Recursive Methods

Mathematical Methods for Economics (771)

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A General Version

Readings

- Krusell, Per. (2014). Real Macroeconomic Theory, Chapter 4 "Dynamic Optimization"
- Judd, Kenneth, L. (1991). "A Review of *Recursive Methods in Economic Dynamics* by Stockey, N., Lucas Jr., R., and Prescott, E.," Journal of Economic Literature, 29(1), 69−77

Additional:

- Namay L. Stockey, Robert E. Lucas Jr., and Edward C. Prescott, 1989. Recursive methods in economic dynamics. Harvard university press (CH 4, 5).

Why Dynamic Programming?

☐ Most macro models involve a dynamic optimization problem and a resulting (infinite) sequence of real numbers that solves it.

- Dynamic decisions are made recursively (time period by time period) and not once-and-for-all. For example, savings between t and $t+1$ are decided on at t, and not 0
- But: the nature/structure of the optimization problem that a decision maker faces does not depend on the period in which they are making their decisions: it is identical (stationary) at every point in time.
- What changes from period to period are the initial conditions $\rightarrow$ the values of the variables that have been determined by the past or by "nature".

☐ The search for a sequence is sometimes impractical, and not always intuitive. An alternative approach is often available: *dynamic programming*

Why Dynamic Programming?

☐ Recursive methods (dynamic programming) is a fundamental tool of dynamic economic analysis:

- Useful conceptually as well as for analytical and, especially, *numerical* computation.
- Allows for comparative dynamic exercises, in dynamic models with uncertainty (e.g., a counterfactual policy intervention)
- Permit the inclusion of the stochastic shocks (i.e., a non-deterministic system).
- But the reduction of a dynamic model to a *recursive model* must be done carefully.

☐ We will go over the basics of this approach.
☐ The focus will be on concepts, as opposed to mathematical rigor or formal proofs

Recursive methods in economic dynamics

In general, we turn away from looking for a sequence of prices and allocations that satisfy equilibrium conditions, and instead look for a collection of policy functions, independent of time, which express current decisions and prices as functions of the state variables, which in turn are sufficient statistics of the past.

— *Judd (1991, p.71)*

Preliminaries

□ We are looking for a function $g(\cdot)$ that does not vary with time: a *decision* or *policy rule*

□ The critical step is defining the state variable

- variables whose values are already determined in period t (predetermined)

□ The choice of control variables can matter in how easily we can solve the model.

- decision variables whose values decision makers explicitly choose in period t with the goal of optimizing their objective function.

□ Whatever the choice, there must be enough constraints (market conditions) such that the values of the rest of the relevant variables in period t are determined.

The canonical problem

We consider the finite horizon dynamic optimization problem:

$$\begin{aligned} \max_{\{k_{t+1}\}_{t=0}^{T}} \quad & \sum_{t=0}^{T} \beta^t F(k_t, k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} \geq 0 \;\forall\; t \end{aligned} \tag{1}$$

where we want to find the sequence $\{k_{t+1}\}_{t+0}^{T}$ that maximizes the objective function (1).

Assume $F(k_t, k_{t+1}) = u(c_t)$ is a concave utility function, and β is the stationary discount factor. Let k_t be the capital stock at *the beginning of period* t, $y_t = f(k_t)$ a neoclassical production function, and c_t consumption in period t chosen at *the end of the period*.

The canonical problem

Then a social planner for this economy will solve the problem

$$\max_{\{c_t\}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{2}$$

where the choice path for c_t is constrained by production possibilities, represented by the law of motion

$$k_{t+1} = \underbrace{y_t - c_t}_{\text{Investment}} = f(k_t) - c_t, \tag{3}$$

where $k_0 > 0$ is the initial endowment of capital.

To make our lives simple we assume:

- the resource constraint (3) is binding;
- c_t and k_t are nonnegative for all t;
- $\lim_{c \to 0} u'(c) = \infty$ (i.e., $c_t = 0$ at any t cannot be optimal). So we ignore $c_t \geq 0$, but we still consider the inequality $k_{t+1} \geq 0$ (Krussel, p.44); and,
- $\delta = 1$ implies that the capital stock depreciates 100%.

