# From points to complexes: a concept of unexpectedness for simplicial complexes

Thiago Holleben

October 4, 2025

# Interpolation problems and unexpectedness

Consider a set of points $X = \{X_1, \ldots, X_s\} \subset \mathbb{P}^n$ and integers $\alpha_1, \ldots, \alpha_s > 0$. Many classical problems in algebraic geometry ask about properties of the hypersurfaces that pass through $X_i$ with multiplicity $\alpha_i$ for all $i$.

# Interpolation problems and unexpectedness

Consider a set of points $X = \{X_1, \ldots, X_s\} \subset \mathbb{P}^n$ and integers $\alpha_1, \ldots, \alpha_s > 0$. Many classical problems in algebraic geometry ask about properties of the hypersurfaces that pass through $X_i$ with multiplicity $\alpha_i$ for all $i$.

In 2018, Cook, Harbourne, Migliore and Nagel consider the problem of computing the dimension of the space of forms of degree $d$ that vanish at a set of points $X$ and vanish at a general point $P$ with multiplicity $m$.

They argue that the expected dimension of this space should be $\max(0, \dim(I_X)_d - \binom{m+n-1}{n})$.

# Interpolation problems and unexpectedness

Consider a set of points $X = \{X_1, \ldots, X_s\} \subset \mathbb{P}^n$ and integers $\alpha_1, \ldots, \alpha_s > 0$. Many classical problems in algebraic geometry ask about properties of the hypersurfaces that pass through $X_i$ with multiplicity $\alpha_i$ for all $i$.

In 2018, Cook, Harbourne, Migliore and Nagel consider the problem of computing the dimension of the space of forms of degree $d$ that vanish at a set of points $X$ and vanish at a general point $P$ with multiplicity $m$.

They argue that the expected dimension of this space should be $\max(0, \dim(I_X)_d - \binom{m+n-1}{n})$. When the actual dimension $\dim(I_X \cap I_P^m)_d$ of this space is strictly larger, they say $X$ admits an unexpected hypersurface of degree $d$

# Unexpected hypersurfaces and Macaulay duality

Due to a result of Emsalem and Iarrobino we know there is a relationship between

unexpected hypersurfaces and multiplication maps in $R/(\ell_1^{a_1}, \ldots, \ell_s^{a_s})$

## WLP and SLP

An artinian algebra $A$ has the weak Lefschetz property (WLP) if the multiplication maps by a general linear form $\times \ell : A_i \to A_{i+1}$ have full rank for every $i$. If the maps $\times \ell^j : A_i \to A_{i+j}$ have full rank for every $i, j$ the algebra has the strong Lefschetz property (SLP)

## Theorem (Cook-Harbourne-Migliore-Nagel, 2018)

*Let $Z = \{X_1, \ldots, X_s\} \subset \mathbb{P}^2$ and $\{L_1, \ldots, L_s\}$ the dual lines. Then $Z$ has an unexpected curve of degree $j+1$ if and only if*

$$\times \ell^2 : (R/(L_1^{j+1}, \ldots, L_s^{j+1}))_{j-1} \to (R/(L_1^{j+1}, \ldots, L_s^{j+1}))_{j+1}$$

*does not have full rank for general $\ell$*

# What is an inverse system and where does it show up

Consider $R = \mathbb{K}[x_1, \ldots, x_n]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$. $S$ is an $R$-mod via contraction:

$$x_1^{a_1} \ldots x_n^{a_n} \circ y_1^{b_1} \ldots y_n^{b_n} = \begin{cases} y_1^{b_1 - a_1} \ldots y_n^{b_n - a_n} & \text{if} \quad b_i \geq a_i \quad \text{for all} \quad i \\ 0 & \text{otherwise} \end{cases}$$

# What is an inverse system and where does it show up

Consider $R = \mathbb{K}[x_1, \ldots, x_n]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$. $S$ is an $R$-mod via contraction:

$$x_1^{a_1} \ldots x_n^{a_n} \circ y_1^{b_1} \ldots y_n^{b_n} = \begin{cases} y_1^{b_1 - a_1} \ldots y_n^{b_n - a_n} & \text{if} \quad b_i \geq a_i \quad \text{for all} \quad i \\ 0 & \text{otherwise} \end{cases}$$

The inverse system of an ideal $I \subset R$ is

$$I^{-1} = \{F \in S \colon g \circ F = 0 \quad \text{for all} \quad g \in I\} \subset S$$

