

Graph Convolutional Neural Networks

沈华伟

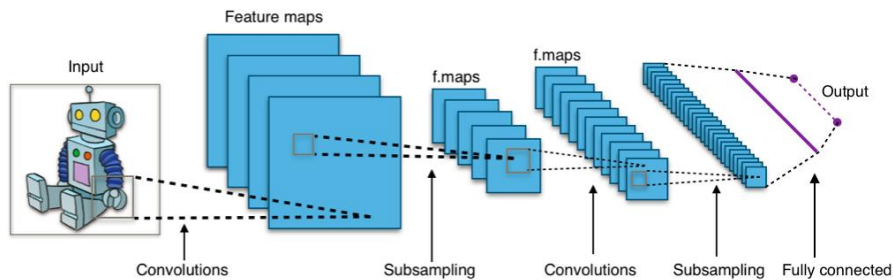
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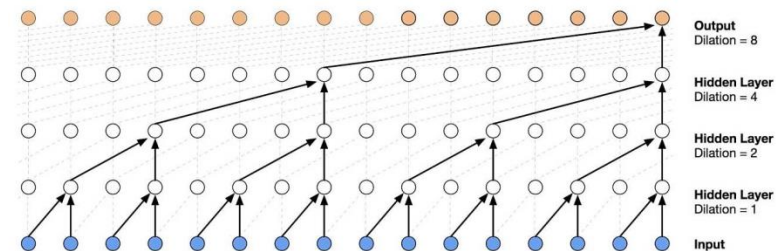
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» Convolutional Neural Network

- Convolutional neural network (CNN) gains great success on Euclidean data, e.g., image, text, audio, and video
 - Image classification, object detection, machine translation



Convolutional neural networks on image

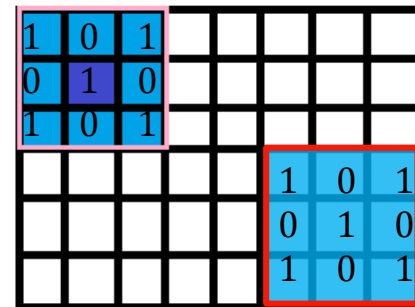
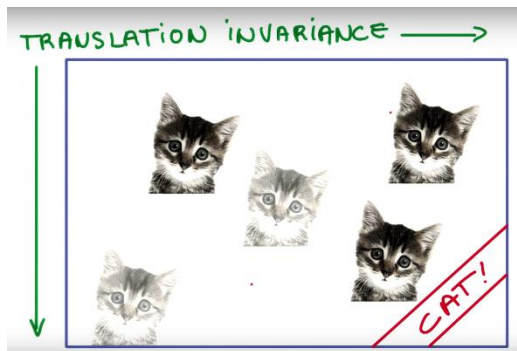


Temporal convolutional network

- The power of CNN lies in
 - its ability to learn **local stationary structures**, via **localized convolution filter**, and compose them to form **multi-scale hierarchical patterns**

» Convolutional Neural Network

- Localized convolutional filters are **translation- or shift-invariant**
 - Which are able to recognize identical features independently of their spatial locations



X-Shape

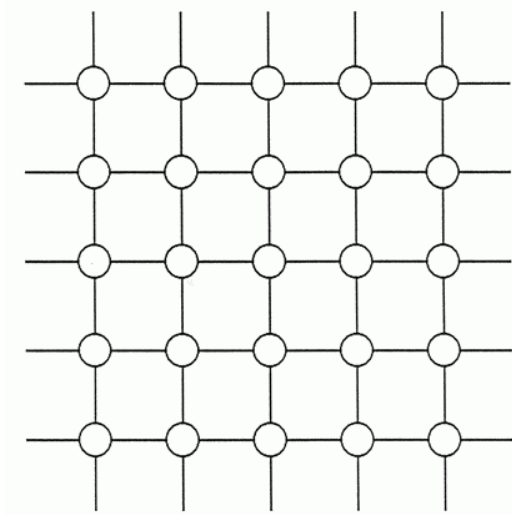
Template Matching

1	0	1
0	1	0
1	0	1

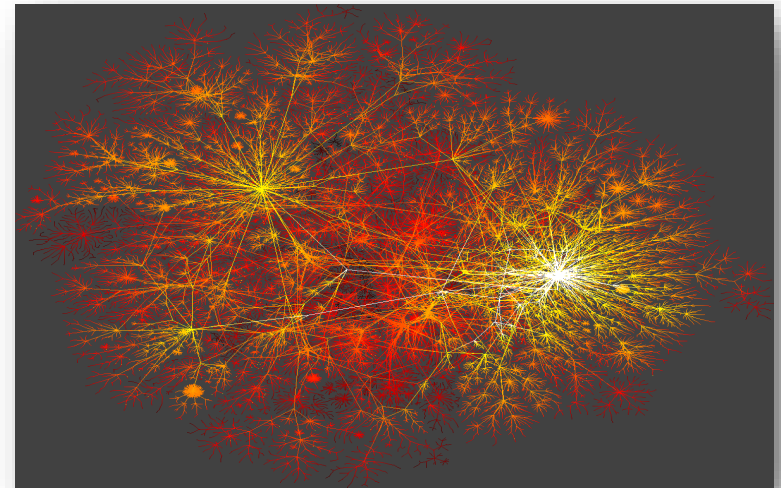
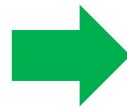
- One interesting problem is **how to generalize convolution to non-Euclidean domain**, e.g., graph?
 - Irregular structure of graph poses challenges for defining convolution for graph data

➤ From CNN to graph CNN

- Convolution is well defined in Euclidean data, grid-like network
- Not straightforward to define convolution on irregular network, widely observed in real world



Grid-like network



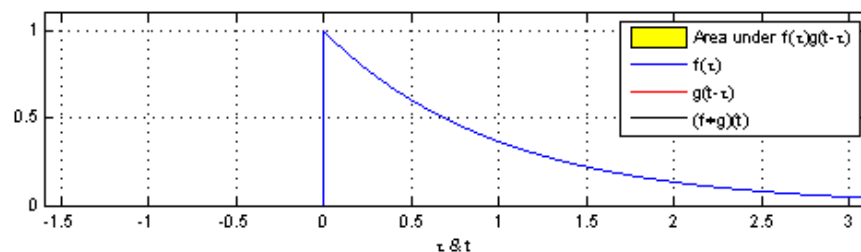
Irregular networks

» Convolution

- Convolution is a mathematical operation on two functions, f and g , to produce a third function h .
 - Defined as the **integral**, in continuous case, or **sum**, in discrete case, of the **product** of the two functions after one is reversed and shifted.

Continuous case

$$h(t) = (f * g)(t) \stackrel{\text{def}}{=} \int f(t - \tau)g(\tau) d\tau$$

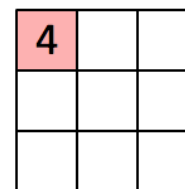
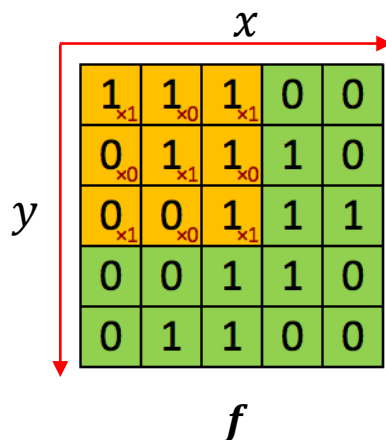


Discrete case

$$h(x, y)$$

$$= (f * g)(x, y)$$

$$\stackrel{\text{def}}{=} \sum_{m,n} f(x - m, y - n)g(m, n)$$



$g =$

$g(1,1)$ 1	$g(0,1)$ 0	$g(-1,1)$ 1
$g(1,0)$ 0	$g(0,0)$ 1	$g(-1,0)$ 0
$g(1,-1)$ 1	$g(0,-1)$ 0	$g(-1,-1)$ 1

➤ Existing methods to define convolution

- **Spectral methods: define convolution in spectral domain**
 - Convolution is defined via graph Fourier transform and convolution theorem.
 - The main challenge is that **convolution filter** defined in spectral domain **is not localized in vertex domain**.
- **Spatial methods: define convolution in the vertex domain**
 - Convolution is defined as a weighted average function over all vertices located in the neighborhood of target vertex.
 - The main challenge is that **the size of neighborhood varies remarkably across nodes**, e.g., power-law degree distribution.

Spectral methods for graph convolutional neural networks

» Spectral methods

■ Given a graph $G = (V, E, W)$

- V is node set with $n = |V|$, E is edge set, and $W \in R^{n \times n}$ is the weighted adjacency matrix
- Each node is associated with d features, and $X \in R^{n \times d}$ is the feature matrix of nodes, each column of X is a signal defined over nodes

■ Graph Laplacian

- $L = D - W$, where D is a diagonal matrix with $D_{ii} = \sum_j W_{ij}$
- Normalized graph Laplacian

$$L = I - D^{-\frac{1}{2}} W D^{-\frac{1}{2}}$$

where I is the identity matrix.

» Graph Fourier Transform

■ Fourier basis of graph G

- The complete set of orthonormal eigenvectors $\{u_l\}_{l=1}^n$ of L , ordered by its non-negative eigenvalues $\{\lambda_l\}_{l=1}^n$
- Graph Laplacian could be diagonalized as

$$L = U\Lambda U^T$$

where $U = [u_1, \dots, u_n]$, and $\Lambda = \text{diag}([\lambda_1, \dots, \lambda_n])$

■ Graph Fourier transform

- Graph Fourier transform of a signal $x \in R^n$ is defined as

$$\hat{x} = U^T x$$

- Graph Fourier inverse transform is

$$x = U\hat{x}$$

➤ Define convolution in spectral domain

- Convolution theorem

- The Fourier transform of a convolution of two signals is the **point-wise product** of their Fourier transforms

- According to convolution theorem, given a signal x as input and the other signal y as filter, graph convolution $*_G$ could be written as

$$x *_G y = U \left((U^T x) \odot (U^T y) \right)$$

Here, the convolution filter in spectral domain is $U^T y$.

