Micro-Foundations in Growth Models

Session 3: The Ramsey-Cass-Koopmans Model

ECO5021F: Macroeconomics
University of Cape Town

Readings

Required

- Romer, D. (2019). Advanced Macroeconomics. Chapter 2, Part A

Recommended

- Econtalk: Romer on Growth (2007); Spence on Growth (2010)
- Econtalk: Greg Mankiw on Gasoline Taxes, Keynes and Macroeconomics, 22 January 2007

Contents:

The RCK Model

- RCK model is considered one of the three basic models in modern macroeconomics
- Extends the Solow model to include **optimising agents (firms and households)**
- The other basic model is the overlapping generations model (which we will cover next week)
- Note: For simplicity, I will use the subscript time notation c_t instead of $c(t)$. The RCK model is still continuous time, so c_t stands for variable c at instant t.

The RCK Model

Basic Assumptions

- Similar assumptions to those of the Solow Model
- However, the RCK economy is inhabited by a large number, H, of infinitely lived individuals that operate in a competitive decentralised economy.
- Labour and Labour augmenting technology still grow exogenously:

$$\frac{\dot{A}_t}{A_t} = g \quad ; \quad A_t = A_0 e^{gt}$$
$$\frac{\dot{L}_t}{L_t} = n \quad ; \quad L_t = L_0 e^{nt}$$

- Big difference: capital accumulation determined by **the interaction of firms and households in the competitive market**

The RCK Model: Firms

Assumptions

The role of the firm

1. There are a large number of identical competitive firms in this economy
2. These firms are profit-maximising
3. And share the same production technology (Cobb-Douglas production function)
4. Technology grows exogenously at the rate of g
5. Firms are owned by the households
6. Goods and factor markets are competitive

The RCK Model: Firms

Behaviour: Factor payments to Capital

In the competitive decentralised economy all factors of production earn their marginal products:

- The marginal product of capital: $\frac{\partial Y_t}{\partial K_t} \equiv \frac{\partial y_t}{\partial k_t} \equiv f'(k_t) = \alpha k_t^{\alpha-1}$
- Usually, the real rate of return on a unit of capital would be given by $f'(k_t) - \delta = r_t$.
- But because we also assume their is no depreciation $\delta = 0$, r_t equals its earnings per unit of time (marginal product).
- r_t can also vary over time, so the cumulative interest function, is

$$R_t = \int_{\tau=0}^{t} r_\tau d\tau$$

- It turns out, the marginal contribution to R_t at instant t is simply the value of r at instant t: $\partial R_t / \partial t = r_t$.

The RCK Model: Firms

Behaviour: Factor payments to Labour

Each worker receives wage W_t per instant of time. If we define $w_t = \frac{W_t}{A_t}$ as the wage per effective worker; each worker receives $A_t w_t$.

Effective labour receives its marginal product:

Class Exercise

The RCK Model: Firms

Behaviour: Factor payments to Labour

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Effective labour receives its marginal product:

Class Exercise

$$
\begin{aligned}
w_t &= \frac{\partial Y_t}{\partial A_t L_t} = (1-\alpha)K_t^{\alpha}(A_t L_t)^{1-\alpha-1} \\
&= K_t^{\alpha}(A_t L_t)^{-\alpha} - \alpha K_t^{\alpha}(A_t L_t)^{-\alpha} \\
&= \left(\frac{K_t}{A_t L_t}\right)^{\alpha} - \alpha K_t^{\alpha}(A_t L_t)^{-\alpha} \cdot \frac{A_t L_t}{K_t} \cdot \frac{K_t}{A_t L_t} \\
&= \left(\frac{K_t}{A_t L_t}\right)^{\alpha} - \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} \cdot \frac{K_t}{A_t L_t} \\
&= k_t^{\alpha} - \alpha k_t^{\alpha-1} k_t \\
&= f(k_t) - f'(k_t) k_t
\end{aligned}
$$

The RCK Model: Households

Assumptions

The role of the household

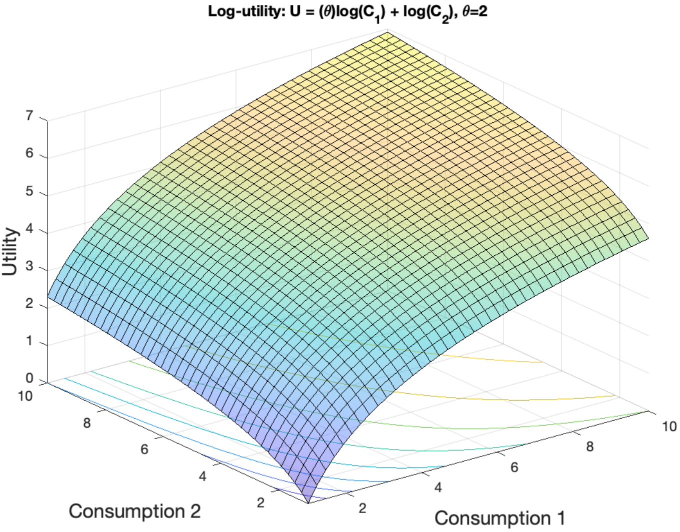
1. There are a large number of identical, infinitely lived households H of *size* L_t which grows at rate n.
2. Each member of the household supplies one unit of labour at every point in time
3. Households rent all their capital to firms at rate r_t
4. Initial capital holdings of K_0/H per household
5. Each individual in the household divides income at each point in time $\leftrightharpoons$ consumption and savings so as to maximise its lifetime utility ...

