Investment Handout

Honours Macroeconomics - Gideon du Rand

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1 Intro

It is crucial to note what we will mean by "capital" in this lecture: physical productive capital

I.e. any physical object that has the following features:

1. It can be used (usually in combination with other things) to produce "goods" that may be either consumption goods or other capital goods
2. It can be used repeatedly, to some extent. I.e. it can be accumulated.

The physical components of a production plant are an example.

We do not mean "financial capital" in any way. I.e. not the concept we mean when we say a bank should be "well capitalized". In this note I will use "financial asset" for that concept.

2 Simple static micro model

Consider a firm that can rent productive capital at rate r_K per unit that produces consumption goods for sale.

Let the profits at a point in time be given by

$$\pi\left(k,\mathbf{X}\right)-r_Kk$$

where $\pi\left(.\right)$ accounts for all other dimensions of optimization over other relevant factors such as the production technology, price of the outputs, quantities of other inputs all collected in the vector $\mathbf{X}$. Think about this as a more complete model where we solve for optimal choices of all variables as functions of some fixed level of capital k and then consider the final problem of selecting the the optimal k.

The necessary optimality condition

$$\frac{\partial\pi\left(k^*,\mathbf{X}\right)}{\partial k}=r_K \tag{1}$$

implicitly defines the optimal choice for k as a function of r_K if it could be costlessly obtained, hence the "desired capital stock" k^*.

For this quantity to be well defined, the the second order sufficient condition for optimality must also hold:

$$\frac{\partial^2 \pi \left(k^*, \mathbf{X}\right)}{\partial k^2} < 0$$

This would be the case if, say, there are diminishing marginal product to capital.

Taking derivatives with respect to r_K on both sides of 1 yields:

$$\begin{aligned}
\frac{\partial \left[\frac{\partial \pi(k^*(r_K),\mathbf{X})}{\partial k}\right]}{\partial r_K} &= 1 \\
\frac{\partial \left[\frac{\partial \pi(k^*(r_K),\mathbf{X})}{\partial k}\right]}{\partial k} \frac{\partial k^*\left(r_K\right)}{\partial r_K} &= 1 \\
\frac{\partial^2 \pi \left(k^*\left(r_K\right), \mathbf{X}\right)}{\partial k^2} \frac{\partial k^*\left(r_K\right)}{\partial r_K} &= 1 \\
\frac{\partial k^*\left(r_K\right)}{\partial r_K} &= \left(\frac{\partial^2 \pi \left(k^*\left(r_K\right), \mathbf{X}\right)}{\partial k^2}\right)^{-1} < 0
\end{aligned}$$

Where the last inequality comes from our assumption that we are considering a firm with a well defined optimal capital stock, and the expression as a whole reflecting the familiar result that the optimal capital stock is decreasing in the rental rate of capital, if this is the case.

This static view is clearly the "best case scenario" in any dynamic world: these conditions pertain only to the dynamic environment where the firm can costlessly and *instantaneously* achieve any desired final capital stock. In such a world, if the rental cost of capital is a time varying process $r_K(t)$, the desired capital could be maintained continuously to respond to any change in $r_K(t)$.

2.1 User cost of capital

Since most firms own the capital they use in production, there is no clear empirical counterpart to $r_K(t)$

The "user cost of capital" is the phrase assigned to the accurate internal measure of the marginal cost of capital for a firm that owns its own capital.

In the typical "classical" view, this should include:

1. The opportunity cost of holding physical capital rather than the best alternative financial asset.
   Suppose the rate on all financial assets is given by r_t and the price of capital $p_{K,t}$ then the per unit opportunity cost of holding stock of capital k_t at moment t is simply $r_t p_{K,t}$
2. The depreciation in value of the capital at constant rate δ implies an instantaneous cost of $\delta p_{K,t}$ per unit of capital

3. The capital losses per unit from holding capital is given by the negative of the rate of change of the price of capital (since gains are negative losses and we are considering the marginal cost of capital):$-\triangle p_{K,t+1} \equiv -(p_{K,t+1} - p_{K,t})$

So that

$$\begin{aligned} r_{K,t} &= p_{K,t}r_t + \delta p_{K,t} - (p_{K,t+1} - p_{K,t}) \\ &= p_{K,t}\left[r_t + \delta - \frac{p_{K,t+1} - p_{K,t}}{p_{K,t}}\right] \end{aligned}$$

