Appendix to
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Abstract

This Appendix contains a detailed derivation of the Generalized New Keynesian (GNK) model with a direct reference to the equations in the published article (indicated as PA). We present the model, derive the steady state and the log-linear equations, and provide background for the dynamic analysis in the PA.

*The views expressed here are solely those of the authors and do not necessarily reflect the position of the Federal Reserve Bank of New York or any other part of the Federal Reserve System.

Contents

1 The GNK Model

1.1 Households

The period utility function of the representative agent is assumed to be separable in consumption (C) and labor (N):

$$U\left(C_t, N_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma} - d_n e^{\varsigma_t} \frac{N_t^{1+\varphi}}{1+\varphi}, \tag{1}$$

where ς_t is a labour supply shock; the period by period budget constraint is given by:

$$P_t C_t + \left(1+i_t\right)^{-1} B_t = W_t N_t + D_t + B_{t-1}, \tag{2}$$

where i_t is the nominal interest rate, B_t is one-period bond holdings, W_t is the nominal wage rate, N_t is the labor input, and D_t is distributed dividends. The representative consumer maximizes the expected discounted intertemporal utility, subject to the budget constraints; this yields the following first-order conditions, where β is the discount factor:

$$\text{Euler equation}: \quad \frac{1}{C_t^\sigma} = \beta E_t \left[\left(\frac{P_t}{P_{t+1}} \right) \left(1+i_t\right) \left(\frac{1}{C_{t+1}^\sigma} \right) \right], \tag{3}$$

$$\text{Labor supply equation:} \quad \frac{W_t}{P_t} = -\frac{U_N}{U_C} = \frac{d_n e^{\varsigma_t} N_t^\varphi}{1/C_t^\sigma} = d_n e^{\varsigma_t} N_t^\varphi C_t^\sigma. \tag{4}$$

1.2 Technology

Final good producers aggregate intermediate goods $Y_{i,t}$ with technology:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{5}$$

where ε indicates the elasticity of substitution among intermediate goods. Their optimal demand for intermediate inputs is equal to $Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\varepsilon} Y_t$.

The production function of intermediate goods producers is:

$$Y_{i,t} = A_t N_{i,t}^{1-\alpha}, \tag{6}$$

where A_t is an exogenous process for the level of technology, which we assume stationary.[1] The labor demand of firm i is:

$$N_{i,t}^d = \left[\frac{Y_{i,t}}{A_t} \right]^{\frac{1}{1-\alpha}}. \tag{7}$$

Total cost and marginal cost, in real term, are respectively:

$$TC_{i,t}^r = \frac{W_t}{P_t} N_{i,t} = \frac{W_t}{P_t} \left[\frac{Y_{i,t}}{A_t} \right]^{\frac{1}{1-\alpha}}, \tag{8}$$

[1] In the PA the production technology is linear in labor ($\alpha = 0$). Here we consider the more general case of decreasing returns to labor ($0 < \alpha < 1$).

and

$$MC^r_{i,t} = \frac{A_t^{\frac{1}{\alpha-1}}}{1-\alpha}\frac{W_t}{P_t}Y_{i,t}^{\frac{\alpha}{1-\alpha}}. \tag{9}$$

Note that marginal costs of firm i depend upon the quantity produced by the firm, given decreasing returns to scale. Since firms charging different prices would produce different levels of output, they have different marginal costs. Substituting the expression for $Y_{i,t}$ in (9), these are:

$$MC^r_{i,t} = \frac{A_t^{\frac{1}{\alpha-1}}}{1-\alpha}\frac{W_t}{P_t}\left[\left(\frac{P_{i,t}}{P_t}\right)^{-\varepsilon}Y_t\right]^{\frac{\alpha}{1-\alpha}}. \tag{10}$$

1.3 Firms' pricing

The pricing model is based on Calvo (1983). In each period there is a fixed probability $1-\theta$ that a firm can re-optimize its nominal price, which we denote by $P^*_{i,t}$. With probability θ, instead, the firm can index its price to the previous period inflation rate: $P_{i,t} = \pi_t^{\varrho}P_{i,t-1}$, where inflation is $\pi_t = \frac{P_t}{P_{t-1}}$ and $\varrho \in [0,1]$ indicates the degree of indexation (e.g., Christiano et al., 2005).[2] The price setting problem is:

$$\max_{P^*_{i,t}} \quad E_t\sum_{j=0}^{\infty}\mathcal{D}_{t,t+j}\theta^j\left[\frac{P^*_{i,t}\Pi^{\varrho}_{t-1,t+j-1}}{P_{t+j}}Y_{i,t+j} - \frac{W_{t+j}}{P_{t+j}}\left(\frac{Y_{i,t+j}}{A_{t+j}}\right)^{\frac{1}{1-\alpha}}\right],$$

subject to the demand constraint:

$$Y_{i,t+j} = \left(\frac{P^*_{i,t}\Pi^{\varrho}_{t-1,t+j-1}}{P_{t+j}}\right)^{-\varepsilon}Y_{t+j},$$

where $\mathcal{D}_{t,t+j} \equiv \beta^j\frac{\lambda_{t+j}}{\lambda_0}$ is the stochastic discount factor, with λ_{t+j} denoting the $t+j$ marginal utility of consumption. $\Pi_{t,t+j}$ indicates cumulative inflation between periods t and $t+j$:

$$\Pi_{t,t+j} = \begin{cases} 1 & \text{for } j=0 \\ \frac{P_{t+1}}{P_t}\frac{P_{t+2}}{P_{t+1}}\times\cdots\times\frac{P_{t+j}}{P_{t+j-1}} & \text{for } j\geqslant 1 \end{cases} \tag{11}$$

The firm's first order condition is:

$$E_t\sum_{j=0}^{\infty}\theta^j\mathcal{D}_{t,t+j}\left[\begin{array}{c}(1-\varepsilon)\left(P^*_{i,t}\right)^{-\varepsilon}\left[\frac{\Pi^{\varrho}_{t-j,t+j-1}}{P_{t+j}}\right]^{1-\varepsilon}Y_{t+j}+ \\ \frac{\varepsilon}{1-\alpha}\left(P^*_{i,t}\right)^{\frac{-\varepsilon-1+\alpha}{1-\alpha}}\frac{W_{t+j}}{P_{t+j}}\left[\frac{Y_{t+j}}{A_{t+j}}\right]^{\frac{1}{1-\alpha}}\left[\frac{\Pi^{\varrho}_{t-j,t+j-1}}{P_{t+j}}\right]^{\frac{-\varepsilon}{1-\alpha}}\end{array}\right] = 0,$$

from which we get:

$$\left(P^*_{i,t}\right)^{1+\frac{\varepsilon\alpha}{1-\alpha}} = \frac{\varepsilon}{\varepsilon-1}\frac{1}{1-\alpha}\frac{E_t\sum_{j=0}^{\infty}\theta^j\mathcal{D}_{t,t+j}\frac{W_{t+j}}{P_{t+j}}\left[\frac{Y_{t+j}}{A_{t+j}}\right]^{\frac{1}{1-\alpha}}\left[\frac{\Pi^{\varrho}_{t-j,t+j-1}}{P_{t+j}}\right]^{\frac{-\varepsilon}{1-\alpha}}}{E_t\sum_{j=0}^{\infty}\theta^j\mathcal{D}_{t,t+j}\left[\frac{\Pi^{\varrho}_{t-j,t+j-1}}{P_{t+j}}\right]^{1-\varepsilon}Y_{t+j}} \tag{12}$$

[2] This is a more general version than the baseline model analyzed in the PA, for which there is no indexation. As we indicate in what follows, the PA results are obtained by setting $\varrho = 0$.

