A Baseline DSGE Model

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1 Introduction

In this online appendix, we present the canonical DSGE model that we estimate in Section 6 of "Estimating DSGE Models: Recent Advances and Future Challenges" in more detail. The basic structure of the economy is as follows. We have a representative household that consumes, saves, holds money, and supplies labor. A final good firm produces output using a continuum of intermediate good producers. These intermediate goods, in turn, are produced by monopolistic competitors by renting capital and labor from the household. The intermediate good producers are subject to nominal price rigidities à la Calvo. The representative household is the owner of all of these firms. The model is closed with a government that sets up monetary and fiscal policy. Since there are trends in the data, we introduce two unit roots, one in the level of neutral technology and one in the investment-specific technology, that induce stochastic long-run growth.

1.1 Representative household

The representative household has a lifetime utility function on consumption, c_t, real money balances, m_t/p_t (where p_t is the price level), and hours worked, l_t:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t d_t \left\{ \log \left(c_t - h c_{t-1} \right) + \upsilon \log \left(\frac{m_t}{p_t} \right) - \varphi_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\}, \tag{1}$$

where β is the discount factor, h controls habit persistence, and ϑ is the inverse of the Frisch labor supply elasticity. We pick a utility function (log utility in consumption) whose marginal relation of substitution between consumption and leisure is linear in consumption to ensure the presence of a balanced growth path (BGP) with constant hours.

The intertemporal preference shock d_t follows a law of motion

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t},$$

where $\varepsilon_{d,t} \sim \mathcal{N}(0,1)$. The labor supply shock φ_t follows a law of motion

$$\log \varphi_t = \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_{\varphi,t},$$

where $\varepsilon_{\varphi,t} \sim \mathcal{N}(0,1)$. We should understand both shocks as a stand-in for more complex mechanisms, such as financial frictions, demographic shifts, or changes in risk attitudes.

The household's budget constraint is:

$$\begin{aligned} c_t + x_t + \frac{m_t}{p_t} + \frac{b_{t+1}}{p_t} + \int q_{t+1,t} a_{t+1} d\omega_{t+1,t} \\ = w_t l_t + \left(r_t u_t - \mu_t^{-1} \Phi\left[u_t\right] \right) k_{t-1} + \frac{m_{t-1}}{p_t} + R_{t-1} \frac{b_t}{p_t} + a_t + T_t + F_t. \end{aligned}$$

In terms of uses –the left-hand side of equation (1.1)– and beyond consumption, the household can save in physical capital, by investing x_t in new capital, holding real money balances, purchasing government debt, b_t, and trading in Arrow securities. More concretely, a_{t+1} is the amount of those securities that pays one unit of consumption in event $\omega_{t+1,t}$ purchased at time t at (real) price $q_{t+1,t}$. This price will be such that, in equilibrium, the net supply of the Arrow securities would be zero and, from the next equation on, we drop the terms associated with these securities.

In terms of resources –the right-hand side of equation (1.1)– the household gets income by renting its labor supply at the real wage w_t and its capital at real rental price r_t. The household chooses the utilization rate of capital, $u_t > 0$, given the depreciation cost $\mu_t^{-1}\Phi\left[u_t\right]$, where μ_t is an investment-specific technological shock that we will introduce below. We interpret $u_t = 1$ as the "normal" utilization rate. The household also has access to its money balances, the government debt (with a nominal gross interest rate of R_t), the Arrow security that pays in the actually realized event, the lump-sum transfers (of taxes if negative) from the government, T_t, and the profits of the economy's firms, F_t.

Capital follows $k_t = \left(1 - \delta\Phi\left[u_t\right]\right) k_{t-1} + \mu_t \left(1 - S\left[\frac{x_t}{x_{t-1}}\right]\right) x_t$, where δ is the depreciation rate when $u_t = 1$ and $S\left[\cdot\right]$ is an adjustment cost function. We assume that $S\left[\Lambda_x\right] = 0$,

$S'[\Lambda_x] = 0$, and $S''[\cdot] > 0$, where Λ_x is the growth rate of investment along the BGP.

The investment-specific technological shock follows a unit-root process

$$\mu_t = \mu_{t-1} \exp\left(\Lambda_\mu + z_{\mu,t}\right) \text{ where } z_{\mu,t} = \sigma_\mu \varepsilon_{\mu,t} \text{ and } \varepsilon_{\mu,t} \sim \mathcal{N}(0,1).$$

The value of μ_t is also the inverse of the relative price of new capital in consumption terms.

The Lagrangian function associated with the household's problem is:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} d_t \left\{ \log\left(c_t - hc_{t-1}\right) + \upsilon \log\left(\frac{m_t}{p_t}\right) - \varphi_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \\ -\lambda_t \left\{ \begin{array}{c} c_t + x_t + \frac{m_t}{p_t} + \frac{b_t}{p_t} \\ -w_t l_t - \left(r_t u_t - \mu_t^{-1} \Phi\left[u_t\right]\right) k_{t-1} - \frac{m_{t-1}}{p_t} - R_{t-1} \frac{b_{t-1}}{p_t} - T_t - \digamma_t \end{array} \right\} \\ -Q_t \left\{ k_t - (1-\delta) k_{t-1} - \mu_t \left(1 - S\left[\frac{x_t}{x_{t-1}}\right]\right) x_t \right\} \end{array} \right]$$

where the household maximizes over c_t, l_t, b_t, u_t, u_t, and x_t, λ_t is the Lagrangian multiplier associated with the budget constraint and Q_t the Lagrangian multiplier associated with installed capital. Also, define the (marginal) Tobin's Q as $q_t = \frac{Q_t}{\lambda_t}$, (loosely the value of installed capital in terms of its replacement cost)

The first-order conditions with respect to c_t, l_t, b_t, u_t, and x_t are:

$$d_t \left(c_t - hc_{t-1}\right)^{-1} - h\beta \mathbb{E}_t d_{t+1} \left(c_{t+1} - hc_t\right)^{-1} = \lambda_t$$

$$\lambda_t = \beta \mathbb{E}_t \left\{ \lambda_{t+1} \frac{R_t}{\Pi_{t+1}} \right\}$$

$$r_t = \mu_t^{-1} \Phi'\left[u_t\right]$$

$$q_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \left((1-\delta) q_{t+1} + r_{t+1} u_{t+1} - \mu_{t+1}^{-1} \Phi\left[u_{t+1}\right] \right) \right\}$$

$$1 = q_t \mu_t \left(1 - S\left[\frac{x_t}{x_{t-1}}\right] - S'\left[\frac{x_t}{x_{t-1}}\right] \frac{x_t}{x_{t-1}}\right) + \beta \mathbb{E}_t q_{t+1} \mu_{t+1} \frac{\lambda_{t+1}}{\lambda_t} S'\left[\frac{x_{t+1}}{x_t}\right] \left(\frac{x_{t+1}}{x_t}\right)^2.$$

The last equation is important. If $S[\cdot] = 0$ (i.e., there are no adjustment costs), we get $q_t = \frac{1}{\mu_t}$, i.e., the marginal Tobin's Q is equal to the replacement cost of capital (the relative price of capital). Furthermore, if $\mu_t = 1$, as in the standard neoclassical growth model, $q_t = 1$.

1.2 The final good producer

There is one final good is produced using intermediate goods with the production function:

$$y_t^d = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{2}$$

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (2), taking as given all intermediate goods prices p_{ti} and the final good price p_t. As a consequence their maximization problem is:

$$\max_{y_{it}} p_t y_t^d - \int_0^1 p_{it} y_{it} di.$$

The input demand functions associated with this problem are, for all i, $y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t^d$, where y_t^d is the aggregate demand. If we combine this input demand functions with the zero profit condition $p_t y_t^d = \int_0^1 p_{it} y_{it} di$, we get:

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}.$$

1.3 Intermediate good producers

Each intermediate good producer i has a technology $y_{it} = A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha}$, where k_{it-1} is the capital rented by the firm, l_{it}^d is the labor input rented, and A_t is the technology level that evolves as a unit-root process $A_t = A_{t-1} \exp\left(\Lambda_A + z_{A,t}\right)$, where $z_{A,t} = \sigma_A \varepsilon_{A,t}$ and $\varepsilon_{A,t} \sim \mathcal{N}(0,1)$.

Also, define $z_t = A_t^{\frac{1}{1-\alpha}} \mu_t^{\frac{\alpha}{1-\alpha}}$, a variable that encodes the joint effect of both technology shocks along the BGP. Simple algebra tells us that $z_t = z_{t-1} \exp\left(\Lambda_z + z_{z,t}\right)$ where $z_{z,t} = \frac{z_{A,t} + \alpha z_{\mu,t}}{1-\alpha}$ and $\Lambda_z = \frac{\Lambda_A + \alpha \Lambda_\mu}{1-\alpha}$.

Intermediate good producers solve a two-stage problem. In the first stage, taken the input prices w_t and r_t as given, firms rent l_{it}^d and k_{it-1} in perfectly competitive factor markets to minimize real cost:

$$\min_{l_{it}^d, k_{it-1}} w_t l_{it}^d + r_t k_{it-1}$$

subject to their supply curve $y_{it} = A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha}$.

The first-order conditions for this problem are:

$$\begin{aligned} w_t &= \varrho \left(1-\alpha\right) A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{-\alpha} \\ r_t &= \varrho \alpha A_t k_{it-1}^{\alpha-1} \left(l_{it}^d\right)^{1-\alpha}, \end{aligned}$$

where ϱ is the Lagrangian multiplier.

The real cost is then:

$$\left(\frac{1}{1-\alpha}\right) w_t l_{it}^d.$$

Given that the firm has constant returns to scale, we can find the real marginal cost mc_t by setting the level of labor and capital equal to the requirements of producing one unit of good $A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = 1$ or:

$$A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} l_{it}^d\right)^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t}\right)^{\alpha} l_{it}^d = 1$$

that implies that:

$$l_{it}^d = \frac{\left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t}\right)^{-\alpha}}{A_t}.$$

Then:

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{w_t^{1-\alpha} r_t^{\alpha}}{A_t}.$$

This marginal cost does not depend on i: all firms receive the same technology shocks and all firms rent inputs at the same price.

In the second stage, intermediate good producers choose the price that maximizes discounted real profits subject to Calvo pricing. In each period, a fraction $1 - \theta_p$ of firms can change their prices. All other firms can only index their prices by past inflation. Indexation is controlled by the parameter $\chi \in [0, 1]$, where $\chi = 0$ is no indexation and $\chi = 1$ is total indexation.

The problem of the firms is then:

$$\max_{p_{it}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} - mc_{t+\tau} \right) y_{it+\tau} \right\}$$

subject to

$$y_{it+\tau} = \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}^d,$$

where the marginal value of a dollar to the household, is treated as exogenous by the firm. Since we have complete markets in securities and the utility function is separable in consumption, $\lambda_{t+\tau}/\lambda_t$ is the correct valuation of future profits.

Substituting the demand curve in the objective function and the previous expression, we get:

$$\max_{p_{it}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left(\left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \frac{p_{it}}{p_t} \right)^{1-\varepsilon} - \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \frac{p_{it}}{p_t} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^d \right\}$$

whose solution p_{it}^* implies the first-order condition:

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left\{ \left((1-\varepsilon) \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{1-\varepsilon} \frac{p_{it}^*}{p_t} + \varepsilon \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^d \right\} = 0,$$

where we have dropped irrelevant constants and used the fact that we are in a symmetric equilibrium.

