Derivations of Consumption Theories

Contents

1 Friedman's Permanent Income Hypothesis

Consider the consumption problem of an individual:

- no uncertainty
- a finite lifetime T
- initial wealth A
- a stream of lifetime income that arrives in each period $\{Y_t\}_{t=1}^{T}$
- the discount rate and the interest rate are both assumed to be zero, but there is a perfect storage technology.

Utility function:

$$\mathcal{U} = \sum_{t=1}^{T} [u(C_t)]$$

$$u'(C_t) > 0$$

$$u''(C_t) < 0$$

Let S_t be the amount of resources transferred from period t to $t+1$ Then the period by period budget constraints are:

$$\begin{aligned}
C_1 &= A + Y_1 - S_1 \\
C_2 &= S_1 + Y_2 - S_2 \\
&\vdots \\
C_t &= S_{t-1} + Y_t - S_t \\
&\vdots \\
C_T &= S_{T-1} + Y_T
\end{aligned}$$

Summing all together gives the Inter-temporal Lifetime Budget constraint:

$$\sum_{t=1}^{T} C_t = A + \sum_{t=1}^{T} Y_t$$

Thus the Lagrangian for this problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} u(C_t) + \lambda \left[A + \sum_{t=1}^{T} Y_t - \sum_{t=0}^{T} C_t \right]$$

FOC for C_t :

$$u'(C_t) = \lambda$$
$$\therefore C_t = C_s = C^* \forall t, s$$

Plugging into budget constraint:

$$\sum_{t=0}^{T} C^* = A + \sum_{t=0}^{T} Y_t$$
$$TC^* = A + \sum_{t=0}^{T} Y_t$$
$$C^* = \frac{1}{T}\left(A + \sum_{t=0}^{T} Y_t\right) = Y^{\text{Permanent}}$$

Or: lifetime resources spread equally across life.
What does this imply for savings?

$$C_t = S_{t-1} + Y_t - S_t$$
$$S_t - S_{t-1} = Y_t - C_t$$
$$\triangle S_t = Y_t - \frac{1}{T}\left(A + \sum_{t=0}^{T} Y_t\right) = Y_t^{\text{Transitory}}$$

For neatness, let's assume $A = 0$:
Thus, by construction for individual i :

$$Y_{it} = Y_i^{\text{Permanent}} + Y_{it}^{\text{Transitory}}$$
$$\sum_{t=0}^{T} Y_{it}^{\text{Transitory}} = 0$$

In an independently drawn sample, it is similarly to be expected that:

$$\sum_{i=0}^{N} Y_{it}^{\text{Transitory}} \approx 0$$

In a regression of the Keynesian consumption function we have:

$$C = a + bY + e$$

So the OLS estimators are:

$$\widehat{b} = \frac{\operatorname{cov}(Y, C)}{\operatorname{var}(Y)}$$
$$\widehat{a} = \bar{C} - \widehat{b}\bar{Y}$$

If the permanent income hypothesis is accurate this becomes:

$$\begin{aligned}\widehat{b} &= \frac{\operatorname{cov}\left(Y^{\text{Permanent}} + Y^{\text{Transitory}}, Y^{\text{Permanent}}\right)}{\operatorname{var}\left(Y^{\text{Permanent}} + Y^{\text{Transitory}}\right)} \\ &= \frac{\operatorname{var}\left(Y^{\text{Permanent}}\right)}{\operatorname{var}\left(Y^{\text{Permanent}}\right) + \operatorname{var}\left(Y^{\text{Transitory}}\right)}\end{aligned}$$

Since by definition $\operatorname{cov}\left(Y^{\text{Transitory}}, Y^{\text{Permanent}}\right) = 0$
Thus in cross sections where we observe several individuals independently at a single moment, all in different life stages and positions and different "short run positions", we expect $\operatorname{var}\left(Y^{\text{Transitory}}\right) \gg 0$ so we expect $0 < \widehat{b} < 1$.

With time series data, however, since we are averaging/aggregating over individuals, we expect $\operatorname{var}\left(Y^{\text{Permanent}}\right) \gg \operatorname{var}\left(Y^{\text{Transitory}}\right) \approx 0$ due to long run growth, which means we expect $\widehat{b} \approx 1$.

This is exactly the pattern found empirically. But this is not strong evidence. The PIH model above requires some extreme assumption on knowledge about the future. The paper by Hall (1978) was the first attempt to fix that in a rational expectations world.

