Micro-Foundations in Growth Models

Session 4: Diamond's overlapping generations (OLG) Model

ECO5021F: Macroeconomics
University of Cape Town

Readings for this week

Required:

- Romer, D. (2019). Advanced Macroeconomics. McGraw-Hill. (5th edition). **Chapter 2.8 - 2.12**.

The Diamond OLG Model

The Diamond OLG Model

- The RCK model and the OLG model have been hugely influential in modern macroeconomics.
- The RCK model is a special **continuous-time** version of the OLG model.
- OLG models, as the name suggests, are inhabited by more than one generation who interact with each other.
- Over time, **new generations replace the older ones**, so that each generation deals with another generation during their lifetime.

The Diamond OLG Model

Features of the model

1. The model facilitates an analysis of the macroeconomic implications of lifetime consumption-savings decisions
2. The model provides a convenient framework for studying the **factors underlying the accumulation of capital**
3. The effect of government policy can easily be traced in OLGs: we are able to **examine the generational incidence of taxation**
4. **Ricardian issues** can be analysed by a straightforward extension to bequests
5. In this model the government outlives any specific generation, breaking the link between the time horizon of government and consumers.
6. The model may be realistic in the sense that the market outcome is not necessarily the social planner's solution.
7. In contrast with the RCK model, **the decentralised equilibrium of the OLG need not be Pareto efficient**

The Diamond OLG Model

Assumptions: the environment

- **Time:** OLG model will be derived in discrete time
- **Population:** Inhabited by individuals that all live for precisely two periods (turnover of individuals in the economy).
- Population grows at rate of n per period:

$$L_t = (1 + n)L_{t-1}$$

- There are L_t young individuals and L_{t-1} old individuals in the economy at any given period.
- **Labour:** Every individual works when young (supplies one unit of labour).
- Income of labour is used to finance private consumption in *both periods* of the individual's life.
- Every individual saves a proportion of income when young and **consumes accumulated wealth when old**.

The Diamond OLG Model

Assumptions: firms and factor costs

- Several competitive firms in the economy produce output according to:

$$Y = F(K_t, A_t L_t)$$

- Production function exhibits **constant returns to scale** and satisfies the **Inada conditions**
- Technology, A, is labour-augmenting and grows at rate of g:

$$A_t = (1+g)A_{t-1}$$

- Competitive market forces ensure firms pay marginal products for all factors of production, which means capital receives a real rental payment of r_t and labour a real wage of w_t:

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= f(k_t) - f'(k_t) \cdot k_t \end{aligned}$$

The Diamond OLG Model

Assumptions: firms and factor costs

Intergenerational dynamics:

- In the first period there is capital stock of K_0
- **Capital is owned equally by all old individuals**
- **Labour is supplied by young**; combines with K to produce output
- Each member of the young generation of every period **saves a portion of its labour income** $(A_t w_t)$ which forms the capital stock of the next period

  (where $w_t = W_t/A_t$ is the wage *per effective worker*)
- Each member of the old generation consume their capital income and their existing wealth

  (i.e., savings from their youth including the rate of return on capital)
- Investment earns a return equal to the rate of return on capital

The Diamond OLG Model

Households

- We need to analyse the situation of the household by considering **the objective function and the constraints on this optimisation**
- The utility function for each individual born in period t:

$$U_t = \frac{(c_t^y)^{1-\theta}}{1-\theta} + \left(\frac{1}{1+\rho}\right) \times \frac{(c_{t+1}^o)^{1-\theta}}{1-\theta}, \quad \theta > 0, \rho > -1$$

- where c_t^y is consumption per effective worker when young, while c_{t+1}^o when old.
- The budget constraint of an individual born at t is given by

$$c_t^y + \frac{1}{1+r_{t+1}} \cdot c_{t+1}^o = A_t w_t$$

- This states that PV lifetime consumption equals lifetime income for every generation * exercise

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Intertemporal optimisation problem

- We need to construct the LaGrangian function and find the first order conditions with respect to c_t^y and c_{t+1}^o

$$\mathcal{L} = \frac{\left(c_t^y\right)^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \cdot \frac{\left(c_{t+1}^o\right)^{1-\theta}}{1-\theta} + \lambda \left[A_t w_t - c_t^y - \frac{1}{1+r_{t+1}} \cdot c_{t+1}^o\right]$$

