

Computable Analysis for Extraction of Certified Programs

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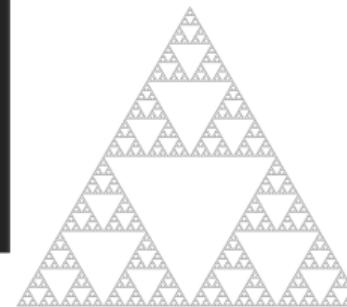
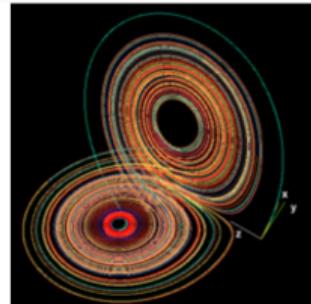
Overview

1. Real numbers and computation in the Rocq proof assistant
2. cAERN
3. Initial Value Problems and the Cauchy-Kovalevskaya theorem

Computation and Formal Proof

A **proof assistant** is an interactive tool for writing and checking formal proofs.

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```



Computation in Rocq

There are several ways to use computation in the Rocq (formerly Coq) proof assistant:



```
Fixpoint div2 (n : nat) :=
  match n with
  | 0 | 1 => 0
  | S (S n') => (div2 n')+1
  end.
```

```
Compute (div2 100). (* = 50 : nat *)
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Prove specification in additional step:

```
Lemma div2_spec : ∀ n, ∃ r, r ≤ 1 ∧ 2*div2 n + r = n.
Proof. ... Qed.
```

Computation in Rocq

Extract a witness from a constructive proof:



```
Lemma half_exists : ∀ n, {m | ∃ r, r ≤ 1 ∧ 2*m + r = n}.
```

Proof.

```
apply (well_founded_induction lt_wf).
intros n H;destruct n.
exists 0;exists 0;split;auto.          (* n = 0 *)
destruct n as [| n'].
exists 0; exists 1;split;auto.          (* n = 1 *)
destruct (H n') as [m P];[lia|].      (* n = n'+2*)
exists (S m).
destruct P as [r [P1 P2]];exists r; split;auto;lia.
```

Defined.

```
Compute (proj1_sig (half_exists 100)). (* = 50 : nat *)
```

Computation in Rocq

Extract a witness from a constructive proof:



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Lemma half_exists : ∀ n, {m | ∃ r, r ≤ 1 ∧ 2*m + r = n}.
```

Proof.

apply (well_1 Universes Prop for logical propositions and Type for computational data:

```
intros n H;de
```

$\exists x : A, P x : \text{Prop}$ (logical existence)

```
exists 0;exis
```

$\{x : A \mid P x\} : \text{Type}$ (computational witness)

```
destruct n as
```

```
exists 0; exi
```

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destruct (H n') as [m P];[lia|]. (* n = n' + 2*)
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exists (S m).
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Program Extraction

Extraction of Program Code (e.g. Ocaml or Haskell):



```
Lemma half_exists : ∀ n, {m | ∃ r, r ≤ 1 ∧ 2*m + r = n}.\nProof. ... Defined.\nExtraction half_exists.
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```
Lemma half_exists : ∀ n, {m | ∃ r, r ≤ 1 ∧ 2*m + r = n}.
```

Proof. ... Defined.

```
Extraction half_exists.
```



```
(** val half_exists : nat -> nat **)
let rec half_exists = function
| 0 -> 0
| S n -> (match n with
            | 0 -> 0
            | S n0 -> let s = half_exists n0 in S s)
```

Computation vs. Program Extraction

Computation in Rocq

- + Easy to use
- + Enables checking examples and testing conjectures quickly
- + Can be used directly in formal proofs

- Can be very slow
- Not suited for large computations

Program Extraction

- + Generates stand-alone programs
- + Typically faster
- + Allows integration of extracted code into existing developments

- Needs trust in extraction process
- May still need manual optimization
- Cannot be used directly within proofs
- Additional effort to compile/run

Real Numbers in Rocq

The Rocq standard library:

- Classical real numbers have been included in the standard library for a long time.

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 - Computation is often very inefficient.
- Equality defined as equivalence relation (`Setoid`).
 - Requires more complex reasoning: Standard tactics can not be used directly, instances must be shown to respect equality.

Real Numbers in Rocq

CoRN (Constructive Coq Repository at Nijmegen) [1]

- Large library for constructive analysis in Rocq.

[1] Cruz-Filipe, Geuvers & Wiedijk, C-CoRN, the Constructive Coq Repository at Nijmegen, Math. Knowledge Manag., 2004.

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- Large library for constructive analysis in Rocq.
- Based on metric spaces and the completion monad.

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- Steeper learning curve.

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Real Numbers in Rocq

Some other libraries support verification of approximate or floating-point computations:

- **Flocq** [2]
 - Formalizes IEEE-style floating-point arithmetic.
 - Offers customizable precision, rounding modes, and proofs of rounding error bounds.
 - Useful for verifying numerical algorithms where machine-level floats are employed.

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 - Offers customizable precision, rounding modes, and proofs of rounding error bounds.
 - Useful for verifying numerical algorithms where machine-level floats are employed.
- **Coq-Interval** [3]
 - An efficient implementation of interval arithmetic for intervals with floating-point end points.
 - Automatically translates expressions over classical reals into intervals, and provides tactics to solve inequalities.
 - Works well for common proof obligations.

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cAERN [4]

- Axiomatically defines a real number type, modeling exact real computation.