➤ Define convolution in spectral domain

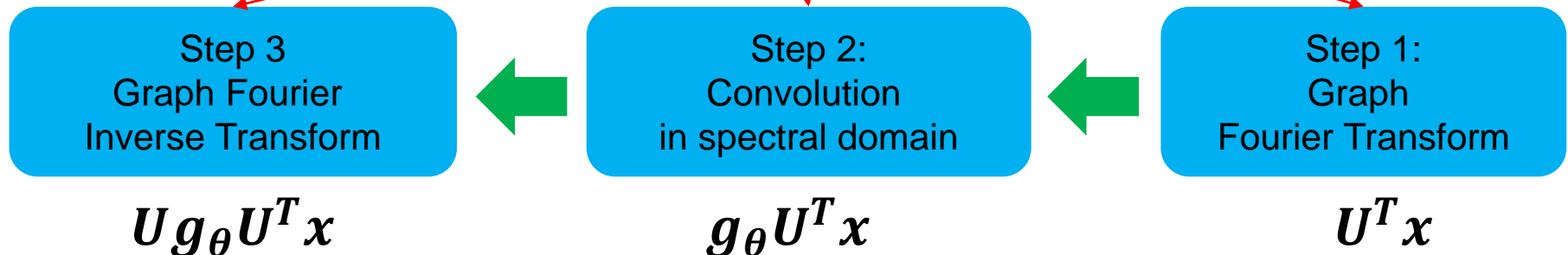
■ Graph convolution in spectral domain

- Let $U^T y = [\theta_0, \dots, \theta_{n-1}]^T$ and $g_\theta = \text{diag}([\theta_0, \dots, \theta_{n-1}])$, we have

$$x *_G y = U \left((U^T x) \odot (U^T y) \right)$$



$$x *_G y = U g_\theta U^T x$$



» Spectral Graph CNN

■ Spectral Graph CNN

$$x_{k+1,j} = h \left(\sum_{i=1}^{f_k} U F_{k,i,j} U^T x_{k,i} \right)$$

$$j = 1, \dots, f_{k+1}$$

Signals in the k -th layer

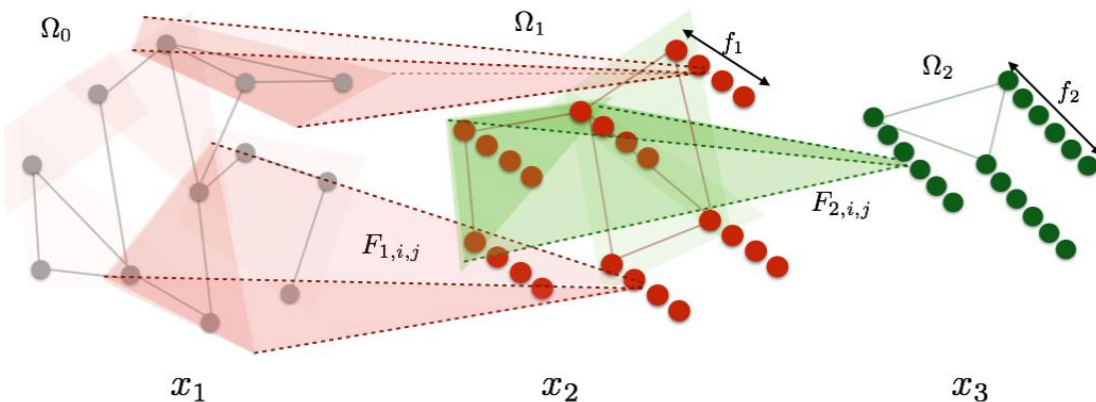
Filter in the k -th layer

Graph Fourier Transform

$$\hat{x} = U^T x$$

Graph Fourier Inverse Transform

$$x = U \hat{x}$$



➤ Shortcomings of Spectral graph CNN

- **Requiring eigen-decomposition of Laplacian matrix**
 - Eigenvectors are explicitly used in convolution
- **High Computational cost**
 - Multiplication with graph Fourier basis U is $O(n^2)$
- **Not localized in vertex domain**

➤ ChebyNet: parameterizing filter

- Parameterizing convolution filter via polynomial approximation

$$g_{\theta} = \text{diag}([\theta_0, \dots, \theta_{n-1}])$$



$$g_{\beta}(\Lambda) = \sum_{k=0}^{K-1} \beta_k \Lambda^k$$

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

- ChebyNet

$$\mathbf{x} *_G \mathbf{y} = \mathbf{U} g_{\beta}(\Lambda) \mathbf{U}^T \mathbf{x} = \sum_{k=0}^{K-1} \beta_k L^k \mathbf{x}$$

The number of free parameters reduces from n to K

» ChebyNet vs. Spectral Graph CNN

- Eigen-decomposition is not required
- Computational cost is reduced from $O(n^2)$ to $O(|E|)$

$$\mathbf{x} *_G \mathbf{y} = \mathbf{U} g_{\beta}(\Lambda) \mathbf{U}^T \mathbf{x} = \sum_{k=0}^{K-1} \beta_k L^k \mathbf{x}$$

- Convolution is localized in vertex domain
 - Convolution is strictly localized in a ball of radius K , i.e., K hops from the central vertex

Is this method good enough? What could we do more?

Our method:
Graph Wavelet Neural Network
(ICLR 2019)

➤ Graph wavelet neural network

- ChebyNet achieves localized convolutional via **restricting the space of graph filters** as a polynomial function of eigenvalue matrix Λ

$$g_{\theta}(\Lambda) = \sum_{k=0}^{K-1} \theta_k \Lambda^k$$

- We focus on the Fourier basis to achieve localized graph convolution

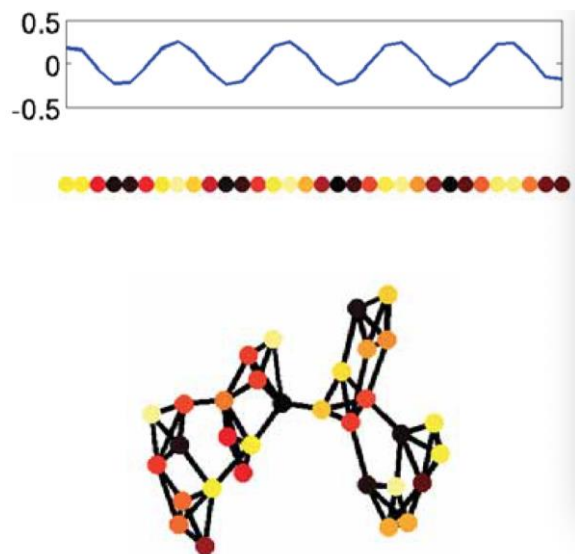
$$x *_G y = U g_{\theta} U^T x$$

- We propose to replace Fourier basis with **wavelet basis**.

Fourier vs. Wavelet

Fourier Basis

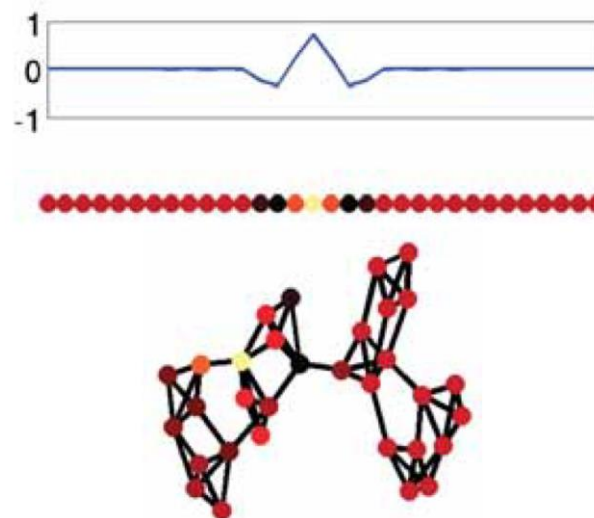
- Dense
- Not localized
- High Computational cost



Fourier basis : U

Wavelet Basis

- Sparse
- Localized
- Low Computational cost



Wavelet basis : $\psi_s = Ue^{\lambda s}U^T$

➤ Graph wavelet neural network

■ Graph Wavelet Neural Network

- Replace graph Fourier transform with graph wavelet transform

Graph Fourier transform

$$\hat{x} = U^T x$$

Inverse Fourier transform

$$x = U \hat{x}$$

Graph Wavelet transform

$$x^* = \psi_S^{-1} x$$

Inverse Wavelet transform

$$x = \psi_S x^*$$

➤ Graph wavelet neural network (GWNN)

■ Graph convolution via wavelet transform

$$x *_{\mathcal{G}} y = U((U^{\top} y) \odot (U^{\top} x)),$$

$$x *_{\mathcal{G}} y = \psi_s((\psi_s^{-1} y) \odot (\psi_s^{-1} x))$$

Replacing basis

■ Graph wavelet neural network

$$x_{k+1,j} = h\left(\sum_{i=1}^p U F_{k,i,j} U^{\top} x_{k,i}\right) \quad \rightarrow \quad x_{k+1,j} = h\left(\sum_{i=1}^p \psi_s F_{k,i,j} \psi_s^{-1} x_{k,i}\right)$$

$$j = 1, \dots, q$$

Parameter complexity : $O(n * p * q)$

➤ Reducing parameter complexity

■ Key idea:

- Detaching graph convolution from feature transformation

$$x_{k+1,j} = h \left(\sum_{i=1}^p \psi_S F_{k,i,j} \psi_S^{-1} x_{k,i} \right) \quad j = 1, \dots, q$$



