

# Graduate Macro Theory II:

## A New Keynesian Model with Price Stickiness

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### 1 Introduction

This set of notes lays out and analyzes the canonical New Keynesian (NK) model. The NK model takes a real business cycle model as its backbone and adds to that sticky prices, a form of nominal rigidity that allows purely nominal shocks to have real effects, and which alters the response of the economy to real shocks in a way that gives rise to a non-trivial role for active stabilization policy.

To get price-stickiness in the model, we have to have firms as price-setters, which means we need to move away from the perfectly competitive benchmark. To do so we introduce monopolistic competition in the way similar to before. We split production into two sectors, where the final goods sector is perfectly competitive and aggregates intermediates into a final good for consumption. This generates a downward-sloping demand for intermediates. There are a continuum of intermediate goods producers who can set their own prices, but take all other prices as given. All the action in the model is at the level of the intermediate producers. We assume that they are not freely able to adjust their prices each period. In particular, the Calvo (1983) assumption posits that each period firms face a fixed probability of being allowed to change their price. This seems a little ridiculous in terms of its realism, but this assumption facilitates aggregation, and this is why it is so popular. With any price rigidity, any firm's price becomes a state variable. With a continuum of intermediate goods firms, we'd have a continuum of state variables. The Calvo (1983) assumption allows us to aggregate out this heterogeneity. Even though it seems somewhat bizarre on its surface, it has some normative implications that seem pretty reasonable (in particular, price stability ends up being an important normative goal).

The basic New Keynesian model that I'll lay out below (and which is laid out in Woodford (2003) and Gali (2007) textbook treatments) has no investment or capital. This simplifies the analysis quite a bit and permits us to get better intuition. It is not a completely innocuous omission, and we'll later look at how the inclusion of capital in the model affects things.

## 2 Households

There is a representative household that consumes, supplies labor, accumulates bonds, holds shares in firms, and accumulates money. It gets utility from holding real balances. Its problem is:

$$\max_{C_t, N_t, B_{t+1}, M_t} E_0 \sum_{t=0}^{\infty} \beta^t \left( \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \ln \left( \frac{M_t}{P_t} \right) \right)$$

Here I have gone ahead and assumed that utility from real balances is logarithmic. As long as real balances are additively separable from consumption and labor, money in the utility function doesn't do much interesting here.<sup>1</sup> The nominal flow budget constraint is:

$$P_t C_t + B_{t+1} + M_t - M_{t-1} \leq W_t N_t + \Pi_t - P_t T_t + (1 + i_{t-1}) B_t$$

Here money is the numeraire, and  $P_t$  is the price of goods in terms of money.  $B_t$  is the stock of nominal bonds a household enters the period with, and they pay out (known as of  $t-1$ ) nominal interest rate  $i_{t-1}$ . The household also enters the period with a stock of money,  $M_{t-1}$ . Note that I'm not being super consistent with timing notation here:  $M_{t-1}$  and  $B_t$  are both known at  $t-1$ . The reason I write it this way is because the aggregate supply of money in period  $t$ ,  $M_t$ , is not going to be predetermined but rather set by a central bank.  $W_t$  is the nominal wage (denominated in units of money, not goods).  $\Pi_t$  denotes (nominal) profits remitted by firms, and  $T_t$  is a lump sum tax paid to a government (the government will have the role of setting the money supply and remitting any seignorage revenue back to the household lump sum).

A Lagrangian for the household is:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \ln \left( \frac{M_t}{P_t} \right) + \lambda_t (W_t N_t + \Pi_t - P_t T_t + (1 + i_{t-1}) B_t - P_t C_t - B_{t+1} - M_t + M_{t-1}) \right]$$

The FOC are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= 0 \Leftrightarrow C_t^{-\sigma} = \lambda_t P_t \\ \frac{\partial \mathcal{L}}{\partial N_t} &= 0 \Leftrightarrow \psi N_t^\eta = \lambda_t W_t \\ \frac{\partial \mathcal{L}}{\partial B_{t+1}} &= 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} (1 + i_t) \\ \frac{\partial \mathcal{L}}{\partial M_t} &= 0 \Leftrightarrow \theta \frac{1}{M_t} = \lambda_t - \beta E_t \lambda_{t+1} \end{aligned}$$

We can eliminate the multiplier and re-write these conditions as:

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<sup>1</sup>If we were to assume that central bank policy focuses on an interest rate rather than a monetary aggregate, as we will do below, then we could ignore money altogether so long as utility from money is separable. This is sometimes referred to as a "cashless" economy.

$$\psi N_t^\eta = C_t^{-\sigma} w_t \quad (1)$$

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma} (1 + i_t) \frac{P_t}{P_{t+1}} \quad (2)$$

$$\theta \left( \frac{M_t}{P_t} \right)^{-1} = \frac{i_t}{1 + i_t} C_t^{-\sigma} \quad (3)$$

### 3 Production

For the production side of things we split into two. There is a representative competitive final goods firm which aggregates intermediate inputs according to a CES technology. To the extent to which the intermediates are imperfect substitutes in the CES aggregator, this generates a downward-sloping demand for each intermediate variety, which gives these intermediate producers pricing power. There are a continuum (large number) of intermediates, so these producers behave as monopolistically competitive (they treat all prices but their own as given). These firms produce output using labor and are subject to an aggregate productivity shock. They are not freely able to adjust prices each period, in a way that we will discuss in more depth below.

#### 3.1 Final Good Producer

The final output good is a CES aggregate of a continuum of intermediates:

$$Y_t = \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \quad (4)$$

Here  $\epsilon > 1$ . The profit maximization problem of the final good firm is:

$$\max_{Y_t(j)} P_t \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

The FOC for a typical variety of intermediate  $j$  is:

$$P_t \frac{\epsilon}{\epsilon-1} \left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}-1} \frac{\epsilon-1}{\epsilon} Y_t(j)^{\frac{\epsilon-1}{\epsilon}-1} = P_t(j)$$

This can be written:

$$\left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} = \frac{P_t(j)}{P_t}$$

Or:

$$\left( \int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon}$$

Making note of the definition of the aggregate final good, we have::

$$Y_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \quad (5)$$

This says that the relative demand for the  $j^{th}$  intermediate is a function of its relative price, with  $\epsilon$  the price elasticity of demand, and is proportional to aggregate output,  $Y_t$ .

To derive a price index, define nominal output as the sum of prices times quantities:

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) dj$$

Plugging in the demand for each variety, we have:

$$P_t Y_t = \int_0^1 P_t(j)^{1-\epsilon} P_t^\epsilon Y_t dj$$

Pulling out of the integral things which don't depend on  $j$ :

$$P_t Y_t = P_t^\epsilon Y_t \int_0^1 P_t(j)^{1-\epsilon} dj$$

Simplifying, we get an expression for the aggregate price level:

$$P_t = \left( \int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}} \quad (6)$$

### 3.2 Intermediate Producers

A typical intermediate producer produces output according to a constant returns to scale technology in labor, with a common productivity shock,  $A_t$ :

$$Y_t(j) = A_t N_t(j) \quad (7)$$

Intermediate producers face a common wage. They are not freely able to adjust price so as to maximize profit each period, but will always act to minimize cost. The cost minimization problem is to minimize total cost subject to the constraint of producing enough to meet demand:

$$\min_{N_t(j)} W_t N_t(j)$$

s.t.

$$A_t N_t(j) \geq \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t$$

The multiplier on the constraint here will have the interpretation as marginal cost – how much will costs change if you are forced to produce an extra unit of output. A Lagrangian is:

$$\mathcal{L} = -W_t N_t(j) + \varphi_t(j) \left( A_t N_t(j) - \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \right)$$

The FOC is:

$$\frac{\partial \mathcal{L}}{\partial N_t(j)} = 0 \Leftrightarrow W_t = \varphi_t(j) A_t$$

Or:

$$\varphi_t = \frac{W_t}{A_t} \quad (8)$$

Here I have dropped the  $j$  reference: marginal cost ( $\varphi_t$ ) is equal to the wage divided by productivity, both of which are common to all intermediate goods firms.

Real flow profit for intermediate producer  $j$  is:

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j)$$

From (8), we know  $W_t = \varphi_t A_t$ . Plugging this into the expression for profits, we get:

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j)$$

Where I have defined  $mc_t \equiv \frac{\varphi_t}{P_t}$  as real marginal cost.

Firms are not freely able to adjust price each period. In particular, each period there is a fixed probability of  $1 - \phi$  that a firm can adjust its price. This means that the probability a firm will be stuck with a price one period is  $\phi$ , for two periods is  $\phi^2$ , and so on. Consider the pricing problem of a firm given the opportunity to adjust its price in a given period. Since there is a chance that the firm will get stuck with its price for multiple periods, the pricing problem becomes dynamic. Firms will discount profits  $s$  periods into the future by  $\widetilde{M}_{t+s} \phi^s$ , where  $\widetilde{M}_{t+s} = \beta^s \frac{u'(C_{t+s})}{u'(C_t)}$  is the stochastic discount factor. Note that discounting is by both the usual stochastic discount factor as well as by the probability that a price chosen in period  $t$  will still be in use in period  $t + s$ . The dynamic problem of an updating firm can be written:

$$\max_{P_t(j)} E_t \sum_{s=0}^{\infty} (\beta \phi)^s \frac{u'(C_{t+s})}{u'(C_t)} \left( \frac{P_t(j)}{P_{t+s}} \left( \frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left( \frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right)$$

Here I have imposed that output will equal demand. Multiplying out, we get:

$$\max_{P_t(j)} E_t \sum_{s=0}^{\infty} (\beta \phi)^s \frac{u'(C_{t+s})}{u'(C_t)} (P_t(j)^{1-\epsilon} P_{t+s}^{\epsilon-1} Y_{t+s} - mc_{t+s} P_t(j)^{-\epsilon} P_{t+s}^{\epsilon} Y_{t+s})$$

The first order condition can be written:

$$(1 - \epsilon)P_t(j)^{-\epsilon}E_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s})P_{t+s}^{\epsilon-1}Y_{t+s} + \epsilon P_t(j)^{-\epsilon-1}E_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s})mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s} = 0$$

Simplifying:

$$P_t(j) = \frac{\epsilon}{\epsilon - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s})mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}}{E_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s})P_{t+s}^{\epsilon-1}Y_{t+s}}$$

First, note that since nothing on the right hand side depends on  $j$ , all updating firms will update to the same reset price, call it  $P_t^{\#}$ . We can write the expression more compactly as:

$$P_t^{\#} = \frac{\epsilon}{\epsilon - 1} \frac{X_{1,t}}{X_{2,t}} \quad (9)$$

Here:

$$X_{1,t} = u'(C_t)mc_t P_t^{\epsilon} Y_t + \phi \beta E_t X_{1,t+1} \quad (10)$$

$$X_{2,t} = u'(C_t)P_t^{\epsilon-1}Y_t + \phi \beta E_t X_{2,t+1} \quad (11)$$

If  $\phi = 0$ , then the right hand side would reduce to  $mc_t P_t = \varphi_t$ . In this case, the optimal price would be a fixed markup,  $\frac{\epsilon}{\epsilon-1}$ , over nominal marginal cost,  $\varphi_t$ .

## 4 Equilibrium and Aggregation

To close the model we need to specify an exogenous process for  $A_t$ , some kind of monetary policy rule to determine  $M_t$ , and a fiscal rule to determine  $T_t$ . Let the aggregate productivity term follow a mean zero AR(1) in the log:

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \quad (12)$$

Let's suppose that the money supply follows an AR(1) in the growth rate, where  $\Delta \ln M_t = \ln M_t - \ln M_{t-1}$ :

$$\Delta \ln M_t = (1 - \rho_m)\pi + \rho_m \Delta \ln M_{t-1} + \varepsilon_{m,t} \quad (13)$$

I have written this process where the mean growth rate of money is equal to  $\pi$ , which will be the steady state level of inflation (this is because real balances will be stationary, so  $M_t$  and  $P_t$  must grow at the same rate in the steady state).  $\varepsilon_{m,t}$  is a monetary shock.

Since the government prints money, it effectively earns some revenue. Right now we're abstracting from the government doing any spending, and for simplicity I'm going to assume that the government does not operate in bond markets (this is innocuous since it raises revenue only through a lump sum tax). The government's budget constraint in nominal terms is then:

$$0 \leq P_t T_t + M_t - M_{t-1}$$

In other words, the change in the stock of money,  $M_t - M_{t-1}$ , is (nominal) revenue for the government. Since it does no spending, at equality lump sum taxes must satisfy:

$$T_t = -\frac{M_t - M_{t-1}}{P_t}$$

So if money growth is positive, e.g.  $M_t > M_{t-1}$ , then lump sum taxes will be negative – the government will be rebating its seignorage revenue to the household lump sum.

