# Investment Handout

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It is crucial to note what we will mean by 'capital' in this lecture: physical productive capital I.e. any physical object that has the following features:

1. It can be used (usually in combination with other things) to produce 'goods' that may be either consumption goods or other capital goods

2. It can be used repeatedly, to some extent–i.e., it can be accumulated.

The physical components of a production plant are an example.
We do not mean 'financial capital' in any way. I.e. not the concept we mean when we say a bank should be 'well capitalized'. In this note I will use 'financial asset' for that concept.

# 1 Simple static micro model

Consider a firm that can rent productive capital at rate $r_K$ per unit that produces consumption goods for sale.

Let the profits at a point in time be given by

$$\pi(k, \mathbf{X}) - r_K k$$

where $\pi(\cdot)$ accounts for all other dimensions of optimization over other relevant factors such as the production technology, price of the outputs, quantities of other inputs all collected in the vector $\mathbf{X}$. Think about this as a more complete model where we solve for optimal choices of all variables as functions of some fixed level of capital $k$ and then consider the final problem of selecting the the optimal $k$.

The necessary optimality condition

$$\frac{\partial \pi\left(k^*, \mathbf{X}\right)}{\partial k} = r_K \tag{1}$$

implicitly defines the optimal choice for $k$ as a function of $r_K$ if it could be costlessly obtained, hence the 'desired capital stock' $k^*$.

For this quantity to be well defined, the the second order sufficient condition for optimality must also hold:

$$\frac{\partial^2 \pi\left(k^*, \mathbf{X}\right)}{\partial k^2} < 0$$

This would be the case if, say, there are diminishing marginal product to capital.

Taking derivatives with respect to $r_K$ on both sides of 1 yields:

$$\frac{\partial\left[\frac{\partial\pi(k^*(r_K),\mathbf{X})}{\partial k}\right]}{\partial r_K}=1$$

$$\frac{\partial\left[\frac{\partial\pi(k^*(r_K),\mathbf{X})}{\partial k}\right]}{\partial k}\frac{\partial k^*\left(r_K\right)}{\partial r_K}=1$$

$$\frac{\partial^2\pi\left(k^*\left(r_K\right),\mathbf{X}\right)}{\partial k^2}\frac{\partial k^*\left(r_K\right)}{\partial r_K}=1$$

$$\frac{\partial k^*\left(r_K\right)}{\partial r_K}=\left(\frac{\partial^2\pi\left(k^*\left(r_K\right),\mathbf{X}\right)}{\partial k^2}\right)^{-1}<0$$

Where the last inequality comes from our assumption that we are considering a firm with a well defined optimal capital stock, and the expression as a whole reflecting the familiar result that the optimal capital stock is decreasing in the rental rate of capital, if this is the case.

This static view is clearly the "best case scenario" in any dynamic world: these conditions pertain only to the dynamic environment where the firm can costlessly and instantaneously achieve any desired final capital stock. In such a world, if the rental cost of capital is a time varying process $r_K(t)$, the desired capital could be maintained continuously to respond to any change in $r_K(t)$.

## 1.1 User cost of capital

Since most firms own the capital they use in production, there is no clear empirical counterpart to $r_K(t)$

The "user cost of capital" is the phrase assigned to the accurate internal measure of the marginal cost of capital for a firm that owns its own capital.

In the typical "classical" view, this should include:

1. The opportunity cost of holding physical capital rather than the best alternative financial asset.

   Suppose the rate on all financial assets is given by $r(t)$ and the price of capital $p_K(t)$ then the per unit opportunity cost of holding stock of capital $k(t)$ at moment $t$ is simply $r(t)p_K(t)$

2. The depreciation in value of the capital at constant rate $\delta$ implies an instantaneous cost of $\delta p_K(t)$ per unit of capital

3. The capital losses per unit from holding capital is given by the negative of the rate of change of the price of capital (since gains are negative losses and we are considering the marginal cost of capital): $-\dot{p}_K(t)\equiv-\frac{\partial p_K(t)}{\partial t}$

So that

$$r_K(t)=p_K(t)\left[r(t)+\delta-\frac{\dot{p}_K(t)}{p_K(t)}\right]$$

If there are taxes of rate $\tau$ on the firm and the firm receives a tax credit of $f$ on the value of the stock of capital it holds the user cost of capital is lower:

$$r_K(t) = (1 - \tau f) p_K(t) \left[ r(t) + \delta - \frac{\dot{p}_K(t)}{p_K(t)} \right]$$

Note that Romer motivates this equation by referring to a tax credit on investment activities, implying that the credit applies to changes in capital stock. I disagree with his wording: - we can see what equation he intends to attain, but if that equation is to be an accurate discription of the story he is telling it must be a tax credit on the value of the current capital stock, since the user cost of capital applies to the entire stock held.

## 1.2 Problems with the baseline model

- Any discrete change in $r_K$ implies a discrete change in $k^*$.
- In continuous time this implies an infinite rate of investment
- even in discrete times it predicts an extreme level of volatility in investment and a large degree of volatility in capital stocks (at least as large as that in interest rates)
- at any point in time the actual investment is limited by the current level of output of the economy, hence investment cannot exceed this
- No allowance for expectations
- current marginal revenue product of capital equated to current marginal user cost
- empirically it is clear that expectations about future costs and returns are crucial determinants of current investment
- Theoretical source: infinitely flexible level of capital stock
- Empirical reality: factories and machines take time to be built and put into operation
- Theoretical solution:
- Adjustment Costs to capital stock
- short cut way to get capital to evolve smoothly
- models trade-offs exogenously: the faster you try to build a new power plant the more resources have to be diverted from all other uses, which rapidly increases the price of doing so and reduces the efficiency of the last unit of resources employed in the change of capital
- Internal Adjustment costs:

- costs of installing new capital and training workers
- External Adjustment costs:
- inelastic supply of capital stock - increased demand bids up price, reduces optimal increment

## 2 A dynamic model of investment with internal adjustment costs

We study a discrete version of the model first assuming no uncertainty, then give the continuous time analogues of the equations that govern dynamics so that we can study a phase diagram like the RCK model.

