Mathematical Physics

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Introduction

Notes from Michael Aizenman's class "Mathematical Physics" at Princeton in Spring 2016.

This class is of interest to both physicists and mathematicians. Several recent Fields medals are for work related to these topics.

I plan to cover the following topics. The focus is on **Topics in Mathematical Statistic Mechanics**.

1. **The statistical mechanic perspective**: systems can be described at the microscopic level with many degrees of freedom. We observe their collective behavior and find emergent behavior.

2. **Thermodynamics principles**: Intellectually preceding statistical mechanics is thermodynamics, a field of physics which emerged through experimental and intellectual work trying to understand what is happening with the transfer of heat. A couple of principles emerged. This framework is more appropriate to the macroscopic description of physical systems. In departure from mechanics, which cares about equalities like $F = ma, E = mc^2$, a unique thing about thermodynamics is that its key principle is in inequality: entropy increases.
$$\Delta S \geq 0$$

3. **The emergence of thermodynamics from statistical mechanics via the equidistribution assumption and the "large deviation theory"**. I would like to discuss the emergence of thermodynamics from statistical mechanics. Mathematicians formalized the theory but the concepts were introduced earlier by physicists. These principles led Boltzmann to introduce these ideas and it took a while for physics to absorb the ideas.[1]

4. **Phase transitions**: When you have a system, say H_2O, which you can control with temperature and pressure, you can induce changes in state. Continuous change in control parameters results in discrete jumps in the result.

5. **Critical phenomena, critical exponents, universality classes**. Phase transitions are fascinating since there are interesting critical phenomena which are characterized

[1] This may have led to his premature death via suicide.

by critical exponents, which turn out (this is one of the suprising discoveries experimentally) to result in universality classes of critical phenomena. Systems are macroscopically different, but the singularities you observe are given by the same power laws.

6. **Exact solution of the 2-D Ising model**. Mathematically, 3 is the hardest dimension to comprehend.

   (a) 1-D is solvable: correlations can be described by Markov chains, and can be computed.

   (b) In 2-D, the conformal group is very important. It gives many constraints on critical behavior, leading to a rich behavior.

   (c) In 3-D, this does not apply except for ongoing work finding consequences in 3-D from results in 2-D.

   (d) Anything with ≥ 4 dimensions gets simpler. High dimensions are characterized by the fact that loop effects do not play a large role. In 1-D a simple random walk is recurrent. In sufficient high dimensions, a simple random walk is not recurrent. The infinite-dimensional case reduces to models on trees; high dimensions exhibit the same behavior as infinite dimensions.

   In coding theory and discrete math people study phase transitions of different graphs and are interested in the same topics.

   The 2D Ising model is usually a highly specialized topic, but we will do it in a way done a bit differently from normal. We would like to do this through "graph zeta functions". What I found fascinating about this topics is that there are a lot of connections to other topics. There are analogues of the Riemann zeta function on graphs, and there is a relationship. It is actually one of the simplest paths to tackle this proof by.

7. **Stochastic geometry behind correlation functions at criticality**: in the Ising model we have a collection of spin variables $\sigma_x = \pm 1$ for $x \in \mathbb{Z}^d$. The spins are correlated in such a way that agreement among neighbors is encouraged. Thus there are correlations which spread through the system.

   It is interesting and powerful to represent this correlation between spins is via a "shadow system" in which you play the following game: decompose the collection of spins at random into connected clusters. You see spin values but you don't see who is connected to whom. An analogy is that students form cliques in class, and then each clique chooses what to do and votes unanimously. So if you just saw the voting pattern, you would see some cliques, but the nature of correlations among the votes become transparent if you know the clusters.

   The states for critical Ising models become larger, but also become fractal. Fractal geometric objects can be used to explain the structure. There is an interesting fractal geometry which tells us about correlation functions.

We will introduce all of this later in much detail from the ground up. This is related to percolation.

8. **Scaling limits**: Finally, if time permits, I would like to discuss scaling limits. You may want to know properties of a substance when you have something like a lattice that is very fine-grained, a statistic mechanical system which macroscopically is described by a myriad of variables. There are techniques of describing the relevant quantities from the macroscopic perspective, and this becomes relevant for describing the critical state. This provides a fascinating link between statistical mechanics and field theory. It's related to quantum field theory, except we are now in the Euclidean regime. There are some related results for quantum spin systems.