Dividing both sides by P_t (and observing that $1+\frac{\varepsilon\alpha}{1-\alpha}=(1-\varepsilon)+\frac{\varepsilon}{1-\alpha}$), we get:

$$\left(\frac{P_{i,t}^{*}}{P_t}\right)^{1+\frac{\varepsilon\alpha}{1-\alpha}}=\frac{\varepsilon}{(\varepsilon-1)(1-\alpha)}\frac{E_t\sum_{j=0}^{\infty}\theta^j\mathcal{D}_{t,t+j}\frac{W_{t+j}}{P_{t+j}}\left[\frac{Y_{t+j}}{A_{t+j}}\right]^{\frac{1}{1-\alpha}}\left[\frac{\Pi_{t-j,t+j-1}^{\varrho}}{\Pi_{t,t+j}}\right]^{\frac{-\varepsilon}{1-\alpha}}}{E_t\sum_{j=0}^{\infty}\theta^j\mathcal{D}_{t,t+j}\left[\frac{\Pi_{t-j,t+j-1}^{\varrho}}{\Pi_{t,t+j}}\right]^{1-\varepsilon}Y_{t+j}}. \tag{13}$$

Note that (13) reduces to equation (14) in the PA when setting $\alpha=\varrho=0$.

The aggregate price level $P_t=\left[\int_0^1 P_{i,t}^{1-\varepsilon}di\right]^{\frac{1}{1-\varepsilon}}$ evolves according to:

$$P_t=\left[\theta\pi_{t-1}^{(1-\varepsilon)\varrho}P_{t-1}^{1-\varepsilon}+(1-\theta)\left(P_{i,t}^{*}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \tag{14}$$

which is equation (16) in the PA when $\varrho=0$. Dividing by P_t and letting $\pi_t=P_t/P_{t-1}$ and relative price $p_{i,t}^{*}=P_{i,t}^{*}/P_t$, we can write (14) as:

$$1=\theta\pi_{t-1}^{(1-\varepsilon)\varrho}\pi_t^{\varepsilon-1}+(1-\theta)\left(p_{i,t}^{*}\right)^{1-\varepsilon}, \tag{15}$$

or:

$$p_{i,t}^{*}=\left[\frac{1-\theta\pi_{t-1}^{(1-\varepsilon)\varrho}\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}, \tag{16}$$

which corresponds to equation (30) in the PA for $\varrho=0$.

1.3.1 Recursive formulation of the optimal price-setting equation

To derive the generalized new Keynesian Phillips curve (GNKPC) we use a recursive formulation of the optimal price-setting equation. First, we write (13) as:

$$\left(p_{i,t}^{*}\right)^{1+\frac{\varepsilon\alpha}{1-\alpha}}=\frac{\varepsilon}{(\varepsilon-1)(1-\alpha)}\frac{\psi_t}{\phi_t}, \tag{17}$$

where, denoting the real wage as $w_t=W_t/P_t$, using the definition of the discount factor $\mathcal{D}_{t,t+j}\equiv\beta^j\frac{\lambda_{t+j}}{\lambda_0}$, and the fact that $\lambda_{t+j}=C_{t+j}^{-\sigma}$ ($=Y_{t+j}^{-\sigma}$ in equilibrium), the auxiliary variables ψ_t and ϕ_t are defined, respectively, as:

$$\psi_t\equiv E_t\sum_{j=0}^{\infty}(\theta\beta)^j\,Y_{t+j}^{\frac{1}{1-\alpha}-\sigma}A_{t+j}^{\frac{-1}{1-\alpha}}w_{t+j}\left[\frac{\Pi_{t-j,t+j-1}^{\varrho}}{\Pi_{t,t+j}}\right]^{\frac{-\varepsilon}{1-\alpha}} \tag{18}$$

$$\phi_t\equiv E_t\sum_{j=0}^{\infty}(\theta\beta)^j\,Y_{t+j}^{1-\sigma}\left[\frac{\Pi_{t-j,t+j-1}^{\varrho}}{\Pi_{t,t+j}}\right]^{1-\varepsilon}. \tag{19}$$

We then rewrite the infinite sums in (18) and (19) *recursively*, as

$$\psi_t=w_tA_t^{\frac{-1}{1-\alpha}}Y_t^{\frac{1}{1-\alpha}-\sigma}+\theta\beta\pi_t^{\frac{-\varrho\varepsilon}{1-\alpha}}E_t\left[\pi_{t+1}^{\frac{\varepsilon}{1-\alpha}}\psi_{t+1}\right], \tag{20}$$

and

$$\phi_t = Y_t^{1-\sigma} + \theta\beta\pi_t^{\varrho(1-\varepsilon)} E_t\left[\pi_{t+1}^{\varepsilon-1}\phi_{t+1}\right]. \tag{21}$$

Defining an economy-wide real marginal costs as:

$$MC_t = \frac{1}{1-\alpha} w_t A_t^{\frac{1}{\alpha-1}} Y_t^{\frac{\alpha}{1-\alpha}}, \tag{22}$$

equations (17), (20) and (21) correspond to (26)-(28) in the PA, by setting $\alpha = \varrho = 0$.

1.4 Aggregation and price dispersion

Using (7) and the expression for intermediate inputs $Y_{i,t}$, aggregate labor demand is

$$N_t^d = \int_0^1 N_{i,t}^d di = \int_0^1 \left(\frac{Y_{i,t}}{A_t}\right)^{\frac{1}{1-\alpha}} di = \underbrace{\int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di}_{s_t} \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} = s_t \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}. \tag{23}$$

Schmitt-Grohé and Uribe (2007) show that the variable s_t is bounded below by one, so that in this model s_t represents the resource costs due to relative price dispersion arising from positive long-run inflation: the higher s_t, the more labor is needed to produce a given level of output. If trend inflation is zero ($\bar{\pi} = 1$), s_t does not affect the real variables up to the first order. In the PA we thus define s_t as the relative price dispersion measure.

As shown by Schmitt-Grohé and Uribe (2007), under the assumptions of the Calvo model s_t can be rewritten as:

$$s_t = (1-\theta)\left(\frac{P_{i,t}^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta(1-\theta)\left(\frac{P_{i,t-1}^*\pi_{t-1}^{\varrho}}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + $$
$$+ \theta^2(1-\theta)\left(\frac{P_{i,t-2}^*\left(\pi_{t-1}^{\varrho}\pi_{t-2}^{\varrho}\right)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \cdots$$

Collecting terms yields:

$$s_t = (1-\theta)\left(\frac{P_{i,t}^*}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta\pi_{t-1}^{-\frac{\varepsilon\varrho}{1-\alpha}}\pi_t^{\frac{\varepsilon}{1-\alpha}} \times$$
$$\times \left\{(1-\theta)\left(\frac{P_{i,t-1}^*}{P_{t-1}}\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta(1-\theta)\left(\frac{P_{i,t-2}^*\left(\pi_{t-2}^{\varrho}\right)}{P_{t-1}}\right)^{-\frac{\varepsilon}{1-\alpha}} + \cdots\right\}.$$

The expression in curly brackets is exactly the definition of s_{t-1}. Thus, it follows that the equation for s_t can be written recursively as:

$$s_t = (1-\theta)\left(p_{i,t}^*\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta\left(\pi_{t-1}^{-\frac{\varepsilon\varrho}{1-\alpha}}\right)\pi_t^{\frac{\varepsilon}{1-\alpha}} s_{t-1}, \tag{24}$$

which corresponds to equation (34) in the PA, when setting $\alpha = \varrho = 0$.

1.5 Monetary policy

The central bank follows a standard Taylor rule, with weight ϕ_π on deviations of inflation from target $\bar{\pi}$ (we assume that target inflation and trend inflation are the same) and weight ϕ_y on output deviations from steady state output $\bar{Y}$:

$$\left(\frac{1+i_t}{1+\bar{\imath}}\right) = \left(\frac{\pi_t}{\bar{\pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_Y} e^{v_t}, \tag{25}$$

where v_t is a monetary policy shock, and ϕ_π, ϕ_Y are non-negative parameters.