This expression nests the usual result in the fully flexible prices case $\theta_p = 0$:

$$p_{it}^* = \frac{\varepsilon}{\varepsilon - 1} p_t mc_{t+\tau},$$

i.e., the price is equal to a mark-up over the nominal marginal cost.

Since we only consider a symmetric equilibrium across firms, $p_{it}^* = p_t^*$ and:

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left\{ \left((1-\varepsilon) \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{1-\varepsilon} \frac{p_t^*}{p_t} + \varepsilon \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^d \right\} = 0.$$

To express the previous first-order condition recursively, we define:

$$g_t^1 = \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{-\varepsilon} mc_{t+\tau} y_{t+\tau}^d$$

and

$$g_t^2 = \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{1-\varepsilon} \frac{p_t^*}{p_t} y_{t+\tau}^d$$

and then the first-order condition is $\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$. We need $(\beta\theta_p)^\tau \lambda_{t+\tau}$ to go to zero sufficiently fast in relation to the rate of inflation for g_t^1 and g_t^2 to be well-defined and stationary.

Then, we can write the g's recursively as:

$$g_t^1 = \lambda_t mc_t y_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{-\varepsilon} g_{t+1}^1$$

and

$$g_t^2 = \lambda_t \Pi_t^* y_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*} \right) g_{t+1}^2$$

where $\Pi_t^* = \frac{p_t^*}{p_t}$.

Given Calvo's pricing, the price index evolves:

$$p_t^{1-\varepsilon} = \theta_p \left(\Pi_{t-1}^{\chi} \right)^{1-\varepsilon} p_{t-1}^{1-\varepsilon} + (1-\theta_p) p_t^{*1-\varepsilon}$$

or, dividing by $p_t^{1-\varepsilon}$,

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{1-\varepsilon} + (1-\theta_p)\, \Pi_t^{*1-\varepsilon}.$$

1.4 The government problem

The government determines monetary and fiscal policy. We assume there is no interaction between monetary and fiscal policy, that is, the results of open market operations are distributed in a lump-sum fashion to the household.

In terms of monetary policy, the government sets the nominal interest rate following a Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\gamma_\Pi} \left(\frac{\frac{y_t^d}{y_{t-1}^d}}{\Lambda_{y^d}} \right)^{\gamma_y} \right)^{1-\gamma_R} \exp(m_t)$$

through open market operations. The variable Π is the target level of inflation (equal to inflation in the BGP), R is the BGP nominal gross return of capital (equal to the BGP real gross returns of capital plus Π), and Λ_{y^d} is the BGP growth rate of y_t^d. The term $m_t = \sigma_m \varepsilon_{mt}$ is a random shock to monetary policy, where $\varepsilon_{mt} \sim \mathcal{N}(0,1)$.

Government consumption is given by $g_t = \widetilde{g}_t z_t$, where $\widetilde{g}_t$ follows an autoregressive process:

$$\log \widetilde{g}_t = (1-\rho_g) \log g + \rho_g \log \widetilde{g}_{t-1} + \sigma_g \varepsilon_{g,t} \text{ where } \varepsilon_{g,t} \sim \mathcal{N}(0,1)$$

Since the economy is growing, the level of real government consumption is multiplied by the term z_t to keep it as a stationary share of output. Government consumption is financed by lump sum taxes that ensure that the deficit is zero.

1.5 Aggregation

First, we derive an expression for aggregate demand:

$$y_t^d = c_t + g_t + x_t + \mu_t^{-1} \Phi [u_t] k_{t-1}.$$

With this value, the demand for each intermediate good producer is:

$$y_{it} = \left(c_t + g_t + x_t + \mu_t^{-1} \Phi [u_t] k_{t-1} \right) \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon}$$

and using the production function is:

$$A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = \left(c_t + g_t + x_t + \mu_t^{-1} \Phi\left[u_t\right] k_{t-1}\right) \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}.$$

Since all the firms have the same optimal capital-labor ratio:

$$\frac{k_{it-1}}{l_{it}^d} = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t}$$

and by market clearing

$$\int_0^1 l_{it}^d di = l_t^d$$

and

$$\int_0^1 k_{it-1} di = u_t k_{t-1},$$

it must be the case that:

$$\frac{k_{it-1}}{l_{it}^d} = \frac{u_t k_{t-1}}{l_t^d}.$$

Then:

$$A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = A_t \left(\frac{k_{it-1}}{l_{it}^d}\right)^{\alpha} \left(l_{it}^d\right) = A_t \left(\frac{u_t k_{t-1}}{l_t^d}\right)^{\alpha} l_{it}^d$$

Integrating out

$$\int_0^1 A_t \left(\frac{u_t k_{t-1}}{l_t^d}\right)^{\alpha} l_{it}^d di = A_t \left(\frac{u_t k_{t-1}}{l_t^d}\right)^{\alpha} \int_0^1 l_{it}^d di = A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha}$$

and we have

$$A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi z_t = \left(c_t + g_t + x_t + \mu_t^{-1} \Phi\left[u_t\right] k_{t-1}\right) \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di.$$

Define $v_t^p = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di$. By the properties of the index under Calvo's pricing

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + \left(1-\theta_p\right) \Pi_t^{*-\varepsilon},$$

we get:

$$c_t + g_t + x_t + \mu_t^{-1} \Phi\left[u_t\right] k_{t-1} = \frac{A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha}}{v_t^p}.$$

2 Equilibrium

A definition of equilibrium in this economy is standard and the symmetric equilibrium policy functions are determined by the following equations:

- The first-order conditions of the household:

$$d_t\left(c_t - hc_{t-1}\right)^{-1} - h\beta\mathbb{E}_t d_{t+1}\left(c_{t+1} - hc_t\right)^{-1} = \lambda_t$$
$$\phi_t\psi(l_t^s)^\theta = w_t\lambda_t$$
$$\lambda_t = \beta\mathbb{E}_t\left\{\lambda_{t+1}\frac{R_t}{\Pi_{t+1}}\right\}$$
$$r_t = \mu_t^{-1}\Phi'\left[u_t\right]$$
$$q_t = \beta\mathbb{E}_t\left\{\frac{\lambda_{t+1}}{\lambda_t}\left(\left(1-\delta\right)q_{t+1} + r_{t+1}u_{t+1} - \mu_{t+1}^{-1}\Phi\left[u_{t+1}\right]\right)\right\}$$
$$1 = q_t\mu_t\left(1 - S\left[\frac{x_t}{x_{t-1}}\right] - S'\left[\frac{x_t}{x_{t-1}}\right]\frac{x_t}{x_{t-1}}\right) + \beta\mathbb{E}_t q_{t+1}\mu_{t+1}\frac{\lambda_{t+1}}{\lambda_t}S'\left[\frac{x_{t+1}}{x_t}\right]\left(\frac{x_{t+1}}{x_t}\right)^2$$

- The firms that can change prices set them to satisfy:

$$g_t^1 = \lambda_t mc_t y_t^d + \beta\theta_p\mathbb{E}_t\left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon}g_{t+1}^1$$
$$g_t^2 = \lambda_t\Pi_t^* y_t^d + \beta\theta_p\mathbb{E}_t\left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon}\left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right)g_{t+1}^2$$
$$\varepsilon g_t^1 = (\varepsilon - 1)g_t^2$$

where they rent inputs to satisfy their static minimization problem:

$$\frac{u_t k_{t-1}}{l_t^d} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$
$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{w_t^{1-\alpha}r_t^\alpha}{A_t}$$

The price level evolves as $1 = \theta_p\left(\frac{\Pi_{t-1}^\chi}{\Pi_t}\right)^{1-\varepsilon} + \left(1-\theta_p\right)\Pi_t^{*1-\varepsilon}$.

- The government follows:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi}\left(\frac{\frac{y_t^d}{y_{t-1}^d}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R}\exp\left(m_t\right)$$
$$\log\widetilde{g}_t = (1-\rho_g)\log g + \rho_g\log\widetilde{g}_{t-1} + \sigma_g\varepsilon_{g,t}$$

- Markets clear:

$$
\begin{aligned}
y_t^d &= \frac{A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha}}{v_t^p} \\
y_t^d &= c_t + g_t + x_t + \mu_t^{-1} \Phi \left[u_t\right] k_{t-1}
\end{aligned}
$$

where

$$
v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{-\varepsilon} v_{t-1}^p + \left(1-\theta_p\right) \Pi_t^{*-\varepsilon}
$$

and $l_t^s = l_t^d$ and $k_t - (1-\delta) k_{t-1} - \mu_t \left(1 - S\left[\frac{x_t}{x_{t-1}}\right]\right) x_t = 0.$

3 Stationary equilibrium

Since we have growth in this model induced by technological change, most of the variables are growing in average. Thus, before solving the model, we need to make all variables stationary.

3.1 Manipulating the equilibrium conditions

First, we work on the first-order conditions of the household:

$$
d_t \left(\frac{c_t}{z_t} - h \frac{c_{t-1}}{z_{t-1}} \frac{z_{t-1}}{z_t} \right)^{-1} - h \beta \mathbb{E}_t d_{t+1} \left(\frac{c_{t+1}}{z_{t+1}} \frac{z_{t+1}}{z_t} - h \frac{c_t}{z_t} \right)^{-1} = \lambda_t z_t
$$

$$
\phi_t \psi (l_t)^{\vartheta} = \frac{w_t}{z_t} \lambda_t z_t
$$

$$
\lambda_t z_t = \beta \mathbb{E}_t \left\{ \lambda_{t+1} z_{t+1} \frac{z_t}{z_{t+1}} \frac{R_t}{\Pi_{t+1}} \right\}
$$

$$
\mu_t r_t = \Phi' \left[u_t\right]
$$

$$
q_t \mu_t = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1} z_{t+1}}{\lambda_t z_t} \frac{z_t}{z_{t+1}} \frac{\mu_t}{\mu_{t+1}} \left[(1-\delta) q_{t+1} \mu_{t+1} + \mu_{t+1} r_{t+1} u_{t+1} - \Phi\left(u_{t+1}\right) \right] \right\}
$$

$$
\begin{aligned}
1 = q_t \mu_t \left(1 - S\left[\frac{\frac{x_t}{z_t}}{\frac{x_{t-1}}{z_{t-1}}} \frac{z_t}{z_{t-1}} \right] - S'\left[\frac{\frac{x_t}{z_t}}{\frac{x_{t-1}}{z_{t-1}}} \frac{z_t}{z_{t-1}} \right] \frac{\frac{x_t}{z_t}}{\frac{x_{t-1}}{z_{t-1}}} \frac{z_t}{z_{t-1}} \right) \\
+ \beta \mathbb{E}_t q_{t+1} \mu_{t+1} \frac{\lambda_{t+1}}{\lambda_t} S'\left[\frac{\frac{x_{t+1}}{z_{t+1}}}{\frac{x_t}{z_t}} \frac{z_{t+1}}{z_t} \right] \left(\frac{\frac{x_{t+1}}{z_{t+1}}}{\frac{x_t}{z_t}} \frac{z_{t+1}}{z_t} \right)^2 .
\end{aligned}
$$

