# Real Business Cycle Notes

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## Relation to Romer:

This model combines several different sections of the notation of Romer. This is because I changed the sequence from what he presents to one I find more logical. As such it may take some work converting these notes to the notation in Romer. This is a critically important part of your training as a formal economist. You need to be able to translate notation to focus on funtamental mathematical differences.

## 1 Introduction

In these notes I build a very basic Real Business Cycle model. This is in the class of Dynamic Stochastic General Equilibrium models, but without any role for prices/money. This is equivalent to assuming perfect markets, perfect information and perfectly flexible prices. We will extend this model to the cannonical New Keynesian DSGE model in a future lecture. It will serve as the *flexible price* benchmark of the final model we will build in this course.

It is called *general equilibrium* because we will consider simultaneously the production side of the economy, which takes the capital and labour stocks available in the economy and converts it into new output; and the consumption side of the economy, which determines the value of the new output and is constrained by the income from capital and labour. The equilibrating mechanism is the market price.

In RBC models, this equilibrating mechanism is perfect. This in turn means that sub-optimal outcomes are not possible. Thus, all fluctuations in observed wellfare relevant variables are optimal.

While this might seem an absurd case to consider, given that many imperfections we know cause sub-optimal outcomes in the real world, it is not. It serves as a very important caution that is related to Hayek's "pretense of knowledge" syndrome that plagues our profession. This model analyzes a situation where all variation is optimal. More realistic models have variation, *some* of which is sub-optimal. Determining which is which is not obvious at all. This is the lesson of the RBC models.

## 2 The intertemporal nature of labour supply

To highlight the interconnectedness of decisions, we study how labour supply decisions depend on interest rates in a 2 period model:

The household makes two decisions: how much to work and how much of labour earnings to save

Objective function:

$$\mathcal{U} = \left[U\left(C_1\right) - V\left(L_1\right)\right] + \beta\left[U\left(C_2\right) - V\left(L_2\right)\right]$$

where

$$\begin{aligned} U'(C) &> 0 \\ U''(C) &< 0 \\ V'(L) &> 0 \\ V''(L) &> 0 \end{aligned}$$

Let

$$
\begin{aligned}
U(C_t) &= \frac{C_t^{1-\theta}}{1-\theta}, \theta > 0 \\
V(L_t) &= \frac{L_t^{\gamma+1}}{\gamma+1}, \gamma > 0
\end{aligned}
$$

Budget Constraint:
Suppose the consumer can save in some instrument with return $r$

$$
\begin{aligned}
C_1 &= W_1 L_1 - S_1 \\
C_2 &= W_2 L_2 + (1+r) S_1
\end{aligned}
$$

$$
C_1 + \frac{1}{1+r} C_2 = W_1 L_1 + \frac{1}{1+r} W_2 L_2
$$

Lagrangian:

$$
\begin{aligned}
\mathcal{L} = & \left[\frac{C_1^{1-\theta}}{1-\theta} - \frac{L_1^{\gamma+1}}{\gamma+1}\right] + \beta\left[\frac{C_2^{1-\theta}}{1-\theta} - \frac{L_2^{\gamma+1}}{\gamma+1}\right] + \\
& \lambda\left[W_1 L_1 + \frac{1}{1+r} W_2 L_2 - C_1 - \frac{1}{1+r} C_2\right]
\end{aligned}
$$

FOCs
$\frac{\partial \mathcal{L}}{\partial L_1} = 0$:

$$
\begin{aligned}
L_1^{\gamma} &= \lambda W_1 \\
\beta L_2^{\gamma} &= \frac{\lambda}{1+r} W_2 \\
\frac{L_2}{L_1} &= \left(\frac{1}{\beta(1+r)} \frac{W_2}{W_1}\right)^{\frac{1}{\gamma}}
\end{aligned}
$$

If $V(L) = L^2$:

$$
\frac{L_2}{L_1} = \frac{1}{\beta(1+r)} \frac{W_2}{W_1}
$$

What is the impact of changes in the interest rate on the temporal balance of labour supply?