In a way we have a mouthful here, and one could probably give a full course on each of these topics. I will try not to be exhaustive, as we may not progress far. Previously I gave a course on random operators. Each week would cover some area of this subject, saying enough so you have a glimpse of the essence and get some comfort, and then moving on. It's now in a book format, and I would count it as a success if out of these lectures, something similar would emerge. The idea is not to be exhaustive, but to give enough key results. There is much more to be said, but I do not suppose to cover it all.

There is a broad spectrum of references, none of which I'll follow exclusively.

- Sacha Friedli and Yvan Velenik's online book project (on mathematical statistical mechanics) at http://www.unige.ch/math/folks/velenik/smbook/index.html. I recommend this for people totally new to the subject.

- David Ruelle formulated models and basic results mathematically (late 70's). His book helped physicists organize their thoughts. It became outdated quickly, but remains a good starting point and reference for the formalism.

Chapter 1

Introduction to statistical mechanics

2-2-16

1 Canonical Ensembles for the Lattice Gas

1.1 Configurations and ensembles

One way to start is with the axioms of statistical mechanics. Instead I'll take a simple problem, see how it works, and present results in that context. There are simple problems that teach us a lot. ("The elementary problems are the most precious, once you absorb them they are part of your makeup.") The simplest is a lattice gas.

Definition 1.1.1 (Lattice gas)**:** A **lattice gas** is a substrate where at each lattice site there may or may not be a particle. This model has been used to describe alloys where you have a substrate which you can draw as a simple lattice (in particular, we will use $\mathbb{Z}^d$ for simplicity). Each point may have a particle of a certain type.

The **configuration** is a function $n : \mathbb{Z}^d \to \{0, 1\}$:

$$n_x = \begin{cases} 1, & x \text{ is occupied,} \\ 0, & x \text{ is vacant.} \end{cases}$$

For now let us ignore energy conservation. Let us suppose the system is neutral, or at some infinite temperature; energy is not an issue.

I'll use L to denote the size of the box we are considering, and $\Lambda \subset \mathbb{Z}^d$ to be a region (subset of the system) and we have m particles. The **configuration space** is the space of possible n's, $\Omega = \{0, 1\}^\Lambda$.

If you have a finite system and energy does not play a role, but there is conservation of the number of particles, if you shake the box the state may change. Essentially each configuration gets equal weight, and particles do not overlap.

Definition 1.1.2 (**Equidistribution assumption**)**:** All particle configurations which have an equal number of particles have equal probability.

This gives rise to the notion of ensembles.

Definition 1.1.3 (Ensemble)**:** An **ensemble** is a probability measure with respect to which you do averages over the configuration space Ω.

First, we define some notation.

Definition 1.1.4 (Indicator function)**:**

$$\mathbb{1}[\text{cond}] := \begin{cases} 1, & \text{condition satisfied} \\ 0, & \text{else} \end{cases}$$

Now we define two specific ensembles, the **microcanonical ensemble** and the **grand canonical ensemble**.

Definition 1.1.5 (Microcanonical ensemble)**:** In the microcanonical ensemble,

$$\mathbb{P}(n_\Omega) = \frac{\mathbb{1}[\sum_{x\in\Omega} n_x = N]}{Z}$$

where Z is a normalization constant.

For every function $f : \Omega \to \mathbb{R}$ assigning a real number to each configuration, define the **microcanonical ensemble average** by

$$\langle f \rangle_{N,n}^{\text{Can}} = \frac{\sum_{n\in\Omega} \mathbb{1}[n_x = N] f(n)}{\sum_{n\in\Omega} \mathbb{1}[\sum n_x = N]}.$$

The ensemble we will focus on for this course is the grand canonical ensemble.

Definition 1.1.6 (Grand canonical ensemble)**:** The **grand canonical ensemble average** is defined as

$$\langle f \rangle_{\mu,\Lambda}^{\text{Gr.C}} = \frac{\sum_{n\in\Omega} e^{-\mu \sum_{x\in\Lambda} n_x} f(n)}{\sum_{n\in\Omega} e^{-\mu \sum_{x\in\Lambda} n_x}}$$

(Later on we will omit superscripts where it is clear.)

We can relate these two notions with the **equivalence principle**.