   This yields instantaneous utility $u(C_t)$ to that individual $\Rightarrow \therefore$ the total utility that the household obtains each instant is $u(C_t)\frac{L_t}{H}$.[1]

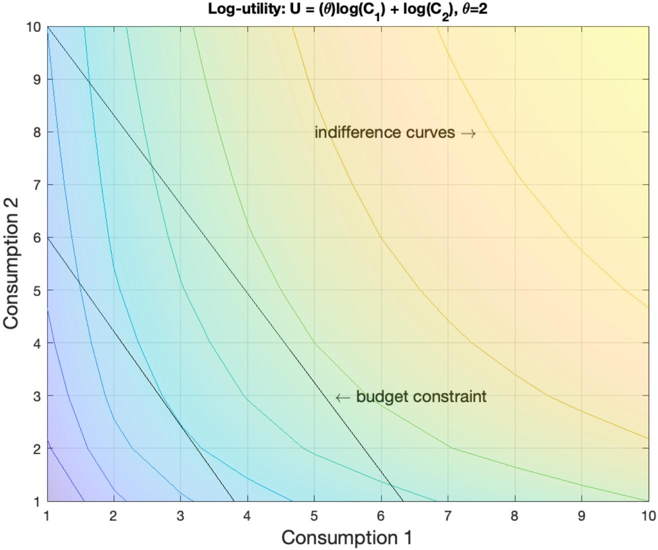
[1] Romer uses L_t/H, but since H never serves any substantive purpose we normalize it to 1.

The RCK Model: Households

...does this look familiar ...?

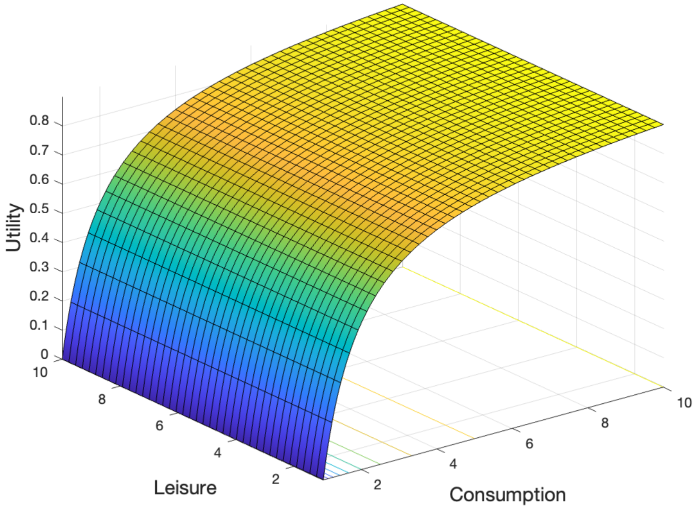


The RCK Model: Households



The RCK Model: Households

…Romer abstracts from the consumption of leisure goods …



The RCK Model: Households

Assumptions: Utility function

In order to maximise lifetime utility, the household needs a utility function:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C_t) L_t dt \ . \tag{1}$$

To ensure a well-defined steady-state the instantaneous utility function takes the form

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta} \ , \qquad \theta > 0 \ , \qquad \rho - n - (1-\theta)g > 0$$

- HHs discount future utility at rate $\rho > 0$.[2]
- Implies: the future is discounted faster than the increase in utility, so that households don't get infinite utility.

[2] Recall: the continuous time version of standard discounting in discrete time is $e^{-\rho t}$.

The RCK Model: Households

Assumptions: Utility function

- $u(\cdot)$ known as *constant-relative-risk-aversion* (CRRA) utility
- θ is the co-efficient of relative risk aversion
- Since there is no uncertainty in this model, the HH's attitude to risk is not directly relevant.
- But θ also determines the HH's willingness to shift consumption over time:
  - the inverse of the elasticity of substitution between consumption at any two points in time ($1/\theta$)

See Problem 2.2 (p.94) and alternative cases: $\theta > 1$; $\theta < 1$; $\theta \to 1$ (p.51)

Class Exercise

(1) Derivative of $u(C_t)$ ... and check what alternative cases imply for utility.
(2) Derive: $-Cu''(C)/u'(C)$

The RCK Model: Households

Behaviour: Budget Constraint

- Recall: the household is endowed with an initial level of capital K_0 and supplies all its capital and labour to firms

  $\Rightarrow$ earns the marginal return on capital and labour.

  $\therefore$ the representative household takes r and w as given.

The RCK Model: Households

Behaviour: Budget Constraint

- Recall: the household is endowed with an initial level of capital K_0 and supplies all its capital and labour to firms

  $\Rightarrow$ earns the marginal return on capital and labour.

