Homework 9

Dec 8, 2021

NOTE: Homework 5 is due Next Friday (Dec. 17, 2021).

1. In the following examples, indicate which statements constitute a simple and which a composite hypothesis:

   (i) When tossing a coin, let X be the r.v. taking value 1 if the head appears and 0 if the tail appears. Then the statement is: The coin is biased.

   (ii) X is a r.v. whose expectation is equal to 5.

2. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from a binomial distribution $B(1, p)$ where p is unknown and $n = 20$. To test: $H_0 : p = 0.2 \leftrightarrow H_1 : p \neq 0.2$, we use the following test function:

$$\varphi(\boldsymbol{X}) = \begin{cases} 1, & \sum\limits_{i=1}^{20} X_i \geq 7 \text{ or } \sum\limits_{i=1}^{20} X_i \leq 1, \\ 0, & \text{otherwise .} \end{cases} \tag{1}$$

   (i) Calculate the power function of the test at $p = 0, 0.1, 0.2, \cdots, 0.9, 1$.

   (ii) Determine the level of significance α and the probability of type II error.

3. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from normal distribution $N(\mu, 9)$ where μ is unknown. Let $\bar{X}$ be the sample mean. To test: $H_0 : \mu = \mu_0 \leftrightarrow H_1 : \mu \neq \mu_0$, the rejection region is

$$D = \left\{ \boldsymbol{X} = (X_1, \cdots, X_n) : |\bar{X} - \mu_0| \geq c \right\}.$$

   (i) Determine c such that the level of significance $\alpha = 0.05$.

   (ii) Determine the power function of the test.

   (iii) How does the power function change with sample size n? Plot the power function for $n = 20, 50, 100$ (suppose that $\mu_0 = 2, \alpha = 0.05$).

4. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from uniform distribution $U(0, \theta)$, where θ is unknown. Let $X_{(n)} = \max\{X_1, \cdots, X_n\}$. For the hypothesis testing problem $H_0 : \theta \leq \theta_0, \leftrightarrow \ H_1 : \theta > \theta_0$, consider the following test function:

$$\varphi(\boldsymbol{x}) = \begin{cases} 1, & X_{(n)} \geq c, \\ 0, & \text{otherwise .} \end{cases} \tag{2}$$

   (i) Calculate the power function of the test $\varphi(\boldsymbol{x})$, and show that it is an increasing function of θ.

   (ii) If $\theta_0 = 1/2$, determine the critical value c such that the level of significance $\alpha = 0.05$.

   (iii) Refer to (ii), plot the power function for $n = 20, 50$.

   (iv) Refer to (ii), determine the sample size n such that the power of the test at $\theta = 3/4$ is at least 0.98.

   (v) Refer to (ii), determine the sample size n such that the type II error of the test is no more than 0.02 at $\theta = 3/4$.

5. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from normal distribution $N(\mu, \sigma^2)$ where μ is unknown and σ is known.

   (i) For testing the hypothesis $H_0 : \mu \leq 0 \quad \leftrightarrow \quad H_1 : \mu > 0$, show that the sample size n can be determined to achieve a given level of significance α and given power $\pi(1)$.

   (ii) What is the numerical value of n for $\alpha = 0.05, \pi(1) = 0.9$ when $\sigma = 1$?

6. Two testers (A and B) analyzed the same product samples and obtained the following results:

tester	sample 1	sample 2	sample 3	sample 4	sample 5	sample 6	sample 7	sample 8
A	4.3	3.2	3.8	3.5	3.5	4.8	3.3	3.9
B	3.7	4.1	3.8	3.8	4.6	3.9	2.8	4.4

   Test whether there is a significant difference (in means) of the analysis results of testers A and B. Take $\alpha = 0.05$.

7. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from binomial distribution $B(1, \theta)$.

   (i) Considering the test problem $H_0 : \theta \leq 0.01, \leftrightarrow \quad H_1 : \theta > 0.01$, construct a test with level of significance $\alpha = 0.05$.

   (ii) Determine the sample size n such that the type II error of the above test is no more than 0.01 at $\theta = 0.08$.

8. Let $X_i, i = 1, \cdots, 90$ and $Y_j, j = 1, \cdots, 100$ be independent r.v.'s from the distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Suppose that the observed values of the sample variance are $s_x^2 = 4$, $s_y^2 = 9$.

   (i) At level of significance $\alpha = 0.05$, test the hypothesis $H_0 : \sigma_1 = \sigma_2, \leftrightarrow \quad H_1 : \sigma_1 \neq \sigma_2$.

   (ii) Find an expression for the computation of the power of the test for $\sigma_1 = 2$ and $\sigma_2 = 3$.