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- Similar operations as in ERC libraries:
 - Exact arithmetic
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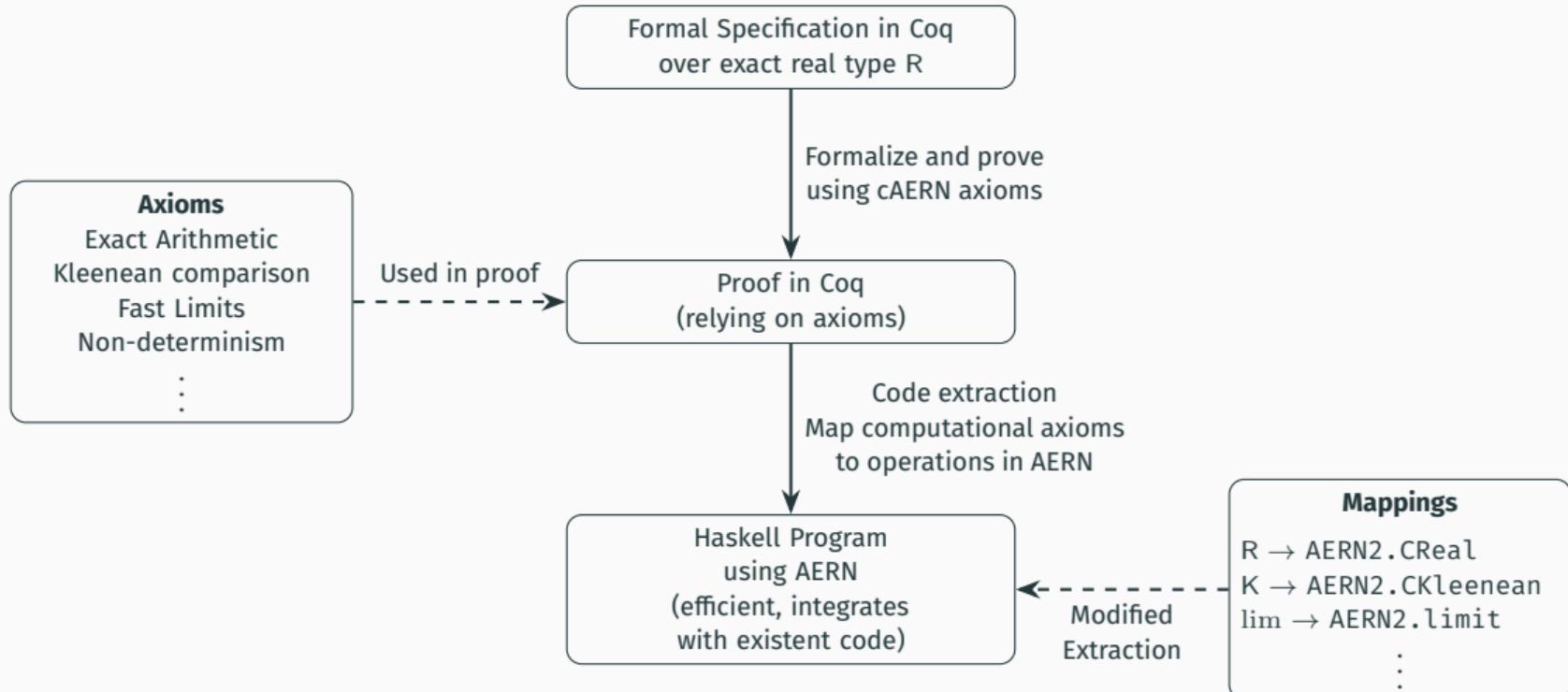
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- Supports Nondeterminism/Multivalued Computation.
- Extraction to Haskell, **on top of the AERN library** for efficient exact real computation.

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cAERN Overview



Code extraction example

```
Lemma real_max_prop :  
  forall x y, {z | (x >= y → z = x)  
                ∧ (x < y → z = y)}.  
  
Proof.  
  intros.  
  apply real_mslimit_P_lt.  
  + (* max is single-valued *)  
    ...  
  + (* construct limit *)  
    intros.  
    apply (mjoin (x>y - prec n)  
                 (y>x - prec n)).  
    ++ intros [c1|c2].  
    +++) (* when x > y - 2-n *)  
    exists x.  
    ...  
    +++) (* when x < y - 2-n *)  
    exists y.  
    ...  
  ++ apply M_split.  
    apply prec_pos.
```

Defined.

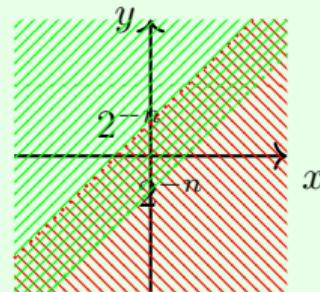
```
real_max_prop ::  
  AERN2.CReal ->  
  AERN2.CReal ->  
  AERN2.CReal  
real_max_prop x y =  
  AERN2.limit (\n ->  
    Prelude.id (\h -> case h of {  
      P.True -> x;  
      P.False -> y})  
  (m_split x y (prec n)))
```

Code extraction example

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Lemma real_max_prop :  
  forall x y, {z | (x  
    ^ (  
      Proof.  
        intros.  
        apply real_mslem:  
        + (* max is sing  
        ...  
        + (* construct l  
          intros.  
          apply (mjoin (x:  
            (y  
            ++ intros [c1|c  
              +++ (* when  
                exists x.  
                ...  
                +++ (* when  
                  exists y.  
                  ...  
                  ++ apply M_split.  
                  apply prec_pos.  
Defined.
```

How to define $\max(x, y)$ without using the representation?

- Apart from primitive operations on reals, cAERN provides a multivalued choice operation. Can be used to defined M -split:



- Use M -split to define a sequence $(z_n)_{n \in \mathbb{N}}$ such that $z_n = x$ if $x - y > 2^{-n}$, $z_n = y$ if $y - x > 2^{-n}$, and any of the two otherwise.
- $(z_n)_{n \in \mathbb{N}}$ is a fast Cauchy sequence converging to $\max(x, y)$.

Code extraction example

```
Lemma real_max_prop :  
  forall x y, {z | (x >= y → z = x)  
                ∧ (x < y → z = y)}.  
  
Proof.  
  intros.  
  apply real_mslimit_P_lt.  
  + (* max is single-valued *)  
    ...  
  + (* construct limit *)  
    intros.  
    apply (mjoin (x>y - prec n)  
                 (y>x - prec n)).  
    ++ intros [c1|c2].  
    +++) (* when x > y - 2-n *)  
    exists x.  
    ...  
    +++) (* when x < y - 2-n *)  
    exists y.  
    ...  
  ++ apply M_split.  
    apply prec_pos.
```

Defined.