1.6 The complete non-linear system

To summarize, the complete non-linear model, imposing equilibrium condition $C_t = Y_t$, is described by equations (3), (4), (16), (17), (20), (21), (23), (24), and (25), which we reproduce here for convenience:

$$\frac{1}{Y_t^\sigma} = \beta E_t \left[\left(\frac{1+i_t}{\pi_{t+1}}\right)\left(\frac{1}{Y_{t+1}^\sigma}\right)\right]$$

$$w_t = d_n e^{\varsigma_t} N_t^\varphi Y_t^\sigma$$

$$p_{i,t}^* = \left[\frac{1-\theta\pi_{t-1}^{(1-\varepsilon)\varrho}\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}$$

$$\left(p_{i,t}^*\right)^{1+\frac{\varepsilon\alpha}{1-\alpha}} = \frac{\varepsilon}{(\varepsilon-1)(1-\alpha)}\frac{\psi_t}{\phi_t}$$

$$\psi_t = w_t A_t^{\frac{-1}{1-\alpha}} Y_t^{\frac{1}{1-\alpha}-\sigma} + \theta\beta\pi_t^{\frac{-\varrho\varepsilon}{1-\alpha}} E_t\left[\pi_{t+1}^{\frac{\varepsilon}{1-\alpha}}\psi_{t+1}\right]$$

$$\phi_t = Y_t^{1-\sigma} + \theta\beta\pi_t^{\varrho(1-\varepsilon)} E_t\left[\pi_{t+1}^{\varepsilon-1}\phi_{t+1}\right]$$

$$N_t = s_t\left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}$$

$$s_t = (1-\theta)\left(p_{i,t}^*\right)^{-\frac{\varepsilon}{1-\alpha}} + \theta\left(\pi_{t-1}^{-\frac{\varepsilon\varrho}{1-\alpha}}\right)\pi_t^{\frac{\varepsilon}{1-\alpha}} s_{t-1}$$

$$\left(\frac{1+i_t}{1+\bar{\imath}}\right) = \left(\frac{\pi_t}{\bar{\pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_Y} e^{v_t}$$

The model has nine endogenous variables: $Y_t, i_t, \pi_t, w_t, N_t, p_{i,t}, \phi_t, \psi_t, s_t$ and three exogenous shocks: A_t, ς_t, v_t.[3]

[3] A DYNARE code to simulate the model is available from the authors upon request.

2 Deterministic steady state

We compute the non-stochastic steady state of this system, characterized by a positive trend inflation $\bar{\pi}$, as:

$$\bar{\pi} = \beta (1 + i), \tag{26}$$

$$w = d_n N^{\varphi} Y^{\sigma}, \tag{27}$$

$$p_i^* = \left[\frac{1 - \theta \bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}, \tag{28}$$

$$(p_i^*)^{1+\frac{\varepsilon\alpha}{1-\alpha}} = \frac{\varepsilon}{(\varepsilon-1)(1-\alpha)} \frac{\psi}{\phi}, \tag{29}$$

$$\psi = \frac{w A^{\frac{-1}{1-\alpha}} Y^{\frac{1}{1-\alpha}-\sigma}}{1 - \theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}}, \tag{30}$$

$$\phi = \frac{Y^{1-\sigma}}{1 - \theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}, \tag{31}$$

$$N = s \left[\frac{Y}{A} \right]^{\frac{1}{1-\alpha}}, \tag{32}$$

$$s = \frac{1-\theta}{1 - \theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}} (p_i^*)^{-\frac{\varepsilon}{1-\alpha}}, \tag{33}$$

and

$$MC = \frac{1}{1-\alpha} w \frac{Y^{\frac{\alpha}{1-\alpha}}}{A^{\frac{1}{1-\alpha}}}. \tag{34}$$

Note that substituting (28) in (33) gives the relation between price dispersion and inflation in steady state:

$$s = \frac{1-\theta}{1 - \theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}} \left[\frac{1 - \theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta} \right]^{\frac{\varepsilon}{(\varepsilon-1)(1-\alpha)}},$$

Setting $\alpha = \varrho = 0$ this is equation (35) in the PA.

Plugging the values of w from (27) and N from (32) into (34) we obtain:

$$MC = \frac{d_n N^{\varphi} Y^{\sigma}}{1-\alpha} \left(\frac{Y^{\alpha}}{A} \right)^{\frac{1}{1-\alpha}} = \frac{d_n s^{\varphi} \left[\frac{Y}{A}\right]^{\frac{\varphi}{1-\alpha}} Y^{\sigma}}{1-\alpha} \left(\frac{Y^{\alpha}}{A} \right)^{\frac{1}{1-\alpha}} = \frac{d_n s^{\varphi}}{1-\alpha} \frac{Y^{\frac{\varphi+\alpha}{1-\alpha}+\sigma}}{A^{\frac{1+\varphi}{1-\alpha}}}. \tag{35}$$

From this equation it follows that:

$$Y^{\frac{\varphi+\alpha}{1-\alpha}+\sigma} = \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}} MC}{d_n s^{\varphi}} = \frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}}}{d_n s^{\varphi} \mu},$$

where we indicated by μ the steady state mark-up ($\mu = 1/MC$). The expression for steady state output is then:

$$Y = \left[\frac{(1-\alpha) A^{\frac{1+\varphi}{1-\alpha}}}{d_n s^{\varphi} \mu} \right]^{\frac{1-\alpha}{\varphi+\sigma+\alpha(1-\sigma)}} \equiv Y(\bar{\pi}). \tag{36}$$

Setting $\alpha = 0$, this is equation (36) in the PA.

In this model superneutrality doesn't hold. We note this by indicating steady state output in (36) as $Y(\bar{\pi})$; this notation will help distinguishing this steady state level of output from the output level in a steady state with zero inflation.

We can also obtain a relation between optimal price and marginal cost in steady state. Using (29) and (27) we get:

$$(p_i^*)^{1+\frac{\varepsilon\alpha}{1-\alpha}} = \frac{\varepsilon}{(\varepsilon-1)(1-\alpha)} \frac{\frac{wA^{\frac{-1}{1-\alpha}}Y^{\frac{1}{1-\alpha}-\sigma}}{1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}}}{\frac{Y^{1-\sigma}}{1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}} = \frac{\varepsilon}{\varepsilon-1}\left(\frac{1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}}\right)MC,$$

where the second equality follows from observing that, from (34): $wA^{\frac{-1}{1-\alpha}}Y^{\frac{1}{1-\alpha}-\sigma} = (1-\alpha)Y^{1-\sigma}MC$. Therefore:

$$p_i^* = \left[\frac{\varepsilon}{\varepsilon-1}\left(\frac{1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}}\right)MC\right]^{\frac{1-\alpha}{1+\alpha(\varepsilon-1)}}.$$

When $\alpha = 0$, the above equation gives the ratio $\frac{p_i^*}{MC}$, as in equation (37) of the PA. Assuming also $\varrho = 0$, we obtain the simple decomposition of the mark-up discussed in the PA:

$$\mu = \frac{1}{MC} = \left(\frac{P}{P_i^*}\right)\left(\frac{p_i^*}{MC}\right) = \underbrace{\left[\frac{1-\theta\bar{\pi}^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{\varepsilon-1}}}_{\text{price adjustment gap}}\underbrace{\left[\frac{\varepsilon}{\varepsilon-1}\frac{1-\beta\theta\bar{\pi}^{\varepsilon-1}}{1-\beta\theta\bar{\pi}^{\varepsilon}}\right]}_{\text{marginal markup}}. \tag{37}$$

As mentioned in footnote 40 of the PA, it is easy to show that with positive trend inflation ($\bar{\pi} > 1$) the *marginal markup* is increasing in β :

$$\begin{aligned}\frac{\partial\left(\frac{1-\beta\theta\bar{\pi}^{\varepsilon-1}}{1-\beta\theta\bar{\pi}^{\varepsilon}}\right)}{\partial\beta} &= \frac{-\theta\bar{\pi}^{\varepsilon-1}\left(1-\beta\theta\bar{\pi}^{\varepsilon}\right)+\theta\bar{\pi}^{\varepsilon}\left(1-\beta\theta\bar{\pi}^{\varepsilon-1}\right)}{\left(1-\beta\theta\bar{\pi}^{\varepsilon}\right)^2} \\ &= \frac{\theta\bar{\pi}^{\varepsilon-1}\left[\bar{\pi}-1\right]}{\left(1-\beta\theta\bar{\pi}^{\varepsilon}\right)^2} > 0.\end{aligned}$$

Hence, the higher the discounting (the lower β), the lower is the marginal mark-up and, other things equal, the lower is the average mark-up and the higher the steady state output.

