Ramsey-Cass-Koopmans Model

Contents

These notes are the bare bones of the derivations for the equations of motion of the Ramsey-Cass-Koopmans and are thus merely an aid, not a replacement for the text in Romer.

1 Continuous time results and conventions

The model is set up in continuous time, as it is simpler to derive the implications. Romer uses the notation $c(t)$ for the variable c at instant t. This is the standard convention in most texts using continuous time. However, to avoid confusion with multiplication and parentheses, in this note, we use the subscript notation. Thus, c_t now stands for variable c at instant t

2 Technology

Production technology:
Output is produced with capital and labour, with labour augmenting technology:

$$Y_t = F\left(K_t, A_t L_t\right)$$

Labour augmenting technology grows at rate g :

$$\frac{\dot{A}_t}{A_t} = g$$
$$A_t = A_0 e^{gt}$$

In intensive form:

$$y_t = f\left(k_t\right)$$

3 Firms

There are an infinite number of perfectly competitive firms that use the same production technology. Equilibrium thus dictates that each factor of production is paid its marginal product.

3.1 Factor Payments to Capital

Marginal product of capital: e.g. the Cobb-Douglass case:

$$
\begin{aligned}
Y_t &= K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha} \\
\frac{\partial Y_t}{\partial K_t} &= \alpha K_t^{\alpha-1} \left(A_t L_t\right)^{1-\alpha} \\
&= \frac{\alpha K_t^{\alpha-1}}{\left(A_t L_t\right)^{\alpha-1}} \\
&= \alpha k_t^{\alpha-1}
\end{aligned}
$$

Note that this is identical to the marginal product of capital per effective worker (using the intensive form):

$$
\begin{aligned}
y_t &= k_t^{\alpha} \\
\frac{\partial y_t}{\partial k_t} &= \alpha k_t^{\alpha-1}
\end{aligned}
$$

It is worthwhile spending some time exploring this result. It is related to the following: a function that is homogenous of degree one in all arguments has partial derivatives that are homogenous of degree zero.

So we can simply work with the intensive form directly:

$$
\frac{\partial y_t}{\partial k_t} \equiv f'\left(k_t\right)
$$

Usually, the real return on a unit of capital would be given by:

$$
r_t = f'\left(k_t\right) - \delta
$$

But since we assume no depreciation, we obtain:

$$
r_t = f'\left(k_t\right)
$$

What would the cumulative return on a unit of capital held for interval of length t be given that it may vary at every instant?

$$
R_t = \int_0^t r_\tau d\tau
$$

Now, what is the marginal impact of changing t ?

The derivative rule that applies here is called the Leibniz Integral Rule:

$$
\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,y)dy\right) = f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx} + \left(\int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}dy\right)
$$

In our case the application of this rule is very simple: neither the integrand (r_τ) nor the lower bound on integration depend on t. Thus $\frac{\partial f(x,y)}{\partial x} = \frac{da(x)}{dx} = 0$.

$$\frac{\partial R_t}{\partial t} = r_t$$

One can also directly reason that this must be the case: R_t is just the accumulated value of all r values between instant 0 and instant t. The marginal contribution to R_t at instant t is simply the value of r at instant $t : r_t$.

3.2 Evolution of Capital

Even though we have not yet considered the household situation, we can describe how capital per effective worker will evolve for any given choice of consumption per effective worker which we denote c_t, since households make only one decision: how much of income per effective worker $f(k_t)$ to consume. The rest is saved by investing in capital.

Thus the addition to capital every instant is $f(k_t) - c_t$.
Without depreciation the amount of capital per effective worker for a given stock k_t decreases at rate $(n+g)$.