In equilibrium, bond-holding is always zero in all periods:  $B_t = 0$ . Using this, plus the relationship between the lump sum tax and money growth derived above, the household budget constraint can be written in real terms:

$$C_t = w_t N_t + \frac{\Pi_t}{P_t}$$

Real dividends received by the household are just the sum of real profits from intermediate goods firms (the final good firm is competitive and earns no profit):

$$\frac{\Pi_t}{P_t} = \int_0^1 \left( \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) \right) dj$$

This can be written:

$$\frac{\Pi_t}{P_t} = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) - w_t \int_0^1 N_t(j) dj$$

Now, market-clearing requires that the sum of labor used by firms equals the total labor supplied by households, so  $\int_0^1 N_t(j) dj = N_t$ . Hence:

$$\frac{\Pi_t}{P_t} = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Throwing this into the household budget constraint, the  $w_t N_t$  terms drop out, leaving:

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj$$

Plug in the demand function for  $Y_t(j)$ :

$$C_t = \int_0^1 P_t(j)^{1-\epsilon} P_t^{\epsilon-1} Y_t dj$$

Bring stuff out of the integral:

$$C_t = P_t^{\epsilon-1} Y_t \int_0^1 P_t(j)^{1-\epsilon} dj$$

Now, since  $P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$ , the terms involving  $P$ s drop out, leaving:

$$C_t = Y_t \tag{14}$$

Now, what is  $Y_t$ ? From the demand for intermediate variety  $j$ , we have:

$$Y_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t$$

Using the production function for each intermediate, this is:

$$A_t N_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t$$

Integrate over  $j$ :

$$\int_0^1 A_t N_t(j) dj = \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj$$

Take stuff out of the integral, with the exception of the price level on the right hand side:

$$A_t \int_0^1 N_t(j) dj = Y_t \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} dj$$

Now define a new variable,  $v_t^p$ , as:

$$v_t^p = \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} dj \tag{15}$$

This is a measure of price dispersion. If there were no pricing frictions, all firms would charge the same price, and  $v_t^p = 1$ . If prices are different, one can show that this expression is bound from below by unity. Using the definition of aggregate labor input, we can therefore write:

$$Y_t = \frac{A_t N_t}{v_t^p} \tag{16}$$

This is the aggregate production function. Since  $v_t^p \geq 1$ , price dispersion results in an output loss – you produce less output than you would given  $A_t$  and aggregate labor input if prices are disperse. This ends up being the gist of why price stability is a good goal.

The full set of equilibrium conditions can then be written:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma} (1 + i_t) \frac{P_t}{P_{t+1}} \tag{17}$$

$$\psi N_t^\eta = C_t^{-\sigma} w_t \tag{18}$$



$$\frac{M_t}{P_t} = \theta \frac{1+i_t}{i_t} C_t^\sigma \quad (19)$$

$$mc_t = \frac{w_t}{A_t} \quad (20)$$

$$C_t = Y_t \quad (21)$$

$$Y_t = \frac{A_t N_t}{v_t^p} \quad (22)$$

$$v_t^p = \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon_p} dj \quad (23)$$

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj \quad (24)$$

$$P_t^\# = \frac{\epsilon_p}{\epsilon_p - 1} \frac{X_{1,t}}{X_{2,t}} \quad (25)$$

$$X_{1,t} = C_t^{-\sigma} mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta E_t X_{1,t+1} \quad (26)$$

$$X_{2,t} = C_t^{-\sigma} P_t^{\epsilon_p-1} Y_t + \phi_p \beta E_t X_{2,t+1} \quad (27)$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \quad (28)$$

$$\Delta \ln M_t = (1 - \rho_m) \pi + \rho_m \Delta \ln M_{t-1} + \varepsilon_{m,t} \quad (29)$$

$$\Delta \ln M_t = \ln M_t - \ln M_{t-1} \quad (30)$$

This is 14 equations in 14 aggregate variables ( $C_t, i_t, P_t, N_t, w_t, M_t, mc_t, A_t, Y_t, v_t^p, P_t^\#, X_{1,t}, X_{2,t}, \Delta \ln M_t$ ). There are two stochastic shocks – the usual productivity shock as well as the monetary shock.

#### 4.1 Re-writing the equilibrium conditions

There are a couple of issues with how I've written these conditions. First, I haven't gotten rid of the heterogeneity – I still have  $j$  indexes showing up. Second, I have the price level showing up, which, as I mentioned above, may not be stationary. Third, I have the nominal money supply showing up, which is not stationary the way I've written the process in terms of money growth. Hence, I want to re-write these conditions (i) only in terms of inflation, eliminating the price level; and (ii) getting rid of the heterogeneity, which the Calvo (1983) assumption allows me to do; and (iii) in terms of real money balances,  $m_t \equiv \frac{M_t}{P_t}$ , instead of nominal money balances.

Define inflation as  $\pi_t = \frac{P_t}{P_{t-1}} - 1$ . The Euler equation can be re-written:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma} (1+i_t)(1+\pi_{t+1})^{-1} \quad (31)$$

The demand for money equation is already written in terms of real balances:

$$m_t = \theta \frac{1+i_t}{i_t} C_t^\sigma \quad (32)$$

Now, we need to get rid of the heterogeneity in the expression for the price level and price dispersion. The expression for the price level is:

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj$$

Now, a fraction  $(1 - \phi)$  of these firms will update their price to the same reset price,  $P_t^\#$ . The other fraction  $\phi$  will charge the price they charged in the previous period. Since it doesn't matter how we "order" these firms along the unit interval, this means we can break up the integral on the right hand side as:

$$P_t^{1-\epsilon} = \int_0^{1-\phi} P_t^{\#,1-\epsilon} dj + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon} dj$$

This can be written:

$$P_t^{1-\epsilon} = (1 - \phi)P_t^{\#,1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon} dj$$

Now, here's the beauty of the Calvo assumption. Because the firms who get to update are randomly chosen, and because there are a large number (continuum) of firms, the integral (sum) of individual prices over some subset of the unit interval will simply be proportional to the integral over the entire unit interval, where the proportion is equal to the subset of the unit interval over which the integral is taken. This means:

$$\int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon} dj = \phi \int_0^1 P_{t-1}(j)^{1-\epsilon} dj = \phi P_{t-1}^{1-\epsilon}$$

This means that the aggregate price level (raised to  $1 - \epsilon$ ) is a convex combination of the reset price and lagged price level (raised to the same power). So:

$$P_t^{1-\epsilon} = (1 - \phi)P_t^{\#,1-\epsilon} + \phi P_{t-1}^{1-\epsilon}$$

In other words, we've gotten rid of the heterogeneity. The Calvo assumption allows us to integrate out the heterogeneity and not worry about keeping track of what each firm is doing from the perspective of looking at the behavior of aggregates. Now, we still have the issue here that we are written in terms of the price level, not inflation. To get it in terms of inflation, divide both sides by  $P_{t-1}^{1-\epsilon}$ , and define  $\pi_t^\# = \frac{P_t^\#}{P_{t-1}} - 1$  as reset price inflation:

$$(1 + \pi_t)^{1-\epsilon} = (1 - \phi)(1 + \pi_t^\#)^{1-\epsilon} + \phi \tag{33}$$

We can also use the Calvo assumption to break up the price dispersion term, by again noting that  $(1 - \phi)$  of firms will update to the same price, and  $\phi$  firms will be stuck with last period's price. Hence:

$$v_t^p = \int_0^{1-\phi} \left( \frac{P_t^\#}{P_t} \right)^{-\epsilon} dj + \int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_t} \right)^{-\epsilon} dj$$

This can be written in terms of inflation by multiplying and dividing by powers of  $P_{t-1}$  where necessary:

$$v_t^p = \int_0^{1-\phi} \left( \frac{P_t^\#}{P_{t-1}} \right)^{-\epsilon} \left( \frac{P_{t-1}}{P_t} \right)^{-\epsilon} dj + \int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} \left( \frac{P_{t-1}}{P_t} \right)^{-\epsilon} dj$$

We can take stuff out of the integral:

$$v_t^p = (1-\phi)(1+\pi_t^\#)^{-\epsilon}(1+\pi_t)^\epsilon + (1+\pi_t)^\epsilon \int_{1-\phi}^1 \left( \frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon} dj$$

By the same Calvo logic, the term inside the integral is just going to be proportional to  $v_{t-1}^p$ . This means we can write the price dispersion term as:

$$v_t^p = (1-\phi)(1+\pi_t^\#)^{-\epsilon}(1+\pi_t)^\epsilon + (1+\pi_t)^\epsilon \phi v_{t-1}^p \quad (34)$$

In other words, we just have to keep track of  $v_t^p$ , not the individual prices.

Now, we need to adjust the reset price expression. First, define two new auxiliary variables as follows:

$$x_{1,t} \equiv \frac{X_{1,t}}{P_t^\epsilon}$$

$$x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon-1}}$$

Dividing both sides of the reset price expressions by the appropriate power of  $P_t$ , we have:

$$x_{1,t} = C_t^{-\sigma} m c_t Y_t + \phi \beta E_t \frac{X_{1,t+1}}{P_t^\epsilon}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta E_t \frac{X_{2,t+1}}{P_t^{\epsilon-1}}$$

Multiplying and dividing the  $t+1$  terms by the appropriate power of  $P_{t+1}$ , we have:

$$x_{1,t} = C_t^{-\sigma} m c_t Y_t + \phi \beta E_t \frac{X_{1,t+1}}{P_{t+1}^\epsilon} \left( \frac{P_{t+1}}{P_t} \right)^\epsilon$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta E_t \frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}} \left( \frac{P_{t+1}}{P_t} \right)^{\epsilon-1}$$

Or, in terms of inflation:

$$x_{1,t} = C_t^{-\sigma} m c_t Y_t + \phi \beta E_t (1 + \pi_{t+1})^\epsilon x_{1,t+1} \quad (35)$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta E_t (1 + \pi_{t+1})^{\epsilon-1} x_{2,t+1} \quad (36)$$

Now, in terms of the reset price expression, since we divided  $X_{1,t}$  by  $P_t^\epsilon$  and divided  $X_{2,t}$  by  $P_t^{\epsilon-1}$ . This means that  $\frac{X_{1,t}}{X_{2,t}} = P_t \frac{x_{1,t}}{x_{2,t}}$ . The reset price expression can now be written:

$$P_t^\# = \frac{\epsilon}{\epsilon - 1} P_t \frac{x_{1,t}}{x_{2,t}}$$

Now, simply divide both sides by  $P_{t-1}$  to have everything in terms of inflation rates:

$$1 + \pi_t^\# = \frac{\epsilon}{\epsilon - 1} (1 + \pi_t) \frac{x_{1,t}}{x_{2,t}} \quad (37)$$

Now, we need to re-write the processes involving money in terms of real balances. We can define:

$$\Delta \ln m_t \equiv \ln m_t - \ln m_{t-1}$$

This is of course equal to:

$$\Delta \ln m_t = \ln m_t - \ln m_{t-1} = \ln M_t - \ln P_t - \ln M_{t-1} + \ln P_{t-1} = \ln M_t - \ln M_{t-1} - \pi_t$$

Hence:

$$\Delta \ln M_t = \Delta \ln m_t + \pi_t$$

This means we can write the process for money growth in terms of real balance growth as:

$$\Delta \ln m_t = (1 - \rho_m) \pi - \pi_t + \rho_m \Delta \ln m_{t-1} + \rho_m \pi_{t-1} + \varepsilon_{m,t} \quad (38)$$

This means the re-written full set of equilibrium conditions is:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma} (1 + i_t) (1 + \pi_{t+1})^{-1} \quad (39)$$

$$\psi N_t^\eta = C_t^{-\sigma} w_t \quad (40)$$

$$m_t = \theta \frac{1 + i_t}{i_t} C_t^\sigma \quad (41)$$

$$m c_t = \frac{w_t}{A_t} \quad (42)$$

$$C_t = Y_t \quad (43)$$

$$Y_t = \frac{A_t N_t}{v_t^p} \quad (44)$$

$$v_t^p = (1 - \phi)(1 + \pi_t^\#)^{-\epsilon}(1 + \pi_t)^\epsilon + (1 + \pi_t)^\epsilon \phi v_{t-1}^p \quad (45)$$

$$(1 + \pi_t)^{1-\epsilon} = (1 - \phi)(1 + \pi_t^\#)^{1-\epsilon} + \phi \quad (46)$$

$$1 + \pi_t^\# = \frac{\epsilon}{\epsilon - 1}(1 + \pi_t) \frac{x_{1,t}}{x_{2,t}} \quad (47)$$

$$x_{1,t} = C_t^{-\sigma} m c_t Y_t + \phi \beta E_t (1 + \pi_{t+1})^\epsilon x_{1,t+1} \quad (48)$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta E_t (1 + \pi_{t+1})^{\epsilon-1} x_{2,t+1} \quad (49)$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \quad (50)$$

$$\Delta \ln m_t = (1 - \rho_m)\pi - \pi_t + \rho_m \Delta \ln m_{t-1} + \rho_m \pi_{t-1} + \varepsilon_{m,t} \quad (51)$$

$$\Delta \ln m_t = \ln m_t - \ln m_{t-1} \quad (52)$$

This is the same set of equations as above, but I have replaced  $P_t$  with  $\pi_t$ ,  $M_t$  with  $m_t$ ,  $P_t^\#$  with  $\pi_t^\#$ , and  $X_{1,t}$  and  $X_{2,t}$  with  $x_{1,t}$  and  $x_{2,t}$ .

