New Keynesian Models

Sang Seok Lee

Department of Economics, Bilkent University

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New Keynesian Theory

- New Keynesian theory builds on RBC theory by adding nominal frictions such as sticky prices and wages
  - This results to "monetary non-neutrality" where monetary policy significantly influences the real economy in the short-run
  - Then, it is possible to discuss the optimal monetary policy design which concerns how monetary policy should be conducted by a monetary authority
- Gordon (1990) discusses the origin of New Keynesian theory and explains how it is different from Neoclassical macroeconomic theory (e.g. RBC theory) in details
- Clarida, Gali, and Gertler (1999) provide a comprehensive overview of New Keynesian literature, including time inconsistency issues such as discretion vs. commitment (see Kydland and Prescott (1979) for introduction to this)

Simple New Keynesian Model

- We will study a simple New Keynesian model by following Walsh (2017) (this is one of standard reference books for monetary theory and policy), but with further simplifications such as taking money out of the model (see Woodford, 2003)
- The model has three sectors: households, firms, and monetary authority (we abstract away government and capital investment)
  - A household allocates its resources across goods and across time
  - A firm chooses how much to produce and what price to set for its good (firms have market power)
  - Monetary authority conducts its policy by setting a nominal interest rate
- Let us start with households
  - We assume the existence of the representative household

Households

- The utility function of the representative household is $E_0\{\sum_{t=0}^{\infty}\beta^t(\frac{C_t^{1-\sigma}-1}{1-\sigma}-\chi\frac{N_t^{1+\eta}}{1+\eta})\}$ where
  - C_t := consumption of a basket of goods in period t
  - N_t := hours worked in period t
- C_t has a Constant Elasticity of Substitution (CES) form: $C_t=[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}}dj]^{\frac{\theta}{\theta-1}}$, where θ is the elasticity of substitution for individual goods and $\theta>1$ (see Dixit and Stiglitz, 1977)
- The household's problem can be broken down into two:
  1. The optimal allocation of a given consumption expenditure across individual goods in the consumption basket
  2. The maximization of the expected discounted life time utility taking the optimal consumption basket as a given

Problem 1

- Objective: $\min_{c_{jt}} \int_0^1 p_{jt} c_{jt} dj$ s.t. $C_t = [\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj]^{\frac{\theta}{\theta-1}}$ where p_{jt} is the price of c_{jt}
- The Lagrangian is $L = \int_0^1 p_{jt} c_{jt} dj + \psi_t (C_t - [\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj]^{\frac{\theta}{\theta-1}})$
  - $L_{c_{jt}} = p_{jt} - \psi_t ([\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj]^{\frac{1}{\theta-1}} c_{jt}^{-\frac{1}{\theta}}) = 0$
  - By definition of C_t, this implies that $p_{jt} - \psi_t C_t^{\frac{1}{\theta}} c_{jt}^{-\frac{1}{\theta}} = 0$
  - Rearranging the above, $c_{jt} = (\frac{p_{jt}}{\psi_t})^{-\theta} C_t$
  - Substituting c_{jt} into C_t, $C_t = [\int_0^1 [(\frac{p_{jt}}{\psi_t})^{-\theta} C_t]^{\frac{\theta-1}{\theta}} dj]^{\frac{\theta}{\theta-1}}$
    $= (\frac{1}{\psi_t})^{-\theta} [\int_0^1 p_{jt}^{1-\theta} dj]^{\frac{\theta}{\theta-1}} C_t$
  - It follows that $\psi_t = [\int_0^1 p_{jt}^{1-\theta} dj]^{\frac{1}{1-\theta}} := P_t$ (the price of the optimal basket of goods in period t)
  - Then, $c_{jt} = (\frac{p_{jt}}{P_t})^{-\theta} C_t$
    - This is the demand curve that firm j faces since the model does not have government expenditure and/or capital investment

Problem 2

- Objective: $\max_{\{C_t,N_t,B_t\}_{t=0}^{\infty}} E_0\{\sum_{t=0}^{\infty}\beta^t(\frac{C_t^{1-\sigma}-1}{1-\sigma}-\chi\frac{N_t^{1+\eta}}{1+\eta})\}$ s.t. $C_t+\frac{B_t}{P_t}=\frac{W_tN_t}{P_t}+(1+R_{t-1})\frac{B_{t-1}}{P_t}+\frac{\Pi_t}{P_t}$ and $B_{-1}>0$ where
  - B_t := nominal bond holding in period t
  - W_t := wage in period t
  - R_t := nominal interest rate between period t and t+1
  - Π_t := profit from firms in period t
- It is straightforward to derive the following first order conditions:
  - $C_t^{-\sigma}=\beta(1+R_t)E_t\{(\frac{P_t}{P_{t+1}})C_{t+1}^{-\sigma}\}$ (Euler equation for consumption (EC))
  - $\chi N_t^{\eta}=C_t^{-\sigma}\frac{W_t}{P_t}$ (intratemporal labor consumption condition (ILC))