Thus we conclude:

$$\dot{k}_t = f(k_t) - c_t - (n+g)k_t$$

3.3 Factor payments to Labour

Each worker receives wage W_t per instant of time. If we define $w_t = \frac{W(t)}{A(t)}$ as the wage per effective worker, each worker receives $A_t w_t$

Effective labour receives its marginal product:

$$\begin{aligned}
w_t &= \frac{\partial Y_t}{\partial A_t L_t} \\
&= (1-\alpha) K_t^\alpha (A_t L_t)^{-\alpha} \\
&= K_t^\alpha (A_t L_t)^{-\alpha} - \alpha K_t^\alpha (A_t L_t)^{-\alpha} \\
&= \left(\frac{K_t}{A_t L_t}\right)^\alpha - \alpha K_t^\alpha (A_t L_t)^{-\alpha} \frac{A_t L_t}{K_t} \frac{K_t}{A_t L_t} \\
&= \left(\frac{K_t}{A_t L_t}\right)^\alpha - \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} \frac{K_t}{A_t L_t} \\
&= k_t^\alpha - \alpha k_t^{\alpha-1} k_t \\
&= f(k_t) - f'(k_t) k_t
\end{aligned}$$

4 Households

4.1 Objective function

Households

- live forever
- discount the future at rate $\rho > 0$. That is: a utility obtained at a moment that is an interval of length t in the future is valued $e^{-\rho t}$ less than the same utility obtained right now. This is simply the continuous time version of standard discounting in discrete time.

The size of the households is L_t (Romer uses $\frac{L_t}{H}$ but since H never serves any substantive purpose we normalize it to 1), which grows at rate n so that if in the 'initial period' the total household size was L_0 we have:

$$\frac{\dot{L}_t}{L_t} = n$$
$$L_t = L_0 e^{nt}$$

Each individual in the household consumes C_t at instant t and this yields instantaneous utility $u(C_t)$ to that individual. Thus the total utility that the household obtains each instant is $u(C_t) L_t$.

The present discounted value of the cumulative utility that it will experience from consumption in each instant over its infinite lifetime is thus:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C_t) L_t dt$$

For the model to have a well defined steady state, the utility function must be of the constant-elasticity of substitution form:

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}$$

4.2 Budget Constraint:

The household is endowed with an initial level of capital K_0 and supplies all its capital and labour to the firms, so earns the marginal return on capital and labour as derived above.

The presented discounted value of consumption must be no larger than the total resources the household will earn in its infinite lifetime:

$$\int_{t=0}^{\infty} e^{-R_t} C_t L_t dt \leq K_0 + \int_{t=0}^{\infty} e^{-R_t} W_t L_t dt$$

Note that the discount factor is the cumulative return on capital, since this is the only asset in this economy that households can use to save with.

This becomes clearer when we rewrite the budget constraint a little:

$$K_0 + \int_{t=0}^{\infty} e^{-R_t} \left(W_t - C_t\right) L_t dt \geq 0$$

where $\left(W_t - C_t\right) L_t$ is the instantaneous level of (dis)saving by the household, which is equivalent to (dis)investments in capital. Thus we can express the level of capital of the household at instant s as:

$$K_s = e^{R_s} K_0 + \int_{t=0}^{s} e^{(R_s - R_t)} \left(W_t - C_t\right) L_t dt$$

Where $e^{R_s} K_0$ is the instant s value of initial capital, and $\left(W_t - C_t\right) L_t$ the instant by instant additions to (or subtractions from) capital which accumulates returns between instant t and s of $e^{(R_s - R_t)}$.

This means

$$e^{-R_s} K_s = K_0 + \int_{t=0}^{s} e^{-R_t} \left(W_t - C_t\right) L_t dt$$

Taking the limit as s approaches ∞ we obtain the budget constraint as above:

$$\lim_{s \to \infty} \left[e^{-R_s} K_s\right] = \lim_{s \to \infty} \left[K_0 + \int_{t=0}^{s} e^{-R_t} \left(W_t - C_t\right) L_t dt\right] \geq 0$$

The condition: $\lim_{s \to \infty} \left[e^{-R_s} K_s\right] \geq 0$ is called a no-Ponzi-game condition. Households cannot end their lives with negative wealth.

