Introduction to DSGE Models

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In this lecture, we combine the work of the last two lectures to obtain the simplest version of the cannonical dynamic stochastic general equilibrium model used in macroeconomic models used to study stabilization policy. The model we will develop does not provide a satisfactory explanation of the nature of dynamics observed the data, but the structure is similar to more complicated models that are currently viewed as the "best" among those models that assume representative agents and firms.

Key new modelling device: Incomplete price adjustment among firms with pricing power

Our major piece of analysis that we will add to the model is several models of incomplete price adjustment among monopolistically competitive firms. We will use the simple iso-elastic demand function we derived in the lecture on nominal rigidities. This will still *look* complicated so we will, soon after stating the full dynamic pricing problem, simplify it by considering a quadratic approximation to the problem. Thus we will "drop" several parts of the problem and maintain only the parts that we need. One could avoid this step, and indeed the papers that this is based on do not use this simplification, but that accumulates a lot of notation that is there for "being analytically precise" that contains adds new/important in terms of economic intuition or that would remain in linear approximation.

We will maintain the following simplifications and short cuts:

- **No capital, no government:** We will continue to abstract from capital and investment, as well as the fiscal side of policy. These can be added using methods similar to the lecture on Investment, but our focus is on a different part of the model today
- **Solving the dynamic pricing problem of the firm in quadratic approximation**
- **Dropping uninteresting constants:** In the final form, we will solve the model via linear approximation in terms of log deviations from steady state: that is, we will express the dynamic behaviour of the aggregate economic concepts as linear functions approximating the transition of these variables back to steady state after a small disturbance. The full non-linear dynamics are not feasible to solve for in closed form, even in the

state-of-the-art models. In deviation from steady state, there can be no constants. We will be a little loose and simply drop these constants from our derivations when they are no longer part of the problem of interest

- **Derive without uncertainty and add it back later:** We do not yet have to truly correct mathematical tools to solve these models under uncertainty. It turns out that the results we obtain by "adding it back later" are identical to what one would obtain using the correct approach, so there is no loss.

1 Production Technology and Labour markets:

In each period t there exists a continuum of consumption goods, indexed by $i \in [0,1]$, each produced by a single firm with monopoly rights to that good, so that firm i can set the price for good i.

- Denote the consumption of good i in period t by C_{it} and the price charged for good i, P_{it}
- Firm i produces only good i using only labour, which is supplied by the representative household to each firm (so each household supplies some amount of labour to each firm).
- Production function:

$$Y_{it} = L_{it}$$

  - Firm i hires labour quantity L_{it} on a perfectly competitive labour market and pays the competitive wage W_{it}

- No government, trade or capital, so in equilibrium:

$$C_{it} = Y_{it} = L_{it}$$

2 Households

2.1 Problem and Optimality Conditions:

We now combine the two components of the consumption problem from the Real Business Cycle model- which gave the intertemporal IS curve and labour supply - and The nominal regidity lecture which gave the demand for individual goods, with an assumed desire for cash holdings.

Lifetime utility is given by:

$$\mathcal{U} = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\theta}}{1-\theta} + \frac{\left(\frac{M_t}{P_t}\right)^{1-\mu}}{1-\mu} - \frac{L_t^{\gamma+1}}{\gamma+1} \right]$$

where

$$\begin{aligned} C_t &= \left[\int_0^1 C_{it}^{\frac{\eta-1}{\eta}} di \right]^{\frac{\eta}{\eta-1}} \\ L_t &= \int_0^1 L_{it} di \\ \theta, \mu &> 0 \\ \gamma, \eta &> 1 \end{aligned}$$

Budget Constraint:

$$M_{t-1} + (1 + i_{t-1}) B_{t-1} + W_t L_t = P_t C_t + M_t + B_t$$

Lagrangian:

$$\begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\theta}}{1-\theta} + \frac{\left(\frac{M_t}{P_t}\right)^{1-\mu}}{1-\mu} - \frac{L_t^{\gamma}}{\gamma} \right] + \\ & \sum_{t=0}^{\infty} \lambda_t \left[M_{t-1} + (1 + i_{t-1}) B_{t-1} + W_t L_t - P_t C_t - M_t - B_t \right] \end{aligned}$$