Firms

Firms in the economy face three constraints:

1. A typical firm j works with a linear production technology $c_{jt} = Z_t N_{jt}$ where N_{jt} is the labor input of firm j and Z_t is the stochastic total factor productivity common to all firms $(E\{Z_t\} = 1)$
2. Firm j faces the downward sloping demand curve $c_{jt} = (\frac{p_{jt}}{P_t})^{-\theta} C_t$
3. In any period, fraction $(1 - \omega)$ of firms are randomly selected and allowed to change their prices. Fraction ω of firms continue charging prices set previously (see Calvo, 1983)

Firms' Problem

- Firm j's cost minimization problem is

$$\min_{N_{jt}}((\frac{W_t}{P_t})N_{jt} + \varphi_t(c_{jt} - Z_t N_{jt}))$$

  where φ_t is the firm's real marginal cost
  - $\varphi_t = MC_t = \frac{W_t/P_t}{Z_t}$ follows from this minimization
- Using the stochastic discount factor $\Delta_{i,t+i} = \beta^i (C_{t+i}/C_t)^{-\sigma}$ (see Campbell, 2000) and $c_{jt} = (\frac{p_{jt}}{P_t})^{-\theta} C_t$, firm j's problem in period t is

$$\max_{p_{jt}} E_t\{\sum_{i=0}^{\infty} \omega^i \triangle_{i,t+i}[(\frac{p_{jt}}{P_{t+i}})c_{jt+i} - MC_{t+i}c_{jt+i}]\}$$

$$= \max_{p_{jt}} E_t\{\sum_{i=0}^{\infty} \omega^i \triangle_{i,t+i}[(\frac{p_{jt}}{P_{t+i}})^{1-\theta} - MC_{t+i}(\frac{p_{jt}}{P_{t+i}})^{-\theta}]C_{t+i}\}$$

Firms' Problem

- All firms adjusting their prices in period t face the same problem so they will set the same price
  - The FOC is

$$E_t\{\sum_{i=0}^{\infty}\omega^i\Delta_{i,t+i}[(1-\theta)(\frac{p_t^*}{P_{t+i}})+\theta MC_{t+i}](\frac{1}{p_t^*})(\frac{p_t^*}{P_{t+i}})^{-\theta}C_{t+i}\}=0$$

    where p_t^* is the optimal price for period t
  - The derivation of the FOC above uses

$$(1-\theta)\left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta}\frac{1}{P_{t+i}}=(1-\theta)\frac{p_{jt}^{-\theta}}{P_{t+i}^{-\theta}}\frac{1}{P_{t+i}}\frac{p_{jt}}{p_{jt}}=(1-\theta)\left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta+1}\frac{1}{p_{jt}}$$

$$-\theta MC_{t+i}(\frac{p_{jt}}{P_{t+i}})^{-\theta-1}\frac{1}{P_{t+i}}=-\theta MC_{t+i}(\frac{p_{jt}}{P_{t+i}})^{-\theta-1}\frac{1}{P_{t+i}}\frac{p_{jt}}{p_{jt}}=-\theta MC_{t+i}(\frac{p_{jt}}{P_{t+i}})^{-\theta}\frac{1}{p_{jt}}$$

Flexible Price Equilibrium

- If the firms can set their prices every period ($\omega = 0$), the FOC above reduces to $(\frac{p_t^*}{P_t}) = (\frac{\theta}{\theta-1})MC_t$
  - The term $(\frac{\theta}{\theta-1})$ reflects the markup over the price which is due to the market power of the firms
  - Because all firms will set the same price, $P_t = p_t^*$
  - So the condition above reduces to $1 = (\frac{\theta}{\theta-1})\frac{W_t/P_t}{Z_t}$
  - Combining with ILC, this becomes $\frac{W_t}{P_t} = \frac{Z_t}{\theta/(\theta-1)} = \frac{\chi N_t^\eta}{C_t^{-\sigma}}$

Flexible Price Equilibrium

- Let us log-linearize $\frac{Z_t}{\theta/(\theta-1)} = \frac{\chi N_t^{\eta}}{C_t^{-\sigma}}$
  - Taking the natural logarithm of the both sides,

$$\ln Z_t - \ln \frac{\theta}{\theta - 1} = \ln \chi + \eta \ln N_t + \sigma \ln C_t$$