4.3 Rewriting in terms of effective labour

Consumption per worker and wage per worker are, respectively, C_t and W_t. We define the "per effective worker versions of these as:

$$c_t = \frac{C_t}{A_t}$$
$$\therefore C_t = A_t c_t$$
$$w_t = \frac{W_t}{A_t}$$
$$\therefore W_t = A_t w_t$$

Thus, using the results that $A_t = A_0 e^{gt}$ and $L_t = L_0 e^{nt}$ we can rewrite the objective and constraint in terms of the variables per effective worker:

First, the objective:

$$
\begin{aligned}
U &= \int_{t=0}^{\infty} e^{-\rho t} u\left(C_t\right) L_t dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\theta}}{1-\theta} L_t dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \frac{\left(A_t c_t\right)^{1-\theta}}{1-\theta} L_t dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \frac{\left(A_0 e^{gt} c_t\right)^{1-\theta}}{1-\theta} L_0 e^{nt} dt \\
&= A_0^{1-\theta} L_0 \int_{t=0}^{\infty} e^{-(\rho-n-(1-\theta)g)t} \frac{c_t^{1-\theta}}{1-\theta} dt \\
&= \int_{t=0}^{\infty} e^{-(\rho-n-(1-\theta)g)t} \frac{c_t^{1-\theta}}{1-\theta} dt
\end{aligned}
$$

Since the constant term $A_0^{1-\theta} L_0$ does not affect behaviour, we normalize it to 1

Next, the constraint:

$$
\begin{aligned}
\int_{t=0}^{\infty} e^{-R_t} C_t L_t dt &\leq K_0 + \int_{t=0}^{\infty} e^{-R_t} W_t L_t dt \\
\int_{t=0}^{\infty} e^{-R_t} A_t L_t c_t dt &\leq K_0 + \int_{t=0}^{\infty} e^{-R_t} A_t L_t w_t dt \\
\int_{t=0}^{\infty} e^{-R_t} A_0 e^{gt} L_0 e^{nt} c_t dt &\leq K_0 + \int_{t=0}^{\infty} e^{-R_t} A_0 e^{gt} L_0 e^{nt} w_t dt \\
\int_{t=0}^{\infty} e^{-R_t+(n+g)t} c_t dt &\leq k_0 + \int_{t=0}^{\infty} e^{-R_t+(n+g)t} w_t dt
\end{aligned}
$$

where the last step collects terms and divides by $A_0 L_0$ and uses $k_0 = \frac{K_0}{A_0 L_0}$.

4.4 Optimization:

Since we are able write everything in terms of a single instant 0 objective in constraint, the Lagrangian for this problem is:

$$
\begin{aligned}
\mathcal{L} = &\int_{t=0}^{\infty} e^{-(\rho-n-(1-\theta)g)t} \frac{c_t^{1-\theta}}{1-\theta} dt + \\
&\lambda \left[k_0 + \int_{t=0}^{\infty} e^{-R_t+(n+g)t} \left(w_t - c_t\right) dt \right]
\end{aligned}
$$

The first order condition for optimality is:

$$
\frac{\partial \mathcal{L}}{\partial c_t} = 0
$$

$$
\hat{\Downarrow}
$$

$$
e^{-(\rho-n-(1-\theta)g)t} c_t^{-\theta} = \lambda e^{-R_t+(n+g)t}
$$

To express this in terms of the growth rate of c_t, we take natural logarithms and time derivatives on both sides:

1. Logarithms:

$$\begin{aligned} -(\rho - n - (1-\theta)g)t - \theta \ln[c_t] &= \ln[\lambda] - R_t + (n+g)t \\ &= \ln[\lambda] - \int_0^t r_\tau d\tau + (n+g)t \end{aligned}$$

2. Time derivatives:

$$\begin{aligned} -(\rho - n - (1-\theta)g) - \theta \frac{\dot{c}_t}{c_t} &= -r_t + (n+g) \\ \frac{\dot{c}_t}{c_t} &= \frac{f'(k_t) - \rho - \theta g}{\theta} \end{aligned}$$

Where the last step rearranges and explicitly uses the return on capital.