

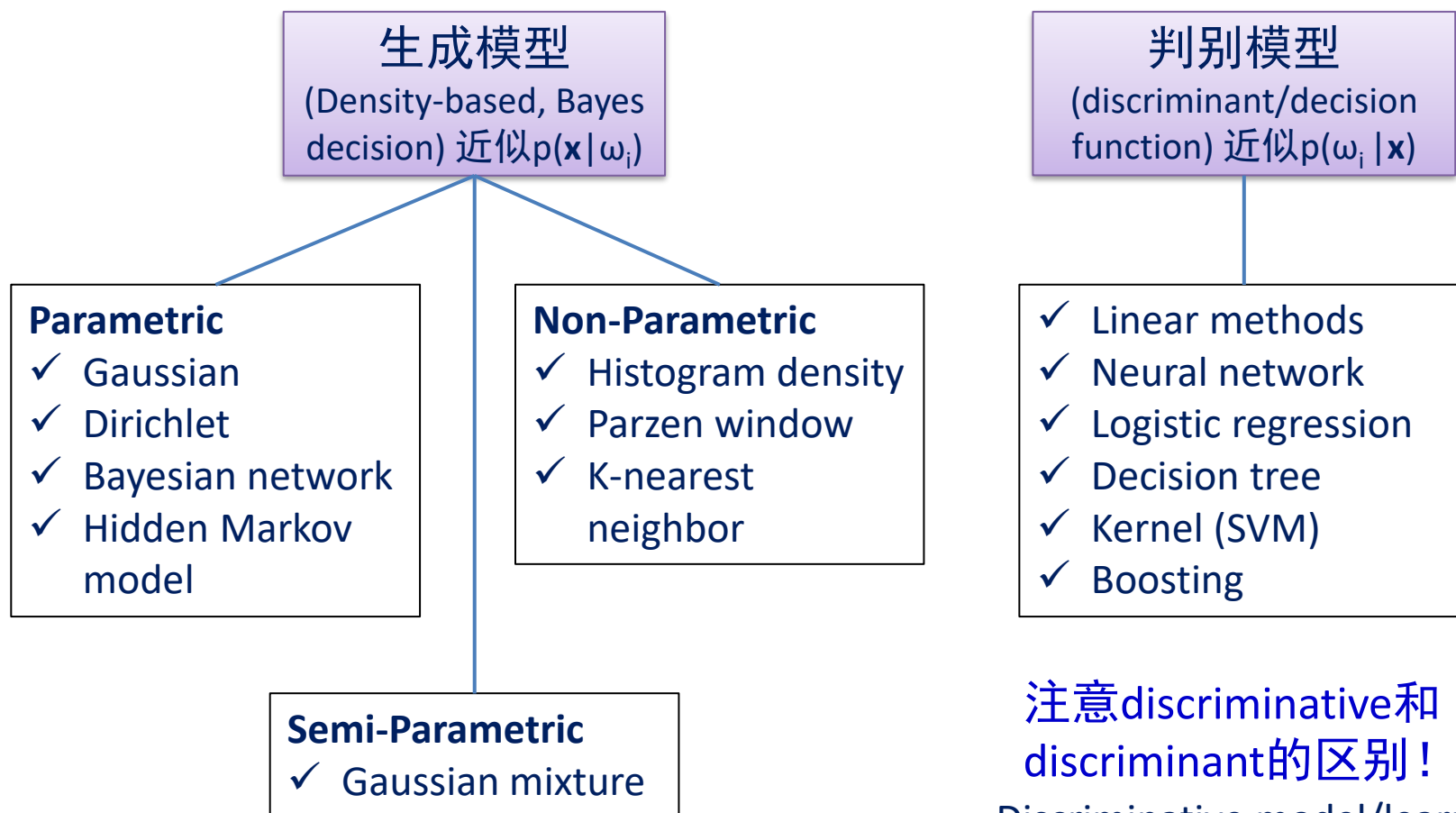
第2章：贝叶斯决策理论

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2020年9月23日

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统计模式识别方法



注意discriminative和discriminant的区别！

Discriminative model/learning
Discriminant function/analysis

提纲

- 问题表示
- 最小错误率决策：2类的例子
- 最小风险决策
- 判别函数和决策面
- 高斯概率密度
- 高斯密度下的判别函数
- 分类错误率

导论：问题表示

- 类别: $\omega_i, i = 1, \dots, c$
- 特征矢量 $\mathbf{x} = [x_1, \dots, x_d] \in R^d$
- 先验概率 $P(\omega_i) \quad \sum_{i=1}^c P(\omega_i) = 1$
- 概率密度函数(条件概率) $p(\mathbf{x} | \omega_i)$
- 后验概率

$$P(\omega_i | \mathbf{x}) = \frac{p(\mathbf{x} | \omega_i)P(\omega_i)}{p(\mathbf{x})} = \frac{p(\mathbf{x} | \omega_i)P(\omega_i)}{\sum_{j=1}^c p(\mathbf{x} | \omega_j)P(\omega_j)}$$

$$\sum_{i=1}^c P(\omega_i | \mathbf{x}) = 1$$

最小错误率决策：2类的例子

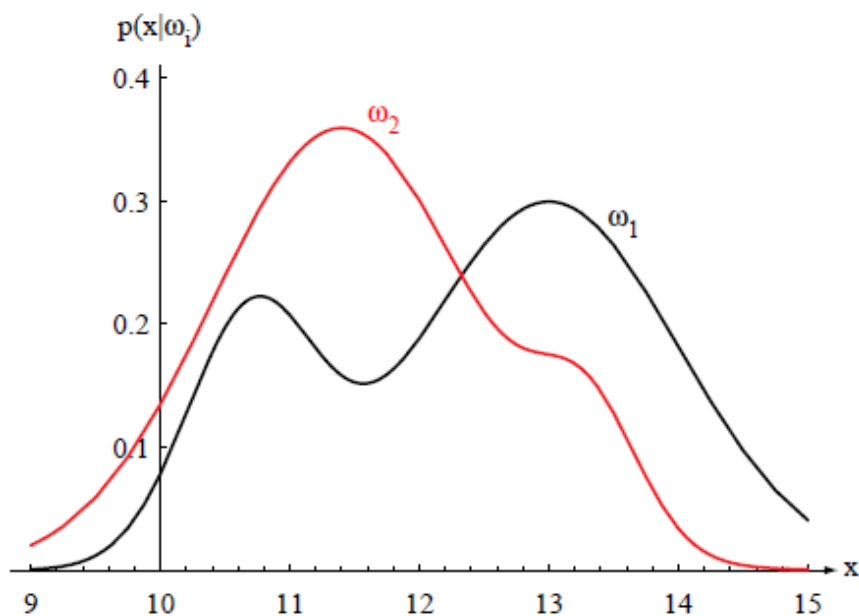
- Salmon (ω_1) and sea bass (ω_2)
- If we have only prior probability
 - 例如，教室门口判断进来的是男生还是女生，没有任何传感器
 - Decide ω_1 if $P(\omega_1) > P(\omega_2)$, otherwise ω_2
 - Minimum error decision

$$P(\text{error}) = \begin{cases} P(\omega_2) & \text{if we decide } \omega_1 \\ P(\omega_1) & \text{if we decide } \omega_2 \end{cases}$$