Definition 1.1.7 (Equivalence principle)**:** Loosely, the equivalence principle says that for any "local function", the microcanonical average is approximately the grand canonical ensemble average

$$\langle f \rangle_{N,\Lambda}^{\text{Can}} \approx \langle f \rangle_{\mu,\Lambda}^{\text{Gr.C}}$$

when we take $\mu = \frac{N}{|\Lambda|}$.

1.2 The equivalence principle

Consider functions which depend only on a system $\Lambda \subset \widetilde{\Lambda}$ of much smaller volume, $|\Lambda| \ll |\widetilde{\Lambda}|$. This is the sense in which the averages match up.

We can think of the difference as the difference between two socities. One is draconian. The first is highly centralized control economy. If you don't fit the build, you get weight 0 and you're thrown out. In the grand canonical, everything goes. Some contribute more than others, and the contributions depend on the parameter μ which is adjustable. The density of particles depends on μ. And there is a value of μ for which the density is equal to $N/|\tilde{\Lambda}|$. Here, the average of the draconian system is equal to the average of the lackadaiscal system, asymptotically.

The micro-canonical ensemble is draconian: the number of particles is prescribed, all other configurations get weight 0. In the grand canonical ensemble, each configuration contributes. There is a value of μ where the density is the same; at that value the local average of the draconian system is asymptotically the same at that of the more relaxed system. This $\approx$ becomes $=$ when you take the thermodynamic limit,

$$\begin{aligned} \widetilde{\Lambda} &\to \mathbb{Z}^d \\ N &\to \infty \\ \frac{N}{|\widetilde{\Lambda}|} &\to \rho. \end{aligned}$$

Now let us see where this comes from. What is the induced distribution of the micro-canonical ensemble on Λ? If all you care about is the number of particles in the average you care about, you just care about getting a small system. If the whole system has a large number of particles, then it does not matter if you focus on a much smaller box since you can trade particles very easily, if the larger volume has a ton of particles. Under the microcanonical ensemble, the probability that $n_{|\Lambda}$ (n restricted to Λ) takes a particular value depends only on one quantity, which is the number of particles the configuration has ($\sum_\Lambda n_x = k$ in the big box). This expression is equal to the cardinality (number) of configurations in the rest of the big box ($n_{|\Lambda^c}$, Λ complement) such that the total number of points in Λ^c is equal to $N - k$. And that of course must be normalized. Now we come to the question of how many configurations there are in the complement of Λ.

We count the number of ways to complete the configuration in $\Lambda^c = \widetilde{\Lambda} \backslash \Lambda$:

$$\frac{|\left\{n_{\Lambda^c} : \sum_{x \in \Lambda^c} n_x = N - k\right\}|}{Z}$$

where Z is a normalization constant.

The number of configurations of M particles in volume V is $\binom{V}{M} = \frac{V!}{M!(V-M)!}$. Using Stirling's approximation

$$\ln(M!) = M(\ln M - 1)(1 + o(1)),$$

After some gymnastics of an elementary nature, there is a fundamental formula. The logarithm of the number of configurations is of tremendous important. Ludwig Boltzmann's

grave has the formula which opened people's eyes to what this mysterious **entropy** S truly is: the logarithm of the number of configurations.

Lemma 1.1.8 (Entropy)**:** Defining

$$s(\rho) = -[\rho \ln \rho + (1-\rho)\ln(1-\rho)],$$

we have

$$\binom{V}{M} = \frac{V!}{M!(V-M)!} =: e^{S(M,V)} \approx e^{Vs(\rho)}$$

where density $\rho = m/V$ and $s(p) = -p\ln(p) - (1-p)\ln(1-p)$.

The ratio ρ varies between 0 and 1. $s(\rho)$ is concave and attains maximum of $\ln 2$ at $\frac{1}{2}$ where it has quadratic behavior.

Proof. Please do this exercise once in your life; it's good to do it once but not too often. □

Shannon also found such a formula for entropy.

The implication is that if you slightly change the density, the number of configurations changes drastically. In physical substances V may be 10^{23}. The change would then be $e^{10^{23}\Delta s}$. In any average over configurations, only those at the peak contribute: "winner takes all."

