Leslie Population Model [using the abstract system steps in Slide 21]

Setup:

- Step 1: $\bar{\bar{z}}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A\bar{\bar{z}}_n = \begin{pmatrix} 1 & 4 \\ 0{,}5 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ ... 13

- Step 2:

Theorem 23.3(c): $\det(A - rI) = 0 \Rightarrow \det\begin{pmatrix} 1-r & 4 \\ 0{,}5 & 0-r \end{pmatrix} = 0$

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

$$r_1 = 2 \quad ; \quad r_2 = -1$$

[Eigen values]

- Step 3:

Theorem 23.3(d)

$r_1 = 2$

$$(A - r_1 I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -1 & 4 \\ 1/2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -v_1 + 4v_2 = 0 \Rightarrow v_1 = 4v_2 \Rightarrow \boxed{\bar{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}}$$

& $r_2 = -1$

$$(A - r_2 I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} 2 & 4 \\ 0{,}5 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2v_1 + 4v_2 = 0 \Rightarrow \boxed{\bar{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

$\{\bar{v}_1, \bar{v}_2\} \Rightarrow$ Eigen-vectors

$$\therefore P = [\bar{v}_1 \ \bar{v}_2] = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det P}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1/6 & 2/6 \\ -1/6 & 4/6 \end{pmatrix}$$

$$D = P^{-1}AP$$

...

$$D = P^{-1} A P$$

Diagonal matrix, D will have the eigenvalues on the diagonal of D

CHECK:

$$D = \overset{P^{-1}}{\begin{pmatrix} 1/6 & 2/6 \\ -1/6 & 4/6 \end{pmatrix}} \overset{A}{\begin{pmatrix} 1 & 4 \\ 0{,}5 & 0 \end{pmatrix}} \overset{P}{\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1/6 & 2/6 \\ -1/6 & 4/6 \end{pmatrix} \begin{pmatrix} 8 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 8/6 + 4/6 & 2/6 - 2/6 \\ -8/6 + 8/6 & -2/6 - 4/6 \end{pmatrix}$$

$$= \begin{pmatrix} 12/6 & 0 \\ 0 & -6/6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \checkmark$$

$r_1 = 2$, $r_2 = -1$

Notice: P^{-1} is the "change of coordinates" (14) or (16) in Leslie Population example.

That is, $\bar{Z} = P^{-1} \bar{z} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1/6 & 1/3 \\ -1/6 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ ... (14)/(16)

& it's inverse transformation is: $\bar{z} = P \bar{Z}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \ldots \quad (15)/(17)$$

The transformed system is now "uncoupled" with D

$\bar{Z}_{n+1} = D \bar{Z}_n$ gives

$$X_{n+1} = 2X_n = 2^n X_0 = 2^n c_1$$

$$Y_{n+1} = -Y_n = (-1)^n Y_0 = (-1)^n c_2$$

CHECK STEPS with the example in textbook.

P.T.O.

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \bar{Z}_{n+1} = D\bar{Z}_n = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix}$$

r_1 r_2 ✓

The transformed system is uncoupled!

- Now solve as two one-dimensional equations:
[start with $n=0$ & iterate forward].

$$X_n = 2^n X_0 = 2^n c_1 \quad \ldots \text{(A)}$$

$$Y_n = (-1)^n Y_0 = (-1)^n c_2 \quad \ldots \text{(B)}$$

- You will now ~~use~~ substitute (A) & (B) back into the change of coordinates [basis] (15)/(17) to express x_n & y_n in terms of X_n & Y_n (and therefore the initial conditions).

$\Rightarrow$ i.e., $\bar{z}_n = P\bar{Z}_n$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^n c_1 \\ (-1)^n c_2 \end{pmatrix}$$

$$= \begin{pmatrix} 4\cdot\boxed{2^n c_1} - 2\cdot\boxed{(-1)^n c_2} \\ \boxed{2^n c_1} + \boxed{(-1)^n c_2} \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 2^n \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2(-1)^n \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(*) WHAT DO YOU SEE? This is the general solution for the system of difference equations $z_{n+1} = Az_n$ [Theorem 23.6]

... P.T.O.

- Using (16)/(14) & (13) we can change system from x,y-variables to X,Y- variables

$[\text{i.e., } \bar{Z} = P^{-1}\bar{z}]$

$$(16)\begin{cases} X_{n+1} = \frac{1}{6}x_{n+1} + \frac{1}{3}y_{n+1} \\ Y_{n+1} = -\frac{1}{6}x_{n+1} + \frac{2}{3}y_{n+1} \end{cases}$$

- Now substitute equations for x_{n+1} & y_{n+1} from (13) into (16).

$z_{n+1} = Az_n$

$$X_{n+1} = \frac{1}{6}(x_n + 4y_n) + \frac{1}{3}\left(\frac{1}{2}x_n\right)$$

$$Y_{n+1} = -\frac{1}{6}(x_n + 4y_n) + \frac{2}{3}\left(\frac{1}{2}x_n\right)$$

- Simplify:

$$X_{n+1} = \frac{2}{6}x_n + \frac{4}{6}y_n = \frac{1}{3}x_n + \frac{2}{3}y_n$$

$$Y_{n+1} = \frac{1}{6}x_n - \frac{2}{3}y_n$$

- Now use the "inverse transformation" $(z = PZ)$ (17) to get $\bar{Z}_{n+1} = \underline{D}\bar{Z}_n$

[recall: we want to check that our definition $D = P^{-1}AP$ holds when we do the transformations step-by-step.] And note: the diagonal matrix D will have the eigenvalues as the diagonal elements].

$$\left.\begin{aligned} X_{n+1} &= \frac{1}{3}(4X_n - 2Y_n) + \frac{2}{3}(X_n + Y_n) = \underline{\underline{2X_n}} \\ Y_{n+1} &= \frac{1}{6}(4X_n - 2Y_n) - \frac{2}{3}(X_n + Y_n) = \underline{\underline{-Y_n}} \end{aligned}\right\} \rightarrow$$

eigenvalue

corresponding eigenvector

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \bar{z}_n = c_1 (r_1)^n \bar{v}_1 + c_2 (r_2)^n \bar{v}_2$$

$$= c_1 (2)^n \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 (-1)^n \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \ldots (18)$$

[CHECK page 1 derivation ... ✓]

- The constants in (18) $\{c_1, c_2\}$ are determined by the exogenous initial conditions x_0 & y_0

We know from (A) & (B) that $X_0 = c_1$ & $Y_0 = c_2$
[Check: subst. $n = 0$ into $X_n = 2^n c_1$ & $Y_n = (-1)^n c_2$]

Using $\bar{z}_n = P \bar{Z}_n$, $n = 0$

$$\therefore \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = P^{-1} \bar{z}_0$$

This system can be solved in the usual way ...

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/6 & 2/6 \\ -1/6 & 4/6 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\therefore$ for given initial conditions x_0 & y_0:

$$\left.\begin{aligned} c_1 &= \tfrac{1}{6} x_0 + \tfrac{1}{3} y_0 \\ c_2 &= -\tfrac{1}{6} x_0 + \tfrac{2}{3} y_0 \end{aligned}\right\}$$

Plugging c_1 & c_2 into (18) will give use the **exact** solution to the system for any n period, $n = 0, 1, \ldots, \infty$