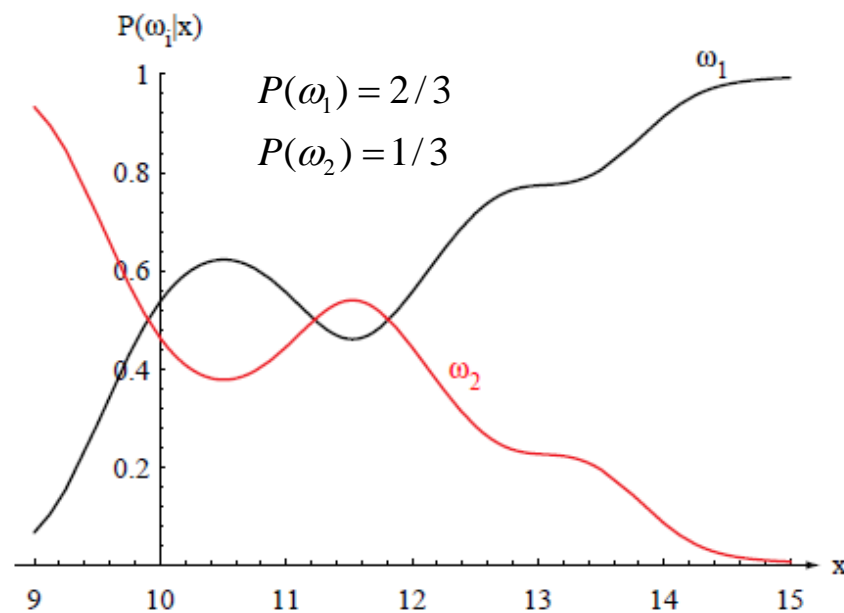
- 教室门口判断性别的例子：错误率？

2类的例子

- 有传感器(特征)的情况
Decision based on posterior probabilities



x轴：一维特征空间



$$P(\omega_i | \mathbf{x}) = \frac{p(\mathbf{x} | \omega_i) P(\omega_i)}{p(\mathbf{x})} = \frac{p(\mathbf{x} | \omega_i) P(\omega_i)}{\sum_{j=1}^c p(\mathbf{x} | \omega_j) P(\omega_j)}$$

- Decision based on posterior probabilities

$$P(error|x) = \begin{cases} P(\omega_1|x) & \text{if we decide } \omega_2 \\ P(\omega_2|x) & \text{if we decide } \omega_1. \end{cases}$$

Decide ω_1 if $P(\omega_1|x) > P(\omega_2|x)$; otherwise decide ω_2

$$P(error|x) = \min [P(\omega_1|x), P(\omega_2|x)].$$

- Evidence (a.k.a. likelihood)

Decide ω_1 if $p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2)$; otherwise decide ω_2

— see
$$P(\omega_i | \mathbf{x}) = \frac{p(\mathbf{x} | \omega_i)P(\omega_i)}{p(\mathbf{x})}$$

教室门口判断性别的例子：用什么传感器(\mathbf{x})?

最小风险决策：贝叶斯决策的一般形式

- 决策代价(loss, cost)

- True class ω_j , decided as α_i $\lambda_{ij} = \lambda(\alpha_i | \omega_j)$

- 有时 λ_{ij} 和 λ_{ji} 相差很大，比如医疗诊断的场合、工业检测、自动商店判断性别

- Condition risk

$$R(\alpha_i | \mathbf{x}) = \sum_{j=1}^c \lambda(\alpha_i | \omega_j) P(\omega_j | \mathbf{x})$$

- Overall (expected) risk

$$R = \int R(\alpha(\mathbf{x}) | \mathbf{x}) p(\mathbf{x}) d\mathbf{x} \quad \alpha(\mathbf{x}) \in \{\alpha_1, \dots, \alpha_c\}$$

- Minimum risk decision (Bayes decision)

$$\arg \min_i R(\alpha_i | x)$$

- Minimum risk decision: 2-class case

- Condition risk

$$R(\alpha_1|\mathbf{x}) = \lambda_{11}P(\omega_1|\mathbf{x}) + \lambda_{12}P(\omega_2|\mathbf{x})$$

$$R(\alpha_2|\mathbf{x}) = \lambda_{21}P(\omega_1|\mathbf{x}) + \lambda_{22}P(\omega_2|\mathbf{x})$$

- Decision rule

$$R(\alpha_1 | x) < R(\alpha_2 | x) \Leftrightarrow (\lambda_{21} - \lambda_{11})P(\omega_1|\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2|\mathbf{x})$$

- Equivalently, decide ω_1 if

$$(\lambda_{21} - \lambda_{11})\underline{p(\mathbf{x}|\omega_1)P(\omega_1)} > (\lambda_{12} - \lambda_{22})p(\mathbf{x}|\omega_2)P(\omega_2)$$

$$\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

(Likelihood ratio)

最小错误率分类

- Zero-one loss

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \quad i, j = 1, \dots, c$$

$$\begin{aligned} R(\alpha_i|\mathbf{x}) &= \sum_{j=1}^c \lambda(\alpha_i|\omega_j) P(\omega_j|\mathbf{x}) \\ &= \sum_{j \neq i} P(\omega_j|\mathbf{x}) \\ &= 1 - P(\omega_i|\mathbf{x}) \end{aligned}$$

- Minimum error decision: Maximum a posteriori (MAP)