```
real_max_prop ::  
  AERN2.CReal ->  
  AERN2.CReal ->  
  AERN2.CReal  
real_max_prop x y =  
  AERN2.limit (\n ->  
    Prelude.id (\h -> case h of {  
      P.True -> x;  
      P.False -> y})  
  (m_split x y (prec n)))
```

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Lemma real_max_prop :  
  forall x y, {z | (x >= y → z = x)  
           ∧ (x < y → z = y)}.
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Proof.

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  + (* construct limit *)  
  intros.
```

```
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```

```
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```

```
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```

```
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```

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```

- + Extracted programs are relatively efficient (comparable to hand-written AERN code).
- + Often more readable than low-level extraction.
- + Extracted code can be combined with non-verified parts of the AERN library.
- + Simple axiomatization, no setoids, proofs similar to using textbook reals.

- No direct computation in the proof assistant.
- Need to trust not only extraction, but also the implementation of ERC operations in AERN.
- (Minor) manual pre-processing of the extracted code is required before running.
- Direct implementation on names can sometimes be easier/more efficient than over generic abstract type.

Ordinary Differential Equations (ODEs)

As a practical application, we consider initial value problems (IVPs) for ordinary differential equations (ODEs). An IVP is an ODE together with an initial condition at

$$\dot{\vec{y}}(t) = \vec{F}(t, \vec{y}(t)), \quad \vec{y}(t_0) = \vec{y}_0, \quad \vec{F} : D \subseteq \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d.$$

A solution is a differentiable function $\vec{y} : I \rightarrow \mathbb{R}^d$, where $I \subseteq \mathbb{R}$ is an open interval containing t_0 , that satisfies the IVP.

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Remarks:

- Higher-order ODEs $\vec{y}^{(k)} = \vec{G}(t, \vec{y}, \dots, \vec{y}^{(k-1)})$ can be rewritten as a first-order ODEs by introducing additional variables for the higher derivatives $\vec{y}^{(2)}, \dots, \vec{y}^{(k)}$.
- Any ODE can be turned into an equivalent autonomous ODE by adding t as additional variable.
- By translation, one can assume $t_0 = 0$ and $\vec{y}_0 = \vec{0}$ without loss of generality.

Computability and Complexity of ODEs

Assume $f : [0, 1] \times [-1, 1] \rightarrow \mathbb{R}$ and consider the IVP $\dot{y}(t) = f(t, y(t)); y(0) = 0$

Assumptions on f	Computability/Complexity of Solution
(Polynomial-time) computable	All solutions can be non-computable [5]
+ Unique solution	Computable, but complexity can be arbitrarily high [6]
+ Lipschitz continuous	PSPACE complete [7]
+ Analytic	Solution is polynomial-time computable [8]

[5] Aberth, *The failure in computable analysis of a classical existence theorem for differential equations*, 1971.

[6] Ko, *On the computational complexity of ordinary differential equations*, Journal of Computer and System Sciences, 1983.

[7] Kawamura, *Lipschitz Continuous Ordinary Differential Equations are Polynomial-Space Complete*, Comp. Complexity, 2010.

[8] Müller & Moiske, *Solving initial value problems in polynomial time*, Proc. of JAIO — Panel '93, 1993.

ODE solving in Proof Assistants

The problem seems quite natural for formal verification in a proof assistant:

- It is a classical topic in analysis with many applications in science and engineering.
- Most ODEs do not have a closed-form solution, therefore numerical methods are required to approximate solutions.
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Reasons include:

- Traditionally more focus on discrete math, logic, programming languages, and finite structures.
- Less developed theory and libraries for real analysis (but growing).
- Constructive reasoning natural in Rocq and can be more challenging.

ODE solving in Rocq

Some prior works in Rocq:

- The CoRN library includes an implementation of the Picard Iteration method. [9]

[9] Makarov & Spitters. *The Picard algorithm for ordinary differential equations in Coq*, Proc. of ITP 2013.

[10] Park & T. A Coq Formalization of Taylor Models and Power Series for Solving ODEs, Proc. of ITP 2024.

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New implementation:

- Formalized in the Rocq proof assistant (approx. 10000 lines of code).
- Supports arbitrary dimension.
- Uses type-classes to support different implementations of real numbers.
- Usable both for program extraction and computation inside Rocq.
- More efficient implementation.

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The Cauchy-Kovalevskaya Theorem

Theorem (ODE version of the Cauchy-Kovalevskaya Theorem)

Let $U \subseteq \mathbb{R}^d$ be open and $f : U \rightarrow \mathbb{R}^d$ analytic. Then the initial value problem

$$\dot{\vec{y}}(t) = \vec{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0$$

has a unique solution which is analytic on an open interval $I \subseteq \mathbb{R}$ containing 0.

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has a unique solution which is analytic on an open interval $I \subseteq \mathbb{R}$ containing 0.

Main idea of the classical proof:

- Expand $\vec{F}(\vec{y})$ in a formal power series around \vec{y}_0 .
- Derive a recurrence for the Taylor coefficients of $\vec{y}(t)$ from the ODE $\dot{\vec{y}} = \vec{F}(\vec{y})$.
- Prove that the formal series has a positive radius of convergence by dominating it by a geometric majorant sequence.

Formal Power Series Solution

Let

$$\dot{\vec{y}}(t) = \vec{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0.$$

We inductively define a sequence of functions $\vec{F}^{[n]} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ of multivariate functions by

$$\vec{F}^{[0]}(\vec{x}) = \vec{x}, \text{ and}$$

$$\vec{F}^{[n+1]}(\vec{x}) = J_{\vec{F}^{[n]}}(\vec{x}) \cdot \vec{f}(\vec{x})$$

Define a function by the formal power series

$$\vec{y}(t) = \sum_{i=0}^{\infty} \frac{1}{n!} \vec{F}^{[n]}(\vec{y}_0) t^n$$

By deriving \vec{y} (as formal power series) and comparing coefficients, we see that \vec{y} satisfies the IVP.