We will simplify the production/profit generating technology to a linear one at firm level and use non-linear capital stock adjustment costs to avoid undefined investment demand. This is for analytic convenience and not realism.

## 3 Market Assumptions

- Consider firms in an industry with $N$ firms where $N$ is large enough for each firm to ignore it's impact on the aggregate levels of variables.
- Demand for the good on aggregate is downward sloping in price but much larger than any one firm can provide.
- The interest rate on financial assets is constant at $r$ per period

Production Technology: Firms operate a production technology that provides cash-flow of $\pi(K_t)$ currency units per unit of capital stock of the individual firm $k_t$, where $K_t$ is the aggregate level of capital in the industry and $\pi'(K_t) < 0$ (this is the downward sloping demand part).

Investment/Capital accumulation Technology: For simplicity, we assume a law of motion for capital given by:

$$k_t = k_{t-1} + I_t$$
$$\Delta k_t = I_t$$

where $I_t$ is the amount of investment done in period $t$ - the sequence of these decisions will be the only choice variables of the firm as we assume some given level of initial capital $k_0 > 0$

Note that this implies two things:

- a zero depreciation rate - again for analytic convenience - adding it will not aid the intuition of the results and come at the cost of more complicated formulae.

- current investment is available for current production - this goes against the "time to build" argument but again yields simpler equilibrium equations at little cost to the central message of the model, since we will be analysing it in continuous time anyway

Additionally we assume that the firm faces capital adjustment costs that are a function of the size of the investment $I_t$. That is, for any choice of $I_t$ the firm must pay cost $C(I_t) \geq 0$ where we assume the following additional structure on this cost function:

$$
\begin{aligned}
C(0) = C'(0) = 0 \\
C''(.) > 0
\end{aligned}
$$

- zero cost and zero marginal cost at zero investment level - so maintaining the status quo capital stock costs nothing.

- positive and increasing marginal cost in the size of investment for any non-zero investment level, whether positive or negative - i.e. it is costly both to increase and to decrease the capital stock at every possible level of capital. (the simple quadratic function $C(I_t) = aI_t^2$ for $a > 0$ would satisfy these restrictions)

- Note for analytic results later that these assumptions imply that the marginal cost of non-zero investment has the same sign as the investment: I.e. reducing capital stock (negative investment) has negative marginal cost, so that the effect on total cost is positive.

We will normalize shortly, but for now let the price of a unit of capital be $p_t^K$ currency units, so that a choice of investment of $I_t$ dollars reduces the available cash-flow by $p_t^K I_t$

Objective of firms: A firm aims to maximize its present discounted market value. We assume that the firm makes only one type of decision: how much of available resources to use to adjust the capital stock of the firm. This in turn means that something happens to the remainder of the resources, and we assume they are the "profits" of the firm that are paid out in each period to the owners of the equity in the firm (left unmodelled). This means that the firms market value is the discounted sum of all future distributions to equity holders of cash-flow.

In any period, the cash-flow not used to adjust the level of capital are given by: $\pi(K_t) k_t - p_t^K I_t - C(I_t)$ and the constraint on the firm in each period is the capital accumulation law.

For simplicity of the results, let us normalize the price of capital to one (or equivalently assume every thing is denominated in units of capital rather than currency).

## 3.1 Solving the optimization problem of the representative firm

The problem of the firm thus can be stated as:

$$\max_{\{I_t\}} \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[\pi\left(K_t\right) k_t - I_t - C\left(I_t\right)\right]$$

$$\text{subject to: } k_t = k_{t-1} + I_t$$

The Lagrangian for this problem is thus:

$$L = \sum_{t=0}^{\infty} \left\{ \frac{1}{(1+r)^t} \left[\pi\left(K_t\right) k_t - I_t - C\left(I_t\right)\right] + \lambda_t \left[k_{t-1} + I_t - k_t\right] \right\}$$

Where $\lambda_t$ is as usual the shadow price of the constraint. In this situation, it turns out to be useful to consider this concept a little further and use a slightly different version of it.

The shadow price of the constraint measures the impact on the maximum value of the objective function of the decision maker of an exogenous relaxation of the constraint by a marginal amount. This means $\lambda_t$ is the marginal impact on the current value of the firm of a marginal increase in the period $t$ capital stock.

Crucially, this marginal impact is in current value terms - i.e. in period 0 units of account (since we are starting at $t = 0$ )

Define the period $t$ value ('future value' in theory-of-interest language) of this marginal contribution to firm value as

$$q_t = (1+r)^t \lambda_t$$

or

$$\lambda_t = \frac{1}{(1+r)^t} q_t$$

Using this we can rewrite the Lagrangian as:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left\{ \left[\pi\left(K_t\right) k_t - I_t - C\left(I_t\right)\right] + q_t \left[k_{t-1} + I_t - k_t\right] \right\}$$

First Order Conditions: