Computable Analysis for Extraction of Certified Programs

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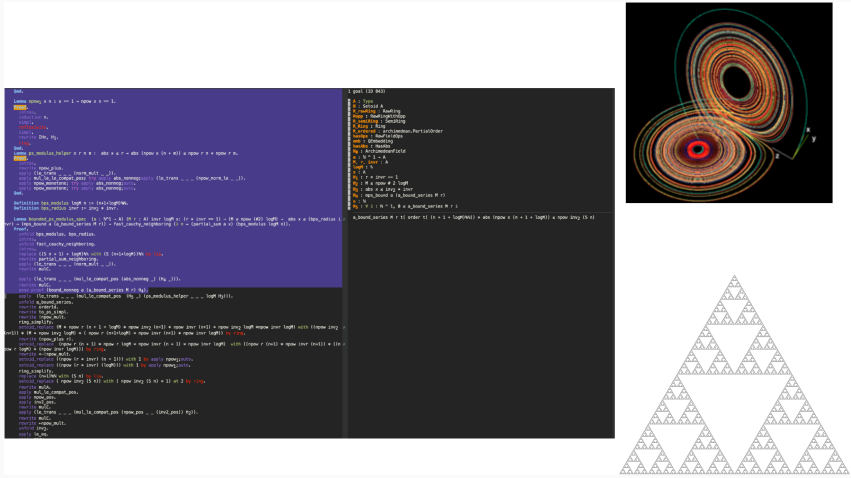
Overview

1. Real numbers and computation in the Rocq proof assistant
2. cAERN
3. Initial Value Problems and the Cauchy-Kovalevskaya theorem

Computation and Formal Proof

A **proof assistant** is an interactive tool for writing and checking formal proofs.



Computation in Rocq

There are several ways to use computation in the Rocq (formerly Coq) proof assistant:

ROCQ

```
Fixpoint div2 (n : nat) :=
  match n with
  | 0 | 1 => 0
  | S (S n') => (div2 n')+1
  end.

Compute (div2 100). (* = 50 : nat *)
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Prove specification in additional step:

```
Lemma div2_spec : ∀ n,  ∃ r, r ≤ 1 ∧ 2*div2 n + r = n.
Proof. ... Qed.
```

Computation in Rocq

Extract a witness from a constructive proof:

ROCQ

```
Lemma half_exists : ∀ n, {m | ∃ r, r ≤ 1 ∧ 2*m + r = n}.
Proof.
  apply (well_founded_induction lt_wf).
  intros n H;destruct n.
  exists 0;exists 0;split;auto.        (* n = 0 *)
  destruct n as [| n'].
  exists 0; exists 1;split;auto.       (* n = 1 *)
  destruct (H n') as [m P];[lia|].     (* n  = n'+2*)
  exists (S m).
  destruct P as [r [P1 P2]];exists r; split;auto;lia.
Defined.
Compute (proj1_sig (half_exists 100)). (* = 50 : nat *)
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Universes **Prop** for logical propositions and **Type** for computational data:

$$\exists x : A,\ P\,x : \text{Prop} \quad \text{(logical existence)}$$

$$\{\, x : A \mid P\,x \,\} : \text{Type} \quad \text{(computational witness)}$$

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Program Extraction

Extraction of Program Code (e.g. Ocaml or Haskell):

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OCaml

```
(** val half_exists : nat -> nat **)
let rec half_exists = function
| O -> O
| S n -> (match n with
          | O -> O
          | S n0 -> let s = half_exists n0 in S s)
```

Computation vs. Program Extraction

Computation in Rocq

+ Easy to use
+ Enables checking examples and testing conjectures quickly
+ Can be used directly in formal proofs

− Can be very slow
− Not suited for large computations

Program Extraction

+ Generates stand-alone programs
+ Typically faster
+ Allows integration of extracted code into existing developments

− Needs trust in extraction process
− May still need manual optimization
− Cannot be used directly within proofs
− Additional effort to compile/run

Real Numbers in Rocq

The Rocq standard library:

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- Equality defined as equivalence relation (Setoid).
  - Requires more complex reasoning: Standard tactics can not be used directly, instances must be shown to respect equality.

Real Numbers in Rocq

CoRN (Constructive Coq Repository at Nijmegen) [1]

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- Steeper learning curve.

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Real Numbers in Rocq

Some other libraries support verification of approximate or floating-point computations:

- **Flocq** [2]
  - Formalizes IEEE-style floating-point arithmetic.
  - Offers customizable precision, rounding modes, and proofs of rounding error bounds.
  - Useful for verifying numerical algorithms where machine-level floats are employed.

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- **Coq-Interval** [3]
  - An efficient implementation of interval arithmetic for intervals with floating-point end points.
  - Automatically translates expressions over classical reals into intervals, and provides tactics to solve inequalities.
  - Works well for common proof obligations.

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- Similar operations as in ERC libraries:
  - Exact arithmetic
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- Soundness of axiomatization via a realizability interpretation.
- Supports Nondeterminism/Multivalued Computation.
- Extraction to Haskell, **on top of the AERN library** for efficient exact real computation.

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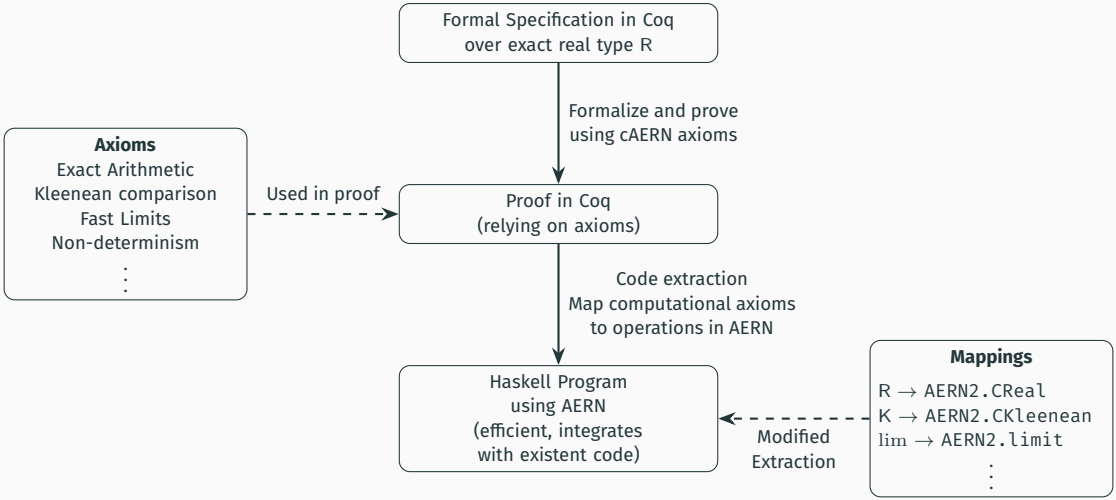
cAERN Overview



Code extraction example

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Lemma real_max_prop :
  forall x y, {z | (x >= y → z = x)
                 ∧ (x < y → z = y)}.
  Proof.
    intros.
    apply real_mslimit_P_lt.
    + (* max is single-valued *)
      ...
    + (* construct limit *)
      intros.
      apply (mjoin (x>y − prec n)
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      ++ intros [c1|c2].
         +++ (* when x > y − 2^−n *)
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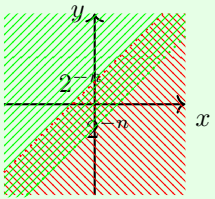
How to define $\max(x, y)$ without using the representation?

- Apart from primitive operations on reals, cAERN provides a multivalued choice operation. Can be used to defined M-split:



- Use M-split to define a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n = x$ if $x - y > 2^{-n}$, $z_n = y$ if $y - x > 2^{-n}$, and any of the two otherwise.
- $(z_n)_{n\in\mathbb{N}}$ is a fast Cauchy sequence converging to $\max(x, y)$.