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$$\dot{y}(t) = F(y(t))$$

$$\ddot{y}(t) = \dot{y}(t) \cdot F'(y(t))$$

$$= F(y(t)) \cdot F'(y(t))$$

⋮

$$\vec{F}^{[0]}(\vec{x}) = \vec{x}, \text{ and}$$

$$\vec{F}^{[n+1]}(\vec{x}) = J_{\vec{F}^{[n]}}(\vec{x}), \quad \text{for } n \geq 0,$$

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The Method of Majorants

- A (single-variate) power series $M(x) = \sum_{k=0}^{\infty} A_k x^k$ is said to majorize a d -variate power series $f(x) = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} x^{\alpha}$ if

$$\forall k \in \mathbb{N}, \quad \sum_{|\alpha|=k} |a_{\alpha}| \leq A_k, \quad \text{where } |\alpha| = \sum_{i=1}^d \alpha_i.$$

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- We are mostly interested in simple (geometric) majorants of the form $\sum_{k=0}^{\infty} M R^k$ for constant $M, R \in \mathbb{R}$.
- In that case, the tail of the series satisfies

$$\left| \sum_{|\alpha|>k} a_{\alpha} x^{\alpha} \right| \leq M \sum_{n=k+1}^{\infty} (R\|\mathbf{x}\|)^n = M \frac{(R\|\mathbf{x}\|)^{k+1}}{1 - R\|\mathbf{x}\|}, \quad (R\|\mathbf{x}\| < 1).$$

Solution Majorants

- We can show that if the right-hand side function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ has a geometric majorant (M, R) then $(1, 2dMR)$ works as a geometric majorant for the solution y , guaranteeing convergence of $y(t)$ for all t with $|t| < \frac{1}{2dMR}$.

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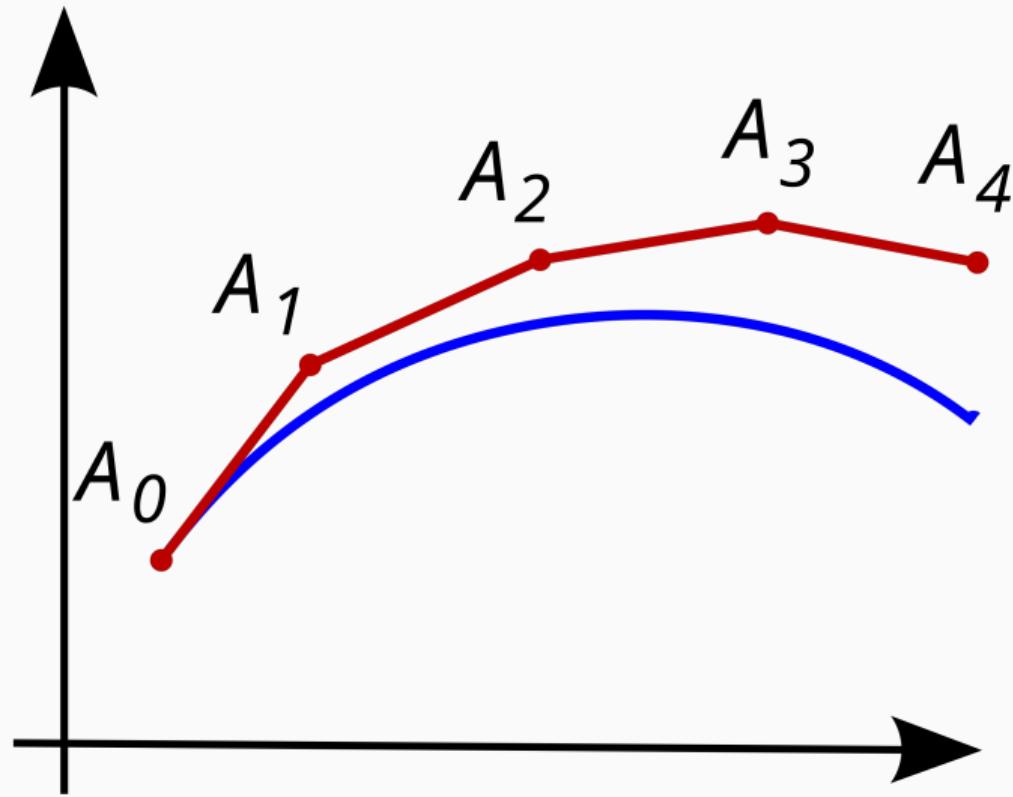
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- We get radius $\frac{1}{4}$ for the solution, while the actual solution $y(t) = \frac{1}{1-t}$ has radius 1.
- On the other hand for

$$\dot{y}(t) = \frac{1}{1-y(t)} = \sum_{k=0}^{\infty} y(t)^k ; y(0) = 0$$

we get radius $\frac{1}{2}$ which is identical to the actual radius of the solution

$$y(t) = 1 - \sqrt{1 - 2t}.$$

Extending the solution



Experiments

- Usable for small radius for Rocq's CauchyReals, but quite slow:



```
(* y'(t) = y(t) ; y(0) = 1 *)
```

```
Definition exp := (pivp_trajectory t(vx) 0 t(1) 1).  
Time Eval vm_compute in (seq_trajectory exp (-10)).
```

Experiments

- Usable for small radii



```
= (0, 1 :: nil)
  :: (1 # 8,
      431017047951035640929
      # 380371209777944985600 :: nil) :: nil
    : list (Q * list Q)
Finished transaction in 0.024 secs (0.023u,0.s) (successful)
```

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```
Definition exp := (pivp_trajectory t(vx) 0 t(1) 1).  
Time Eval vm_compute in (seq_trajectory exp (-26)).
```

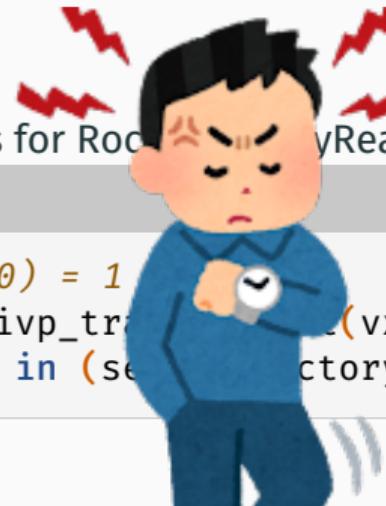
Experiments

- Usable for small radius for Rocq_real on \mathbb{R} , but quite slow:



