\Omega_-)\} \subseteq \text{sh}(\Omega)$$

where the last union is disjoint since the dimensions are different. Therefore, we can now use this set inclusion to complete the induction using the principle of inclusion-exclusion:

$$\begin{aligned}
|\text{sh}(\Omega)| &\geq |\text{sh}(\Omega_+) \cup \text{sh}(\Omega_-)| + |\text{sh}(\Omega_+) \cap \text{sh}(\Omega_-)| \qquad \text{(disjoint sets)} \\
&= |\text{sh}(\Omega_+)| + |\text{sh}(\Omega_-)| - |\text{sh}(\Omega_+) \cap \text{sh}(\Omega_-)| + |\text{sh}(\Omega_+) \cap \text{sh}(\Omega_-)| \\
&= |\text{sh}(\Omega_+)| + |\text{sh}(\Omega_-)| \\
&\geq |\Omega_+| + |\Omega_-| = |\Omega|
\end{aligned}$$

which completes the induction as desired. □

Corollary 2.1.20. *If $|\Omega| \geq 2^{n-1}$ then there exists $\sigma \subseteq \{1, \cdots, n\}$ with $|\sigma| \geq \lceil \frac{n+1}{2} \rceil \geq \frac{n}{2}$ such that $\text{Proj}_{\mathbb{R}^\sigma} \Omega = \{\pm 1\}^\sigma$. If you have half of the points in terms of cardinality, you get half of the dimension: At least.*

We will primarily use the corollary.

2-15-16

Last time we left off with the proof of the Sauer-Shelah lemma. To remind you, we were finding ways to find interesting subsets where matrices behave well.

Now recall we had a linear algebraic fact which I owe you; I will prove it in an analytic way.

The lemma we will prove follows from the following general theorem: Link back to previous reference to the Ky-Fan maximum principle

Theorem 2.1.21. *Ky-Fan maximum principle.*
thm:eignmax Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for every orthonormal basis $u_1, \cdots, u_n \in \mathbb{R}^n$, there exists a permutation π such that

$$f(\langle Bu_1, u_1 \rangle, \cdots, \langle Bu_n, u_n \rangle) \leq f(\lambda\pi(1), \cdots, \lambda\pi(n))$$

Essentially, we're saying using the eigenbasis maximizes the convex function.

Proof. We need for every $i < j$, then $t \to f(x_1, \cdots, x_i + t, x_{i+1}, \cdots, x_{j-1}, x_j - t, x_{j+1}, \cdots, x_n)$ is convex as a function of t. Then if f is smooth, we will have

$$\frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial x_j^2} - 2\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$$

In other words, you just need that the Hessian is positive semidefinite (above, we wrote the determinant for all pairs x_i, x_j - the above condition is equivalent).

You may assume that f is smooth. Without loss of generality, we can give a strict inequality instead:

$$\frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial x_j^2} - 2\frac{\partial^2 f}{\partial x_i \partial x_j} > 0$$

since you can just take some $\epsilon > 0$ and perturb $f(x) + \epsilon \|x\|_2^2$. Then in the inequality in Theorem 2.1.21 is also perturbed by this slight change, and taking $\epsilon \to 0$ gives you the desired inequality.

Now let $u_1, \cdots, u_n$ be an orthonormal basis at which $f(\langle Bu_1, u_1\rangle, \cdots, \langle Bu_n, u_n\rangle)$ attains its maximum. Then for u_i, u_j, we want to rotate in the $i - j$ plane by angle θ. Since u_i, u_j span a two dimensional subspace, recall the 2-dimensional rotation matrix. Let

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}; u_{i;j} = \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

Multiplying, we get

$$R_\theta u_{i;j} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} \cos(\theta)u_i + \sin(\theta)u_j \\ \sin(\theta)u_i - \cos(\theta)u_j \end{bmatrix} = \begin{bmatrix} (R_\theta u_{i;j})_1 \\ (R_\theta u_{i;j})_2 \end{bmatrix}$$