FOC's

$\frac{\partial \mathcal{L}}{\partial L_t} = 0$:

$$\begin{aligned} -\beta^t L_t^{\gamma} + \lambda_t &= 0 \\ \beta^t L_t^{\gamma} &= \lambda_t \qquad (1) \end{aligned}$$

$\frac{\partial \mathcal{L}}{\partial C_t} = 0$

$$\begin{aligned} \beta^t C_t^{-\theta} - \lambda_t P_t &= 0 \\ \frac{\beta^t C_t^{-\theta}}{P_t} &= \lambda_t \qquad (2) \\ \frac{\beta^{t+1} C_{t+1}^{-\theta}}{P_{t+1}} &= \lambda_{t+1} \qquad (3) \end{aligned}$$

$\frac{\partial \mathcal{L}}{\partial M_t} = 0$

$$\beta^t M_t^{-\mu} \left(\frac{1}{P_t} \right)^{1-\mu} = \lambda_t - \lambda_{t+1} \qquad (4)$$

$\frac{\partial \mathcal{L}}{\partial B_t} = 0$

$$\lambda_t = (1 + i_t)\, \lambda_{t+1} \qquad (5)$$

$$\lambda_{t+1} = \frac{1}{(1 + i_t)} \lambda_t$$

These conditions yield:

2.1.1 the labor supply function:

Combine equations 1 and 2 from the static condition that sets the marginal benefit to the household from supplying labour equal to the marginal disutility of providing it:

$$\frac{W_t}{P_t} = \frac{L_t^{\gamma}}{C_t^{-\theta}}$$

2.1.2 the consumption Euler equation:

Combine Equations:
2,3 and 5

$$\begin{aligned} \frac{\beta^t C_t^{-\theta}}{P_t} &= (1 + i_t)\, \frac{\beta^{t+1} C_{t+1}^{-\theta}}{P_{t+1}} \\ C_t^{-\theta} &= (1 + i_t)\, \frac{P_t}{P_{t+1}} \beta C_{t+1}^{-\theta} \\ &= \frac{1 + i_t}{1 + \pi_{t+1}} \beta C_{t+1}^{-\theta} \qquad (6) \\ C_t^{-\theta} &= (1 + r_t)\, \beta C_{t+1}^{-\theta} \qquad (7) \end{aligned}$$

2.1.3 The Money demand function:

5into 4

$$\beta^t \left(\frac{M_t}{P_t}\right)^{-\mu} \frac{1}{P_t} = \frac{i_t}{(1+i_t)}\lambda_t$$

sub in 2

$$\begin{aligned}
\beta^t \left(\frac{M_t}{P_t}\right)^{-\mu} \frac{1}{P_t} &= \frac{i_t}{(1+i_t)} \frac{\beta^t C_t^{-\theta}}{P_t} \\
\left(\frac{M_t}{P_t}\right)^{-\mu} &= \frac{i_t}{(1+i_t)} C_t^{-\theta} \\
\frac{M_t}{P_t} &= C_t^{\frac{\theta}{\mu}} \left(\frac{1+i_t}{i_t}\right)^{\frac{1}{\mu}} \qquad (8)
\end{aligned}$$

2.2 Approximate Equilibrium

In this simple model, the equilibrium conditions are that markets must clear:

$$C_t = Y_t = L_t$$

2.2.1 The intertemporal IS curve:

The Euler equation from above:

$$C_t^{-\theta} = (1+r_t)\,\beta C_{t+1}^{-\theta}$$

In steady state:

$$C_t = C_{t+1} = C^*$$

$$\begin{aligned}
C^{*-\theta} &= (1+r^*)\,\beta C^{*-\theta} \\
-\theta \ln C^* &= \ln(1+r^*) + \ln\beta - \theta \ln C^* \\
(1+r^*) &= \frac{1}{\beta}
\end{aligned}$$

$$\ln(1+r^*) = -\ln\beta$$

In equilibrium around the steady state:
From the Euler equation 7, take logs solve for $\ln C_t$:

$$
\begin{aligned}
C_t^{-\theta} &= (1+r_t)\,\beta C_{t+1}^{-\theta} \\
-\theta \ln C_t &= \ln(1+r_t) + \ln\beta - \theta \ln C_{t+1} \qquad (9) \\
\ln C_t &= -\frac{1}{\theta}\ln(1+r_t) - \frac{1}{\theta}\ln\beta + \ln C_{t+1} \qquad (10)
\end{aligned}
$$