- The first order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t^y} &= \left(c_t^y\right)^{-\theta} + \lambda(-1) = 0 \\
&\implies \left(c_t^y\right)^{-\theta} = \lambda \\
\frac{\partial \mathcal{L}}{\partial c_{t+1}^o} &= \frac{1}{1+\rho} \cdot \left(c_{t+1}^o\right)^{-\theta} + \lambda \left[-\frac{1}{1+r_{t+1}}\right] = 0 \\
&\implies \frac{\left(c_{t+1}^o\right)^{-\theta}}{1+\rho} = \frac{\lambda}{1+r_{t+1}}
\end{aligned}$$

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Intertemporal optimisation problem

- By combining the two first order equations we get an equation similar to the **Euler equation** from the RCK model:

$$\frac{c^o_{t+1}}{c^y_t} = \left(\frac{1 + r_{t+1}}{1 + \rho}\right)^{\frac{1}{\theta}} \tag{1}$$

- From this equation we can see that consumption is increasing over time if the rate of interest exceeds the subjective discount rate (and vice versa)
- The parameter, θ, determines the willingness of the individual to switch consumption between periods, in response to variations in the interest and discount rate.

The Diamond OLG Model

The savings rate

- We will derive the savings rate in a two-step procedure:
- First, we find an expression for consumption by young individuals born in period t, in terms of wage income and the parameters of the model
- Then we use this expression to calculate a savings rate for young individuals as a function of the parameters of the model (**and particularly, the interest rate**).
- We have $c_t^y = A_t w_t - \frac{1}{1+r_{t+1}} \cdot c_{t+1}^o$ and $c_{t+1}^o = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \cdot c_t^y$
- By substitution,

$$c_t^y = A_t w_t - \frac{1}{1+r_{t+1}} \cdot \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \cdot c_t^y$$

The Diamond OLG Model

The savings rate

$$c_t^y = A_t w_t - \frac{1}{1+r_{t+1}} \cdot \left(\frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}} \cdot c_t^y$$

We can simplify this further,

$$c_t^y \left[1 + \frac{1}{1+r_{t+1}} \cdot \left(\frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}} \right] = A_t w_t$$

$$c_t^y \left[1 + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} \right] = A_t w_t$$

$$c_t^y \left[\frac{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} \right] = A_t w_t$$

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The savings rate

$$\begin{aligned}
s\left(r_{t+1}\right) &= \frac{A_t w_t - c_t^y}{A_t w_t} \\
&= \frac{A_t w_t - \left[\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r_{t+1})^{\frac{1-\theta}{\theta}}}\right]\cdot A_t w_t}{A_t w_t} \\
&= 1-\left[\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r_{t+1})^{\frac{1-\theta}{\theta}}}\right] \\
&= \frac{(1+\rho)^{\frac{1}{\theta}}+(1+r_{t+1})^{\frac{1-\theta}{\theta}}-(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r_{t+1})^{\frac{1-\theta}{\theta}}} \\
&= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r_{t+1})^{\frac{1-\theta}{\theta}}}
\end{aligned}$$

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The savings rate

$$s(r_{t+1}) = \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \tag{2}$$

- This equation tells us that the rate of savings is a function of the interest rate
- Savings will be an increasing function of the interest rate when $\theta < 1$.
- However, when $\theta > 1$, the rate of savings is decreasing function of the interest rate
- The relationship between $s(r)$ and r is best explained by the opposing income and substitution effects of interest on savings

* exercise

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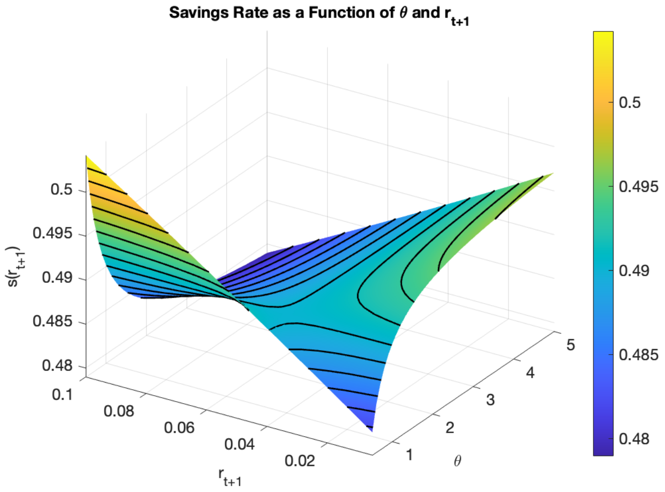
The savings rate