## 5 The Steady State

Let's solve for the non-stochastic steady state of the model. I'm going to use variables without a subscript to denote non-stochastic steady state values.

Steady state  $A = 1$ . Since output and consumption are always equal, it must also be that  $Y = C$ . Steady state inflation is equal to the exogenous target,  $\pi$ . From the re-written AR(1) in growth rates for real balances, in steady state we have:

$$\begin{aligned} \Delta \ln m &= (1 - \rho_m)\pi - (1 - \rho_m)\pi + \rho_m \Delta \ln m \\ (1 - \rho_m)\Delta \ln m &= 0 \\ \Delta \ln m &= 0 \end{aligned}$$

This means that real money balances are stationary in the steady state.

From the Euler equation, we have:

$$1 + i = \frac{1}{\beta}(1 + \pi)$$

In approximate terms, this would say  $i \approx \rho + \pi$ , where  $\beta = \frac{1}{1+\rho}$ , so  $\rho$  has the interpretation as the discount *rate* (whereas  $\beta$  is a discount factor). From the price evolution equation, we can derive the steady state expression for reset price inflation:

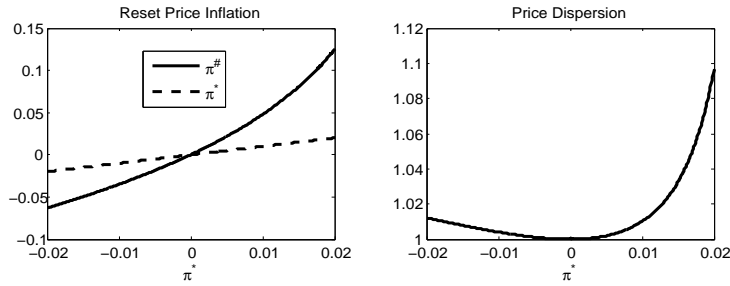
$$1 + \pi^\# = \left( \frac{(1 + \pi)^{1-\epsilon} - \phi}{1 - \phi} \right)^{\frac{1}{1-\epsilon}}$$

If  $\pi = 0$ , then  $\pi^\# = \pi$  because the right hand side is just 1. If  $\pi > 0$ , then  $\pi^\# > \pi$ , and if

$\pi < 0$ , then  $\pi^\# < \pi$ . Given this, we can solve for steady state price dispersion:

$$(1 - (1 + \pi)^\epsilon \phi) v^p = (1 - \phi) \left( \frac{1 + \pi}{1 + \pi^\#} \right)^\epsilon$$

If  $\pi = 0$ , then we have  $v^p = 1$ . If  $\pi \neq 0$ , then  $v^p > 1$ . In the figure below, I plot in the left panel steady state reset price inflation,  $\pi^\#$ , as a function of steady state inflation,  $\pi^*$ . For comparison I plot steady state inflation against itself as well (dashed line). We can see that steady state reset price inflation is less than steady state inflation for negative steady state inflation and greater than steady state inflation for positive steady state inflation. In the right panel I plot steady state price dispersion as a function of trend inflation. This bottoms out at one, but is greater than one for steady state inflation not equal to zero (though price dispersion increases faster as steady state inflation moves away from zero on the positive end than on the negative end). I computed this graph using  $\epsilon = 10$  and  $\phi = 0.75$ .



Given all this, we can solve for the steady state ratio of  $x_1/x_2$  as:

$$\frac{x_1}{x_2} = \frac{1 + \pi^\#}{1 + \pi} \frac{\epsilon - 1}{\epsilon}$$

Now, we also know that:

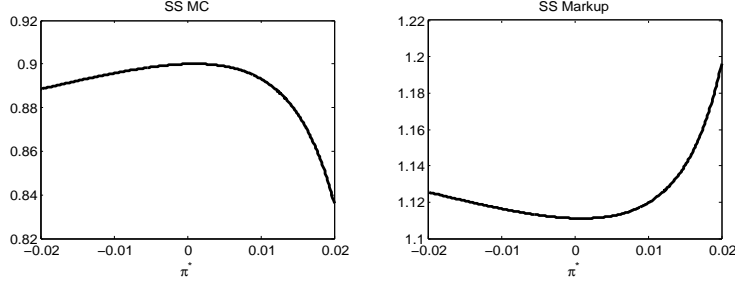
$$\frac{x_1}{x_2} = mc \frac{1 - \phi\beta(1 + \pi)^{\epsilon-1}}{1 - \phi\beta(1 + \pi)^\epsilon}$$

This means we can solve for steady state marginal cost as:

$$mc = \frac{1 - \phi\beta(1 + \pi)^\epsilon}{1 - \phi\beta(1 + \pi)^{\epsilon-1}} \frac{1 + \pi^\#}{1 + \pi} \frac{\epsilon - 1}{\epsilon} \quad (53)$$

Real marginal cost is equal to the inverse price markup. If  $\pi = 0$ , this is just equal to  $\frac{\epsilon-1}{\epsilon}$ . In other words, if steady state inflation is zero, then the steady state markup will be what it would be if prices were flexible. If  $\pi \neq 0$ , then  $mc < \frac{\epsilon-1}{\epsilon}$ , which means that the steady state markup will be higher than it would if inflation were zero. The figure below plots the steady state real marginal cost and steady state price markup as a function of steady state inflation. Here I used a value of

$\beta = 0.99$  along with the values of  $\epsilon$  and  $\phi$  used above.



Once we know steady state marginal cost, then we know the steady state real wage:  $w = mc$ . The lower is marginal cost, the bigger is the wedge between the wage and the marginal product of labor (i.e. the more distorted the economy is). Take this to the static labor supply condition, imposing the equality between  $Y = C$ :

$$\psi N^\eta = Y^{-\sigma} mc$$

Here I have imposed that  $A = 1$ . Now, from the production function, we know that  $Y = N/v^p$ . Plugging this in and simplifying, we can solve for  $N$ :

$$\begin{aligned} \psi N^\eta &= N^{-\sigma} (v^p)^\sigma mc \\ N &= \left( \frac{1}{\psi} (v^p)^\sigma mc \right)^{\frac{1}{\eta+\sigma}} \end{aligned}$$

Given this, we now have  $Y$  we can solve for steady state  $m$ :

$$m = \theta \frac{1+i}{i} Y^\sigma$$

## 6 The Flexible Price Equilibrium

A useful concept that will come in handy, particularly when thinking about welfare, is a hypothetical equilibrium allocation in which prices are flexible, which corresponds to the case when  $\phi = 0$ . Because there is no endogenous state variable in this model when prices are flexible, we can actually solve for the flexible price equilibrium by hand. I use superscript  $f$  to denote the hypothetical flexible price allocation.

When  $\phi = 0$ , we have  $\pi^\# = \pi$  regardless of what  $\pi$  is. Then going to the price dispersion expression, when  $\phi = 0$  we have:

$$v_t^{f,p} = \left( \frac{1 + \pi^\#}{1 + \pi} \right)^{-\epsilon} = 1$$

In other words, if prices are flexible, all firms charge the same prices, and price dispersion is at its lower bound of 1. By combining the reset price inflation term with the auxiliary variables  $x_{1,t}$  and  $x_{2,t}$ , we get that  $mc_t^f = \frac{\epsilon-1}{\epsilon}$ , and is therefore constant. Since marginal cost is the inverse price markup, this just says that if prices are flexible, firms will set price equal to a fixed markup over marginal cost (which we've already seen before in a flexible price case with monopolistic competition). This means that  $w_t^f = \frac{\epsilon-1}{\epsilon} A_t$ . Plugging this into the static labor FOC (along with the market-clearing condition that  $Y_t = C_t$ ), we see:

$$\psi \left( N_t^f \right)^\eta = \left( Y_t^f \right)^{-\sigma} \frac{\epsilon-1}{\epsilon} A_t$$

Using the fact that  $Y_t^f = A_t N_t^f$ , we have:

$$\begin{aligned} \psi \left( N_t^f \right)^\eta &= A_t^{-\sigma} \left( N_t^f \right)^{-\sigma} \frac{\epsilon-1}{\epsilon} A_t \\ N_t^f &= \left( \frac{1}{\psi} \frac{\epsilon-1}{\epsilon} A_t^{1-\sigma} \right)^{\frac{1}{\sigma+\eta}} \end{aligned}$$

This means that flexible price output is:

$$Y_t^f = \left( \frac{1}{\psi} \frac{\epsilon-1}{\epsilon} \right)^{\frac{1}{\sigma+\eta}} A_t^{\frac{1+\eta}{\sigma+\eta}} \quad (54)$$

There is something interesting here which is worth mentioning. If  $\sigma = 1$ , then  $N_t^f$  is a constant and not a function of  $A_t$ . In other words, if prices were flexible and  $\sigma = 1$  (log utility over consumption), labor hours would not react to fluctuations in  $A_t$ . What is driving this is that, if  $\sigma = 1$ , then preferences are consistent with King, Plosser, Rebelo (1988) preferences, in which the income and substitution effects of changes in  $A_t$  exactly offset. When there is capital in the model, this offset only occurs in the long run, so that labor hours are constant in the long run, but not in the short run as capital adjusts to steady state. Without capital, the cancellation of income and substitution effects holds at all times.

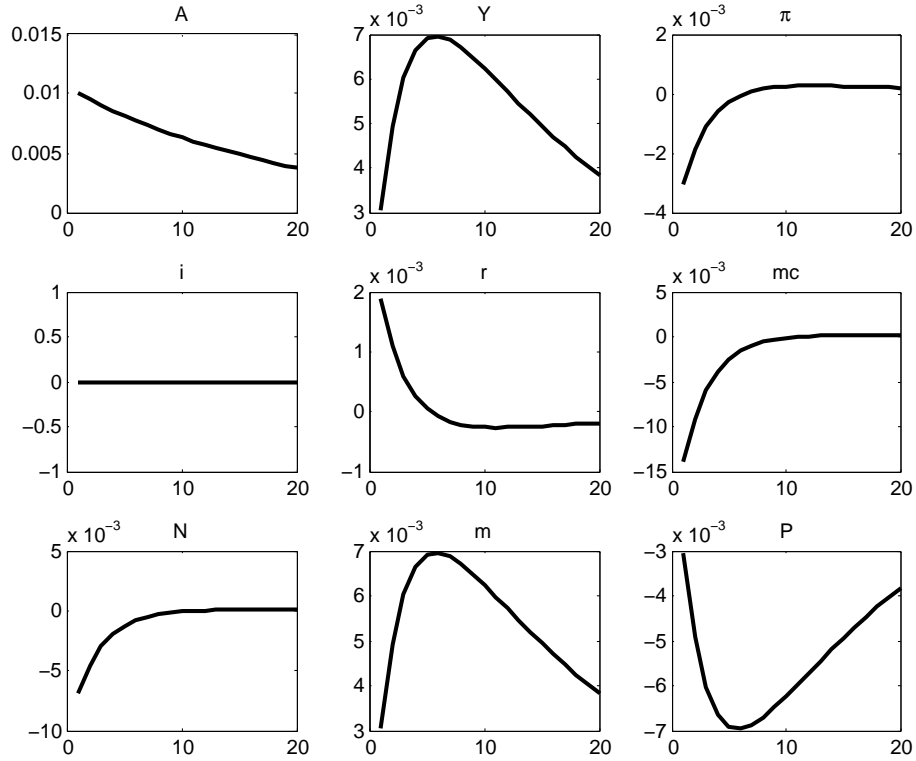
Note also that flexible price output does not depend on anything nominal. This is because, with flexible prices, nominal shocks have no real effects.

## 7 Quantitative Analysis

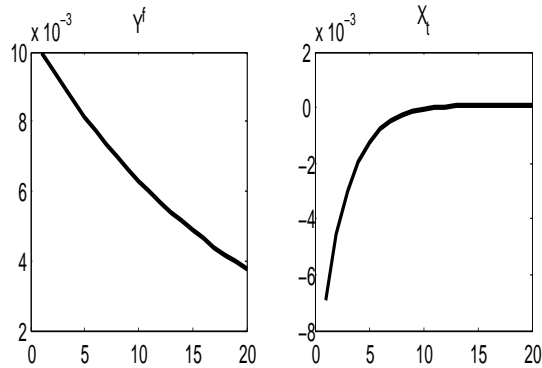
I solve the model quantitatively in Dynare using a first order approximation about the steady state. I use the following parameter values (more on this later):  $\phi = 0.75$ ,  $\sigma = 1$ ,  $\eta = 1$ ,  $\psi = 1$ ,  $\epsilon = 10$ ,  $\theta = 1$ ,  $\rho_a = 0.95$ ,  $\rho_m = 0.0$ , and  $\pi = 0$ . I assume that the standard deviation of both shocks are 0.01.