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cAERN

+ Extracted programs are relatively efficient (comparable to hand-written AERN code).
+ Often more readable than low-level extraction.
+ Extracted code can be combined with non-verified parts of the AERN library.
+ Simple axiomatization, no setoids, proofs similar to using textbook reals.

− No direct computation in the proof assistant.
− Need to trust not only extraction, but also the implementation of ERC operations in AERN.
− (Minor) manual pre-processing of the extracted code is required before running.
− Direct implementation on names can sometimes be easier/more efficient than over generic abstract type.

Ordinary Differential Equations (ODEs)

As a practical application, we consider initial value problems (IVPs) for ordinary differential equations (ODEs). An IVP is an ODE together with an initial condition at

$$\dot{\vec{y}}(t) = \vec{F}(t, \vec{y}(t)), \quad \vec{y}(t_0) = \vec{y}_0, \quad \vec{F} : D \subseteq \mathbb{R}^{d+1} \to \mathbb{R}^d.$$

A solution is a differentiable function $\vec{y} : I \to \mathbb{R}^d$, where $I \subseteq \mathbb{R}$ is an open interval containing t_0, that satisfies the IVP.

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Remarks:

- Higher-order ODEs $\vec{y}^{(k)} = \vec{G}(t, \vec{y}, \ldots, \vec{y}^{(k-1)})$ can be rewritten as a first-order ODEs by introducing additional variables for the higher derivatives $\vec{y}^{(2)}, \ldots, \vec{y}^{(k)}$.
- Any ODE can be turned into an equivalent autonomous ODE by adding t as additional variable.
- By translation, one can assume $t_0 = 0$ and $\vec{y}_0 = \vec{0}$ without loss of generality.

Computability and Complexity of ODEs

Assume $f : [0,1] \times [-1,1] \to \mathbb{R}$ and consider the IVP $\dot{y}(t) = f(t, y(t)); y(0) = 0$

Assumptions on f	**Computability/Complexity of Solution**
(Polynomial-time) computable	All solutions can be non-computable [5]
+ Unique solution	Computable, but complexity can be arbitrarily high [6]
+ Lipschitz continuous	PSPACE complete [7]
+ Analytic	Solution is polynomial-time computable [8]

[5] Aberth, *The failure in computable analysis of a classical existence theorem for differential equations*, 1971.
[6] Ko, *On the computational complexity of ordinary differential equations*, Journal of Computer and System Sciences, 1983.
[7] Kawamura, *Lipschitz Continuous Ordinary Differential Equations are Polynomial-Space Complete*, Comp. Complexity, 2010.
[8] Müller & Moiske, *Solving initial value problems in polynomial time*, Proc. of JAIIO — Panel '93, 1993.

ODE solving in Proof Assistants

The problem seems quite natural for formal verification in a proof assistant:

- It is a classical topic in analysis with many applications in science and engineering.
- Most ODEs do not have a closed-form solution, therefore numerical methods are required to approximate solutions.
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Reasons include:

- Traditionally more focus on discrete math, logic, programming languages, and finite structures.
- Less developed theory and libraries for real analysis (but growing).
- Constructive reasoning natural in Rocq and can be more challenging.

ODE solving in Rocq

Some prior works in Rocq:

- The CoRN library includes an implementation of the Picard Iteration method. [9]

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- A few other works deal with specific systems or algorithms, but do not formalize a general theory.

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[10] Park & T. *A Coq Formalization of Taylor Models and Power Series for Solving ODEs*, Proc. of ITP 2024.

ODE solving in Rocq

Some prior works in Rocq:

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New implementation:

- Formalized in the Rocq proof assistant (approx. 10000 lines of code).
- Supports arbitrary dimension.
- Uses type-classes to support different implementations of real numbers.
- Usable both for program extraction and computation inside Rocq.
- More efficient implementation.