Subtract log steady state euler equation from both sides:

$$
\ln C_t - \ln C^* = -\frac{1}{\theta}\left(\ln(1+r_t) - \ln(1+r^*)\right) - \frac{1}{\theta}\ln\beta + \frac{1}{\theta}\ln\beta + \ln C_{t+1} - \ln C^*
$$

Equilibrium Condition:

$$
C_t = Y_t
$$

Approximation/redefinition (Note: Romer follows a slightly different way of motivating this - the details are irrelevant for us at this stage):

$$
\begin{aligned}
r_t &\equiv \ln(1+r_t) - \ln(1+r^*) \\
y_t &\equiv \ln Y_t - \ln Y^*
\end{aligned}
$$

Plug into 10:

$$
\ln Y_t - \ln Y^* = -\frac{1}{\theta} r_t + \ln Y_{t+1} - \ln Y^* \qquad (11)
$$

$$
y_t = -\frac{1}{\theta} r_t + y_{t+1} \qquad (12)
$$

Now that things are linear, we can add back uncertainty (again: we would have obtained this result anyway, but with much more work):

$$
y_t = -\frac{1}{\theta}\left(i_t - E_t \pi_{t+1}\right) + E_t y_{t+1}
$$

2.2.2 The LM curve:

Plug equilibrium condition into 8:

$$
\begin{aligned}
\frac{M_t}{P_t} &= Y_t^{\frac{\theta}{\mu}} \left(\frac{1+i_t}{i_t}\right)^{\frac{1}{\mu}} \\
Y_t &= \left(\frac{M_t}{P_t}\right)^{\frac{\mu}{\theta}} \left(\frac{i_t}{1+i_t}\right)^{\frac{1}{\theta}}
\end{aligned}
$$

2.2.3 Labour Market equilibrium:

Using the equilibrium condition $C_t = Y_t = L_t$:

$$\frac{W_t}{P_t} = \frac{L_t^{\gamma}}{C_t^{-\theta}}$$

$$\frac{W_t}{P_t} = Y_t^{\theta+\gamma}$$

Or in log-deviations from steady state:

$$w_t - p_t = (\theta + \gamma)\, y_t$$

3 The Dynamic price setting problem of the Firm

As in lecture 8, the demand for each type of good in any period t is given by:

$$Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\eta} Y_t$$

the real profit in any specific period is:

$$\begin{aligned}
\frac{R_{it}}{P_t} &= \frac{P_{it}}{P_t} Y_{it} - \frac{W_t}{P_t} Y_{it} \\
&= \left(\frac{P_{it}}{P_t}\right)^{1-\eta} Y_t - \frac{W_t}{P_t}\left(\frac{P_{it}}{P_t}\right)^{-\eta} Y_t \\
&= Y_t P_t^{\eta-1}\left(P_{it}^{1-\eta} - W_t P_{it}^{-\eta}\right)
\end{aligned}$$

Dynamic Price Setting problem under “stochastically” fixed prices

- We will be considering the problem of a firm setting a price that may remain in place for a number of periods we will normalize the date of the decision to “period 0”. Annoyingly, Romer chooses to lable this P_i rather than P_{i0}

- The problem the firm faces is to set a single price to maximize profits over some time horizon. The most important model we will discuss uses a mechanism that makes it possible for a price to stay fixed forever. To deal with this:

  - let q_t be the probability that P_i set in period 0 is still in effect in period t
  - let λ_t the discount factor that translates period t profits into period 0 units.