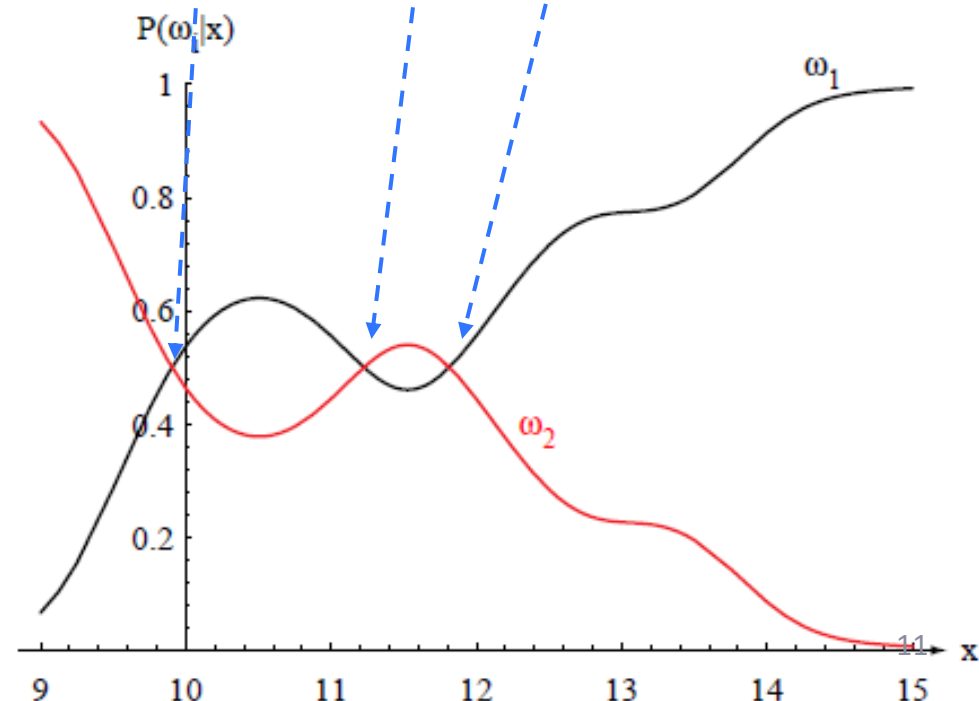
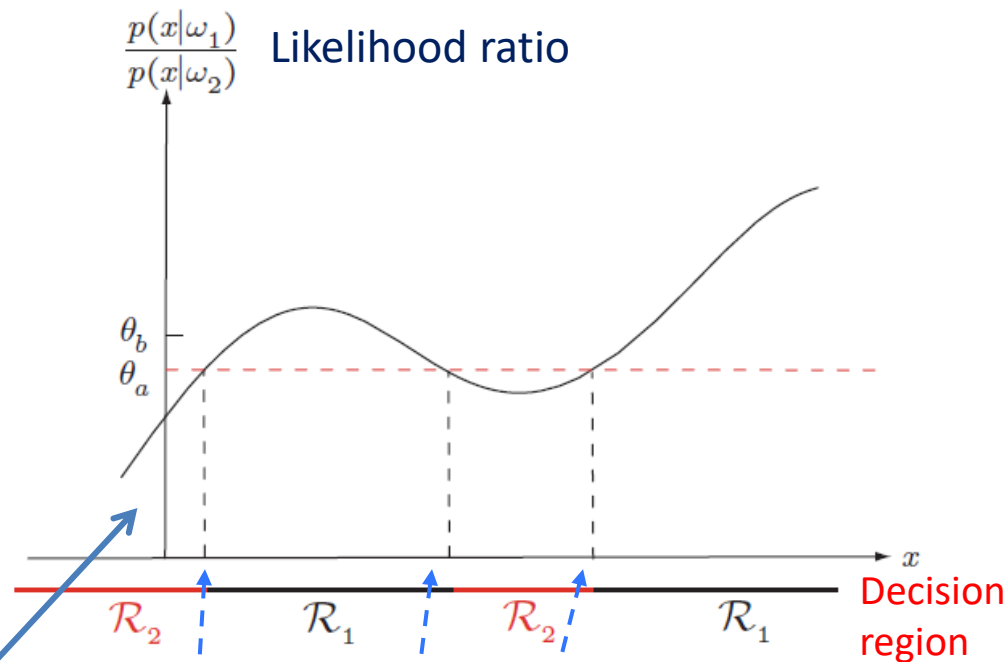
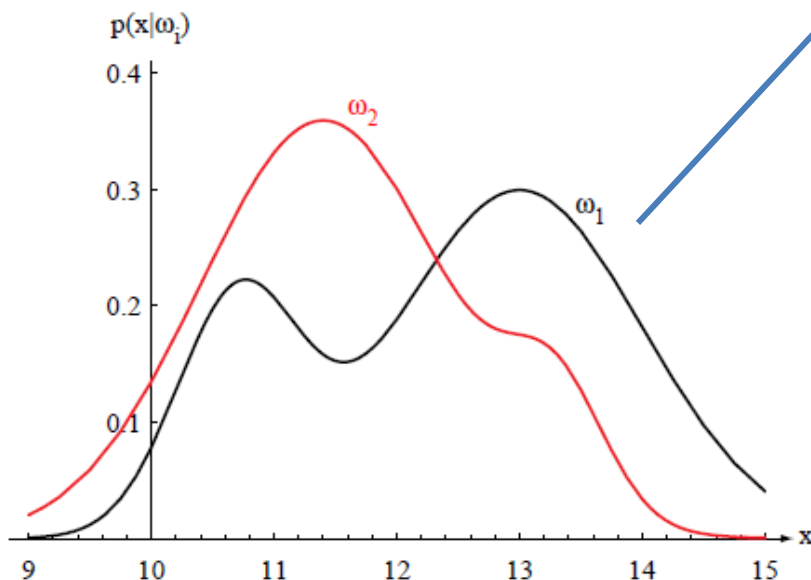
Decide ω_i if $P(\omega_i|\mathbf{x}) > P(\omega_j|\mathbf{x})$ for all $j \neq i$

- 2-class case
 - decide ω_1 if

$$\frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

0-1 loss

$$\frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{P(\omega_2)}{P(\omega_1)}$$



带拒识的决策

- 为什么要拒识？错误识别可能带来严重后果
 - 比如医疗诊断，金额识别

Ambiguity Rejection

6087 1027

6/0?

1985 579

7/1?

4/9?

67814 42

6/5?

42/32/312?

Distance Rejection

In fact, the Tories made it worse now for the sick and needy than Labour had to make it in 1950. And as a percentage of social service expenditure, health had fallen from 28.5 to 23.1 per cent.

English is outlier for a digit recognizer

経営不振に陥っているソニーのパソコン事業は国内投資ファンドが買い取り、開発と製造をてがける新会社バイオをつくった。ソニーの国内向け通販サイトと同社の直営店を通じ、個人から注文をとってきた。

These are outlier for an English recognizer



带拒识的决策

- Formulation (Problem 13, Chapter 2)
 - C+1 classes

$$\lambda(\alpha_i | \omega_j) = \begin{cases} 0, & i = j \\ \lambda_s, & i \neq j \\ \lambda_r, & \text{reject} \end{cases} \quad \lambda_r < \lambda_s$$

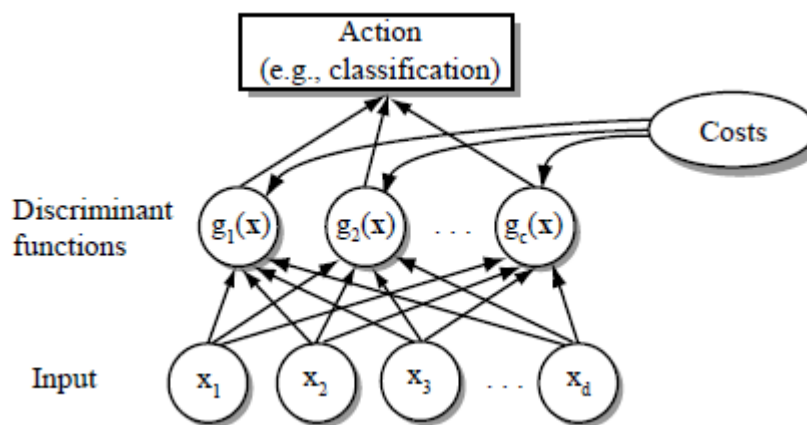
$$R(\alpha_i | \mathbf{x}) = \sum_{j=1}^c \lambda(\alpha_i | \omega_j) P(\omega_j | \mathbf{x})$$

$$\Rightarrow R_i(\mathbf{x}) = \begin{cases} \lambda_s [1 - P(\omega_i | \mathbf{x})], & i = 1, \dots, c \\ \lambda_r, & \text{reject} \end{cases}$$

$$\arg \min_i R_i(\mathbf{x}) = \begin{cases} \arg \max_i P(\omega_i | \mathbf{x}), & \text{if } \max_i P(\omega_i | \mathbf{x}) > 1 - \lambda_r / \lambda_s \\ \text{reject}, & \text{otherwise} \end{cases}$$

判别函数、决策面

- 判别函数(Discriminant Function)
 - 表征模式属于每一类的广义似然度 $g_i(\mathbf{x})$, $i=1, \dots, c$
 - 分类决策 $\arg \max_i g_i(\mathbf{x})$
 - E.g., conditional risk $g_i(\mathbf{x}) = -R(\alpha_i | \mathbf{x})$
 - Posterior probability $g_i(\mathbf{x}) = P(\omega_i | \mathbf{x})$
 - Likelihood $g_i(\mathbf{x}) = p(\mathbf{x} | \omega_i)P(\omega_i)$
 $g_i(\mathbf{x}) = \log p(\mathbf{x} | \omega_i) + \log P(\omega_i)$