```
(*  $y'(t) = y(t)$  ;  $y(0) = 1$ )
```

```
Definition exp := (pivp_trap (fun t y : R => y) 0 t(1) 1).  
Time Eval vm_compute in (set_option pp.simpl false) tactic exp (-26)).
```



Experiments

- Usable for small radii



```
= (0, 1 :: nil)
  :: (1 # 8,
      1550369632718466639670111564068092367629304358686077653993
      # 1368196398735263607727770629765114882112637001269248000000
      :: nil) :: nil
      : list (Q * list Q)
Finished transaction in 95.896 secs (88.74u,6.45s) (successful)
```

(* $y'(t) = y(t)$; $y(0) = 1$ *)

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Time Eval vm_compute in (seq_trajectory exp (-26)).
```

- A bit faster using CoRN, but still only usable for very small interval.
- Much faster using interval arithmetic (without explicit convergence).

Demo

```
(* interval solvers with different parameters *)
Module IIVP3_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 3%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP3_params.

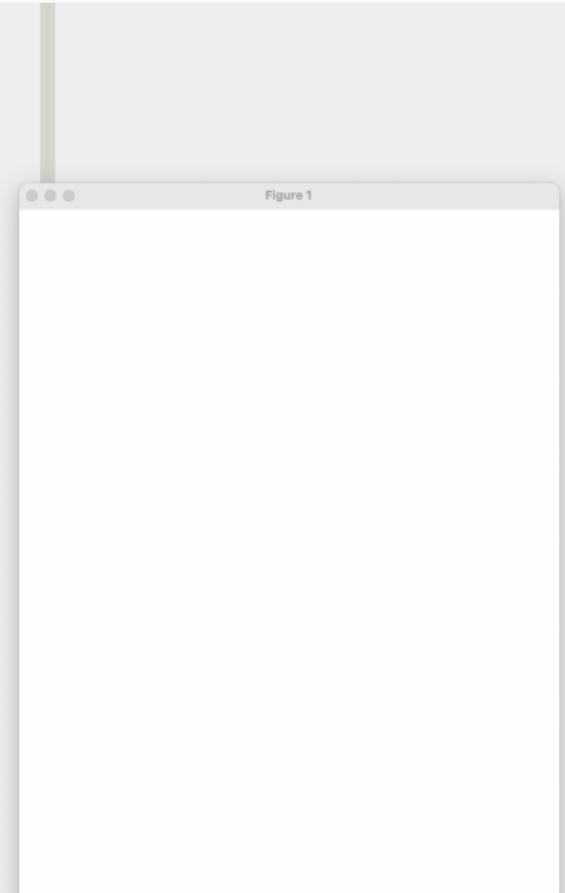
Module IIVP10_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 10%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP10_params.

Module IIVP15_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 15%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP15_params.

Module IIVP3 := IIVP IIVP3_params.
Module IIVP10 := IIVP IIVP10_params.
Module IIVP15 := IIVP IIVP15_params.

(* exponential function  $y' = y$  *)
Definition exp_t := (IIVP10.itrajectory t(vx) 1 0 5.0).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "time_series"
    "Exponential Function"
    (cons ("e^t" : string) nil)
    exp_t).
```



Demo

```
(* interval solvers with different parameters *)
Module IIVP3_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 3%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP3_params.

Module IIVP10_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 10%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP10_params.

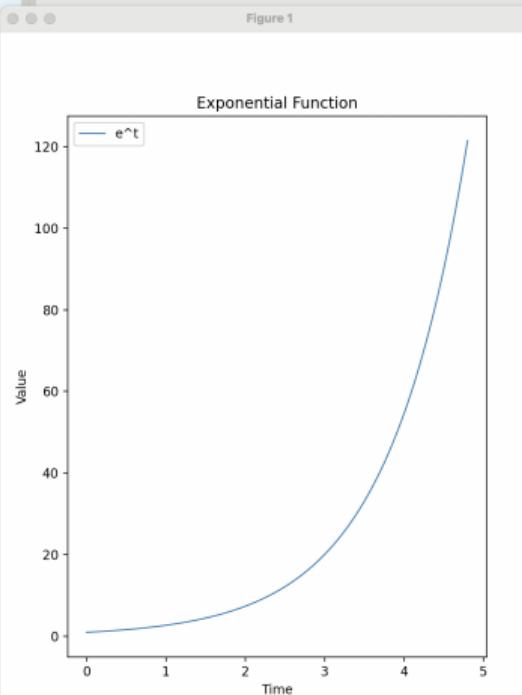
Module IIVP15_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 15%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP15_params.

Module IIVP3 := IIVP IIVP3_params.
Module IIVP10 := IIVP IIVP10_params.
Module IIVP15 := IIVP IIVP15_params.

(* exponential function  $y' = y$  *)
Definition exp_t := (IIVP10.itrajectory t(vx) 1 0 5.0).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "time_series"
    "Exponential Function"
    (cons ("e^t" : string) nil)
    exp_t)[]
```

```
Warning: Declaring a scope implicitly is deprecated; use in advance an explicit "Declare Scope bigQ_scope."
[undeclared-scope,deprecated-since=8.10,deprecated,default]
Warning: Output directory is unset. Using "/Users/holgerthies/Downloads/bigQ_scope". Use command line option "-output-directory" to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using "/Users/holgerthies/work/taylor_rcaps". Use command line option "-output-directory" to set a default directory.
[default-output-directory,filesystem,default]
Finished transaction in 4.001 secs (3.611u,0.31s) (successful)
```