2 Adding interest and discount rates

Let's augment the model with:

1. a known CRRA utility function:

$$u\left(C_t\right) = \frac{C_t^{1-\theta}}{1-\theta}$$

2. A time preference parameter: future utility is discounted at rate $\rho > 0$ for each period, so that the intertemporal discount factor becomes $\beta = \frac{1}{1+\rho}$. What that means is that our agents prefer utility today over tomorrow. E.g. the utility value of some level of consumption C in period t is higher than the current utility value of C tomorrow: $u\left(C_t = C\right) > \frac{1}{1+\rho} u\left(C_{t+1} = C\right)$. This implies the objective function that the consumer wants to maximize across her lifetime is by choice of consumption path $\left\{C_\tau\right\}_{\tau=1}^T$:

$$\mathcal{U}\left(\left\{C_\tau\right\}_{\tau=1}^T\right) = \sum_{t=1}^{T} \left(\frac{1}{1+\rho}\right)^t \frac{C_t^{1-\theta}}{1-\theta}$$

3. a fixed market interest rate $r > 0$, so that the intratemporal budget constraints become:

$$
\begin{aligned}
C_1 &= A + Y_1 - S_1 \\
C_2 &= (1+r)S_1 + Y_2 - S_2 \\
&\vdots \\
C_t &= (1+r)S_{t-1} + Y_t - S_t \\
&\vdots \\
C_T &= (1+r)S_{T-1} + Y_T
\end{aligned}
$$

successively solving for S_t in each period yields the intertemporal budget constraint:

$$\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^t C_t = A + \sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^t Y_t$$

With these changes, the Lagrangian becomes:

$$\mathcal{L} = \sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \frac{C_t^{1-\theta}}{1-\theta} + \lambda\left[A + \sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^t Y_t - \sum_{t=0}^{T}\left(\frac{1}{1+r}\right)^t C_t\right]$$

This yields first order conditions (for C_t and C_{t+1})

$$
\begin{aligned}
\left(\frac{1}{1+\rho}\right)^t C_t^{-\theta} &= \lambda\left(\frac{1}{1+r}\right)^t \\
\left(\frac{1}{1+\rho}\right)^{t+1} C_{t+1}^{-\theta} &= \lambda\left(\frac{1}{1+r}\right)^{t+1}
\end{aligned}
$$

Take the ration of these:

$$
\begin{aligned}
\left(\frac{1}{1+\rho}\right)\frac{C_{t+1}^{-\theta}}{C_t^{-\theta}} &= \left(\frac{1}{1+r}\right) \\
C_{t+1} &= \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\theta}} C_t
\end{aligned}
$$

If $r > \rho, C_{t+1} > C_t$

3 Hall's random walk result - adding uncertainty

Consider the consumption problem of an individual

- uncertainty
- a finite lifetime T
- initial wealth A

- a stream of lifetime income that arrives in each period $\{Y_t\}_{t=1}^{T}$
- The discount and interest rates are both assumed to be zero, but there is a perfect storage technology.

Utility function:

$$\mathcal{U} = E_1 \left[\sum_{t=1}^{T} \left(C_t - \frac{a}{2} C_t^2 \right) \right]$$

$$a > 0$$

$$u'(C_t) = 1 - aC_t$$

Budget Constraint:

$$\sum_{t=1}^{T} E_1 [C_t] = A + \sum_{t=1}^{T} E_1 [Y_t]$$

$$\mathcal{L} = E_1 \left[\sum_{t=1}^{T} \left(C_t - \frac{a}{2} C_t^2 \right) \right] + \lambda \left[A + \sum_{t=1}^{T} Y_t - \sum_{t=0}^{T} C_t \right]$$

Same procedure solving for optimal C_1 relative to C_t via the FOC's yields:

$$1 - aC_1 = \lambda = E_1 [1 - aC_t]$$

$$\therefore C_1 = E_1 [C_t]$$

Plugging into the budget constraint yields:

$$C_1 = \frac{1}{T} \left(A + \sum_{t=1}^{T} E_1 [Y_t] \right)$$

So the optimal decision in the first period is to consume fraction $\frac{1}{T}$ of expected lifetime resources.

When we consider the decision in any period t based on the information available at that moment:

$$\therefore C_{t-1} = E_{t-1} [C_t]$$

This means that changes in consumption in this model should be unpredictable.

Using the definition of expectation we can write:

$$C_t = E_{t-1} [C_t] + e_t$$

$$\text{where}$$

$$E_{t-1} [e_t] = 0$$

Using $C_{t-1} = E_{t-1} [C_t]$ we get:

$$C_t = C_{t-1} + e_t$$

Which means consumption must be a Martingale, called a Random Walk if e_t is i.i.d.

Now note that the precise random walk result relies on Hall's choice of utility function that has some dubious properties.