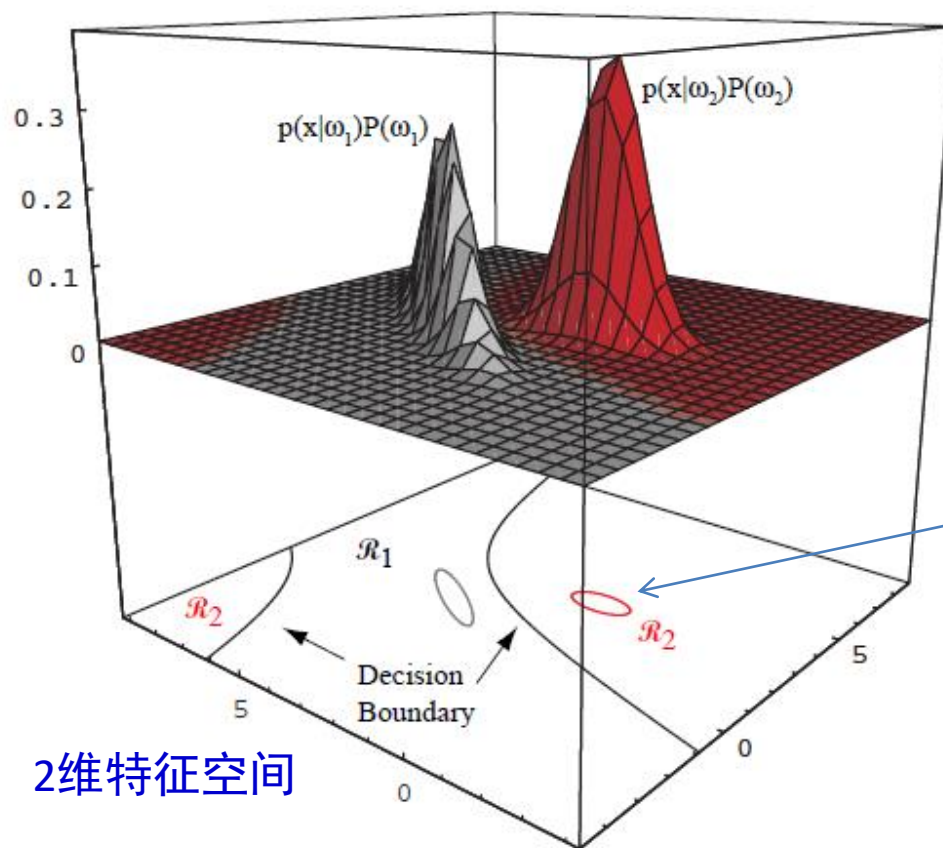
- 决策面(Decision surface)
 - 特征空间中二类判别函数相等的点的集合

$$g(\mathbf{x}) \equiv g_1(\mathbf{x}) - g_2(\mathbf{x}) \quad g(\mathbf{x}) = P(\omega_1|\mathbf{x}) - P(\omega_2|\mathbf{x}) = 0$$

$$g(\mathbf{x}) = \ln \frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} + \ln \frac{P(\omega_1)}{P(\omega_2)}$$

有判别函数就可以分类了，为什么还来求决策面？

答：加深对特征空间和分类器性质的理解



正态分布下的一个例子

Density
1/e
ellipse

2维特征空间

贝叶斯决策用于模式分类

- Bayes决策的关键
 - 类条件概率密度估计
 - 先验概率：从训练样本估计或假设等概率
 - 决策代价 $[\lambda_{ij}]$ ，一般为0-1代价
- 分类器设计
 - 收集训练样本
 - 用每一类的样本估计类条件概率密度 $p(\mathbf{x} | \omega_i)$
 - 估计类先验概率
 - 模型参数集： $\{p(\mathbf{x} | \omega_i, \theta_i), P(\omega_i)\}, i = 1, \dots, c$
- 分类过程
 - 计算测试样本 \mathbf{x} 属于每一类的后验概率
 - 最大后验概率/最小风险决策

概率密度估计方法

- 参数法：假定概率密度函数形式

$$p(\mathbf{x} | \omega_i) = p(\mathbf{x} | \theta_i)$$

- Distribution: Gaussian, Gamma, Bernouli
- Parameter estimation: maximum-likelihood (ML), Bayesian estimation
- 非参数法：可以表示任意概率分布，无函数形式
 - Parzen window, k-NN
 - 需要保存所有或大部分样本
- Semi-parametric, 近似任意概率分布，有函数形式
 - Distribution: Gaussian mixture (GM)
 - Estimation: expectation-maximization (EM)

高斯密度函数

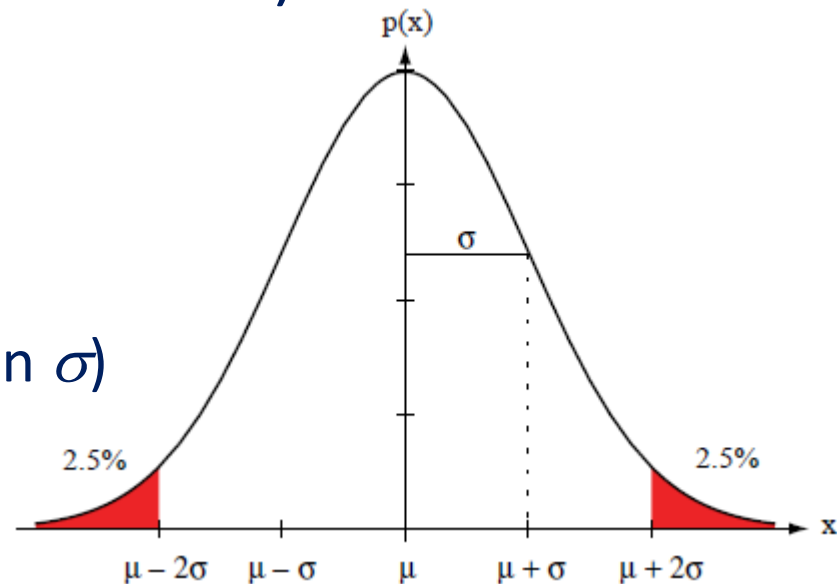
- Gaussian density (normal distribution)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right]$$

- Mean μ
- Variance σ^2 (standard deviation σ)

$$\mu \equiv \mathcal{E}[x] = \int_{-\infty}^{\infty} xp(x) dx$$

$$\sigma^2 \equiv \mathcal{E}[(x - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 p(x) dx$$



$$H(p(x)) = - \int p(x) \ln p(x) dx$$

- 在给定均值和方差的所有分布中，正态分布的熵最大(Problem 20, Chapter 2)
- 根据Central Limit Theorem，大量独立随机变量之和趋近正态分布
- 实际环境中，很多类别的特征分布趋近正态分布

- Multivariate normal density $p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

- 公式要牢记

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp \left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

- Mean $\boldsymbol{\mu} \equiv \mathcal{E}[\mathbf{x}] = \int \mathbf{x} p(\mathbf{x}) d\mathbf{x} \quad \mu_i = \mathcal{E}[x_i]$