  - $\ln Z_t \approxeq \ln Z + \frac{Z_t - Z}{Z} = \ln Z + \hat{z}_t$ (Z denotes the steady state value of Z and $\hat{z}_t$ the percentage deviation from Z)
  - $\ln \frac{\theta}{\theta-1}$ and $\ln \chi$ remain the same because they are constants
  - $\eta \ln N_t \approxeq \eta \ln N^f + \eta \hat{n}_t^f$ (N^f denotes the flexible price equilibrium steady state of N and $\hat{n}_t^f$ the percentage deviation from N^f)
  - $\sigma \ln C_t \approxeq \sigma \ln C^f + \sigma \hat{c}_t^f$
  - Using $\ln Z - \ln \frac{\theta}{\theta-1} = \ln \chi + \eta \ln N^f + \sigma \ln C^f$, the log-linear approximation becomes $\hat{z}_t = \eta \hat{n}_t^f + \sigma \hat{c}_t^f$

Flexible Price Equilibrium

- Because all firms demand the same amount of labor input, N_{jt} is the same $\forall j$
- Using $c_{jt} = Z_t N_{jt}$, $c_{jt} = (\frac{p_{jt}}{P_t})^{-\theta} C_t$, and the labor market equilibrium condition, $C_t = Z_t N_t$
  - Log-linearizing this gives $\hat{c}_t^f = \hat{n}_t^f + \hat{z}_t$
- Because the demand in the goods market consists of the household consumption alone, $\hat{c}_t^f = \hat{y}_t^f$ by the goods market equilibrium condition where Y_t is the aggregate output in period t

Flexible Price Equilibrium

- We derived two equations: $\hat{z}_t = \eta \hat{n}_t^f + \sigma \hat{y}_t^f$ and $\hat{y}_t^f = \hat{n}_t^f + \hat{z}_t$
- Combining the two yields $\hat{y}_t^f = (\frac{1+\eta}{\sigma+\eta})\hat{z}_t$ which is called "the flexible price equilibrium level of output"
  - This is also called "the natural level of output"

Sticky Price

- Let us consider the sticky price case ($\omega > 0$)
- Price index $P_t = [\int_0^1 p_{jt}^{1-\theta} dj]^{\frac{1}{1-\theta}}$ evolves according to

$$P_t^{1-\theta} = (1-\omega)(p_t^*)^{1-\theta} + \omega P_{t-1}^{1-\theta}$$

- Let us log-linearize the above around the flexible price equilibrium steady state with zero inflation rate
  - Let $Q_t := \frac{p_t^*}{P_t}$. The steady state value of this is 1
  - Dividing the both sides of the price index above by $P_t^{1-\theta}$, $1 = (1-\omega)Q_t^{1-\theta} + \omega(P_{t-1}/P_t)^{1-\theta}$ follows
  - Taking the natural logarithm of the both sides, $0 = \ln((1-\omega)Q_t^{1-\theta} + \omega(P_{t-1}/P_t)^{1-\theta})$

Sticky Price

$$0 = \ln((1-\omega)Q_t^{1-\theta} + \omega(P_{t-1}/P_t)^{1-\theta})$$

- The approximation of the left hand side is 0 because it is a constant
- The right hand side is approximated by $\ln(1) + (1-\omega)(1-\theta)(\frac{Q_t-1}{1}) + \omega(1-\theta)(\frac{(P_{t-1}/P_t)-1}{1}) = (1-\theta)[(1-\omega)\hat{q}_t - \omega\pi_t]$ where $(\frac{Q_t-1}{1}) = \hat{q}_t$ and $\frac{P_t-P_{t-1}}{P_{t-1}} = \pi_t$
- Overall, the approximation gives $0 = (1-\omega)\hat{q}_t - \omega\pi_t$
  - Rearranging this, $\hat{q}_t = (\frac{\omega}{1-\omega})\pi_t$

Sticky Price

$$E_t\{\sum_{i=0}^{\infty}\omega^i\Delta_{i,t+i}[(1-\theta)(\frac{p_t^*}{P_{t+i}})+\theta MC_{t+i}](\frac{1}{p_t^*})(\frac{p_t^*}{P_{t+i}})^{-\theta}C_{t+i}\}=0$$

- Using the definition of $\Delta_{i,t+i}=\beta^i(C_{t+i}/C_t)^{-\sigma}$,

$$E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i(\frac{C_{t+i}}{C_t})^{-\sigma}(\theta-1)(\frac{p_t^*}{P_{t+i}})(\frac{1}{p_t^*})(\frac{p_t^*}{P_{t+i}})^{-\theta}C_{t+i}\}$$

$$=E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i(\frac{C_{t+i}}{C_t})^{-\sigma}\theta MC_{t+i}](\frac{1}{p_t^*})(\frac{p_t^*}{P_{t+i}})^{-\theta}C_{t+i}\}$$