2.1 The cost of price dispersion in steady state

From (32):

$$Y = \frac{A}{s^{1-\alpha}}N^{1-\alpha} = \tilde{A}N^{1-\alpha}.$$

where we defined $\tilde{A} = \frac{A}{s^{1-\alpha}}$. A change in s causes a variation of $\tilde{A}$. By definition, denoting by $\Delta\tilde{A}$ the percentage change in $\tilde{A}$, we have that a change in s from s_0 to s_1 yields:

$$\Delta\tilde{A} = \frac{\frac{\bar{A}}{s_1^{1-\alpha}} - \frac{\bar{A}}{s_0^{1-\alpha}}}{\frac{\bar{A}}{s_0^{1-\alpha}}} = \left(\frac{s_0}{s_1}\right)^{1-\alpha} - 1 = \left(\frac{1}{1+\Delta s}\right)^{1-\alpha} - 1, \tag{38}$$

by using the fact that $\Delta s = \frac{s_1 - s_0}{s_0}$, hence $\frac{s_1}{s_0} = 1 + \Delta s$. Expression (38) allows to map percentage changes in the steady state value s, which are due to changes in trend inflation, into an equivalent percentage change in aggregate productivity. If $s_0 = 1$, so that $\Delta s = s_1 - 1$:

$$\Delta \tilde{A} = \left(\frac{1}{s_1}\right)^{1-\alpha} - 1.$$

If , in addition, $\alpha = 0$:

$$\Delta \tilde{A} = \frac{1}{s_1} - 1,$$

showing that, as discussed in footnote 37 of the PA, decreasing returns to labor ($0 < \alpha < 1$) make price dispersion more sensitive to changes in trend inflation, and therefore deliver a larger equivalent change in productivity for any change in s.

3 The log-linear GNK model

We now take a log-linear approximation around the deterministic steady state defined above for a generic trend inflation rate $\bar{\pi}$. Defining by $\hat{x}_t$ the log deviation of any variable x_t from its steady state, from (3), (4), (23), and (25), we obtain, respectively:

$$\hat{Y}_t = E_t \hat{Y}_{t+1} - \sigma^{-1} \left(\hat{\imath}_t - E_t \hat{\pi}_{t+1}\right), \tag{39}$$

$$\hat{w}_t = \varphi \hat{N}_t + \sigma \hat{Y}_t + \varsigma_t, \tag{40}$$

$$\hat{N}_t = \hat{s}_t + \left(\frac{1}{1-\alpha}\right) \left[\hat{Y}_t - \hat{A}_t\right], \tag{41}$$

$$\hat{\imath}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{Y}_t + v_t. \tag{42}$$

To obtain an expression for $\hat{s}_t$ we start by log-linearizing (24) to obtain:

$$\hat{s}_t = \left[\frac{-\frac{\varepsilon}{1-\alpha}\left(1-\theta\right)\left(p_i^*\right)^{-\frac{\varepsilon}{1-\alpha}}}{s}\right] \hat{p}_{i,t}^* + \left[\frac{-\frac{\varepsilon\varrho}{1-\alpha}\theta\bar{\pi}^{-\frac{\varepsilon\varrho}{1-\alpha}}\bar{\pi}^{\frac{\varepsilon}{1-\alpha}}s}{s}\right]\hat{\pi}_{t-1} +$$
$$\left[\frac{\frac{\varepsilon}{1-\alpha}\theta\bar{\pi}^{-\frac{\varepsilon\varrho}{1-\alpha}}\bar{\pi}^{\frac{\varepsilon}{1-\alpha}}s}{s}\right]\hat{\pi}_t + \left[\frac{\theta\bar{\pi}^{-\frac{\varepsilon\varrho}{1-\alpha}}\bar{\pi}^{\frac{\varepsilon}{1-\alpha}}s}{s}\right]\hat{s}_{t-1}.$$

Next, substituting the expression for steady state price dispersion s from (33), we obtain:

$$\hat{s}_t = \left[-\frac{\varepsilon}{1-\alpha}\left(1 - \theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right)\right]\hat{p}_{i,t}^* + \left[\theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[-\frac{\varepsilon\varrho}{1-\alpha}\hat{\pi}_{t-1} + \frac{\varepsilon}{1-\alpha}\hat{\pi}_t + \hat{s}_{t-1}\right]. \tag{43}$$

Next, we log-linearize expression (17) to get:

$$\left(1 + \frac{\varepsilon\alpha}{1-\alpha}\right)\hat{p}_{i,t}^* = \hat{\psi}_t - \hat{\phi}_t, \tag{44}$$

where expressions for $\hat{\psi}_t$ and $\hat{\phi}_t$ are obtained by log-linearizing equations (20) and (21), respectively.

From (20) we get:

$$\begin{aligned}\hat{\psi}_t &= \left[\frac{wA^{\frac{-1}{1-\alpha}}\bar{Y}^{\frac{1}{1-\alpha}-\sigma}}{\psi}\right]\hat{w}_t - \left[\frac{1}{1-\alpha}\frac{wA^{\frac{-1}{1-\alpha}}\bar{Y}^{\frac{1}{1-\alpha}-\sigma}}{\psi}\right]\hat{A}_t + \\ &+ \left[\left(\frac{1}{1-\alpha}-\sigma\right)\frac{wA^{\frac{-1}{1-\alpha}}\bar{Y}^{\frac{1}{1-\alpha}-\sigma}}{\psi}\right]\hat{Y}_t \\ &- \left[\frac{\varrho\varepsilon\theta\beta}{1-\alpha}\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\hat{\pi}_t + \left[\frac{\varepsilon\theta\beta}{1-\alpha}\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]E_t\hat{\pi}_{t+1} + \left[\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]E_t\hat{\psi}_{t+1}.\end{aligned}$$

Substituting in the value for ψ from (30), this expression becomes:

$$\begin{aligned}\hat{\psi}_t &= \left[1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[\hat{w}_t - \left(\frac{1}{1-\alpha}\right)\hat{A}_t + \left(\frac{1}{1-\alpha}-\sigma\right)\hat{Y}_t\right] \\ &+ \left[\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[E_t\hat{\psi}_{t+1} + \left(\frac{\varepsilon}{1-\alpha}\right)E_t\hat{\pi}_{t+1} - \left(\frac{\varrho\varepsilon}{1-\alpha}\right)\hat{\pi}_t\right].\end{aligned} \tag{45}$$

Analogously, from (21) we get:

$$\begin{aligned}\hat{\phi}_t &= \left[(1-\sigma)\frac{\bar{Y}^{1-\sigma}}{\bar{\phi}}\right]\hat{Y}_t + \left[\varrho(1-\varepsilon)\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\hat{\pi}_t \\ &+ \left[(\varepsilon-1)\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]E_t\hat{\pi}_{t+1} + \left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]E_t\hat{\phi}_{t+1}\end{aligned}$$

which, upon substituting in the value of steady state ϕ from (31), becomes:

$$\hat{\phi}_t = \left(1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)(1-\sigma)\hat{Y}_t + \left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\left[\varrho(1-\varepsilon)\hat{\pi}_t + E_t\hat{\phi}_{t+1} + (\varepsilon-1)E_t\hat{\pi}_{t+1}\right]. \tag{46}$$

Finally, the log-linearization of equation (16) gives:

$$\begin{aligned}0 &= \left[\theta(1-\varepsilon)\varrho\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\hat{\pi}_{t-1} + \left[\theta(\varepsilon-1)\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\hat{\pi}_t \\ &+ \left[(1-\theta)(1-\varepsilon)(p_i^*)^{1-\varepsilon}\right]\hat{p}_{i,t}^*,\end{aligned}$$

which, substituting p_i^* from (28), becomes:

$$\left[\theta(1-\varepsilon)\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right][\hat{\pi}_t - \varrho\hat{\pi}_{t-1}] = \left[\left(1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)(1-\varepsilon)\right]\hat{p}_{i,t}^*$$

or

$$\hat{p}_{i,t}^* = \frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{\left(1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)}[\hat{\pi}_t - \varrho\hat{\pi}_{t-1}]. \tag{47}$$

To sum up, the log-linearized GNK model consists of equations (39), (40), (41), (43), (47), (44), (45), (46), and (42).