- Covariance matrix

$$\boldsymbol{\Sigma} \equiv \mathcal{E}[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^t] = \int (\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^t p(\mathbf{x}) d\mathbf{x}$$

$$\sigma_{ij} = \mathcal{E}[(x_i - \mu_i)(x_j - \mu_j)]$$

If x_i and x_j are statistically independent, $\sigma_{ij} = 0$

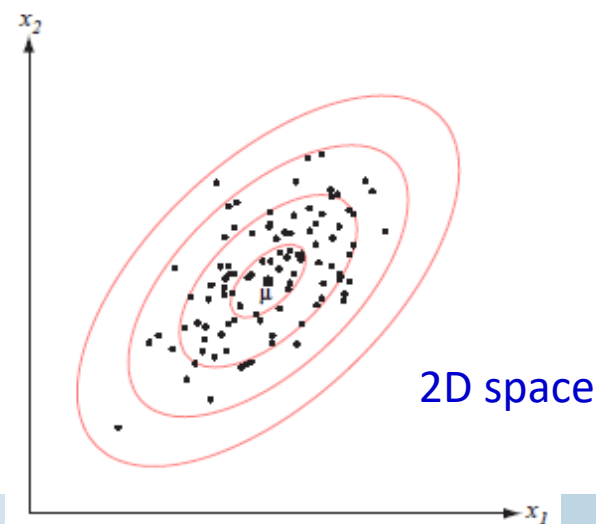
$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{bmatrix}$$

- 等密度点轨迹: hyperellipsoid

- 特殊情况下为圆形或超球面

- Mahalanobis distance

$$r^2 = (\mathbf{x} - \boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})$$



Break

协方差矩阵的性质

- 实对称矩阵
- Eigenvalues & eigenvectors (本征值, 本征向量)

$$\Sigma \phi_i = \lambda_i \phi_i \quad \Phi = [\phi_1 \phi_2 \cdots \phi_d] \quad \Lambda = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_d]$$

– Orthonormal

$$\begin{aligned} \Phi^T \Phi &= I \\ \Phi^T &= \Phi^{-1} \end{aligned} \quad \Longleftrightarrow \quad \phi_i^T \phi_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

– 矩阵表示

$$\Sigma \Phi = \Phi \Lambda \quad \Longleftrightarrow \quad \Sigma = \Phi \Lambda \Phi^T \quad \Longleftrightarrow \quad \Sigma = \sum_{i=1}^d \lambda_i \phi_i \phi_i^T$$

- 矩阵对角化

$$\Phi^T \Sigma \Phi = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix} \quad \Longleftrightarrow \quad \phi_i^T \Sigma \phi_i = \lambda_i$$

- 应用：Principal component analysis (PCA)

- 一种降维(特征提取)方法
- 将随机矢量投影到低维子空间，使子空间投影的重建误差最小
- 选择本征值最大的 m ($m < d$)个本征向量作为子空间的基(basis)

线性空间中正交变换($\Phi^T \Phi = I$) 不影响欧氏距离

$$\sum_{j=1}^d [(\mathbf{x} - \mu)^T \phi_j]^2 = \|\mathbf{x} - \mu\|^2$$

子空间投影 $\mu + \sum_{j=1}^m [(\mathbf{x} - \mu)^T \phi_j] \phi_j$

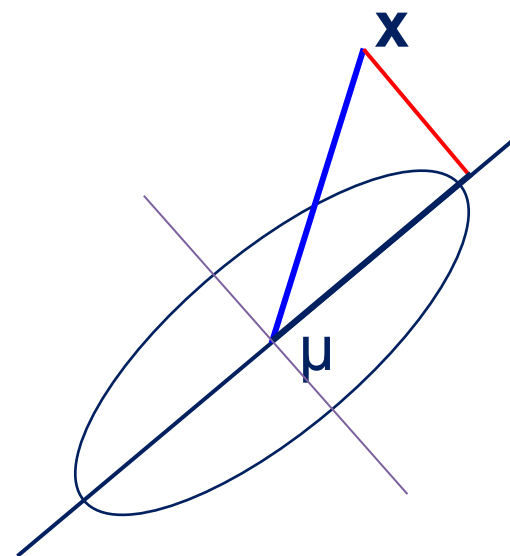
投影重建误差

$$r_E = \|\mathbf{x} - \mu\|^2 - \sum_{j=1}^m [(\mathbf{x} - \mu)^T \phi_j]^2 = \sum_{j=m+1}^d [(\mathbf{x} - \mu)^T \phi_j]^2$$

期望

$$\mathcal{E}(r_E) = \mathcal{E} \left\{ \sum_{j=m+1}^d [(\mathbf{x} - \mu)^T \phi_j]^2 \right\} = \mathcal{E} \left\{ \sum_{j=m+1}^d \phi_j^T (\mathbf{x} - \mu)(\mathbf{x} - \mu)^T \phi_j \right\}$$

$$= \sum_{j=m+1}^d \phi_j^T \mathcal{E}[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T] \phi_j = \sum_{j=m+1}^d \phi_j^T \Sigma \phi_j = \sum_{j=m+1}^d \lambda_j$$



$$\min \sum_{j=m+1}^d \lambda_j \text{ or } \max \sum_{j=1}^m \lambda_j$$

意味着取 λ_j 最大的
 m 个本征向量作为
子空间基(basis)



线性变换的高斯分布

- 线性变换 $y = A^t x$
 - $A^t A = 1$: 正交变换(坐标轴旋转)
 - 变换后的分布仍为正态分布

$$p(y) \sim N(A^t \mu, A^t \Sigma A)$$

- Diagonalization

$$A = \Phi$$

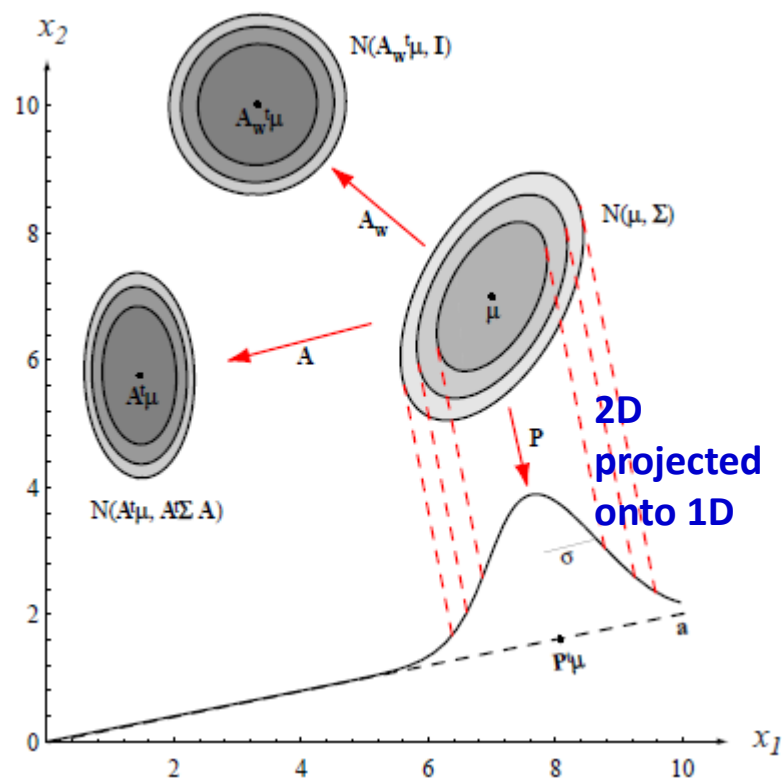
$$A^t \Sigma A = \Lambda$$

- Whitening transform

$$A_w = \Phi \Lambda^{-1/2}$$

$$A_w^t \Sigma A_w = \Lambda^{-1/2} \Phi^t \Sigma \Phi \Lambda^{-1/2}$$

$$= \Lambda^{-1/2} \Lambda \Lambda^{-1/2} = I$$



相关知识：特征提取/降维

- Dimensionality reduction
- Feature extraction
 - Feature generation: original data $\mathbf{d} \rightarrow \mathbf{x}$
 - Linear feature extraction $\mathbf{x} = \mathbf{A}^T \mathbf{d}$
- Feature selection (for reduction and performance)
 - Feature subset selection: a learning/optimization problem
- Feature transform (for extraction or reduction)
 - Linear transform $\mathbf{y} = \mathbf{A}^T \mathbf{x}$
 - Nonlinear transform: may increase dimensionality, e.g. kernel PCA, kernel LDA
- Handcrafted feature
- Feature learning
 - Automatic feature generation, e.g. convolutional neural network (CNN)