If there are taxes of rate τ on the firm and the firm receives a tax credit of f on the value of the stock of capital it holds the user cost of capital is lower:

$$r_{K,t} = (1 - \tau f)\, p_{K,t}\left[r_t + \delta - \frac{p_{K,t+1} - p_{K,t}}{p_{K,t}}\right]$$

Note that Romer motivates this equation by referring to a tax credit on *investment* activities, implying that the credit applies to changes in capital stock. I disagree with his wording: - we can see what equation he intends to attain, but if that equation is to be an accurate discription of the story he is telling it must be a tax credit on the value of the current capital stock, since the user cost of capital applies to the entire stock held.

2.2 Problems with the baseline model

- Any discrete change in r_K implies a discrete change in k^*.
  - In continuous time this implies an infinite rate of investment
  - even in discrete times it predicts an extreme level of volatility in investment and a large degree of volatility in capital stocks (at least as large as that in interest rates)
  - at any point in time the actual investment is limited by the current level of output of the economy, hence investment cannot exceed this
- No allowance for expectations
  - current marginal revenue product of capital equated to current marginal user cost
  - empirically it is clear that expectations about future costs and returns are crucial determinants of current investment
- Theoretical source: infinitely flexible level of capital stock
- Empirical reality: factories and machines take time to be built and put into operation

- Theoretical solution:
  - Adjustment Costs to capital stock
    - short cut way to get capital to evolve smoothly
    - models trade-offs exogenously: the faster you try to build a new power plant the more resources have to be diverted from all other uses, which rapidly increases the price of doing so and reduces the efficiency of the last unit of resources employed in the change of capital
  - Internal Adjustment costs:
    - costs of installing new capital and training workers
  - External Adjustment costs:
    - inelastic supply of capital stock - increased demand bids up price, reduces optimal increment

3 A dynamic model of investment with internal adjustment costs

We study a discrete version of the model first assuming no uncertainty, then give the continuous time analogues of the equations that govern dynamics so that we can study a phase diagram like the RCK model.

We will simplify the production/profit generating technology to a linear one at firm level and use non-linear capital stock adjustment costs to avoid undefined investment demand. This is for analytic convenience and not realism.

Market Assumptions:

- Consider firms in an industry with N firms where N is large enough for each firm to ignore it's impact on the aggregate levels of variables.
- Demand for the good on aggregate is downward sloping in price but much larger than any one firm can provide.
- The interest rate on financial assets is constant at r per period

Production Technology: Firms operate a production technology that provides cash-flow of $\pi(K_t)$ currency units per unit of capital stock of the individual firm k_t, where K_t is the aggregate level of capital in the industry and $\pi'(K_t) < 0$ (this is the downward sloping demand part).

Investment/Capital accumulation Technology: For simplicity, we assume a law of motion for capital given by:

$$\begin{aligned} k_t &= k_{t-1} + I_t \\ \triangle k_t &= I_t \end{aligned}$$

where I_t is the amount of investment done in period t - the sequence of these decisions will be the only choice variables of the firm as we assume some given level of initial capital $k_0 > 0$

Note that this implies two things:

- a zero depreciation rate - again for analytic convenience - adding it will not aid the intuition of the results and come at the cost of more complicated formulae.
- current investment is available for current production - this goes against the "time to build" argument but again yields simpler equilibrium equations at little cost to the central message of the model, since we will be analysing it in continuous time anyway

Additionally we assume that the firm faces capital adjustment costs that are a function of the size of the investment I_t. That is, for any choice of I_t the firm must pay cost $C(I_t) \geq 0$ where we assume the following additional structure on this cost function:

$$\begin{aligned} C(0) &= C'(0) = 0 \\ C''(.) &> 0 \end{aligned}$$

- zero cost and zero marginal cost at zero investment level - so maintaining the *status quo* capital stock costs nothing.
- positive and increasing marginal cost in the size of investment for any non-zero investment level, whether positive or negative - i.e. it is costly both to increase *and* to decrease the capital stock at every possible level of capital. (the simple quadratic function $C(I_t) = aI_t^2$ for $a > 0$ would satisfy these restrictions)
- Note for analytic results later that these assumptions imply that the marginal cost of non-zero investment has the same sign as the investment: I.e. reducing capital stock (negative investment) has negative marginal cost, so that the effect on total cost is positive.