# What is an inverse system and where does it show up

Consider $R = \mathbb{K}[x_1, \dots, x_n]$ and $S = \mathbb{K}[y_1, \dots, y_n]$. $S$ is an $R$-mod via contraction:

$$x_1^{a_1} \dots x_n^{a_n} \circ y_1^{b_1} \dots y_n^{b_n} = \begin{cases} y_1^{b_1 - a_1} \dots y_n^{b_n - a_n} & \text{if} \quad b_i \geq a_i \quad \text{for all} \quad i \\ 0 & \text{otherwise} \end{cases}$$

The inverse system of an ideal $I \subset R$ is

$$I^{-1} = \{F \in S \colon g \circ F = 0 \quad \text{for all} \quad g \in I\} \subset S$$

- If $I^{-1}$ is cyclic, $R/I$ is artinian and Gorenstein. Moreover, it is a bijection
- $\dim I^{-1}_{-j} = \dim(R/I)_j$

When $I^{-1} = (F)$ we call $F$ the Macaulay dual generator of $I$

# Where does it show up

1. The inverse systems of artinian reductions of (CM) squarefree monomial ideals can be interpreted as "stresses" that model continuous motions of a graph that perserve edge lengths

# Where does it show up

1. The inverse systems of artinian reductions of (CM) squarefree monomial ideals can be interpreted as "stresses" that model continuous motions of a graph that perserve edge lengths
2. Emsalem and Iarrobino's result!

# Where does it show up

1. The inverse systems of artinian reductions of (CM) squarefree monomial ideals can be interpreted as "stresses" that model continuous motions of a graph that perserve edge lengths
2. Emsalem and Iarrobino's result!

**A way to think of this framework**

1: Find an algebra of a specific form failing WLP/SLP

2: Use inverse systems to translate the failure to an interesting geometric property

# When life gives you an algebra failing the WLP...

Consider a squarefree monomial ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$ and $J = I + (x_1^a, \dots, x_n^a)$. It is known that WLP/SLP for monomial ideals can be checked by taking $\ell = x_1 + \cdots + x_n$.

Our framework is:

1. Find $A = R/J$ failing the WLP for special $J$ (and $I$)
2. Apply inverse systems (somehow!) to translate failure to an interesting combinatorial property

## … make a regular sequence!

Let $I$ be a Gorenstein squarefree monomial ideal and $J = I + (x_1^a, \ldots, x_n^a)$. Assume $A = R/J$ fails the WLP due to surjectivity in degree $d - 1 \to d$

### The idea

If $\times\ell$ is not surjective, $\times\ell^T$ is not injective. A polynomial $F \in \ker \times\ell^T$ should be the Macaulay dual generator of $I + \theta$, where $\theta$ is a system of parameters of $I$

# … make a regular sequence!

Let $I$ be a Gorenstein squarefree monomial ideal and $J = I + (x_1^a, \dots, x_n^a)$. Assume $A = R/J$ fails the WLP due to surjectivity in degree $d - 1 \to d$

**The idea**

If $\times\ell$ is not surjective, $\times\ell^T$ is not injective. A polynomial $F \in \ker \times\ell^T$ should be the Macaulay dual generator of $I + \theta$, where $\theta$ is a system of parameters of $I$

We call a system of parameters arising this way an unexpected system of parameters of $I$

# Sometimes the unexpected should be expected

## Theorem (H, 2025+)

*Let $I$ be a Gorenstein* squarefree monomial ideal of codimension $\leq n-2$. Then the elementary symmetric polynomials form an unexpected system of parameters of $I$. In particular,*

$$\frac{R}{I+(x_1^{d+2},\ldots,x_n^{d+2})}$$

*fails the WLP, where $d+1 = \dim R/I$*

# Sometimes the unexpected should be expected

## Theorem (H, 2025+)

*Let $I$ be a Gorenstein* squarefree monomial ideal of codimension $\leq n-2$. Then the elementary symmetric polynomials form an unexpected system of parameters of $I$. In particular,*

$$\frac{R}{I + (x_1^{d+2}, \dots, x_n^{d+2})}$$

*fails the WLP, where $d+1 = \dim R/I$*

## Corollary (Miró-Roig, Migliore, Nagel - 2010)

*For every $n$, the algebra*

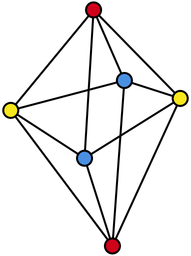
$$\frac{R}{(x_1 \dots x_n, x_1^n, \dots, x_n^n)}$$

*fails the WLP.* *(Take $I = (x_1 \dots x_n)$)*

# A combinatorial meaning to linear unexpectedness

A famous class of simplicial complexes in combinatorics is the class of balanced complexes: these are complexes that can be colored in such a way that every maximal face has exactly one vertex of each color.