By combining equations (40), (47), (44), (46), (41), and (43), one can substitute out the variables $\hat{N}_t, \hat{p}_{i,t}^*, \hat{w}_t$ and $\hat{\phi}_t$, and write the system in more compact form as a

5-equation system in the variables $\hat{Y}_t, \hat{\pi}_t, \hat{\psi}_t, \hat{s}_t$ and $\hat{\imath}_t$, driven by the three exogenous shocks: $\hat{A}_t, \varsigma_t$, and v_t.

In the next two subsections we show that the supply side of the model can be alternatively written as a GNKPC in terms of marginal costs, in terms of output, or in terms of output gap.

3.1 The GNKPC in terms of marginal costs

To ease notation, we define the quasi-difference of inflation as $\Delta_t \equiv \hat{\pi}_t - \varrho\hat{\pi}_{t-1}$. Equations (47), (45) and (46) can then be written, respectively, as:

$$\hat{p}^*_{i,t} = \frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}\Delta_t \tag{48}$$

$$\begin{aligned}\hat{\psi}_t = &\left[1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[\hat{w}_t - \left(\frac{1}{1-\alpha}\right)\hat{A}_t + \left(\frac{1}{1-\alpha}-\sigma\right)\hat{Y}_t\right] \\ &+\left[\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[E_t\hat{\psi}_{t+1} + \left(\frac{\varepsilon}{1-\alpha}\right)E_t\Delta_{t+1}\right]\end{aligned} \tag{49}$$

$$\hat{\phi}_t = \left(1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)(1-\sigma)\hat{Y}_t + \left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\left[E_t\hat{\phi}_{t+1} + (\varepsilon-1)E_t\Delta_{t+1}\right] \tag{50}$$

Substituting (48) into (44) and rearranging, we obtain:

$$\hat{\phi}_t = \hat{\psi}_t - \left(1+\frac{\varepsilon\alpha}{1-\alpha}\right)\frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}\Delta_t.$$

Using this expression to substitute $\hat{\phi}_t$ and $\hat{\phi}_{t+1}$ in (50), and rearranging, we get:

$$\begin{aligned}\hat{\psi}_t = &\left(1+\frac{\varepsilon\alpha}{1-\alpha}\right)\frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}\Delta_t + \left(1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)(1-\sigma)\hat{Y}_t \\ &+\left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\left[E_t\hat{\psi}_{t+1} - \left(1+\frac{\varepsilon\alpha}{1-\alpha}\right)\frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}E_t\Delta_{t+1} + (\varepsilon-1)E_t\Delta_{t+1}\right].\end{aligned} \tag{51}$$

Now note that, from (49), $\hat{\psi}_t$ is function of the real wage (a component of the marginal cost); substituting (51) in (49) and rearranging, we obtain:

$$\begin{aligned}&\left(1+\frac{\varepsilon\alpha}{1-\alpha}\right)\frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}\Delta_t = \left(1-\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)(\sigma-1)\hat{Y}_t + \\ &+\left[1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[\hat{w}_t - \frac{1}{1-\alpha}\hat{A}_t + \left(\frac{1}{1-\alpha}-\sigma\right)\hat{Y}_t\right] \\ &+\left[\left(\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right) - \left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\right]E_t\hat{\psi}_{t+1} + \\ &+\left(\frac{\varepsilon}{1-\alpha}\right)\left[\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}} - \theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]E_t\Delta_{t+1} \\ &+\left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\left(1+\frac{\varepsilon\alpha}{1-\alpha}\right)\frac{1}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}E_t\Delta_{t+1}.\end{aligned} \tag{52}$$

From (22), the log-linear expression for the marginal cost is:

$$\widehat{mc}_t = \hat{w}_t + \frac{1}{\alpha - 1}\hat{A}_t + \frac{\alpha}{1-\alpha}\hat{Y}_t, \tag{53}$$

so that the expression in the second square brackets of (52) is:

$$\hat{w}_t - \frac{1}{1-\alpha}\hat{A}_t + \left(\frac{1}{1-\alpha} - \sigma\right)\hat{Y}_t = \widehat{mc}_t + (1-\sigma)\,\hat{Y}_t.$$

Substituting this expression in (52) and, recalling that $\Delta_t \equiv \hat{\pi}_t - \varrho\hat{\pi}_{t-1}$,we obtain an expression of the dynamics of inflation in terms of marginal costs.

In the case of the PA, $\alpha = \varrho = 0$, this expression is:

$$\begin{aligned}\hat{\pi}_t = {} & \frac{\left(1-\theta\bar{\pi}^{\varepsilon-1}\right)\left(1-\theta\beta\bar{\pi}^{\varepsilon}\right)}{\theta\bar{\pi}^{\varepsilon-1}}\widehat{mc}_t \\ & + \beta\left[1+\varepsilon\left(\bar{\pi}-1\right)\left(1-\theta\bar{\pi}^{\varepsilon-1}\right)\right]E_t\hat{\pi}_{t+1} \\ & + \beta\left[1-\bar{\pi}\right]\left[1-\theta\bar{\pi}^{\varepsilon-1}\right]\left[(1-\sigma)\,\hat{Y}_t - E_t\hat{\psi}_{t+1}\right],\end{aligned} \tag{54}$$

which is equation (41) in the PA.

Furthermore, using expression (53) for the marginal costs, (51) can be written as:

$$\hat{\psi}_t = \left[1-\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[\widehat{mc}_t + (1-\sigma)\,\hat{Y}_t\right] + \left[\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[E_t\hat{\psi}_{t+1} + \left(\frac{\varepsilon}{1-\alpha}\right)E_t\Delta_{t+1}\right], \tag{55}$$

which, when $\alpha = \varrho = 0$, is equation (42) in the PA:

$$\hat{\psi}_t = \left[1-\theta\beta\bar{\pi}^{\varepsilon}\right]\left[\widehat{mc}_t + (1-\sigma)\,\hat{Y}_t\right] + \theta\beta\bar{\pi}^{\varepsilon}\left[E_t\hat{\psi}_{t+1} + \varepsilon E_t\hat{\pi}_{t+1}\right]. \tag{56}$$

Finally, combining (43) and (48), we obtain an expression for price dispersion as function of inflation and past price dispersion:

$$\begin{aligned}\hat{s}_t &= \left[-\frac{\varepsilon}{1-\alpha}\left(1-\theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right)\right]\left(\frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}\right)\Delta_t + \left[\theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right]\left[\frac{\varepsilon}{1-\alpha}\Delta_t + \hat{s}_{t-1}\right] \\ &= \frac{\varepsilon}{1-\alpha}\left[\theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}} - \left(1-\theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right)\frac{\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}}\right]\Delta_t + \theta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\hat{s}_{t-1}.\end{aligned}$$

For $\alpha = \varrho = 0$, and observing that in that case $\Delta_t = \widehat{\pi}_t$, this expression simplifies to:

$$\hat{s}_t = \left[\frac{\varepsilon\theta\bar{\pi}^{\varepsilon-1}}{1-\theta\bar{\pi}^{\varepsilon-1}}\left(\bar{\pi}-1\right)\right]\widehat{\pi}_t + \theta\bar{\pi}^{\varepsilon}\hat{s}_{t-1}, \tag{57}$$

which is expression (45) in the PA.