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The Cauchy-Kovalevskaya Theorem

Theorem (ODE version of the Cauchy-Kovalevskaya Theorem)
Let $U \subseteq \mathbb{R}^d$ be open and $f : U \to \mathbb{R}^d$ analytic. Then the initial value problem

$$\dot{\vec{y}}(t) = \vec{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0$$

has a unique solution which is analytic on an open interval $I \subseteq \mathbb{R}$ containing 0.

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Main idea of the classical proof:

- Expand $\vec{F}(\vec{y})$ in a formal power series around $\vec{y_0}$.
- Derive a recurrence for the Taylor coefficients of $\vec{y}(t)$ from the ODE $\dot{\vec{y}} = \vec{F}(\vec{y})$.
- Prove that the formal series has a positive radius of convergence by dominating it by a geometric majorant sequence.

Formal Power Series Solution

Let

$$\dot{\vec{y}}(t) = \vec{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0.$$

We inductively define a sequence of functions $\vec{F}^{[n]} : \mathbb{R}^d \to \mathbb{R}^d$ of multivariate functions by

$$\vec{F}^{[0]}(\vec{x}) = \vec{x}, \text{ and}$$
$$\vec{F}^{[n+1]}(\vec{x}) = J_{\vec{F}^{[n]}}(\vec{x}) \cdot \vec{f}(\vec{x})$$

Define a function by the formal power series

$$\vec{y}(t) = \sum_{i=0}^{\infty} \frac{1}{n!} \vec{F}^{[n]}(\vec{y_0}) t^n$$

By deriving $\vec{y}$ (as formal power series) and comparing coefficients, we see that $\vec{y}$ satisfies the IVP.

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$$\dot{y}(t) = F(y(t))$$
$$\ddot{y}(t) = \dot{y}(t) \cdot F'(y(t))$$
$$= F(y(t)) \cdot F'(y(t))$$
$$\vdots$$

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The Method of Majorants

- A (single-variate) power series $M(x) = \sum_{k=0}^{\infty} A_k x^k$ is said to majorize a d-variate power series $f(x) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha x^\alpha$ if

$$\forall k \in \mathbb{N}, \quad \sum_{|\alpha|=k} \left|a_\alpha\right| \ \leq \ A_k, \quad \text{where } |\alpha| = \sum_{i=1}^{d} \alpha_i.$$

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- We are mostly interested in simple (geometric) majorants of the form $\sum_{k=0}^{\infty} MR^k$ for constant $M, R \in \mathbb{R}$.
- In that case, the tail of the series satisfies

$$\Big|\sum_{|\alpha|>k} a_\alpha\, x^\alpha\Big| \;\leq\; M \sum_{n=k+1}^{\infty} (R\|x\|)^n \;=\; M\,\frac{(R\|x\|)^{k+1}}{1-R\|x\|}, \quad (R\|x\| < 1).$$

Solution Majorants

- We can show that if the right-hand side function $f : \mathbb{R}^d \to \mathbb{R}^d$ has a geometric majorant (M, R) then $(1, 2dMR)$ works as a geometric majorant for the solution y, guaranteeing convergence of $y(t)$ for all t with $|t| < \frac{1}{2dMR}$.

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- Example:

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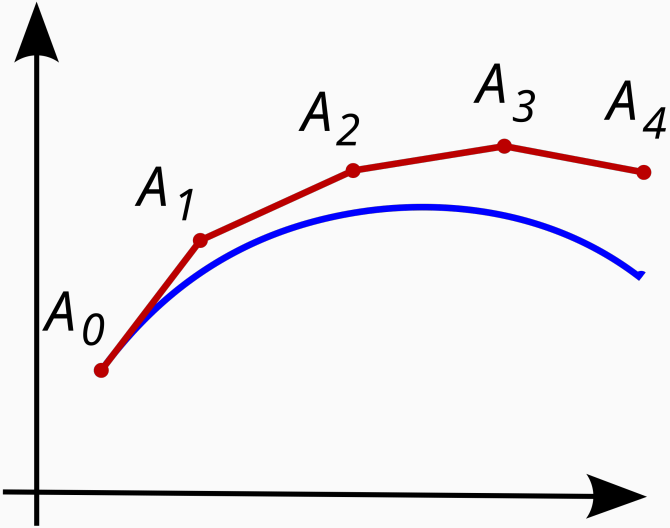
- On the other hand for