* Since the firms ultimately belong to the households, Romer mentions that this discount factor should be what is known as the subjective discount factor derived from the utility function:

$$\Lambda_t = \frac{\beta^t U'(C_t)}{U'(C_0)}$$

* This is an equilibrium result. One usually starts the firm off using the real interest rate as the discount factor as this is the effective rate at which it can transfer wealth across time by market forces. In equilbrium, of course, the market and subjective discount factors should be the same:

$$\Lambda_t = \frac{\beta^t U'(C_t)}{U'(C_0)} = \prod_{s=0}^{t} \frac{1}{1+r_s}$$

The expected profit in period t is:

$$E_0 \left[q_t \Lambda_t \frac{R_{it}}{P_t} \right]$$

Using these definitions, the problem facing a firm in period zero is:

$$\max_{P_i} \sum_{t=0}^{\infty} E_0 \left[q_t \Lambda_t \frac{R_{it}}{P_t} \right]$$

Adding the definition of real profits from above:

$$\max_{P_i} \sum_{t=0}^{\infty} E_0 \left[q_t \Lambda_t Y_t P_t^{\eta-1} \left(P_i^{1-\eta} - W_t P_i^{-\eta} \right) \right]$$

The last additional bit we will add will be convenient when we start using approximations to simplify the problem: Let us rewrite the profit in terms of the desired price in each period rather than wage.

Recall from lecture 8 that if the firm is able to set it's price in every period (i.e. perfectly flexible price scenario) the problem is entirely static (i.e. in each period the optimal price is a function of only variables from that period).

$$\frac{P_{it}^*}{P_t} = \frac{\eta}{\eta-1} \frac{W_t}{P_t}$$

Recalling the observation that the right hand side variables do not vary by i and are constants from the point of view of the firm's price setting decision, we can drop the i index: $P_{it}^* = P_t^*$. The optimal price P_t^* to set in any period t is a markup over marginal costs, and we can thus rewrite the equation to eliminate:

$$\begin{aligned} \frac{P_t^*}{P_t} &= \frac{\eta}{\eta-1} \frac{W_t}{P_t} \\ P_t^* &= \frac{\eta}{\eta-1} W_t \\ W_t &= \frac{\eta-1}{\eta} P_t^* \end{aligned}$$

So the firms multiperiod price setting problem becomes:

$$\max_{P_i} \sum_{t=0}^{\infty} E_0 \left[q_t \Lambda_t Y_t P_t^{\eta-1} P_i^{-\eta} \left(P_i - \frac{\eta-1}{\eta} P_t^* \right) \right]$$

It is not obvious how to solve this problem in closed form in its full non-linear form so we will make a number of assumptions to simplify it to a point where we can find an approximate solution.

Why? because the time paths of λ_t (e.g.) depend on consumption path which in turn depends on price path

3.1 Approximation of the solution to the firms problem

We will make two assumptions that will allow us to use Taylor's approximation theorem to give some characterization of solution to the problem:

1. β is very close to one and inflation is low (this means the subjective discount factor λ_t is close to constant in steady state); and
2. the economy stays close to its flexible price equilibrium

The latter assumption is typical in complicated non-linear models that are solved in approximate form: the approximations are only valid close to the point around which the model approximation is constructed (the steady state) so our predictions are only valid in a small neighbourhood of that point. Hence we "assume" that this is all that ever happens. The financial crisis is a good example of how wrong this assumption can be in the real world.

Since we will approximate, we are not particularly interested in the exact nonlinear structure relating the desired price in each period $P_t^*$and the price set in period zero P_i. Additionally, as is also standard, it is convenient for interpretations to consider all variables in log deviation from steady state. Here normalizations means that the log steady state variables are all zero, so we need only consider the logs of the variables.

This means the firms problem can be written as:

$$\max_{p_i} \sum_{t=0}^{\infty} E_0 \left[q_t \Lambda_t Y_t P_t^{\eta-1} F(p_i, p_t^*) \right]$$

Now since we assumed that the economy stays near its flexible price equilibrium:

- the variation in $\lambda_t Y_t P_t^{\eta-1}$ will be small relative to the deviation of any individual firm's deviation from it's preferred price, so we will ignore it

$$\max_{p_i} \sum_{t=0}^{\infty} E_0 \left[q_t \Lambda_t Y_t P_t^{\eta-1} F(p_i, p_t^*) \right] \approx \max_{p_i} \sum_{t=0}^{\infty} E_0 \left[q_t F(p_i, p_t^*) \right]$$