Then, we replace f with $g(\theta) =$

$$f\left(\langle Bu_1, u_1\rangle, \cdots, \langle B\left(R_\theta u_{i;j}\right)_1, (R_\theta u_{i;j})_1\rangle, \langle B\left(R_\theta u_{i;j}\right)_2, (R_\theta u_{i;j})_2\rangle, \cdots, \langle Bu_n, u_n\rangle\right)$$

where we keep all other dot products the same. Then we can show $g'(0) = 0, g''(0) \leq 0$ since g attains its maximum at $\theta = 0$. Expanding out the rotated dot products explicitly in $g(\theta)$, we get

$$\cos^2(\theta)\langle Bu_i, u_i\rangle + \sin^2(\theta)\langle Bu_j, u_j\rangle + \sin(2\theta)\langle Bu_i, u_j\rangle$$

and

$$\sin^2(\theta)\langle Bu_i, u_i\rangle + \cos^2(\theta)\langle Bu_j, u_j\rangle - \sin(2\theta)\langle Bu_i, u_j\rangle$$

Then we can mechanically take the derivatives at Take derivatives with respect to what exactly? We get

$$0 = g'(0) = 2\langle Bu_i, u_j\rangle(f_{x_i} - f_{x_j})$$

$$0 \geq g''(0) = 2\left(\langle Bu_j, u_j\rangle - \langle Bu_i, u_i\rangle\right)\left(f_{x_i} - f_{x_j}\right) + 4\langle Bu_i, u_j\rangle^2\left(f_{x_i x_i} + f_{x_j x_j} - 2f_{x_i x_j}\right)$$

and this implies that $\langle Bu_i, u_j\rangle = 0$, which implies that for all i, $Bu_i = \mu_i \mu_i$. Thus any function applied to a vector of dot products is maximized at eigenvalues. $\square$

Example 2.1.22: $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions in Theorem **??** and $(u_1, \cdots, u_n)$, $v_1, \cdots, v_n)$ are two orthonormal bases of $\mathbb{R}^n$. Then for every $A : \mathbb{R}^n \to \mathbb{R}^n$, there exists $\pi \in S_n$, $(\epsilon_1, \cdots, \epsilon_n) \in \{\pm 1\}^n$ such that

$$f(\langle Au_1, v_1\rangle, \langle Au_2, v_2\rangle, \cdots, \langle Au_n, v_n\rangle) \leq f(\epsilon_1 S_{\pi(1)}(A), \cdots, \epsilon_n S_{\pi(n)}(A))$$

Then show that choosing u, v as the singular values maximizes f.

To solve this problem, you can rotate both in the same direction and take derivatives, and also rotate them in opposite directions and take derivatives to get enough information to prove that the singular values are the maximum.

Essentially, a lot of the inequalities you find in these books follow from this. For instance, if you want to prove that the Schatten p-norm is a norm, it follows directly from this fact.

Corollary 2.1.23. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ that is invariant under premutations and sign:*

$$\|(x_1, \cdots, x_n)\| = \|(\epsilon_1 x_{\pi(1)}, \cdots, \epsilon_n x_{\pi(n)})$$

for all $\epsilon \in \{\pm 1\}^n$ and $\pi \in S_n$ (In the literature, we call this a symmetric norm). This induces a norm on matrices $M_{m\times n}(\mathbb{R})$ with

$$\|A\| = \|(S_{\pi(1)}(A), \cdots, S_{\pi(n)}(A)\|)$$

Then for A, B matrices, we want to primarily show the triangle inequality:

$$\|A + B\| \leq \|A\| + \|B\|$$

$$A + B = \sqrt{(A+B)^*(A+B)} \cdot W$$

where W is orthogonal matrix (this is just polar decomposition), and

$$|A + B| = \sqrt{(A+B)^*(A+B)}$$

Let $u_1, \cdots, u_n$ be an orthonormal basis of $A + B$. Then,

$$\|A + B\| = \|\langle |A+B|u_1, u_1\rangle, \cdots, \langle |A+B|u_n, u_n\rangle\| = \|(\langle (A+B)u_i, Wu_i\rangle)_{i=1}^n\|$$

$$\leq \|(\langle (A+B)u_i, Wu_i\rangle)_{i=1}^n\| + \|(Bu_i, Wu_i)\| \leq \|A\| + \|B\|$$

So basically, the Schatten p-norm is a norm because it obeys rotational symmetries.

