Smooth Infinitesimal Analysis

A Modern Reformulation of Calculus

Holden Swindell

University of Wisconsin-Madison

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The Derivative

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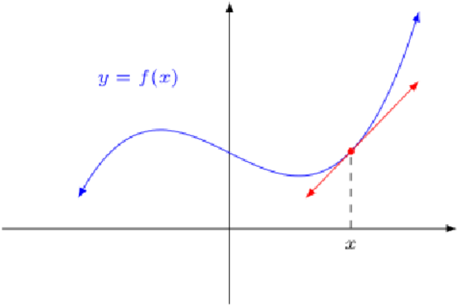

Pop Quiz!

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What is the formal definition of the derivative?

You have 5 minutes. Yes, Honorlock is required - which means no hats, no hoodies, no blinking, and no breathing (you could be cheating).

Pop Quiz - Solution

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Definition: Classical Derivative I

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. The derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

Taking a Closer Look

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Definition: Classical Derivative II

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. The derivative of f at a is the value $f'(a) = L$ such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $h \neq 0$, if $|h| < \delta$, then

$$\left| \frac{f(a+h) - f(a)}{h} - L \right| < \varepsilon,$$

provided such an L exists.

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Issues beyond elegance...

Pathological Function Wall of Shame

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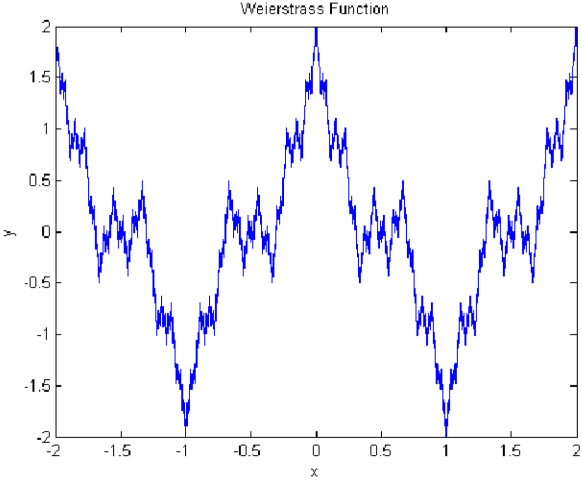
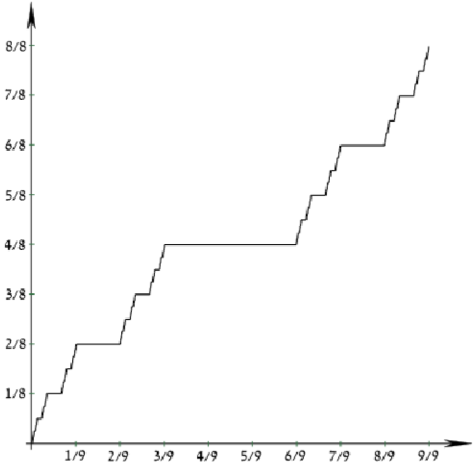
Continuous everywhere, differentiable nowhere



Pathological Function Wall of Shame

Continuous everywhere, zero derivative almost everywhere, but still manages to get from 0 to 1



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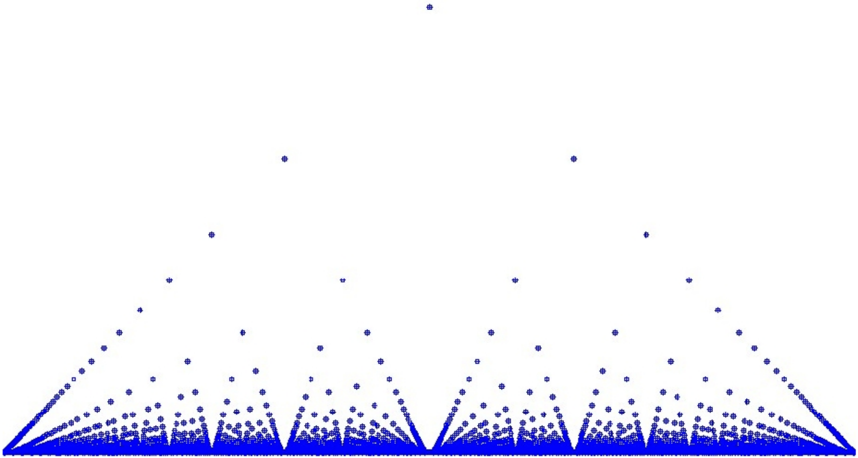

The Derivative (now in n dimensions!)