  $\therefore$ the representative household takes r and w as given.
- The HHs budget ("wallet") places a constraint on the optimisation:

$$\underbrace{\int_{t=0}^{\infty} e^{-R_t} C_t L_t dt}_{\text{PV(Consumption)}} \quad \leq \quad \underbrace{K_0 + \int_{t=0}^{\infty} e^{-R_t} W_t L_t dt}_{\text{Initial Wealth + PV(Income)}} \qquad (2)$$

The present *discounted* value of consumption should be smaller or equal to the sum of the households initial wealth and present *discounted* value of household income

The RCK Model: Households

Behaviour: Budget Constraint

- What is the discount factor?
  - Only asset in this economy that households can use to save
  - $\therefore$ the cumulative return on capital

The RCK Model: Households

Behaviour: Budget Constraint

This becomes clearer when we rewrite the budget constraint a little:

$$K_0 + \int_{t=0}^{\infty} e^{-R_t}(W_t - C_t)L_t dt \quad \geq \quad 0$$

- where $(W_t - C_t)L_t$ is the instantaneous level of (dis)saving by the household, which is equivalent to (dis)investments in capital.
- we can $\therefore$ express the level of capital of the HH at instant s as:

$$K_s \quad = \quad e^{R_s} K_0 + \int_{t=0}^{s} e^{R_s - R_t}(W_t - C_t)L_t dt \; ,$$

... or, equivalently by discounting ...

$$e^{-R_s} K_s \quad = \quad K_0 + \int_{t=0}^{s} e^{-R_t}(W_t - C_t)L_t dt$$

The RCK Model: Households

Behaviour: Budget Constraint

Taking the limit as s approaches ∞ we obtain the budget constraint as above:

$$\lim_{s \to \infty} [e^{-R_s} K_s] \quad = \quad \lim_{s \to \infty} \left[K_0 + \int_{t=0}^{s} e^{-R_t} (W_t - C_t) L_t dt \right] \quad \geq \quad 0$$

- The condition: $\lim_{s \to \infty} [e^{-R_s} K_s] \geq 0$ is called a *no-Ponzi-game condition*.

  i.e., households cannot end their lives with negative wealth.

The RCK Model: Households

Rewriting in terms of effective labour

- As with firms, we need to re-write the variables in the objective function and the budget constraint in terms of effective labour
- Recall: consumption per worker and wage per worker are, respectively, C_t and W_t.
- The *per effective worker* versions of these are:

$$c_t = \frac{C_t}{A_t} \quad , \therefore \quad C_t = A_t c_t$$
$$w_t = \frac{W_t}{A_t} \quad , \therefore \quad W_t = A_t w_t$$

$\therefore$ using the results that $A_t = A_0 e^{gt}$ and $L_t = L_0 e^{nt}$ we can rewrite the objective and constraint in *per effective worker* terms . . .

The RCK Model: Households

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Equation (1): the household's utility (objective) function

$$
\begin{aligned}
U &= \int_{t=0}^{\infty} e^{-\rho t} u(C_t) L_t dt = \int_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\theta}}{1-\theta} L_t dt \\
&= \int_{t=0}^{\infty} e^{-\rho t} \frac{(A_t c_t)^{1-\theta}}{1-\theta} L_t dt = \int_{t=0}^{\infty} e^{-\rho t} \frac{(A_0 e^{gt} c_t)^{1-\theta}}{1-\theta} L_0 e^{nt} dt \\
&= A_0^{1-\theta} L_0 \int_{t=0}^{\infty} e^{-(\rho - n - (1-\theta)g)t} \frac{c_t^{1-\theta}}{1-\theta} dt \\
&\equiv B \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt \ , \qquad (3)
\end{aligned}
$$

where $\beta = (\rho - n - (1-\theta)g) > 0$. And since the constant term B does not affect behaviour, it can be normalized to 1.

The RCK Model: Households

Behaviour: Maximisation Problem

Equation (2): the household's budget constraint, follows similarly ...

$$
\begin{aligned}
\int_{t=0}^{\infty} e^{-R_t} C_t L_t dt &\leq K_0 + \int_{t=0}^{\infty} e^{-R_t} W_t L_t dt \\
&\vdots \\
\int_{t=0}^{\infty} e^{-R_t+(n+g)t} c_t dt &\leq k_0 + \int_{t=0}^{\infty} e^{-R_t+(n+g)t} w_t dt \ , \qquad (4)
\end{aligned}
$$

where we use the fact that $k_0 = K_0/A_0 L_0$.

We can now solve the HH's optimisation problem:

- to choose the path of c_t to maximize life-time utility, (3), subject to the budget constraint, (4).

The RCK Model: Households

Behaviour: Optimality Condition

Since the marginal utility of consumption is always positive, the HH satisfies it's budget constraint with equality.