Remember this theorem! For many many results, you simply need to apply the right convex function to get the result.

Our lemma follows from setting $f(x) = \sum_{i=1}^k x_i$.

Lemma 2.1.24. *For every $A : \mathbb{R}^m \to \mathbb{R}^n$, and every orthogonal projection $P : \mathbb{R}^n \to \mathbb{R}$ of rank k,*

$$\mathrm{Tr}(A^*AP) \leq \sum_{i=1}^k s_i(A)^2$$

where the $s_i(A)$ are the square roots of the eigenvalues of $B = A^*A$.

Proof. Take an orthonormal basis of P $u_1, \cdots, u_n$ such that $u_1, \cdots, u_k$ is a basis of the range of P. Then

$$\mathrm{Tr}(BP) = \sum_{j=1}^{k} \langle Be_j, e_j \rangle \le \sum_{i=1}^{k} s_i(B) = \sum_{i=1}^{k} s_i(A)^2$$

□

Now we need another geometric lemma for the proof of change name of theorem gen-srank to restricted invertibility principle? fix reference Theorem **??**, the restricted invertibility principle.

Lemma 2.1.25. *Step* 1. lem:step1 *Fix $m, n, r \in \mathbb{N}$. Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator with rank$(A) \ge r$. For every $\tau \subseteq \{1, \cdots, m\}$, denote*

$$E_\tau = (\mathrm{span}((Ae_j)_{j \in \tau}))^\perp$$

Then there exists $\tau \subseteq \{1, \cdots, m\}$ with $|\tau| = r$ such that for all $j \in \tau$,

$$\|\mathrm{Proj}_{E_{\tau \setminus \{j\}}} Ae_j\|_2 \ge \frac{1}{\sqrt{m}} \left(\sum_{i=1}^{m} s_i(A)^2 \right)^{1/2}$$

Basically we're taking the projection of the j^{th} column onto the orthogonal completement of the span of the subspace of all columns in the set except for the j^{th} one, and bounding the norm of that by a dimension term and the square root of the sum of the eigenvalues. (this is sharp asymptotically, and may in fact even be sharp as written too - I need to check Check this?).

Proof. For every $\tau \subseteq \{1, \cdots, m\}$, denote

$$K_\tau = \mathrm{conv}\left(\{\pm Ae_j\}_{j \in \tau} \right)$$

Essentially, you want to make this convex hull have big volume, and once you do that, you will get all these inequalities for free. Let $\tau \subseteq \{1, \cdots, m\}$ be such that $\mathrm{vol}_r(K_\tau)$ is maximized among all subsets of size r. We know that $\mathrm{vol}_r(K_\tau) > 0$. Observe that for any $\beta \subseteq \{1, \cdots, m\}$ of size $r-1$ and $i \notin \beta$, we have

$$K_{\beta \cup \{i\}} = \mathrm{conv}\left(K_\beta \cup \{\pm Ae_i\} \right)$$

So all you're doing is having found E_β and a point Ae_i, for the convex hull you get a double cone $E_\beta, +Ae_i, -Ae_i$ where E_β is the orthogonal complement of the space spanned by β. So what is the height of this cone? This is just $\|\mathrm{Proj}_{E_\beta} Ae_i\|_2$. Therefore, the r-dimensional volume is given by

$$\mathrm{vol}_r(K_{\beta \cup \{i\}}) = 2 \cdot \frac{\mathrm{vol}_{r-1}(K_\beta) \cdot \|\mathrm{Proj}_{E_\beta} Ae_i\|_2}{r}$$