高斯密度下的判别函数

- 判别函数 $g_i(\mathbf{x}) = \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i)$

$$p(\mathbf{x} | \omega_i) = \frac{1}{(2\pi)^{d/2} |\Sigma_i|^{1/2}} \exp \left[-\frac{1}{2} (\mathbf{x} - \mu_i)^T \Sigma_i^{-1} (\mathbf{x} - \mu_i) \right]$$

$$g_i(\mathbf{x}) = -\frac{1}{2} (\mathbf{x} - \mu_i)^T \Sigma_i^{-1} (\mathbf{x} - \mu_i) - \frac{d}{2} \ln 2\pi - \frac{1}{2} \ln |\Sigma_i| + \ln P(\omega_i)$$

- Quadratic discriminant function (QDF)
- 在不同covariance假设条件下得到一些特殊形式

- Case 1: $\Sigma_i = \sigma^2 I$

(去掉与类别无关项)

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x} - \boldsymbol{\mu}_i\|^2}{2\sigma^2} + \ln P(\omega_i)$$

– Euclidean distance $\|\mathbf{x} - \boldsymbol{\mu}_i\|^2 = (\mathbf{x} - \boldsymbol{\mu}_i)^t (\mathbf{x} - \boldsymbol{\mu}_i)$

- Nearest mean/distance classifier

– 展开二次式 $(\mathbf{x} - \boldsymbol{\mu}_i)^t (\mathbf{x} - \boldsymbol{\mu}_i)$

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2} [\mathbf{x}^t \mathbf{x} - 2\boldsymbol{\mu}_i^t \mathbf{x} + \boldsymbol{\mu}_i^t \boldsymbol{\mu}_i] + \ln P(\omega_i)$$

– 忽略与类别无关项，得到线性判别函数

$$g_i(\mathbf{x}) = \mathbf{w}_i^t \mathbf{x} + w_{i0}$$

$$\mathbf{w}_i = \frac{1}{\sigma^2} \boldsymbol{\mu}_i \quad w_{i0} = \frac{-1}{2\sigma^2} \boldsymbol{\mu}_i^t \boldsymbol{\mu}_i + \ln P(\omega_i)$$

- Case 1: $\Sigma_i = \sigma^2 I$ (continued)
 - 二类决策面(判别函数相等的点构成)

$$g_i(\mathbf{x}) = g_j(\mathbf{x})$$

$$\Rightarrow \mathbf{w}^t(\mathbf{x} - \mathbf{x}_0) = 0 \quad \mathbf{w} = \mu_i - \mu_j$$

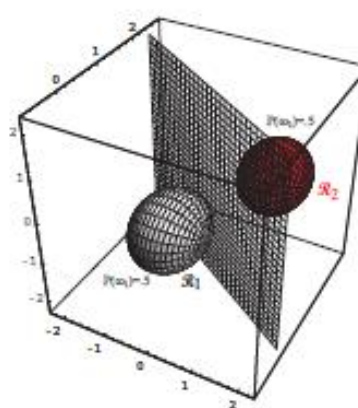
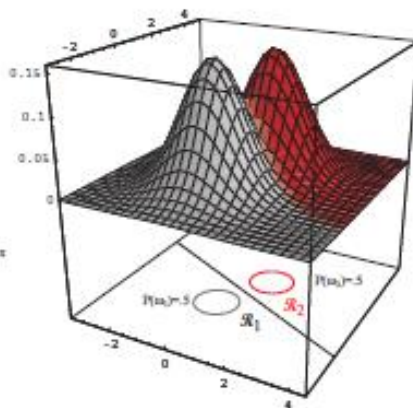
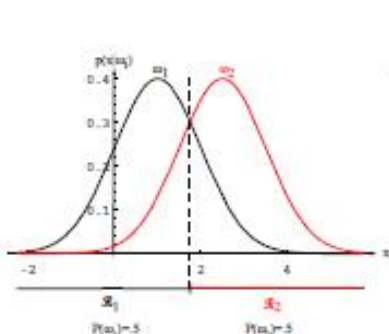
W: 法向量
平面与 $\mu_1 - \mu_2$ 垂直

$$\mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)} (\mu_i - \mu_j)$$

位置移向先验
概率小的类别

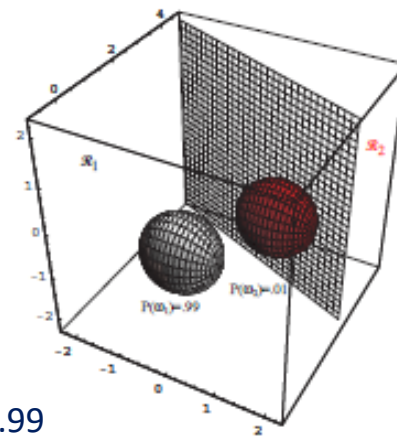
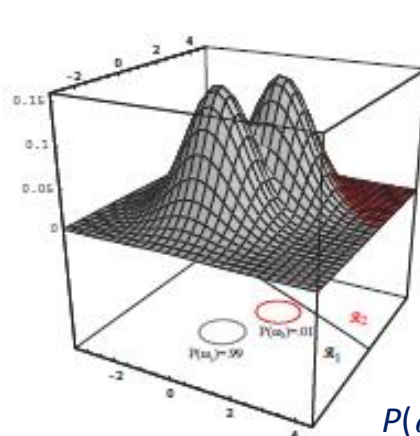
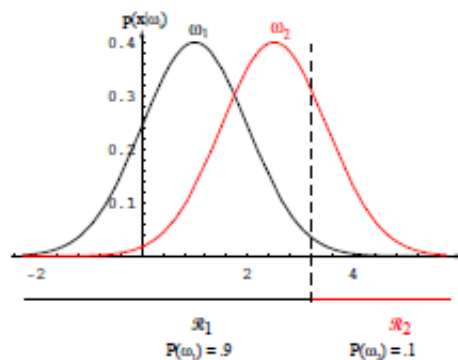
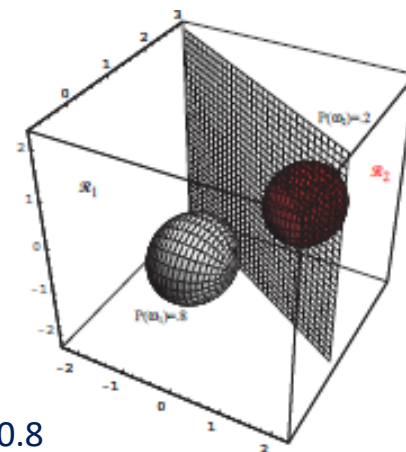
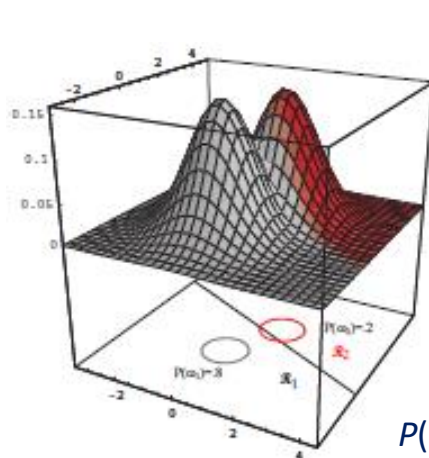
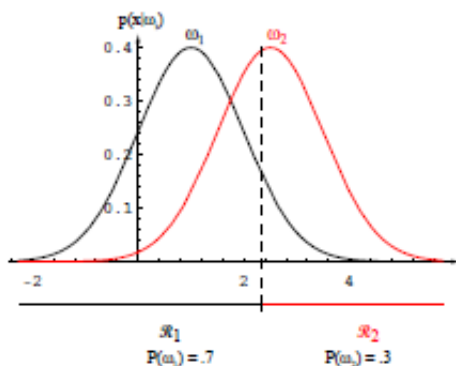
- 1D, 2D, 3D的情况