# 3 The full model

As usual, this micro founded macro model consists of two decision makers (we leave out the government sector for simplicity):

1. Households, who have to choose how much to work for a given wage; and how much of current wealth to consume (or save).
2. Firms, who have to choose how much capital to rent and labour to hire, at current and wage and rate of return on capital, in order to maximize profits from sales.

In general equilibrium, their interaction will determine the real prices in this economy: the equilibrium real wage and the equilibrium real rate of return on capital.

### 3.1 Firms

We start with firms, because in this model, firms hold no assets - they rent their labour and capital from the households. Thus they face a period - by - period problem.

Moreover, to simplify the analysis, we assume that firms are perfectly competitive, thus they operate with the constraint of making zero profit.

Lastly, we assume that firms operate a Cobb-Douglas, constant returns to scale production with labour augmenting technology:

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

$$\begin{aligned} \ln A_t &= a_0 + gt + a_t \\ a_t &= \rho a_{t-1} + \varepsilon_t \end{aligned}$$

$$\begin{aligned} -1 &< \rho < 1 \\ \varepsilon_t &\sim N\left(0, \sigma^2\right) \end{aligned}$$

The firms pay $r_t$ for each unit of $K_t$ rented, and $W_t$ for each unit of $L_t$ hired. Thus the costs of a firm are given by:

$$r_t K_t + W_t L_t$$

Thus the Lagrangian cost minimization problem is:

$$\mathcal{L}\left(L_t, K_t\right) = r_t K_t + W_t L_t + \mu \left(Y_t - K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}\right)$$

The FOCs with respect to capital and labour are:

$$r_t = \mu \alpha K_t^{\alpha-1} \left(A_t L_t\right)^{1-\alpha}$$

$$W_t = \mu \left(1-\alpha\right) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha}$$

Rewriting these equation (just algebra):

$$\begin{aligned} r_t &= \mu \alpha K_t^{\alpha-1} \left(A_t L_t\right)^{1-\alpha} \\ &= \mu \alpha \frac{K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}}{K_t} \\ &= \mu \alpha \frac{Y_t}{K_t} \end{aligned}$$

$$\begin{aligned} W_t &= \mu \left(1-\alpha\right) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} \\ &= \mu \left(1-\alpha\right) \frac{K_t^{\alpha} A_t^{1-\alpha} L_t^{1-\alpha}}{L_t} \\ &= \mu \left(1-\alpha\right) \frac{Y_t}{L_t} \end{aligned}$$

Zero Profit implies:

$$
\begin{aligned}
Y_t &= r_t K_t + W_t L_t \\
&= \left(\mu\alpha \frac{Y_t}{K_t}\right) K_t + \left(\mu \left(1-\alpha\right) \frac{Y_t}{L_t}\right) L_t \\
&= \mu Y_t
\end{aligned}
$$

So we conclude $\mu = 1$ and the equations above can be simplified:

$$
r_t = \alpha K_t^{\alpha-1} \left(A_t L_t\right)^{1-\alpha} = \alpha \frac{Y_t}{K_t}
$$

$$
W_t = \left(1-\alpha\right) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} = \left(1-\alpha\right) \frac{Y_t}{L_t}
$$

Note that perfect competition implies that each factor of production receives their "fair share" of the output produced:

$$
\begin{aligned}
r_t K_t &= \alpha Y_t \\
W_t L_t &= \left(1-\alpha\right) Y_t
\end{aligned}
$$

And that the capital-labour ratio is a function of the relative productivity and relative price of the two factors:

$$
\frac{K_t}{L_t} = \frac{\alpha}{\left(1-\alpha\right)} \frac{W_t}{r_t}
$$

A key point for our purpose at this stage is this: the wage and the rental rate of capital are both *stochastic*, depending on the random outcome of the technology shock

## 3.2 Households

There is uncertainty about the future:

Objective Function:

$$
\mathcal{U}_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{\beta^t \left[U\left(C_t, L_t\right)\right]\right\}
$$

For simplicity, we impose a functional form that is easy to work with:

$$
\mathcal{U} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t + b \ln\left(1 - L_t\right)\right]
$$

Note: we assume each consumer has 1 unit of time each period that it can split between leisure and labour.