Then $|\tau| = r$ is the maximizing subset of F_Ω. Any $j \in \tau$ and $i \in \{1, \cdots, m\}$. Then choosing $\beta = \tau \setminus \{j\}$ (we're choosing the maximizer). Insert line comparing volumes for j and i here? Then we get that

$$\|\mathrm{Proj}_{E_{\tau \setminus \{j\}}} Ae_j\|_2 \ge \|\mathrm{Proj}_{E_{\tau \setminus \{j\}}} Ae_i\|_2$$

for every $j \in \tau$ and $i \in \{1, \cdots, m\}$. But then

$$\|\text{Proj}_{E_{\tau\setminus\{j\}}} A\|_{S_2}^2 = \sum_{i=1}^{m} \|\text{Proj}_{E_{\tau\setminus\{j\}}} Ae_i\|_2^2 \leq m\|\text{Proj}_{E_{\tau\setminus\{j\}}} Ae_j\|_2^2$$

Then, for all $j \in \tau$,

$$\|\text{Proj}_{E_{\tau\setminus\{j\}}} Ae_j\|_2 \geq \frac{1}{\sqrt{m}} \|\text{Proj}_{E_{\tau\setminus\{j\}}} A\|_{S_2}$$

Let's denote $P = \text{Proj}_{E_{\tau\setminus\{j\}}}$, where P is an orthogonal projection of rank $r - 1$. Then,

$$\begin{aligned}\|PA\|_{S_2}^2 &= \text{Tr}((PA)^*(PA)) = \text{Tr}(A^*P^*PA) = \text{Tr}(A^*PA) = \text{Tr}(AA^*P) \\ &= \text{Tr}(AA^*) - \text{Tr}(AA^*(I-P)) \geq \sum_{i=1}^{m} s_i(A)^2 - \sum_{i=1}^{r+1} s_i(A)^2 = \sum_{i=r}^{m} s_i(A)^2\end{aligned}$$

since $I - P$ is a projection of rank $m - r + 1$. So the maximum this could be is the tail, which is the variational argument we proved earlier. And this is what we claimed. □

In our proof of the restricted invertibility principle, this is the first step. Before proving it, let me just tell you what the second step looks like.

Lemma 2.1.26. *Step 2.* lem:step2 *For $k, m, n \in \mathbb{N}$, $A : \mathbb{R}^m \to \mathbb{R}^n$, rank$(A) > k$. Let $\omega = \{1, \cdots, m\}$ with $|w| = \text{rank}(A)$ such that $\{Ae_j\}_{j\in\omega}$ are linearly independent. Denote for every $j \in \Omega$*

$$F_j = E_{\omega\setminus\{j\}} = \left(\text{span}(Ae_i)_{i\in\omega\setminus\{j\}}\right)$$

Then there exists $\sigma \subseteq \omega$ with $|\omega| \geq k$ such that

$$\|\,(AJ_\sigma)^{-1}\,\|_{S_\infty} \leq \frac{\sqrt{\text{rank}(A)}}{\sqrt{\text{rank}(A) - k}} \cdot \max_{j\in\omega} \sqrt{\|\text{Proj}_{F_j} Ae_j\|}$$

Proof. Next Time. TODO □

Most of the work is in the second step. First we pass to a subset where we have some information about the shortest possible orthogonal project. But Step 1 saves us by bounding what this can be. Here we use the Grothendieck inequality, Sauer-Shelah, etc. Everything: It's simple, but it kills the restricted invertibility principle.

Theorem 2.1.27. *Step 1 and Step 2 imply the Restricted Invertibility Principle.*

Proof. Take $A : \mathbb{R}^m \to \mathbb{R}^n$. By Step 1, we can find subset $\tau \subseteq \{1, \cdots, m\}$ with $|\tau| = r$. For all $j \in \tau$, we can find

$$\|\text{Proj}_{E_{\tau\setminus\{j\}}} Ae_j\|_2 \geq \frac{1}{\sqrt{m}} \left(\sum_{i=r}^{m} s_i(A)^2\right)^{1/2}$$

Now we apply Step 2 to AJ_τ, using $\omega = \tau$, and find the further subset $\sigma \subseteq \tau$ such that

$$\|\,(AJ_\sigma)^{-1}\,\|_{S_\infty} \leq \min_{k<r<\text{rank}(A)} \sqrt{\frac{mr}{(k-r)\sum_{i=r}^{m} s_i(A)^2}}$$

which we get by plugging directly in mr for the rank and using Step 1 to get the denominator. □