- We can use a second order approximation of $F(p_i, p_t^*)$ arround the flexible price equililbrium $p_i = p_t^*$ where F is maximized:

$$F(p_i, p_t^*) \approx F(p_t^*, p_t^*) + \frac{\partial F(p_t^*, p_t^*)}{\partial p_i}(p_i - p_t^*) + \frac{1}{2}\frac{\partial^2 F(p_t^*, p_t^*)}{\partial p_i^2}(p_i - p_t^*)^2$$

- At the max:

$$\begin{aligned} \frac{\partial F(p_t^*, p_t^*)}{\partial p_i} &= 0 \\ \frac{\partial^2 F(p_t^*, p_t^*)}{\partial p_i^2} &< 0 \end{aligned}$$

- So

$$F(p_i, p_t^*) \approx \underbrace{A}_{>0} + \underbrace{B}_{<0}(p_i - p_t^*)^2$$

- Thus:

$$\arg\max_{p_i} \sum_{t=0}^{\infty} E_0\left[q_t \Lambda_t Y_t P_t^{\eta-1} F(p_i, p_t^*)\right] \approx \arg\min_{p_i} \sum_{t=0}^{\infty} E_0\left[q_t (p_i - p_t^*)^2\right]$$

- Additionally, we will ignore the expectation operator and add it back later

So the problem of the firm can be reduced to:

$$\begin{aligned} & \arg\min_{p_i} \sum_{t=0}^{\infty}\left[q_t (p_i - p_t^*)^2\right] \\ = \quad & \arg\min_{p_i} \sum_{t=0}^{\infty}\left[q_t \left(p_i^2 - 2p_i p_t^* + p_t^{*2}\right)\right] \\ = \quad & \arg\min_{p_i}\left[\sum_{t=0}^{\infty} q_t p_i^2 - 2\sum_{t=0}^{\infty} q_t p_i p_t^* + \sum_{t=0}^{\infty} q_t p_t^{*2}\right] \\ = \quad & \arg\min_{p_i}\left[p_i^2 \sum_{t=0}^{\infty} q_t - 2p_i \sum_{t=0}^{\infty} q_t p_t^* + \sum_{t=0}^{\infty} q_t p_t^{*2}\right] \end{aligned}$$

FOC:

$$\begin{aligned} 2p_i \sum_{t=0}^{\infty} q_t &= 2\sum_{t=0}^{\infty} q_t p_t^* \\ p_i &= \sum_{t=0}^{\infty} \frac{q_t}{\sum_{s=0}^{\infty} q_s} p_t^* \\ &= \sum_{t=0}^{\infty} \omega_t p_t^* \end{aligned}$$

So the optimal price to set for a firm who knows the price might be in effect with some probability in any of the future periods is simple the probability weighted average of all future desired prices. If there is uncertainty, we will again depend on the approximate nature of our solution to use:

$$p_i = \sum_{t=0}^{\infty} \omega_t E_0 \left[p_t^* \right]$$

The last steps is to relate this to the typical aggregate variables that we have empirical measures for

Using:

$$\begin{aligned} \frac{W_t}{P_t} &= BY_t^{\theta+\gamma-1} \\ w_t &= p_t + (\theta + \gamma)\, y_t \end{aligned}$$

and:

$$\begin{aligned} P_t^* &= \frac{\eta}{\eta - 1} W_t \\ p_t^* &= w_t \end{aligned}$$

and:

$$\begin{aligned} Y_t &= \frac{M_t}{P_t} \\ y_t &= m_t - p_t \end{aligned}$$

Combining:

$$\begin{aligned} p_t^* &= w_t \\ &= p_t + (\theta + \gamma)\, y_t \\ &= p_t + (\theta + \gamma)\,(m_t - p_t) \\ &= \phi m_t + (1 - \phi)\, p_t \end{aligned}$$

Where we have defined $\phi = (\theta + \gamma) > 0$.

