Tutorial 2: Asset price bubbles

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1.

Question 1

Question 1

Consider a stock that pays dividends of D_t and whose price is P_t in period t. Consumers are risk neutral* with discount rate r, so their objective function is:

$$E_t\left[\sum_{s=0}^{\infty}\frac{C_{t+s}}{(1+r)^s}\right]$$

1. Use a calculus of variations argument to show that equilibrium requires $P_t = E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right]$.

   Assume that if the stock is sold, it happens after the dividend for that period is paid out.

*risk neutral → linear utility function

Question 1

Calculus of variations: marginal changes should be utility neutral along the equilibrium path (MC = MU). No Lagrangian this time ☹

Marginal change → very small reduction in consumption and very small increase in the asset.

Consider an infinitesimal adjustment to consumption of dC. If the agent buys an infinitesimal unit of the asset:

- consumption (and given linearity, utility) will reduce in period t by $P_t dC$ to buy more of the stock.
- this will yield uncertain dividend D_{t+1} in $t+1$ and the asset's selling price will be an uncertain P_{t+1}. In expectation, the net present value of the change to future utility is thus $E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right]dC$

Question 1

- Loss in present utility = $P_t dC$
- Net present value of the increase in future utility = $E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right]dC$

If the agent is optimizing, then these trade-offs must be equal in equilibrium

$$\begin{aligned} P_t dC &= E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right]dC \\ P_t &= E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right] \end{aligned}$$

2.

Question 2

Question 2

2. Iterate the expression in (1) forward to derive an expression for P_t in terms of only future dividends and the interest rate, using the following *no-bubbles* condition:

$$\lim_{s\to\infty} E_t\left[\frac{P_{t+s}}{(1+r)^s}\right]=0$$

and the law of iterated expectations: $E_t\left[E_{t+1}\left[x_{t+s}\right]\right]=E_t\left[x_{t+s}\right]$.

Question 2

Iterating the optimality condition one period into the future:

$$\begin{aligned} P_t &= E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right] \\ &= E_t\left[\frac{D_{t+1}}{(1+r)}\right] + E_t\left[\frac{\boxed{P_{t+1}}}{(1+r)}\right] \\ &= E_t\left[\frac{D_{t+1}}{(1+r)}\right] + E_t\left[\frac{1}{(1+r)}\boxed{E_{t+1}\left[\frac{D_{t+2}+P_{t+2}}{(1+r)}\right]}\right] \end{aligned}$$

The law of iterated expectations says: $E_t\left[E_{t+1}\left[x_{t+s}\right]\right] = E_t\left[x_{t+s}\right]$.

Question 2

Therefore:

$$\begin{aligned} P_t &= E_t\left[\frac{D_{t+1}}{(1+r)}\right] + E_t\left[\frac{D_{t+2}}{(1+r)^2}\right] + E_t\left[\frac{P_{t+2}}{(1+r)^2}\right] \\ &= \sum_{j=1}^{s} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] + E_t\left[\frac{P_{t+s}}{(1+r)^s}\right] \\ &= \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] + \underbrace{\boxed{\lim_{s\to\infty} E_t\left[\frac{P_{t+s}}{(1+r)^s}\right]}}_{= 0 \text{ (No bubbles condition)}} \\ &= \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] \end{aligned}$$

3.

Question 3

Question 3

$$\lim_{s\to\infty} E_t\left[\frac{P_{t+s}}{(1+r)^s}\right] = 0$$

3. Give a clear description of the intuitive meaning of the *no-bubbles* condition.

The no-bubbles condition (a Transversality condition) says that the present discounted value of the asset price infinitely far in the future must be zero.

- This means that one cannot permanently borrow against the value of the asset (which would be a Ponzi scheme) in the future and so obtain infinite utility.
- In mathematical terms it means that if there is any expected permanent growth in P_t it must be slower than the rate at which the discount factor $\left[\frac{1}{(1+r)^s}\right]$ shrinks to zero.

4.

Question 4

Question 4(a)

4. Now we relax the *no-bubbles* assumption.

a) **Deterministic bubbles:** Suppose that P_t equals the expression you derived in (2) plus $(1 + r)^t\, b$ where $b > 0$. Show that this expression still satisfies the consumer's optimality condition $P_t = E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right]$ and interpret what this means.

[Hint: start with the new expression, lead it forward one period and take conditional expectation E_t].