The firms that can change prices set them to satisfy:

$$g_t^1 = \lambda_t z_t mc_t \frac{y_t^d}{z_t} + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$
$$g_t^2 = \lambda_t z_t \Pi_t^* \frac{y_t^d}{z_t} + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$
$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2,$$

where they rent inputs to satisfy their static minimization problem:

$$\frac{u_t}{l_t^d}\frac{k_{t-1}}{z_{t-1}\mu_{t-1}} = \frac{\alpha}{1-\alpha}\frac{w_t}{z_t}\frac{1}{r_t\mu_t}\frac{z_t}{z_{t-1}}\frac{\mu_t}{\mu_{t-1}}$$
$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{\left(\frac{w_t}{z_t}\right)^{1-\alpha}(r_t\mu_t)^{\alpha}\frac{z_t^{1-\alpha}}{\mu_t^{\alpha}}}{A_t}.$$

The price level evolves as:

$$1 = \theta_p \left(\frac{\Pi_{t-1}^\chi}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p)\,\Pi_t^{*1-\varepsilon}.$$

The government follows:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi}\left(\frac{\frac{\frac{y_t^d}{z_t}}{\frac{y_{t-1}^d}{z_{t-1}}}\frac{z_t}{z_{t-1}}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R}\exp(m_t)$$
$$\log\widetilde{g}_t = (1-\rho_g)\log g + \rho_g \log\widetilde{g}_{t-1} + \sigma_g\varepsilon_{g,t}.$$

Markets clear:

$$\frac{y_t^d}{z_t} = \frac{\mu_{t-1}^{\alpha} z_{t-1}^{\alpha}\frac{A_t}{z_t}\left(u_t\frac{k_{t-1}}{z_{t-1}\mu_{t-1}}\right)^{\alpha}\left(l_t^d\right)^{1-\alpha}}{v_t^p}$$

but since $\mu_{t-1}^{\alpha} z_{t-1}^{\alpha} = \frac{z_{t-1}}{A_{t-1}}$, we have

$$\frac{y_t^d}{z_t} = \frac{\frac{z_{t-1}}{A_{t-1}}\frac{A_t}{z_t}\left(u_t\frac{k_{t-1}}{z_{t-1}\mu_{t-1}}\right)^{\alpha}\left(l_t^d\right)^{1-\alpha}}{v_t^p}$$

$$\frac{y_t^d}{z_t} = \frac{c_t}{z_t} + \frac{x_t}{z_t} + \widetilde{g}_t + \frac{z_{t-1}}{z_t}\frac{\mu_{t-1}}{\mu_t}\Phi[u_t]\frac{k_{t-1}}{z_{t-1}\mu_{t-1}}$$

where $v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\,\Pi_t^{*-\varepsilon}$ and

$$l_t = l_t^d$$

$$\frac{k_t}{z_t\mu_t}\frac{z_t\mu_t}{z_{t-1}\mu_{t-1}} - (1-\delta)\frac{k_{t-1}}{z_{t-1}\mu_{t-1}} - \frac{\mu_t}{\mu_{t-1}}\frac{z_t}{z_{t-1}}\left(1 - S\left[\frac{\frac{x_t}{z_t}z_t}{\frac{x_{t-1}}{z_{t-1}}z_{t-1}}\right]\right)\frac{x_t}{z_t} = 0.$$

3.2 Change of variables

We now redefine that variables to obtain a system on stationary variables that we can easily manipulate. Hence, we define $\widetilde{c}_t = \frac{c_t}{z_t}$, $\widetilde{\lambda}_t = \lambda_t z_t$, $\widetilde{r}_t = r_t\mu_t$, $\widetilde{q}_t = q_t\mu_t$, $\widetilde{x}_t = \frac{x_t}{z_t}$, $\widetilde{w}_t = \frac{w_t}{z_t}$, $\widetilde{w}_t^* = \frac{w_t^*}{z_t}$, $\widetilde{k}_t = \frac{k_t}{z_t\mu_t}$, and $\widetilde{y}_t^d = \frac{y_t^d}{z_t}$.

Then, the set of equilibrium conditions are:

- The first-order conditions of the household:

$$d_t\left(\widetilde{c}_t - h\widetilde{c}_{t-1}\frac{z_{t-1}}{z_t}\right)^{-1} - h\beta\mathbb{E}_t d_{t+1}\left(\widetilde{c}_{t+1}\frac{z_{t+1}}{z_t} - h\widetilde{c}_t\right)^{-1} = \widetilde{\lambda}_t$$

$$\phi_t\psi(l_t)^\theta = \widetilde{w}_t\widetilde{\lambda}_t$$

$$\widetilde{\lambda}_t = \beta\mathbb{E}_t\{\widetilde{\lambda}_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}\}$$

$$\widetilde{r}_t = \Phi'\left[u_t\right]$$

$$\widetilde{q}_t = \beta\mathbb{E}_t\left\{\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{z_t}{z_{t+1}}\frac{\mu_t}{\mu_{t+1}}\left((1-\delta)\,\widetilde{q}_{t+1} + \widetilde{r}_{t+1}u_{t+1} - \Phi\left(u_{t+1}\right)\right)\right\}$$

$$1 = \widetilde{q}_t\left(1 - S\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right] - S'\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right]\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right)$$
$$+\beta\mathbb{E}_t\widetilde{q}_{t+1}\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{z_t}{z_{t+1}}S'\left[\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t}\frac{z_{t+1}}{z_t}\right]\left(\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t}\frac{z_{t+1}}{z_t}\right)^2$$

- The firms that can change prices set them to satisfy:

$$g_t^1 = \widetilde{\lambda}_t mc_t\widetilde{y}_t^d + \beta\theta_p\mathbb{E}_t\left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

$$g_t^2 = \widetilde{\lambda}_t\Pi_t^*\widetilde{y}_t^d + \beta\theta_p\mathbb{E}_t\left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon}\left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$

$$\varepsilon g_t^1 = (\varepsilon-1)g_t^2$$

where they rent inputs to satisfy their static minimization problem:

$$\frac{u_t \widetilde{k}_{t-1}}{l_t^d} = \frac{\alpha}{1-\alpha} \frac{\widetilde{w}_t}{\widetilde{r}_t} \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}}$$
$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} (\widetilde{w}_t)^{1-\alpha} \widetilde{r}_t^{\alpha}$$

- The price level evolves $1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p) \Pi_t^{*1-\varepsilon}$.
- The government follows:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{\widetilde{y}_t^d}{\widetilde{y}_{t-1}^d} \frac{z_t}{z_{t-1}}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R} \exp(m_t)$$
$$\log \widetilde{g}_t = (1-\rho_g) \log g + \rho_g \log \widetilde{g}_{t-1} + \sigma_g \varepsilon_{g,t}.$$

- Markets clear:

$$\widetilde{y}_t^d = \widetilde{c}_t + \widetilde{x}_t + \widetilde{g} + \frac{z_{t-1}}{z_t} \frac{\mu_{t-1}}{\mu_t} \Phi[u_t] \widetilde{k}_{t-1}$$
$$\widetilde{y}_t^d = \frac{\frac{A_t}{A_{t-1}} \frac{z_{t-1}}{z_t} \left(u_t \widetilde{k}_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha}}{v_t^p}$$

where

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p) \Pi_t^{*-\varepsilon}$$

and $l_t = l_t^d$ and $\widetilde{k}_t \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}} - (1-\delta) \widetilde{k}_{t-1} - \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}} \left(1 - S\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}} \frac{z_t}{z_{t-1}}\right]\right) \widetilde{x}_t = 0$.

4 The steady state

Now, we will find the deterministic steady-state of the model. First, let $\widetilde{z} = \exp(\Lambda_z)$, $\widetilde{\mu} = \exp(\Lambda_\mu)$, and $\widetilde{A} = \exp(\Lambda_A)$. Also, given the definition of $\widetilde{c}$, $\widetilde{x}_t$, $\widetilde{w}_t$, $\widetilde{w}_t^*$, and $\widetilde{y}_t^d$, we have that $\Lambda_c = \Lambda_x = \Lambda_w = \Lambda_{w^*} = \Lambda_{y^d} = \Lambda_z$.

Then, in steady state, the first-order conditions of the household can be written as:

$$\frac{1}{\widetilde{c} - \frac{h}{\widetilde{z}}\widetilde{c}} - h\beta \frac{1}{\widetilde{c}\widetilde{z} - h\widetilde{c}} = \widetilde{\lambda}$$

$$\psi(l)^{\theta} = \widetilde{w}\widetilde{\lambda}$$

$$1 = \beta \frac{1}{\widetilde{z}} \frac{R}{\Pi}$$

$$\widetilde{r} = \Phi' [1]$$

$$\widetilde{q} = \beta \frac{1}{\widetilde{z}\widetilde{\mu}} \left((1-\delta)\, \widetilde{q} + \widetilde{r}u - \Phi [1] \right)$$

$$1 = \widetilde{q} \left(1 - S [\widetilde{z}] - S' [\widetilde{z}]\, \widetilde{z}\right) + \beta \frac{\widetilde{q}}{\widetilde{z}} S' [\widetilde{z}]\, \widetilde{z}^{2},$$

the first-order conditions of the firm as:

$$g^{1} = \widetilde{\lambda} mc \widetilde{y}^{d} + \beta\theta_{p} \left(\frac{\Pi^{\chi}}{\Pi} \right)^{-\varepsilon} g^{1}$$

$$g^{2} = \widetilde{\lambda} \Pi^{*} \widetilde{y}^{d} + \beta\theta_{p} \left(\frac{\Pi^{\chi}}{\Pi} \right)^{1-\varepsilon} g^{2}$$

$$\varepsilon g^{1} = (\varepsilon - 1) g^{2}$$

$$\frac{u\widetilde{k}}{l^{d}} = \frac{\alpha}{1-\alpha} \frac{\widetilde{w}}{\widetilde{r}} \widetilde{z}\widetilde{\mu}$$

$$mc = \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \widetilde{w}^{1-\alpha} \widetilde{r}^{\alpha}$$

the law of motion for prices as:

$$1 = \theta_{p} \left(\frac{\Pi^{\chi}}{\Pi} \right)^{1-\varepsilon} + (1-\theta_{p})\, \Pi^{*1-\varepsilon}$$

and the market clearing conditions as:

$$\widetilde{c} + \widetilde{x} + \widetilde{g} = \widetilde{y}^{d}$$

$$v^{p} \widetilde{y}^{d} = \frac{A}{z} (u\widetilde{k})^{\alpha} (l^{d})^{1-\alpha}$$

$$l = l^{d}$$

$$v^{p} = \theta_{p} \left(\frac{\Pi^{\chi}}{\Pi} \right)^{-\varepsilon} v^{p} + (1-\theta_{p})\, \Pi^{*-\varepsilon}$$

$$\widetilde{k}\widetilde{z}\widetilde{\mu} - (1-\delta)\, \widetilde{k} - \widetilde{z}\widetilde{\mu} \left(1 - S [z]\right) \widetilde{x} = 0.$$