$$\dot{y}(t) = \frac{1}{1 - y(t)} = \sum_{k=0}^{\infty} y(t)^k \; ; \; y(0) = 0$$

we get radius $\frac{1}{2}$ which is identical to the actual radius of the solution

$$y(t) = 1 - \sqrt{1 - 2t}.$$

Extending the solution



Experiments

- Usable for for small radius for Rocq's CauchyReals, but quite slow:

ROCQ

```
(* y'(t) = y(t) ; y(0) = 1 *)
Definition exp := (pivp_trajectory t(vx) 0 t(1) 1).
Time Eval vm_compute in (seq_trajectory exp (-10)).
```

Experiments

- Usable for for small radi

```
     = (0, 1 :: nil)
       :: (1 # 8,
           43101704795103564092 9
           # 38037120977794498560 0 :: nil) :: nil
     : list (Q * list Q)
Finished transaction in 0.024 secs (0.023u,0.s) (successful)
```

ROCQ

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(* y'(t) = y(t) ; y(0) = 1 *)
Definition exp := (pivp_trajectory t(vx) 0 t(1) 1).
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(* y'(t) = y(t) ; y(0) = 1 *)
Definition exp := (pivp_trajectory t(vx) 0 t(1) 1).
Time Eval vm_compute in (seq_trajectory exp (-26)).
```

Experiments

- Usable for for small radius for Roc yReals, but quite slow:

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(* y'(t) = y(t) ; y(0) = 1
Definition exp := (pivp_tr (vx) 0 t(1) 1).
Time Eval vm_compute in (se ctory exp (-26)).
```

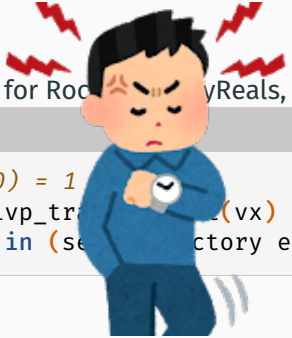

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Time Eval vm_compute in (seq_trajectory exp (-26)).
```

```
     = (0, 1 :: nil)
       :: (1 # 8,
           155036963271846663967011156406809236762930435868607765399 3
           # 136819639873526360772777062976511488211263700126924800000 0
           :: nil) :: nil
     : list (Q * list Q)
Finished transaction in 95.896 secs (88.74u,6.45s) (successful)
```

Experiments

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(* y'(t) = y(t) ; y(0) = 1 *)
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- A bit faster using CoRN, but still only usable for very small interval.
- Much faster using interval arithmetic (without explicit convergence).

Demo

```
(* interval solvers with different parameters *)
Module IIVP3_params  <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 3%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP3_params.

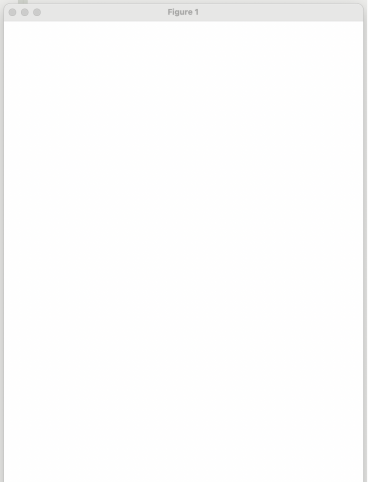
Module IIVP10_params  <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 10%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP10_params.