$T \in R^{q \times p}$
with $p * q$ parameters

$$y_{k,j} = \sum_{i=1}^p T_{ji} x_{k,i}$$

Feature transformation



$$x_{k+1,j} = h(\psi_S F_k \psi_S^{-1} y_{k,j})$$

Graph convolution

F_k is a diagonal matrix
with n parameters

The number of parameters reduces from $O(n * p * q)$ to $O(n + p * q)$

GWNN vs. ChebyNet

■ Benchmark datasets

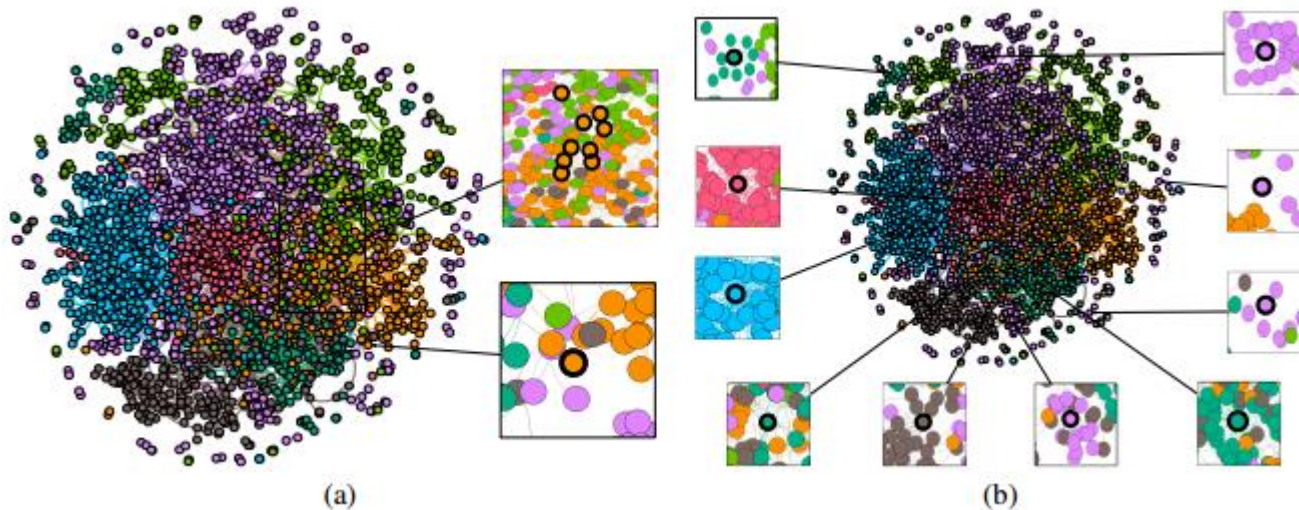
Dataset	Nodes	Edges	Classes	Features	Label Rate
Citeseer	3,327	4,732	6	3,703	0.036
Cora	2,708	5,429	7	1,433	0.052
Pubmed	19,717	44,338	3	500	0.003

■ Results at the task of node classification

Method	Cora	Citeseer	Pubmed
MLP	55.1%	46.5%	71.4%
ManiReg	59.5%	60.1%	70.7%
SemiEmb	59.0%	59.6%	71.7%
LP	68.0%	45.3%	63.0%
DeepWalk	67.2%	43.2%	65.3%
ICA	75.1%	69.1%	73.9%
Planetoid	75.7%	64.7%	77.2%
Spectral CNN	73.3%	58.9%	73.9%
ChebyNet	81.2%	69.8%	74.4%
GWNN	82.8%	71.7%	79.1%

➤ Graph wavelet neural network

- Each Graph wavelet offers us a local view, i.e., from a center node, about the proximity for each pair of nodes



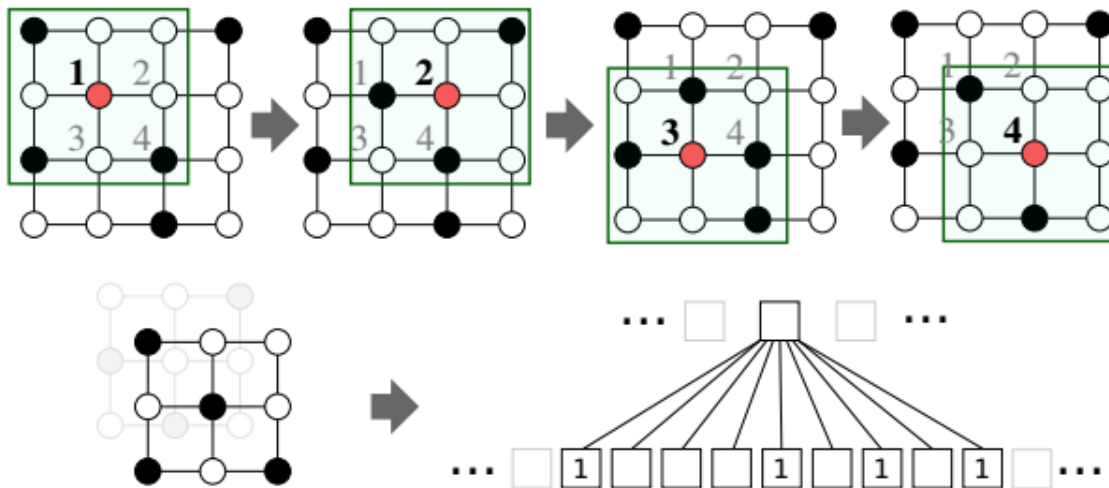
Wavelet offers us a better basis for defining graph convolutional networks in spectral domain

Spatial methods for graph convolutional neural networks

➤ Spatial Methods for Graph CNN

■ By analogy

- ❑ What can we learn from the architecture of standard convolutional neural network?



1. Determine Neighborhood



2. Impose an order in neighborhood

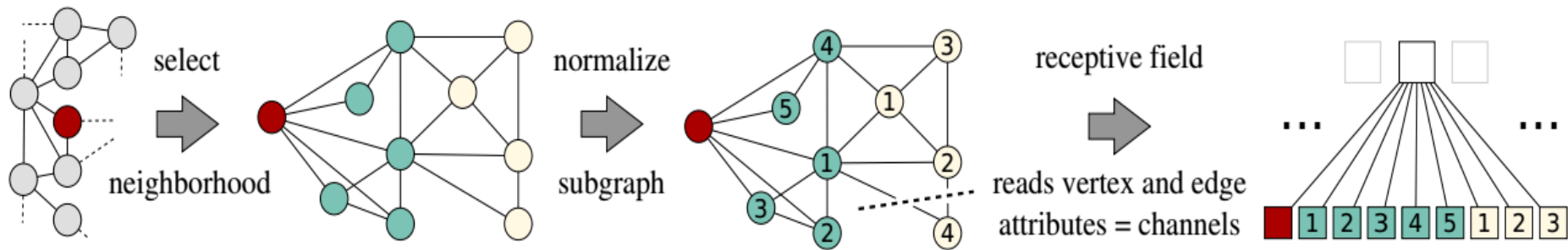


3. Parameter sharing

➤ Spatial Methods for Graph CNN

■ By analogy

- ❑ For each node, select the fixed number of nodes as its neighboring nodes, according to certain proximity metric
- ❑ Impose an order according to the proximity metric
- ❑ Parameter sharing



1. Determine
Neighborhood

2. Impose an order in
neighborhood

3. Parameter sharing

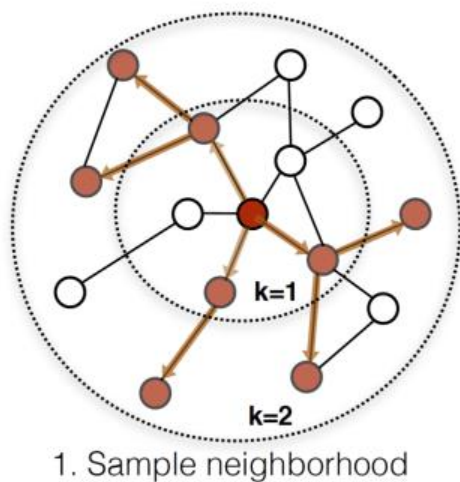
➤ Spatial Methods for Graph CNN

■ GraphSAGE

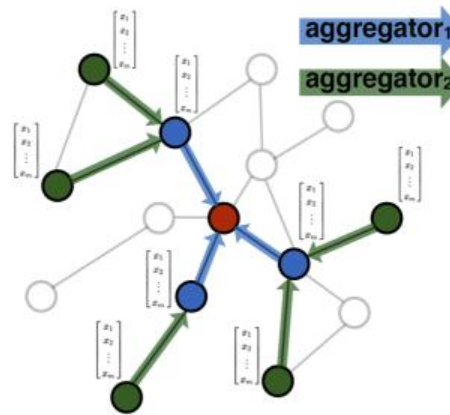
- ❑ Sampling neighbors
- ❑ Aggregating neighbors

$$a_v^{(k)} = \text{AGGREGATE}^{(k)} \left(\left\{ h_u^{(k-1)} : u \in \mathcal{N}(v) \right\} \right)$$

$$h_v^{(k)} = \text{COMBINE}^{(k)} \left(h_v^{(k-1)}, a_v^{(k)} \right)$$



1. Sample neighborhood



2. Aggregate feature information from neighbors

GraphSAGE: Inductive Learning

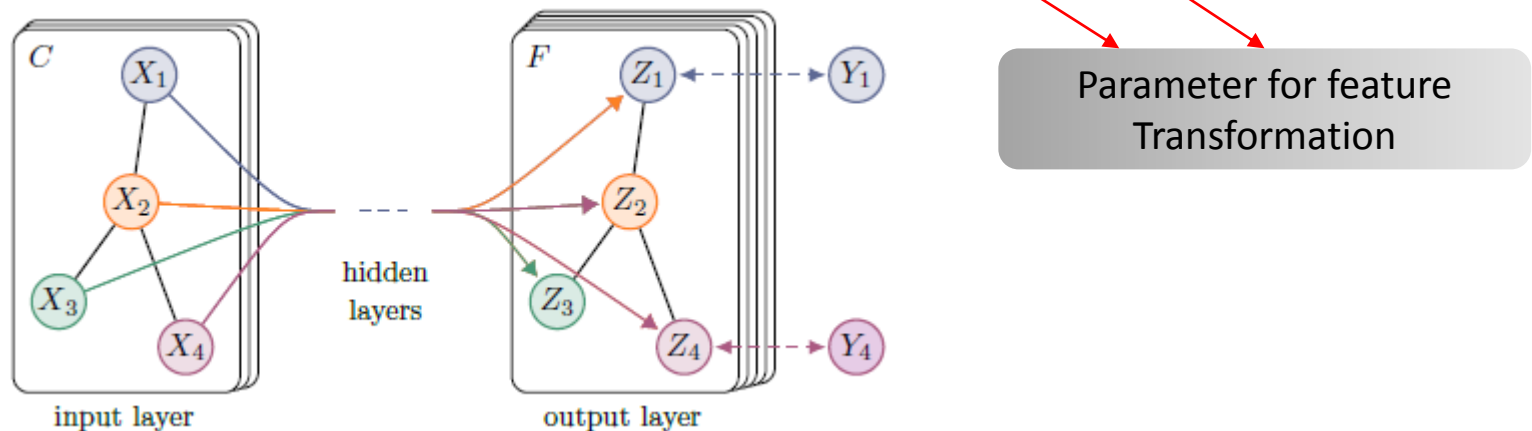
General framework of graph neural networks:
Aggregate the information of neighboring nodes to update the representation of center node

