Hansens' RBC model

Please make sure you've read Uhlig's article, especially section 4. Take special note of: the timing convention; that R_t is simply a definition; and, how to obtain the steady state.

$$\max E \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\eta}}{1-\eta} - AN_t \right) \tag{1}$$

s.t.

$$\begin{aligned}
C_t + I_t &= Y_t && (2)\\
K_t &= I_t + (1-\delta)K_{t-1} && (3)\\
Y_t &= F(Z_t, K_{t-1}, N_t) = Z_t K_{t-1}^{\rho} N_t^{1-\rho} && (4)\\
\log Z_t &= (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t \ , \quad \epsilon_t \sim i.i.d.\mathcal{N}(0, \sigma^2) && (5)
\end{aligned}$$

We assume that (2), (3), and (4) are binding. (5) is exogenous and follows an AR(1) process, so we can ignore it; but this uncertainty necessitates the expectations operator in the FOCs for variables in $t+1$.

- Control variables: K_t; N_t
- Endogenous state variables: K_{t-1}
- Exogenous/Stochastic state variables: Z_{t-1}

1 The value function

Let $U(C_t, N_t) = \frac{C_t^{1-\eta}}{1-\eta} - AN_t$ be utility in period t. The value function follows:

$$\begin{aligned}
V(K_{t-1}, Z_{t-1}) &= \max_{C_t, I_t, Y_t, K_t, N_t} U(C_t, N_t) + \beta E_t[V(K_t, Z_t)] \ \text{ s.t. } (2), (3), (4)\\
V(K_{t-1}, Z_{t-1}) &= \max_{C_t, I_t, Y_t, K_t, N_t} \left(\frac{C_t^{1-\eta}}{1-\eta} - AN_t \right) + \beta E_t[V(K_t, Z_t)] \ \text{ s.t. } (2), (3), (4)\\
V(K_{t-1}, Z_{t-1}) &= \max_{I_t, Y_t, K_t, N_t} \left(\frac{(Y_t - I_t)^{1-\eta}}{1-\eta} - AN_t \right) + \beta E_t[V(K_t, Z_t)] \ \text{ s.t. } (3), (4)\\
V(K_{t-1}, Z_{t-1}) &= \max_{K_t, N_t} \left(\frac{(Z_t K_{t-1}^{\rho} N_t^{1-\rho} - K_t + (1-\delta)K_{t-1})^{1-\eta}}{1-\eta} - AN_t \right) + \beta E_t[V(K_t, Z_t)]
\end{aligned}$$

2 The first-order conditions

Don't panic. You can use notation to save time and space. Let $\frac{\partial U(\cdot)}{\partial C} = U_c(\cdot) = C_t^{-\eta}$ be the marginal utility of consumption with respect to C.

FOC N_t:

$$\frac{\partial V(\cdot_t)}{\partial N_t} \quad : \quad \frac{\partial U(C_t,N_t)}{\partial C_t}\frac{\partial C_t}{\partial Y_t}\frac{\partial Y_t}{\partial N_t} + \frac{\partial U(C_t,N_t)}{\partial N_t} = 0 \qquad \text{[chain rule]}$$

$$\frac{(1-\eta)C_t^{1-\eta-1}}{1-\eta}(1-\rho)Z_t K_{t-1}^{\rho} N_t^{1-\rho-1} - A = 0$$

$$C_t^{-\eta}(1-\rho)Z_t K_{t-1}^{\rho} N_t^{1-\rho}\frac{1}{N_t} - A = 0$$

$$\therefore \qquad A = C_t^{-\eta}(1-\rho)\frac{Y_t}{N_t} \tag{6}$$

We can re-state (6) as

$$AC_t^{\eta} = (1-\rho)\frac{Y_t}{N_t}$$

which implies that the marginal rate of substitution between labour and consumption (U_n/U_c) equals the marginal product of labour $F_n(\cdot)$. Note: no envelope theorem needed (why?).