Module IIVP15_params  <: IIVP_PARAMS.
  Definition prec := 30%positive. (* interval precision *)
  Definition order := 15%N. (* taylor expansion order *)
  Definition max_steps := 1000%N. (* max number of iterations *)
  Definition step_factor := (Q2Fa 0.25) (* factor of max step size for each step *).
End IIVP15_params.

Module IIVP3  := IIVP IIVP3_params.
Module IIVP10  := IIVP IIVP10_params.
Module IIVP15  := IIVP IIVP15_params.

(* exponential function y' = y *)
Definition exp_t := (IIVP10.itrajectory t(vx) 1 0 5.0).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "time_series"
     "Exponential Function"
     (cons ("e^t" : string) nil)
     exp_t).
```



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Time Redirect "data" Eval vm_compute in
  (print_trajectory
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     "Exponential Function"
     (cons ("e^t" : string) nil)
     exp_t).
```
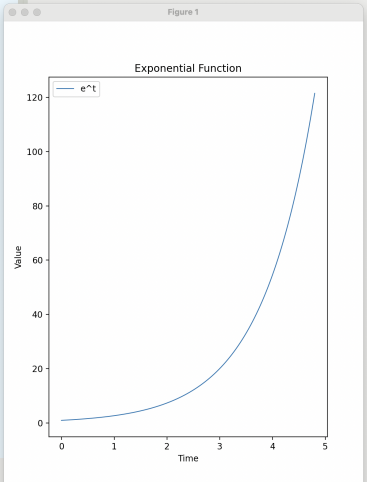

```
Warning: Declaring a scope implicitly is deprecated; use in advance an
explicit "Declare Scope bigQ_scope.".
[undeclared-scope,deprecated-since-8.10,deprecated,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]

Finished transaction in 4.001 secs (3.611u,0.31s) (successful)
```



Demo

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Module IIVP3_params  <: IIVP_PARAMS.
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End IIVP3_params.

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Time Redirect "data" Eval vm_compute in
  (print_trajectory
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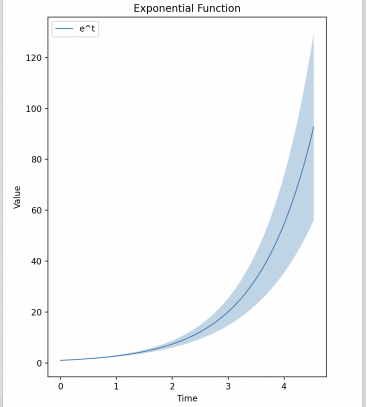
     "Exponential Function"
     (cons ("e^t" : string) nil)
     exp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]

Finished transaction in 2.954 secs (2.604u,0.309s) (successful)
```

Figure 1



Demo

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End IIVP15_params.

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(* exponential function y' = y *)
Definition exp_t := (IIVP10.itrajectory t(vx) 1 0 5.0).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "step_size"
     "Exponential Function"
     (cons ("e^t" : string) nil)
     exp_t).
```
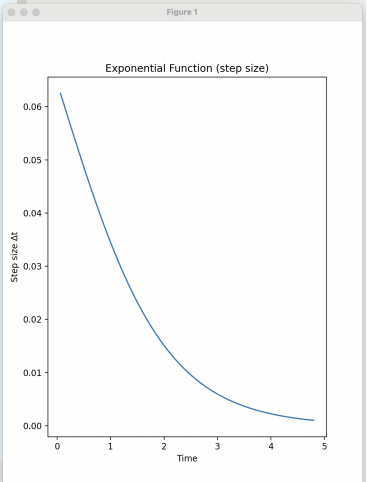
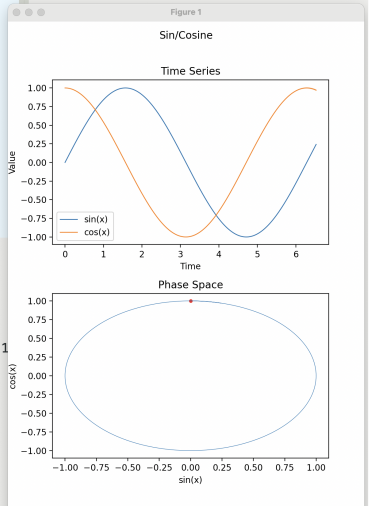

