Diamond's Overlapping Generations Model

Contents

These notes are the bare bones of the derivations for the equations of motion of the Overlapping Generations Model and are thus merely an aid, not a replacement for the text in Romer.

1 Firms and Production Technology

Firms are identical to the Ramsey-Cass-Koopmans model. Since the firm's problem is static, it is identical in continuous and discrete time. Only the assumptions and results are stated here:

- Production is constant returns to scale in Capital and Effective Labour $(A_t L_t)$

$$Y_t = F\left(K_t, A_t L_t\right)$$

- Technology grows at rate g per period, so in discrete time this means:

$$A_{t+1} = (1+g)A_t$$

- In intensive form (dividing by $A_t L_t$):

$$\begin{aligned} y_t &= f\left(k_t\right) \\ f'\left(k_t\right) &> 0 \\ f''\left(k_t\right) &< 0 \end{aligned}$$

- Firms are perfectly competitive, make no profits, and pay factors their marginal products:

$$\begin{aligned} r_t &= f'\left(k_t\right) \\ w_t &= f\left(k_t\right) - f'\left(k_t\right) k_t \end{aligned}$$

2 Consumers

Each period L_t consumers are born who live for two periods. The rate of population growth is n per period so that $L_{t+1} = (1+n)L_t$.

In the first period of life, consumers are young:

- each young consumer supplies a unit of labour to perfectly competitive firms, earning wage $W_t = A_t w_t$ where w_t is wage per effective worker as before;

- wage income $A_t w_t$ is split between consumption while young $C_{y,t}$ and saving in capital which will be employed in the following period, so we denote it as $K_{i,t+1}$ where the i subscript denotes that this is the capital holding of an individual consumer.

In the second period of life, consumers are old:

- They do not work;
- earn a total of $(1 + r_{t+1}) K_{i,t+1}$ on their savings which they totally consume $C_{o,t+1}$, and then die.

2.1 Budget Constraint

The budget constraint in the first period of life is:

$$C_{y,t} = A_t w_t - K_{i,t+1}$$

and in the second period:

$$C_{o,t+1} = (1 + r_{t+1}) K_{i,t+1}$$

Thus the intertemporal budget constraint is:

$$C_{y,t} + \frac{1}{1 + r_{t+1}} C_{o,t+1} = A_t w_t$$

Note that this has the standard interpretation: present discounted value of lifetime consumption must be equal to present discounted value of life time income (which is trivial in this model).

To complete the model, we assume that there is a population of old consumers in the initial period that own and consume from equal shares an initial capital stock K_0.

2.2 Objective function:

Consumers care about their subjectively discounted present value of utility:

$$U(C_{y,t}, C_{o,t+1}) = \frac{C_{y,t}^{1-\theta} - 1}{1 - \theta} + \frac{1}{1 + \rho} \frac{C_{o,t+1}^{1-\theta} - 1}{1 - \theta}$$

Where $\theta > 0$ is the inverse of the elasticity of substitution as before, and $\rho > 0$ is the subjective discount factor.

2.3 Optimizing Behaviour:

The Lagrangian for this problem is given by:

$$\mathcal{L} = \frac{C_{y,t}^{1-\theta} - 1}{1 - \theta} + \frac{1}{1 + \rho} \frac{C_{o,t+1}^{1-\theta} - 1}{1 - \theta} + \lambda \left[A_t w_t - C_{y,t} - \frac{1}{1 + r_{t+1}} C_{o,t+1} \right]$$

In this setup, we now have two choice variables (since the spread of lifetime consumption determines optimal saving. Any alternative form that is equivalent could have been used)

The first order optimality conditions are therefore:

$$\frac{\partial \mathcal{L}}{\partial C_{y,t}} = 0$$

$$\Updownarrow$$

$$C_{y,t}^{-\theta} = \lambda$$

and

$$\frac{\partial \mathcal{L}}{\partial C_{o,t+1}} = 0$$

$$\Updownarrow$$

$$\frac{1}{1+\rho} C_{o,t+1}^{-\theta} = \lambda \frac{1}{1+r_{t+1}}$$

Combining these conditions and rearranging:

$$\begin{aligned}
\frac{1}{1+\rho} C_{o,t+1}^{-\theta} &= C_{y,t}^{-\theta} \frac{1}{1+r_{t+1}} \\
\left(\frac{C_{o,t+1}}{C_{y,t}}\right)^{-\theta} &= \frac{1+\rho}{1+r_{t+1}} \\
\frac{C_{o,t+1}}{C_{y,t}} &= \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}
\end{aligned}$$

This is called an Euler equation and is a standard result in all dynamic optimal consumption problems in macroeconomics. Note that it is simply the discrete equivalent of the optimal growth rate of consumption in the RCK model.