» Spatial Methods for Graph CNN

■ GCN: Graph Convolution Network

- ❑ **Aggregating information from neighborhood via a normalized Laplacian matrix**
- ❑ **Shared parameters are from feature transformation**
- ❑ **A reduced version of ChebyNet**

$$Z = f(X, A) = \text{softmax}\left(\hat{A} \text{ReLU}\left(\hat{A}X \underbrace{W^{(0)}}_{\text{input}}\right) \underbrace{W^{(1)}}_{\text{output}}\right)$$



➤ Spatial Methods for Graph CNN

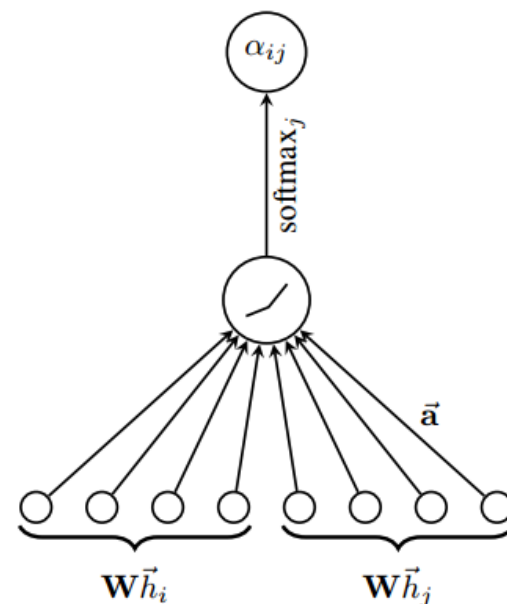
■ GAT: Graph Attention Network

- ❑ Learning the aggregation matrix, i.e., Laplacian matrix in GCN, via attention mechanism
- ❑ Shared parameters contain two parts
 - Parameters for feature transformation
 - Parameters for attention

Parameter for feature Transformation

$$\alpha_{ij} = \frac{\exp \left(\text{LeakyReLU} \left(\vec{a}^T [\mathbf{W} \vec{h}_i \| \mathbf{W} \vec{h}_j] \right) \right)}{\sum_{k \in \mathcal{N}_i} \exp \left(\text{LeakyReLU} \left(\vec{a}^T [\mathbf{W} \vec{h}_i \| \mathbf{W} \vec{h}_k] \right) \right)}$$

Parameter of Attention mechanism



Attention Mechanism in GAT

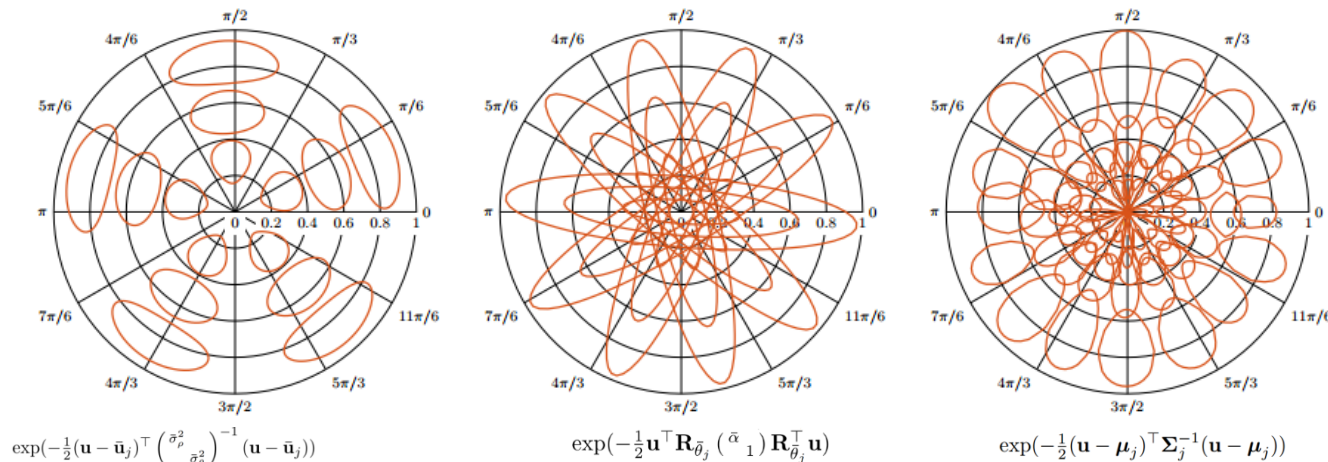
➤ Spatial Methods for Graph CNN

■ MoNet: A general framework for spatial methods

- ❑ Define **multiple kernel functions**, parameterized or not, to measure the similarity between target node and other nodes
- ❑ Convolution kernels are the **weights** of these kernel functions

$$(f \star g)(x) = \sum_{j=1}^J g_j D_j(x) f$$

Convolution kernel



Our method:

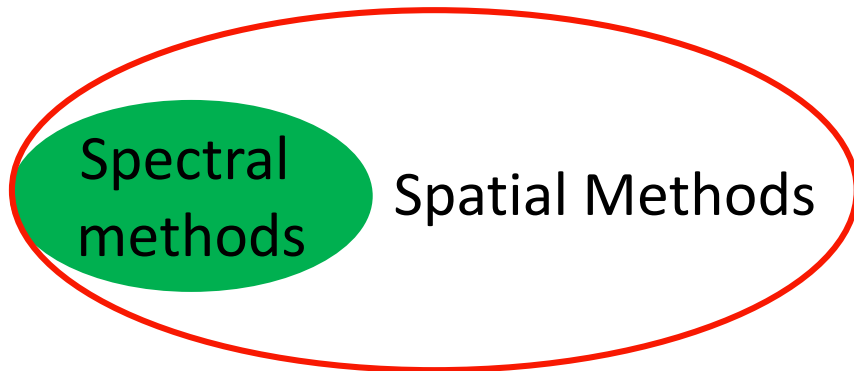
**Graph Convolutional Networks using Heat Kernel
for Semi-supervised Learning**

(IJCAI 2019)

➤ Spectral methods vs. Spatial methods

■ Connections

- ❑ Spectral methods are special cases of spatial methods



$$(f \star g)(x) = \sum_{j=1}^J g_j D_j(x) f$$

Kernel function :
Characterizing the
similarity or distance
among nodes

■ Difference

- ❑ Spectral methods define kernel functions via an explicit space transformation, i.e., projecting into spectral space
- ❑ Spatial methods directly define kernel functions

➤ Spectral methods: Recap

■ Spectral CNN

$$y = U g_{\theta} U^T x = (\theta_1 \boxed{u_1 u_1^T} + \theta_2 \boxed{u_2 u_2^T} + \cdots + \theta_n \boxed{u_n u_n^T}) x$$

■ ChebyNet

$$y = (\theta_0 I + \theta_1 L + \theta_2 L^2 + \cdots + \theta_{K-1} L^{K-1}) x$$

■ GCN

$$y = \theta(I - L)x$$

Question:

Why GCN with less parameters performs better than ChebyNet?

➤ Graph Signal Processing: filter

- Smoothness of a signal x over graph is measured by

$$x^T L x = \sum_{(u,v) \in E} A_{uv} \left(\frac{x_u}{\sqrt{d_u}} - \frac{x_v}{\sqrt{d_v}} \right)^2$$

$\lambda_i = u_i^T L u_i$ can be viewed as the frequency of u_i

- Basic filters

- $u_i u_i^T$ ($1 \leq i \leq n$) are a set of basic filters
- For a graph signal x , the basic filter $u_i u_i^T$ only allows the component with frequency λ_i passes

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n,$$

$$u_i u_i^T x = \alpha_i u_i$$

➤ Combined filters: High-pass vs. Low-pass

■ Combined filters

- A linear combination of basic filters

$$\theta_1 u_1 u_1^T + \theta_2 u_2 u_2^T + \cdots + \theta_n u_n u_n^T$$

- L^k is a combined filter with the coefficients $\{\lambda_i^k\}_{i=1}^n$
- L^k assign high coefficients to basic filters with high-frequency, i.e., L^k is a high-pass filter

■ GCN only consider $k = 0$ and $k = 1$, avoiding the boosting effect to basic filters with high-frequency

- Behaving as a low-pass combined filter
- Explaining why GCN performs better than ChebyNet

» Our method: GraphHeat

■ Low-pass combined filters

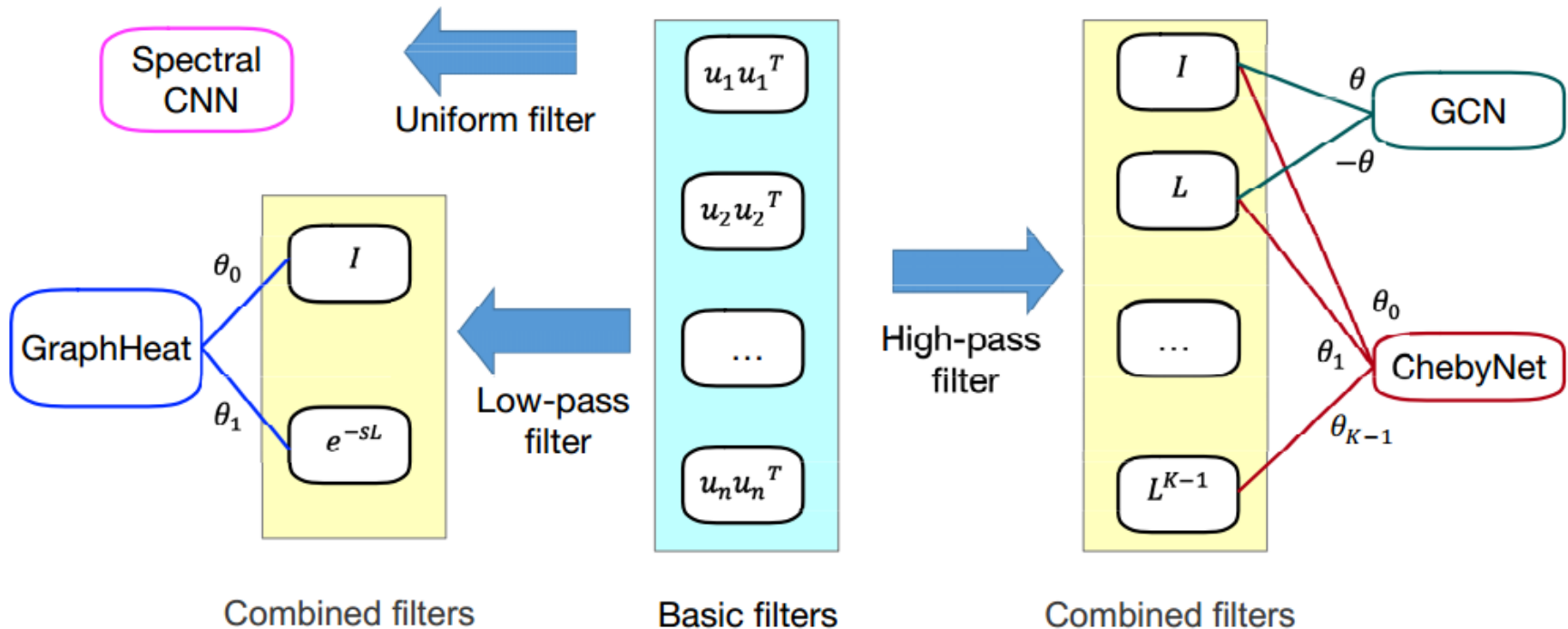
- $\{e^{-skL}\}$, where s is scaling parameter, and k is order
- e^{-sL} is heat kernel over graph, which defines the similarity among nodes via heat diffusion over graph

$$e^{-sL} = Ue^{-s\Lambda}U^T, \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