With a more general utility function such as a CES function, we would have had a non-linear marginal utility:

$$u'(C_{t-1}) = E_{t-1}\left[u'(C_t)\right] \neq u'(E_{t-1}[C_t])$$

So that marginal utility would be predicted as a random walk. The random walk in consumption hypothesis is a first order approximation of this result, and would be accurate if deviations from it were small (and as we will see in a bit, if interest rates and discount rates were approximately equal).

$$C_2 = \frac{1}{T-1}\left(A_1 + \sum_{t=2}^{T} E_1[Y_t]\right)$$

4 Consumption and Risky Assets

4.1 Optimal Investment Behaviour:

Consider an infinitely lived consumer with discount rate ρ that maximizes a concave utility function by consuming and investing in a set of risky assets.

Asset i is described by its price in period t, P_t^i, and its stream of uncertain payoffs: $D_{t+1}^i, D_{t+2}^i, D_{t+3}^i, \ldots$

An optimizing agent that reduces C_t by a marginal amount dC to invest in this asset can buy $P_t^i dC$ units of the asset, which will decrease her period t utility by $u'(C_t)\, P_t^i dC$.

Her investment will increase her utility in all future periods $t+k$ by $u'(C_{t+k})\, D_{t+k}^i dC$.
If she is optimizing these changes must be equal in expectation:

$$u'(C_t)\, P_t^i = E_t\left[\sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k} u'(C_{t+k})\, D_{t+k}^i\right]$$

Suppose she holds the asset for only one period and define the gross return on the asset $R_{t+1}^i = \frac{P_{t+1}^i + D_{t+1}^i}{P_t^i}$

Note:

- Romer assumes D includes the selling price, but it is more standard to treat them individually.

- Romer uses the net return $r^i_{t+1} = R^i_{t+1} - 1$, but it is good practice to be equally comfortable with both.

Then the optimality condition becomes:

$$u'(C_t) = \frac{1}{(1+\rho)} E_t\left[u'(C_{t+1}) R^i_{t+1}\right]$$

Now recall that for two random variables, A and $B, E(AB) = E(A)E(B) +$ Cov(A, B) so the optimality condition can be expressed as:

$$\begin{aligned} u'(C_t) &= \frac{1}{(1+\rho)}\left\{E_t\left[u'(C_{t+1})\right]E_t\left[R^i_{t+1}\right] + \mathrm{Cov}_t\left[u'(C_{t+1}), R^i_{t+1}\right]\right\} \\ &\approx \frac{1}{(1+\rho)}\left\{E_t\left[u'(C_{t+1})\right]E_t\left[R^i_{t+1}\right] - a\,\mathrm{Cov}_t\left[C_{t+1}, R^i_{t+1}\right]\right\} \end{aligned}$$

Where the second line uses the fact that marginal utility is a decreasing function of consumption.

What does this mean?

A consumer cares only about whether the asset is correlated with her own consumption process. A new asset that tends to give high returns when consumption is low and vice versa would be worth investing in.

If a new asset comes into play that has a negative correlation with consumption, it is optimal to invest in it (shift portfolio weights to it)

As this occurs, consumption becomes more correlated with that asset, so the negative correlation reduces. It is optimal to increase investment in this asset until that covariance is zero. What does this mean for a new asset positively correlated with consumption?

4.2 The Consumption CAPM model

If every person is the same, it must be that the equilibrium price is also determined by this optimizing behaviour:

$$P^i_t = E_t\left[\sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k} \frac{u'(C_{t+k})}{u'(C_t)} D^i_{t+k}\right]$$

The term $\frac{1}{(1+\rho)^k}\frac{u'(C_{t+k})}{u'(C_t)}$ is called the equilibrium subjective stochastic discount factor.

In terms of expected returns:

$$\begin{aligned} E_t\left[R^i_{t+1}\right] &= \frac{1}{E_t\left[u'(C_{t+1})\right]}\left\{(1+\rho)u'(C_t) + a\,\mathrm{Cov}_t\left[C_{t+1}, R^i_{t+1}\right]\right\} \\ E_t\left[1 + r^i_{t+1}\right] &= \frac{(1+\rho)u'(C_t)}{E_t\left[u'(C_{t+1})\right]} + \frac{a\,\mathrm{Cov}_t\left[C_{t+1}, 1 + r^i_{t+1}\right]}{E_t\left[u'(C_{t+1})\right]} \end{aligned}$$

Now suppose there is a risk free asset with certain return $\bar{r}_{t+1}$. This means it has zero covariance with consumption:

$$1 + \bar{r}_{t+1} = \frac{(1+\rho)u'(C_t)}{E_t\left[u'(C_{t+1})\right]}$$

The difference in net return between a risky asset and a safe asset is:

$$E_t\left[r^i_{t+1}\right] - \bar{r}_{t+1} = \frac{a\,\mathrm{Cov}_t\left[C_{t+1}, 1 + r^i_{t+1}\right]}{E_t\left[u'(C_{t+1})\right]}$$

That is: in equilibrium, the excess return of an asset over the risk free rate is proportional to its covariance with consumption.