Algebraically, we can view the coloring as a finer grading of the Stanley-Reisner ideal: variables with the same color have the same degree.



# A combinatorial meaning to linear unexpectedness

A famous class of simplicial complexes in combinatorics is the class of balanced complexes: these are complexes that can be colored in such a way that every maximal face has exactly one vertex of each color.

Algebraically, we can view the coloring as a finer grading of the Stanley-Reisner ideal: variables with the same color have the same degree.

**Theorem (H, 2025+)**

*If a simplicial complex is homeomorphic to a sphere (stronger than Gorenstein*!), it admits an unexpected lsop if and only if it is balanced*

Proof: uses results from next talk!

# What is the intuition for unexpected sops?

One way to think of a sop $\theta$ of $I \subset R$ being unexpected, is if the following two conditions are satisfied:

1. The socle degree of $R/(I, \theta)$ is $d$
2. $x_i^a \in I + \theta$ for some $a \ll d$ and every $i$

# What is the intuition for unexpected sops?

One way to think of a sop $\theta$ of $I \subset R$ being unexpected, is if the following two conditions are satisfied:

1. The socle degree of $R/(I, \theta)$ is $d$
2. $x_i^a \in I + \theta$ for some $a \ll d$ and every $i$

From this perspective, we get to the following question that is interesting on its own

## Question

Given a squarefree monomial ideal $I \subset R$ and a pair of integers $(d, a)$, when is there a sop $\theta$ of $I$ such that

1. socle degree of $R/(I, \theta)$ is $d$
2. $x_i^a \in I + \theta$ for every $i$

# What is the intuition for unexpected sops?

One way to think of a sop $\theta$ of $I \subset R$ being unexpected, is if the following two conditions are satisfied:

1. The socle degree of $R/(I, \theta)$ is $d$
2. $x_i^a \in I + \theta$ for some $a \ll d$ and every $i$

From this perspective, we get to the following question that is interesting on its own

**Question**

Given a squarefree monomial ideal $I \subset R$ and a pair of integers $(d, a)$, when is there a sop $\theta$ of $I$ such that

1. socle degree of $R/(I, \theta)$ is $d$
2. $x_i^a \in I + \theta$ for every $i$
3. Bonus: $x_1 + \cdots + x_n \in I + \theta$

General sops are really bad to guarantee condition 2 for pairs $a \ll d$

The WLP comes into play when we add condition 3

# Maybe it really should be unexpected

**Theorem (H, 2025+)**

*Let $\Delta$ be a collapsible simplicial complex of dimension $d$ and Stanley-Reisner ideal $I_\Delta$. Then there does not exist an unexpected sop $\theta$ of $I_\Delta$ where $x_i^a \in I_\Delta + \theta$ and the socle degree of $R/(I_\Delta, \theta)$ is $t > d(a-1)$*

This holds because the collapsible property allows us to show a multiplication map has full rank.

# Maybe it really should be unexpected

**Theorem (H, 2025+)**

*Let $\Delta$ be a collapsible simplicial complex of dimension $d$ and Stanley-Reisner ideal $I_\Delta$. Then there does not exist an unexpected sop $\theta$ of $I_\Delta$ where $x_i^a \in I_\Delta + \theta$ and the socle degree of $R/(I_\Delta, \theta)$ is $t > d(a-1)$*

This holds because the collapsible property allows us to show a multiplication map has full rank.

Moreover, in this special case, the rank of the matrix turns out to be the analytic spread of a special squarefree monomial ideal.

# Two seemingly distinct questions

Let $F$ be a polynomial of degree $d$ in $n$ variables and $\nabla \cdot F = F_{x_1} + \cdots + F_{x_n}$

**Question 1**

For which pairs $d, n$ is there a solution $F$ of $\nabla \cdot F = 0$ such that no monomial in $F$ is divisible by $x_i^a$?

**Question 2**

Let $A = R/I$ be an artinian Gorenstein algebra such that $x_i^a \in I$ for every $i$ and $x_1 + \cdots + x_n \in I$. What is the highest possible socle degree of $A$?

What if we add "simplicial restrictions" to the questions above?