- After some manipulations,

$$(\theta-1)\,(p_t^*)^{-\theta}\,E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}P_{t+i}^{\theta-1}\}=\theta\,(p_t^*)^{-\theta-1}\,E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}MC_{t+i}P_{t+i}^{\theta}\}$$

$$p_t^*\,E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}P_{t+i}^{\theta-1}\}=\frac{\theta}{\theta-1}E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}MC_{t+i}P_{t+i}^{\theta}\}$$

Sticky Price

$$p_t^* E_t\{\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} P_{t+i}^{\theta-1}\} = \frac{\theta}{\theta-1} E_t\{\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} MC_{t+i} P_{t+i}^{\theta}\}$$

- Using the definition of $Q_t = \frac{p_t^*}{P_t}$ and setting $\frac{\theta}{\theta-1} = \mu$,

$$E_t\{\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} P_{t+i}^{\theta-1}\} P_t^{1-\theta} P_t^{\theta} Q_t = \mu E_t\{\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} MC_{t+i} P_{t+i}^{\theta}\}$$

$$E_t\{\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} (\frac{P_{t+i}}{P_t})^{\theta-1}\} Q_t = \mu E_t\{\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} MC_{t+i} (\frac{P_{t+i}}{P_t})^{\theta}\}$$

  - In the flexible price equilibrium steady state with zero inflation, the above reduces to $Q = \mu MC = 1$

- Let us log-linearize the above around the flexible price equilibrium steady state with zero inflation
  - Taking the natural logarithm of the left hand side, $E_t\{\ln(\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} (\frac{P_{t+i}}{P_t})^{\theta-1})\} + \ln Q_t$ (we used the technique $\ln E\{x\} = E\{\ln x\}$)

Sticky Price

$$E_t\{\ln(\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}(\frac{P_{t+i}}{P_t})^{\theta-1})\}+\ln Q_t$$

- First term:

$$\ln(\frac{C^{1-\sigma}}{1-\omega\beta})+\frac{(1-\sigma)C^{-\sigma}}{C^{1-\sigma}/(1-\omega\beta)}\sum_{i=0}^{\infty}\omega^i\beta^i E_t\{C_{t+i}-C\}$$

$$+\frac{(\theta-1)C^{1-\sigma}/P}{C^{1-\sigma}/(1-\omega\beta)}\sum_{i=0}^{\infty}\omega^i\beta^i E_t\{P_{t+i}-P\}-\frac{(\theta-1)C^{1-\sigma}/P}{C^{1-\sigma}/(1-\omega\beta)}\sum_{i=0}^{\infty}\omega^i\beta^i(P_t-P)$$

$$=\ln(\frac{C^{1-\sigma}}{1-\omega\beta})+(1-\sigma)(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i E_t\{\hat{c}_{t+i}\}$$

$$+(\theta-1)(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i E_t\{\hat{p}_{t+i}\}-(\theta-1)(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i\hat{p}_t$$

$$=\ln(\frac{C^{1-\sigma}}{1-\omega\beta})+(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i[(1-\sigma)E_t\{\hat{c}_{t+i}\}+(\theta-1)(E_t\{\hat{p}_{t+i}\}-\hat{p}_t)]$$

- Second term:

$$\ln(1)+(\frac{Q_t-1}{1})=\hat{q}_t$$

Sticky Price

$$E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}(\frac{P_{t+i}}{P_t})^{\theta-1}\}Q_t = \mu E_t\{\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}MC_{t+i}(\frac{P_{t+i}}{P_t})^{\theta}\}$$

- Now, let us deal with the right hand side of the equation above
- Taking the natural logarithm of the right hand side and applying the usual technique,

$$\ln\mu + E_t\{\ln(\sum_{i=0}^{\infty}\omega^i\beta^i C_{t+i}^{1-\sigma}MC_{t+i}(\frac{P_{t+i}}{P_t})^{\theta})\}$$

Sticky Price

$$\ln \mu + E_t\{\ln(\sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} MC_{t+i} (\frac{P_{t+i}}{P_t})^{\theta})\}$$