let the price of a unit of capital be $p_{K,t}$ currency units, so that a choice of investment of I_t units of capital reduces the available cash-flow by $p_{K,t}I_t + C(I_t)$

Objective of firms: A firm aims to maximize its present discounted market value. We assume that the firm makes only one type of decision: how much of available resources to use to adjust the capital stock of the firm. This in turn means that something happens to the remainder of the resources, and we assume they are the "profits" of the firm that are paid out in each period to the owners of the equity in the firm (left unmodelled). This means that the firms market value is the discounted sum of all future distributions to equity holders of cash-flow.

In any period, the cash-flow not used to adjust the level of capital are given by: $\pi(K_t) k_t - p_{K,t} I_t - C(I_t)$ and the constraint on the firm in each period is the capital accumulation law.

For simplicity we assume the price of capital is fixed p_K. In the final lecture, we will explicitly consider the equilibrium market price of capital.

3.1 Solving the optimization problem of the representative firm

The problem of the firm thus can be stated as:

$$\max_{\{I_t\}} \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[\pi(K_t) k_t - p_K I_t - C(I_t)\right]$$

$$\text{subject to: } k_t = k_{t-1} + I_t$$

The Lagrangian for this problem is thus:

$$L = \sum_{t=0}^{\infty} \left\{ \frac{1}{(1+r)^t} \left[\pi(K_t) k_t - p_K I_t - C(I_t)\right] + \lambda_t \left[k_{t-1} + I_t - k_t\right] \right\}$$

Where λ_t is as usual the shadow price of the constraint. In this situation, it turns out to be useful to consider this concept a little further and use a slightly different version of it.

The shadow price of the constraint measures the impact on the maximum value of the objective function of the decision maker of an exogenous relaxation of the constraint by a marginal amount. This means λ_t is the marginal impact on the *current* value of the firm of a marginal increase in the *period* t capital stock.

Crucially, this marginal impact is in current value terms - i.e. in period 0 units of account (since we are starting at $t = 0$)

Define the period t value ("future value" in theory-of-interest language) of this marginal contribution to firm value as

$$\begin{aligned} q_t &= (1+r)^t \lambda_t \\ &\text{or} \\ \lambda_t &= \frac{1}{(1+r)^t} q_t \end{aligned}$$

Using this we can rewrite the Lagrangian as:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \{[\pi(K_t) k_t - p_K I_t - C(I_t)] + q_t [k_{t-1} + I_t - k_t]\}$$

First Order Conditions:

It will be convenient to treat both the investment rate (I_t) and the capital stock (k_t) as choice variables and solving the resulting condition in terms of q_t which has a nice interpretation in this model: the marginal value to the firm (in units of capital in period t) of an increase in period t capital.

FOC with respect to I_t

$\frac{\partial \mathcal{L}}{\partial I_t} = 0$

$$\begin{aligned} \frac{1}{(1+r)^t} [-p_K - C'(I_t^*) + q_t] &= 0 \\ q_t - p_K &= C'(I_t^*) \end{aligned}$$

Interpretation:

- Start at $I_t^* = 0$ - I.e. when optimal investment is zero, or that the capital stock is at its optimal level. The equation above implies this occurs when: $q_t = p_K$ that is the marginal value to the firm of a marginal unit of capital stock is exactly equal to the selling/buying price in the market, so the firm would not gain anything by changing its capital stock.
- If $q_t > p_K$, capital is more valuable internally than on the market, so the firm should increase its capital stock, i.e. choose a positive investment level $I_t^* > 0$;
- If $q_t < p_K$, capital is more valuable on the market than its contribution to firm value, so the firm should sell off some of its capital, i.e. $I_t^* < 0$.
- In short the condition says that investment should be chosen so that the marginal cost of the investment equals the marginal benefit (the marginal increase in the value of the firm).