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]

Finished transaction in 3.742 secs (3.427u,0.289s) (successful)
```



Demo

```
Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "time_series"
     "Exponential Function"
     (cons ("e^t" : string) nil)
     exp_t).

(* plot types :
 "time_series",
"phase_space",
"both", "scatter",
"step_size",
"width" *)

(*sin/cosine y1'= y2; y2'=-y1 *)
Definition sin_cos_t := (IIVP10.itrajectory t(vy;-vx) t(0;1) 0 6.5).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "both"
     "Sin/Cosine"
     (("sin(x)" : string) :: ("cos(x)" : string) :: nil)
     sin_cos_t).

(* y1' = y2;
   y2' = mu * (1 - y1^2)y_2 - y_1;
   y0 = (0,0.1) *)
Definition vdp_example := (vdp_ivp  0.5).

Definition vdp_t := (IIVP15.itrajectory vdp_example.(ivp_rhs) vdp_example.(ivp_y0) 0 1
*).


Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "both"
     "Van der Pol Oscillator"
     (("$x$" : string) :: ("$\dot x$" : string) :: nil)
     vdp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]

Finished transaction in 1.336 secs (1.19u,0.125s) (successful)
```

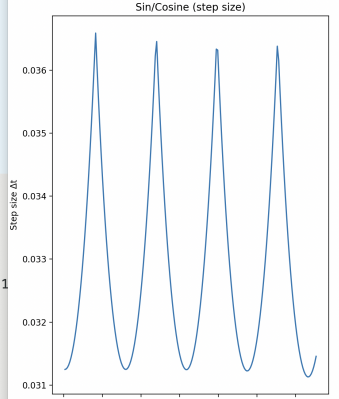


Demo

```
Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "time_series"
     "Exponential Function"
     (cons ("e^t" : string) nil)
     exp_t).

(* plot types :
 "time_series",
"phase_space",
"both", "scatter",
"step_size",
"width" *)

(*sin/cosine y1'= y2; y2'=-y1 *)
Definition sin_cos_t := (IIVP10.itrajectory t(vy;-vx) t(0;1) 0 6.5).

Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "step_size"
     "Sin/Cosine"
     (("sin(x)" : string) :: ("cos(x)" : string) :: nil)
     sin_cos_t).

(* y1' = y2;
   y2' = mu * (1 - y1^2)y_2 - y_1;
   y0 = (0,0.1) *)
Definition vdp_example := (vdp_ivp  0.5).

Definition vdp_t := (IIVP15.itrajectory vdp_example.(ivp_rhs) vdp_example.(ivp_y0) 0 1
*).


Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "both"
     "Van der Pol Oscillator"
     (("$x$" : string) :: ("$\dot x$" : string) :: nil)
     vdp_t).
```

```
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]
Warning: Output directory is unset, using
"/Users/holgerthies/work/taylor_rocqs". Use command line option
"-output-directory to set a default directory.
[default-output-directory,filesystem,default]

Finished transaction in 0.768 secs (0.678u,0.089s) (successful)
```

Figure 1



Demo

```
 "time_series",
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"both", "scatter",
"step_size",
"width" *)

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).


Time Redirect "data" Eval vm_compute in
  (print_trajectory
     "both"
     "Van der Pol Oscillator"
     (("$x$" : string) :: ("$\dot x$" : string) :: nil)
     vdp_t).