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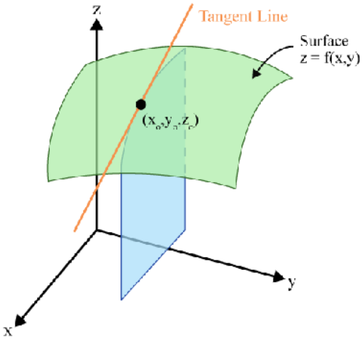
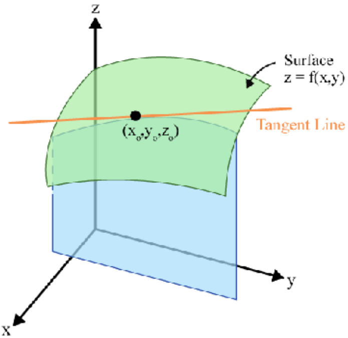


Slope of the surface in the x-direction



Slope of the surface in the y-direction

Higher-Dimensional Issues

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- All partial derivatives exist $\neq$ differentiable!

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$$\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$$

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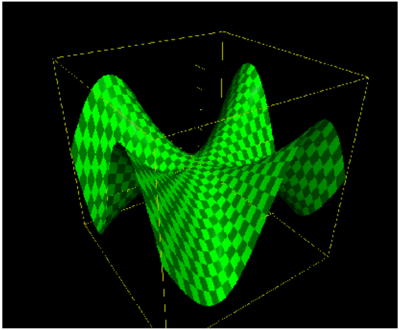

Takeaways

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Main Idea

This notion of "differentiable" (in any number of dimensions) is suboptimal due to the existence of pathological functions.

TL;DR: $\mathbb{R}$ is not built for differentiation.

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Question:

Is there an alternative notion of "differentiable" that better aligns with our intuitive sense of smoothness?

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Is there an alternative notion of "differentiable" that better aligns with our intuitive sense of smoothness?

Answer:

Yes!

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Intuition

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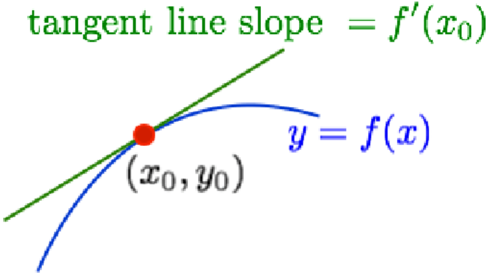
A differentiable function should well-approximated by a line near any point.



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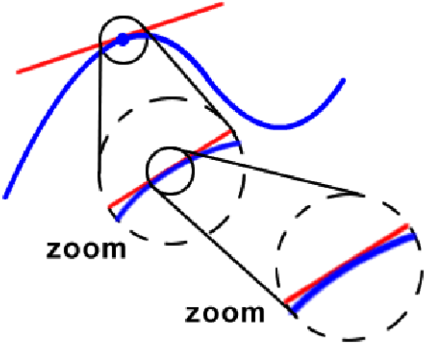

The Synthetic Real Line

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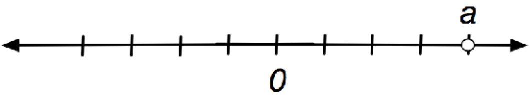

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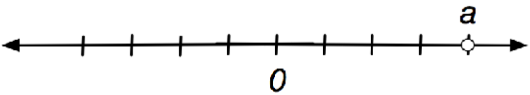

(If you know some abstract algebra, $\mathfrak{L}$ is a commutative, ordered, nonzero ring.)

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- Proof doesn't work in $\mathfrak{L}$ (might not have division) - we could have nonzero infinitesimals

Microstraightness

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Definition: Smooth Line Object

A line object $\mathfrak{L}$ is smooth if it satisfies the "microstraightness axiom": for every function $f : D \to \mathfrak{L}$, there exists a unique $h \in \mathfrak{L}$ such that $f(\varepsilon) = f(0) + \varepsilon h$ for all $\varepsilon \in D$.

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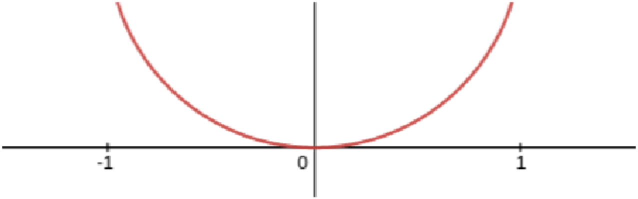

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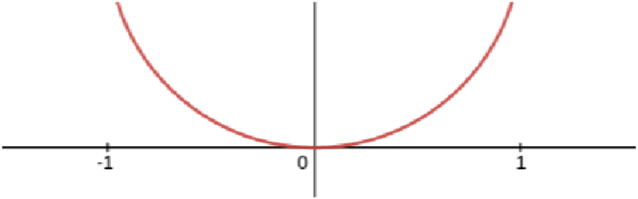

- What about $f(0) = 1$, $f(x) = 0$ for $x \neq 0$? - intuitionistic logic!