To find the steady-state, we need to choose functional forms for $\Phi[\cdot]$ and $S[\cdot]$. For $\Phi[u]$ we pick: $\Phi[u] = \Phi_1(u-1) + \frac{\Phi_2}{2}(u-1)^2$. Since, in the steady state, $u = 1$, then $\widetilde{r} = \Phi'[1] = \gamma_1$ and $\Phi[1] = 0$. The investment adjustment cost function is $S\left[\frac{x_t}{x_{t-1}}\right] = \frac{\kappa}{2}(\frac{x_t}{x_{t-1}} - \Lambda_x)^2$. Then, along the BGP, $S[z] = S'[\Lambda_x] = 0$. Using this two expressions, we can rearrange the system of equations that determine the steady state as:

$$(1 - h\beta/\widetilde{z})\frac{1}{1-\frac{h}{\widetilde{z}}}\frac{1}{\widetilde{c}} = \widetilde{\lambda}$$

$$\psi(l)^{\theta} = \widetilde{w}\widetilde{\lambda}$$

$$R = \frac{\Pi\widetilde{z}}{\beta}$$

$$\widetilde{r} = \gamma_1$$

$$\widetilde{r} = \left(1 - \frac{\beta}{\widetilde{z}\widetilde{\mu}}(1-\delta)\right)/\frac{\beta}{\widetilde{z}\widetilde{\mu}}$$

$$\left(1 - \beta\theta_p\Pi^{(1-\chi)\varepsilon}\right)g^1 = \widetilde{\lambda}mc\widetilde{y}^d$$

$$\left(1 - \beta\theta_p\Pi^{-(1-\chi)(1-\varepsilon)}\right)g^2 = \widetilde{\lambda}\Pi^*\widetilde{y}^d$$

$$\varepsilon g^1 = (\varepsilon - 1)g^2$$

$$\frac{\widetilde{k}}{l^d} = \frac{\alpha}{1-\alpha}\frac{\widetilde{w}}{\widetilde{r}}\widetilde{z}\widetilde{\mu}$$

$$mc = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\widetilde{w}^{1-\alpha}\widetilde{r}^{\alpha}$$

$$\frac{1 - \theta_p\Pi^{-(1-\chi)(1-\varepsilon)}}{1-\theta_p} = \Pi^{*1-\varepsilon}$$

$$\widetilde{c} + \widetilde{x} + \widetilde{g} = \widetilde{y}^d$$

$$v^p\widetilde{y}^d = \frac{\widetilde{A}}{\widetilde{z}}(\widetilde{k})^{\alpha}(l^d)^{1-\alpha}$$

$$l = l^d$$

$$\frac{1 - \theta_p\Pi^{(1-\chi)\varepsilon}}{1-\theta_p}v^p = \Pi^{*-\varepsilon}$$

$$\widetilde{k} = \frac{\widetilde{z}\widetilde{\mu}}{\widetilde{z}\widetilde{\mu} - (1-\delta)}\widetilde{x}.$$

First, notice that there is some restrictions on γ_1

$$\tilde{r} = \frac{1 - \frac{\beta}{\widetilde{z}\widetilde{\mu}}(1-\delta)}{\frac{\beta}{\widetilde{z}\widetilde{\mu}}} = \gamma_1$$

and that the nominal interest rate is:

$$R = \frac{\Pi \widetilde{z}}{\beta}$$

The relationship between inflation and optimal relative prices is:

$$\Pi^* = \left(\frac{1 - \theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{1 - \theta_p} \right)^{\frac{1}{1-\varepsilon}}.$$

The expression for the dispersion of prices is given by:

$$v^p = \frac{1 - \theta_p}{1 - \theta_p \Pi^{(1-\chi)\varepsilon}} \Pi^{*-\varepsilon}.$$

We can calibrate ψ such that in steady state $l = 1$ (the utility function we selected does not have natural units for labor, so $l = 1$ is just a normalization). Labor demand must equal labor supply, $l^d = 1$.

Once we have l^d, we can solve for wages, capital, investment, output, and consumption. Wages are given by:

$$\frac{\left(1 - \beta\theta_p \Pi^{(1-\chi)\varepsilon}\right)}{\left(1 - \beta\theta_p \Pi^{-(1-\chi)(1-\varepsilon)}\right)} \frac{\varepsilon - 1}{\varepsilon} \Pi^* = mc$$
$$mc = \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \widetilde{w}^{1-\alpha} \widetilde{r}^{\alpha}$$

That allows us to find $\widetilde{k} = \frac{\alpha}{1-\alpha} \frac{\widetilde{w}}{\widetilde{r}} \widetilde{z} \widetilde{\mu}$.

We also have:

$$\widetilde{y}^d = \frac{\frac{\widetilde{A}}{\widetilde{z}} (\widetilde{k})^{\alpha} (l^d)^{1-\alpha}}{v^p}.$$

But, since in steady-state $\widetilde{k} = \frac{\widetilde{z}\widetilde{\mu}}{\widetilde{z}\widetilde{\mu} - (1-\delta)} \widetilde{x}$, it is the case that:

$$\widetilde{c} + \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}} \widetilde{k} + \widetilde{g} = \widetilde{y}^d = \frac{\frac{\widetilde{A}}{\widetilde{z}} (\widetilde{k})^{\alpha}}{v^p}.$$

Finally:

$$\widetilde{c} \quad = \quad \frac{\frac{\widetilde{A}}{\widetilde{z}} \widetilde{k}^{\alpha} - \phi}{v^p} - \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}} \widetilde{k} - \widetilde{g}.$$

5 Log-linear approximations

For each variable var_t, we define $\widehat{var}_t = \log var_t - \log var$, where var is the steady-state value for the variable var_t. Then, we can write $var_t = var \exp^{\widehat{var}_t}$.

We start by log-linearizing the marginal utility of consumption:

$$d_t \left(\widetilde{c}_t - h\widetilde{c}_{t-1}\frac{z_{t-1}}{z_t} \right)^{-1} - h\beta \mathbb{E}_t d_{t+1} \left(\widetilde{c}_{t+1}\frac{z_{t+1}}{z_t} - h\widetilde{c}_t \right)^{-1} = \widetilde{\lambda}_t. \tag{3}$$

It is helpful to define the auxiliary variable $aux_t = d_t \left(\widetilde{c}_t - h\widetilde{c}_{t-1}\frac{z_{t-1}}{z_t} \right)^{-1}$. Then, we have that:

$$aux_t - h\beta \mathbb{E}_t \frac{1}{\widetilde{z}_{t+1}} aux_{t+1} = \widetilde{\lambda}_t$$

where $\widetilde{z}_{t+1} = \frac{z_{t+1}}{z_t}$.

Then, (3) can be written as:

$$aux \exp^{\widehat{aux}_t} - \frac{h\beta}{z} aux \mathbb{E}_t \exp^{\widehat{aux}_{t+1} - \widehat{\widetilde{z}}_{t+1}} = \widetilde{\lambda} e^{\widehat{\widetilde{\lambda}}_t}$$

which can be log-linearized as:

$$aux \left(\widehat{aux}_t - \frac{h\beta}{z} \mathbb{E}_t (\widehat{aux}_{t+1} - \widehat{\widetilde{z}}_{t+1}) \right) = \widetilde{\lambda} \widehat{\widetilde{\lambda}}_t. \tag{4}$$

Using the following two steady state relationship that $aux(1 - \frac{h\beta}{\widetilde{z}}) = \widetilde{\lambda}$ and that $\mathbb{E}_t \widehat{\widetilde{z}}_{t+1} = 0$, we can write:

$$\widehat{aux}_t - \frac{h\beta}{z} \mathbb{E}_t \widehat{aux}_{t+1} = \left(1 - \frac{h\beta}{z} \right) \widehat{\widetilde{\lambda}}_t.$$

Next, we log-linearize the auxiliary variable:

$$aux e^{\widehat{aux}_t} = d \exp^{\widehat{d}_t} \left(\widetilde{c} \exp^{\widehat{\widetilde{c}}_t} - \frac{h}{\widetilde{z}} \widetilde{c} \exp^{\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t} \right)^{-1}.$$

The log-linear approximation is:

$$\begin{aligned} aux \widehat{aux}_t &= d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-1} \widehat{d}_t - d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-2} \widetilde{c} \widehat{\widetilde{c}}_t \\ &+ d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-2} \frac{h\widetilde{c}}{\widetilde{z}} \left(\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t \right). \end{aligned}$$

Making use of the fact that $aux = d\left(\widetilde{c} - \frac{h}{\widetilde{z}}\widetilde{c}\right)^{-1}$, we find:

$$aux\widehat{aux}_t = aux\left(\widehat{d}_t - \left(\widetilde{c} - \frac{h}{\widetilde{z}}\widetilde{c}\right)^{-1}\widetilde{c}\widehat{\widetilde{c}}_t + \left(\widetilde{c} - \frac{h}{\widetilde{z}}\widetilde{c}\right)^{-1}\frac{h\widetilde{c}}{\widetilde{z}}\left(\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t\right)\right),$$

that simplifies to:

$$aux\widehat{aux}_t = aux\left(\widehat{d}_t - \left(1 - \frac{h}{\widetilde{z}}\right)^{-1}\widehat{\widetilde{c}}_t + \left(1 - \frac{h}{\widetilde{z}}\right)^{-1}\frac{h}{\widetilde{z}}\left(\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t\right)\right)$$

$$\widehat{aux}_t = \widehat{d}_t - \left(1 - \frac{h}{\widetilde{z}}\right)^{-1}\left(\widehat{\widetilde{c}}_t - \frac{h\widehat{\widetilde{c}}_{t-1}}{\widetilde{z}} + \frac{h\widehat{\widetilde{z}}_t}{\widetilde{z}}\right). \tag{5}$$

Putting the (4) and (5) together:

$$\left(1 - \frac{h\beta}{z}\right)\widehat{\widetilde{\lambda}}_t =$$
$$\widehat{d}_t - \left(1 - \frac{h}{\widetilde{z}}\right)^{-1}\left(\widehat{\widetilde{c}}_t - \frac{h\widehat{\widetilde{c}}_{t-1}}{\widetilde{z}} + \frac{h\widehat{\widetilde{z}}_t}{\widetilde{z}}\right) - \frac{h\beta}{z}\mathbb{E}_t\left\{\widehat{d}_{t+1} - \left(1 - \frac{h}{\widetilde{z}}\right)^{-1}\left(\widehat{\widetilde{c}}_{t+1} - \frac{h\widehat{\widetilde{c}}_t}{\widetilde{z}} + \frac{h\widehat{\widetilde{z}}_{t+1}}{\widetilde{z}}\right)\right\}.$$

After some algebra, we arrive to the final expression:

$$(1 - \frac{h\beta}{z})\widehat{\widetilde{\lambda}}_t = \widehat{d}_t - \frac{h\beta}{z}\mathbb{E}_t\widehat{d}_{t+1} - \frac{1 + h^2/z^2\beta}{\left(1 - \frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_t + \frac{h}{\widetilde{z}\left(1 - \frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_{t-1} + \frac{\beta h/z}{\left(1 - \frac{h}{\widetilde{z}}\right)}\mathbb{E}_t\widehat{\widetilde{c}}_{t+1} - \frac{h}{\widetilde{z}\left(1 - \frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{z}}_t.$$

Now, we log-linearize the Euler equation:

$$\widetilde{\lambda}_t = \beta\mathbb{E}_t\{\widetilde{\lambda}_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}\}.$$

To do so, we write the expression as

$$\widetilde{\lambda}e^{\widehat{\widetilde{\lambda}}} = \beta\mathbb{E}_t\{\widetilde{\lambda}e^{\widehat{\widetilde{\lambda}}_{t+1}}\frac{1}{\widetilde{z}e^{z_{z,t}}}\frac{Re^{\widehat{R}_t}}{\Pi e^{\widehat{\Pi}_{t+1}}}\}$$

By using the fact that

$$R = \frac{\Pi\widetilde{z}}{\beta}$$

we simplify to:

$$e^{\widehat{\widetilde{\lambda}}} = \mathbb{E}_t\{e^{\widehat{\widetilde{\lambda}}_{t+1}}\frac{1}{e^{z_{z,t}}}\frac{e^{\widehat{R}_t}}{e^{\widehat{\Pi}_{t+1}}}\}.$$