3.2 The GNKPC in terms of output

To express the GNKPC as an inflation-ouput relationship, we first substitute the expressions for labor input (41), into that for real wage, (40), to obtain:

$$\widehat{w}_t = \varphi \hat{s}_t + \left(\frac{\varphi}{1-\alpha} + \sigma \right) \hat{Y}_t - \frac{\varphi}{1-\alpha} \hat{A}_t + \varsigma_t. \tag{58}$$

We then go back to eq. (52), and, using (58), observe that the term in square brackets is

$$\hat{w}_t - \frac{1}{1-\alpha} \hat{A}_t + \left(\frac{1}{1-\alpha} - \sigma \right) \hat{Y}_t = \varphi \hat{s}_t + \frac{\varphi + 1}{1-\alpha} \left(\hat{Y}_t - \hat{A}_t \right) + \varsigma_t$$

so that eq. (52) can be rewritten as:

$$\begin{aligned}
&\left(1 + \frac{\varepsilon\alpha}{1-\alpha}\right) \frac{\theta \bar{\pi}^{(\varepsilon-1)(1-\varrho)}}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}} \Delta_t = \left(1 - \theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right)(\sigma - 1)\hat{Y}_t + \\
&+ \left[1 - \theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right] \left[\varphi\hat{s}_t + \frac{\varphi+1}{1-\alpha}\left(\hat{Y}_t - \hat{A}_t\right) + \varsigma_t\right] \\
&+ \left[\left(\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}}\right) - \left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right]\right] E_t\hat{\psi}_{t+1} + \\
&+ \left(\frac{\varepsilon}{1-\alpha}\right) \left[\theta\beta\bar{\pi}^{\frac{\varepsilon(1-\varrho)}{1-\alpha}} - \theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right] E_t\Delta_{t+1} \\
&+ \left[\theta\beta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}\right] \left(1 + \frac{\varepsilon\alpha}{1-\alpha}\right) \frac{1}{1-\theta\bar{\pi}^{(\varepsilon-1)(1-\varrho)}} E_t\Delta_{t+1}.
\end{aligned}$$

When $\alpha = \varrho = 0$, $\Delta_t = \widehat{\pi}_t$, and this expression simplifies to:

$$\begin{aligned}
\frac{\theta\bar{\pi}^{\varepsilon-1}}{1-\theta\bar{\pi}^{\varepsilon-1}} \widehat{\pi}_t &= \left(1 - \theta\beta\bar{\pi}^{\varepsilon-1}\right)(\sigma-1)\hat{Y}_t + \left[1 - \theta\beta\bar{\pi}^{\varepsilon}\right] \left[\varphi\hat{s}_t + (\varphi+1)\left(\hat{Y}_t - \hat{A}_t\right) + \varsigma_t\right] \\
&+ \left[\theta\beta\bar{\pi}^{\varepsilon-1}\right][\bar{\pi} - 1] E_t\hat{\psi}_{t+1} + \varepsilon\left[\theta\beta\bar{\pi}^{\varepsilon-1}\right][\bar{\pi}-1] E_t\widehat{\pi}_{t+1} + \frac{\theta\beta\bar{\pi}^{\varepsilon-1}}{1-\theta\bar{\pi}^{\varepsilon-1}} E_t\widehat{\pi}_{t+1}.
\end{aligned}$$

Adding and subtracting $(1 - \theta\beta\bar{\pi}^{\varepsilon})(\sigma - 1)\hat{Y}_t$ to the right hand side yields:

$$\begin{aligned}
\frac{\theta\bar{\pi}^{\varepsilon-1}}{1-\theta\bar{\pi}^{\varepsilon-1}} \widehat{\pi}_t &= \left(1 - \theta\beta\bar{\pi}^{\varepsilon-1}\right)(1-\bar{\pi})(\sigma-1)\hat{Y}_t + \left[1 - \theta\beta\bar{\pi}^{\varepsilon}\right] \left[\varphi\hat{s}_t + (\varphi+\sigma)\hat{Y}_t - (\varphi+1)\hat{A}_t + \varsigma_t\right] \\
&+ \left[\theta\beta\bar{\pi}^{\varepsilon-1}\right][\bar{\pi}-1] E_t\hat{\psi}_{t+1} + \theta\beta\bar{\pi}^{\varepsilon-1}\left[\varepsilon(\bar{\pi}-1) + \frac{1}{1-\theta\bar{\pi}^{\varepsilon-1}}\right] E_t\widehat{\pi}_{t+1},
\end{aligned}$$

which can be written as:

$$\begin{aligned}
\widehat{\pi}_t &= \lambda(\bar{\pi})\hat{Y}_t + b_1(\bar{\pi}) E_t\widehat{\pi}_{t+1} + \kappa(\bar{\pi})\left[\varphi\hat{s}_t - (\varphi+1)\hat{A}_t + \varsigma_t\right] \\
&+ b_2(\bar{\pi})\left((1-\sigma)\hat{Y}_t - E_t\hat{\psi}_{t+1}\right).
\end{aligned} \tag{59}$$

This is equation (43) in the PA, where the coefficients are defined as follows: $\lambda(\bar{\pi}) \equiv \kappa(\bar{\pi})(\varphi + \sigma)$, $\kappa(\bar{\pi}) \equiv \frac{(1-\theta\bar{\pi}^{\varepsilon-1})(1-\theta\beta\bar{\pi}^{\varepsilon})}{\theta\bar{\pi}^{\varepsilon-1}}$, $b_1(\bar{\pi}) \equiv \beta\left(1 + \varepsilon(\bar{\pi}-1)\left(1-\theta\bar{\pi}^{\varepsilon-1}\right)\right)$, and $b_2(\bar{\pi}) \equiv \left[\beta\left(1-\theta\bar{\pi}^{\varepsilon-1}\right)\right][1-\bar{\pi}]$.

3.3 The GNKPC in terms of output gap

In order to define an output gap we need to define the flexible price equilibrium in this model. First we define the steady state output under flexible prices as $\bar{Y}$; from (36), since under flexible prices $s = 1$ and $\theta -> 0$, we derive:

$$\bar{Y} = \left[\frac{(1-\alpha)}{d_n \mu} A^{\frac{\varphi+1}{1-\alpha}}\right]^{\frac{1-\alpha}{\varphi+\alpha+\sigma(1-\alpha)}}, \tag{60}$$

where now $\mu = \frac{\varepsilon}{\varepsilon-1}$. In the steady state under flexible prices output does not depend on inflation, and it is also equal to output in the zero inflation steady state of a sticky price model.

Taking shocks into account, the flexible price equilibrium is:

$$Y_t^n = \bar{Y} \left(\frac{A_t}{A}\right)^{\frac{\varphi+1}{\varphi+\alpha+\sigma(1-\alpha)}} \left(e^{\varsigma_t}\right)^{\frac{\alpha-1}{\varphi+\alpha+\sigma(1-\alpha)}},$$

where we assumed $\bar{\varsigma} = 0$. Taking logs:

$$\log Y_t^n - \log \bar{Y} = \frac{\varphi+1}{\varphi+\alpha+\sigma(1-\alpha)} \hat{A}_t - \frac{1-\alpha}{\varphi+\alpha+\sigma(1-\alpha)} \varsigma_t. \tag{61}$$

We define the output gap in a zero-inflation steady state as $\tilde{Y}_t = \log Y_t - \log Y_t^n$. We can then decompose the log-deviation of output from steady state in our model: $\hat{Y}_t = \log Y_t - \log Y(\bar{\pi})$, where $Y(\bar{\pi})$ is the steady state level of output (as derived in expression (36) above), as follows:

$$\begin{aligned}
\hat{Y}_t &= \log Y_t - \log Y(\bar{\pi}) = (\log Y_t - \log Y_t^n) + (\log Y_t^n - \log \bar{Y}) + (\log \bar{Y} - \log Y(\bar{\pi})) \\
&= \tilde{Y}_t + \frac{\varphi+1}{\varphi+\alpha+\sigma(1-\alpha)} \hat{A}_t - \frac{1-\alpha}{\varphi+\alpha+\sigma(1-\alpha)} \varsigma_t - \tilde{Y},
\end{aligned} \tag{62}$$

where we have indicated the long-run output gap by $\tilde{Y} = \log Y(\bar{\pi}) - \log \bar{Y}$.[4] This equation corresponds to equation (63) in the PA.

