Linearization

Mathematical Methods for Economics (771)

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Contents:

Readings

- Dejong, D. N. and C. Dave, 2007. Structural Macroeconometrics. Princeton University Press. Chapter 2: Approximating and solving DSGE models;
- A Toolkit for analyzing nonlinear dynamic stochastic models easily (by Uhlig Harald).

Introduction

A DSGE model consists of a system of nonlinear difference equations. It typically includes three components (DeJong and Dave (2007:11)):

1. the decision problem – a characterization of the environment in which decision makers reside;
2. a set of decision rules that dictate their behaviour (e.g., FOCs);
3. a characterization of the uncertainty they face in making decisions (e.g., stochastic shocks, risk aversion & information asymmetry).

Introduction

There generally exists no closed-form solution for such (non-linear) problems.

$\therefore$ we must result to numerical and/or approximation techniques $\rightarrow$ construct a linear approximation of the model.

There are varied approaches to approximate a nonlinear system linearly (see, e.g., Judd (1998).), such as

- local approximation (eg. Taylor series and Padé expansions);
- interpolation;
- regression.

In this session, we study the log-linearization method only.

- the standard method;
- a simpler method (Uhlig's method);

Introduction

We first:

(1) take natural logs of the system of non-linear difference equations

We then:

(2) linearize the logged difference equations about a particular point (usually a steady state). And simplify until we have a system of linear difference equations:

- variables are expressed as (percentage) deviations about their steady state values;
- directly amenable to empirical implementation

Introduction

Why this method?

- Linearization is nice because we know how to work with linear difference equations;
- Log-difference terms are nice because they provide natural interpretations of the units (i.e., everything is in percentage terms)

Since the principle of both methods that we focus on here is to use a Taylor-series expansion around the steady state to replace all equations by approximations, let's first review the theorem of Taylor series approx.

Taylor Series Approximation

Taylor's theorem for a univariate function $f(x)$:

$$
\begin{aligned}
f(x) \quad = \quad & f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 \\
+ \quad & \frac{f^{(3)}(x^*)}{3!}(x - x^*)^3 + \ldots + \frac{f^n(x^*)}{n!}(x - x^*)^n + R_{n+1}(x)
\end{aligned} \tag{1}
$$

where we refer to $R_{n+1}(x)$ as the residual term.

Taylor's theorem says that one can use derivative information at a single point to construct a polynomial approximation of a function at a point (x^*)—usually a steady state. It is the theoretical basis for all local approximations.

Taylor Series Approximation

We go one step further:

For a function that is sufficiently smooth, the higher order derivatives will be small, and the function can be well approximated (about x^*) linearly as:

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) \qquad (2)$$

The theorem also applies to multivariate functions:

For example, the first order approximation of $f(x, y)$ about the point (x^*, y^*) is:

$$f(x, y) \approx f(x^*, y^*) + f_x(x^*, y^*)(x - x^*) + f_y(x^*, y^*)(y - y^*) \qquad (3)$$

The Standard Method

The log-linearization method is based on the assumption that if the system is not too far from the steady state, the linear version result closely approximates the original system.

Consider a model that consists of a set of nonlinear equations of the following general form,

$$F(x_t) = \frac{G(x_t)}{H(x_t)} \tag{4}$$

Taking the logarithms,

$$ln(F(x_t)) = ln(G(x_t)) - ln(H(x_t)) \tag{5}$$

Again, by applying the first-order Taylor Series approximation, we can linearize (5) around its steady state x^{ss}.

The Standard Method

The log linearized (5),

$$
\begin{aligned}
ln(F(x^{ss})) + \frac{F'(x^{ss})}{F(x^{ss})}(x_t - x^{ss}) \quad &= \quad [ln(G(x^{ss}) + \frac{G'(x^{ss})}{G(x^{ss})}(x_t - x^{ss})] - [ln(H(x^{ss})) \\
&\quad + \frac{H'(x^{ss})}{H(x^{ss})}(x_t - x^{ss})]
\end{aligned} \tag{6}
$$

To simplify the model, we can eliminate the steady state of the model from (6),

$$
\frac{F'(x^{ss})}{F(x^{ss})}(x_t - x^{ss}) = \frac{G'(x^{ss})}{G(x^{ss})}(x_t - x^{ss}) - \frac{H'(x^{ss})}{H(x^{ss})}(x_t - x^{ss}) \tag{7}
$$

The model is now linear in x_t.

The Standard Method

The above procedure applies equally well in multivariate cases. To summarize, the 3-step procedure for log-linearizing is:

1. Take logs
2. Do a first order Taylor series expansion about a point (the steady state)
3. Simplify so that everything is expressed in $\%$ deviations from steady state

Uhlig's Method

Uhlig's method does not require explicit differentiation and the linearized model is expressed in terms of log differences of the variables.

Suppose X_t be the vector of variables, X^{ss} their steady state, and x_t the vector of log-deviations:

$$x_t = lnX_t - lnX^{ss} \tag{8}$$

That is, x_t denotes the percentage deviation from their steady state levels. Alternatively[1]

$$X_t = X^{ss} e^{x_t} \approx X^{ss}(1 + x_t) \tag{9}$$

1. Given (8)

Uhlig's Method

Four steps (Uhlig, 1995:5):

1. multiply out everything in each equation;
2. replace a variable X_t with $X_t = X^{ss}e^{x_t}$;
3. take logarithms, where both sides of an equation only involve products, or use the following three building blocks[2]:

$$e^{x_t + ay_t} \approx 1 + x_t + ay_t \tag{10}$$

$$x_t y_t \approx 0 \tag{11}$$

$$E_t[ae^{x_{t+1}}] \approx E_t[ax_{t+1}] \tag{12}$$

4. drop constants in each equation in the end.

2. E.g., $e^{x_t} \approx 1 + x_t$; $aX_t = aX^{ss}e^{x_t}$;
$(X_t + a)Y_t = X^{ss}Y^{ss}e^{x_t} + X^{ss}Y^{ss}e^{y_t} + aY^{ss}y_t$

Examples

Log-linearize the following using both the Taylor method and Uhlig's method, and compare your answers:

(1) The resource constraint

$$Y_t = C_t + I_t \ ,$$

indicating that output can be either consumed or invested.

(2) The Cobb-Douglas production function

$$Y_t = a_t K_t^{\alpha} N_t^{1-\alpha} \ ,$$

where output is produced by use of capital and labour and is subject to a productivity shock.

Hansen's real business cycle model (Uhlig, pp. 5-7)

With the tools you've learned from dynamic programming, you can now simulate a dynamic stochastic general equilibrium model.

We will work use Hansen's real business cycle model (Uhlig, pp. 5-7).

See the tutorial on recursive methods and linearization on SUNLearn.

For the final question you will be given step-by-step instructions to simulate the model and obtain impulse response functions (of the variables) in response to a 1 percentage point increase in total factor productivity (see figure in the next slide).

Both Dynare and Octave are free software. Octave is basically a free version of matlab.

Impulse response functions