We can therefore setup the Lagrangian function:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\beta t} \frac{c_t^{1-\theta}}{1-\theta} dt \quad + \quad \lambda \left[k_0 + \int_{t=0}^{\infty} e^{-R_t + (n+g)t} (w_t - c_t) dt \right] .$$

The first order condition for optimality is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= 0 \\ &\Updownarrow \\ e^{-\beta t} c_t^{-\theta} &= \lambda e^{-R_t + (n+g)t} \end{aligned} \tag{5}$$

The RCK Model: Households

Behaviour: Optimality Condition

To see what Eq. (5) implies for the behaviour of consumption, we can express it in terms of growth rates ...

First, take natural logs of both sides, and then take the time derivative to get ...[3]

$$-\beta - \theta \frac{\dot{c}_t}{c_t} \quad = \quad -r_t + (g+n)$$

Using the definition of β and the fact that $r_t = f'(k_t)$ we get:

$$\frac{\dot{c}_t}{c_t} \quad = \quad \frac{r_t - \rho - \theta g}{\theta} = \frac{f'(k_t) - \rho - \theta g}{\theta} \tag{6}$$

[3] Recall that $R_t = \int_{\tau=0}^{t} r_\tau d\tau$ and, therefore at every instant t, $\frac{\partial R_t}{\partial t} = r_t$.

The RCK Model: Households

Behaviour: Optimality Condition

- Eq. (6) is known as **the Euler equation**

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - \rho - \theta g}{\theta}$$

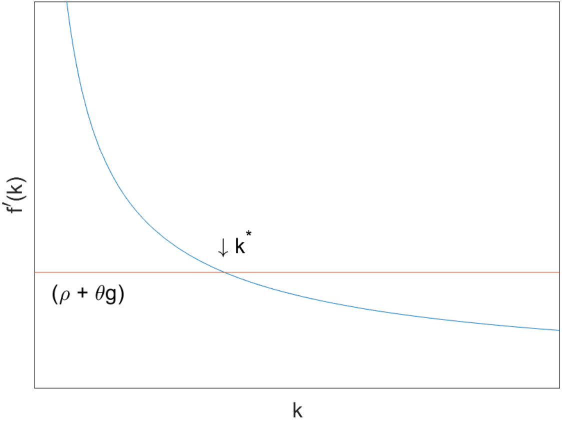
- It describes the evolution of consumption, c_t, over time
  - the optimal *inter-temporal* consumption path of households (see p. 56)

$$\frac{\dot{C}_t}{C_t} = \frac{\dot{A}_t}{A_t} + \frac{\dot{c}_t}{c_t} = g + \frac{r_t - \rho - \theta g}{\theta} = \frac{r_t - \rho}{\theta}$$

- Similar to the equation of motion for k, we will be able to track the behaviour for this equation in the two-dimensional (k, c) space.
- $\dot{c}_t$ will be zero whenever $f'(k_t) = \rho + \theta g$
- The point where $\dot{c}_t = 0$ will also deliver our steady state level of capital stock, k^*

The RCK Model: Dynamics of the Economy

The dynamics of consumption per effective worker, c_t



- With diminishing marginal returns to capital, when $k_t > k^*$ then $f'(k_t) < \rho + \theta g$ and vice versa.
- This gives us the direction of the arrows for the phase diagram

For intuition, write the Euler equation (6) terms of consumption per worker):

$$\frac{\dot{C}_t}{C_t} = \frac{f'(k_t) - \rho}{\theta} = \frac{r_t - \rho}{\theta}$$

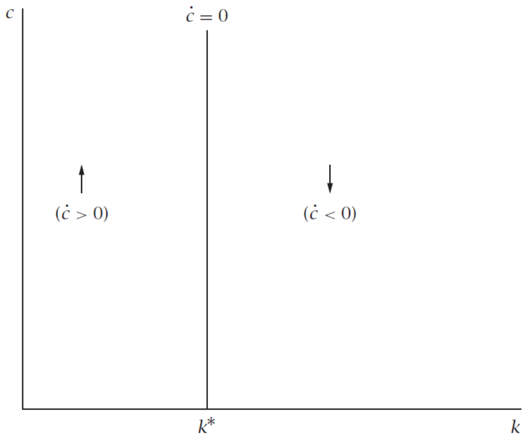
- if $r_t > \rho$, $\dot{C}_t$ is growing; if $r_t < \rho$, $\dot{C}_t$ is shrinking.
- the smaller is θ the more sensitive are the changes in consumption in response to differences between the real interest rate and the discount rate. (the less marginal utility changes as consumption changes.)

Intuitively:

- If $f'(k_t) = r_t > \rho$, the economy is generating a high return on capital relative to households' preference for present consumption.
- i.e., the economy is productive enough to support increasing consumption levels, leading to growth in aggregate consumption.
- households can therefore save more *and* consume more in the future because the high capital investment returns outweigh the preference for immediate consumption.
- households smooth consumption over lifetime: higher capital investment returns $\rightarrow$ higher income and thus higher consumption $\rightarrow$ increase in consumption over time.