We now derive the GNKPC in terms of output gap for the case of the PA, where $\alpha = \varrho = 0$. Starting from equation (59), we substitute in it the expression for $\hat{Y}_t$:

$$\begin{aligned}
\hat{\pi}_t &= \beta \left[1 + \varepsilon (\bar{\pi} - 1) \left(1 - \theta \bar{\pi}^{\varepsilon-1}\right)\right] E_t \hat{\pi}_{t+1} \\
&+ \frac{\left(1 - \theta \bar{\pi}^{\varepsilon-1}\right)\left(1 - \theta \beta \bar{\pi}^{\varepsilon}\right)}{\theta \bar{\pi}^{\varepsilon-1}} \left[\varphi \hat{s}_t - (\varphi+1) \hat{A}_t + (\varphi+\sigma) \left(\tilde{Y}_t + \frac{\varphi+1}{\sigma+\varphi} \hat{A}_t - \frac{1}{\varphi+\sigma} \varsigma_t - \tilde{Y}\right) + \varsigma_t\right] \\
&+ \beta \left(1 - \theta \bar{\pi}^{\varepsilon-1}\right) (1 - \bar{\pi}) \left[(1-\sigma) \left(\tilde{Y}_t + \frac{\varphi+1}{\sigma+\varphi} \hat{A}_t - \frac{1}{\varphi+\sigma} \varsigma_t - \tilde{Y}\right) - E_t \hat{\psi}_{t+1}\right]
\end{aligned}$$

and, rearranging, we get:

$$\begin{aligned}
\hat{\pi}_t &= \kappa(\bar{\pi}) \varphi \hat{s}_t + \lambda(\bar{\pi}) \left(\tilde{Y}_t - \tilde{Y}\right) + b_1(\bar{\pi}) E_t \hat{\pi}_{t+1} + \\
&+ b_2(\bar{\pi}) \left[\left(\tilde{Y}_t - \tilde{Y} + \frac{\varphi+1}{\sigma+\varphi} \hat{A}_t - \frac{1}{\varphi+\sigma} \varsigma_t\right)(1-\sigma) - E_t \hat{\psi}_{t+1}\right],
\end{aligned} \tag{63}$$

[4] The expression for $\tilde{Y}$ can be easily derived from expressions (36) and (60).

where $\lambda(\bar{\pi}), \kappa(\bar{\pi}), b_1(\bar{\pi})$, and $b_2(\bar{\pi})$ were defined previously. This is equation (64) in the PA.

Similarly, upon substitution of $\hat{Y}_t$, the expression for $\hat{\psi}_t$ becomes:

$$\hat{\psi}_t = (1-\theta\beta\bar{\pi}^{\varepsilon})\left(\varphi\hat{s}_t + (\varphi+1)\left(\tilde{Y}_t - \tilde{Y} + \frac{1-\sigma}{\sigma+\varphi}\hat{A}_t\right) - \frac{1-\sigma}{\varphi+\sigma}\varsigma_t\right) \qquad (64)$$
$$+\,\theta\beta\bar{\pi}^{\varepsilon}\left[E_t\hat{\psi}_{t+1} + \varepsilon E_t\hat{\pi}_{t+1}\right],$$

which is (65) in the PA.

Note that the slope of the GNKPC depends on $\kappa(\bar{\pi}) = \frac{\left(1-\theta\bar{\pi}^{\varepsilon-1}\right)\left(1-\theta\beta\bar{\pi}^{\varepsilon}\right)}{\theta\bar{\pi}^{\varepsilon-1}}$, which is decreasing with trend inflation since $\theta\beta\bar{\pi}^{\varepsilon} < 1$ and $\theta\bar{\pi}^{\varepsilon-1} < 1$.

3.4 The log-linear model in the zero-inflation steady state

When prices are stable (i.e. at zero inflation: $\bar{\pi} = 1$), it is straightforward to show that the standard results are obtained. The log-linear model in this case (assuming also no indexation: $\varrho = 0$), simplifies to:

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma^{-1}\left(\hat{\imath}_t - E_t\hat{\pi}_{t+1}\right)$$
$$\widehat{w}_t = \varphi\widehat{N}_t + \sigma\hat{Y}_t + \varsigma_t$$
$$\hat{p}^*_{i,t} = \frac{\theta}{1-\theta}\hat{\pi}_t$$
$$\left(1 + \frac{\varepsilon\alpha}{1-\alpha}\right)\hat{p}^*_{i,t} = \hat{\psi}_t - \hat{\phi}_t$$

$$\hat{\psi}_t = [1-\theta\beta]\left[\hat{w}_t - \left(\frac{1}{1-\alpha}\right)\hat{A}_t + \left(\frac{1}{1-\alpha} - \sigma\right)\hat{Y}_t\right]$$
$$+\,[\theta\beta]\left[\hat{\psi}_{t+1} + \left(\frac{\varepsilon}{1-\alpha}\right)\hat{\pi}_{t+1}\right]$$

$$\hat{\phi}_t = (1-\theta\beta)(1-\sigma)\hat{Y}_t + [\theta\beta]\left[\hat{\phi}_{t+1} + (\varepsilon-1)\hat{\pi}_{t+1}\right]$$
$$\hat{N}_t = \hat{s}_t + \left(\frac{1}{1-\alpha}\right)\left[\hat{Y}_t - \hat{A}_t\right]$$
$$\hat{s}_t = \left[-\frac{\varepsilon}{1-\alpha}(1-\theta)\right]\hat{p}^*_{i,t} + [\theta]\left[\frac{\varepsilon}{1-\alpha}\hat{\pi}_t + \hat{s}_{t-1}\right]$$
$$\hat{\imath}_t = \phi_\pi\hat{\pi}_t + \phi_y\hat{Y}_t.$$

Price dispersion becomes (as in eq. (46) in the PA):

$$\hat{s}_t = \left[-\frac{\varepsilon\theta}{1-\alpha}\right]\hat{\pi}_t + \frac{\varepsilon\theta}{1-\alpha}\hat{\pi}_t + \theta\hat{s}_{t-1} = \theta\hat{s}_{t-1}$$

so the dynamics of $\hat{s}_t$ does not affect first order dynamics.

The NKPC in terms of marginal costs is:

$$\hat{\pi}_t = \left[\left(\frac{1-\alpha}{1-\alpha+\varepsilon\alpha}\right)\frac{(1-\theta)(1-\theta\beta)}{\theta}\right]\widehat{mc}_t + \beta E_t\hat{\pi}_{t+1} = \lambda\widehat{mc}_t + \beta E_t\hat{\pi}_{t+1}.$$

Using the expression for $\widehat{mc}_t$ derived by combining (53) and (58), the NKPC in terms of output becomes:

$$\hat{\pi}_t = \lambda\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\hat{Y}_t - \lambda\frac{\varphi+1}{1-\alpha}\hat{A}_t + \lambda\varsigma_t + \beta E_t\hat{\pi}_{t+1}.$$

Finally, with the usual substitutions, the NKPC in terms of output gap becomes:

$$\hat{\pi}_t = \kappa\tilde{Y}_t + \beta\hat{\pi}_{t+1},$$

where $\kappa = \lambda\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)$, and $\tilde{Y}_t = -\frac{\varphi+1}{\sigma(1-\alpha)+\varphi+\alpha}\hat{A}_t + \hat{Y}_t + \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\varsigma_t$.