Impulse responses to the productivity shock are shown below.





There are a couple of interesting things to point out here. Output responds very little on impact, and significantly less than the increase in  $A_t$ . Indeed, we actually see a fairly large *decline* in  $N_t$  when  $A_t$  goes up. Inflation falls. The response of the price level (which I compute by cumulating the response of inflation) is roughly the mirror image of the output response. The nominal interest rate does not move at all at any horizon, though the real interest rate increases. Real marginal cost falls, which suggests that the real wage rises by less than  $A_t$  (effectively, firms charge bigger markups).



Above I plot the impulse response of the flexible price level of output and a new variable I

call the “output gap,” defined as  $\ln X_t = \ln Y_t - \ln Y_t^f$ . Because output responds significantly less than the flexible price level of output to the productivity shock, we see a large negative output gap opening up following the positive productivity shock.

What’s going on here? If  $\phi = 0$ , we see that output would respond significantly more to the productivity shock than in the baseline case I used where  $\phi = 0.75$ . What is going on here? When prices are sticky, output becomes (partially) “demand-determined,” and with exogenous money supply the way I have it here, price rigidity prevents demand from rising sufficiently when “supply” increases, so output rises by “too little” relative to what would happen with flexible prices. An easy way to see this is to look at the money demand relationship. In logs, we have:

$$\ln m_t = \ln \theta + \ln(1 + i_t) - \ln i_t + \sigma \ln Y_t$$

To the extent to which the nominal interest rate doesn’t move (which it in fact doesn’t here)<sup>2</sup>, the movement in output must be proportional to the movement in real balances. Since I’ve assumed that  $M_t$  is set exogenously, the only way  $m_t$  can move is through changes in  $P_t$ . Hence, as we can see in the IRFs, the output movement ends up just being the mirror image of the movement in  $P_t$ . And since prices are sticky,  $P_t$  can’t move enough relative to what it would do under price flexibility. Hence,  $m_t$  fails to increase sufficiently, and  $Y_t$  can’t rise as much as it would if prices were flexible.

There is another way to see how price rigidity effectively limits the demand increase, resulting in a response of output that is too small relative to what would happen in the absence of price rigidity. If prices were flexible, in the period of the shock,  $P_t$  would immediately fall (so  $m_t$  could rise), but would then start to rise. This means that expected inflation would actually rise. Given a fixed nominal interest rate (via the logic above), this means that the real interest rate would fall if prices were flexible.<sup>3</sup> With price stickiness, in contrast, inflation falls, and stays persistently low (basically, waves of firms come each period and cut their prices, so inflation stays low for a while). This means that expected inflation falls, not rises as it would if prices were flexible. This means that the real interest rate rises when  $A_t$  increases, which works to choke off demand.

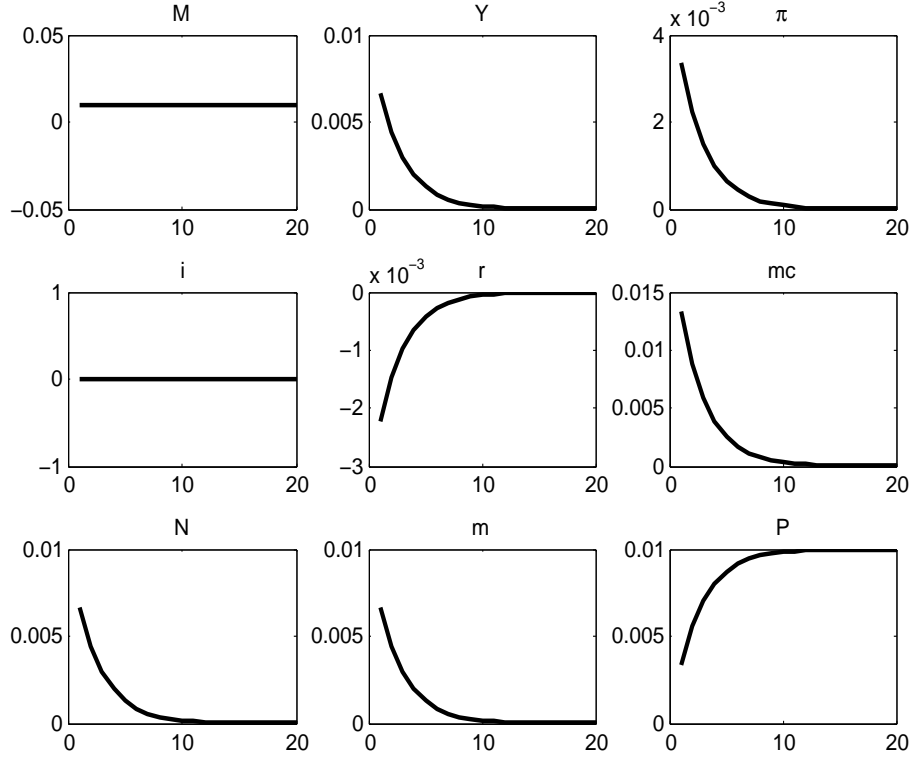
Next, consider a shock to the money supply. Since I have assumed that  $\rho_m = 0$ , nominal money follows a random walk, so the shock results in a one time permanent level shift in  $M_t$ . Here, we

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<sup>2</sup>To see why this happens, go back to the FOC from the household’s problem, you can write:  $\lambda_t = \theta \frac{1}{M_t} + \beta E_t \lambda_{t+1}$ . Solving this forward, you’d get:  $\lambda_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{\theta}{M_{t+j}}$ . If  $M_t$  doesn’t respond to a shock, then  $\lambda_t$  can’t either. But from the first order condition for bonds,  $\lambda_t = \beta E_t \lambda_{t+1} (1 + i_t)$ . If  $\lambda_t$  and  $\lambda_{t+1}$  don’t react to the productivity shock (which they won’t if  $M_t$  is fixed), then  $i_t$  cannot react to the shock. Note that this result would not hold generally for specifications of utility from real balances which are not logarithmic or which are non-separable from the other arguments of utility. To see this, suppose that utility over real balances were still separable but instead showed up as  $\theta \left(\frac{M_t}{P_t}\right)^{1-\zeta} - 1$ . Then the FOC for the choice of  $M_t$  would be:  $\lambda_t = \theta M_t^{-\zeta} P_t^{\zeta-1} + \beta E_t \lambda_{t+1}$ . If  $\zeta = 1$  (log utility), then the  $P_t$  term drops out, and you get the result that  $\lambda_t$  can only move if  $M_t$  moves or is expected to move, which means that  $i_t$  cannot change if you get a productivity shock and the money supply is held constant. But if  $\zeta \neq 1$ , then  $P_t$  shows up; since  $P_t$  change, then  $\lambda_t$  and  $E_t \lambda_{t+1}$  will both change, which will mean that  $i_t$  will potentially change.

<sup>3</sup>In a log-linear version, it is straightforward to show that the hypothetical real interest rate if prices were flexible (sometimes called the “natural rate of interest”) would just be proportional to expected productivity growth, which is negative after a productivity shock given that I have assumed the process for productivity is a stationary AR(1).

observe that  $Y_t$ ,  $N_t$ , and  $\pi_t$  all rise. There is a temporary rise in  $m_t$ .  $mc_t$  rises, which means that  $w_t$  rises (since  $A_t$  is fixed): this is necessary to get workers to work more. The real interest rate falls, though again the nominal interest rate doesn't move.<sup>4</sup> Evidently, having sticky prices allows the nominal monetary shock to have real effects.

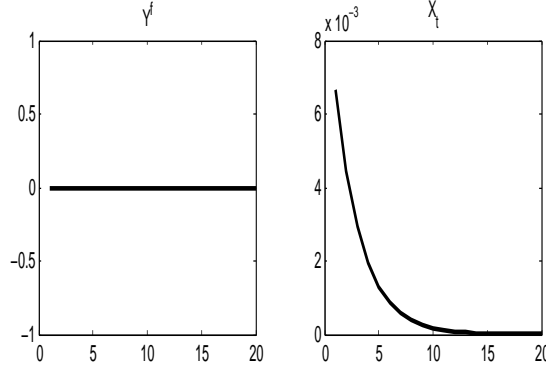


What is going on here? There are again a couple of ways to see this. Focusing on the money demand relationship, we again have the result that, for a fixed nominal interest rate, real balances and real GDP move together. When  $M_t$  increases, if prices were flexible  $P_t$  would increase by the same amount, so real balances wouldn't change, and hence  $Y_t$  wouldn't change. But with sticky prices,  $P_t$  can't increase sufficiently, so  $m_t$  rises, and therefore so too does output. Another way to see what is going on is by focusing on the real interest rate. If prices were flexible, the one time increase in  $M_t$  would be met by a one time permanent increase in  $P_t$ , so  $E_t P_{t+1} = P_t$ , and therefore expected inflation would not react. With expected inflation fixed, and the nominal rate fixed, there would be no effect on the real interest rate. But with price stickiness, because not all firms can immediately adjust their prices, the aggregate price level adjusts slowly, and in particular  $E_t P_{t+1} > P_t$ , so expected inflation rises. Higher expected inflation with a fixed nominal rate means

<sup>4</sup>We can see why the nominal interest rate doesn't move again via the first order conditions. We again must have:  $\lambda_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{\theta}{M_{t+j}}$ . Since  $M_t$  follows a random walk,  $\lambda_t$  and  $\lambda_{t+1}$  will both fall when  $M_t$  goes up, but by the same amount. Since  $\lambda_t = \beta E_t \lambda_{t+1} (1 + i_t)$ , this again implies that  $i_t$  will not react. If  $\rho_m > 0$ ,  $\lambda_t$  and  $\lambda_{t+1}$  would react differently because  $E_t M_{t+j} \neq M_t$ , and the nominal interest rate would move. The same caveat would apply as in the above footnote if utility over real balances were not logarithmic.

a lower real interest rate, which stimulates expenditure and results in the output increase.

Below I show the impulse response of the flexible price level of output and the output gap to the monetary policy shock. Since the flexible price level of output does not react, the response of the gap is identical to the response of output.



## 8 Log-Linearization

It is very common to see the basic New Keynesian model presented in log-linear form. The equations turn out to be pretty intuitive. It ends up being a decent amount of work, but there are some important payoffs to going through the hard work of linearizing the equations by hand. It turns out that life is much easier if we do the linearization about a steady state with  $\pi = 0$  (i.e. a “zero inflation steady state”).

Start with the Euler equation, going ahead and imposing the accounting identity that  $C_t = Y_t$ . We have:

$$\begin{aligned} -\sigma \ln Y_t &= \ln \beta - \sigma E_t \ln Y_{t+1} + i_t - E_t \pi_{t+1} \\ -\sigma \tilde{Y}_t &= -\sigma E_t \tilde{Y}_{t+1} + \tilde{i}_t - E_t \tilde{\pi}_{t+1} \end{aligned}$$

Where  $\tilde{Y}_t = \frac{Y_t - Y}{Y}$ ,  $\tilde{i}_t = i_t - i$ , and  $\tilde{\pi}_t = \pi_t - \pi$ . In other words, the variables already in rate form (interest rate and inflation) are expressed as absolute deviations, and variables not already in rate form as percent (log) deviations. We can re-write this as:

$$\tilde{Y}_t = E_t \tilde{Y}_{t+1} - \frac{1}{\sigma} (\tilde{i}_t - E_t \tilde{\pi}_{t+1}) \quad (55)$$

This is sometimes called the “New Keynesian IS Curve.” This is a bit of a misnomer: in old Keynesian models, the IS curve stands for “Investment = Saving,” and there is no investment in this model. Nevertheless, the idea is to show that there exists an inverse relationship between demand for current spending and the real interest rate. This expression is “New” in the sense

that it is forward-looking: current demand depends not just on the real interest rate but also on expected future income.