The RCK Model: Dynamics of the Economy

The dynamics of consumption per effective worker, c



$$\frac{\dot{c}_t}{c_t} = \frac{\overbrace{f'(k_t)}^{r_t} - (\rho + \theta g)}{\theta}$$

$$\begin{cases} \text{When } k_t > k^* \text{ then } \dot{c}_t \text{ is negative} \\ \text{When } k_t < k^* \text{ then } \dot{c}_t \text{ is positive} \end{cases}$$

The RCK Model: Dynamics of the Economy

The dynamics of capital per effective worker, k

As in the Solow model, $\dot{k}_t =$ actual investment $-$ break even investment:

$$\dot{k}_t = f(k_t) - c_t - (n+g)k_t \ ,$$

where households save by investing in capital what they don't consume: $f(k_t) - c_t$, and we are assuming that there is no depreciation, $\delta = 0$.

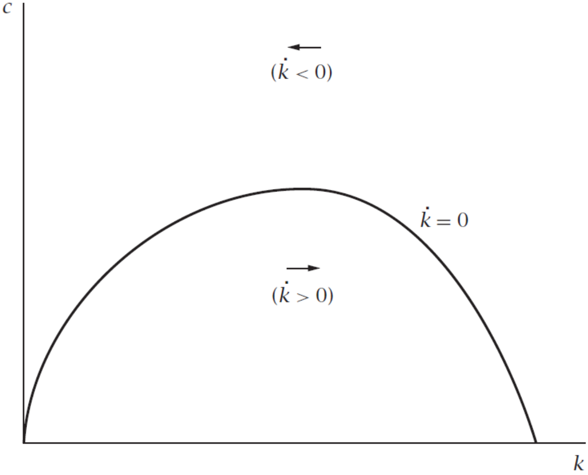
- Steady state must be characterized by $\dot{k}_t = 0$
- Find the level of consumption per effective worker that would make this true for a given k_t:

$$c_t = f(k_t) - (n+g)k_t \ .$$

$\frac{\partial c}{\partial k}$: c_t is increasing in k_t until $f'(k_t) = (n+g)$ (the golden-rule level of k) and then decreasing

The RCK Model: Dynamics of the Economy

The dynamics of capital per effective worker, k



$$\dot{k}_t = f(k_t) - c_t - (n+g)k_t \quad \begin{cases} \text{If } c_t > \dot{k}_t = 0 \text{ locus, then } \dot{k}_t \text{ is negative} \\ \text{If } c_t < \dot{k}_t = 0 \text{ locus, then } \dot{k}_t \text{ is positive} \end{cases}$$

The RCK Model: Dynamics and Steady-State

The phase diagram

Important:

- Capital is a **state variable**.
  - It can only be changed incrementally over time by (dis)investing.
  - It *cannot* “jump” (change by discrete amounts).
- Consumption is a **control variable.**
  - It *can* jump by discrete amounts.
  - But consumers do not like consumption variability, so only shocks makes it jump
  - Consumers will never *plan* to make consumption jump

The RCK Model: Dynamics and Steady-State

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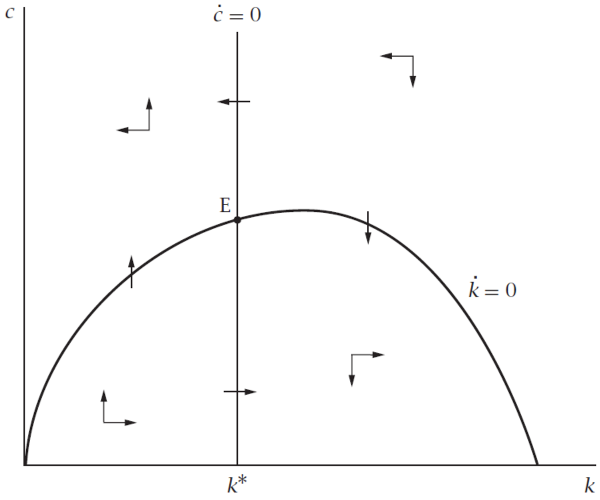
How do we interpret the implications of the dynamic evolution values we found? Graphically via a phase diagram

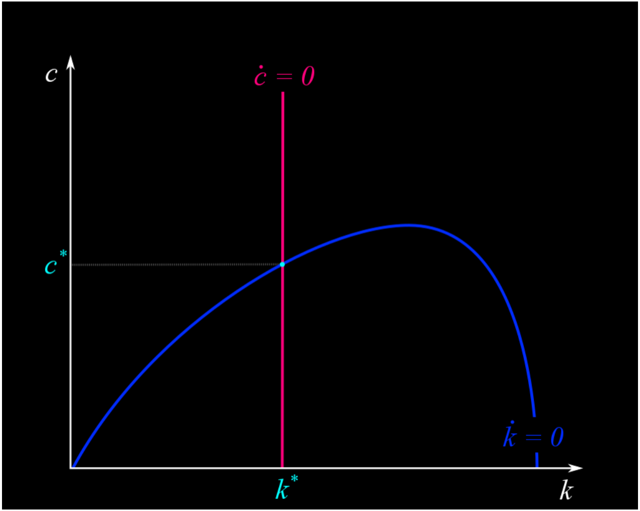
- Locus where k is constant ($\dot{k}_t = 0$)
- Locus where c is constant ($\dot{c}_t = 0$)
- Steady State
- Dynamics to steady state: Key point: initial K is given, initial C is a choice