What is the probability of observing k particles in the small box given n in the big box?

$$\mathbb{P}(n_\Lambda) \approx \frac{e^{|\Lambda^c| s\left(\frac{N-k}{|\widetilde{\Lambda}|-|\Lambda|}\right)}}{Z}$$

where Z is a normalizing constant. The probability is propotional in exponent to the volume of the complement multiplied times the entropy of the density, which we saw in the exercise above. Then we re-write the density of the small box in terms of the overall density:

$$\frac{N-k}{|\widetilde{\Lambda}|-|\Lambda|} = \rho - \underbrace{\left(\rho - \frac{N}{|\widetilde{\Lambda}|-|\Lambda|}\right)}_{\to 0 \text{ independent of } k} - \frac{k}{|\widetilde{\Lambda}|-|\Lambda|}$$

The entropy is infinitely differentiable except at the endpoints, so we can just expand. Whatever you see inside affects the number outside, but does not effect the density outside. This explains why we only have a tiny correction to ρ. Thus

$$s\left(\frac{N-k}{|\widetilde{\Lambda}|-|\Lambda|}\right) \approx s(\rho) - s'(\rho)\frac{k}{|\Lambda^c|}$$

Changing k by a little bit affects how many particles are outside but not so much the density outside: the correction term is small. Hence for $\sum_{x\in\Lambda} n_x$,

$$\mathbb{P}(n_\Lambda) \approx \frac{e^{|\Lambda^c| s(\rho)} e^{-s'(\rho)k}}{Z}.$$

$e^{|\Lambda^c|s(\rho)}$ is a huge factor but it does not vary with k so we can omit it. Writing out Z as the sum of the numerators over all n_Λ, and letting $\mu = s'(\rho)$, we get that this equals

$$= \frac{e^{-\mu k}}{\sum_{n' \in \Omega_\Lambda} e^{-\mu \sum_{x \in \Lambda} n'_x}},$$

which is the grand canonical ensemble average. So all you have to do is be sure to pick μ as the derivative of the thermodynamic function at the correct density. That is how we derive the canonical ensemble in this case. The rest of the system acts on the small system as a "particle (heat) bath."

Here we used very explicit machinery, namely, the Stirling formula. We want a general expression that doesn't rely on the Stirling formula because in more complicated models, we will not have the luxury of using the Stirling approximation. We'll do this in the next section.

Then we will be able to consider models where there are more energy constraints. The general procedure is first make a list of energy constraint. There is a generalization of the equivalence principle where

- in the micro-canonical ensemble we average over configurations where the constraints have prescribed values. (We considered the special case where just the number of particles was prescribed.)

- in the grand canonical ensemble, we add an energy term to the exponential: $e^{-\mu N(n) - \beta \mathcal{E}(n)}$ where $N(n)$ is the number of particles, $-\mu N(n)$ is the Gibbs factor, and $\mathcal{E}(n)$ the energy.

How can we construct an alternative method without the Stirling formula?
2-4-16

1.3 Generalizing Ensemble Analysis to Harder Cases

Previously we used very specific machinery to derive the grand canonical ensemble for the lattice gas; how might we derive similar expressions for more complicated systems? We would like to avoid using the Stirling formula, which only happened to work in the simple case. To derive the entropy bypassing Stirling's approximation, you may proceed by analyzing partition functions.

Define the **partition function** for the grand canonical ensemble as the (unnormalized) sum of the likelihoods over all configurations. At each site there are 2 possibilities and they

are independent, so the sum factors into a product.[1]

$$\begin{aligned} Z_\Lambda &= \sum_{n\in\Omega} e^{-\mu N(n)} \\ &= \sum_{n\in\Omega}\prod_{x\in\Lambda} e^{-\mu \mathbb{1}[n_x=1]} \\ &= \prod_{x\in\widetilde{\Lambda}} (1+e^{-\mu}) \\ &= (1+e^{-\mu})^{|\Lambda|} \\ \frac{1}{|\Lambda|}\ln Z_\Lambda &= 1+e^{-\mu} \end{aligned} \tag{1.1}$$

eq:z1

So what does this have to do with entropy? We can calculate Z_Λ a different way, as follows. Note that the only thing that matters in the summand is the number of particles in n, so let's group the summands by this. Letting $\rho = \frac{N}{|\Lambda|}$, and using the fact that the number of states with density ρ is $e^{|\Lambda| s(\rho)}$,[2]