Derivatives!

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Definition: Synthetic Derivative

The derivative of $f : \mathfrak{L} \to \mathfrak{L}$ at $a \in \mathfrak{L}$ is the unique $h \in \mathfrak{L}$ such that $f(a + \varepsilon) = f(a) + \varepsilon h$ for all $\varepsilon \in D$. We denote this $f'(a) = h$.

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Consequence: Smoothness

All functions $\mathfrak{L} \to \mathfrak{L}$ are smooth!

Basic First Consequences

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- Let's find the derivative of x^2.

$$(x+\varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2 = x^2 + 2x\varepsilon \implies (x^2)' = 2x.$$

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- Let's prove the sum rule.

$$\begin{aligned}(f+g)(x+\varepsilon) &= f(x+\varepsilon) + g(x+\varepsilon) \\ &= f(x) + f'(x)\varepsilon + g(x) + g'(x)\varepsilon \\ &= (f(x)+g(x)) + (f'(x)+g'(x))\varepsilon\end{aligned}$$

$$\implies (f+g)'(x) = f'(x) + g'(x).$$

Basic First Consequences (cont.)

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- Let's prove the product rule.

$$
\begin{aligned}
(fg)(x+\varepsilon) &= f(x+\varepsilon)g(x+\varepsilon) \\
&= (f(x)+f'(x)\varepsilon)(g(x)+g'(x)\varepsilon) \\
&= f(x)g(x)+f'(x)g(x)\varepsilon+f(x)g'(x)\varepsilon+f'(x)g'(x)\varepsilon^2 \\
&= f(x)g(x)+(f'(x)g(x)+f(x)g'(x))\varepsilon
\end{aligned}
$$

$$\implies (fg)'(x) = f'(x)g(x)+f(x)g'(x).$$

Optimization

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Definition: Stationary Point

We say $f : \mathfrak{L} \to \mathfrak{L}$ has a stationary point at $a \in \mathfrak{L}$ if for all $\varepsilon \in D$, $f(a + \varepsilon) = f(a)$.

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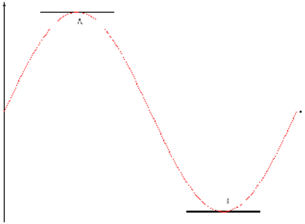

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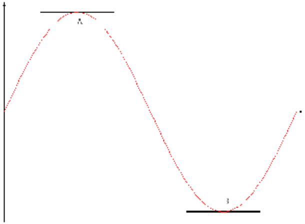

- Let's prove that $f'(a) = 0$ at a stationary point.

$$f(a+\varepsilon) = f(a) \implies f(a) + f'(a)\varepsilon = f(a) \implies f'(a)\varepsilon = 0$$

$$\therefore f'(a) = 0.$$

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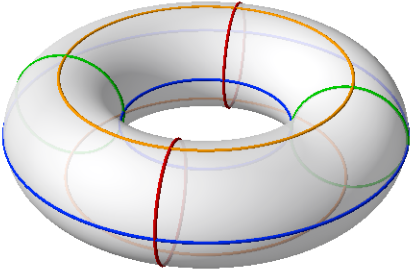

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(If you know category theory, we need models for a *smooth topos*.)

Applications

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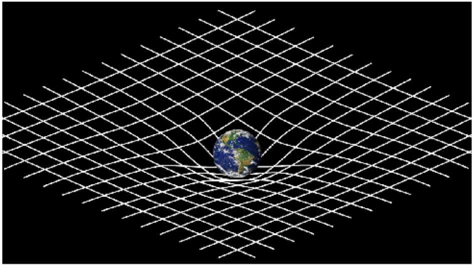
- Physics! (mechanics, GR, fundamental particles)



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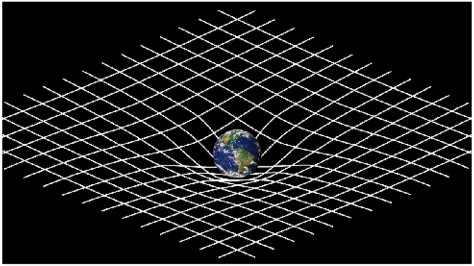

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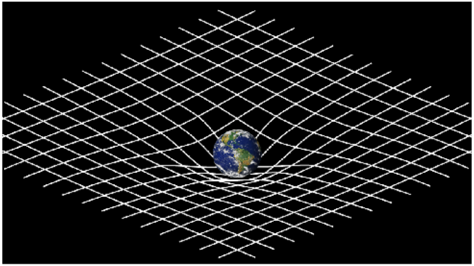

- More generally: applied calculus (engineering, economics, etc.)
- My favorite: pedagogical

Thanks!

Thanks for Coming!