FOC with respect to k_t

The Lagragian is an infinite series of objective functions and capital accumulation constraints in every period. To see why the result below holds, expand the summation explicitly for a few of these terms, specifically for arbitrary periods t and $t+1$:

$$\begin{aligned} \cdots &+ \frac{1}{(1+r)^t} [\pi(K_t) k_t - p_K I_t - C(I_t)] + \frac{1}{(1+r)^t} q_t [k_{t-1} + I_t - k_t] \\ &+ \frac{1}{(1+r)^{t+1}} [\pi(K_{t+1}) k_{t+1} - p_K I_{t+1} - C(I_{t+1})] + \frac{1}{(1+r)^{t+1}} q_{t+1} [k_t + I_{t+1} - k_{t+1}] + \cdots \end{aligned}$$

The variable k_t occurs in *one* element of the sequence of objective functions (the one for period t), but *two* elements of the sequence of capital accumulation contraints:

- in period t, k_t is the "new" capital stock
- in period $t+1$, k_t is the "old" capital stock.

When we take the derivative with respect to k_t then, get the following result: $\frac{\partial \mathcal{L}}{\partial k_t} = 0$

$$\begin{aligned}
\frac{1}{(1+r)^t}\left[\pi\left(K_t\right) - q_t\right] + \frac{1}{(1+r)^{t+1}}\left[q_{t+1}\right] &= 0 \\
(1+r)\,\pi\left(K_t\right) &= (1+r)\,q_t - q_{t+1} \\
\pi\left(K_t\right) &= \frac{1}{(1+r)}\left(rq_t - \triangle q_t\right)
\end{aligned}$$

where we have defined $\triangle q_t = q_{t+1} - q_t$

The left hand side is the marginal revenue product of capital and the right hand side captures the marginal user cost of capital - the interest foregone from holding capital minus the capital gains.

Alternatively, rewrite the equation to

$$q_t = \pi\left(K_t\right) + \frac{1}{(1+r)} q_{t+1}$$

One can interpret this as saying that the firm is optimizing if it holds a capital stock whose contribution to the value of the firm is equal to the current period contribution of that capital ($\pi\left(K_t\right)$) plus its discounted contribution to the valuation of the firm next period ($\frac{1}{(1+r)} q_{t+1}$). Since that valuation next period will include the valuation the following period we can use iterative limits to see what this implies

Iterating this equation (repeatedly substituting for q_{t+1}using the same equation):

$$\begin{aligned}
q_t &= \pi\left(K_t\right) + \frac{1}{(1+r)}\left(\pi\left(K_{t+1}\right) + \frac{1}{(1+r)} q_{t+2}\right) \\
&= \pi\left(K_t\right) + \frac{1}{(1+r)}\pi\left(K_{t+1}\right) + \frac{1}{(1+r)^2} q_{t+2} \\
&= \pi\left(K_t\right) + \frac{1}{(1+r)}\pi\left(K_{t+1}\right) + \frac{1}{(1+r)^2}\left(\pi\left(K_{t+2}\right) + \frac{1}{(1+r)} q_{t+3}\right) \\
&= \pi\left(K_t\right) + \frac{1}{(1+r)}\pi\left(K_{t+1}\right) + \frac{1}{(1+r)^2}\pi\left(K_{t+2}\right) + \frac{1}{(1+r)^3} q_{t+3} \\
&= \sum_{s=0}^{T-1}\left[\frac{1}{(1+r)^s}\pi\left(K_{t+s}\right)\right] + \frac{1}{(1+r)^T} q_{t+T}
\end{aligned}$$

But our firm is infinitely lived, so the RHS must hold for all t. So taking limits as $T \to \infty$ gives us:

$$\begin{aligned} q_t &= \lim_{T\to\infty} \sum_{s=0}^{T-1} \left[\frac{1}{(1+r)^s} \pi(K_{t+s}) \right] + \lim_{T\to\infty} \frac{1}{(1+r)^T} q_{t+T} \\ &= \sum_{s=0}^{\infty} \left[\frac{1}{(1+r)^s} \pi(K_{t+s}) \right] + \lim_{T\to\infty} \frac{1}{(1+r)^T} q_{t+T} \end{aligned}$$

How do we use this to learn about the nature of the solution to the problem?