- The basic filter $u_i u_i^T$ ($1 \leq i \leq n$) has the coefficient $e^{-s\lambda_i}$, suppressing signals with high-frequency

GraphHeat vs. baseline methods

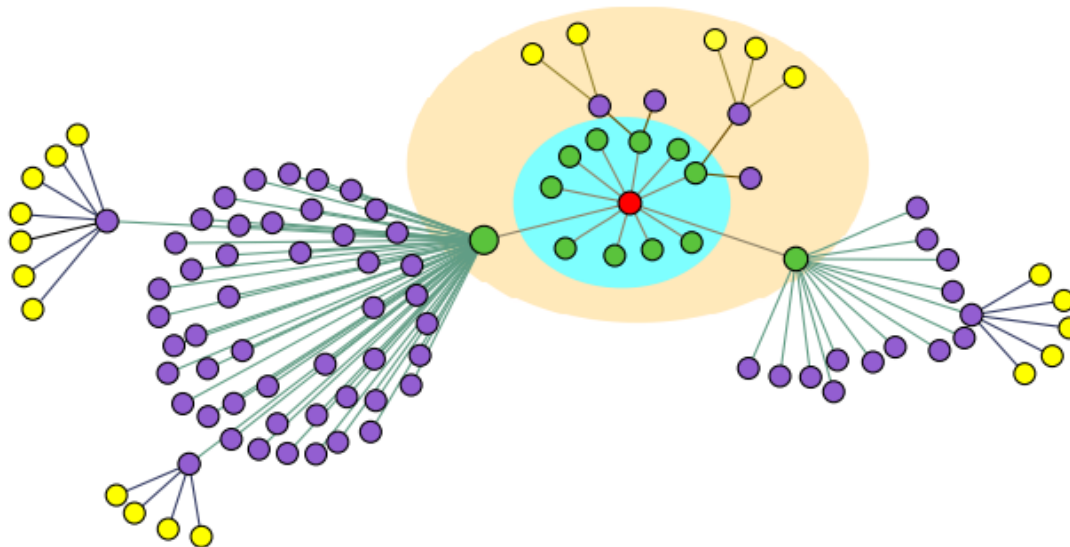
■ Compared with baseline methods



GraphHeat vs. baseline methods

■ Neighborhood

- GCN and ChebyNet determine neighborhood according to the hops away from center node, i.e., in an order-style
 - Nodes in different colors
- GraphHeat determines neighborhood according to the similarity function by heat diffusion over graph
 - Nodes in different circles



Experimental results

Results at the task of node classification

Method	Cora	Citeseer	Pubmed
MLP	55.1%	46.5%	71.4%
ManiReg	59.5%	60.1%	70.7%
SemiEmb	59.0%	59.6%	71.7%
LP	68.0%	45.3%	63.0%
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ICA	75.1%	69.1%	73.9%
Planetoid	75.7%	64.7%	77.2%
ChebyNet	81.2%	69.8%	74.4%
GCN	81.5%	70.3%	79.0%
MoNet	81.7 \pm 0.5%	—	78.8 \pm 0.3%
GAT	83.0 \pm 0.7%	72.5 \pm 0.7%	79.0 \pm 0.3%
GraphHeat	83.7%	72.5%	80.5%

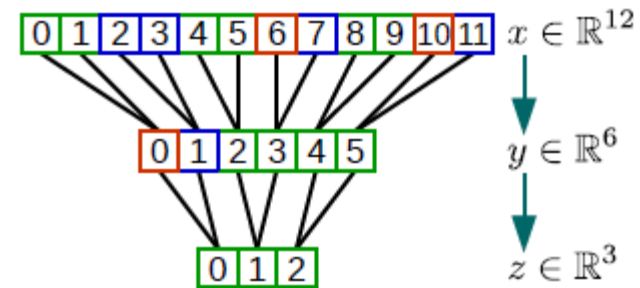
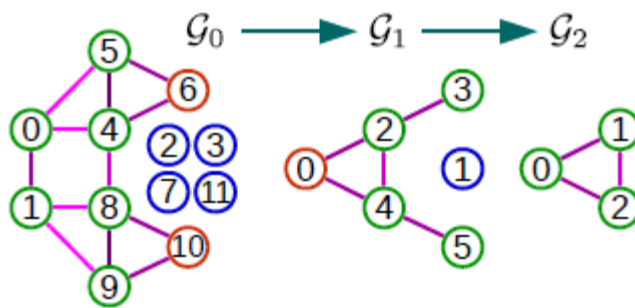
GraphHeat achieves state-of-the-art performance on the task of node classification on the three benchmark datasets

Graph Pooling

Graph Pooling via graph coarsening

■ Graph coarsening

- Merging nodes into clusters and take each cluster as a super node

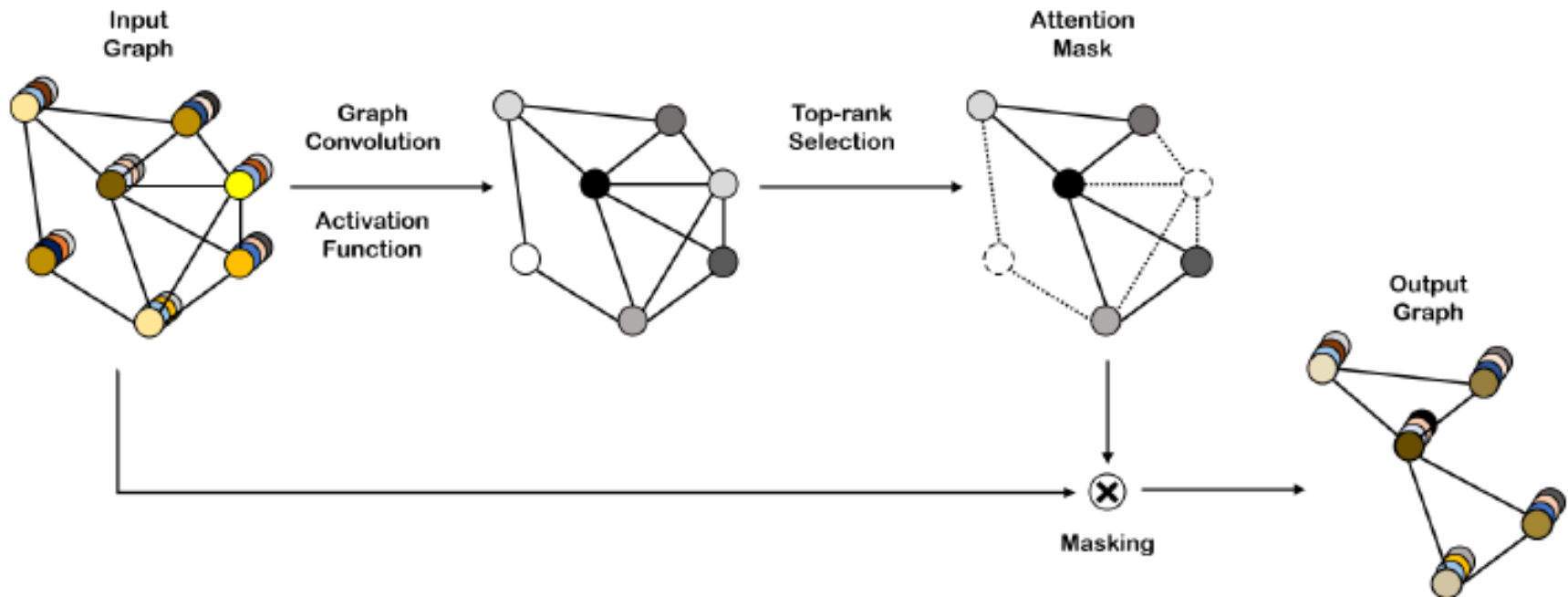


- Node merging could be done a priori or during the training process of graph convolutional neural networks, e.g, DiffPooling

Graph pooling via node selection

■ Node selection

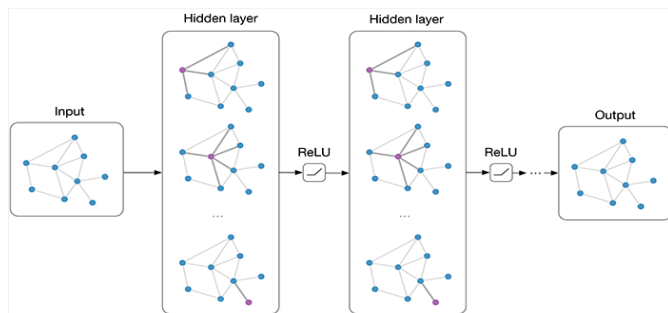
- ❑ Learn a metric to quantify the importance of nodes and select several nodes according to the learned metric



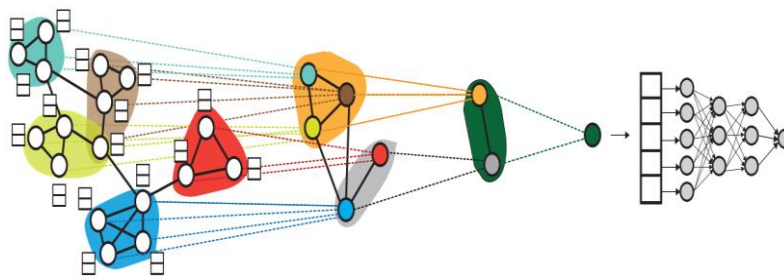
Expressive Power of Graph Neural Networks

Graph Neural Networks

- Graph neural networks (GNNs) gained remarkable success
 - Achieving state-of-the-art empirical performance in node classification, link prediction, and graph classification.