The RCK Model: Dynamics of the Economy

The dynamics of c and k

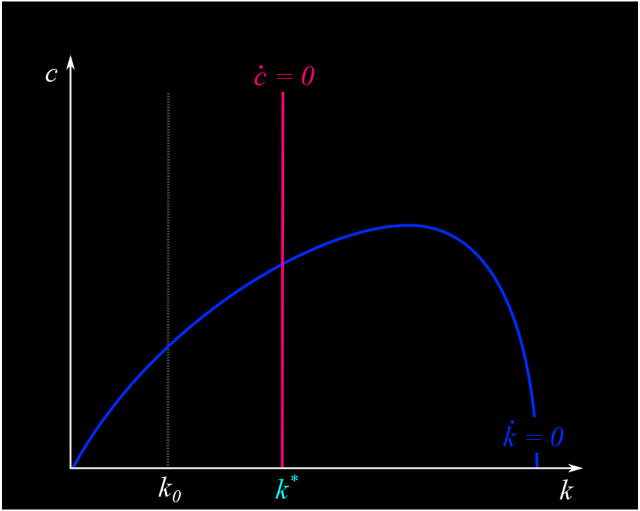


The RCK Model: Steady State



The RCK Model: Dynamics of the Economy

The behaviour of c and k for various initial values of c



The RCK Model: Dynamics of the Economy

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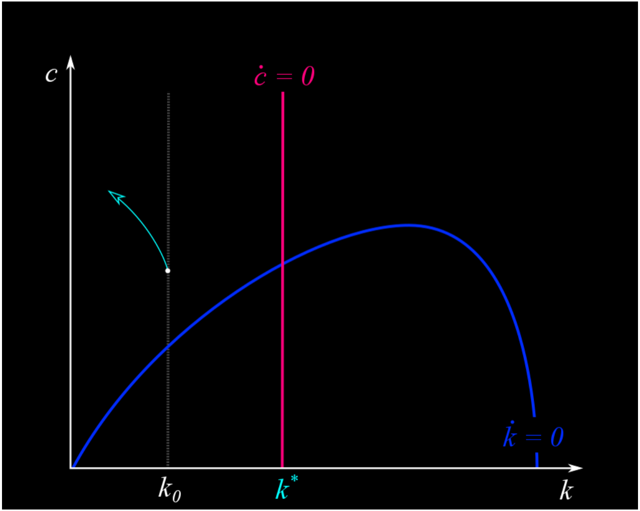

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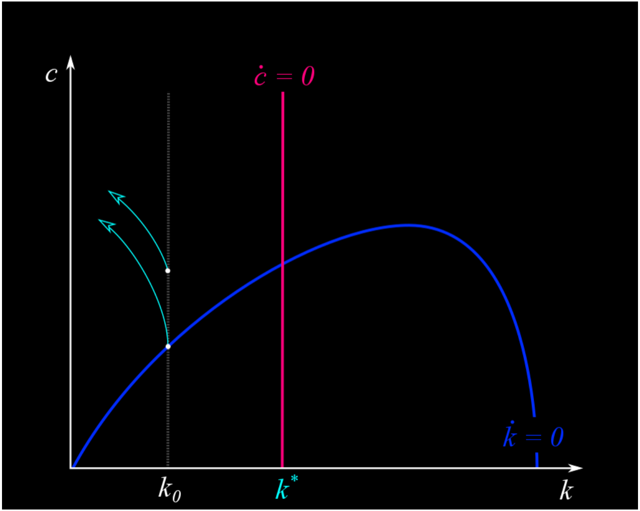

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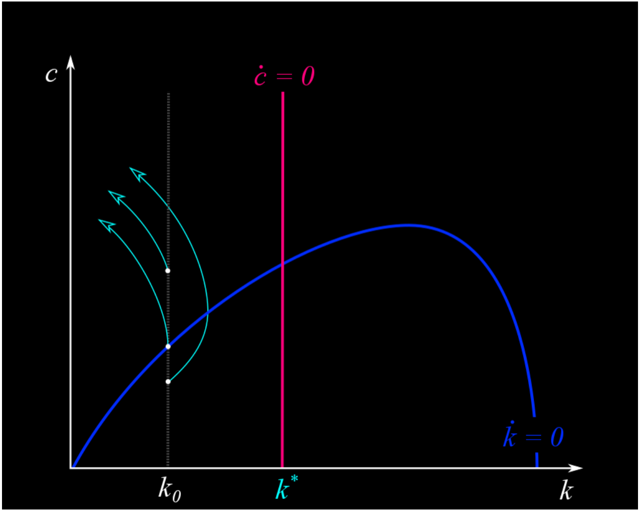

