Exercise for General PhysicsII-set9

1. Given the definition of a physical operator's commutator: [A,B]=AB-BA (A,B are operators or matrix under provided basis, so order is important). Please show the following relations are correct: a) [A, B+C]=[A,B]+[A,C]. b)[A,BC]=B[A,C]+[A,B]C c)[AB,C]=A[B,C]+[A,C]B
2. Let's X represents the operator for measuring the position, and P the operator for momentum measurement. I will give you the following results: [X, X]=0, [P,P]=0, [X,P]=i$\hbar$. (You may wonder what the number means, since commutator is arithmetic among operators. General convention is to omit the identity matrix in the expression, so 0 means 0I which is a null matrix. etc.). Based on the above result, show the following: a) [X, f(X)]=0. (hint, f(X) Taylor expansion in power series of X) b) Hamiltonian is defined as $H=\frac{P^2}{2m}+V(X)$, (P,X are operators). Show that a) [X,H]=$\frac{i\hbar P}{m}$ (You may need result of problem 1); b) [P,H]=f(X)= $-i\hbar\frac{d(V(X))}{dX}$
(Taylor expansion here would help, here X is an operator, d(V(X))/dX is derivative of V treating X as variable, its result will be another function of operator X depending on function form of V). From this result, and what I derived in lecture: $\frac{d<A>}{dt}=\frac{1}{i\hbar}<[A,H]>$, you will see the relation （Ehrenfest Theorem） between Quantum Mechanics and Classical Mechanics. Please check it yourself though this is not required.
3. Prove that eigenvalues for Hermite operators are real, and eigenvalues associated with different eigenvalues are orthoganol. (I derived in lecture, but do it yourself or your ways)
4. For an operator A, I had shown that it could be expressed as $A=\sum_i a_i |u_i><u_i|$, where a_i is the eigenvalue and $|u_i>$ is the corresponding eigenvector of A. Prove that for the function of operator f(A), it can be expressed as: $f(A)=\sum_i f(a_i)|u_i><u_i|$ (Hint: Use Taylor expansion of f(A))

5. Griffiths P3.7.

(a) Suppose that $f(x), g(x)$ are two eigenfunctions of an operator $\hat{Q}$, with same eigenvalue q. Show that any linear combination of f and g is an eigenfunction of $\hat{Q}$ with same eigenvalue q.

(b) Check that $f(x)=e^x, g(x)=e^{-x}$ are eigenfunctions of the operator $\frac{d^2}{dx^2}$, with same eigenvalue. Construct two linear combinations of f and g that are orthogonal eigenfucntions on interval (-1.1).

6. Sequential measurements. An operator $\hat{A}$ representing observable A, has two normalized

eigenstates $|\psi_1>, |\psi_2>$ with eigenvalues a_1, a_2 respectively. Operator $\hat{B}$, representing observable B, has two normalized eigenstates $|\phi_1>, |\phi_2>$, with eigenvalues b_1, b_2. The eigenstates are related by:

$$|\psi_1>= (3|\phi_1> +4|\phi_2>)/5, \quad |\psi_2>= (4|\phi_1> -3|\phi_2>)/5$$

(a) When observable A is measured and the value a_1 is obtained. What is the state of the system immediately after the measurement?

(b) If B is now measured, what are the possible results, and what are their probabilities?

(c) Right after the measurement of B, A is measured again. What is the probability of getting a_1?

7. Consider a system whose H is given by $\hat{H} = a(|\phi_1><\phi_2| + |\phi_2><\phi_1|)$, *a* is a real number with unit in energy, and $|\phi_1>, |\phi_2>$ are normalized eigenstates of an observable A that has no degenerate eigenvalues.

(a) Is H a projection operator? What about $a^{-2}\hat{H}^2$? Note: for a projector, the necessary and sufficient condition is: $\hat{P}^2 = \hat{P}$

(b) Are the $|\phi_1>, |\phi_2>$ eigenstates of H?

(c) Calculate the commutator $[\hat{H}, |\phi_1><\phi_1|]$ and $[\hat{H}, |\phi_2><\phi_2|]$

(d) Find the eigenvalues and eigenvectors of H.

8. Griffiths P3.4.

(a) Show the sum of two Hermitian operators is Hermitian.

(b) Suppose $\hat{Q}$ is Hermitian, and α is a complex number. Under what condition (on α) is $\alpha\hat{Q}$ Hermitian too?

(c) When is the product of two Hermitian operators Hermitian?

(d) Show that the position operators and the Hamiltonian operator $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$ are Hermitian. (you may use the fact that P operator is an Hermitian, whose proof is in my note p523-524)

9. Consider a one-dimensional particle which moves along x-axis and the Hamilton operator H (expression in space) is: $\hat{H} = -\varepsilon\frac{d^2}{dx^2} + 16\varepsilon\hat{X}^2$, where ε is some real constant with

dimensions of energy.

(a) Is $\psi(x) = Ae^{-2x^2}$, where A is a normalization constant needs to be determined, an eigenfunction of H? If yes, what is the eigenvalue? (The integral: $\int_{-\infty}^{\infty} e^{-4x^2} dx = \sqrt{\pi}/2$)

(b) What is the probability of finding the particle along the negative x-axis?

(c) For the wave function $\phi(x) = 2x\psi(x)$, is it an eigenfunction and what is the eigenvalue if it is? Are the $\psi(x), \phi(x)$ orthogonal?

10. Consider a system (a 3-level system) whose state and two observables A, B are given (under 3 orthogonal certain basis) by:

$$|\psi> = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(Q0) What are the eigenstates and eigenvalues for A, B?

(a) For the given state $|\psi>$, what is the probability that a measurement of A yields value -1? ((b), (c) below are referring to state $|\psi>$ too)

(b) Let's carry out a set of two measurements where B is measured first and immediately afterwards, A is measured. Find the probability of obtaining a value of 0 for B and 1 for A.

(c) If we measured A first and then B immediately afterwards, what is the probability of obtaining a value of 1 for A and 0 for B?

(d) Do the two observable A, B commute?

11. Griffiths P 3.37.

The Hamiltonian for a three-level system is represented by a matrix:

$$H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$$, where a, b and c are real numbers.

(a) If the system starts out in the state: $|\psi(0)> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, what is $|\psi(t)>$?

(b) If the system starts out in the state: $|\psi(0)> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, what is $|\psi(t)>$?