$$\begin{aligned} Z_\Lambda &= \sum_{n\in\Omega} e^{-\mu N(n)} \\ &\approx \sum_{\rho\in\frac{1}{|\Lambda|}\mathbb{Z}} e^{-\mu N(n)} e^{|\Lambda| s(\rho)} \\ &= \sum_{\rho\in\frac{1}{|\Lambda|}\mathbb{Z}} e^{|\Lambda|[s(\rho)-\mu\rho]} \\ &\approx e^{|\Lambda| \max_{\rho\in[0,1]}[s(\rho)-\mu\rho]}. \end{aligned} \tag{1.2}$$

eq:z2

To get the last step, note that the error we are making by focusing on the maximal point rather than counting with multiplicity the near-maximal value is at most a factor equal to the volume, since we can upper bound our error by taking all points in Λ instead of merely an ε-ball around the maximum. "In analysis, it often pays to avoid being excessively generous." Since $\frac{1}{|\Lambda|}\ln|\Lambda| \to 0$, this term is negligible. The expression $\max_{\rho\in[0,1]}[s(\rho)-\mu\rho]$ is called the **Legendre transform** of the entropy and denoted by $s^*(\rho)$.

Matching (1.1) and (1.2), we find

$$s^*(\mu) := \max_{0\le\rho\le 1}[s(\rho)-\mu\rho] = \ln(1+e^{-\mu})$$

Using this, we can derive the expression the formula for $s(\rho)$, using the inverse Legendre transform.

[1] Note that Λ in the last section is the micro-canonical ensemble, but here it is the grand canonical ensemble.

[2] Rather than getting this with $s(\rho) = -[\rho\ln(\rho) - (1-\rho)\ln(1-\rho)]$ by the Stirling approximation in the last section, we can take this as the *definition* of s here, and use it to solve for s.

To check that taking the maximum in (1.2) was legit, note that[3]

$$\max_\rho[s(\rho)-\mu\rho] \le \frac{1}{|\Lambda|}\ln Z_\Lambda \le \max_\rho[s(\rho)-\mu\rho] + \underbrace{\frac{1}{|\Lambda|}\ln|\Lambda|}_{\to 0}$$

$$\implies \lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\ln Z_\Lambda = \sup_{0\le\rho\le 1}[s(\rho)-\rho\mu].$$

We can verify that $s(\rho) = -[\rho\ln\rho + (1-\rho)\ln(1-\rho)]$ does indeed make (**??**) hold. We find the maximum (critical point) or $s(\rho)-\mu\rho$ by setting the derivative to 0.[4] We use the trick

$$[x(\ln x - 1)]' = \ln x.$$

We can subtract 1 from each of the logs changing the expression by a constant. Thus

$$s'(\rho) = -\ln\rho + \ln(1-\rho) - \mu \implies \frac{\rho}{1-\rho} = e^{-\mu}.$$

We can use this to solve for ρ and find $s^*(\mu)$ (do this yourself).

In general, a micro-canonical ensemble specifies all the conserved quantities: Particles, energies, and whatever else there is. The grand canonical ensemble also generalizes by changing the factor to $e^{-\mu N}e^{\beta H}$. Such factors are referred to as **Gibbs factors** or Gibbs measures.

2 Concavity and the Legendre transform

2.1 Basic concavity results

Definition 1.2.1: A function on $\mathbb{R}^k$ is **concave** if for any $x_0, x_1 \in \mathbb{R}^k$, $0 \le \lambda \le 1$,

$$F(\lambda x_1 + (1-\lambda)x_0) \ge \lambda F(x_1) + (1-\lambda)F(x_0).$$

For a **convex** function, the same inequality with the sign flipped holds. A negative convex function is concave. Concavity (convexity) means if you draw a chord between two points, it will lie below (above) the curve.[5]

For a strictly concave function, a maximum, whenever it exists, is unique.

Theorem 1.2.2. *For any concave function on* $\mathbb{R}$,

[3] Mathematicians are paranoid, so we use sup instead of max. For many practical purposes they are the same.

[4] Don't differentiate in public.

[5] "As Richard Feynmann pointed out, getting the sign right is the hardest thing."