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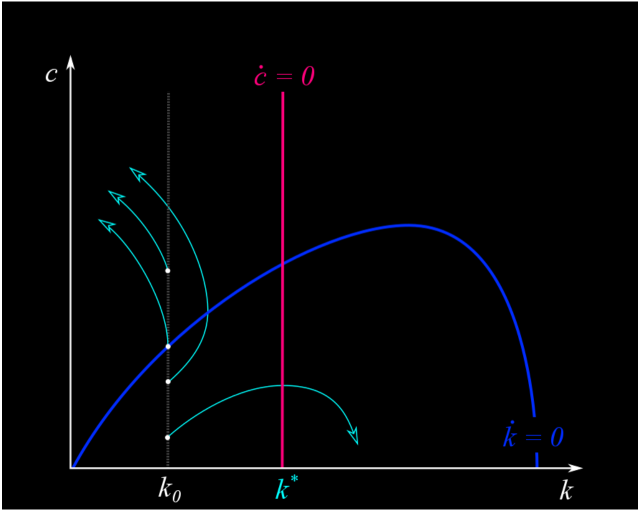

The RCK Model: Dynamics of the Economy

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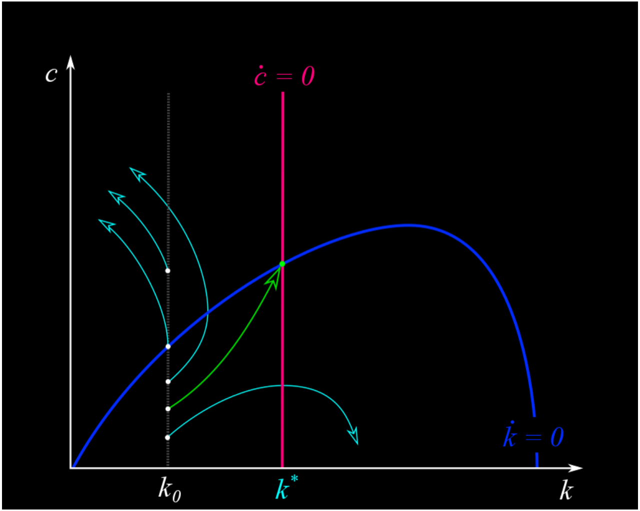
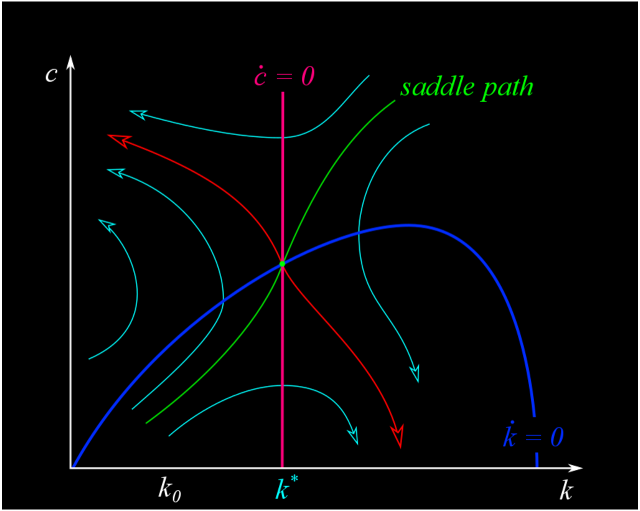


The RCK Model: Dynamics of the Economy

The Saddle Path



The RCK Model: Dynamics of the Economy

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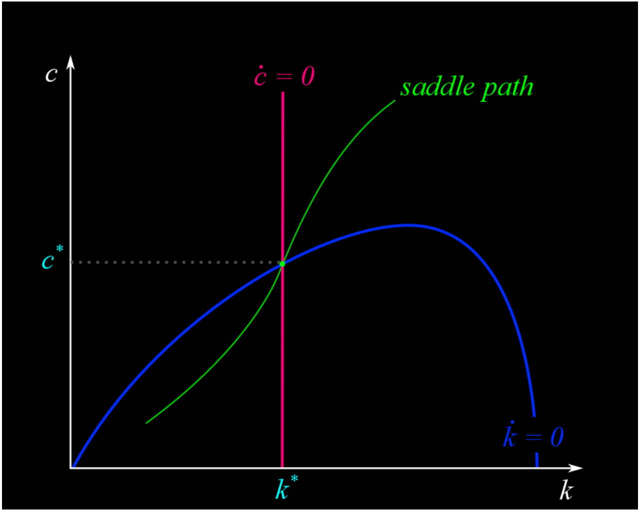
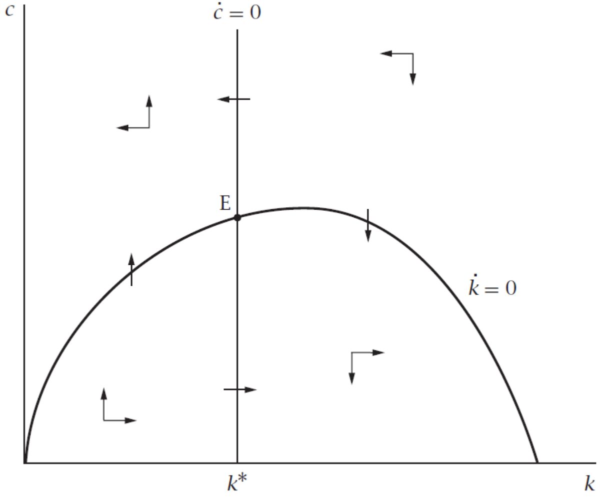