Next, log-linearize the static labor demand specification. This expression is already log-linear, and works out to be:

$$\eta \tilde{N}_t = -\sigma \tilde{Y}_t + \tilde{w}_t \quad (56)$$

From the marginal cost relationship, we can eliminate the wage:

$$\tilde{w}_t = \tilde{m}c_t + \tilde{A}_t \quad (57)$$

Plugging this in:

$$\eta \tilde{N}_t = -\sigma \tilde{Y}_t + \tilde{m}c_t + \tilde{A}_t$$

Log-linearize the production function:

$$\tilde{Y}_t = \tilde{A}_t + \tilde{N}_t - \tilde{v}_t^p$$

Now, what is  $\tilde{v}_t^p$ ? Let's take logs and go from there:

$$\ln v_t^p = \ln \left( (1 - \phi)(1 + \pi_t^\#)^{-\epsilon} (1 + \pi_t)^\epsilon + (1 + \pi_t)^\epsilon \phi v_{t-1}^p \right)$$

Now, from our discussion above, we know that  $v^p = 1$  when  $\pi = 0$ . Totally differentiating:

$$\begin{aligned} \tilde{v}_t^p = \frac{1}{1} \left( -\epsilon(1 - \phi)(1 + \pi^\#)^{-\epsilon-1} (1 + \pi)^\epsilon (\pi_t^\# - \pi^\#) + \epsilon(1 - \phi)(1 + \pi^\#)^{-\epsilon} (1 + \pi)^{\epsilon-1} (\pi_t - \pi) \dots \right. \\ \left. \dots \epsilon(1 + \pi)^{\epsilon-1} \phi v_{t-1}^p (\pi_t - \pi) + (1 + \pi)^\epsilon \phi (v_{t-1}^p - v^p) \right) \end{aligned}$$

Using now known facts about the steady state, this reduces to:

$$\tilde{v}_t^p = -\epsilon(1 - \phi)\tilde{\pi}_t^\# + \epsilon(1 - \phi)\tilde{\pi}_t + \epsilon\phi\tilde{\pi}_t + \phi\tilde{v}_{t-1}^p$$

This can be written:

$$\tilde{v}_t^p = -\epsilon(1 - \phi)\tilde{\pi}_t^\# + \epsilon\tilde{\pi}_t + \phi\tilde{v}_{t-1}^p$$

Now, log-linearize the equation for the evolution of inflation:

$$\begin{aligned} (1 - \epsilon)\pi_t &= \ln \left( (1 - \phi)(1 + \pi_t^\#)^{1-\epsilon} + \phi \right) \\ (1 - \epsilon)(\pi_t - \pi) &= (1 + \pi)^{\epsilon-1} \left( (1 - \epsilon)(1 - \phi)(1 + \pi^\#)^{-\epsilon} (\pi_t^\# - \pi^\#) \right) \end{aligned}$$

In the last line above, the  $(1 + \pi)^{\epsilon-1}$  shows up because the term inside parentheses is equal to  $(1 + \pi)^{1-\epsilon}$  evaluated in the steady state, and when taking the derivative of the log this term gets inverted evaluated at that point. Using facts about the zero inflation steady state, we have:

$$(1 - \epsilon)\tilde{\pi}_t = (1 - \epsilon)(1 - \phi)\tilde{\pi}_t^\#$$

Or:

$$\tilde{\pi}_t = (1 - \phi)\tilde{\pi}_t^\# \quad (58)$$

In other words, actual inflation is just proportional to reset price inflation, where the constant is equal to the fraction of firms that are updating their prices. This is pretty intuitive. Now, use this in the expression for price dispersion:

$$\tilde{v}_t^p = \epsilon \left( \tilde{\pi}_t - (1 - \phi)\tilde{\pi}_t^\# \right) + \phi\tilde{v}_{t-1}^p$$

But from above, the first term drops out, so we are left with:

$$\tilde{v}_t^p = \phi\tilde{v}_{t-1}^p \quad (59)$$

If we are approximating about the zero inflation steady state in which  $v^p = 1$ , then we're starting from a position in which  $\tilde{v}_{t-1}^p = 0$ , so this means that  $\tilde{v}_t^p = 0$  at all times. In other words, about a zero inflation steady state, price dispersion is a second order phenomenon, and we can just ignore it in a first order approximation about a zero inflation steady state.

Given this, the log-linearized production function is just:

$$\tilde{Y}_t = \tilde{A}_t + \tilde{N}_t \quad (60)$$

Now, plug this in to eliminate  $\tilde{N}_t$  from the log-linearized static labor supply condition from above:

$$\eta \left( \tilde{Y}_t - \tilde{A}_t \right) = -\sigma\tilde{Y}_t + \tilde{m}c_t + \tilde{A}_t$$

Simplifying a little bit, we get:

$$(\sigma + \eta)\tilde{Y}_t - (1 + \eta)\tilde{A}_t = \tilde{m}c_t$$

From above, we had solved for an expression for the flexible price level of output as:

$$Y_t^f = \left( \frac{1}{\psi} \frac{\epsilon - 1}{\epsilon} \right)^{\frac{1}{\sigma + \eta}} A_t^{\frac{1 + \eta}{\sigma + \eta}}$$

This is already log-linear, so we have:

$$\tilde{Y}_t^f = \frac{1+\eta}{\sigma+\eta} \tilde{A}_t \quad (61)$$

This means we can write:

$$\tilde{A}_t = \frac{\sigma+\eta}{1+\eta} \tilde{Y}_t^f$$

Plugging this in above, we get:

$$\tilde{mc}_t = (\sigma + \eta) (\tilde{Y}_t - \tilde{Y}_t^f) \quad (62)$$

In other words, real marginal cost is proportional to the output gap,  $\tilde{X}_t = \tilde{Y}_t - \tilde{Y}_t^f$ . Recall that real marginal cost is the inverse price markup. So if the gap is zero (output is equal to what it would be with flexible prices), then markups are equal to the desired fixed steady state markup of  $\frac{\epsilon}{\epsilon-1}$ . If the output gap is positive, then real marginal cost is above its steady state, so markups are lower than desired (equivalently, the economy is less distorted). The converse is true when the gap is negative.

Now, let's log-linearize the reset price expression. This is multiplicative, and so is already in log-linear form. We have:

$$\tilde{\pi}_t^\# = \tilde{\pi}_t + \tilde{x}_{1,t} - \tilde{x}_{2,t} \quad (63)$$

Now we need to log-linearize the auxiliary variables. Imposing the identity that  $Y_t = C_t$ , we have:

$$\ln x_{1,t} = \ln (Y_t^{1-\sigma} mc_t + \phi\beta E_t(1 + \pi_{t+1})^\epsilon x_{1,t+1})$$

Totally differentiating:

$$\frac{x_{1,t} - x_1}{x_1} = \frac{1}{x_1} ((1-\sigma)Y^{1-\sigma}mc(Y_t - Y) + Y^{1-\sigma}(mc_t - mc) + \epsilon\phi\beta(1+\pi)^{\epsilon-1}x_1(\pi_{t+1} - \pi) + \phi\beta(1+\pi)^\epsilon(x_{1,t+1} - x_1))$$

Distributing the  $\frac{1}{x_1}$  and multiplying, dividing where necessary to get in to percent deviation terms, and making use of the continued assumption of linearization about a zero inflation steady state, we have:

$$\tilde{x}_{1,t} = \frac{(1-\sigma)Y^{1-\sigma}mc}{x_1} \tilde{Y}_t + \frac{Y^{1-\sigma}mc}{x_1} \tilde{mc}_t + \epsilon\phi\beta E_t \tilde{\pi}_{t+1} + \phi\beta E_t \tilde{x}_{1,t+1}$$

Now, with zero steady state inflation, we know that  $x_1 = \frac{Y^{1-\sigma}mc}{1-\phi\beta}$ . This simplifies the first two terms:

$$\tilde{x}_{1,t} = (1 - \sigma)(1 - \phi\beta)\tilde{Y}_t + (1 - \phi\beta)\tilde{m}c_t + \epsilon\phi\beta E_t\tilde{\pi}_{t+1} + \phi\beta E_t\tilde{x}_{1,t+1} \quad (64)$$

Now, log-linearize  $x_{2,t}$ :

$$\ln x_{2,t} = \ln (Y_t^{1-\sigma} + \phi\beta E_t(1 + \pi_{t+1})^{\epsilon-1}x_{2,t+1})$$

Totally differentiating:

$$\frac{x_{2,t} - x_2}{x_2} = \frac{1}{x_2} ((1 - \sigma)Y^{-\sigma}(Y_t - Y) + (\epsilon - 1)\phi\beta(1 + \pi)^{\epsilon-2}x_2(\pi_{t+1} - \pi) + \phi\beta(1 + \pi)^{\epsilon-1}(x_{2,t+1} - x_2))$$

Distributing the  $x_2$ , multiplying and dividing by appropriate terms, and making use of the fact that  $\pi =$ , we have:

$$\tilde{x}_{2,t} = \frac{(1 - \sigma)Y^{1-\sigma}}{x_2}\tilde{Y}_t + (\epsilon - 1)\phi\beta E_t\tilde{\pi}_{t+1} + \phi\beta E_t\tilde{x}_{2,t+1}$$

Since  $x_2 = \frac{Y^{1-\sigma}}{1-\phi\beta}$ , this can be written:

$$\tilde{x}_{2,t} = (1 - \sigma)(1 - \phi\beta)\tilde{Y}_t + (\epsilon - 1)\phi\beta E_t\tilde{\pi}_{t+1} + \phi\beta E_t\tilde{x}_{2,t+1} \quad (65)$$

Now, subtracting  $\tilde{x}_{2,t}$  from  $\tilde{x}_{1,t}$ , we have:

$$\tilde{x}_{1,t} - \tilde{x}_{2,t} = (1 - \phi\beta)\tilde{m}c_t + \phi\beta E_t\tilde{\pi}_{t+1} + \phi\beta E_t(\tilde{x}_{1,t+1} - \tilde{x}_{2,t+1})$$

From above, we also know that:

$$\tilde{x}_{1,t} - \tilde{x}_{2,t} = \tilde{\pi}_t^\# - \tilde{\pi}_t$$

But  $\tilde{\pi}_t^\# = \frac{1}{1-\phi}\tilde{\pi}_t$ , so we must also have:

$$\tilde{x}_{1,t} - \tilde{x}_{2,t} = \frac{\phi}{1-\phi}\tilde{\pi}_t$$

Make this substitution above:

$$\frac{\phi}{1-\phi}\tilde{\pi}_t = (1 - \phi\beta)\tilde{m}c_t + \phi\beta E_t\tilde{\pi}_{t+1} + \phi\beta E_t\left(\frac{\phi}{1-\phi}E_t\tilde{\pi}_{t+1}\right)$$

Multiplying through:

$$\tilde{\pi}_t = \frac{(1 - \phi)(1 - \phi\beta)}{\phi}\tilde{m}c_t + (1 - \phi)\beta E_t\tilde{\pi}_{t+1} + \phi\beta E_t\tilde{\pi}_{t+1}$$

Or:



$$\tilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} \tilde{m}c_t + \beta E_t \tilde{\pi}_{t+1} \quad (66)$$

This expression is sometimes called the New Keynesian Phillips Curve. It is “new” in the sense of being forward-looking. It is a Phillips Curve in the sense of showing a relationship between some real measure,  $\tilde{m}c_t$ , and inflation,  $\tilde{\pi}_t$ . It is also common to see the Phillips Curve expressed in terms of the output gap, using the relationship between real marginal cost and the gap that we derived above:

$$\tilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} (\sigma + \eta) (\tilde{Y}_t - \tilde{Y}_t^f) + \beta E_t \tilde{\pi}_{t+1} \quad (67)$$

Using the terminal condition that inflation will return to steady state eventually, we can solve the NKPC forward to get:

$$\tilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} \sum_{j=0}^{\infty} \beta^j \tilde{m}c_{t+j} \quad (68)$$

In other words, current inflation is proportional to the present discounted value of expected real marginal cost. This expression is actually pretty intuitive. Real marginal cost is the inverse price markup. In the model without price rigidity, firms desire constant markups. If expected future marginal cost is high, then firms will have low markups. Firms given the option of updating prices today will try to increase price today (since they may be stuck with that price in the future) to hit their desired price markup (and vice-versa), putting upward pressure on current inflation (and vice versa). The “slope” of the Phillips Curve is decreasing in  $\phi$ : when  $\phi$  is large, the coefficient on marginal cost (or the gap) is small, suggesting that real movements put little upward pressure on inflation. When  $\phi$  is small, the Phillips Curve is steep. In the limiting case, as prices become perfectly flexible ( $\phi \rightarrow \infty$ ), the Phillips Curve becomes vertical, which means  $\tilde{m}c_t = 0$  and  $\tilde{Y}_t = \tilde{Y}_t^f$  (e.g. we would be at the flexible price allocation).

The expressions for  $A_t$  and money growth are already log-linear, so we have:

$$\tilde{A}_t = \rho_a \tilde{A}_{t-1} + \varepsilon_{a,t} \quad (69)$$

$$\Delta \tilde{m}_t = -\tilde{\pi}_t + \rho_m \tilde{\pi}_{t-1} + \rho_m \Delta \tilde{m}_{t-1} + \varepsilon_{m,t} \quad (70)$$

$$\Delta \tilde{m}_t = \tilde{m}_t - \tilde{m}_{t-1} \quad (71)$$

Lastly, we need to log-linearize the money demand expression:

$$\ln m_t = \ln \theta + i_t - \ln i_t + \sigma \ln Y_t$$

Totally differentiating:

$$\tilde{m}_t = \tilde{i}_t - \frac{1}{i} \tilde{i}_t + \sigma \tilde{Y}_t$$

Recall that  $\tilde{i}_t = i_t - i$ , where  $i = \frac{1}{\beta} - 1$  since steady state inflation is zero. Hence we can write this as:

$$\tilde{m}_t = \tilde{i}_t - \frac{\beta}{1 - \beta} \tilde{i}_t + \sigma \tilde{Y}_t$$

Or:

$$\tilde{m}_t = \left(1 - \frac{\beta}{1 - \beta}\right) \tilde{i}_t + \sigma \tilde{Y}_t \quad (72)$$

This is intuitive. Demand for real balances is decreasing in the real interest rate and increasing  $Y_t$ .