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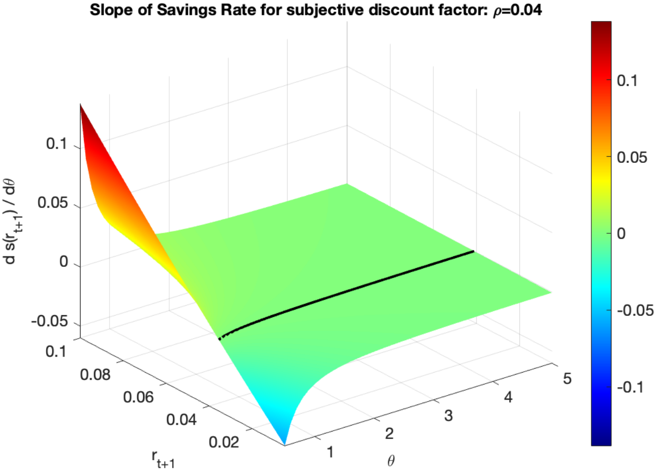
The savings rate



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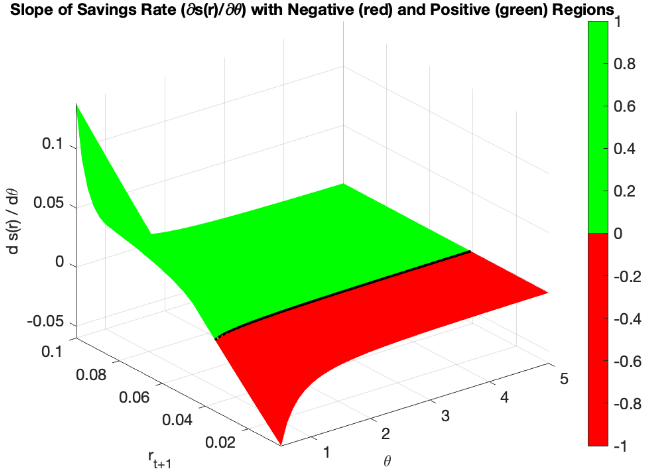
The savings rate



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Income and Substitution Effect

- **Income effect:** When interest rate rises, capital earns a greater return, and so does savings of the individual
  - Individual can achieve a comparable amount of consumption when old by saving less as a young person; leaving more for consumption when young
- **Substitution effect:** A rise in the interest rate makes the trade-off between first and second period consumption more favourable for second period consumption

$$\frac{c^o_{t+1}}{c^y_t} = \left(\frac{1 + r_{t+1}}{1 + \rho}\right)^{\frac{1}{\theta}}$$

- Low θ (e.g. 0.5), **substitution effect** dominates: HH is willing to shift consumption across time; reacts strongly to interest rate changes.
- High θ (e.g. 5), **income effect** dominates: HH resists shifting consumption, prefers smoothing; small adjustment in savings.

The Diamond OLG Model

The equation of motion of k (capital accumulation)

The savings rate (2) we derived allows us to proceed with the analysis of capital accumulation in the OLG model.

- Capital stock in any period is determined by the savings of the young generation in the previous period:* $K_{t+1} = s\left(r_{t+1}\right) A_t w_t L_t$
- Dividing both sides by $A_{t+1}L_{t+1}$, gives us:

$$\begin{aligned} \frac{K_{t+1}}{A_{t+1}L_{t+1}} &= \frac{A_t L_t}{A_{t+1}L_{t+1}} s\left(r_{t+1}\right) w_t \\ k_{t+1} &= \frac{1}{(1+n)(1+g)} s\left(r_{t+1}\right) \overbrace{\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]}^{w_t} \end{aligned} \tag{3}$$

- This is the general solution for the implicit evolution of capital per effective worker.
- We make *two assumptions to ensure global stability of the equilibrium* . . .

The Diamond OLG Model

Savings Rate Under Log Utility

1. The **utility function is logarithmic**: $u(c) = \ln c$, implying $\theta = 1$

- Euler equation (1) simplifies to:

$$\frac{c^o_{t+1}}{c^y_t} = \frac{1 + r_{t+1}}{1 + \rho}$$

  where the intertemporal elasticity of substitution ($1/\theta$) is 1

- HH responds to interest rate changes with **exact offsetting income and substitution effects**
- This implies a constant savings rate (2):

$$s = \frac{1}{2 + \rho}$$

- **This savings rate is independent of the interest rate!**

2. The **production function is Cobb-Douglas**: $f(k) = k^\alpha$

- This means the dynamic equation for capital accumulation takes the following form:

$$\begin{aligned} k_{t+1} &= \frac{1}{(1+n)(1+g)} s\left(r_{t+1}\right)\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right] \\ &= \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha)k_t^\alpha \\ &= Dk_t^\alpha \end{aligned}$$