- If $\lim_{T\to\infty} \frac{1}{(1+r)^T} q_{t+T} = 0$ then the q_t term means exactly what we claimed - it is the marginal contribution to the lifetime stream of cash-flows (which after all investments and costs are taken into account is the lifetime stream of marginal profits) of the capital stock in period t.
- If, say, $\lim_{T\to\infty} \frac{1}{(1+r)^T} q_{t+T} > 0$, since $\lim_{T\to\infty} \frac{1}{(1+r)^T} = 0$ it must mean that $\lim_{T\to\infty} q_{t+T} = \infty$ - i.e. that the marginal contribution of capital must diverge - That cannot be optimal - to put infinite value at the end of time - that means that some part of capital was never employed for profit and that cannot be profit maximizing. If the limit is negative, it means the firm borrows infinitely to ever expand the capital stock which cannot be an equilibrium either.
- Another way of seeing it is that, since $\lambda_t = \frac{1}{(1+r)^t} q_t$ the condition is that $\lim_{T\to\infty} \lambda_T = 0$ i.e. that the present value of capital stock in the indefinite future must go to zero, which is the same as saying that it is not optimal (or well defined) to let the capital stock explode.

Hence for optimality and a well defined solution to the system of capital stock and marginal value of capital, we must have $\lim_{T\to\infty} \frac{1}{(1+r)^T} q_{t+T} = 0$. This is called a transversality condition and a similar one is always part of the full description of an infinite horizon problem. We shall use it to motivate why the saddle path of the continuous time version of the model is the only sensible characterization of the model equilibrium.

Another interpretation of q_t Given our transversality condition, we have

$$q_t = \lim_{T\to\infty} \sum_{s=0}^{T-1} \left[\frac{1}{(1+r)^s} \pi(K_{t+s}) \right]$$

a marginal increase in the capital stock raises the value of the firm by q_t, thus it is the marginal market value of capital for this firm. If this exceeds the replacement cost of capital (p_K), then it is worthwhile for the firm to invest in more capital.

This is called "Tobin's q", and note carefully that is the value of the *marginal* unit of capital.

In typical empirical papers, all we can measure is the average value of q - the total market value of the firm devided by the total replacement value of capital.

In our situation (with convex adjustment costs) marginal q is always below average q.

In real world situations, a firm that is large relative to its market, will face a downward sloping demand curve for its product, thus even with a linear production function will face diminishing returns to capital as the output price has to fall to sell more goods, thus one expects its marginal q to be smaller than average q. A firm with a large stock of obsolete capital, on the other hand, may have a marginal q greater than average q.

This becomes important when one wants to take the model to the data. Romer gives a coverage of several seminal contributions.

3.2 The continuous time case

We could have used continuous time methods to arrive at equivalents of the above, but one can easily see the result by the following arguments:

suppose the "distance" between periods is not 1 but $\triangle t$ and we denote the time index in parentheses rather than subscripts.

- The first becomes (using the definition of investment as the "per fraction $\triangle t$ period" growth in capital):

$$q(t) - p_K = C'\left(\frac{k(t+\triangle t) - k(t)}{\triangle t}\right)$$

- The second becomes (scaling all discounting and increments for the size of the factional change in period):

$$\pi(K(t)) = \left[\frac{1}{(1+r)}\right]^{\triangle t}\left(rq(t) - \frac{q(t+\triangle t) - q(t)}{\triangle t}\right)$$

letting $\triangle t \to 0$ (the definition of continuous time) gives:

$$\begin{aligned} q(t) - p_K &= C'\left(\dot{k}(t)\right) \\ \pi(K(t)) &= rq(t) - \dot{q}(t) \end{aligned}$$

3.3 Market Equilibrium

In equilibrium, all N firms act in this way, so we have $K(t) = Nk(t)$ and that the marginal value of capital to any firm is identical.

The aggregate versions of the equations are (since N does not change):

$$\begin{aligned} q(t) - p_K &= C'\left(\frac{\dot{K}(t)}{N}\right) \\ \pi(K(t)) &= rq(t) - \dot{q}(t) \end{aligned}$$

We will use a phase diagram to depict the predictions of this model.
In steady state:

$$\begin{aligned}\dot{K}(t) = 0 \quad &\Leftrightarrow \quad q(t) = p_K \\ \dot{q}(t) = 0 \quad &\Leftrightarrow \quad q(t) = \frac{\pi(K(t))}{r}\end{aligned}$$

Out of steady state:

$$\begin{aligned}\pi(K(t)) &= rq(t) - \dot{q}(t) \\ \dot{q}(t) &= rq(t) - \pi(K(t))\end{aligned}$$