Defining the savings rate $s\left(r_{t+1}\right)$ as the fraction of income is not consumed when young:

$$C_{y,t} = \left(1 - s\left(r_{t+1}\right)\right) A_t w_t$$

Plugging this into the budget constraint allows us to find the savings rate as a function of the parameters of the model:

$$C_{y,t} + \frac{1}{1+r_{t+1}} C_{o,t+1} = A_t w_t$$

$$
\begin{aligned}
C_{y,t} + \frac{1}{1+r_{t+1}}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} C_{y,t} &= A_t w_t \\
\left[1 + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}}\right] C_{y,t} &= A_t w_t \\
\left[\frac{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}}\right] C_{y,t} &= A_t w_t \\
C_{y,t} &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t \\
&\Downarrow \\
s\left(r_{t+1}\right) &= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}
\end{aligned}
$$

How does the interest rate affect the savings rate? There are income and substitution effects at play:

- Income/Wealth effect: the consumer saves a positive amount, so when the interest rate increases each unit of saving pays off more, so the consumer is richer - this makes her want to consume more in both periods, i.e. save less in the first period
- Substitution Effect: when the interest rate increases, consumer earlier becomes relatively more expensive, so the consumer prefers to shift consumption to the second period, or equivalently, save more

The relative strength of these effects depend on θ :

- If $\theta = 1$ (so that utility is logarithmic),

$$
\begin{aligned}
s\left(r_{t+1}\right) &= \frac{1}{(1+\rho)^1 + 1} \\
&= \frac{1}{2+\rho}
\end{aligned}
$$

This is a situation where the income and substitution effects exactly cancel out:

$$
s'\left(r_{t+1}\right)\big|_{\theta=1} = 0
$$

- When $\theta > 1$, the consumer wants a smoother consumption profile, hence will sacrifice some average income to make the variation of income smaller. This implies that the income effect dominates, and the savings rate decreases in the interest rate:

$$s'(r_{t+1})|_{\theta>1} < 0$$

- When $\theta < 1$, the consumer is willing to accept a more variable consumption profile in order to obtain a higher average consumption level. This implies that the substitution effect dominates, and the savings rate increases in the interest rate:

$$s'(r_{t+1})|_{\theta<1} > 0$$

Graphically: see Figure 1 (see also 3-D graphic of savings function and it's derivative in the slides)

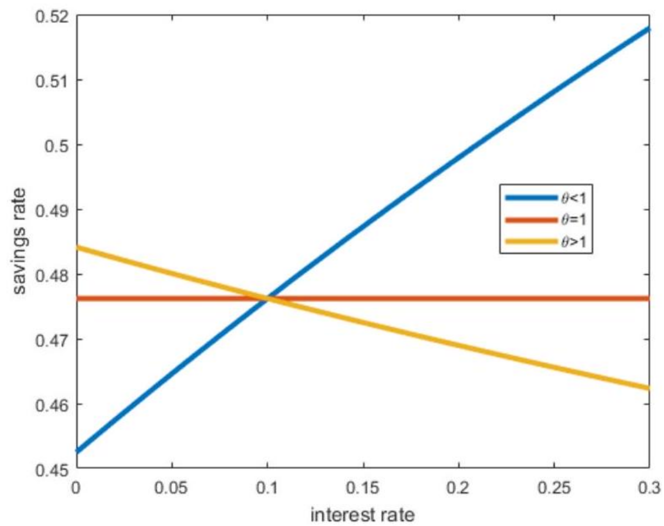

Figure 1: Savings rate as a function of interest rate for different θs

3 Dynamics of the Economy

3.1 Equation of motion for capital per effective worker:

Since all consumers are identical, we can aggregate over their decisions to find the behaviour of aggregate capital.