Now, it is easy to show that:

$$\widehat{\widetilde{\lambda}}_t = \mathbb{E}_t\{\widehat{\widetilde{\lambda}}_{t+1} + \widehat{R}_t - \widehat{\Pi}_{t+1}\}.. \tag{6}$$

Next, we log-linearize the static consumption/labor optimal condition $\phi_t\psi(l_t)^\theta = \widetilde{w}_t\widetilde{\lambda}_t$:

$$\widehat{\phi}_t + \theta\widehat{l}_t = \widehat{\widetilde{w}}_t + \widehat{\widetilde{\lambda}}_t. \tag{7}$$

Let us now consider $\widetilde{r}_t = \Phi'[u_t]$. First, we write $\widetilde{r}e^{\widehat{\widetilde{r}}_t} = \Phi'\left[u\exp^{\widehat{u}_t}\right]$, where the log-linear approximation is:

$$\widetilde{r}\widehat{\widetilde{r}}_t = \Phi''[u]\,u\widehat{u}_t.$$

Since $\widetilde{r} = \Phi'[u]$, then:

$$\widehat{\widetilde{r}}_t = \frac{\Phi''[u]\,u}{\Phi'[u]}\widehat{u}_t,$$

or

$$\widehat{\widetilde{r}}_t = \frac{\Phi_2}{\Phi_1}\hat{u}_t. \tag{8}$$

The next equation relates the shadow price of capital to the return on investment:

$$\widetilde{q}_t = \beta\mathbb{E}_t\left\{\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{1}{\widetilde{z}_{t+1}}\frac{1}{\widetilde{\mu}_{t+1}}\left((1-\delta)\,\widetilde{q}_{t+1} + \widetilde{r}_{t+1}u_{t+1} - \Phi\left[u_{t+1}\right]\right)\right\}$$

where $\widetilde{\mu}_{t+1} = \frac{\mu_{t+1}}{\mu_t}$. We can we write this expression as

$$\widetilde{q}e^{\widehat{\widetilde{q}}_t} = \frac{\beta}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\exp^{\Delta\widehat{\widetilde{\lambda}}_{t+1}-\widehat{\widetilde{z}}_{t+1}-\widehat{\widetilde{\mu}}_{t+1}}\left\{\begin{array}{c}(1-\delta)\,\widetilde{q}e^{\widehat{\widetilde{q}}_{t+1}}+\\ \widetilde{r}u\exp^{\widehat{\widetilde{r}}_{t+1}+\widehat{u}_{t+1}}-\Phi\left[u\exp^{\widehat{u}_{t+1}}\right]\end{array}\right\}.$$

Log-linearization delivers:

$$\begin{aligned}\widetilde{q}\widehat{\widetilde{q}}_t \;&=\; \frac{\beta}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\left(\triangle\widehat{\widetilde{\lambda}}_{t+1} - \widehat{\widetilde{z}}_{t+1} - \widehat{\widetilde{\mu}}_{t+1}\right)\left((1-\delta)\,\widetilde{q} + \widetilde{r}u - \Phi[u]\right)\\ &\quad+\frac{\beta}{\widetilde{z}\widetilde{\mu}}\left(\mathbb{E}_t\,(1-\delta)\,\widetilde{q}\widehat{\widetilde{q}}_{t+1} + \widetilde{r}u\left(\widehat{\widetilde{r}}_{t+1} + \widehat{u}_{t+1}\right) - \Phi[u]\,u\widehat{u}_{t+1}\right).\end{aligned}$$

Making use of the following steady-state relationships: $u = 1$, $\Phi[u] = 0$, $\widetilde{r} = \Phi'[u]$, $\widetilde{q} = 1$ and $1 = \frac{\beta}{\widetilde{z}\widetilde{\mu}}\widetilde{r} + \frac{\beta}{\widetilde{z}\widetilde{\mu}}(1-\delta)$, and $\mathbb{E}_t\left(-\widehat{\widetilde{z}}_{t+1} - \widehat{\widetilde{\mu}}_{t+1}\right) = 0$, the previous expression simplifies to:

$$\widehat{\widetilde{q}}_t = \left((1-\delta)+\widetilde{r}\right)\frac{\beta}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\Delta\widehat{\widetilde{\lambda}}_{t+1} + \frac{\beta\,(1-\delta)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1} + \frac{\beta}{\widetilde{z}\widetilde{\mu}}\widetilde{r}u\mathbb{E}_t\widehat{\widetilde{r}}_{t+1},$$

that implies:

$$\widehat{\widetilde{q}}_t = \mathbb{E}_t \Delta \widehat{\widetilde{\lambda}}_{t+1} + \frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}} \mathbb{E}_t \widehat{\widetilde{q}}_{t+1} + \left(1 - \frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right) \mathbb{E}_t \widehat{\widetilde{r}}_{t+1}. \tag{9}$$

The next equation to log-linearize is:

$$1 = \widetilde{q}_t \left(1 - S\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}} \frac{z_t}{z_{t-1}}\right] - S'\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}} \frac{z_t}{z_{t-1}}\right] \frac{\widetilde{x}_t}{\widetilde{x}_{t-1}} \frac{z_t}{z_{t-1}}\right) + \beta \mathbb{E}_t \widetilde{q}_{t+1} \frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t} \frac{z_t}{z_{t+1}} S'\left[\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t} \frac{z_{t+1}}{z_t}\right] \left(\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t} \frac{z_{t+1}}{z_t}\right)^2$$

which can be rearranged as:

$$\begin{aligned} 1 \quad = \quad & \widetilde{q} \exp^{\widehat{\widetilde{q}}_t} \left(1 - S\left[\widetilde{z} \exp^{\Delta \widehat{\widetilde{x}}_t + \widehat{\widetilde{z}}_t}\right] - S'\left[\widetilde{z} \exp^{\Delta \widehat{\widetilde{x}}_t + \widehat{\widetilde{z}}_t}\right] \widetilde{z} \exp^{\Delta \widehat{\widetilde{x}}_t + \widehat{\widetilde{z}}_t}\right) + \\ & + \beta \frac{\widetilde{q}}{\widetilde{z}} \mathbb{E}_t \exp^{\widehat{\widetilde{q}}_t - \widehat{\widetilde{z}}_{t+1} + \Delta \widehat{\widetilde{\lambda}}_{t+1}} S''\left[\widetilde{z} \exp^{\Delta \widehat{\widetilde{x}}_{t+1} + \widehat{\widetilde{z}}_{t+1}}\right] \widetilde{z}^2 \exp^{2\left(\Delta \widehat{\widetilde{x}}_{t+1} + \widehat{\widetilde{z}}_{t+1}\right)}. \end{aligned}$$

Taking the log-linear approximation (and using the fact that $\widetilde{q} = 1$) we get:

$$0 = \widehat{\widetilde{q}}_t - S''[\widetilde{z}]\, \widetilde{z}^2 \left(\Delta \widehat{\widetilde{x}}_t + \widehat{\widetilde{z}}_t\right) + \frac{\beta}{z} S''[\widetilde{z}]\, \widetilde{z}^3 \mathbb{E}_t \left(\Delta \widehat{\widetilde{x}}_{t+1} + \widehat{\widetilde{z}}_{t+1}\right).$$

Reorganizing:

$$\kappa \widetilde{z}^2 \left(\Delta \widehat{\widetilde{x}}_t + \widehat{\widetilde{z}}_t\right) = \widehat{\widetilde{q}}_t + \beta \kappa \widetilde{z}^2 \mathbb{E}_t \Delta \widehat{\widetilde{x}}_{t+1}, \tag{10}$$

where κ comes from the adjustment cost function.

Let us log-linearize the law of motion for g_t^1 and g_t^2. First consider

$$g_t^1 = \widetilde{\lambda}_t mc_t \widetilde{y}_t^d + \beta \theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

that can be rewritten as:

$$g^1 \exp^{\widehat{g}_t^1} = \widetilde{\lambda} mc \widetilde{y}^d \exp^{\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d} + \beta \theta_p g^1 \Pi^{\varepsilon(1-\chi)} \mathbb{E}_t \exp^{\varepsilon(\widehat{\Pi}_{t+1} - \chi \widehat{\Pi}_t) + \widehat{g}_{t+1}^1}.$$

If we log-linearize that last expression, we get:

$$g^1 \widehat{g}_t^1 = \widetilde{\lambda} mc \widetilde{y}^d \left(\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d\right) + \beta \theta_p g^1 \Pi^{\varepsilon(1-\chi)} \mathbb{E}_t \left(\varepsilon(\widehat{\Pi}_{t+1} - \chi \widehat{\Pi}_t) + \widehat{g}_{t+1}^1\right).$$

Since, in steady state $1 - \beta \theta_p \Pi^{\varepsilon(1-\chi)} = \frac{\widetilde{\lambda} mc \widetilde{y}^d}{g^1}$, we have:

$$\widehat{g}_t^1 = \left(1 - \beta \theta_p \Pi^{\varepsilon(1-\chi)}\right) \left(\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d\right) + \beta \theta_p \Pi_t^{\varepsilon(1-\chi)} \mathbb{E}_t \left(\varepsilon(\widehat{\Pi}_{t+1} - \chi \widehat{\Pi}_t) + \widehat{g}_{t+1}^1\right). \tag{11}$$

Let us now consider:

$$g_t^2 = \widetilde{\lambda}_t \Pi_t^* \widetilde{y}_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2,$$

that can rewritten as:

$$g^2 \exp^{\widehat{g}_t^2} = \widetilde{\lambda}\Pi^*\widetilde{y}^d \exp^{\widehat{\widetilde{\lambda}}_t + \widehat{\Pi}_t^* + \widehat{\widetilde{y}}_t^d} + \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} g^2 \mathbb{E}_t \exp^{-(1-\varepsilon)\left(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t\right) - \left(\widehat{\Pi}_{t+1}^* - \widehat{\Pi}_t^*\right) + \widehat{g}_{t+1}^2}.$$

If we log-linearize that last expression, we get:

$$\begin{aligned} g^2 \widehat{g}_t^2 &= \widetilde{\lambda}\Pi^*\widetilde{y}^d \left(\widehat{\widetilde{\lambda}}_t + \widehat{\Pi}_t^* + \widehat{\widetilde{y}}_t^d\right) \\ &\quad + \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} g^2 \mathbb{E}_t \left(-(1-\varepsilon)\left(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t\right) - \left(\widehat{\Pi}_{t+1}^* - \widehat{\Pi}_t^*\right) + \widehat{g}_{t+1}^2\right). \end{aligned}$$

It can be shown that:

$$1 - \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} = \frac{\widetilde{\lambda}\Pi^*\widetilde{y}^d}{g^2},$$

therefore:

$$\begin{aligned} \widehat{g}_t^2 &= \left(1 - \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}\right) \left(\widehat{\widetilde{\lambda}}_t + \widehat{\Pi}_t^* + \widehat{\widetilde{y}}_t^d\right) \\ &\quad + \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} \mathbb{E}_t \left(-(1-\varepsilon)\left(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t\right) - \left(\widehat{\Pi}_{t+1}^* - \widehat{\Pi}_t^*\right) + \widehat{g}_{t+1}^2\right). \end{aligned} \tag{12}$$