The canonical problem

We now have a consumption-savings decision problem, with the following Lagrangian function:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t [u[f(k_t) - k_{t+1}] + \mu k_{t+1}] \tag{4}$$

The next step involves taking the derivative w.r.t the decision variable k_{t+1}. The first-order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &: \quad -\beta^t u'(c_t) + \beta^t \mu_t + \beta^{t+1} u'(c_{t+1}) f'(k_{t+1}) = 0, \quad t = 0, \ldots, T-1 \\
\frac{\partial \mathcal{L}}{\partial k_{T+1}} &: \quad -\beta^T u'(c_T) + \beta^T \mu_T = 0, \quad \text{for period } T
\end{aligned}$$

Finally, the Kuhn-Tucker conditions include:

$$\begin{aligned}
\mu_t k_{t+1} &= 0, \quad t = 0, \ldots, T \\
k_{t+1} &\geq 0, \quad t = 0, \ldots, T \\
\mu_t &\geq 0, \quad t = 0, \ldots, T
\end{aligned}$$

The canonical problem

The summary statement of the first-order conditions is then the “Euler equation”:

$$
\begin{aligned}
u'[f(k_t) - k_{t+1}] &= \beta u'[f(k_{t+1}) - k_{t+2}] f'(k_{t+1}), \quad t = 0, \ldots, T-1 \\
& k_0 \text{ given}, \; k_{T+1} = 0,
\end{aligned}
$$

where the capital sequence is what we need to solve for.[1]

The following conditions ensure a unique solution, such that the FOCs are sufficient:[2]

- The objective $U = \sum_{t=0}^{T} u(c_t)$ is (strictly) concave
- The constraint (choice) set is convex in $\{c_t, k_{t+1}\}$

1 Recall the optimization problem (1).

2 That is, computing the equilibrium policy function in a recursive model is valuable because it is a sufficient description of equilibrium, and from it one can derive any economic quantity (Judd, 1991). Note: From $\partial \mathcal{L} / \partial k_{T+1}$, and since $u'(c) > 0 \; \forall \; c$, we conclude that $\mu_T > 0$. This implies that $k_{T+1} = 0$: the consumer leaves no capital for after the last period.

The canonical problem

Let's interpret the key equation for optimization, the Euler equation:

$$\underbrace{u'(c_t)}_{\substack{\text{Utility lost if you} \\ \text{invest "one" more unit}}} = \underbrace{\beta u'(c_{t+1})}_{\substack{\text{Utility increase} \\ \text{next period per unit}}} \cdot \underbrace{f'(k_{t+1})}_{\substack{\text{Return on the} \\ \text{invested unit}}} \tag{5}$$

Thus, because of the concavity of u, equalizing the marginal cost of saving (LHS) to the marginal benefit of saving (RHS) is a condition for an optimum.

How do the primitives affect savings behaviour?
Three components:

(i) Consumption "smoothing" via strictly concave utility function u

(ii) Impatience via discount factor β

(iii) Income and substitution effects via the return to savings $f'(k_{t+1}) = R_t$

Example 4.1: Logarithmic utility (p. 47)

Example 4.2: CIES (constant intertemporal elasticity of substitution) utility function (p. 49)

Infinite Horizon & Sufficient conditions

Why should macroeconomists study the case of an infinite horizon?

- Altruism: people do not live forever, but they may care about their descendants
- Simplicity: with a long time horizon, finite- and infinite-horizon models show very similar results. Infinite horizon models are stationary in nature.

The infinite horizon only requires one additional condition to that in the finite case: the *transversality condition*. Both ensure the capital stock is zero in the limit. See Proposition 4.4, p. 56:

An alternative approach

- Our approach up to now has been to look for a sequence of real numbers $\{k_{t+1}\}_0^\infty$ that generates an optimal consumption plan.
- The solution was a difference (functional) equation: the Euler equation.
- The search for a sequence is sometimes impractical, and not always intuitive.
- An alternative approach that is intuitive and useful for both analytic *and* numerical computation, is dynamic programming using *recursive methods*

The value function

Using the canonical (neoclassical) model as an example, assume we can derive the individual's discounted value of utility in period t as:

$$V(k_t) \equiv \max_{\{k_{t+1+i}\}_{i=0}^{\infty}} \sum_{i=0}^{\infty} \beta^i u[f(k_{t+i}) - k_{t+1+i}] \tag{6}$$

Given the current state (k_t), $V(k_t)$ gives the supremum over all possible policies of the present values of current and future utility.