This means we can re-write the complete (though reduced, because I've eliminated a lot of the extraneous variables) log-linearized system of equations as:

$$\tilde{Y}_t = E_t \tilde{Y}_{t+1} - \frac{1}{\sigma} \left( \tilde{i}_t - E_t \tilde{\pi}_{t+1} \right) \quad (73)$$

$$\tilde{\pi}_t = \frac{(1 - \phi)(1 - \phi\beta)}{\phi} (\sigma + \eta) \left( \tilde{Y}_t - \tilde{Y}_t^f \right) + \beta E_t \tilde{\pi}_{t+1} \quad (74)$$

$$\tilde{Y}_t^f = \frac{1 + \eta}{\sigma + \eta} \tilde{A}_t \quad (75)$$

$$\tilde{A}_t = \rho_a \tilde{A}_{t-1} + \varepsilon_{a,t} \quad (76)$$

$$\Delta \tilde{m}_t = -\tilde{\pi}_t + \rho_m \tilde{\pi}_{t-1} + \rho_m \Delta \tilde{m}_{t-1} + \varepsilon_{m,t} \quad (77)$$

$$\Delta \tilde{m}_t = \tilde{m}_t - \tilde{m}_{t-1} \quad (78)$$

$$\tilde{m}_t = \left(1 - \frac{\beta}{1 - \beta}\right) \tilde{i}_t + \sigma \tilde{Y}_t \quad (79)$$

This is seven equations in seven variables  $(\tilde{Y}_t, \tilde{i}_t, \tilde{\pi}_t, \tilde{Y}_t^f, \tilde{A}_t, \Delta \tilde{m}_t, \tilde{m}_t)$ . We have an “aggregate demand” expression given by the linearized Euler equation, an “aggregate supply” relationship given by the Phillips Curve, a productivity shock, a money supply relationship, a money demand relationship, and two auxiliary expressions defining growth in real balances and the flexible price level of output. Note that you won't get *exactly* the same output as Dynare will give you from this linearization. It will be very close. The reason is that I approximated  $\ln(1 + i_t) = i_t$ , whereas when Dynare does the approximation it won't use that extra term in doing the log-linearization.

## 9 A Taylor Rule Formulation

In the model as laid out so far, whether in log-linearized form or not, I have characterized monetary policy with an exogenous rule for money growth. This doesn't seem to square particularly well with

actual central bank practice, where central bankers tend to think of policy in terms of interest rates, not monetary aggregates per se.

For reasons that will soon become clearer, an exogenous interest rate rule will lead to an indeterminacy in the model. An interest rate specification of policy needs to feature nominal interest rates reacting, and reacting sufficiently, to *endogenous* variables like inflation and output. The most popular interest rate rule is somewhat generically called a Taylor rule, after John Taylor. It takes a form similar to the following:

$$i_t = (1 - \rho_i)i + \rho_i i_{t-1} + (1 - \rho_i)(\phi_\pi(\pi_t - \pi) + \phi_x(\ln X_t - \ln X)) + \varepsilon_{i,t} \quad (80)$$

$i$  is the steady state interest,  $\pi$  is an exogenous steady state inflation target,  $\ln X_t$  is the output gap,  $\ln X$  is the steady state output gap, and  $\varepsilon_{i,t}$  is a monetary policy shock, analogous to the  $\varepsilon_{m,t}$  in the money growth specification.  $\rho_i$  is a smoothing parameter, and  $\phi_\pi$  and  $\phi_x$  are non-negative coefficients. Assume that  $\phi_\pi > 1$  (we'll talk about why later). The policy rule is one of partial adjustment – it says that the current nominal rate is a convex combination of the lagged nominal rate and the current target rate, where the current target rate is a linear function of the deviations of inflation and the output gap from target (where I have implicitly assumed that the targets are the long run steady state levels). Note that there is no mention of money in this policy rule specification. I can effectively replace the money growth process above with this rule. Given the chosen nominal interest rate, the central bank will implicitly print the requisite amount of money to meet money demand at that interest rate. Given the specification for money we have used – where money enters the utility function in an additively separable fashion – we could actually not talk about money at all, and consider the economy to be “cashless.”

The full set of equilibrium conditions can be written:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma} (1 + i_t)(1 + \pi_{t+1})^{-1} \quad (81)$$

$$\psi N_t^\eta = C_t^{-\sigma} w_t \quad (82)$$

$$m_t = \theta \frac{1 + i_t}{i_t} C_t^\sigma \quad (83)$$

$$mc_t = \frac{w_t}{A_t} \quad (84)$$

$$C_t = Y_t \quad (85)$$

$$Y_t = \frac{A_t N_t}{v_t^p} \quad (86)$$

$$v_t^p = (1 - \phi)(1 + \pi_t^\#)^{-\epsilon}(1 + \pi_t)^\epsilon + (1 + \pi_t)^\epsilon \phi v_{t-1}^p \quad (87)$$

$$(1 + \pi_t)^{1-\epsilon} = (1 - \phi)(1 + \pi_t^\#)^{1-\epsilon} + \phi \quad (88)$$

$$1 + \pi_t^\# = \frac{\epsilon}{\epsilon - 1}(1 + \pi_t) \frac{x_{1,t}}{x_{2,t}} \quad (89)$$

$$x_{1,t} = C_t^{-\sigma} m c_t Y_t + \phi \beta E_t (1 + \pi_{t+1})^\epsilon x_{1,t+1} \quad (90)$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta E_t (1 + \pi_{t+1})^{\epsilon-1} x_{2,t+1} \quad (91)$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \quad (92)$$

$$i_t = (1 - \rho_i) i + \rho_i i_{t-1} + (1 - \rho_i) (\phi_\pi (\pi_t - \pi) + \phi_x (\ln X_t - \ln X)) + \varepsilon_{i,t} \quad (93)$$

Here I have gotten rid of the money growth specification, and since I've done that I no longer need to keep  $\Delta \ln m_t$  as a variable, so this is actually one fewer equation and one fewer unknown than I previously had. As I said above, I really don't even need to keep track of  $m_t$  anymore either, but I'll keep it in because that turns out to be instructive.

One point that I should mention up front. One might be tempted to think that the steady state log output gap is zero, meaning that  $Y = Y^f$ . This will only be the case if  $\pi = 0$ , otherwise  $Y < Y^f$ . From above, we know that:

$$Y^f = \left( \frac{1}{\psi} \frac{\epsilon - 1}{\epsilon} \right)^{\frac{1}{\sigma + \eta}} \quad (94)$$

For the sticky price economy, we have:

$$N = \left( \frac{1}{\psi} (v^p)^\sigma m c \right)^{\frac{1}{\eta + \sigma}} \quad (95)$$

We know that:

$$Y = \frac{N}{v^p}$$

Steady state output is then:

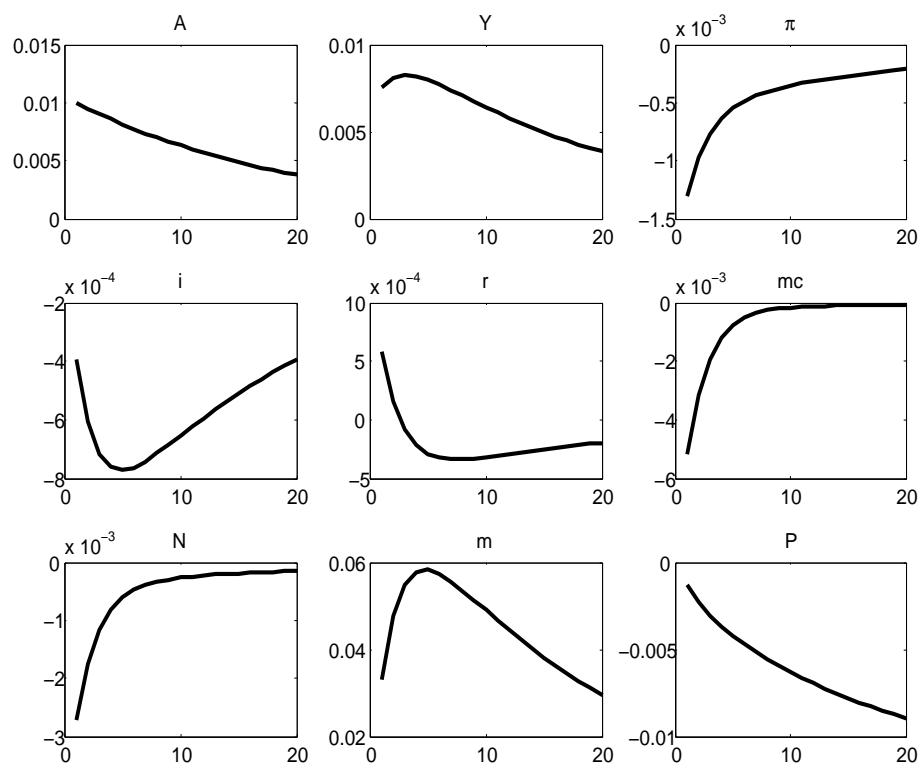
$$Y = \left( \frac{1}{\psi} \right)^{\frac{1}{\sigma + \eta}} (v^p)^{-\frac{\eta}{\eta + \sigma}} m c^{\frac{1}{\eta + \sigma}} \quad (96)$$

We also know:

$$m c = \frac{1 - \phi \beta (1 + \pi)^\epsilon}{1 - \phi \beta (1 + \pi)^{\epsilon-1}} \frac{1 + \pi^\#}{1 + \pi} \frac{\epsilon - 1}{\epsilon}$$

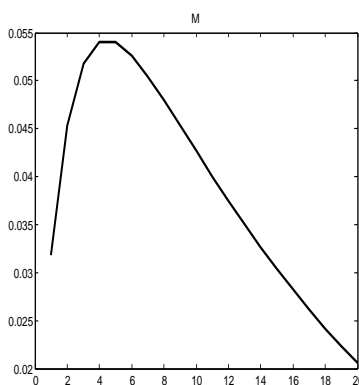
Now, if  $\pi = 0$ , then  $m c = \frac{\epsilon-1}{\epsilon}$  and  $v^p = 1$ , so this reduces to the same expression as  $Y^f$ , so we'll have  $Y = Y^f$ . But if  $\pi > 0$ , you can show that  $m c < \frac{\epsilon-1}{\epsilon}$  and we know that  $v^p > 1$ . Since the exponent on  $m c$  is positive, and the exponent on  $v^p$  negative, this means that  $\pi > 0$  will mean  $Y < Y^f$ , which means the steady state output gap will be negative,  $\ln X = \ln Y - \ln Y^f < 0$ .

I solve the model in Dynare using the coefficients  $\rho_i = 0.8$ ,  $\phi_\pi = 1.5$ , and  $\phi_x = 0$ . Below are the impulse responses to a technology shock under the Taylor rule formulation:

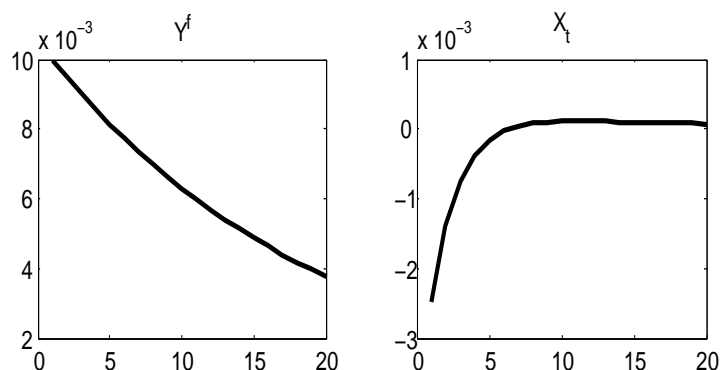


If you compare these to what we had earlier, you'll notice that they are substantively different. In particular, the impact increase in output under the Taylor rule is much larger than under the exogenous money growth rule; hence we also see a smaller drop in hours on impact, a smaller increase in the real interest rate, and a smaller drop in inflation. We also see the nominal interest rate moving. Also, the response of the price level here seems to be more or less permanent, whereas in the money growth rule case it seemed to be mean-reverting.