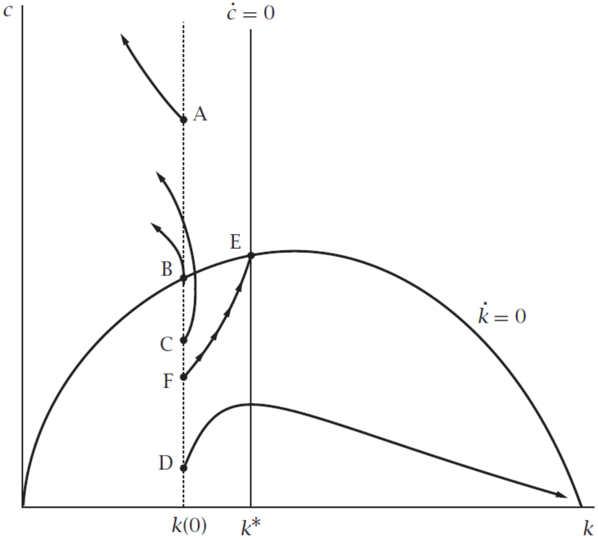
The RCK Model: Dynamics of the Economy

The dynamics of c and k



The RCK Model: Dynamics of the Economy

The behaviour of c and k for various initial values of c



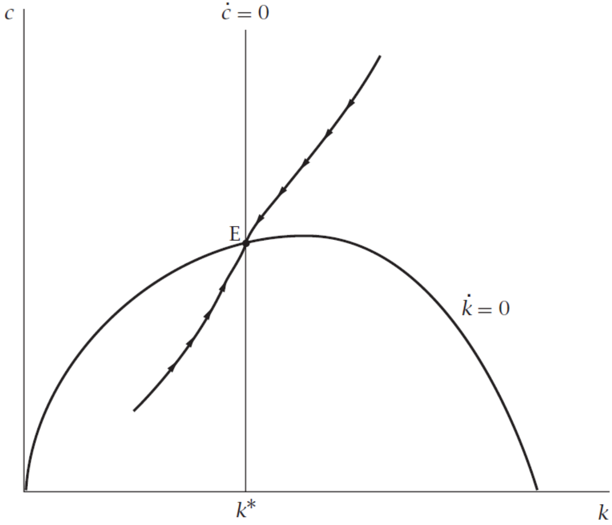
The RCK Model: Dynamics of the Economy

The Saddle Path

- Trajectories of **A**, **B**, **C**, **D** all satisfy the equation of motion.
- However, we eliminate them as potential initial values considering their implications for the capital stock and the budget constraint.
- From our **assumptions**, we don't allow capital stock to be negative or explosive.
- There is a function which relates each possible starting value of k to a unique starting value for c so that economy moves to equilibrium
- This function is called the **saddle path**.

The RCK Model: Dynamics of the Economy

The Saddle Path



The RCK Model: Dynamics of the Economy

The Saddle Path

- In the Solow Model we also obtained an equilibrium but there was **no guarantee that this equilibrium would be a desirable result.**
- However, in this model we have allowed households to **choose their saving rate at each point in time**.
- When the model is on a saddle path and eventually on its BGP, then it meets the requirements of the **first welfare theorem**
  - any competitive equilibrium, where markets are complete, leads to a Pareto efficient allocation of resources.
- The conditions of the first welfare theorem ensures that the balanced growth path is **Pareto optimal**

The RCK Model: Dynamics of the Economy

The Balanced Growth Path

- When the economy reaches point **E**, it will behave in the same manner as the Solow model:
  - Capital, output, and consumption per unit of effective worker will remain constant
  - The total capital stock, total output and total consumption will grow at the rate of $n + g$
  - Capital per worker, output per worker and consumption per worker grow at the rate of g
- The central result of the Solow model, regarding the drivers of economic growth **does not depend on the assumption of a constant and exogenous savings rate**.
- The importance of this model lies in its flexibility to answer many analytical questions relating to an economy's long run evolution

Application

- Since we now have forward looking decision makers, we have additional types of exogenous changes that we can analyse
- Importantly, two additional dimensions are interesting:

  An exogenous change to one of our parameters may be:
  - Permanent or Temporary
  - Anticipated or Unanticipated
- Unanticipated permanent increase in subjective discount factor
- Anticipated increase in subjective discount factor
- Permanent fall in technological growth rate
- Adding government to the model

Welfare question (problem set)

- In the Solow-Swan model, it was possible to “over-save.” i.e. have a steady state capital per effective worker that is larger than the one that maximizes consumption per effective worker
- This is called Dynamic Inefficiency *[Why? Hint: Pareto]*
- Is the same possible in the RCK model?