(* exact reals *)


Require Import Coq.Reals.Abstract.ConstructiveReals.
From Coq.Reals Require Import ConstructiveCauchyReals.
From Coq.Reals.Cauchy Require Import ConstructiveRcomplete.
Require Import coqreals.
```

```
Warning: Output directory is unset, using
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[default-output-directory,filesystem,default]
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[default-output-directory,filesystem,default]

Finished transaction in 17.335 secs (17.019u,0.282s) (successful)
```
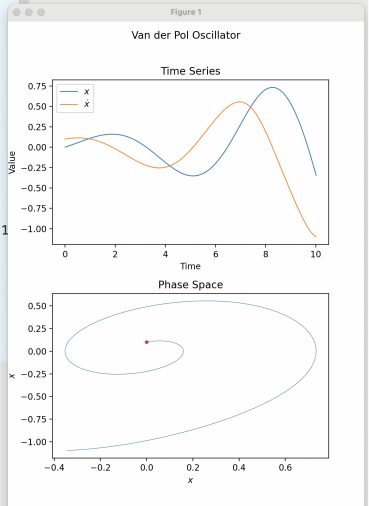
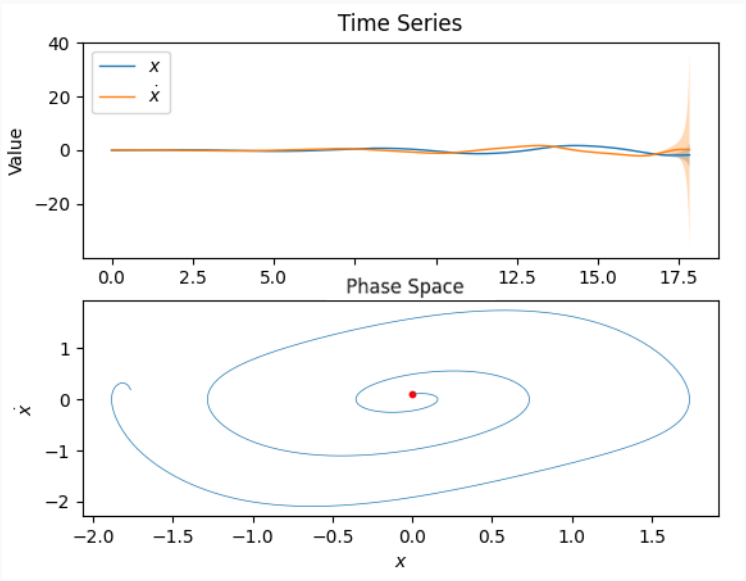
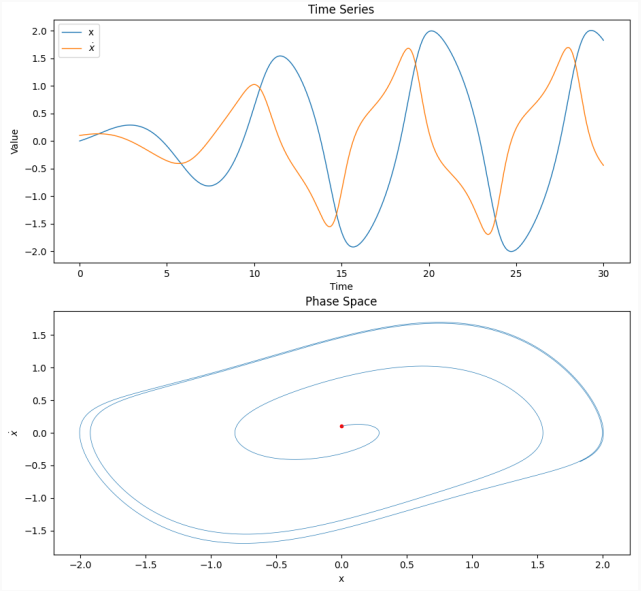



Demo


Time Series
Value
40
20
0
−20
x
$\dot{x}$
0.0
2.5
5.0
12.5
15.0
17.5
Phase Space
$\dot{x}$
1
0
−1
−2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

Demo



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