This is a **first-order nonlinear difference equation** with **globally stable dynamics** because:

- $D > 0,$ where $D = (1-\alpha)/(1+n)(1+g)(2+\rho)$
- $0 < \alpha < 1$, so k^α is concave
- A unique steady state exists, and the system **monotonically converges** toward it

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Capital Accumulation

$$k_{t+1} = \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha)k_t^{\alpha}$$

- According to the equation above, k_{t+1} is an increasing function of k_t (but at a decreasing rate, WHY?)
- We plot this functional relationship on a graph in (k_t, k_{t+1}) space
- $45°$ line is the locus where $\dot{k} = 0$
- Assume economy is at k_0, which is greater than k^*, and where $k_t > k_{t+1}$.
- This means that k_1 in the next period will be smaller than k_0
- Erosion of k will continue until the economy reaches k^*

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Capital Accumulation

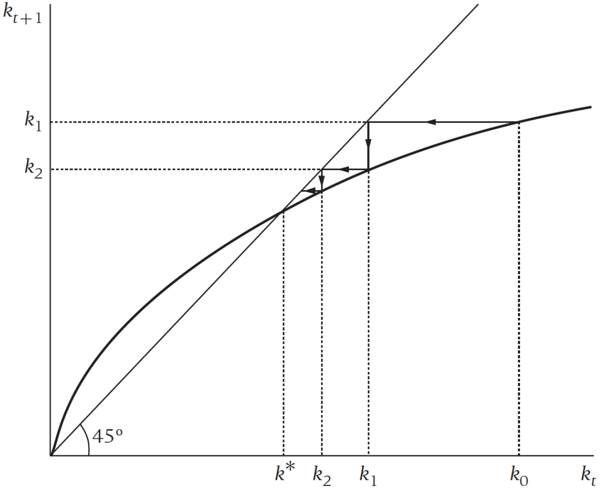

FIGURE 2.10 The dynamics of k

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Fall in the Discount Rate

- Assume economy is at the balanced growth path at k^* and discount rate falls from ρ to ρ'; where $\rho > \rho'$
- This change in ρ *increases the slope* of the function k_{t+1}

  i.e., the young *save a greater fraction of their labour income*
- New balanced growth path is to the right of the previous BGP, i.e. $k^{*\prime} > k^*$ k rises monotonically from old value of k^*, to the new one (where $k_t > k_{t+1}$)
- Similar result to that of Solow and RCK models.

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Fall in the Discount Rate

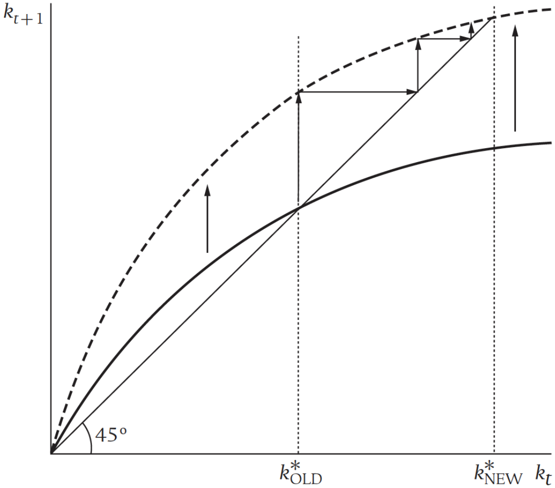


FIGURE 2.11 The effects of a fall in the discount rate

The Diamond OLG Model

Dynamic (In)Efficiency

- A significant difference between the Solow and RCK models was that the BGP in the RCK model was always a social optimum.
- The OLG model **does not** guarantee a socially optimal solution
- Reason for this is that the utility of the unborn generations are not properly discounted in the market at time t, when the living make their optimising decision
- A Pareto efficient result, according to Romer, requires competitive markets, absence of externalities and a **large, but finite, number of agents**.
- In the OLG model, the last condition is violated
- The constantly growing population offers the opportunity to a social planner to **raise the consumption of all generations** through inter-generational transfer
- What does this mean?