Lastly then, the optimal price to set in period 0 is a function of all expected future prices and money aggregates in which there is a probability that the price will still be in effect:

$$p_i = \sum_{t=0}^{\infty} \omega_t E_0 \left[\phi m_t + (1 - \phi)\, p_t \right]$$

4 Different Models

4.1 Fischer Model

In any period, half the prices were set two periods ago, call this $p_t^2 = p_{t|t-2}$ the other half one period ago ($p_t^1 = p_{t|t-1}$), so that the average price in period t is:

$$p_t = \frac{1}{2}\left(p_{t|t-1} + p_{t|t-2}\right)$$

To plug this into the general function above, note:

In each period a single firm must set two prices, and it knows when those prices are going to be valid

so in period $t-2$ a firm sets price for $t-1$ knowing that $\omega_{t-1} = 1$ and $\omega_{s>t-1} = 0$ and for price in $t+1$

Thus we know:

$$\begin{aligned}
p_{t|t-1} &= E_{t-1}\left[\phi m_t + (1-\phi)\, p_t\right] \\
&= E_{t-1}\left[\phi m_t + (1-\phi)\frac{1}{2}\left(p_{t|t-1} + p_{t|t-2}\right)\right] \\
&= \phi E_{t-1} m_t + (1-\phi)\frac{1}{2}\left(p_{t|t-1} + p_{t|t-2}\right)
\end{aligned}$$

$$\begin{aligned}
p_{t|t-2} &= E_{t-2}\left[\phi m_t + (1-\phi)\, p_t\right] \\
&= E_{t-2}\left[\phi m_t + (1-\phi)\frac{1}{2}\left(p_{t|t-1} + p_{t|t-2}\right)\right] \\
&= \phi E_{t-2} m_t + (1-\phi)\frac{1}{2}\left(E_{t-2}\left[p_{t|t-1}\right] + p_{t|t-2}\right)
\end{aligned}$$

Solving for $p_{t|t-1}$ and $p_{t|t-2}$ and using $y_t = m_t - p_t$.

$$\begin{aligned}
p_t &= E_{t-2} m_t + \frac{\phi}{1+\phi}\left(E_{t-1} m_t - E_{t-2} m_t\right) \\
y_t &= \frac{1}{1+\phi}\left(E_{t-1} m_t - E_{t-2} m_t\right) + \left(m_t - E_{t-1} m_t\right)
\end{aligned}$$

4.2 Calvo model

Each period a firm has probability α of getting to change its price, regardless of how long it has been since it last changed its price.

- By the law of large numbers, this means each period exactly a fraction α of firms do set their prices anew

- Firms are chosen at random
- This means we have a simple and deterministic expression for $q_t{=}(1-\alpha)^t$
- In this model, it turns out to be simple to allow for $\beta \neq 1$

Recall the optimal price, allowing for an arbitrary $0 < \beta < 1$:

$$\begin{aligned} p_{it} &= \sum_{j=0}^{\infty} \frac{\beta^j q_j}{\sum_{s=0}^{\infty} \beta^s q_s} E_t\left[p_{t+j}^*\right] \\ &= \sum_{t=0}^{\infty} \frac{\beta^j (1-\alpha)^j}{\sum_{s=0}^{\infty} \beta^s (1-\alpha)^s} E_t\left[p_{t+j}^*\right] \end{aligned}$$

- a convergent geometric series: $a^0 + a + a^2 + ... = (1-a)$ if $|a| < 1$

$$p_{it} = [1 - \beta(1-\alpha)] \sum_{j=0}^{\infty} \beta^j (1-\alpha)^j E_t\left[p_{t+j}^*\right]$$

Since all firms face the same aggregate problem, they will choose the same price, let x_t be the price chosen by firms that get to change their prices.

Thus this means the average price in any period t is:

$$p_t = \alpha x_t + (1-\alpha) p_{t-1}$$

Some rearrangements we will need:

Subtracting p_{t-1} from both sides yields (remember - we are working in log deviations from steady state):

$$p_t - p_{t-1} = \ln P_t - \ln P_{t-1} = \ln \frac{P_t}{P_{t-1}} = \ln(1+\pi_t) \approx \pi_t$$

So:

$$\begin{aligned} p_t - p_{t-1} = \pi_t &= \alpha x_t + (1-\alpha) p_{t-1} - p_{t-1} \\ &= \alpha (x_t - p_{t-1}) \end{aligned}$$