Node classification



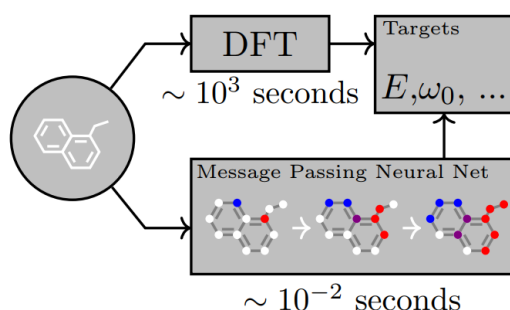
Graph classification

- The design of new GNNs is mostly based on
 - empirical intuition, heuristics and experimental trial-and-error.
- We lack theoretical understanding of the properties and limitations of GNNs.
 - One fundamental problem is the expressive power of GNNs

Power of GNNs: an empirical view

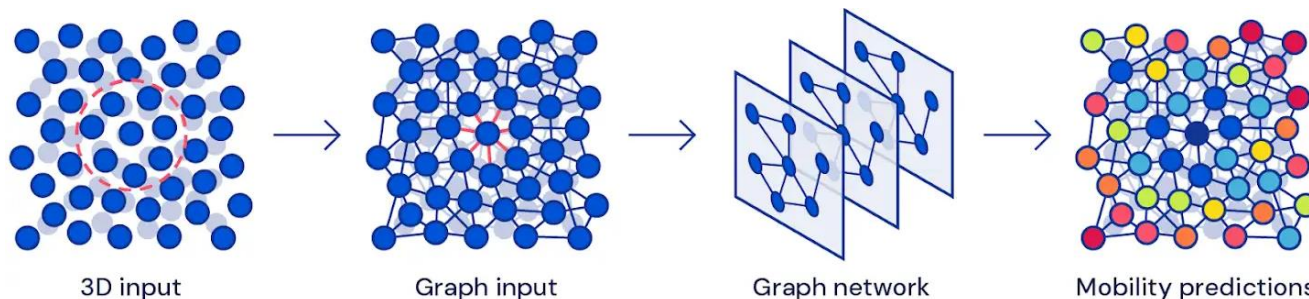
■ GNNs for quantum chemistry

- ❑ Predict the quantum properties of molecules using GNNs
- ❑ Traditional methods, i.e., DFT (Density Functional Theory), is computationally expensive



- ✓ Predict DFT to within **chemical accuracy** on **11 out of 13 targets** with molecule topology and spatial information as input
- ✓ Predict DFT to within chemical accuracy on **5 out of 13 targets** while operating on the topology of the molecule alone

■ Modeling dynamics of glassy systems using GNNs



» About Expressive Power

- **Number of possible distinct occurrences**
 - **Expressive power of n bits is 2^n**
 - For example, 5 bits has the ability to distinguish 32 distinct states
 - Given no more than 32 bottles of water and one of them is poisonous, at most 5 mice are required to identify which bottle is poisonous.
- **Approximation capability**
 - **Expressive power of 1-layer perceptron**
 - 1-layer perceptron is not a universal approximator
 - For example, XOR cannot be approximated by 1-layer perceptron.
 - **Multi-layer perceptron offers us a universal approximator**
 - Universal approximation theorem
 - Expressive power scales **exponentially** with the number of layers

➤ Two typical tasks for GNNs

■ Denote a graph with $G = (V, E, W, X)$

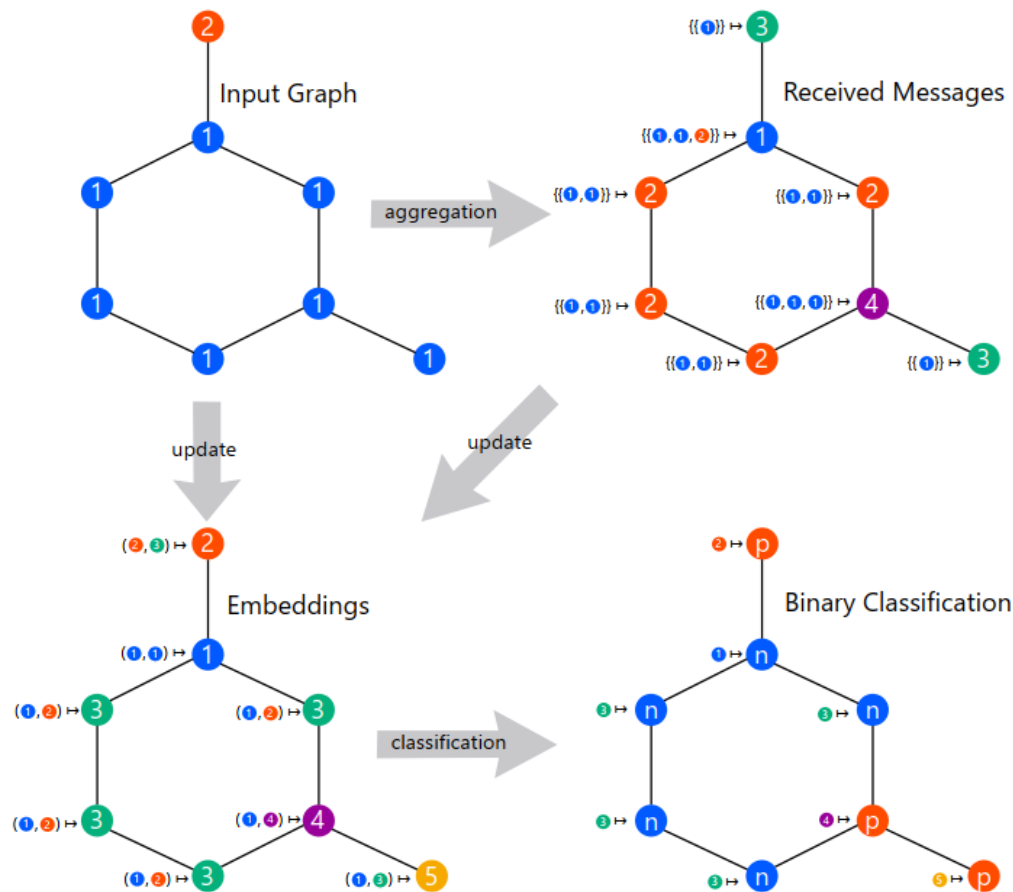
- V is the node set with $n = |V|$, E is the edge set, and $W \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix
- Each node is associated with d features, and $X \in \mathbb{R}^{n \times d}$ is the feature matrix of nodes

■ Two typical tasks for graph

- **Node classification**: Each node $v \in V$ has a label y_v , and the goal is to learn a representation vector h_v such that v 's label can be predicted as $y_v = f(h_v)$
- **Graph classification**: Given a set of graphs $\{G_1, \dots, G_N\}$ with labels $\{y_1, \dots, y_N\}$, the goal is to learn a representation vector h_G for each graph G such that we can predict the label of graph as $y_G = g(h_G)$

➤ GNNs for node classification

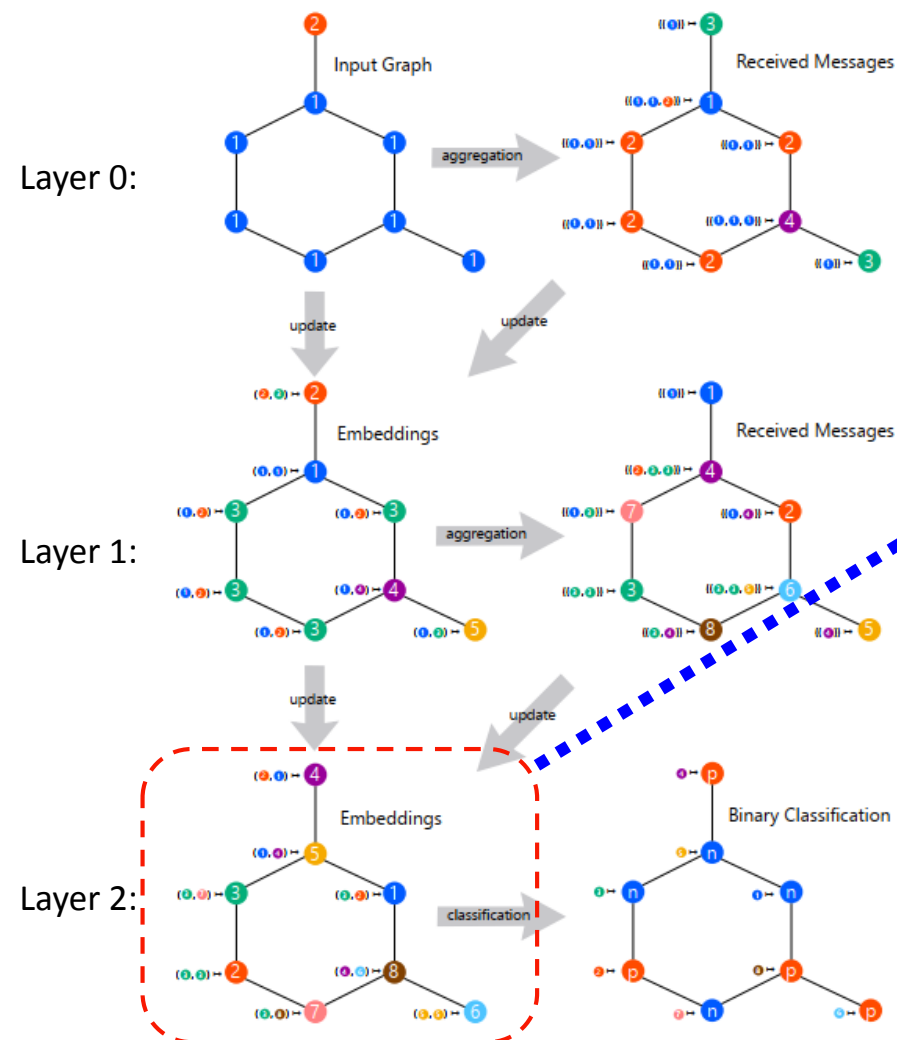
■ Example: 1-layer GCN



Limited expressive power of 1-layer GCN: it cannot fully distinguish all nodes

➤ GNNs for node classification

■ Example: 2-layer GCN



- ✓ 2-layer GCN can **fully** distinguish all nodes in this toy example.
- ✓ Depth of GNNs matters.