From (2): $$P_t = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] \boxed{+ (1+r)^t\, b}$$

Question 4(a)

The new expression is:

$$P_t = \sum_{j=1}^{\infty} E_t \left[\frac{D_{t+j}}{(1+r)^j} \right] + (1+r)^t\, b$$

lead it forward 1 period

$$P_{t+1} = \sum_{j=1}^{\infty} E_{t+1} \left[\frac{D_{t+1+j}}{(1+r)^j} \right] + (1+r)^{t+1}\, b$$

take conditional expectation E_t

$$\begin{aligned} E_t\left[P_{t+1}\right] &= E_t \left(\sum_{j=1}^{\infty} E_{t+1} \left[\frac{D_{t+1+j}}{(1+r)^j} \right] + (1+r)^{t+1}\, b \right) \\ &= \sum_{j=1}^{\infty} E_t \left[\frac{D_{t+1+j}}{(1+r)^j} \right] + (1+r)^{t+1}\, b \end{aligned}$$

Question 4(a)

From the previous slide:

$$E_t\left[P_{t+1}\right] = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^j}\right] + (1+r)^{t+1} b$$

Now divide through by $(1+r)$ and then add $E_t\left(\frac{D_{t+1}}{1+r}\right)$ to both sides. Note carefully how the exponents change:

$$E_t\left[\frac{P_{t+1}}{1+r}\right] = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^{j+1}}\right] + (1+r)^{t} b$$

$$E_t\left[\frac{P_{t+1}}{1+r}\right] + E_t\left(\frac{D_{t+1}}{1+r}\right) = E_t\left(\frac{D_{t+1}}{1+r}\right) + \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^{j+1}}\right] + (1+r)^{t} b$$

$$E_t\left[\frac{P_{t+1}+D_{t+1}}{1+r}\right] = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^{j}}\right] + (1+r)^{t} b$$

Question 4(a)

Started with this: $$P_t = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] + (1+r)^t b$$

$$E_t\left[\frac{P_{t+1}+D_{t+1}}{1+r}\right] = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] + (1+r)^t b$$

Interpretation:

- we showed there exists more than one solution to the asset price process that satisfies optimality.
- The no-bubbles case is only one of many solutions (technically infinite number: a different solution for every possible $b > 0$). This happens here because the optimality condition is a linear difference equation, which can have many solutions that are equally valid.

Interpreting the economics: both a bubble and a no bubble price path may satisfy in period optimality.

∴ merely insisting on optimal behaviour in any given period **does not** on its own rule out the possibility of non-fundamental price movements in asset markets.

Question 4(b)(i)

b) **Stochastic bursting bubbles:** Suppose that P_t equals the expression you derived in (2) plus q_t where

$$q_t = \begin{cases} \frac{(1+r)q_{t-1}}{\alpha} & \text{with probability } \alpha \\ 0 & \text{with probability } (1-\alpha) \end{cases}$$

From (2): $$P_t = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] \boxed{+\, q_t}$$

i. Again, show that this new expression for P_t still satisfies $P_t = E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right]$

Question 4(b)(i)

Similar to 4(a):

$$P_t = \sum_{j=1}^{\infty} E_t \left[\frac{D_{t+j}}{(1+r)^j} \right] + q_t$$

lead it forward 1 period

$$\therefore P_{t+1} = \sum_{j=1}^{\infty} E_{t+1} \left[\frac{D_{t+1+j}}{(1+r)^j} \right] + q_{t+1}$$

take conditional expectation E_t

$$\begin{aligned} E_t \left(P_{t+1} \right) &= E_t \left(\sum_{j=1}^{\infty} E_{t+1} \left[\frac{D_{t+1+j}}{(1+r)^j} \right] + q_{t+1} \right) \\ &= \sum_{j=1}^{\infty} E_t \left[\frac{D_{t+1+j}}{(1+r)^j} \right] + \underline{E_t q_{t+1}} \end{aligned}$$

Question 4(b)(i)

Difference to 4(a): now q_{t+1} is not deterministic.

But, it has a known and simple probability description so we can find $E_t q_{t+1}$ (note carefully how the piecewise description of q_t is employed):

$$E_t\left(P_{t+1}\right) = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^j}\right] + E_t q_{t+1}$$

$$E_t\left(P_{t+1}\right) = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^j}\right] + (1-\alpha)\cdot 0 + \alpha\frac{(1+r)\,q_t}{\alpha}$$

From the previous slide:

Question 4(b)

$$E_t\left(P_{t+1}\right) = \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^j}\right] + (1-\alpha)\cdot 0 + \alpha\frac{(1+r)\,q_t}{\alpha}$$

Rest similar to 4(a): divide through by $(1+r)$ and add $E_t\left(\frac{D_{t+1}}{1+r}\right)$ to both sides.

$$\begin{aligned}
E_t\left(P_{t+1}\right) &= \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^j}\right] + (1+r)\,q_t \\
E_t\left(\frac{P_{t+1}}{1+r}\right) + E_t\left(\frac{D_{t+1}}{1+r}\right) &= E_t\left(\frac{D_{t+1}}{1+r}\right) + \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+1+j}}{(1+r)^{j+1}}\right] + q_t \\
P_t = E_t\left[\frac{D_{t+1}+P_{t+1}}{(1+r)}\right] &= \sum_{j=1}^{\infty} E_t\left[\frac{D_{t+j}}{(1+r)^j}\right] + q_t
\end{aligned}$$

Again, we have shown that this stochastic bubble process also satisfies the difference equation that characterizes optimality.