The Diamond OLG Model

Dynamic (In)Efficiency

- A social planner with infinite planning horizon will be in a position to improve on the decentralised equilibrium
  - **Decentralised (competitive) equilibrium** in this model is **not Pareto efficient**
- This is because the **BGP may exceed the golden rule** level of capital stock in the OLG model
- Analytically, using log-utlity and $g = 0$:

$$
\begin{aligned}
k^* &= \frac{1}{(1+n)(2+\rho)}(1-\alpha)k^{*\alpha} \\
k^{*1-\alpha} &= \frac{1}{(1+n)(2+\rho)}(1-\alpha) \\
k^* &= \left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{\frac{1}{1-\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
k^* &= \left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{\frac{1}{1-\alpha}} \\
f'(k^*) &= \alpha k^{*\alpha-1} = \alpha\left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{\frac{\alpha-1}{1-\alpha}} \\
&= \alpha\left[\frac{(1+n)(2+\rho)}{(1-\alpha)}\right] \\
&= \frac{\alpha}{(1-\alpha)}(1+n)(2+\rho)
\end{aligned}
$$

- With $g = 0$, the golden rule capital stock satisfies: $f'(k_{\mathrm{GR}}) = n$ [Recall: $c = f(k) - nk \Rightarrow$ set: $\partial c/\partial k = 0$]
- The value of k^* could be larger or smaller than k_{GR}

The Diamond OLG Model

Dynamic (in)efficiency

$$f'\left(k^{*}\right)=\frac{\alpha}{(1-\alpha)}(1+n)(2+\rho)$$

- For α sufficiently small, $f'(k^{*})<f'\left(k_{\mathrm{GR}}\right)$, the capital stock on the balanced growth path exceeds the golden-rule level: $k^{*}>k_{\mathrm{GR}}$
- Suppose we are in a position where k^{*} exceeds k_{GR}, then the social planner will arrange for **more resources to be devoted to consumption from the initial period onwards**
- In other words, the social planner will arrange for a **permanent lowering of the savings rate**, which means greater consumption in every subsequent period, and a fall in k
- In fact, the social planner could arrange for k to fall until it **reached the golden rate capital stock**

The Diamond OLG Model

Efficiency (when $k^* > k_{\mathrm{GR}}$)

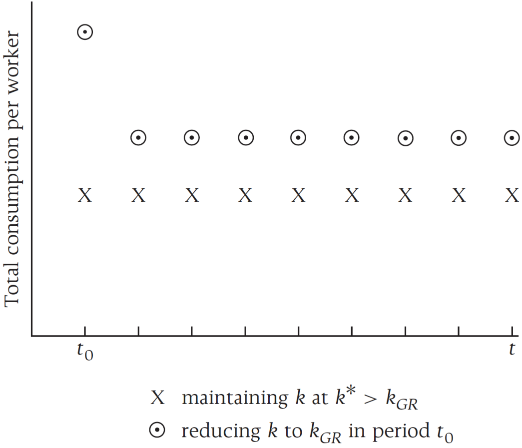
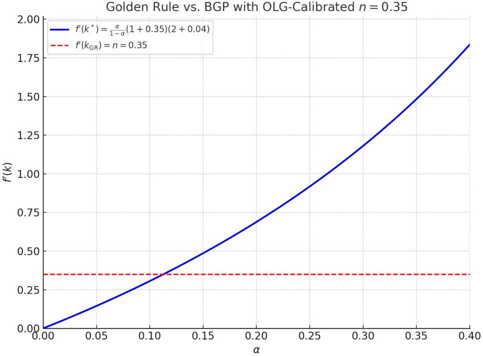


FIGURE 2.13 How reducing k to the golden-rule level affects the path of consumption per worker

Over- or under-accumulation?



- The red line intersects the blue curve at a very low value: $\alpha^{\mathrm{GR}} = 0.11$ (for given values of $n = 0.35$* and $\rho = 0.04$).
- For most realistic values of $\alpha \in (0.2, 0.4)$, we have: $f'(k^*) > f'(k_{\mathrm{GR}})$, which implies: $k^* < k_{\mathrm{GR}}$

$\Rightarrow$ economies are typically **under-accumulating** capital.

Key Takeaways from the Diamond OLG Model

- The Diamond OLG model introduces **generational turnover** and shows how individual savings behaviour impacts long-run capital accumulation.
- **Log utility** ($\theta = 1$) delivers a constant savings rate:

  $$s = \frac{1}{2+\rho}$$

  independent of the interest rate, due to offsetting income and substitution effects.
- The model highlights a potential for **dynamic inefficiency**:

  $$k^* \gtrless k_{\mathrm{GR}} \quad \text{depending on the value of } \alpha$$

  In practice, we typically find $k^* < k_{\mathrm{GR}}$: economies **under-accumulate** capital.
- Unlike the RCK model, **the decentralised equilibrium may not be Pareto efficient**, opening the door for **policy intervention** (e.g. pay-as-you-go pensions, intergenerational transfers).