- First term: Its approximation is itself as $\ln \mu$ is a constant
- Second term:

$$\ln(\frac{C^{1-\sigma}MC}{1-\omega\beta}) + \frac{(1-\sigma)C^{-\sigma}MC}{(C^{1-\sigma}MC)/(1-\omega\beta)} \sum_{i=0}^{\infty} \omega^i \beta^i E_t\{C_{t+i} - C\}$$

$$+\frac{C^{1-\sigma}}{(C^{1-\sigma}MC)/(1-\omega\beta)} \sum_{i=0}^{\infty} \omega^i \beta^i E_t\{MC_{t+i} - MC\}$$

$$+\frac{(\theta C^{1-\sigma}MC)/P}{(C^{1-\sigma}MC)/(1-\omega\beta)} \sum_{i=0}^{\infty} \omega^i \beta^i E_t\{P_{t+i} - P\} - \frac{(\theta C^{1-\sigma}MC)/P}{(C^{1-\sigma}MC)/(1-\omega\beta)} \sum_{i=0}^{\infty} \omega^i \beta^i (P_t - P)$$

$$= \ln(\frac{C^{1-\sigma}}{1-\omega\beta}) + \ln MC + (1-\sigma)(1-\omega\beta) \sum_{i=0}^{\infty} \omega^i \beta^i E_t\{\hat{c}_{t+i}\} + (1-\omega\beta) \sum_{i=0}^{\infty} \omega^i \beta^i E_t\{\hat{mc}_{t+i}\}$$

$$+\theta(1-\omega\beta) \sum_{i=0}^{\infty} \omega^i \beta^i E_t\{\hat{p}_{t+i}\} - \theta(1-\omega\beta) \sum_{i=0}^{\infty} \omega^i \beta^i \hat{p}_t$$

$$= \ln(\frac{C^{1-\sigma}}{1-\omega\beta}) + \ln MC + (1-\omega\beta) \sum_{i=0}^{\infty} \omega^i \beta^i [(1-\sigma)E_t\{\hat{c}_{t+i}\} + E_t\{\hat{mc}_{t+i}\} + \theta(E_t\{\hat{p}_{t+i}\} - \hat{p}_t)]$$

Sticky Price

- The left hand side:

$$\ln\left(\frac{C^{1-\sigma}}{1-\omega\beta}\right)+(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i[(1-\sigma)E_t\{\hat{c}_{t+i}\}+(\theta-1)(E_t\{\hat{p}_{t+i}\}-\hat{p}_t)]+\hat{q}_t$$

- The right hand side:

$$\ln\mu MC+\ln\left(\frac{C^{1-\sigma}}{1-\omega\beta}\right)+(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i[(1-\sigma)E_t\{\hat{c}_{t+i}\}+E_t\{\hat{mc}_{t+i}\}+\theta(E_t\{\hat{p}_{t+i}\}-\hat{p}_t)]$$

- Equating the two and using $\mu MC = 1$ result to

$$\hat{q}_t+\hat{p}_t=(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i(E_t\{\hat{mc}_{t+i}\}+E_t\{\hat{p}_{t+i}\})$$

Sticky Price

$$\hat{q}_t + \hat{p}_t = (1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i(E_t\{\hat{mc}_{t+i}\} + E_t\{\hat{p}_{t+i}\})$$

- The above can be written as

$$\hat{q}_t + \hat{p}_t = (1-\omega\beta)(\hat{mc}_t + \hat{p}_t) + \omega\beta(E_t\{\hat{q}_{t+1}\} + E_t\{\hat{p}_{t+1}\})$$

Why is it the case? Note that

$$E_t\{\hat{q}_{t+1}\} + E_t\{\hat{p}_{t+1}\} = (1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i(E_t\{\hat{mc}_{t+1+i}\} + E_t\{\hat{p}_{t+1+i}\})$$

Multiplying the both sides by $\omega\beta$

$$\omega\beta(E_t\{\hat{q}_{t+1}\} + E_t\{\hat{p}_{t+1}\}) = (1-\omega\beta)\sum_{i=1}^{\infty}\omega^i\beta^i(E_t\{\hat{mc}_{t+i}\} + E_t\{\hat{p}_{t+i}\})$$

which corresponds to $i \geq 1$ of $(1-\omega\beta)\sum_{i=0}^{\infty}\omega^i\beta^i(E_t\{\hat{mc}_{t+i}\} + E_t\{\hat{p}_{t+i}\})$

Sticky Price

- Rearranging $\hat{q}_t + \hat{p}_t = (1-\omega\beta)(\hat{mc}_t + \hat{p}_t) + \omega\beta(E_t\{\hat{q}_{t+1}\} + E_t\{\hat{p}_{t+1}\})$,

$$\hat{q}_t = (1-\omega\beta)\hat{mc}_t + \omega\beta(E_t\{\hat{q}_{t+1}\} + E_t\{\pi_{t+1}\})$$

  where $\pi_{t+1} = \hat{p}_{t+1} - \hat{p}_t$
- Putting $\hat{q}_t = (\frac{\omega}{1-\omega})\pi_t$ from the price index and $\hat{q}_t = (1-\omega\beta)\hat{mc}_t + \omega\beta(E_t\{\hat{q}_{t+1}\} + E_t\{\pi_{t+1}\})$ together,

$$(\frac{\omega}{1-\omega})\pi_t = (1-\omega\beta)\hat{mc}_t + \frac{\omega\beta}{1-\omega}E_t\{\pi_{t+1}\}$$