The value function in period $t+1$:

$$V(k_{t+1}) = \max_{\{k_{t+1+i}\}_{i=1}^{\infty}} \sum_{i=1}^{\infty} \beta^{i-1} u[f(k_{t+i}) - k_{t+1+i}] \tag{7}$$

We now separate the period t problem (6) from that of future periods …

The value function

...using maximisation-by-steps:

$$
\begin{aligned}
V(k_t) &= \max_{k_{t+1}\in[0,f(k_t)]} \Big\{ \underbrace{u[f(k_t)-k_{t+1}]}_{i=0} \\
&\quad + \max_{\{k_{t+1+i}\}_{i=1}^{\infty}} \sum_{i=1}^{\infty} \beta^i u[f(k_{t+i})-k_{t+1+i}] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
V(k_t) &= \max_{k_{t+1}\in[0,f(k_t)]} \Big\{ u[f(k_t)-k_{t+1}] \\
&\quad + \beta \max_{\{k_{t+1+i}\}_{i=1}^{\infty}} \sum_{i=1}^{\infty} \beta^{i-1} u[f(k_{t+i})-k_{t+1+i}] \Big\}
\end{aligned} \tag{8}
$$

By definition of (7), (8) equals:

$$
V(k_t) = \max_{k_{t+1}\in[0,f(k_t)]} \{u[f(k_t)-k_{t+1}] + \beta V(k_{t+1})\}
$$

The value function

$$V(k_t) = \max_{k_{t+1} \in [0, f(k_t)]} \{u[f(k_t) - k_{t+1}] + \beta V(k_{t+1})\} \tag{9}$$

- (9) is the dynamic programming formulation.
- It presents exactly the same problem as that shown in (6), but written in a recursive form;
- It is known as the Bellman equation, and it is a functional equation: the unknown is a function V
- If we find a V that satisfies (9) for any value of k_t, then all the maximizations on the RHS are well-defined
- The decision rule for $k_{t+1} = g(k_t)$, alluded to earlier, follows:

$$g(k_t) = \underset{k_{t+1}}{\operatorname{argmax}} \{u[f(k_t) - k_{t+1}] + \beta V(k_{t+1})\} \tag{10}$$

- To proceed, we need to assume that the value function $V(\cdot)$ exists, that a maximum exists and it is unique;
- Moreover, we need the *envelope theorem* to derive the functional Euler equation.

Example 4.5 Solving a parametric dynamic programming problem (work from the “guess” that the value function has the form $V(k) = a + b \log k$ to obtain the decision rule: $k' = \alpha\beta A k^{\alpha}$).

Simple steps to solving a dynamic optimization problem using the envelope theorem

$$V(k_t) = \max_{c_t, k_{t+1}} \left\{ u(c_t) + \beta V(k_{t+1}) \right\}, \quad \text{s.t. } c_t = f(k_t) - k_{t+1} \tag{11}$$

Assume the constraint binds, as before, and take FOC w.r.t k_{t+1}:

$$\begin{aligned}
\frac{\partial V(k_t)}{\partial k_{t+1}} \quad &: \quad \underbrace{\frac{\partial u(c_t)}{\partial c_t} \frac{\partial c_t}{\partial k_{t+1}}}_{\text{Chain rule}} + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = 0 \\
\therefore \quad & u'(c_t)(-1) + \beta V'(k_{t+1}) = 0 \\
\therefore \quad & u'(c_t) = \beta V'(k_{t+1}) \qquad (12)
\end{aligned}$$

We need to find $V'(k_{t+1})$ ...

Simple steps to solving a dynamic optimization problem using the envelope theorem

We can use the envelope theorem, by taking FOC w.r.t k_t and then iterating forward by one period:

$$\begin{aligned}
\frac{\partial V(k_t)}{\partial k_t} &: \quad \frac{\partial u(c_t)}{\partial c_t}\frac{\partial c_t}{\partial f(k_t)}\frac{\partial f(k_t)}{\partial k_t} + \beta\frac{\partial V(k_{t+1})}{\partial k_t} \\
&\therefore \quad u'(c_t)(1)f'(k_t) + \beta(0) \\
\therefore \frac{\partial V(k_{t+1})}{\partial k_{t+1}} &= V'(k_{t+1}) = u'(c_{t+1})f'(k_{t+1})
\end{aligned} \tag{13}$$