Note that is easy to show that $\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$ log-linearizes to:

$$\widehat{g}_t^1 = \widehat{g}_t^2. \tag{13}$$

The relationship between the capital-labor ratio and the real wage-real interest rate

$$\frac{u_t \widetilde{k}_{t-1}}{l_t^d} = \frac{\alpha}{1-\alpha} \frac{\widetilde{w}_t}{\widetilde{r}_t} \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}}.$$

becomes:

$$\widehat{u}_t + \widehat{\widetilde{k}}_{t-1} - \widehat{l}_t^d = \widehat{\widetilde{w}}_t - \widehat{\widetilde{r}}_t + \widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t. \tag{14}$$

Let us log-linearize the marginal cost

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} (\widetilde{w}_t)^{1-\alpha} \widetilde{r}_t^\alpha$$

to get

$$\widehat{mc}_t = (1-\alpha)\widehat{\widetilde{w}}_t + \alpha \widehat{\widetilde{r}}_t. \tag{15}$$

Now we concentrate on the aggregate price law of motion:

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{1-\varepsilon} + (1-\theta_p)\, \Pi_t^{*1-\varepsilon}$$

that can be rewritten as:

$$1 = \theta_p \Pi^{-(1-\varepsilon)(1-\chi)} \exp^{-(1-\varepsilon)(\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1})} + (1-\theta_p)\,(\Pi^*)^{(1-\varepsilon)} \exp^{(1-\varepsilon)\widehat{\Pi}_t^*}$$

that log-linearizes to:

$$\frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)\,(\Pi^*)^{(1-\varepsilon)}} (\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1}) = \widehat{\Pi}_t^*. \tag{16}$$

The Taylor rule log-linearizes to:

$$\widehat{R}_t = \gamma_R \widehat{R}_{t-1} + (1-\gamma_R) \left(\gamma_\Pi \widehat{\Pi}_t + \gamma_y (\triangle \widehat{\widetilde{y}}_t^d + \widehat{\widetilde{z}}_t) \right) + \widehat{m}_t. \tag{17}$$

(the government consumption rule is already linear).

The market clearing conditions:

$$l_t = l_t^d,$$

$$\widetilde{c}_t + \widetilde{x}_t + \widetilde{g}_t + \frac{\Phi\,[u_t]\, \widetilde{k}_{t-1}}{\widetilde{\mu}_t \widetilde{z}_t} = \widetilde{y}_t^d,$$

and

$$v_t^p \widetilde{y}_t^d = \frac{\widetilde{A}_t}{\widetilde{z}_t} \left(u_t \widetilde{k}_{t-1} \right)^{\alpha} \left(l_t^d \right)^{1-\alpha} - \phi$$

can be written as:

$$l \exp^{\widehat{l}_t} = l^d \exp^{\widehat{l}_t^d},$$

$$\widetilde{c} \exp^{\widehat{\widetilde{c}}_t} + \widetilde{x} \exp^{\widehat{\widetilde{x}}_t} + \widetilde{g} \exp^{\widehat{\widetilde{g}}_t} + \frac{\Phi \left[\widetilde{u} \exp^{\widehat{\widetilde{u}}_t} \right] \widetilde{k} \exp^{\widehat{\widetilde{k}}_{t-1}}}{\widetilde{z} \widetilde{\mu} \exp^{\widehat{\widetilde{\mu}}_t + \widehat{\widetilde{z}}_t}} = \widetilde{y}^d \exp^{\widehat{\widetilde{y}}_t^d},$$

and

$$v^p \widetilde{y}^d \exp^{\widehat{v}_t^p + \widehat{\widetilde{y}}_t^d} = \frac{\widetilde{A}}{\widetilde{z}} \left(u \widetilde{k} \right)^{\alpha} \left(\widetilde{l}^d \right)^{1-\alpha} \exp^{\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha \left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1} \right) + (1-\alpha) \widehat{l}_t^d} - \phi$$

respectively, where $\widetilde{A}_{t+1} = \frac{A_{t+1}}{A_t}$.

Log-linearizin,g we get:

$$\widehat{l}_t = \widehat{l}_t^d, \tag{18}$$

$$\tilde{c}\widehat{\tilde{c}}_t + \tilde{x}\widehat{\tilde{x}}_t + \tilde{g}\widehat{\tilde{g}}_t + \frac{\gamma_1 \widetilde{k}}{\widetilde{z}\widetilde{\mu}}\widehat{u}_t = \tilde{y}^d \widehat{\widetilde{y}}^d_t, \tag{19}$$

and

$$\left(\tilde{y}^d v^p\right)\left(\widehat{v}^p_t + \widehat{\widetilde{y}}^d_t\right) = \frac{\widetilde{A}}{\widetilde{z}}\left(u\widetilde{k}\right)^{\alpha}\left(\widetilde{l}^d\right)^{1-\alpha}\left(\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1}\right) + (1-\alpha)\,\widehat{l}^d_t\right). \tag{20}$$

Let us consider $v^p_t = \theta_p\left(\frac{\Pi^{\chi}_{t-1}}{\Pi_t}\right)^{-\varepsilon} v^p_{t-1} + (1-\theta_p)\,\Pi^{*-\varepsilon}_t$, that can be written as

$$v^p \exp^{\widehat{v}^p_t} = \theta_p \Pi^{\varepsilon(1-\chi)} v^p \exp^{\varepsilon(\widehat{\Pi}_t - \chi\widehat{\Pi}_{t-1}) + \widehat{v}^p_{t-1}} + (1-\theta_p)\,\Pi^{*-\varepsilon}\exp^{\varepsilon\widehat{\Pi}^*_t}.$$

Using

$$\left(1-\theta_p\Pi^{\varepsilon(1-\chi)}\right)v^p = (1-\theta_p)\,\Pi^{*-\varepsilon}$$

we get:

$$\widehat{v}^p_t = \theta_p\Pi^{\varepsilon(1-\chi)}\left(\varepsilon(\widehat{\Pi}_t - \chi\widehat{\Pi}_{t-1}) + \widehat{v}^p_{t-1}\right) - \left(1-\theta_p\Pi^{\varepsilon(1-\chi)}\right)\varepsilon\widehat{\Pi}^*_t. \tag{21}$$

Finally, let us log-linearize the law of motion of capital:

$$\tilde{k}_t\widetilde{z}_t\widetilde{\mu}_t = (1-\delta)\,\tilde{k}_{t-1} + \widetilde{\mu}_t\widetilde{z}_t\left(1 - S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\widetilde{z}_t\right]\right)\tilde{x}_t.$$

If we rearrange terms, we get:

$$\tilde{k}\widetilde{z}\widetilde{\mu}\exp^{\widehat{\tilde{k}}_t + \widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t} = (1-\delta)\,\tilde{k}\exp^{\widehat{\tilde{k}}_{t-1}} + \widetilde{z}\widetilde{\mu}\exp^{\widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t}\left(1 - S\left[z e^{\widehat{\tilde{z}}_t + \triangle\widehat{\tilde{x}}_t}\right]\right)\tilde{x}\exp^{\widehat{\tilde{x}}_t}.$$

Log-linearizing:

$$\tilde{k}\widetilde{z}\widetilde{\mu}\left(\widehat{\tilde{k}}_t + \widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t\right) = (1-\delta)\,\tilde{k}\widehat{\tilde{k}}_{t-1} + \widetilde{z}\widetilde{\mu}\tilde{x}(\widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t + \widehat{\tilde{x}}_t).$$

Using $\tilde{k} = \frac{\widetilde{z}\widetilde{\mu}}{\widetilde{z}\widetilde{\mu} - (1-\delta)}\tilde{x}$, we can rearrange the previous expression to get:

$$\widehat{\tilde{k}}_t + \widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t = \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\tilde{k}}_{t-1} + \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}(\widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t + \widehat{\tilde{x}}_t)$$

or

$$\widehat{\tilde{k}}_t = \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\tilde{k}}_{t-1} + \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\tilde{x}}_t - \frac{1-\delta}{\widetilde{z}\widetilde{\mu}}\left(\widehat{\tilde{z}}_t + \widehat{\tilde{\mu}}_t\right). \tag{22}$$

6 System of linear stochastic difference equations

We now collect the whole system of linear stochastic difference equations.

Equation 1 The first equation is:

$$\frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)\left(\Pi^*\right)^{(1-\varepsilon)}}(\widehat{\Pi}_t - \chi\widehat{\Pi}_{t-1}) = \widehat{\Pi}_t^*$$

In order to make notation for compact, define $a_2 = \frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)(\Pi^*)^{(1-\varepsilon)}}$. Substituting for a_2:

$$a_2\widehat{\Pi}_t - a_2\chi\widehat{\Pi}_{t-1} - \widehat{\Pi}_t^* = 0. \tag{23}$$

Equation 2 The second equation is

$$\widehat{\widehat{r}}_t = \phi_u \hat{u}_t,$$

where $\phi_u = \Phi_2/\Phi_1$. Then,

$$-\widehat{\widehat{r}}_t + \phi_u \hat{u}_t = 0. \tag{24}$$

Equation 3 The third equation is

$$\widehat{g}_t^1 = \widehat{g}_t^2$$

Rearranging:

$$\widehat{g}_t^1 - \widehat{g}_t^2 = 0. \tag{25}$$

Equation 4 The fourth equation is:

$$\widehat{u}_t + \widehat{\widehat{k}}_{t-1} - \widehat{l}_t^d = \widehat{\widehat{w}}_t - \widehat{\widehat{r}}_t + \widehat{\widehat{z}}_t + \widehat{\widehat{\mu}}_t$$

Rearranging:

$$\widehat{u}_t + \widehat{\widehat{r}}_t + \widehat{\widehat{k}}_{t-1} - \widehat{l}_t^d - \widehat{\widehat{w}}_t - \widehat{\widehat{z}}_t - \widehat{\widehat{\mu}}_t. \tag{26}$$

Equation 5 The fifth equation is:

$$\widehat{mc}_t = (1-\alpha)\widehat{\widehat{w}}_t + \alpha\widehat{\widehat{r}}_t$$

Rearranging:

$$(1-\alpha)\widehat{\widehat{w}}_t + \alpha\widehat{\widehat{r}}_t - \widehat{mc}_t = 0. \tag{27}$$

Equation 6 The sixth equation is:

$$\widehat{R}_t = \gamma_R \widehat{R}_{t-1} + (1-\gamma_R)\left(\gamma_\Pi \widehat{\Pi}_t + \gamma_y(\triangle \widehat{\widetilde{y}}^d_t + \widehat{\widetilde{z}}_t)\right) + \widehat{m}_t$$

Rearranging:

$$-\widehat{R}_t + \gamma_R \widehat{R}_{t-1} + (1-\gamma_R)\gamma_\Pi \widehat{\Pi}_t + (1-\gamma_R)\gamma_y \widehat{\widetilde{z}}_t + (1-\gamma_R)\gamma_y \widehat{\widetilde{y}}^d_t - (1-\gamma_R)\gamma_y \widehat{\widetilde{y}}^d_{t-1} + \widehat{m}_t = 0. \quad (28)$$

Equation 7 The seventh equation is:

$$\tilde{c}\widehat{\widetilde{c}}_t + \tilde{x}\widehat{\widetilde{x}}_t + \tilde{g}\widehat{\widetilde{g}}_t + \frac{\gamma_1 \widetilde{k}}{\widetilde{z}\widetilde{\mu}}\widehat{u}_t = \tilde{y}^d \widehat{\widetilde{y}}^d_t$$

or, rearranging:

$$\tilde{c}\widehat{\widetilde{c}}_t + \tilde{x}\widehat{\widetilde{x}}_t + \tilde{g}\widehat{\widetilde{g}}_t + \frac{\gamma_1 \widetilde{k}}{\widetilde{z}\widetilde{\mu}}\widehat{u}_t - \tilde{y}^d \widehat{\widetilde{y}}^d_t = 0. \quad (29)$$