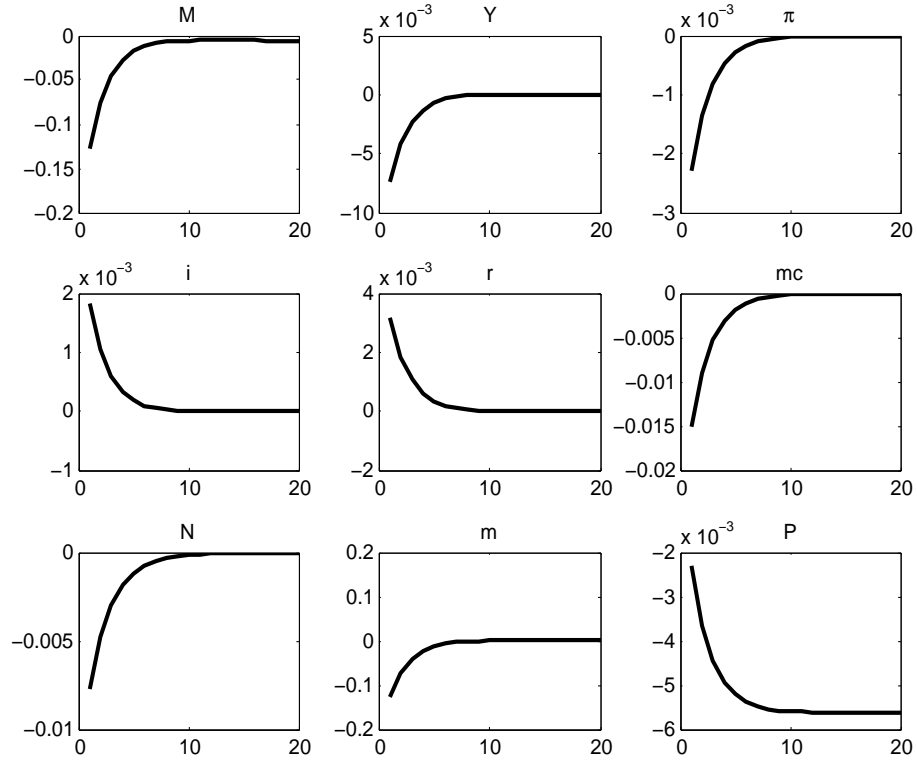
To get a better sense of these differences, below I plot the impulse response of the nominal supply to the technology shock. Note that the nominal money supply didn't respond in the previous case.



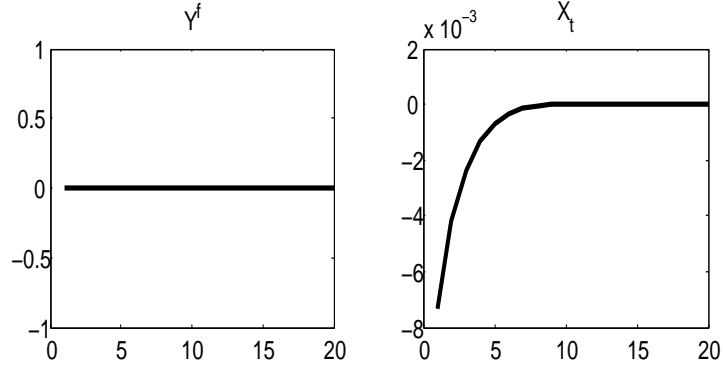
Here we see that the nominal money supply rises rather significantly. In other words, under the Taylor rule the money supply is effectively endogenous, and the central bank reacts to the increased productivity by accommodating it and increasing the money supply. This increase in the money supply helps real balances increase – now we don't have to simply rely on prices falling to get real balances to go up, so output can expand by more than it would if the money supply were fixed. This endogenous response of the money supply is what allows output to rise by more than it did under the exogenous money rule process earlier. Accordingly, we see a smaller drop in the output gap in response to the technology shock.



Next, consider a positive shock to the Taylor rule, which raises the nominal interest rate. This coincides with a decline in the money supply, an increase in the real interest rate, and a decline in economic activity. The channels at play for why this nominal shock has real effects are the same as above when we thought about the nominal shock in terms of the money supply. There are two ways to think about. First, the decrease in the money supply is matched by a less than proportional decrease in the price level because of price stickiness; this means that real balances decline, which via the basic logic above necessitates a decline in output. It also has effect of raising the real interest rate. The nominal rate rises, and because of price stickiness expected inflation does not rise enough, so the real rate rises, which leads to a reduction in demand.



Below are the impulse responses of the flexible price level of output as well as the output gap.



In log-linear terms, the Taylor rule is just:

$$\tilde{i}_t = \rho_i \tilde{i}_{t-1} + (1 - \rho_i) \left( \phi_\pi \tilde{\pi}_t + \phi_x \tilde{X}_t \right) + \varepsilon_{i,t} \quad (97)$$

We can the write the linearized model as:

$$\tilde{Y}_t = E_t \tilde{Y}_{t+1} - \frac{1}{\sigma} \left( \tilde{i}_t - E_t \tilde{\pi}_{t+1} \right) \quad (98)$$

$$\tilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}(\sigma + \eta) \left( \tilde{Y}_t - \tilde{Y}_t^f \right) + \beta E_t \tilde{\pi}_{t+1} \quad (99)$$

$$\tilde{Y}_t^f = \frac{1+\eta}{\sigma+\eta} \tilde{A}_t \quad (100)$$

$$\tilde{A}_t = \rho_a \tilde{A}_{t-1} + \varepsilon_{a,t} \quad (101)$$

$$\tilde{i}_t = \rho_i \tilde{i}_{t-1} + (1 - \rho_i) \left( \phi_\pi \tilde{\pi}_t + \phi_x \tilde{X}_t \right) + \varepsilon_{i,t} \quad (102)$$

Sometimes you will see different ways of writing the model out, and it is useful to review them here. First, note that we can eliminate  $\tilde{A}_t$  and just write the model in terms of  $\tilde{Y}_t^f$ . Let  $\omega = \frac{1+\eta}{\sigma+\eta}$ :

$$\begin{aligned} \tilde{Y}_t^f &= \omega \tilde{A}_t \\ \tilde{Y}_t^f &= \omega \left( \rho_a \tilde{A}_{t-1} + \varepsilon_{a,t} \right) \\ \tilde{Y}_t^f &= \omega \left( \rho_a \frac{1}{\omega} \tilde{Y}_{t-1}^f + \varepsilon_{a,t} \right) \\ \tilde{Y}_t^f &= \rho_a \tilde{Y}_{t-1}^f + \omega \varepsilon_{a,t} \end{aligned} \quad (103)$$

In other words, we can think about the flexible price level of output is effectively being exogenous, obeying the same AR(1) process as  $\tilde{A}_t$ , but with the shock scaled by the factor  $\omega$ .

Now, let's write the Euler/IS equation in terms of the output gap,  $\tilde{X}_t$ , instead of output. We can do this by subtracting  $\tilde{Y}_t^f$  and  $E_t \tilde{Y}_{t+1}^f$  from both sides:

$$\begin{aligned} \tilde{Y}_t - \tilde{Y}_t^f - E_t \tilde{Y}_{t+1}^f &= -\tilde{Y}_t^f + E_t \tilde{Y}_{t+1} - E_t \tilde{Y}_{t+1}^f - \frac{1}{\sigma} \left( \tilde{i}_t - E_t \tilde{\pi}_{t+1} \right) \\ \tilde{X}_t &= E_t \tilde{X}_{t+1} + E_t \tilde{Y}_{t+1}^f - \tilde{Y}_t^f - \frac{1}{\sigma} \left( \tilde{i}_t - E_t \tilde{\pi}_{t+1} \right) \end{aligned}$$

Now, from the Fisher relationship, we know that  $\tilde{r}_t = \tilde{i}_t - E_t \tilde{\pi}_{t+1}$ . Now, let's consider a hypothetical allocation in which prices are flexible (e.g.  $\phi = 0$ ). Then we know by construction that  $\tilde{X}_t = 0$ . This means we can solve for a hypothetical flexible price real interest rate (sometimes called the “Wicksellian natural rate of interest” as:

$$\begin{aligned} 0 &= E_t \tilde{Y}_{t+1}^f - \tilde{Y}_t^f - \frac{1}{\sigma} \tilde{r}_t^f \\ \tilde{r}_t^f &= \sigma \left( E_t \tilde{Y}_{t+1}^f - \tilde{Y}_t^f \right) \end{aligned} \quad (104)$$

In words, the “natural rate” of interest is proportional to the expected growth rate of the flexible price level of output. We can use this to write the Euler equation as:



$$\tilde{X}_t = E_t \tilde{X}_{t+1} - \frac{1}{\sigma} \left( \tilde{i}_t - E_t \tilde{\pi}_{t+1} - \tilde{r}_t^f \right) \quad (105)$$

In other words, the current output gap equals the expected future output gap minus  $\frac{1}{\sigma}$  times the “real interest rate gap” – the gap between the actual real interest rate and the flexible price real interest rate. Holding  $E_t \tilde{X}_{t+1}$  fixed, if the real interest rate gap is positive (actual real rate “too high”) then the gap will be negative, and vice-versa.

Since  $E_t \tilde{Y}_t^f = \rho_a \tilde{Y}_t^f$ , this further reduces to:

$$\tilde{r}_t^f = \sigma(\rho_a - 1) \tilde{Y}_t^f \quad (106)$$

Now, plug in the AR(1) process for  $\tilde{Y}_t^f$  that we just derived:

$$\begin{aligned} \tilde{r}_t^f &= \sigma(\rho_a - 1) \left( \rho_a \tilde{Y}_{t-1}^f + \omega \varepsilon_{a,t} \right) \\ \tilde{r}_t^f &= \sigma(\rho_a - 1) \left( \rho_a \frac{1}{\sigma(\rho_a - 1)} \tilde{r}_{t-1}^f + \omega \varepsilon_{a,t} \right) \\ \tilde{r}_t^f &= \rho_a \tilde{r}_{t-1}^f + \sigma(\rho_a - 1) \omega \varepsilon_{a,t} \end{aligned} \quad (107)$$

With this, we can reduce the entire log-linearized system to:

$$\tilde{X}_t = E_t \tilde{X}_{t+1} - \frac{1}{\sigma} \left( \tilde{i}_t - E_t \tilde{\pi}_{t+1} - \tilde{r}_t^f \right) \quad (108)$$

$$\tilde{\pi}_t = \frac{(1 - \phi)(1 - \phi\beta)}{\phi} (\sigma + \eta) \tilde{X}_t + \beta E_t \tilde{\pi}_{t+1} \quad (109)$$

$$\tilde{i}_t = \rho_i \tilde{i}_{t-1} + (1 - \rho_i) \left( \phi_\pi \tilde{\pi}_t + \phi_x \tilde{X}_t \right) + \varepsilon_{i,t} \quad (110)$$

$$\tilde{r}_t^f = \rho_a \tilde{r}_{t-1}^f + \sigma(\rho_a - 1) \omega \varepsilon_{a,t} \quad (111)$$

This is sometimes called the “three equation New Keynesian model.” This may look odd, since there are actually four equations, but only three of these equations describe endogenous variables: the first equation is the Euler/IS/demand relationship, the second is the Phillips Curve, and the third is a policy rule. These three equations are what make up the “three equation model.” The fourth equation is an exogenous process for  $\tilde{r}_t^f$  (which, again, we derived from a process for productivity).

## 10 Slight Detour: The Method of Undetermined Coefficients

Consider the small scale model above. It features two forward-looking jump variables ( $\tilde{\pi}_t$  and  $\tilde{X}_t$ ) and two state variables (one endogenous, the interest rate,  $\tilde{i}_t$ , and one exogenous, the natural rate of interest,  $\tilde{r}_t^f$ ). We could solve for the policy functions mapping the states into the jump variables

using the methodology we laid out in class. Dynare will do this for us.

Another solution methodology, which is more intuitive, is to use the method of undetermined coefficients. This involves postulating (really, guessing) that the policy functions are linear, imposing that, and then solving a system of equations for the policy rule coefficients. In a small scale model without many state variables, this is often pretty straightforward and will allow us to get analytical policy functions, which is nice. Let's try that out here.

