Statistical Inference - Homework 2

Nov. 18, 2021

NOTE: Homework 2 is due Nov. 26, 2021.

1. Let $X_1, \cdots, X_m$ *i.i.d.* $\sim N(a, \sigma^2)$, $Y_1, \cdots, Y_n$ *i.i.d.* $\sim N(b, \sigma^2)$ and X_i's and Y_j's are independent. Let $\bar{X} = \sum_{i=1}^m X_i/m$, $\bar{Y} = \sum_{j=1}^n Y_j/n$, and
$$S^2 = \frac{1}{n+m-2}\left[\sum_{i=1}^m (X_i - \bar{X})^2 + \sum_{j=1}^n (Y_j - \bar{Y})^2\right].$$
Show that $(\bar{X}, \bar{Y}, S^2)$ is a sufficient and complete statistic of (a, b, σ^2).

2. Let $X_1, \cdots, X_n$ be a random sample from the distribution with p.d.f.
$$f(x;\theta) = \frac{1}{2\theta}\exp\left\{-\frac{|x|}{\theta}\right\}, \quad -\infty < x < +\infty,\ \theta > 0.$$
Show that $T = \sum_{i=1}^n |X_i|$ is a sufficient and complete statistic of θ.

3. Let r.v.'s $X_1, \cdots, X_n$ *i.i.d.* $\sim N(\theta, \theta^2)$, is $\bar{X}$ a sufficient statistic of θ?

4. Let $X_1, \cdots, X_n$ be a random sample from two parameter exponential distribution with p.d.f.
$$f(x;\lambda,\mu) = \lambda^{-1}\exp\left\{-\frac{x-\mu}{\lambda}\right\} I_{\{x>\mu\}},$$
where $0 < \lambda < +\infty$, $-\infty < \mu < +\infty$ are two unknown parameters. Show that
   (i) $\left(X_{(1)}, \sum_{i=1}^n X_{(i)}\right)$ is sufficient for (λ, μ);
   (ii) $X_{(1)}$ is independent of $\sum_{i=1}^n (X_i - X_{(1)})$.

5. Let $X_1, \cdots, X_n$ be a random sample from $U(\theta_1, \theta_2)$. Prove $(X_{(1)}, X_{(n)})$ are sufficient and complete statistics.

6. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from exponential distribution with p.d.f.
$$f(x;\theta) = \exp\{-(x-\theta)\} I_{\{x>\theta\}}, \quad -\infty < \theta < +\infty$$
(1) Derive the moment estimate of θ and show that it is unbiased.
(2) Calculate the variance of the moment estimate.

7. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from Gamma distribution with parameters α, β and p.d.f.
$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$
Derive the moment estimates of α and β.

8. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample be a random sample from a population with the pmf
$$P_\theta(X = x) = \theta^x(1-\theta)^{1-x}, \quad x = 0, 1, 0 \le \theta \le 0.5.$$
Find the method of moments estimator of θ. Find the mean squared errors of the estimator.