- 当 $P(\omega_1) = P(\omega_2)$ ，决策面为二类均值的等分面



— 当先验概率变化，决策面发生平移

移向概率
小的类别



- Case 2: $\Sigma_i = \Sigma$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x} - \mu_i)^t \Sigma_i^{-1}(\mathbf{x} - \mu_i) - \frac{d}{2} \ln 2\pi - \frac{1}{2} \ln |\Sigma_i| + \ln P(\omega_i)$$

$$\Rightarrow g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x} - \mu_i)^t \Sigma^{-1}(\mathbf{x} - \mu_i) + \ln P(\omega_i)$$

– 展开二次式 $(\mathbf{x} - \mu_i)^t \Sigma^{-1}(\mathbf{x} - \mu_i)$

线性判别函数! $g_i(\mathbf{x}) = \mathbf{w}_i^t \mathbf{x} + w_{i0}$

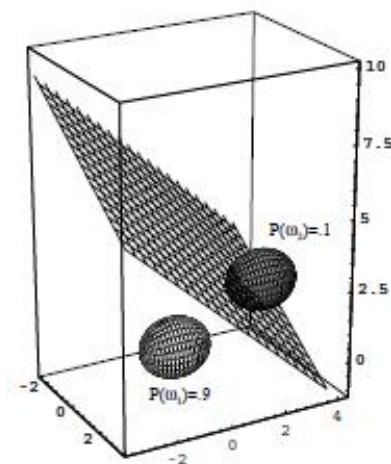
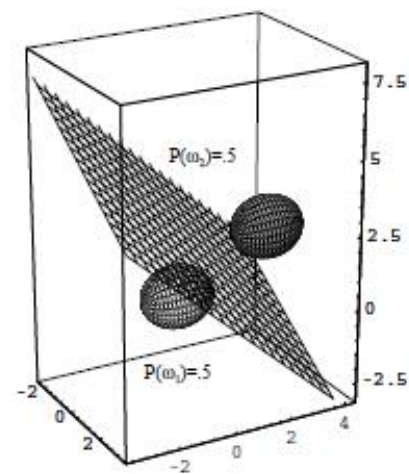
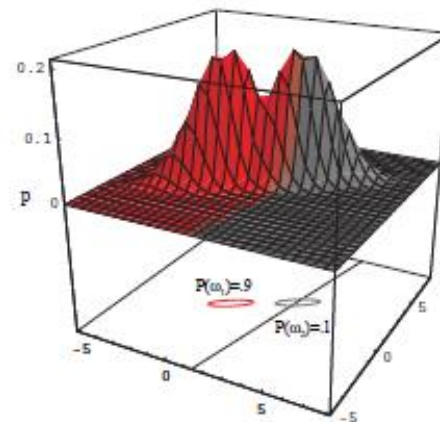
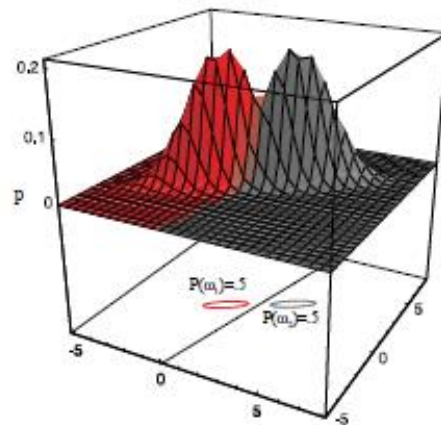
$$\mathbf{w}_i = \Sigma^{-1} \mu_i \quad w_{i0} = -\frac{1}{2} \mu_i^t \Sigma^{-1} \mu_i + \ln P(\omega_i)$$

– 二类决策面 $g_i(\mathbf{x}) = g_j(\mathbf{x})$

$$\Rightarrow \mathbf{w}^t (\mathbf{x} - \mathbf{x}_0) = 0 \quad \mathbf{w} = \Sigma^{-1}(\mu_i - \mu_j)$$

$$\mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\ln [P(\omega_i)/P(\omega_j)]}{(\mu_i - \mu_j)^t \Sigma^{-1}(\mu_i - \mu_j)} (\mu_i - \mu_j)$$

- 注意跟 $\mu_1 - \mu_2$ 的关系，决策面不一定与之垂直
- 当 $P(\omega_1) = P(\omega_2)$ ，决策面经过 $(\mu_1 + \mu_2)/2$



$$P(\omega_1) = P(\omega_2)$$

$$P(\omega_1) \neq P(\omega_2)$$

- Case 3: $\Sigma_i =$ arbitrary

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1} (\mathbf{x} - \boldsymbol{\mu}_i) - \frac{d}{2} \ln 2\pi - \frac{1}{2} \ln |\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

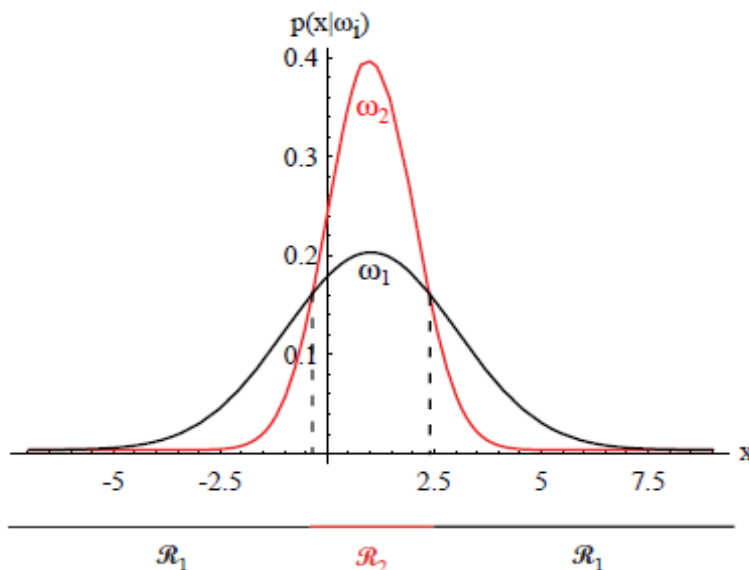
$$g_i(\mathbf{x}) = \mathbf{x}^t \mathbf{W}_i \mathbf{x} + \mathbf{w}_i^t \mathbf{x} + w_{i0}$$

$$\mathbf{W}_i = -\frac{1}{2} \boldsymbol{\Sigma}_i^{-1} \quad \mathbf{w}_i = \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i$$