Equation 8 The eighth equation is:

$$(\tilde{y}^d v^p)\left(\widehat{v}^p_t + \widehat{\widetilde{y}}^d_t\right) = \frac{\widetilde{A}}{\widetilde{z}}\left(u\widetilde{k}\right)^\alpha \left(\widetilde{l}^d\right)^{1-\alpha}\left(\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1}\right) + (1-\alpha)\,\widetilde{l}^d_t\right).$$

Define the parameter $produc = \frac{\widetilde{A}}{\widetilde{z}}\left(u\widetilde{k}\right)^\alpha \left(\widetilde{l}^d\right)^{1-\alpha}$. This is defined as

$$produc = \frac{\widetilde{A}}{\widetilde{z}}\left\{\exp\left[\log(u)\right]\exp\left[\log(\widetilde{k})\right]\right\}^\alpha \left\{\exp\left[\log(\widetilde{l}^d)\right]\right\}^{1-\alpha}.$$

In terms of our code, we have that $\tilde{y}^d = \exp\left[\log\left(\tilde{y}^d\right)\right]$, $v^p = \exp\left[\log\left(v^p\right)\right]$. Substituting:

$$(\tilde{y}^d v^p)\left(\widehat{v}^p_t + \widehat{\widetilde{y}}^d_t\right) = produc\left(\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1}\right) + (1-\alpha)\,\widetilde{l}^d_t\right).$$

Rearranging:

$$\begin{aligned}(\tilde{y}^d v^p)\widehat{v}^p_t + (\tilde{y}^d v^p)\widehat{\widetilde{y}}^d_t - (produc)\widehat{\widetilde{A}}_t + (produc)\widehat{\widetilde{z}}_t - (\alpha)(produc)\widehat{u}_t \\ -(\alpha)(produc)\widehat{\widetilde{k}}_{t-1} - (1-\alpha)\,(produc)\widetilde{l}^d_t = 0. \quad (30)\end{aligned}$$

Equation 9 The ninth equation is:

$$\widehat{v}^p_t = \theta_p \Pi^{\varepsilon(1-\chi)}\left(\varepsilon(\widehat{\Pi}_t - \chi\widehat{\Pi}_{t-1}) + \widehat{v}^p_{t-1}\right) - \left(1 - \theta_p \Pi^{\varepsilon(1-\chi)}\right)\varepsilon\widehat{\Pi}^*_t$$

Define $a_3 = \beta\theta_p\Pi^{\varepsilon(1-\chi)}$. Then, $\theta_p\Pi^{\varepsilon(1-\chi)} = \frac{a_3}{\beta}$. This type of parameter definition will become clearer when we analyze the price setting equations. Then, substituting:

$$\widehat{v}_t^p = \frac{a_3}{\beta}\varepsilon\widehat{\Pi}_t - \chi\varepsilon\frac{a_3}{\beta}\widehat{\Pi}_{t-1} + \frac{a_3}{\beta}\widehat{v}_{t-1}^p - \left(1 - \frac{a_3}{\beta}\right)\varepsilon\widehat{\Pi}_t^*$$

And rearranging:

$$\frac{a_3\varepsilon}{\beta}\widehat{\Pi}_t - \frac{a_3\varepsilon\chi}{\beta}\widehat{\Pi}_{t-1} + \frac{a_3}{\beta}\widehat{v}_{t-1}^p - \left(1 - \frac{a_3}{\beta}\right)\varepsilon\widehat{\Pi}_t^* - \widehat{v}_t^p = 0. \tag{31}$$

Equation 10 The tenth equation is:

$$\widehat{\widetilde{k}}_t = \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\widetilde{k}}_{t-1} + \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\widetilde{x}}_t - \frac{1-\delta}{\widetilde{z}\widetilde{\mu}}\left(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t\right),$$

which we rearrange to:

$$\frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\widetilde{k}}_{t-1} + \left[1 - \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right]\widehat{\widetilde{x}}_t - \frac{1-\delta}{\widetilde{z}\widetilde{\mu}}\left(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t\right) - \widehat{\widetilde{k}}_t = 0. \tag{32}$$

Equation 11 The eleventh equation is:

$$\widehat{\widetilde{z}}_t = \frac{\widehat{\widetilde{A}}_t + \alpha\widehat{\widetilde{\mu}}_t}{1-\alpha}$$

which we rearrange to:

$$\frac{1}{1-\alpha}\widehat{\widetilde{A}}_t + \frac{\alpha}{1-\alpha}\widehat{\widetilde{\mu}}_t - \widehat{\widetilde{z}}_t = 0. \tag{33}$$

Equation 12 The twelfth equation is:

$$\widehat{\phi}_t + \theta\widehat{l}_t = \widehat{\widetilde{w}}_t + \widehat{\widetilde{\lambda}}_t$$

which we rearrange to:

$$\widehat{\phi}_t + \theta\widehat{l}_t - \widehat{\widetilde{w}}_t - \widehat{\widetilde{\lambda}}_t = 0.$$

Equation 13 The thirteenth equation is:

$$(1 - \frac{h\beta}{z})\widehat{\widetilde{\lambda}}_t = \widehat{d}_t - \frac{h\beta}{z}\mathbb{E}_t\widehat{d}_{t+1} - \frac{1 + h^2/z^2\beta}{\left(1 - \frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_t + \frac{h}{\widetilde{z}\left(1 - \frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_{t-1} + \frac{\beta h/z}{\left(1 - \frac{h}{\widetilde{z}}\right)}\mathbb{E}_t\widehat{\widetilde{c}}_{t+1} - \frac{h}{\widetilde{z}\left(1 - \frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{z}}_t$$

Rearranging:

$$\widehat{d}_t - \frac{h\beta}{\widetilde{z}}\mathbb{E}_t\widehat{d}_{t+1} - \frac{1+h^2/\widetilde{z}^2\beta}{\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_t + \frac{b}{\widetilde{z}\left(1-\frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_{t-1} + \frac{\beta h/\tilde{z}}{\left(1-\frac{b}{\widetilde{z}}\right)}\mathbb{E}_t\widehat{\widetilde{c}}_{t+1} - \frac{b}{\widetilde{z}\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{z}}_t - \left(1-\frac{h\beta}{\widetilde{z}}\right)\widehat{\widetilde{\lambda}}_t = 0. \tag{34}$$

Equation 14 The fourteenth equation is:

$$\widehat{\widetilde{\lambda}}_t = \mathbb{E}_t\{\widehat{\widetilde{\lambda}}_{t+1} + \widehat{R}_t - \widehat{\Pi}_{t+1}\}$$

which we rearrange to:

$$\mathbb{E}_t\{\widehat{\widetilde{\lambda}}_{t+1} - \widehat{\widetilde{\lambda}}_t + \widehat{R}_t - \widehat{\Pi}_{t+1}\} = 0. \tag{35}$$

Equation 15 The fifteenth equation is:

$$\widehat{\widetilde{q}}_t = \mathbb{E}_t\Delta\widehat{\widetilde{\lambda}}_{t+1} + \frac{\beta\left(1-\delta\right)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1} + \left(1-\frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right)\mathbb{E}_t\widehat{\widetilde{r}}_{t+1}$$

which we rearrange to:

$$\mathbb{E}_t\widehat{\widetilde{\lambda}}_{t+1} - \widehat{\widetilde{\lambda}}_t + \frac{\beta\left(1-\delta\right)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1} + \left(1-\frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right)\mathbb{E}_t\widehat{\widetilde{r}}_{t+1} - \widehat{\widetilde{q}}_t = 0. \tag{36}$$

Equation 16 The sixteenth equation is:

$$\kappa\widetilde{z}^2\left(\Delta\widehat{\widetilde{x}}_t + \widehat{\widetilde{z}}_t\right) = \widehat{\widetilde{q}}_t + \beta\kappa\widetilde{z}^2\mathbb{E}_t\Delta\widehat{\widetilde{x}}_{t+1}$$

which we rearrange by undoing the first-difference operator:

$$\widehat{\widetilde{q}}_t + \beta\kappa\widetilde{z}^2\mathbb{E}_t\widehat{\widetilde{x}}_{t+1} - (1+\beta)\kappa\widetilde{z}^2\widehat{\widetilde{x}}_t + \kappa\widetilde{z}^2\widehat{\widetilde{x}}_{t-1} - \kappa\widetilde{z}^2\widehat{\widetilde{z}}_t = 0. \tag{37}$$

Equation 17 The seventeenth first equation is

$$\widehat{g}_t^1 = \left(1-\beta\theta_p\Pi^{\varepsilon(1-\chi)}\right)\left(\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d\right) + \beta\theta_p\Pi_t^{\varepsilon(1-\chi)}\mathbb{E}_t\left(\varepsilon(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t) + \widehat{g}_{t+1}^1\right)$$

As before, define $a_3 = \beta\theta_p\Pi^{\varepsilon(1-\chi)}$. Then:

$$(1-a_3)\,\widehat{\widetilde{\lambda}}_t + (1-a_3)\,\widehat{mc}_t + (1-a_3)\,\widehat{\widetilde{y}}_t^d + \varepsilon a_3\mathbb{E}_t\widehat{\Pi}_{t+1} - \chi\varepsilon a_3\widehat{\Pi}_t + a_3\mathbb{E}_t\widehat{g}_{t+1}^1 - \widehat{g}_t^1 = 0. \tag{38}$$

Equation 18 The eighteenth equation is

$$\begin{aligned}
\widehat{g}_t^2 &= \left(1-\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}\right)\left(\widehat{\widetilde{\lambda}}_t+\widehat{\Pi}_t^*+\widehat{\widetilde{y}}_t^d\right) \\
&\quad +\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}\mathbb{E}_t\left(-(1-\varepsilon)\left(\widehat{\Pi}_{t+1}-\chi\widehat{\Pi}_t\right)-\left(\widehat{\Pi}_{t+1}^*-\widehat{\Pi}_t^*\right)+\widehat{g}_{t+1}^2\right)
\end{aligned}$$

Define: $a_7=\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}$. Substituting:

$$(1-a_7)\,\widehat{\widetilde{\lambda}}_t+\widehat{\Pi}_t^*+(1-a_7)\,\widehat{\widetilde{y}}_t^d+(\varepsilon-1)a_7\mathbb{E}_t\widehat{\Pi}_{t+1}-\chi(\varepsilon-1)a_7\widehat{\Pi}_t-a_7\mathbb{E}_t\widehat{\Pi}_{t+1}^*+a_7\mathbb{E}_t\widehat{g}_{t+1}^2-\widehat{g}_t^2=0.$$

Shocks The preference shocks has the following structure:

$$\begin{aligned}
\widehat{d}_t &= \rho_d\widehat{d}_{t-1}+\varepsilon_{d,t} \\
\widehat{\varphi}_t &= \rho_\varphi\widehat{\varphi}_{t-1}+\varepsilon_{\varphi,t}.
\end{aligned}$$

The government expenditure process is

$$\widehat{\widetilde{g}}_t=\rho_g\widehat{\widetilde{g}}_{t-1}+\sigma_g\varepsilon_{g,t}.$$