FOC K_t:

$$\frac{\partial V(\cdot_t)}{\partial K_t} \quad : \quad \frac{\partial U(C_t,N_t)}{\partial C_t}\left[\underbrace{\frac{\partial C_t}{\partial Y_t}\frac{\partial Y_t}{\partial K_t}}_{=0} + \frac{\partial C_t}{\partial I_t}\frac{\partial I_t}{\partial K_t}\right] + \beta E_t\frac{\partial V(\cdot_{t+1})}{\partial K_t} = 0 \qquad \text{[chain rule]}$$

$$C_t^{-\eta}(-1) + \beta E_t \frac{\partial V(\cdot_{t+1})}{\partial K_t} = 0 \tag{7}$$

Envelope theorem:

$$\frac{\partial V(\cdot_t)}{\partial K_{t-1}} \quad : \quad \frac{\partial U(C_t,N_t)}{\partial C_t}\left[\frac{\partial C_t}{\partial Y_t}\frac{\partial Y_t}{\partial K_{t-1}} + \frac{\partial C_t}{\partial I_t}\frac{\partial I_t}{\partial K_{t-1}}\right] + \underbrace{\beta E_t \frac{\partial V(\cdot_{t+1})}{\partial K_{t-1}}}_{=0} \tag{8}$$

$$C_t^{-\eta}\left[\rho\frac{Z_t K_{t-1}^{\rho} N_t^{1-\rho}}{K_{t-1}} + (1-\delta)\right] \tag{9}$$

$$C_t^{-\eta}[\rho\frac{Y_t}{K_{t-1}} + (1-\delta)] \tag{10}$$

Iterate forward one period:

$$\frac{\partial V(\cdot_{t+1})}{\partial K_t} \quad : \quad C_{t+1}^{-\eta}[\rho\frac{Y_{t+1}}{K_t} + (1-\delta)] = C_{t+1}^{-\eta}R_{t+1} \tag{11}$$

Substitute (11) into (7) to get:

$$C_t^{-\eta} = \beta E_t\left[C_{t+1}^{-\eta}R_{t+1}\right]$$

$$1 = \beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \tag{12}$$

(12) gives the Euler equation which governs the intertemporal consumption path (and therefore capital). In a competitive version of this model (i.e., market economy not social planner), this is an asset pricing equation by which the real return R_t on an asset equates with the marginal product of capital (i.e., the return on investment in production). Note also that we assume certainty equivalence: $E[XY] = E[X]E[Y]$, which implies that $cov[XY] = 0$ (or is constant, since it will fall away after log-linearization). For small deviations from steady-state, a linear approximation is good.

3 Stead-states

Obtain steady-states by dropping t subscripts from FOCs and constraints and then simplify. Although there is know "rule" per se, some standard parameters are left to be calibrated (i.e., given values) and some are determined endogenously and some are substituted out completely (see sections 4 and 6 below). We are left with *implied* three steady-state conditions (for the real rate of return, the physical capital stock to GDP ratio, and the consumption to GDP ratio) and five parameters to calibrate (excluding the size of the stochastic shock).

$$R = \frac{1}{\beta}$$

$$\frac{K}{Y} = \frac{\rho}{R - (1 - \delta)}$$

$$\frac{C}{Y} = 1 - \delta \, \frac{K}{Y}$$

4 System of log-linearized equations

Follow the steps of Taylor and Uhlig to obtain the following log-linearized equations . [Hint: you will need to use the steady-state conditions to simplify the coefficients].

$$y_t = \frac{C}{Y} \, c_t + \delta \, \frac{K}{Y} \, i_t \tag{13}$$

$$k_t = (1 - \delta) \; k_{t-1} + \delta \, i_t \tag{14}$$

$$y_t = z_t + \rho \, k_{t-1} + (1 - \rho) \, n_t \tag{15}$$

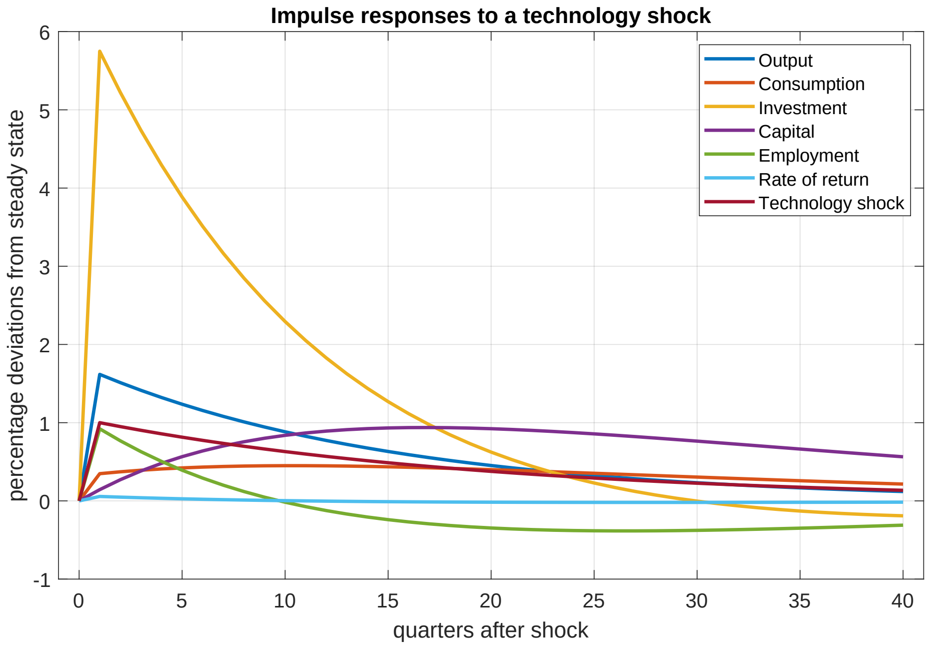
$$z_t = \psi \, z_{t-1} + \epsilon_{zt} \tag{16}$$