1. *The directional derivatives* $F'(x \pm 0)$ *exist at all* $x \in \mathbb{R}$. *The* ***directional derivative*** *is defined as*

$$F'(x+0) = \lim_{\varepsilon \to 0^+} \frac{F(x+\varepsilon) - F(x)}{\varepsilon}$$
$$F'(x-0) = \lim_{\varepsilon \to 0^-} \frac{F(x+\varepsilon) - F(x)}{\varepsilon}.$$

2. $F'(x-0) \geq F'(x+0)$ *and* $F'(x \pm 0)$ *are decreasing.*

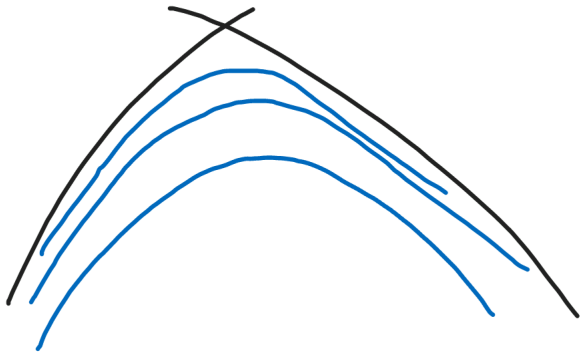
3. *For all but countably many values* $x \in \mathbb{R}$, $F'(x-0) = F'(x+0)$, *i.e.,* F *is differentiable at* x.

4. *Let* F_n *be a sequence of concave functions which converge pointwise: for all* x, $\lim_{N\to\infty} F_N(x) =: \widetilde{F}(x)$ *exists. Then*

   (a) $\widetilde{F}$ *is concave.*

   (b) *At points of differentiability of* $\widetilde{F}$, *the derivatives also converge,*[6]

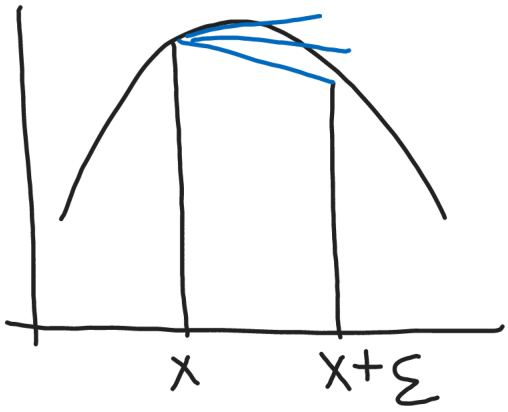
$$F'(x \pm 0) \to \widetilde{F}'(x).$$



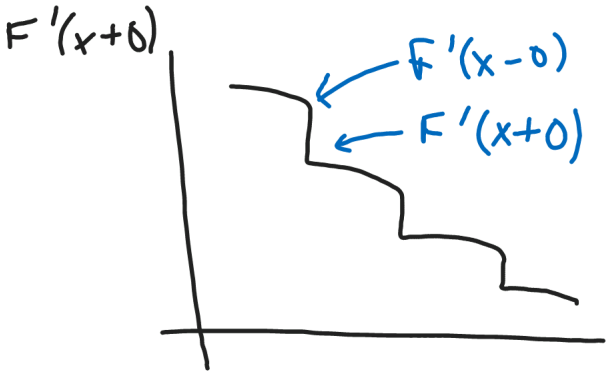
Much of this generalizes to directional derivatives in n dimensions.

Proof. For (1) note that concavity implies that the slope of the line between $x, x+\varepsilon$ is increasing as $\varepsilon \to 0^+$.

[6]Finite energy functions are always smooth, but their limit can have discontinuous derivative.



To prove (3), consider the graph of the derivative $F'(x+0)$. It's a monotone decreasing function: it starts somewhere and goes down. At points where $F'(x+0)$ is continuous, we have $F'(x+0) = F'(x-0)$. The discontinuities, or "steps," are the points where $F'(x+0) \neq F'(x-0)$. Now we use the fact that the sum of any uncountable collection of nonzero numbers is ∞. Applying this to the steps, we find that the number of steps is countable, i.e., the total collection of discontinuity points has to be countable.[7] [8]



□

2.2 Concave properties of the Legendre transform

Definition 1.2.3: The **Legendre transform** of a function is defined as

$$(TG)(y) = \inf_x [y \cdot x - G(x)] = -\sup_x [G(x) - y \cdot x]$$

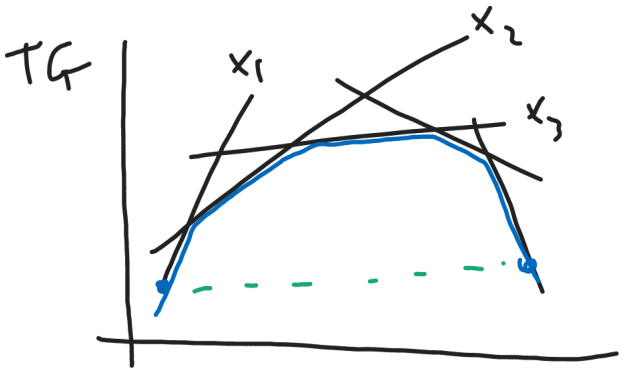
[7] More rigorously, for every nonzero interval $[F'(x+0), F'(x-0)]$, we can associate with it a rational number. Then note $\mathbb{Q}$ is countable.