  - Multiplying the both sides by $\frac{1-\omega}{\omega}$,

$$\pi_t = \tilde{\kappa}\hat{mc}_t + \beta E_t\{\pi_{t+1}\}$$

    where $\tilde{\kappa} = \frac{(1-\omega)(1-\omega\beta)}{\omega}$

Sticky Price

$$\pi_t = \tilde{\kappa}\hat{mc}_t + \beta E_t\{\pi_{t+1}\}$$

where $\tilde{\kappa} = \frac{(1-\omega)(1-\omega\beta)}{\omega}$

- Let us manipulate the above further. Recall that $MC_t = \frac{W_t/P_t}{Z_t}$
- Log-linearizing this yields $\hat{mc}_t = \hat{w}_t - \hat{p}_t - \hat{z}_t$
  - $\hat{y}_t = \hat{z}_t + \hat{n}_t$ (try this on your own!)
  - Using ILC and the goods market equilibrium condition, $\hat{w}_t - \hat{p}_t = \eta\hat{n}_t + \sigma\hat{y}_t$
  - Using these two and the definition of $\hat{y}_t^f = (\frac{1+\eta}{\sigma+\eta})\hat{z}_t$,

$$\hat{mc}_t = \hat{w}_t - \hat{p}_t - \hat{z}_t = \eta\hat{n}_t + \sigma\hat{y}_t - \hat{z}_t$$

$$= \eta\hat{z}_t - \eta\hat{z}_t + \eta\hat{n}_t + \sigma\hat{y}_t - \hat{z}_t$$

$$= (\sigma+\eta)\hat{y}_t - (1+\eta)\hat{z}_t$$

$$= (\sigma+\eta)[\hat{y}_t - (\frac{1+\eta}{\sigma+\eta})\hat{z}_t] = \gamma(\hat{y}_t - \hat{y}_t^f)$$

  where $\gamma := \sigma + \eta$
- Letting $x_t = \hat{y}_t - \hat{y}_t^f$,

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa x_t$$

  where $\kappa := \gamma\tilde{\kappa}$

Aggregate Demand

- $C_t^{-\sigma} = \beta(1 + R_t)E_t\{(\frac{P_t}{P_{t+1}})C_{t+1}^{-\sigma}\}$ (EC)
- EC can be log-linearized as $\hat{y}_t = E_t\{\hat{y}_{t+1}\} - (\frac{1}{\sigma})(\hat{R}_t - E_t\{\pi_{t+1}\})$ since $\hat{c}_t = \hat{y}_t$
  - In terms of x_t, this becomes

    $$x_t = E_t\{x_{t+1}\} - (\frac{1}{\sigma})(\hat{R}_t - E_t\{\pi_{t+1}\}) + u_t$$

    where $u_t = E_t\{\hat{y}_{t+1}^f\} - \hat{y}_t^f$ depends only on $\hat{z}_t$ (thus stochastic) other than fixed parameters
    - u_t is often called a demand shock
  - Some authors use the notation

    $$x_t = E_t\{x_{t+1}\} - (\frac{1}{\sigma})(\hat{R}_t - E_t\{\pi_{t+1}\} - r_t^n)$$

    where $r_t^n = \sigma u_t = \sigma(E_t\{\hat{y}_{t+1}^f\} - \hat{y}_t^f)$ is called the natural rate of interest

Complete Model

- $x_t = E_t\{x_{t+1}\} - (\frac{1}{\sigma})(\hat{R}_t - E_t\{\pi_{t+1}\}) + u_t$ is referred to as the forward-looking IS curve
- $\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa x_t$ is called the New Keynesian Phillips curve
- These two are very reminiscent of the AD-AS framework, except that they are microfounded (and also forward-looking) whereas the AD-AS is terribly ad-hoc
- Let us close the model by specifying monetary policy
  - Let us use a simple interest rate rule of the form $\hat{R}_t = \delta\pi_t + v_t$ where v_t is a monetary policy shock
  - The case with $\delta > 1$ is called the Taylor principle which is also the determinacy condition (i.e., existence and uniqueness of stationary equilibrium). See Bullard and Mitra (2002) (more on this in the problem set)

State Space Representation

- To keep things simple, let us suppose $Z_t = E\{Z_t\} = 1\ \forall t$. Then, $u_t = 0$ $\forall t$ by its definition
- Also, let $v_t = \rho v_{t-1} + \varepsilon_t$, where ε_t follows an iid White noise process
- Then, the state space representation of the model takes the following form:

$$\begin{bmatrix} 1 & 0 & 0 \\ -\sigma^{-1} & 1 & \sigma^{-1} \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} v_t \\ E_t\{x_{t+1}\} \\ E_t\{\pi_{t+1}\} \end{bmatrix}$$

$$= \begin{bmatrix} \rho & 0 & 0 \\ 0 & 1 & \delta\sigma^{-1} \\ 0 & -\kappa & 1 \end{bmatrix} \begin{bmatrix} v_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \varepsilon_t$$

- Note that v_t is a predetermined variable and x_t and π_t are jump variables

Blanchard and Kahn (1980)

- The state space representation has the form $A_0 x_{t+1} = A_1 x_t + B_0 \epsilon_t$ where A_0, A_1, and B_0 are coefficient matrices, x_t is a column vector of predetermined and jump variables, and ϵ_t is a column vector of iid White noise processes
- Assuming A_0 is invertible, rewrite the above as $x_{t+1} = A x_t + B\epsilon_t$ where $A = A_0^{-1} A_1$ and $B = A_0^{-1} B_0$
  - We invert A_0 instead of A_1 to present a different way of doing the same thing
- Let w_t denote a column vector of predetermined variables (v_{t-1}) and y_t a column vector of jump variables (x_t and π_t)
  - Then, we can write the above as $\begin{bmatrix} w_{t+1} \\ E_t\{y_{t+1}\} \end{bmatrix} = A \begin{bmatrix} w_t \\ y_t \end{bmatrix} + B\epsilon_t$
- After the Jordan decomposition, $\begin{bmatrix} w_{t+1} \\ E_t\{y_{t+1}\} \end{bmatrix} = P\Lambda P^{-1} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + B\epsilon_t$
  - This can be rewritten as $P^{-1} \begin{bmatrix} w_{t+1} \\ E_t\{y_{t+1}\} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + R\epsilon_t$ where $R = P^{-1}B$

Blanchard and Kahn (1980)

- The Blanchard-Khan condition here is that the number of unstable eigenvalues (whose moduli are greater than one) must be equal to the number of jump variables in order to obtain the saddle path of the problem (why is the definition of stability different here?)
- P^{-1}, Λ (Λ_1 is stable and Λ_2 is not), and R can be partitioned so that unstable eigenvalues are associated with jump variables
  - That is, $\begin{bmatrix} P_{11}^* & P_{12}^* \\ P_{21}^* & P_{22}^* \end{bmatrix} \begin{bmatrix} w_{t+1} \\ E_t\{y_{t+1}\} \end{bmatrix}$ $= \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11}^* & P_{12}^* \\ P_{21}^* & P_{22}^* \end{bmatrix} \begin{bmatrix} w_t \\ y_t \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \epsilon_t$
- Let $\begin{bmatrix} \tilde{w}_t \\ \tilde{y}_t \end{bmatrix} = \begin{bmatrix} P_{11}^* & P_{12}^* \\ P_{21}^* & P_{22}^* \end{bmatrix} \begin{bmatrix} w_t \\ y_t \end{bmatrix}$
  - Then, the system becomes $E_t\left\{\begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix}\right\} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{w}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \epsilon_t$
- The above is decoupled as $E_t\{\tilde{w}_{t+1}\} = \Lambda_1 \tilde{w}_t + R_1 \epsilon_t$ and $E_t\{\tilde{y}_{t+1}\} = \Lambda_2 \tilde{y}_t + R_2 \epsilon_t$

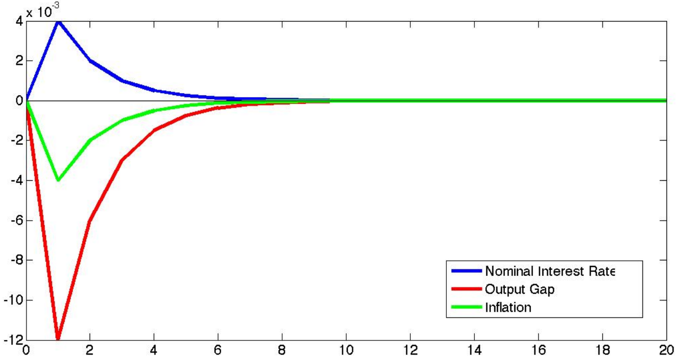
Blanchard and Kahn (1980)

- Solving $E_t\{\tilde{y}_{t+1}\} = \Lambda_2 \tilde{y}_t + R_2 \epsilon_t$ for $\tilde{y}_t$ forward,
  $\tilde{y}_t = -\Lambda_2^{-1} R_2 \epsilon_t + \lim_{j\to\infty} (\Lambda_2^{-1})^j E_t\{\tilde{y}_{t+j}\} = -\Lambda_2^{-1} R_2 \epsilon_t$
  - $\tilde{y}_t = P_{21}^* w_t + P_{22}^* y_t = -\Lambda_2^{-1} R_2 \epsilon_t$
    $\Rightarrow y_t = -P_{22}^{*-1} P_{21}^* w_t - P_{22}^{*-1} \Lambda_2^{-1} R_2 \epsilon_t$
- $\tilde{w}_t = P_{11}^* w_t + P_{12}^* y_t = P_{11}^* w_t + P_{12}^* (-P_{22}^{*-1} P_{21}^* w_t - P_{22}^{*-1} \Lambda_2^{-1} R_2 \epsilon_t)$
  $= (P_{11}^* - P_{12}^* P_{22}^{*-1} P_{21}^*) w_t - P_{12}^* P_{22}^{*-1} \Lambda_2^{-1} R_2 \epsilon_t$
  - $E_t\{\tilde{w}_{t+1}\} = \Lambda_1 \tilde{w}_t + R_1 \epsilon_t$ and $E_t\{w_{t+1}\} = w_{t+1}$
    $\Rightarrow w_{t+1} = (P_{11}^* - P_{12}^* P_{22}^{*-1} P_{21}^*)^{-1} \Lambda_1 (P_{11}^* - P_{12}^* P_{22}^{*-1} P_{21}^*) w_t +$
    $(P_{11}^* - P_{12}^* P_{22}^{*-1} P_{21}^*)^{-1} (R_1 - \Lambda_1 P_{12}^* P_{22}^{*-1} \Lambda_2^{-1} R_2) \epsilon_t$
    - Refer to the RBC slides for the reasoning behind this
- Again, we have linear solutions. These are computationally cheap to work with
- Let us try this on Matlab, R, and Julia

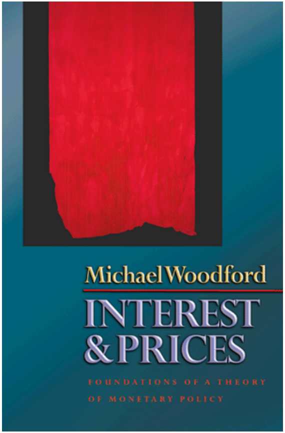
Solution to Simple New Keynesian Model

- We obtain the following solutions:
  - $v_t = .5v_{t-1} + \varepsilon_t$
  - $x_t = -.6v_{t-1} - 1.2\varepsilon_t$
  - $\pi_t = -.2v_{t-1} - .4\varepsilon_t$
- Using the solutions above, one can carry out simulation and compute various statistics of interest
- In monetary policy literature, impulse response functions are frequently used to analyze model dynamics
  - What they do is that they map out responses of model variables over time in terms of percentage deviations from their steady state values when one of the variables is hit by an one-time impulse or shock
  - For instance, we can discuss how an one-time monetary policy shock of size .01 changes the nominal interest rate, the output gap, and the inflation from their steady state values using the solutions above
- Let us do this on Matlab

Impulse Response Functions



Further Reading on New Keynesian Theory



The Bible of New Keynesian Theory!

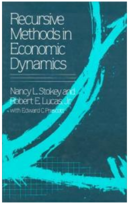
Other Recommended Readings

- Smets and Wouters (2003) present a medium-scale DSGE model which forms the basis of large-scale DSGE models of most central banks
  - The model features real frictions such as habit formation in consumption and adjustment costs in investment in addition to nominal price and wage rigidities
  - It is estimated using seven euro area macroeconomic time series (real GDP, consumption, investment, employment, real wages, inflation and the nominal short-term interest rate) using Bayesian methods and validation techniques (see Fernandez-Villaverde, 2009)
  - It performs as well as conventional empirical models such as Vector Autoregression (VAR) models in out-of-the-sample forecast

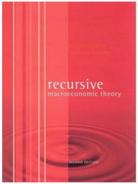
Other Recommended Readings

- Iacoviello (2005) and Iacoviello and Neri (2010) incorporate the financial accelerator of Kiyotaki and Moore (1997) into a New Keynesian model to demonstrate how housing market movements contribute to business cycles
  - Their model reflects the view that housing market is central to understanding business cycles (see Leamer (2007) who declares that "housing is the business cycle") and shows that housing market spillovers are considerably huge, mainly falling on consumption

Other Useful Reference Books

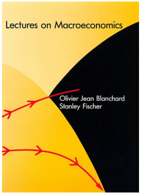


For a Rigorous Treatment of Dynamic Optimization Theory!

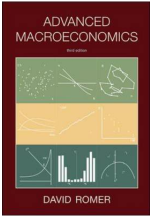


For Methods of Modern Macroeconomics!

Other Useful Reference Books



For Various Topics in Macroeconomic Theory!



For an Overview of Modern Macroeconomic Theory!