Question 4(b)(ii)

$$q_t = \begin{cases} \frac{(1+r)q_{t-1}}{\alpha} & \text{with probability } \alpha \\ 0 & \text{with probability } (1-\alpha) \end{cases}$$

ii. If there is a bubble in period t (i.e. that $q_t > 0$), what is the probability that the bubble has burst by period $t + s$ (i.e. that $q_{t+s} = 0$)?

Note: $q_j = 0$ implies that $q_{k>j} = 0$, so in this model, once a bubble has burst, it is gone forever.

If $q_t > 0$, then $q_{t+1} = \frac{(1+r)q_t}{\alpha} > 0$ with probability α. $\therefore$ the bubble will not have burst in period $t+1$ with probability α.

$q_{t+2} = \frac{(1+r)^2 q_t}{\alpha^2} > 0$ with probability α^2

$\therefore q_{t+s} > 0$ with probability α^s

Question 4(b)(ii)

$$q_t = \begin{cases} \frac{(1+r)q_{t-1}}{\alpha} & \text{with probability } \alpha \\ 0 & \text{with probability } (1-\alpha) \end{cases}$$

How likely is it that the bubble has burst by period $t + 1$?

= the probability that it has not burst by period t (= 1 by assumption) times the probability that it bursts in period $t + 1$, $(1 - \alpha)$.

Probability that the bubble **has already burst** by (i.e. in any period up to) period $t + 3$?

The probability that it burst in period $t + 1$, $(1 - \alpha)$, plus the probability that it burst in $t + 2$, $\alpha(1 - \alpha)$, plus the probability that it only burst in $t + 3$, $\alpha^2(1 - \alpha)$.

Therefore:

$$\begin{aligned} \Pr(q_{t+s} = 0) &= (1-\alpha)\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{s-1}\right) \\ &= (1-\alpha)\sum_{j=0}^{s-1}\alpha^j \end{aligned}$$

Question 4(b)(iii)

iii. What is the limit of this probability in (ii) as $s \to \infty$? Interpret this result.

$$\begin{aligned}
\lim_{s\to\infty}\left[\Pr\left(q_{t+s}=0\right)\right] &= \lim_{s\to\infty}\left[(1-\alpha)\sum_{j=0}^{s-1}\alpha^j\right] \\
&= (1-\alpha)\lim_{s\to\infty}\left[\sum_{j=0}^{s-1}\alpha^j\right] \\
&= (1-\alpha)\left[\frac{1}{1-\alpha}\right] \\
&= 1
\end{aligned}$$

Question 4(b)(iii)

iii. What is the limit of this probability in (ii) as $s \to \infty$? Interpret this result.

$$\lim_{s\to\infty}\left[\Pr\left(q_{t+s}=0\right)\right] = \lim_{s\to\infty}\left[(1-\alpha)\sum_{j=0}^{s-1}\alpha^j\right]$$

$$= (1-\alpha)\left(1+\alpha+\alpha^2+\ldots+\alpha^{s-1}\right)$$

$$= (1-\alpha)\sum_{j=0}^{s-1}\alpha^j$$

$$1 + 0.5 + 0.5^2 + 0.5^3 + 0.5^4 + 0.5^5 + \ldots = 1 + 0.5 + 0.25 + 0.125 + 0.0625 + 0{,}03125 + \ldots = 1{,}984375$$

$$= (1-\alpha)\lim_{s\to\infty}\left[\sum_{j=0}^{s-1}\alpha^j\right]$$

$$= (1-\alpha)\left[\frac{1}{1-\alpha}\right]$$

$$= 1$$

Question 4(b)(iii)

Interpretation: if there currently is a bubble, and an equal, strictly positive probability that a bubble may burst in any period, then the bubble will eventually burst with certainty → in this model, all asset price bubbles eventually burst.

- The likelihood of a bubble bursting may not be purely exogenous, and it is unlikely to be identical in all periods.
- But the above result will extend as follows: as long as there is high enough probability in every period that the bubble may burst, exogenous or endogenous, the bubble must eventually burst.

Thank you!