Budget Constraint:

Consumers own their labour resources as well as the capital stock in the economy, so their available resources are given by the wage income from supplying labour and income from renting out capital:

$$
\text{income: } W_t L_t + r_t K_t
$$

They have to decide how much of the available resources in each period to consume and how much to invest to increase the capital stock:

$$
\text{expenses: } C_t + I_t
$$

The capital stock depreciates at rate $\delta$ each period, so the capital accumulation equation (often called the *law of motion* of capital) is given by:

$$\begin{aligned} K_{t+1} &= (1-\delta) K_t + I_t \\ I_t &= K_{t+1} - (1-\delta) K_t \end{aligned}$$

Thus the full budget constrain in period $t$ of the household is given by the requirement that expenses must equal income:

$$\begin{aligned} C_t + I_t &= W_t L_t + r_t K_t \\ C_t + K_{t+1} - (1-\delta) K_t &= W_t L_t + r_t K_t \\ C_t + K_{t+1} &= W_t L_t + (1 + r_t - \delta) K_t \end{aligned}$$

Lagrangian:

$$\begin{aligned} \mathcal{L}(C_t, L_t, K_{t+1}) &= \mathbb{E}_0 \textstyle\sum_{t=0}^{\infty} \quad \{\beta^t [\ln C_t + b \ln (1 - L_t)] \\ &\qquad\qquad\quad + \lambda_t [W_t L_t + (1 + r_t - \delta) K_t - C_t - K_{t+1}]\} \end{aligned}$$

Lagrangian with the summation unpacked (to assist in finding the first order conditions):

$$\begin{aligned} \mathcal{L}(C_t, L_t, K_{t+1}) = \mathbb{E}_0 \quad & \{[\ln C_0 + b \ln (1 - L_0)] + \ldots + \beta^t [\ln C_t + b \ln (1 - L_t)] + \ldots \\ & + \lambda_0 [W_0 L_0 + (1 + r_0 - \delta) K_0 - C_0 - K_1] \\ & \vdots \\ & + \lambda_t [W_t L_t + (1 + r_t - \delta) K_t - C_t - K_{t+1}] \\ & + \lambda_{t+1} [W_{t+1} L_{t+1} + (1 + r_{t+1} - \delta) K_{t+1} - C_{t+1} - K_{t+2}] \\ & + \ldots\} \end{aligned}$$

FOC's
$\frac{\partial \mathcal{L}}{\partial C_t} = 0$

$$\begin{aligned} \beta^t C_t^{-1} - \lambda_t &= 0 && (1) \\ \beta^t C_t^{-1} &= \lambda_t && (2) \\ \beta^{t+1} C_{t+1}^{-1} &= \lambda_{t+1} && (3) \end{aligned}$$

$\frac{\partial \mathcal{L}}{\partial L_t} = 0$:

$$\begin{aligned} -\beta^t \frac{b}{1 - L_t} + \lambda_t W_t &= 0 \\ \beta^t \frac{b}{1 - L_t} &= \lambda_t W_t && (4) \end{aligned}$$

$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0$

$$\lambda_t = \mathbb{E}_t [(1 + r_{t+1} - \delta) \lambda_{t+1}] \qquad (5)$$

Combining the first order conditions allows us to find the standard optimality conditions that should be familiar from previous studies: the Euler equation, which determines the optimal evolution of consumption, and the optimal labour supply function.

**Euler equation:**
Combine equations 2,3 and 5:

$$C_t^{-1} = \mathbb{E}_t\left[\beta\left(1 + r_{t+1} - \delta\right) C_{t+1}^{-1}\right]$$

**Labour supply:**
Take the ratio of equations 2 and 4

$$\frac{C_t^{-1}}{\frac{b}{1-L_t}} = \frac{1}{W_t}$$

Note that the left hand side is the marginal rate of substitution of consumption for labour - the utility trade-off from increasing labour in order to increase consumption at the margin (marginal utility of consumption over marginal (dis-)utility of labour. The right hand side is the market trade-off: increasing labour increases income by $W_t$ per unit, which can be used to purchase consumption at $P_t = 1$ per unit.

Rearranging gives the labour supply function:

$$\begin{aligned} \frac{C_t}{1 - L_t} &= \frac{W_t}{b} \\ L_t &= \frac{W_t - bC_t}{W_t} \end{aligned}$$