Consider the "three equation" model with the exogenous process for  $\tilde{r}_t^f$ . To make life easy, let's assume that  $\rho_i = 0$ . This means that  $\tilde{i}_t$  is no longer a state; indeed, it becomes redundant and can be substituted out entirely. Doing so, the remaining system can be written as follows:

$$\tilde{X}_t = E_t \tilde{X}_{t+1} - \frac{1}{\sigma} \left( \phi_\pi \tilde{\pi}_t + \phi_x \tilde{X}_t - E_t \tilde{\pi}_{t+1} - \tilde{r}_t^f \right) \quad (112)$$

$$\tilde{\pi}_t = \kappa \tilde{X}_t + \beta E_t \tilde{\pi}_{t+1} \quad (113)$$

$$\tilde{r}_t^f = \rho_a \tilde{r}_{t-1}^f + \sigma(\rho_a - 1) \omega \varepsilon_{a,t} \quad (114)$$

To keep notation tight, I have defined  $\kappa = \frac{(1-\phi)(1-\phi\beta)}{\phi}(\sigma + \eta)$  as the slope on the Phillips Curve. The only state variable is  $\tilde{r}_t^f$ . Let's guess that the policy functions are linear:

$$\begin{aligned} \tilde{X}_t &= \lambda_1 \tilde{r}_t^f \\ \tilde{\pi}_t &= \lambda_2 \tilde{r}_t^f \end{aligned}$$

Plug these in to the Euler/IS and Phillips Curves:

$$\begin{aligned} \lambda_1 \tilde{r}_t^f &= \lambda_1 \rho_a \tilde{r}_t^f - \frac{1}{\sigma} \left( \phi_\pi \lambda_2 \tilde{r}_t^f + \phi_x \lambda_1 \tilde{r}_t^f - \lambda_2 \rho_a \tilde{r}_t^f - \tilde{r}_t^f \right) \\ \sigma \lambda_1 \tilde{r}_t^f - \sigma \lambda_1 \rho_a \tilde{r}_t^f + \phi_\pi \lambda_2 \tilde{r}_t^f + \phi_x \lambda_1 \tilde{r}_t^f - \lambda_2 \rho_a \tilde{r}_t^f - \tilde{r}_t^f &= 0 \\ (\sigma \lambda_1 - \sigma \lambda_1 \rho_a + \phi_\pi \lambda_2 + \phi_x \lambda_1 - \lambda_2 \rho_a - 1) \tilde{r}_t^f &= 0 \\ \Rightarrow \sigma \lambda_1 - \sigma \lambda_1 \rho_a + \phi_\pi \lambda_2 + \phi_x \lambda_1 - \lambda_2 \rho_a - 1 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_2 \tilde{r}_t^f &= \kappa \lambda_1 \tilde{r}_t^f + \beta \lambda_2 \rho_a \tilde{r}_t^f \\ \lambda_2 \tilde{r}_t^f - \kappa \lambda_1 \tilde{r}_t^f - \beta \lambda_2 \rho_a \tilde{r}_t^f &= 0 \\ (\lambda_2 - \kappa \lambda_1 - \beta \lambda_2 \rho_a) \tilde{r}_t^f &= 0 \\ \Rightarrow \lambda_2 - \kappa \lambda_1 - \beta \lambda_2 \rho_a &= 0 \end{aligned}$$

Here I have made use of the fact that  $E_t \tilde{r}_{t+1}^f = \rho_a \tilde{r}_t^f$ . The above amounts to two equations in two unknowns ( $\lambda_1$  and  $\lambda_2$ ). We can solve for these coefficients. From the second expression, we have:

$$\lambda_1 = \frac{1 - \beta\rho_a}{\kappa} \lambda_2$$

Simplify the first expression somewhat:

$$(\sigma(1 - \rho_a) + \phi_x) \lambda_1 + (\phi_\pi - \rho_a) \lambda_2 - 1 = 0$$

Plug in for  $\lambda_1$ :

$$\left( (\sigma(1 - \rho_a) + \phi_x) (1 - \beta\rho_a) \frac{1}{\kappa} + \phi_\pi - \rho_a \right) \lambda_2 = 1$$

So:

$$\lambda_2 = \left( (\sigma(1 - \rho_a) + \phi_x) (1 - \beta\rho_a) \frac{1}{\kappa} + \phi_\pi - \rho_a \right)^{-1}$$

This can be re-written:

$$\lambda_2 = \frac{\kappa}{(\sigma(1 - \rho_a) + \phi_x) (1 - \beta\rho_a) + \kappa(\phi_\pi - \rho_a)} \quad (115)$$

Then we have:

$$\lambda_1 = \frac{1 - \beta\rho_a}{(\sigma(1 - \rho_a) + \phi_x) (1 - \beta\rho_a) + \kappa(\phi_\pi - \rho_a)} \quad (116)$$

So the policy functions are:

$$\tilde{X}_t = \frac{1 - \beta\rho_a}{(\sigma(1 - \rho_a) + \phi_x) (1 - \beta\rho_a) + \kappa(\phi_\pi - \rho_a)} \tilde{r}_t^f \quad (117)$$

$$\tilde{\pi}_t = \frac{\kappa}{(\sigma(1 - \rho_a) + \phi_x) (1 - \beta\rho_a) + \kappa(\phi_\pi - \rho_a)} \tilde{r}_t^f \quad (118)$$

You can verify that you get the same thing by sticking the linearized equations into Dynare and letting it do the work for you.

## 11 Another Detour: Calibrating the Calvo Parameter $\phi$

How does one come up with a reasonable value for  $\phi$ ? This is a really important parameter in the model – the bigger it is (the stickier are prices), the bigger will be the effects of nominal shocks and the more distorted will be the response of variables to real shocks.

It turns out that there exists a close mapping between  $\phi$  and the expected duration of a price change. Consider a firm that gets to update its price in a period. In expectation, how long will it be stuck with that price? The probability of getting to adjust its price one period from now is  $1 - \phi$ . The probability of adjusting in two periods is  $\phi(1 - \phi)$ :  $\phi$  is the probability it doesn't adjust after one period, and  $1 - \phi$  is the probability it can adjust in two periods. The probability of adjusting in three periods is  $\phi^2(1 - \phi)$ :  $\phi^2$  is the probability it gets to the third period with its

initial price, and  $1 - \phi$  is the probability it can adjust in that period. And so on. So the expected duration of a price chosen today is:

$$\text{Expected Duration} = (1 - \phi) \sum_{j=1}^{\infty} \phi^{j-1} j$$

The summation part on the inside can be written:

$$\begin{aligned} S &= 1 + 2\phi + 3\phi^2 + 4\phi^3 + \dots \\ S\phi &= \phi + 2\phi^2 + 3\phi^3 + \dots \\ (1 - \phi)S &= 1 + \phi + \phi^2 + \phi^3 + \dots \\ (1 - \phi)S &= \frac{1}{1 - \phi} \\ S &= \frac{1}{(1 - \phi)^2} \end{aligned}$$

The second to last line above uses the fact that  $1 + \phi + \phi^2 + \dots = \frac{1}{1 - \phi}$  as long as  $\phi < 1$ . Combining this with the  $1 - \phi$  outside the summation term above, we have:

$$\text{Expected Duration} = \frac{1}{1 - \phi}$$

Bils and Klenow (2004, *JPE*) analyze micro data on pricing and computing the average length of time between prices changes. Though there is substantial heterogeneity across types of goods (e.g. the price of newspapers rarely changes, while gasoline changes daily), for most goods, prices change on average once every six months, which would suggest that  $\phi \approx 1/2$  at a quarterly frequency (average duration of two quarters). For these models to produce realistic responses to monetary policy shocks you need  $\phi$  much higher (more like the 0.75 value I've been using). So an important area of research essentially involves ways to “flatten” the Phillips Curve without assuming counterfactually large levels of price rigidity.

## 12 Alternative Price Stickiness Model: Rotemberg (1982) Pricing

We have thus far modeled price stickiness via the Calvo (1983) assumption, wherein each period firms face a constant hazard of being able to adjust their price. This means that in equilibrium firms are heterogeneous, but the model is rigged in such a way that aggregation works out nicely. An alternative pricing assumption is based on Rotemberg (1982). In the Rotemberg model, firms face a quadratic cost of price adjustment. In equilibrium they all end up behaving identically. To a first order approximation about a zero inflation steady state, the Rotemberg and Calvo models can be parameterized to be identical. They are not identically to order higher than one, and they have different implications for micro data (e.g. in the Calvo model only a fraction of firms will adjust their price in a given period, but in the Rotemberg model all firms will be changing their prices

each period).

In the Rotemberg model, there are many intermediate goods firms. Firms face the same demand curve from the final good firm as earlier. They produce output according to  $Y_t = A_t N_t(j)$ , just as in the Calvo model. Cost-minimization implies that real marginal cost is  $mc_t = \frac{w_t}{A_t}$ , where  $w_t \equiv W_t/P_t$  is the real wage, common to all firms.

Nominal flow profit for producer  $j$  is given by:

$$\Pi_t(j) = P_t(j) \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - W_t N_t(j) - \frac{\psi}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 P_t Y_t \quad (119)$$

The parameter  $\psi \geq 0$  measures the cost of price adjustment, and it is measured in units of the final good. We can write the profit function in real dollars as:

$$\Pi_t(j) = P_t(j) \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} - w_t N_t(j) - \frac{\psi}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \quad (120)$$

Since  $w_t = A_t mc_t$ , we can write this as:

$$\Pi_t(j) = P_t(j) \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} - mc_t A_t N_t(j) - \frac{\psi}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \quad (121)$$

But this is just:

$$\Pi_t(j) = P_t(j) \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} - mc_t A_t Y_t(j) - \frac{\psi}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \quad (122)$$

Plugging in the demand function for variety  $j$ , we get:

$$\Pi_t(j) = P_t(j) \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} - mc_t A_t P_t(j)^{-\epsilon} P_t^\epsilon Y_t - \frac{\psi}{2} \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \quad (123)$$

Each period, firms choose price to maximize the expected present discounted value of flow profit, where discounting is by the household's stochastic discount factor. The optimality condition for price-setting can be written:

$$\begin{aligned} (\epsilon - 1) \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_t} = \\ \epsilon mc_t \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon-1} \frac{Y_t}{P_t} - \psi \left( \frac{P_t(j)}{P_{t-1}(j)} - 1 \right) \frac{Y_t}{P_{t-1}(j)} + \\ \beta \psi E_t \left[ \frac{u_C(C_{t+1}, N_{t+1})}{u_C(C_t, N_t)} \left( \frac{P_{t+1}(j)}{P_t(j)} - 1 \right) \left( \frac{P_{t+1}(j)}{P_t(j)} \right) \left( \frac{Y_{t+1}}{P_t(j)} \right) \right] \end{aligned} \quad (124)$$

In equilibrium all firms behave identically. This means that they all charge the same price and produce the same output. The optimality condition for prices can be written in terms of inflation rates as:

$$(\epsilon - 1) = \epsilon mc_t - \psi(1 + \pi_t)\pi_t + \beta E_t \frac{u_C(C_{t+1}, N_{t+1})}{u_C(C_t, N_t)} \psi(1 + \pi_{t+1})\pi_{t+1} \frac{Y_{t+1}}{Y_t} \quad (125)$$

If we assume that  $u_C = C_t^{-1}$ , and in equilibrium know that  $Y_t = C_t$ , then these terms cancel, leaving:

$$(\epsilon - 1) = \epsilon mc_t - \psi(1 + \pi_t)\pi_t + \beta E_t \psi(1 + \pi_{t+1})\pi_{t+1} \quad (126)$$

Now, let's log-linearize this:

$$\ln(\epsilon - 1) = \ln(\epsilon mc_t - \psi(1 + \pi_t)\pi_t + \beta E_t \psi(1 + \pi_{t+1})\pi_{t+1})$$

$$0 = \frac{1}{\epsilon - 1} [\epsilon dmc_t - \psi(1 + \pi)d\pi_t - \psi\pi d\pi_t + \beta\psi(1 + \pi)d\pi_{t+1} + \beta\psi\pi d\pi_{t+1}]$$

Now, if we are linearizing about the point  $\pi = 0$ , then this simplifies greatly to:

$$0 = \frac{\epsilon}{\epsilon - 1} dmc_t - \frac{\psi}{\epsilon - 1} \tilde{\pi}_t + \beta \frac{\psi}{\epsilon - 1} E_t \tilde{\pi}_{t+1}$$

Now, what is steady state  $mc$ ? It is  $\frac{\epsilon-1}{\epsilon}$ . So we can write this as:

$$0 = \tilde{mc}_t - \frac{\psi}{\epsilon - 1} \tilde{\pi}_t + \beta \frac{\psi}{\epsilon - 1} E_t \tilde{\pi}_{t+1}$$

Or, re-arranging terms:

$$\tilde{\pi}_t = \frac{\epsilon - 1}{\psi} \tilde{mc}_t + \beta E_t \tilde{\pi}_{t+1} \quad (127)$$

If  $\psi = (\epsilon - 1) \frac{\phi}{(1-\phi)(1-\phi\beta)}$ , then this Phillips Curve is identical to what we had in the Calvo model.

The aggregate resource constraint comes out to:

$$Y_t = C_t + \frac{\psi}{2} \pi_t^2 Y_t \quad (128)$$

When you linearize this, you get:

$$\begin{aligned} \ln Y_t &= \ln \left[ C_t + \frac{\psi}{2} \pi_t^2 Y_t \right] \\ \tilde{Y}_t &= \frac{dC_t}{Y} + \frac{\psi}{2} \pi_t^2 \tilde{Y}_t + 2\psi\pi\tilde{\pi}_t \end{aligned}$$

If we are linearizing about the point  $\pi = 0$ , then the latter terms drop out, and  $C = Y$ , leaving:

$$\tilde{Y}_t = \tilde{C}_t \quad (129)$$

The rest of the equilibrium conditions of the model (Euler equation, labor supply condition,

money demand condition, and policy rule) are all the same.