4 Macroeconomic dynamics

For this part we used in the PA a simplified version of the baseline model where, in addition to the assumptions of constant return to scale and no indexation ($\alpha = \varrho = 0$), we also assumed $\sigma = 1$ and $\varphi = 0$. With these assumptions the log-linear equations (59), (45) and (39) simplify to:

$$\widehat{\pi}_t = \lambda(\bar{\pi})\left[\hat{Y}_t - \hat{A}_t + \varsigma_t\right] + b_1(\bar{\pi})E_t\hat{\pi}_{t+1} - b_2(\bar{\pi})E_t\hat{\psi}_{t+1} \tag{65}$$

$$\hat{\psi}_t = (1-\theta\beta\bar{\pi}^{\varepsilon})\left(\hat{Y}_t - \hat{A}_t + \varsigma_t\right) + \theta\beta\bar{\pi}^{\varepsilon}E_t\left[\hat{\psi}_{t+1} + \varepsilon\hat{\pi}_{t+1}\right] \tag{66}$$

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - (\hat{\imath}_t - E_t\hat{\pi}_{t+1}). \tag{67}$$

These are equations (53) to (55) in the PA. The policy rule (42) is unchanged, and the notation for the coefficients is as indicated previously; with the additional assumptions above it is also the case that $\lambda(\bar{\pi}) = \kappa(\bar{\pi})$. Note that because of the assumption of linear utility in labor ($\varphi = 0$), price dispersion, $\hat{s}_t$, does not affect the dynamics of the above system. Technology, labor supply, and monetary shocks are specified as AR(1) processes, respectively as:

$$\begin{aligned}
\hat{A}_t &= \rho_A\hat{A}_{t-1} + u_{At} \qquad & u_{At} &\sim i.i.d.N(0,1)\\
\varsigma_t &= \rho_\zeta\varsigma_{t-1} + u_{\zeta t} \qquad & u_{\zeta t} &\sim i.i.d.N(0,1)\\
\nu_t &= \rho_v\nu_{t-1} + u_{\nu t} \qquad & u_{\nu t} &\sim i.i.d.N(0,1),
\end{aligned}$$

which are equations (50) to (52) in the PA.

The system can be easily solved with the method of undetermined coefficient. The solution is given by the two equations:

$$\hat{\pi}_t = \frac{\lambda(\bar{\pi})}{1+\lambda(\bar{\pi})\frac{\phi_\pi}{1+\phi_y}}\left(\varsigma_t - \hat{A}_t - \frac{1}{1+\phi_y}v_t\right), \tag{68}$$

which is equation (57) in the PA, and:

$$\hat{Y}_t = \frac{1}{\frac{1+\phi_y}{\phi_\pi} + \lambda(\bar{\pi})}\left(\lambda(\bar{\pi})\left(\hat{A}_t - \varsigma_t\right) - \frac{1}{\phi_\pi}v_t\right) \tag{69}$$

which is equation (58) in the PA.

To look at the effects of trend inflation on the impact of the shocks on output and inflation, denoting by π_i and y_i, for $i = A, \varsigma, v$, the impact of shock i on variables π and Y, respectively, we compute the effect on the technology shocks impact as:

$$\frac{\partial \pi_A}{\partial \bar{\pi}} = \frac{\partial \frac{\lambda(\bar{\pi})}{1+\lambda(\bar{\pi})\frac{\phi_\pi}{1+\phi_y}}}{\partial \bar{\pi}} = \frac{1}{\left(1+\lambda(\bar{\pi})\frac{\phi_\pi}{1+\phi_y}\right)^2}\frac{\partial \lambda(\bar{\pi})}{\partial \bar{\pi}} < 0$$

and

$$\frac{\partial |y_A|}{\partial \bar{\pi}} = \frac{\partial \frac{\lambda(\bar{\pi})}{\frac{1+\phi_y}{\phi_\pi}+\lambda(\bar{\pi})}}{\partial \bar{\pi}} = \frac{\frac{1+\phi_y}{\phi_\pi}}{\left(\frac{1+\phi_y}{\phi_\pi}+\lambda(\bar{\pi})\right)^2}\frac{\partial \lambda(\bar{\pi})}{\partial \bar{\pi}} < 0.$$

It follows that an increase in $\bar{\pi}$ reduces the impact effect of a technology (labor supply) shock on inflation and output because it flattens the Phillips curve by reducing $\lambda(\bar{\pi})$. Note that the effect of ς is just the opposite of the one of A.

For the monetary shocks we compute:

$$\frac{\partial |\pi_v|}{\partial \bar{\pi}} = \frac{\partial \frac{\lambda(\bar{\pi})}{1+\phi_y+\phi_\pi\lambda(\bar{\pi})}}{\partial \bar{\pi}} = \frac{1+\phi_y}{\left(1+\phi_y+\lambda(\bar{\pi})\phi_\pi\right)^2}\frac{\partial \lambda(\bar{\pi})}{\partial \bar{\pi}} < 0,$$

so that an increase in trend inflation decreases the absolute value of the negative response of inflation. Moreover, since $y_v = \frac{-1}{1+\phi_y+\phi_\pi\lambda(\bar{\pi})}$, and $\frac{\partial \lambda(\bar{\pi})}{\partial \bar{\pi}} < 0$, it follows that an increase in trend inflation increases the absolute value of the negative response of output.

5 The optimal trend inflation

In this section we derive the results of Yun (2005) in our GNK model, that we analyze in section 3.5 of the PA. Again assuming, as in the PA, constant return to scale and no indexation ($\alpha = \varrho = 0$), substituting the expression for $p_{i,t}^*$ in (15) into (24) yields:

$$s_t = (1-\theta)\left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right)^{\frac{-\varepsilon}{1-\varepsilon}} + \theta\pi_t^{\varepsilon}s_{t-1}. \tag{70}$$

Following Yun (2005), the Ramsey problem can be defined as maximizing the utility function of the representative consumer under the resource constraint, $A_tN_t = C_ts_t$ and the evolution of price dispersion s_t described in (70). The first-order condition with respect to inflation is thus (it just involves (70)):

$$0 = (1-\theta)\left(\frac{-\varepsilon}{1-\varepsilon}\right)\left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right)^{\frac{-1}{1-\varepsilon}}\frac{-\theta\left(\varepsilon-1\right)\pi_t^{\varepsilon-2}}{1-\theta}+\theta\varepsilon\pi_t^{\varepsilon-1}s_{t-1}$$

$$0 = -\varepsilon\theta\pi_t^{\varepsilon-2}\left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right)^{\frac{-1}{1-\varepsilon}}+\theta\varepsilon\pi_t^{\varepsilon-1}s_{t-1} = -\left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right)^{\frac{-1}{1-\varepsilon}}+\pi_t s_{t-1}.$$

Hence:

$$\left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}} = \pi_t s_{t-1}, \tag{71}$$

which is equation (2.5) in Yun (2005), p. 94. Substituting this equation into (70) we get:

$$s_t = \pi_t^{\varepsilon}s_{t-1}\left[\theta+(1-\theta)\,s_{t-1}^{\varepsilon-1}\right]. \tag{72}$$

Noting that (71) can be written as:

$$1 = \pi_t^{\varepsilon-1}\left[\theta+(1-\theta)\,s_{t-1}^{\varepsilon-1}\right], \tag{73}$$

and dividing (72) by (73) we get:

$$s_t = \frac{\pi_t^{\varepsilon}s_{t-1}\left[\theta+(1-\theta)\,s_{t-1}^{\varepsilon-1}\right]}{\pi_t^{\varepsilon-1}\left[\theta+(1-\theta)\,s_{t-1}^{\varepsilon-1}\right]} = \pi_t s_{t-1},$$

or:

$$\pi_t = \frac{s_t}{s_{t-1}}. \tag{74}$$

This is equation (2.6) in Yun p. 95, and (68) in the PA. Plugging this equation into (73) gives equation (2.7) in Yun and (69) in the PA:

$$s_t = s_{t-1}\left[\theta+(1-\theta)\,s_{t-1}^{\varepsilon-1}\right]^{\frac{1}{1-\varepsilon}}. \tag{75}$$

Now recall equation (24), which under $\varrho = 0$ is:

$$s_t = (1-\theta)\left[p_{i,t}^{*}\right]^{-\varepsilon}+\theta\pi_t^{\varepsilon}s_{t-1},$$

and use equation (74) to write:

$$s_t\frac{\left(1-\theta\pi_t^{\varepsilon-1}\right)}{(1-\theta)} = \left[p_{i,t}^{*}\right]^{-\varepsilon},$$

and equation (71) to get:

$$s_t\left(\pi_t s_{t-1}\right)^{\varepsilon-1} = \left[p_{i,t}^{*}\right]^{-\varepsilon}.$$

Using again (74), it then follows:

$$p_{i,t}^{*} = \frac{1}{s_t}, \tag{76}$$

which is equation (2.12) in Yun (2005), p. 96, and equation (70) in the PA.