Demo

```
(* interval solvers with different parameters *)
Module IIVP3_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 3%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP3_params.

Module IIVP10_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 10%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP10_params.

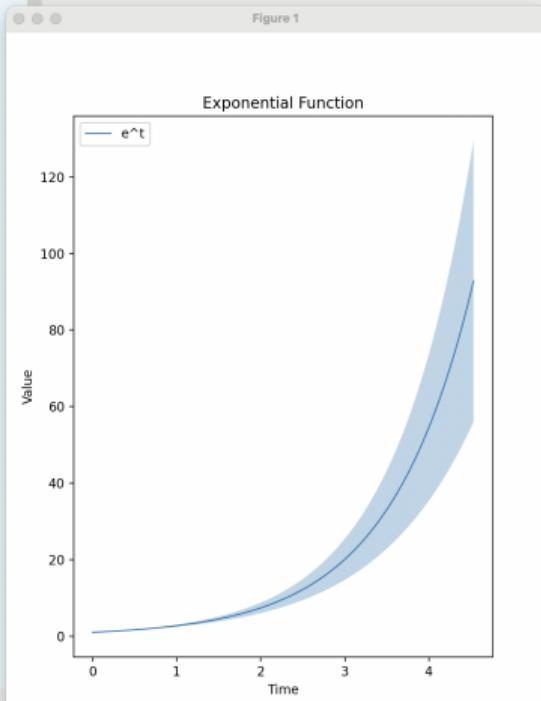
Module IIVP15_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 15%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP15_params.

Module IIVP3 := IIVP IIVP3_params.
Module IIVP10 := IIVP IIVP10_params.
Module IIVP15 := IIVP IIVP15_params.

(* exponential function  $y' = y$  *)
Definition exp_t := (IIVP3.itrajectory t(vx) 1 0 5.0).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "time_series"
    "Exponential Function"
    (cons ("e^t" : string) nil)
    exp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcaps". Use command line option
"-output-directory" to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcaps". Use command line option
"-output-directory" to set a default directory.
[default-output-directory,filesystem,default]
Finished transaction in 2.954 secs (2.604u,0.309s) (successful)
```



Demo

```
(* interval solvers with different parameters *)
Module IIVP3_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 3%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP3_params.

Module IIVP10_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 10%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP10_params.

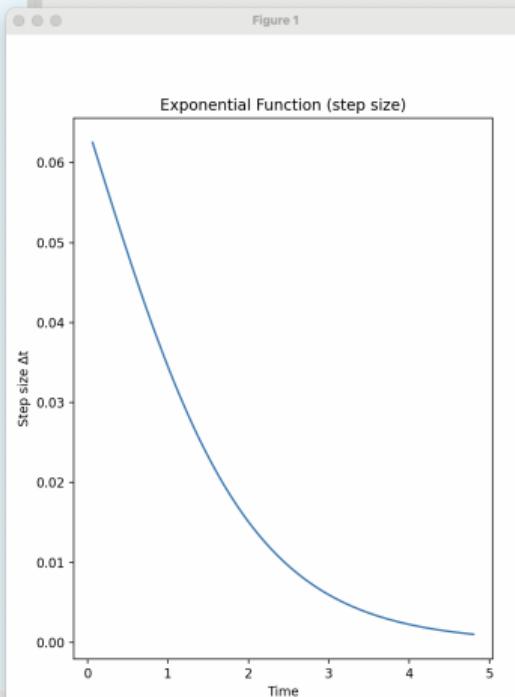
Module IIVP15_params <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 15%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP15_params.

Module IIVP3 := IIVP IIVP3_params.
Module IIVP10 := IIVP IIVP10_params.
Module IIVP15 := IIVP IIVP15_params.

(* exponential function  $y' = y$  *)
Definition exp_t := (IIVP10.itrajectory t(vx) 1 0 5.0).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "step_size"
    "Exponential Function"
    (cons ("e^t" : string) nil)
    exp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcaps". Use command line option
"-output-directory" to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcaps". Use command line option
"-output-directory" to set a default directory.
[default-output-directory,filesystem,default]
Finished transaction in 3.742 secs (3.427u,0.289s) (successful)
```



Demo

```
Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "time_series"
    "Exponential Function"
    (cons ("e^t" : string) nil)
    exp_t).

(* plot types :
  "time_series",
  "phase_space",
  "both", "scatter",
  "step_size",
  "width" *)

(*sin/cosine y1'=y2; y2'=-y1 *)
Definition sin_cos_t := (IIVP10.itrajectory t(vy;-vx) t(0;1) 0 6.5).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "both"
    "Sin/Cosine"
    ((("sin(x)" : string) :: ("cos(x)" : string) :: nil)
     sin_cos_t).

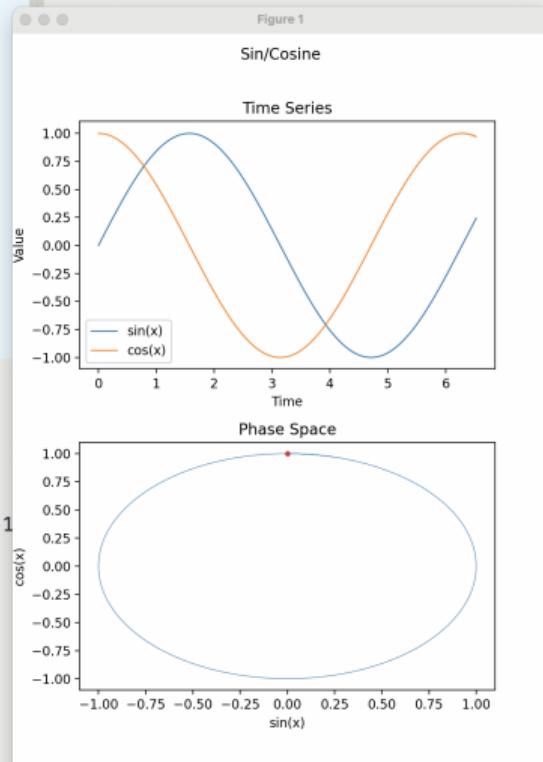
(* y1' = y2;
   y2' = mu * (1 - y1^2)y_2 - y_1;
   y0 = (0,0.1) *)
Definition vdp_example := (vdp_ivp 0.5).

Definition vdp_t := (IIVP15.itrajectory vdp_example.(ivp_rhs) vdp_example.(ivp_y0) 0 1
   ).
```



```
Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "both"
    "Van der Pol Oscillator"
    (("$x$" : string) :: ("$dot x$" : string) :: nil)
    vdp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcqs". Use command line option
"--output-directory" to set a default directory.
[default=~/Desktop/filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcqs". Use command line option
"--output-directory" to set a default directory.
[default=~/Desktop/filesystem,default]
Finished transaction in 1.336 secs (1.19u,0.125s) (successful)
```