Alternatively:

$$\begin{aligned} (x_t - p_{t-1}) &= \frac{\pi_t}{\alpha} \\ (x_{t+1} - p_t) &= \frac{\pi_{t+1}}{\alpha} \\ E_t\left[(x_{t+1} - p_t)\right] &= E_t\left[\frac{\pi_{t+1}}{\alpha}\right] \\ E_t\left[x_{t+1}\right] - p_t &= \frac{E_t\left[\pi_{t+1}\right]}{\alpha} \end{aligned}$$

Let's unpack x_t a bit:

$$
\begin{aligned}
x_t &= [1-\beta(1-\alpha)]\sum_{j=0}^{\infty}\beta^j(1-\alpha)^j E_t\left[p^*_{t+j}\right] \\
&= [1-\beta(1-\alpha)]\beta^0(1-\alpha)^0 E_t[p^*_t] + [1-\beta(1-\alpha)]\sum_{j=1}^{\infty}\beta^j(1-\alpha)^j E_t\left[p^*_{t+j}\right] \\
&= [1-\beta(1-\alpha)]p^*_t + \beta(1-\alpha)[1-\beta(1-\alpha)]\sum_{j=0}^{\infty}\beta^j(1-\alpha)^j E_t\left[p^*_{t+1+j}\right] \\
&= [1-\beta(1-\alpha)]p^*_t + \beta(1-\alpha)E_t[x_{t+1}]
\end{aligned}
$$

Subtract p_t from both sides, and add and subtract p_{t-1} on the LHS:

$$
\begin{aligned}
x_t - p_{t-1} - p_t + p_{t-1} &= [1-\beta(1-\alpha)]p^*_t + \beta(1-\alpha)E_t[x_{t+1}] - [1-\beta(1-\alpha)+\beta(1-\alpha)]p_t \\
(x_t - p_{t-1}) - (p_t - p_{t-1}) &= [1-\beta(1-\alpha)](p^*_t - p_t) + \beta(1-\alpha)(E_t[x_{t+1}] - p_t)
\end{aligned}
$$

Using results from above, and $(p^*_t - p_t) = \phi y_t$ (Romer equation 6.58, p. 272) that is, if prices are sticky, the optimal price deviates from the aggregate price proportionally to the output gap: if output is above steady state - high relative demand, it is optimal to charge higher than average price, and vice versa.

$$
\begin{aligned}
\frac{\pi_t}{\alpha} - \pi_t &= [1-\beta(1-\alpha)]\phi y_t + \beta(1-\alpha)\frac{E_t[\pi_{t+1}]}{\alpha} \\
(1-\alpha)\pi_t &= \alpha[1-\beta(1-\alpha)]\phi y_t + \beta(1-\alpha)E_t[\pi_{t+1}] \\
\pi_t &= \frac{\alpha}{(1-\alpha)}[1-\beta(1-\alpha)]\phi y_t + \beta E_t[\pi_{t+1}]
\end{aligned}
$$

And thus, defining $\kappa = \frac{\alpha}{(1-\alpha)}[1-\beta(1-\alpha)]\phi$ yields the New Keynesian Phillips Curve: Note that this is a Supply side concept - it comes from the optimal pricing behaviour of producers of goods.

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1}$$

5 Typical DSGE model in solved form

The standard form in which the implications of a DSGE model are analysed is in a system of dynamic linear equations that describe the relationships between a small number of aggregate variables.

These usually consist of an IS curve (the demand side), a Philips Curve (the supply side) and a monetary policy function, called a Taylor rule, each augmented with some stochastic component:

$$
\begin{aligned}
y_t &= -\frac{1}{\theta}\left(i_t - E_t\pi_{t+1}\right) + E_t y_{t+1} + u_t^d \\
\pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t^s \\
i_t &= \psi_y E_t y_{t+1} + \psi_\pi E_t \pi_{t+1} + u_t^m
\end{aligned}
$$

This system can be solved using linear algebra and its predictions for the dynamics of the three variables can be studied and evaluated against real data

This is a delicate problem, and while extremely popular, does not work very well in this simplified form.

There are obviously much larger versions of them (from 10 - 30 variables) which work better in normal times but the crisis showed that this cannot be the exclusive thing to look at.