The following shocks are i.i.d.:

$$\begin{aligned}
\widehat{\widetilde{\mu}}_t &= z_{\mu,t} \\
\widehat{\widetilde{A}}_t &= z_{A,t} \\
m_t &= \sigma_m\varepsilon_{m,t}.
\end{aligned}$$

6.1 Solving the model

Now, let

$$state_t=\left(\widehat{\Pi}_t,\widehat{g}_t^1,\widehat{g}_t^2,\widehat{\widetilde{k}}_t,\widehat{R}_t,\widehat{\widetilde{y}}_t^d,\widehat{\widetilde{c}}_t,\widehat{v}_t^p,\widehat{\widetilde{q}}_t,\widehat{\widetilde{x}}_t,\widehat{\widetilde{\lambda}}_t,\widehat{\widetilde{z}}_t\right)',$$

$$nstate_t=\left(\widehat{\widetilde{r}}_t,\widehat{u}_t,\widehat{\Pi}_t^*,\widehat{mc}_t,\widehat{l}_t,\widehat{\widetilde{w}}_t\right)',$$

$$exo_t=\left(z_{\mu,t},\widehat{d}_t,\widehat{\varphi}_t,z_{A,t},m_t\right)',$$

and

$$\varepsilon_t=\left(\varepsilon_{\mu,t},\varepsilon_{d,t},\varepsilon_{\varphi,t},\varepsilon_{A,t},\varepsilon_{m,t},\varepsilon_{g,t}\right)'.$$

Then, we write the system defined above as:

$$0 = AA * state_t + BB * state_{t-1} + CC * nstate_t + DD * exo_t,$$

$$0 = \mathbb{E}_t \left(\begin{array}{c} FF * state_{t+1} + GG * state_t + HH * state_{t-1} \\ +JJ * nstate_{t+1} + KK * nstate_t + LL * exo_{t+1} + MM * exo_t \end{array} \right),$$

and

$$exo_{t+1} = NN * exo_t + \Sigma^{1/2} * \varepsilon_{t+1} \text{ with } \mathbb{E}_t \varepsilon_{t+1} = 0.$$

The matrices in this notation are:

$$AA = \begin{pmatrix}
a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1-\gamma_R)\gamma_\Pi & 0 & 0 & 0 & -1 & (1-\gamma_R)\gamma_y & 0 & 0 & 0 & 0 & 0 & (1-\gamma_R)\gamma_y \\
0 & 0 & 0 & 0 & 0 & -\tilde{y}^d & \tilde{c} & 0 & 0 & \tilde{x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{y}^d v^p & 0 & \tilde{y}^d v^p & 0 & 0 & 0 & produc \\
a_3\varepsilon/\beta & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1-\frac{(1-\delta)}{\tilde{z}\tilde{\mu}} & 0 & -\frac{(1-\delta)}{\tilde{z}\tilde{\mu}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}$$

$$BB = \begin{pmatrix}
-\chi a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_R & -(1-\gamma_R)\gamma_y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha(produc) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-a_3\varepsilon\chi}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_3}{\beta} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-\delta}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$
CC = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 \\
-1 & \phi_u & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & -1 \\
\alpha & 0 & 0 & -1 & 0 & (1-\alpha) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\gamma_1 \tilde{k}}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 \\
0 & -\alpha(produc) & 0 & 0 & -(1-\alpha)(produc) & 0 \\
0 & 0 & -(1-\frac{a_3}{\beta})\varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \theta & -1
\end{pmatrix}
$$

$$
DD = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & g \\
0 & 0 & 0 & -produc & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1-\delta}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 & 0 \\
\frac{\alpha}{1-\alpha} & 0 & 0 & \frac{1}{1-\alpha} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

$$
FF = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta h/\tilde{z}}{1-\frac{h}{\tilde{z}}} & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta(1-\delta)}{\tilde{z}\tilde{\mu}} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\kappa\tilde{z}^2 & 0 & 0 \\
a_3\varepsilon & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_7(\varepsilon-1) & 0 & a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{pmatrix}
$$

$$
GG = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & GG_{1,7} & 0 & 0 & 0 & -(1-h\beta/\tilde{z}) & GG_{1,12} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & GG_{4,11} & 0 & -\kappa\tilde{z}^2 \\
-a_3\varepsilon\chi & -1 & 0 & 0 & 0 & 1-a_3 & 0 & 0 & 0 & 0 & 1-a_3 & 0 \\
-a_7(\varepsilon-1)\chi & 0 & -1 & 0 & 0 & 1-a_7 & 0 & 0 & 0 & 0 & 1-a_7 & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
GG_{1,7} &= -\frac{1+\beta h^2/\widetilde{z}^2}{1-\frac{h}{\tilde{z}}} \\
GG_{1,12} &= -\frac{h}{\tilde{z}(1-\frac{h}{\tilde{z}})} \\
GG_{4,11} &= -(1+\beta)\kappa\tilde{z}^2
\end{aligned}
$$

$$
HH = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{h}{\tilde{z}(1-\frac{h}{\tilde{z}})} & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa\tilde{z}^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
JJ = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{\beta(1-\delta)}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a_7 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
KK = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-a_3 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
LL = \begin{pmatrix} 0 & -\beta h/\tilde{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
MM = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
NN = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_d & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_g \end{pmatrix}
$$

$$
\Sigma = \begin{pmatrix} \sigma_\mu^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_d^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_\varphi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_A^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_m^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_g^2 \end{pmatrix}
$$

7 State space representation

A standard solution method gives us for the previous system of stochastic difference equations gives us:

$$
state_t = PP * state_{t-1} + QQ * exo_t
$$

and

$$
nstate_t = RR * state_{t-1} + SS * exo_t.
$$

and

$$
exo_t = NN * exo_{t-1} + \Sigma^{1/2} * \varepsilon_t.
$$

Our observables are inflation, the federal funds rate, and the first differences of real wages, output, the relative price of investment, and labor; or in our notation:

$$\mathbb{Y}_t = \left(\log \Pi_t, \log R_t, \triangle \log w_t, \triangle \log y_t, \log l_t, \triangle \log \mu_t^{-1}\right)'$$

Therefore, to write the likelihood function, we need to write the model in the following state space form:

$$\begin{aligned} S_t &= F * S_{t-1} + G * \varepsilon_t \\ obs_t &= H_0 + H_1 * S_t \end{aligned}$$

where $S_t = (state_{t-2}, exo_t, exo_{t-1})$.

Hence,

$$F = \begin{bmatrix} PP & 0 & QQ \\ 0 & NN & 0 \\ 0 & I & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} * \Sigma^{1/2}.$$

Building the H matrix requires more work. Note that:

$$\begin{aligned} \widehat{\Pi}_t &= PP\,(1,:) * state_{t-1} + QQ\,(1,:) * exo_t = \\ & PP\,(1,:) * PP * state_{t-2} + QQ\,(1,:) * exo_t + PP\,(1,:) * QQ * exo_{t-1}, \end{aligned}$$

$$\begin{aligned} \widehat{R}_t &= PP\,(5,:) * state_{t-1} + QQ\,(5,:) * exo_t = \\ & PP\,(5,:) * PP * state_{t-2} + QQ\,(5,:) * exo_t + PP\,(5,:) * QQ * exo_{t-1}, \end{aligned}$$

$$\begin{aligned} \widehat{\widehat{w}}_t - \widehat{\widehat{w}}_{t-1} &= RR\,(6,:) * state_{t-1} + SS\,(6,:) * exo_t - RR\,(6,:) * state_{t-2} - SS\,(2,:) * exo_{t-1} = \\ & RR\,(6,:) * PP * state_{t-2} + RR\,(6,:) * QQ * exo_{t-1} + \\ & SS\,(6,:) * exo_t - RR\,(6,:) * state_{t-2} - SS\,(2,:) * exo_{t-1} = \\ & (RR\,(6,:) * PP - RR\,(6,:)) * state_{t-2} + SS\,(6,:) * exo_t + \\ & (RR\,(6,:) * QQ - SS\,(6,:)) * exo_{t-1} \end{aligned}$$

$$\begin{aligned}
\widehat{\widetilde{y}}_t - \widehat{\widetilde{y}}_{t-1} &= PP\,(6,:) * state_{t-1} + QQ\,(6,:) * exo_t - PP\,(6,:) * state_{t-2} - QQ\,(6,:) * exo_{t-1} = \\
&\quad (PP\,(6,:) * PP - PP\,(6,:)) * state_{t-2} + QQ\,(6,:) * exo_t + \\
&\quad (PP\,(6,:) * QQ - QQ\,(6,:)) * exo_{t-1}
\end{aligned}$$

$$\begin{aligned}
\widehat{l}_t &= RR\,(5,:) * state_{t-1} + SS\,(5,:) * exo_t = \\
&\quad RR\,(5,:) * PP * state_{t-2} + SS\,(5,:) * exo_t + RR\,(5,:) * QQ * exo_{t-1}
\end{aligned}$$

$$\widehat{\widetilde{\mu}}_t = (1\ 0\ 0\ 0\ 0) exo_t.$$

Also, remember that $\widehat{\Pi}_t = \log \Pi_t - \log \Pi_{ss}$, $\widehat{R}_t = \log R_t - \log R_{ss}$, $\widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1} = \triangle \log \widetilde{w}_t = \triangle \log w_t - \triangle \log z_t$, $\widehat{\widetilde{y}}_t - \widehat{\widetilde{y}}_{t-1} = \triangle \log \widetilde{y}_t = \triangle \log y_t - \triangle \log z_t.$, $\widehat{l}_t = \log l_t - \log l_{ss} = \log l_t$, $\widehat{\widetilde{\mu}}_t = \log \frac{\mu_t}{\mu_{t-1}} - \Lambda_\mu$, $\triangle \log \mu_t^{-1} = -\log \frac{\mu_t}{\mu_{t-1}}$, $\Rightarrow \triangle \log \mu_t^{-1} = -\widehat{\widetilde{\mu}}_t - \Lambda_\mu$.

Since $\triangle \log z_t = \frac{\alpha \Lambda_\mu + \Lambda_A}{1-\alpha} + \frac{\alpha z_{\mu,t} + z_{A,t}}{1-\alpha}$, we have:

$$obs_t = H_0 + H_1 * S_t.$$

where

$$H_0 = \begin{bmatrix} \log \Pi_{ss} \\ \log R_{ss} \\ \frac{\alpha \Lambda_\mu + \Lambda_A}{1-\alpha} \\ \frac{\alpha \Lambda_\mu + \Lambda_A}{1-\alpha} \\ 0 \\ -\Lambda_\mu \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} PP\,(1,:)\, PP\,(1,:) & QQ\,(1,:) & PP\,(1,:)\, QQ\,(1,:) \\ PP\,(5,:)\, PP\,(5,:) & QQ\,(5,:) & PP\,(5,:)\, QQ\,(5,:) \\ RR\,(6,:)\,(RR\,(6,:) - I) & SS\,(6,:) & SS\,(6,:)\,(RR\,(6,:) - I) \\ PP\,(6,:)\,(PP\,(6,:) - I) & QQ\,(6,:) & QQ\,(6,:)\,(PP\,(6,:) - I) \\ RR\,(5,:)\, RR\,(5,:) & SS\,(5,:) & RR\,(5,:)\, SS\,(5,:) \\ 0 & (-1\ 0\ 0\ 0\ 0\ 0) & 0 \end{bmatrix}.$$