Demo

```
Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "time_series"
    "Exponential Function"
    (cons ("e^t" : string) nil)
    exp_t).

(* plot types :
  "time_series",
  "phase_space",
  "both", "scatter",
  "step_size",
  "width" *)

(*sin/cosine y1'=y2; y2'=-y1 *)
Definition sin_cos_t := (IIVP10.itrajectory t(vy;-vx) t(0;1) 0 6.5).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "step_size"
    "Sin/Cosine"
    ((("sin(x)" : string) :: ("cos(x)" : string) :: nil)
    sin_cos_t).

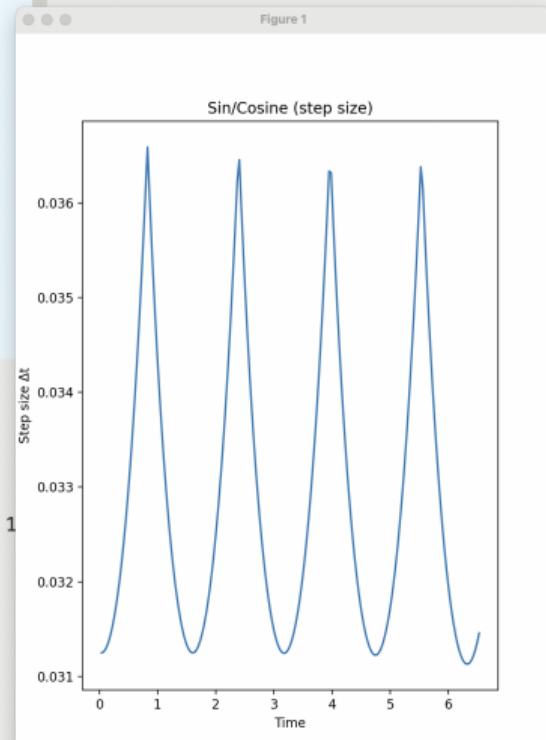
(* y1' = y2;
   y2' = mu * (1 - y1^2)y_2 - y_1;
   y0 = (0,0.1) *)
Definition vdp_example := (vdp_ivp 0.5).

Definition vdp_t := (IIVP15.itrajectory vdp_example.(ivp_rhs) vdp_example.(ivp_y0) 0 1
   ).
```



```
Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "both"
    "Van der Pol Oscillator"
    (("$x$" : string) :: ("$dot x$" : string) :: nil)
    vdp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcqs". Use command line option
"--output-directory" to set a default directory.
[default=taylor_rcqs,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcqs". Use command line option
"--output-directory" to set a default directory.
[default=taylor_rcqs,filesystem,default]
Finished transaction in 0.768 secs (0.678u,0.089s) (successful)
```