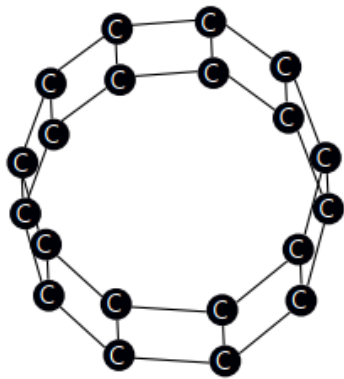
Can the expressive power of GNNs be improved **infinitely** via improving their depth? Is there a bound?

» GNNs for graph classification

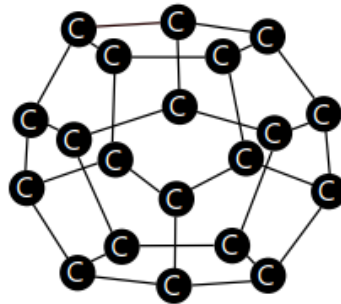
- GNN is a function: $G \rightarrow R^d$
 - Mapping a graph, where node feature is from a **countable** space, into a **real-valued** vector with infinite expressivity
- The expressive power of GNNs lies in their capability to distinguish different graphs
 - Different graphs should have different representations in R^d
- Two key factors
 - Node Feature
 - Feature transformation with neural networks, e.g., MLP
 - + Graph Structure
 - **Graph isomorphism**

» GNNs for graph classification

■ Examples to show the limited expressive power of GNNs

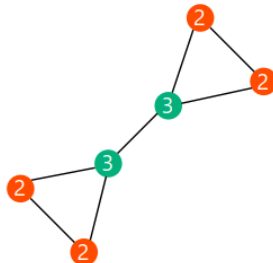
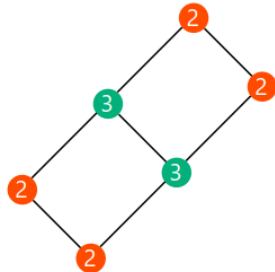


(a) Decaprismane.



(b) Dodecahedrane.

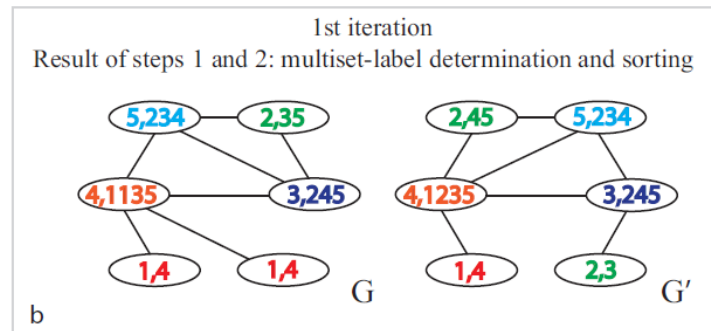
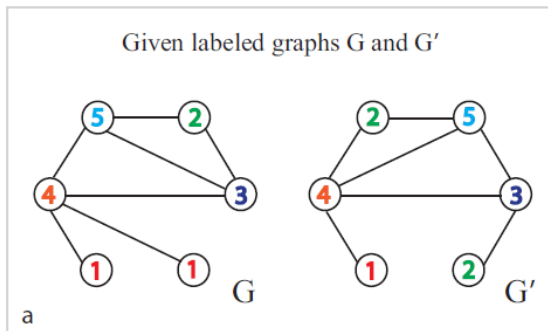
- ✓ Regular graphs
- ✓ Identical node labels



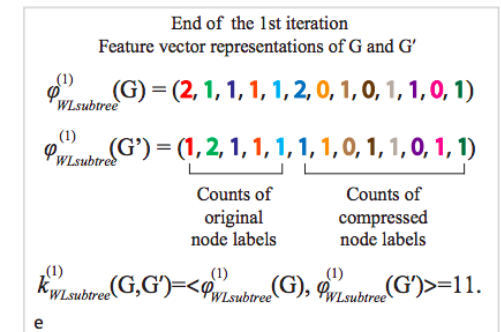
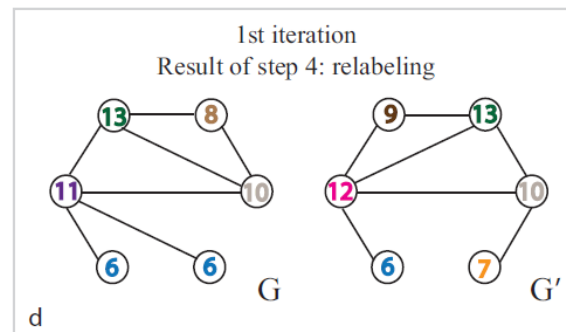
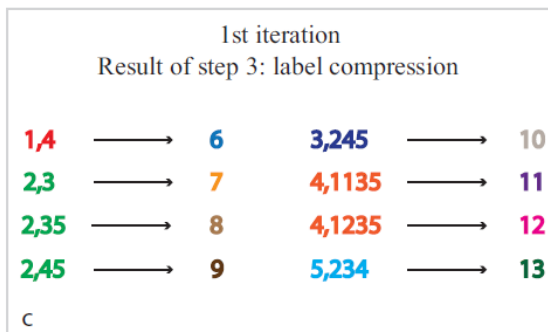
- ✓ Regular graphs
- ✓ Node labels are not identical

➤ Weisfeiler-Lehman Isomorphism Test

- WL test is widely used to judge whether two graphs, labeled or unlabeled, are **topologically identical**, or how they are **topologically similar**.

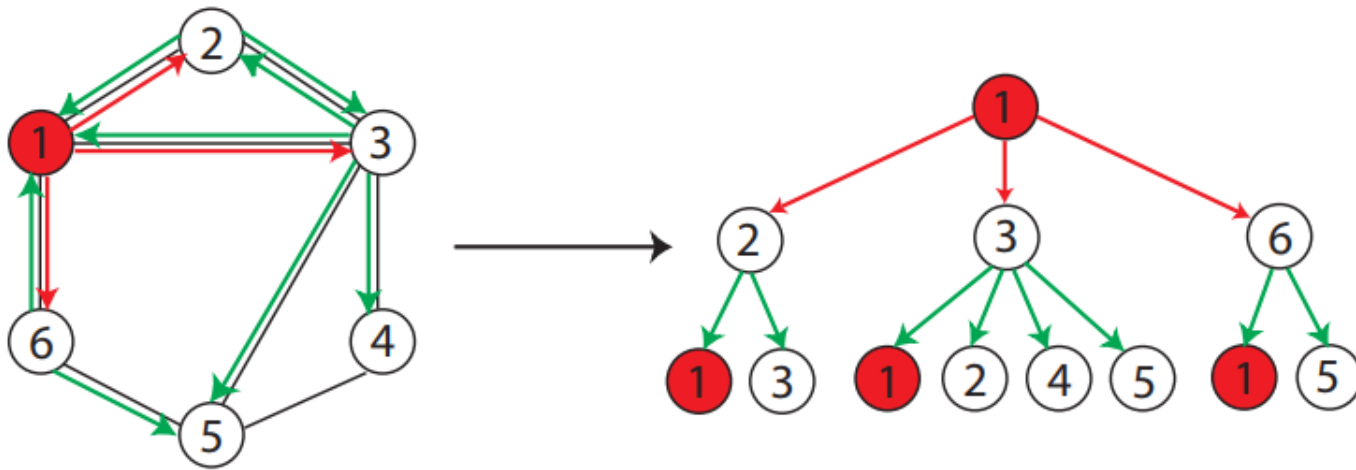


Example: similarity measurement.



WL Test: Subtree-based Graph Kernel

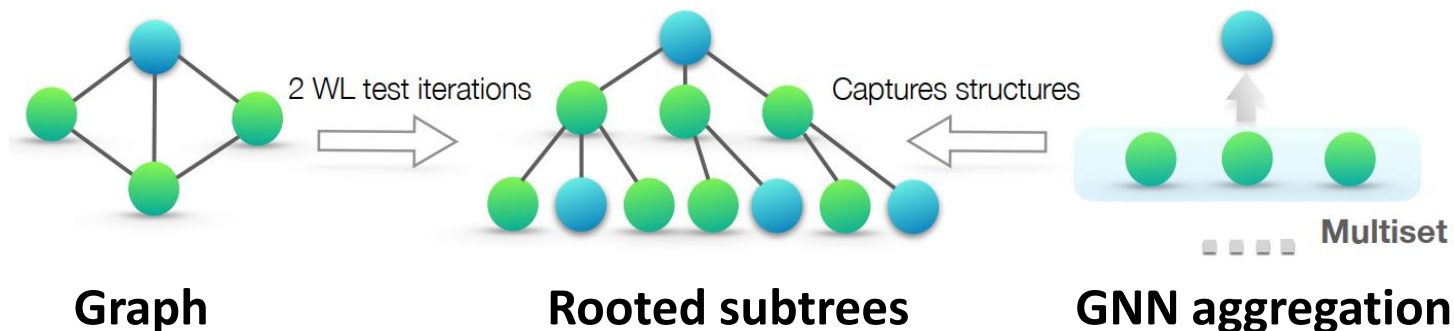
- A node's label at the k -th iteration of WL test represents a subtree structure of height k rooted at the node



For WL test, graph features are essentially counts of different rooted subtrees in the graph.

➤ Connection Between GNN and WL Test

- GNNs **recursively** update each node's feature vector to capture network structure and features of neighboring nodes, i.e., **rooted subtree** – **following the practice of WL test**.
- Features of neighboring nodes form a **multiset**
 - The same element can appear multiple times since different nodes can have identical feature vectors.
 - A multiset is denoted as a 2-tuple $X = (S, m)$, where S is the underlying set with distinct elements, and $m: S \rightarrow \mathbb{N}_{\geq 1}$ gives the multiplicity of elements.