$$\eta \, c_t = y_t - n_t \tag{17}$$

$$0 = E_t[\eta \, (c_t - c_{t+1})] + r_{t+1} \tag{18}$$

$$r_t = \rho \, \frac{Y}{K} \; (y_t - k_{t-1}) \tag{19}$$

5 Impulse response functions



6 DYNARE and LaTeX definitions

Table 1: Endogenous

Variable	**LaTeX**	**Description**
`y`	y	output
`c`	c	consumption
`n`	n	hours
`i`	i	investment
`k`	k	capital
`r`	r	real rate
`z`	z	TFP

Table 2: Exogenous

Variable	**LaTeX**	**Description**
`epsilon`	ϵ_z	TFP shock

Table 3: Parameters

Variable	**LaTeX**	**Description**
`rho`	ρ	capital share
`bet`	β	discount factor
`delt`	δ	depreciation rate
`psi`	ψ	persistence TFP shock
`eta`	η	risk aversion
`sigmae`	σ_e	i.i.d TFP shock

7 The full DYNARE .mod file

```
//Basic RBC model (see also, Sims RBC stylized facts)

var y ${y}$ (long_name='output')
    c ${c}$ (long_name='consumption')
    n ${n}$ (long_name='hours')
    i ${i}$ (long_name='investment')
    k ${k}$ (long_name='capital')
    r ${r}$ (long_name='real rate')
    z ${z}$ (long_name='TFP');

varexo epsilon ${\epsilon_z}$ (long_name='TFP shock');

parameters rho     ${\rho}$  (long_name='capital share')
           bet      ${\beta}$   (long_name='discount factor')
           delt     ${\delta}$  (long_name='depreciation rate')
           psi      ${\psi}$  (long_name='persistence TFP shock')
           eta      ${\gamma}$  (long_name='risk aversion')
           sigmae   ${\sigma_{e}}$ (long_name='i.i.d TFP shock');

% comment:  (1/eta) = intertemporal elasticity of substitution

rho = 0.33;
bet = 0.99;
delt = 0.025;
psi = 0.95;
eta = 2;
sigmae = 0.01;

model(linear);

#R = 1/bet;
#KY = rho/(R - (1-delt));
#CY = 1 - KY*delt;

[name='Resource constraint']
y = CY*c + delt*KY*i;

[name='Euler equation']
(1/eta)*r(+1) = c(+1) - c;

[name='Labor FOC']
eta*c = y - n;

[name='real interest rate/firm FOC capital']
r = rho*(1/KY)*(y - k(-1));

[name='Law of motion capital']
k = (1-delt)*k(-1) + delt*i;

[name='production function']
y = z + rho*k(-1) + (1-rho)*n;

[name='exogenous TFP process']
z = psi*z(-1) + epsilon;
```

```
end;

shocks;
var epsilon = sigmae^2;
end;

steady;
check;

stoch_simul(order=1, periods=150, nodisplay);

%stoch_simul(order=1); // for theoretical moments

%-----------------------------------------------------------------
% Plot IRFs to a 1.0% s.d. technology shock
%-----------------------------------------------------------------

figure
plot([0:options_.irf], [0 oo_.irfs.y_epsilon]*100)
hold
plot([0:options_.irf], [0 oo_.irfs.c_epsilon]*100)
plot([0:options_.irf], [0 oo_.irfs.i_epsilon]*100)
plot([0:options_.irf], [0 oo_.irfs.k_epsilon]*100)
plot([0:options_.irf], [0 oo_.irfs.n_epsilon]*100)
plot([0:options_.irf], [0 oo_.irfs.r_epsilon]*100)
plot([0:options_.irf], [0 oo_.irfs.z_epsilon]*100)
title('Impulse responses to a technology shock')
legend('Output','Consumption','Investment','Capital','Employment','Rate of return','Techn
ylabel('percentage deviations from steady state')
xlabel('quarters after shock')
```