Substitute (13) into (12) to get the functional Euler equation, as in our "canonical problem":

$$\underbrace{u'(c_t)}_{\substack{\text{Utility lost if you} \\ \text{invest "one" more unit}}} = \underbrace{\beta u'(c_{t+1})}_{\substack{\text{Utility increase} \\ \text{next period per unit}}} \cdot \underbrace{f'(k_{t+1})}_{\substack{\text{Return on the} \\ \text{invested unit}}} \tag{14}$$

A General Version (similiar to Krussel, Ch 4.3.)

In a general form, for the model economy, the social-planning problem or, equivalently, the competitive equilibrium involves solving the following dynamic programming problem:

$$V(x_t, z_t) = \max_{y_t} [F(x_t, y_t, z_t) + \beta E_t V(x_{t+1}, z_{t+1})] \quad (15)$$

$$s.t. \quad x_{t+1} = G(x_t, y_t, z_t) \quad (16)$$

where,

- x_t: a vector of state variables in t;
- y_t: a vector of control variables in t;
- z_t: a vector of stochastic state variables in t;
- F(·,·): objective function to be maximized;
- (16): budget constraint. We include the expectations operator because of the presence of uncertainty z_t.

A General Version

The solution to this problem:

$$y_t = H(x_t, z_t) \tag{17}$$

(17) is the so called policy function (or decision rule) which describes how the control variable behaves as a function of the state variables in t.

Since the policy function optimizes the choice of the control variables for every permitted value of x_t, it must fulfill the following condition:

$$V(x_t, z_t) = F(x_t, H(x_t, z_t), z_t) + \beta E_t V(G(x_t, H(x_t, z_t), z_t), z_{t+1}) \tag{18}$$

A General Version

To find the policy function $H(x_t, z_t)$, we need the FOCs of (15) and its envelope condition,

$$\frac{\partial V(\cdot_t)}{\partial y_t} \quad : \quad 0 = F_y(x_t, y_t, z_t) + \beta E_t[\overbrace{V_x(G(x_t, y_t, z_t), z_{t+1}) G_y(x_t, y_t, z_t)}^{\frac{\partial V(x_{t+1}, z_{t+1})}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial G(\cdot)} \frac{\partial G(\cdot)}{\partial y_t}}] \tag{19}$$

$$\underset{\text{satisfies}}{H(\cdot_t)} \quad : \quad 0 = F_y(H) + \beta E_t[V_x(G(H))G_y(H))] \tag{20}$$

But, $V_x(\cdot_{t+1})$ is unknown

$$\frac{\partial V(\cdot_t)}{\partial x_t} = F_x + \beta E_t V_x G_x + H_x \underbrace{[F_y + \beta E_t V_x G_y]}_{=0 \text{ by } (20)} \tag{21}$$

where,

- $F_y(x_t, y_t, z_t), F_x(x_t, y_t, z_t)$: vector of derivatives of the objective function w.r.t. the control variables and state variables;
- $V_x(G(x_t, y_t, z_t), z_{t+1})$: vector of derivatives of the objective function w.r.t. the state variables in $t + 1$;
- $G_y(x_t, y_t, z_t), G_x(x_t, y_t, z_t)$: vector of derivatives of the budget constraints w.r.t. the control variables and state variables.

A General Version

□ Optimizing (15) such that (16) holds, implies that $G_x(\cdot) = 0$, and (21), iterated forward, becomes:

$$\therefore \frac{\partial V(\cdot_{t+1})}{\partial x_{t+1}} \quad : \quad V_x(x_{t+1}, z_{t+1}) = F_x(x_{t+1}, y_{t+1}, z_{t+1}) \tag{22}$$

Therefore, the FOCs (19) give the following functional Euler equation:

$$0 = F_y(x_t, y_t, z_t) + \beta E_t[F_x(G(x_t, y_t, z_t), y_{t+1}, z_{t+1}) G_y(x_t, y_t, z_t)] \tag{23}$$

Solving for y_t gives the policy function (17).

Self-study

Work through examples 4.1 (p. 47), 4.2 (p. 49), 4.5. (p. 63), and Ch 4.3 (pp. 67-69).