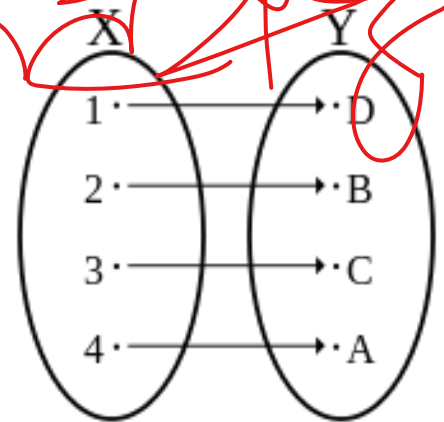
➤ Connection Between GNN and WL Test

- WL test provides **an upper bound** for the expressive power of the aggregation-based GNNs.
 - A maximally powerful GNN never map two different neighborhoods to the same representation, i.e., **the aggregation function over multiset is injective.**

BAD NEWS

For popular GNNs, like GCN and GraphSAGE, their aggregation functions are inherently **not injective.**

集合不能两两映射

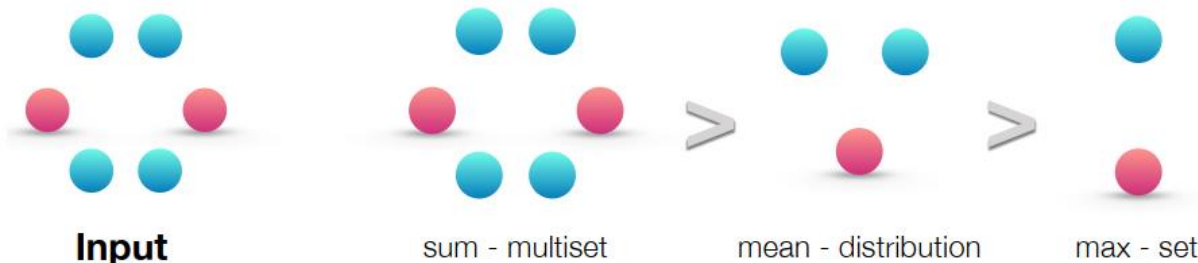


$$f: X \rightarrow Y$$

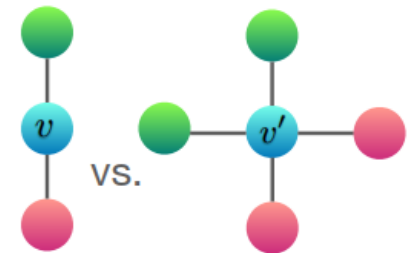
Injective function

Expressivity Limit of GCN and GraphSAGE

- Aggregation function **is not injective**
 - 1-layer perceptron is not sufficient
 - 1-layer perceptron is not a universal approximator
 - Mean and max pooling is not injective
 - Mean pooling learns distributions
 - Max pooling learns sets with distinct elements



Comparing expressive power of sum, mean, max pooling



Max and mean both fail

➤ Graph Isomorphism Network

- Basic idea: compose a universal **injective aggregation function** over a node and the multiset of its neighbors

$$h_v^{(k)} = \phi \left(h_v^{(k-1)}, f \left(\left\{ h_u^{(k-1)} : \forall u \in \mathcal{N}(v) \right\} \right) \right)$$

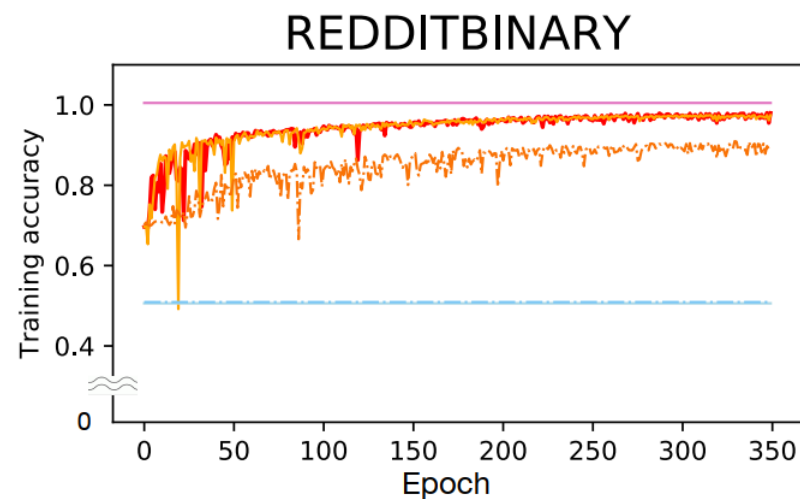
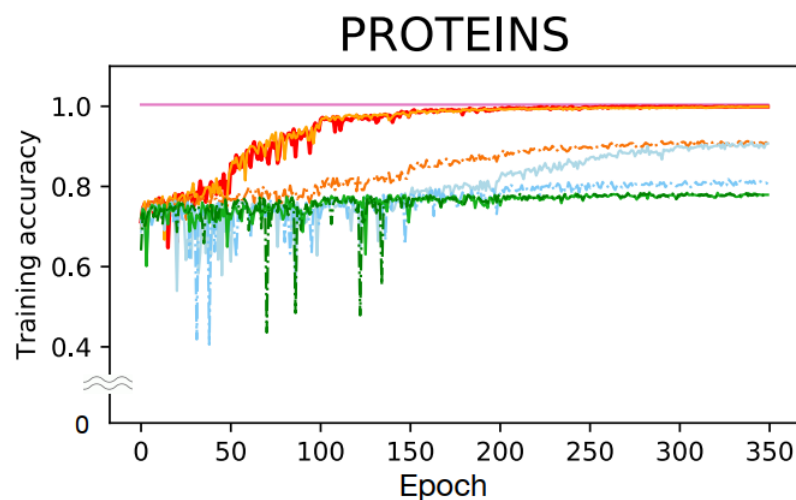
- Graph isomorphism network

$$h_v^{(k)} = \underbrace{\text{MLP}^{(k)}}_{\textcircled{1}} \left((1 + \epsilon^{(k)}) \cdot h_v^{(k-1)} + \underbrace{\sum_{u \in \mathcal{N}(v)}}_{\textcircled{2}} h_u^{(k-1)} \right)$$

- ① Multiple-layer perceptron offers universal approximator.
- ② Sum pooling offers injective condition.

Experimental Validation

- Validating expressive power or representation capability
 - ❑ Metric: **training accuracy**
 - ❑ Task: graph classification
 - ❑ Datasets: bioinformatics and social networks



— WL subtree kernel — Sum -- MLP (GIN-0) - - Sum -- 1-layer - - Mean -- 1-layer (GCN) - - Max -- 1-layer (GraphSAGE)

Performance on Graph Classification

- Does the high expressive power of GNN imply good performance on down-stream task, e.g., graph classification?
 - Metric: **test accuracy**

Datasets	Datasets	IMDB-B	IMDB-M	RDT-B	RDT-M5K	COLLAB	MUTAG	PROTEINS	PTC	NCI1
	# graphs	1000	1500	2000	5000	5000	188	1113	344	4110
	# classes	2	3	2	5	3	2	2	2	2
	Avg # nodes	19.8	13.0	429.6	508.5	74.5	17.9	39.1	25.5	29.8
Baselines	WL subtree	73.8 \pm 3.9	50.9 \pm 3.8	81.0 \pm 3.1	52.5 \pm 2.1	78.9 \pm 1.9	90.4 \pm 5.7	75.0 \pm 3.1	59.9 \pm 4.3	86.0 \pm 1.8 *
	DCNN	49.1	33.5	–	–	52.1	67.0	61.3	56.6	62.6
	PATCHYSAN	71.0 \pm 2.2	45.2 \pm 2.8	86.3 \pm 1.6	49.1 \pm 0.7	72.6 \pm 2.2	92.6 \pm 4.2 *	75.9 \pm 2.8	60.0 \pm 4.8	78.6 \pm 1.9
	DGCNN	70.0	47.8	–	–	73.7	85.8	75.5	58.6	74.4
	AWL	74.5 \pm 5.9	51.5 \pm 3.6	87.9 \pm 2.5	54.7 \pm 2.9	73.9 \pm 1.9	87.9 \pm 9.8	–	–	–
GNN variants	SUM-MLP (GIN-0)	75.1 \pm 5.1	52.3 \pm 2.8	92.4 \pm 2.5	57.5 \pm 1.5	80.2 \pm 1.9	89.4 \pm 5.6	76.2 \pm 2.8	64.6 \pm 7.0	82.7 \pm 1.7
	SUM-MLP (GIN- ϵ)	74.3 \pm 5.1	52.1 \pm 3.6	92.2 \pm 2.3	57.0 \pm 1.7	80.1 \pm 1.9	89.0 \pm 6.0	75.9 \pm 3.8	63.7 \pm 8.2	82.7 \pm 1.6
	SUM-1-LAYER	74.1 \pm 5.0	52.2 \pm 2.4	90.0 \pm 2.7	55.1 \pm 1.6	80.6 \pm 1.9	90.0 \pm 8.8	76.2 \pm 2.6	63.1 \pm 5.7	82.0 \pm 1.5
	MEAN-MLP	73.7 \pm 3.7	52.3 \pm 3.1	50.0 \pm 0.0	20.0 \pm 0.0	79.2 \pm 2.3	83.5 \pm 6.3	75.5 \pm 3.4	66.6 \pm 6.9	80.9 \pm 1.8
	MEAN-1-LAYER (GCN)	74.0 \pm 3.4	51.9 \pm 3.8	50.0 \pm 0.0	20.0 \pm 0.0	79.0 \pm 1.8	85.6 \pm 5.8	76.0 \pm 3.2	64.2 \pm 4.3	80.2 \pm 2.0
	MAX-MLP	73.2 \pm 5.8	51.1 \pm 3.6	–	–	–	84.0 \pm 6.1	76.0 \pm 3.2	64.6 \pm 10.2	77.8 \pm 1.3
	MAX-1-LAYER (GraphSAGE)	72.3 \pm 5.3	50.9 \pm 2.2	–	–	–	85.1 \pm 7.6	75.9 \pm 3.2	63.9 \pm 7.7	77.7 \pm 1.5

High expressive power **does not always** bring good performance
Note that: low expressive power always implies bad performance

» Conclusions

- **WL test provides an upper bound for the expressive power of the aggregation-based GNNs.**
- **Graph isomorphism network is composed as a maximally powerful aggregation-based GNN.**
 - **1-layer perceptron → Multiple-layer perceptron**
 - **Mean, max pooling → Sum pooling**

➤ Is expressive power necessary?

- Expressive power offers us a **theoretical** guide for understanding the capability of GNNs.
 - Whether, and to what degree, are GNNs **universal approximator** to functions mapping graphs to real-valued vector?
 - *For graph classification, No!!!*
 - *For node classification, it is almost.*
- For specific tasks, it is not **practically** necessary to seek high expressive power for performance improvement
 - This partly explains why GCN, GraphSAGE works well although their expressive power is less than GIN.
 - What we really need is a universal function that can **map similar objects** (nodes or graphs) to **close representations**, facilitating down-stream tasks.

Thank you for your attentions!