Total aggregate capital in period $t+1$ is the individual savings by each consumer $S_{i,t}$ times the size of the population:

$$\begin{aligned} K_{t+1} &= S_{i,t} L_t \\ &= s(r_{t+1}) A_t w_t L_t \end{aligned}$$

Since technology grows at rate g per period and the size of the population by rate n, we can state $A_t = \frac{A_{t+1}}{(1+g)}$ and $L_t = \frac{L_{t+1}}{(1+n)}$. Plugging this into the equation for capital:

$$K_{t+1} = s\left(r_{t+1}\right) w_t \frac{A_{t+1}}{(1+g)} \frac{L_{t+1}}{(1+n)}$$
$$\frac{K_t}{A_{t+1} L_{t+1}} = k_{t+1} = \frac{1}{(1+n)(1+g)} s\left(r_{t+1}\right) w_t$$

Using the results from before that $r_{t+1} = f'\left(k_{t+1}\right)$ and $w_t = f'\left(k_t\right) - f'\left(k_t\right) k_t$:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s\left(f'\left(k_{t+1}\right)\right)\left(f\left(k_t\right) - f'\left(k_t\right) k_t\right)$$

This implicitly defines the evolution of capital per effective worker. "Implicitly" because k_{t+1} appears on both sides of the equation, and we cannot find a closed form solution for it without knowing the functional form of $f(.)$.

3.2 Parametric Example:

To see the implications for the steady state and dynamics of the economy, we will impose two additional simplifications:

1. Cobb Douglas production: $y_t = f\left(k_t\right) = k_t^\alpha$
2. Log utility (that is we consider the case where $\theta \rightarrow 1$)

Above we showed that when $\theta = 1$, the savings rate is independent of the interest rate (or equivalently, the marginal return to capital per effective worker):

$$s\left(f'\left(k_{t+1}\right)\right) = \frac{1}{2+\rho}$$

Using $f\left(k_t\right) = k_t^\alpha$:

$$\begin{aligned} w_t &= f\left(k_t\right) - f'\left(k_t\right) k_t \\ &= k_t^\alpha - \alpha k_t^{\alpha-1} k_t \\ &= k_t^\alpha - \alpha k_t^\alpha \\ &= (1-\alpha) k_t^\alpha \end{aligned}$$

Thus we obtain the dynamic equation:

$$k_{t+1} = \frac{1-\alpha}{(1+n)(1+g)(2+\rho)} k_t^\alpha$$

where

$$\frac{1-\alpha}{(1+n)(1+g)(2+\rho)} < 1$$

Steady state occurs where capital per effective worker does not change:

$$k_{t+1} = k_t = k^*$$

In this simple example we can easily solve for this value (which we will use for comparing the welfare implications of the three growth models):

$$\begin{aligned}
k^* &= \frac{1-\alpha}{(1+n)(1+g)(2+\rho)} (k^*)^{\alpha} \\
(k^*)^{1-\alpha} &= \frac{1-\alpha}{(1+n)(1+g)(2+\rho)} \\
k^* &= \left(\frac{1-\alpha}{(1+n)(1+g)(2+\rho)} \right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

A Appendix: A result for the CES utility function

The constant elasticity of consumption utility function is given by:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta} \text{ , with } \theta > 0$$

We cannot evaluate the function when $\theta = 1$, as the denominator is zero. Notice, however, that the numerator is also zero here. This result allows us to use l'Hopital's rule (which has an interesting history you should look up) to determine $\lim_{\theta \to 1} u(c)$.[1]
l'Hopital's rule states, for a quotient of the form $\frac{f(x)}{g(x)}$, if

- $\lim_{x \to x_0} f(x) = 0$ and $\lim_{x \to x_0} g(x) = 0$, or
- $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} g(x) = \infty$, and
- $g'(x) \neq 0$, and
- $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

In our case the the first condition holds already, and the derivative of the numerator with respect to θ is:

$$\frac{\partial}{\partial \theta} \left(c^{1-\theta} - 1 \right) = -\ln(c) c^{1-\theta}$$

The derivative of the numerator with respect to θ is:

$$\frac{\partial}{\partial \theta}(1-\theta) = -1$$

so the third condition holds, and lastly we can directly evaluate the limit of the quotient of derivatives:

$$\begin{aligned} \lim_{\theta \to 1} \frac{-\ln(c) c^{1-\theta}}{-1} &= \lim_{\theta \to 1} \left(\ln(c) c^{1-\theta} \right) \\ &= \ln(c) \end{aligned}$$

Thus we have shown that

$$\lim_{\theta \to 1} \frac{c^{1-\theta} - 1}{1-\theta} = \ln(c)$$

Thus logarithmic utility (and more generally, Cobb-Douglas utility) is a special case of CES utility.

[1]This example is just a simple version, the actual rule is a little more general.