Demo

```
"time_series",
"phase_space",
"both", "scatter",
"step_size",
"width" *)

(*sin/cosine y1' = y2; y2'=-y1 *)
Definition sin_cos_t := (IIVP10.itrajectory t(vy;-vx) t(0;1) 0 6.5).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "time_series"
    "Sin/Cosine"
    ("("sin(x)" : string) :: ("cos(x)" : string) :: nil)
    sin_cos_t).

(* y1' = y2;
   y2' = mu * (1 - y1^2)y_2 - y_1;
   y0 = (0,0.1) *)
Definition vdp_example := (vdp_ivp 0.5).

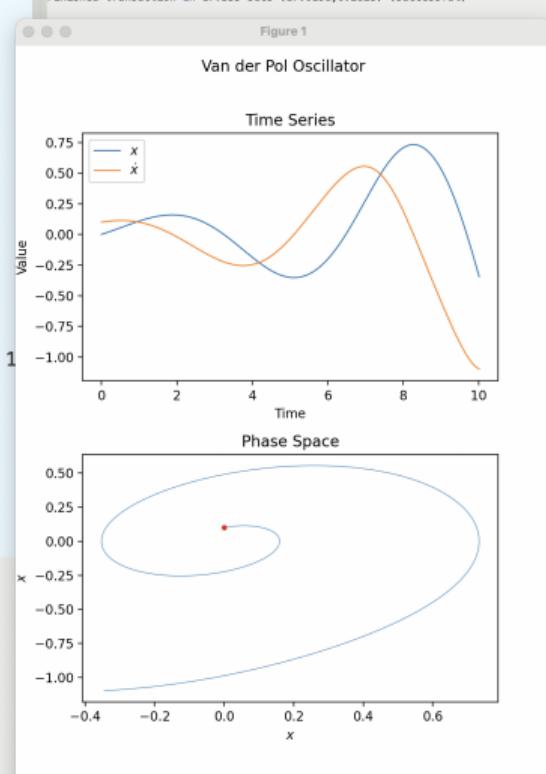
Definition vdp_t := (IIVP15.itrajectory vdp_example.(ivp_rhs) vdp_example.(ivp_y0) 0 1
*).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
    "both"
    "Van der Pol Oscillator"
    ("$x$ : string) :: ("$\dot{x}$ : string) :: nil)
  vdp_t.

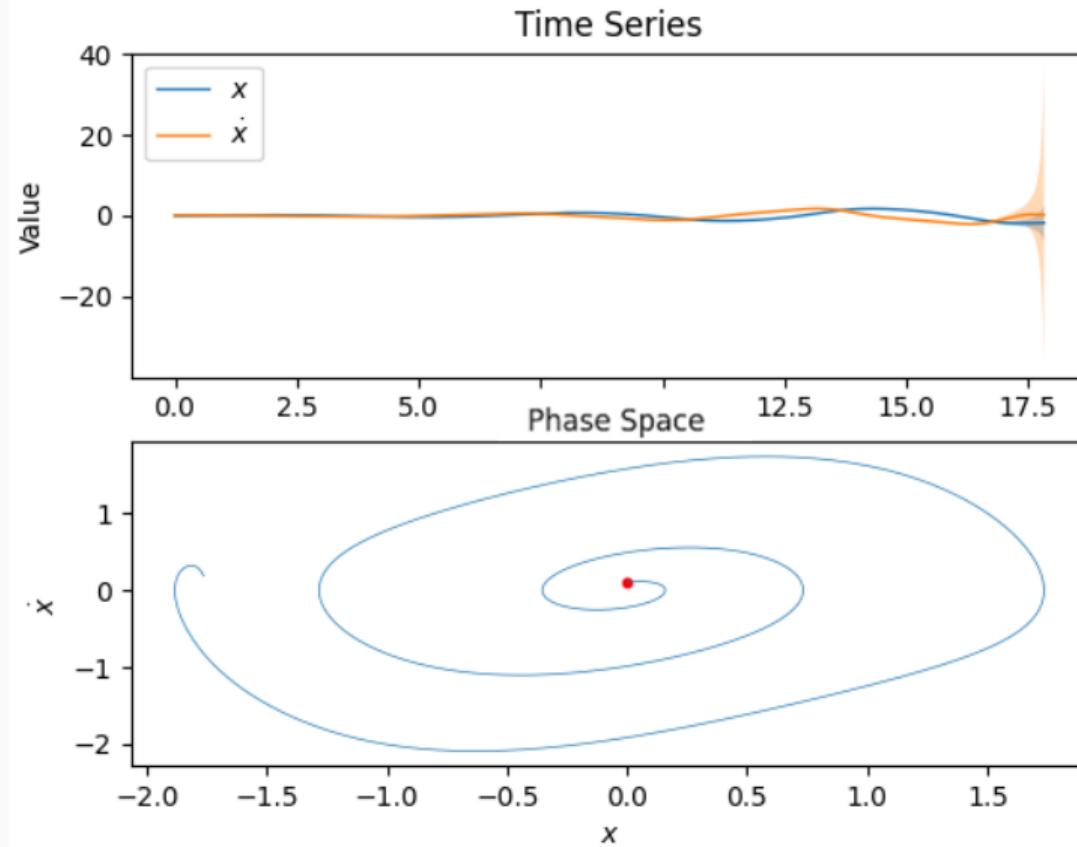
(* exact reals *)

Require Import Coq.Reals.Abstract.ConstructiveReals.
From Coq.Reals Require Import ConstructiveCauchyReals.
From Coq.Reals.Cauchy Require Import ConstructiveRcomplete.
Require Import cogreals.
```

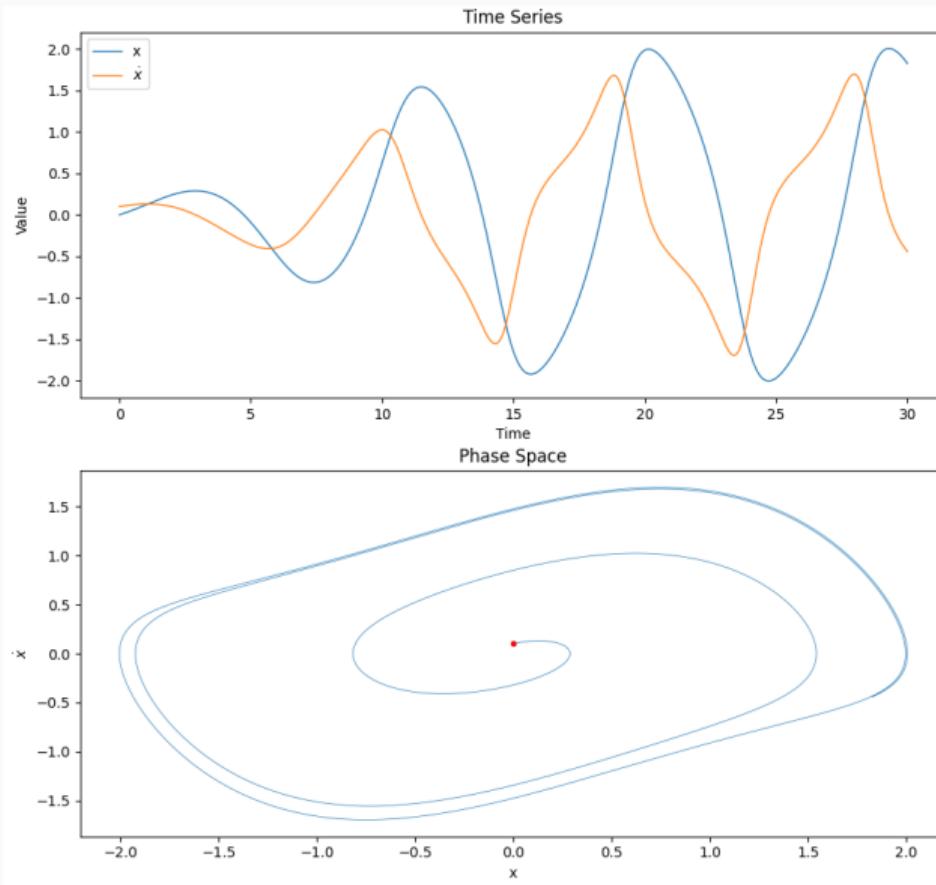
```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcqs". Use command line option
"-output-directory" to set a default directory.
[default=auto, redirect, filesystem, default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rcqs". Use command line option
"-output-directory" to set a default directory.
[default=auto, redirect, filesystem, default]
Finished transaction in 17.335 secs (17.019u,0.282s) (successful)
```



Demo



Demo



Summary and Future Work

- Formalized a constructive version of the Cauchy-Kovalevskaya theorem in Rocq, allowing to compute solution to analytic IVPs on small time intervals.

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