[8] The set of discontinuities can still be devilishly dense, ex. all the rational numbers. In physics it used to be thought that weird functions with Cantor-like sets of discontinuties could not occur, but there are materials whose free energy discontinuities are dense in certain areas.

Theorem 1.2.4. *For any function G, TG is concave.*

Proof. An efficient way to think about this is the following: for each value of the parameter x, as a function of y this is a linear function. For each x we get a linear function. Define the transform by taking the infimum over that.

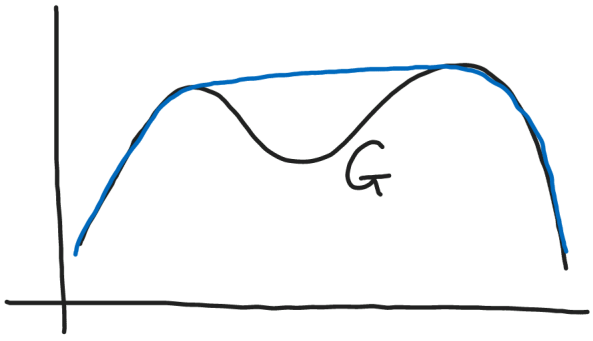
Take 2 points and draw the chord between them. For each linear function the chord lies below it.[9]



□

Definition 1.2.5: The **concave hull** $\widetilde{G}$ of G is defined as the smallest concave function that is at least the function value at every point:

$$\widetilde{G}(x) = \inf \{F(x) : F \text{ concave}, \forall u,\ F(u) \geq G(u)\}.$$



Theorem 1.2.6. *For* concave G,

$$T(TG) = G.$$

In general, $T(TG)$ is the concave hull of G.

[9] I.e., we use the following : Let $\mathcal{F}$ be a collection of concave (e.g. linear) functions. Then $\inf_{f\in\mathcal{F}} f(x)$ is concave.

Proof for G differentiable. Use the fact that if G is differentiable, then

$$\inf_x [y \cdot x - G(x)]$$

occurs at $y = G'(x)$. □

Note that we plotted the function and the dual function on the same graph. However, they have inverse units, for example, energy and inverse temperature. You get a lot of insights into physics if you keep track of the units.

Recall $\rho = \frac{N}{|\Lambda|}$. If all $2^{|\Lambda|}$ configurations are given equal weight, the typical value or ρ, the particles per unit volume is $\frac{1}{2}$. The Law of Large Numbers says that with probability 1 the ratio tends to $\frac{1}{2}$. Is it possible that the density is $\frac{1}{3}$? Yes, but the probability of such a density is given by the entropy: it's exponentially small, $e^{|\Lambda|[s\left(\frac{1}{3}\right)-\ln 2]}$. Anything other than $\frac{1}{2}$ is a large deviation; they occur with exponentially small probability. We want to quantify the probability of large deviation events. Here is a language that people found useful.

Definition 1.2.7: A sequence of probability measures on $\mathbb{R}$ is said to satisfy a **large deviation principle** with speed $\{a_N\}$, and rate function $I(x)$ if for each $x \in \mathbb{R}$, $\varepsilon > 0$,

$$-\inf_{|u-x|<\varepsilon} I(u) \leq \underline{\lim}_{N\to\infty} \frac{1}{a_N} \ln \mathbb{P}_N((x-\varepsilon, x+\varepsilon]) \leq \overline{\lim}_{N\to\infty} \frac{1}{a_N} \ln \mathbb{P}_N((x-\varepsilon, x+\varepsilon]) \leq -\inf_{|x-u|\leq\varepsilon} I(u)$$

For us, $I(x) = \ln 2 - s(x)$.
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3 A variational characterization of Gibbs States

4 1^{st} order phase transitions