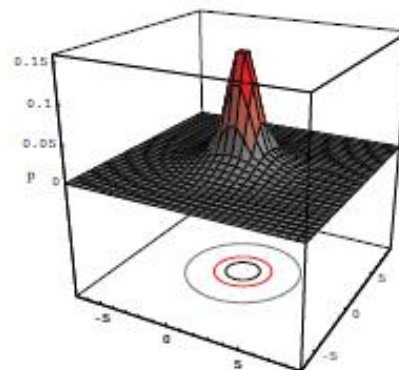
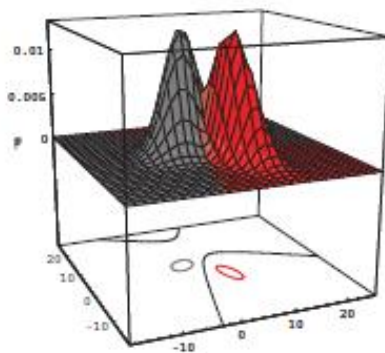
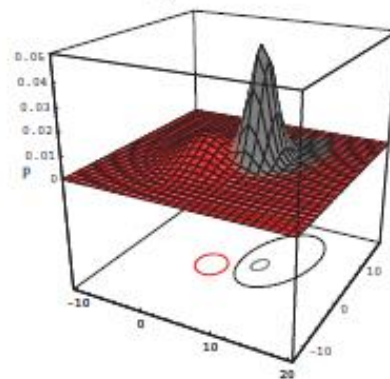
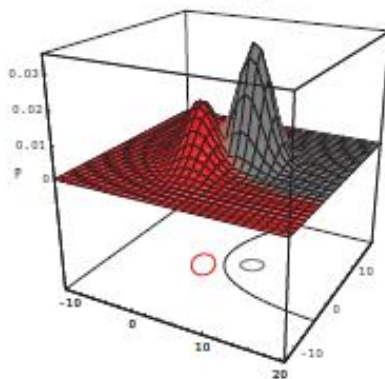
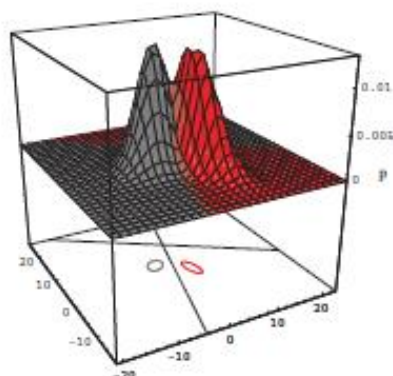
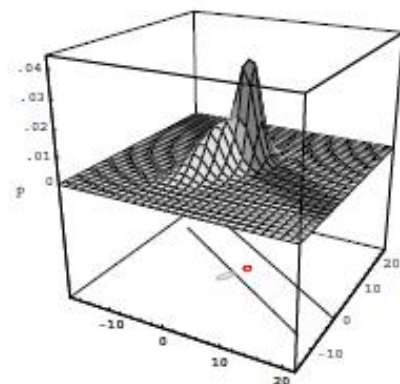
$$w_{i0} = -\frac{1}{2} \boldsymbol{\mu}_i^t \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i - \frac{1}{2} \ln |\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

– 二类决策面: $g_1(\mathbf{x}) = g_2(\mathbf{x})$, hyperquadrics

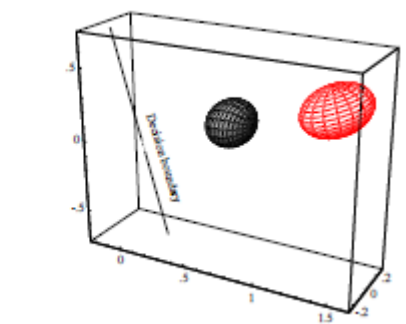
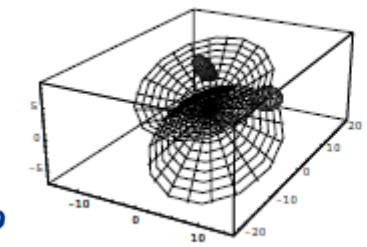
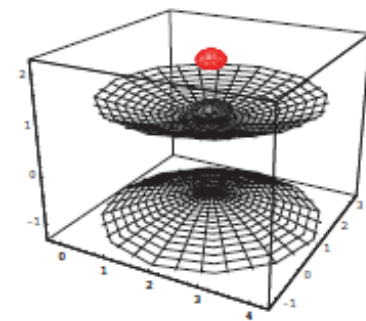
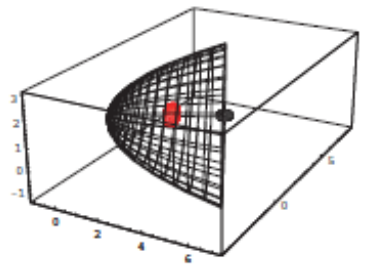
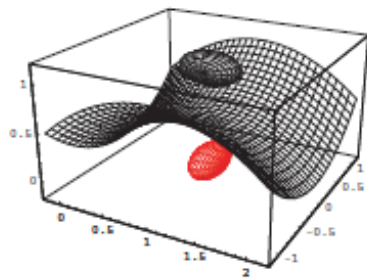
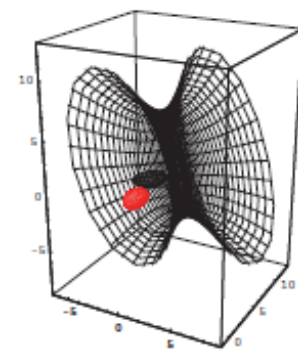
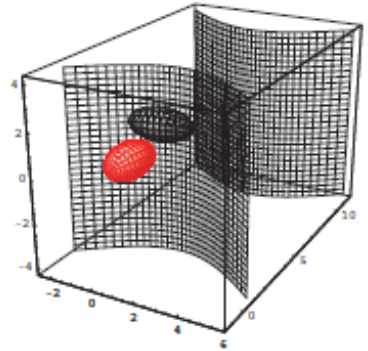
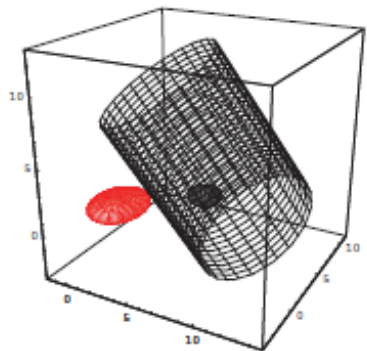
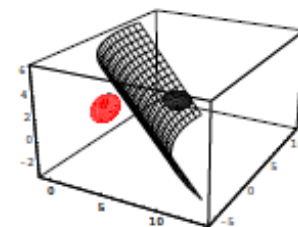
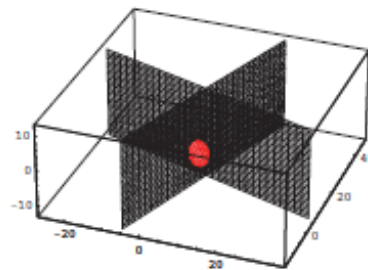
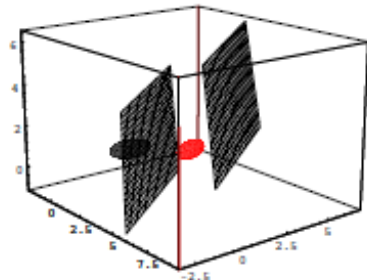
- 等均值的情况下, 1D的例子



2D的例子 (z轴是概率密度)



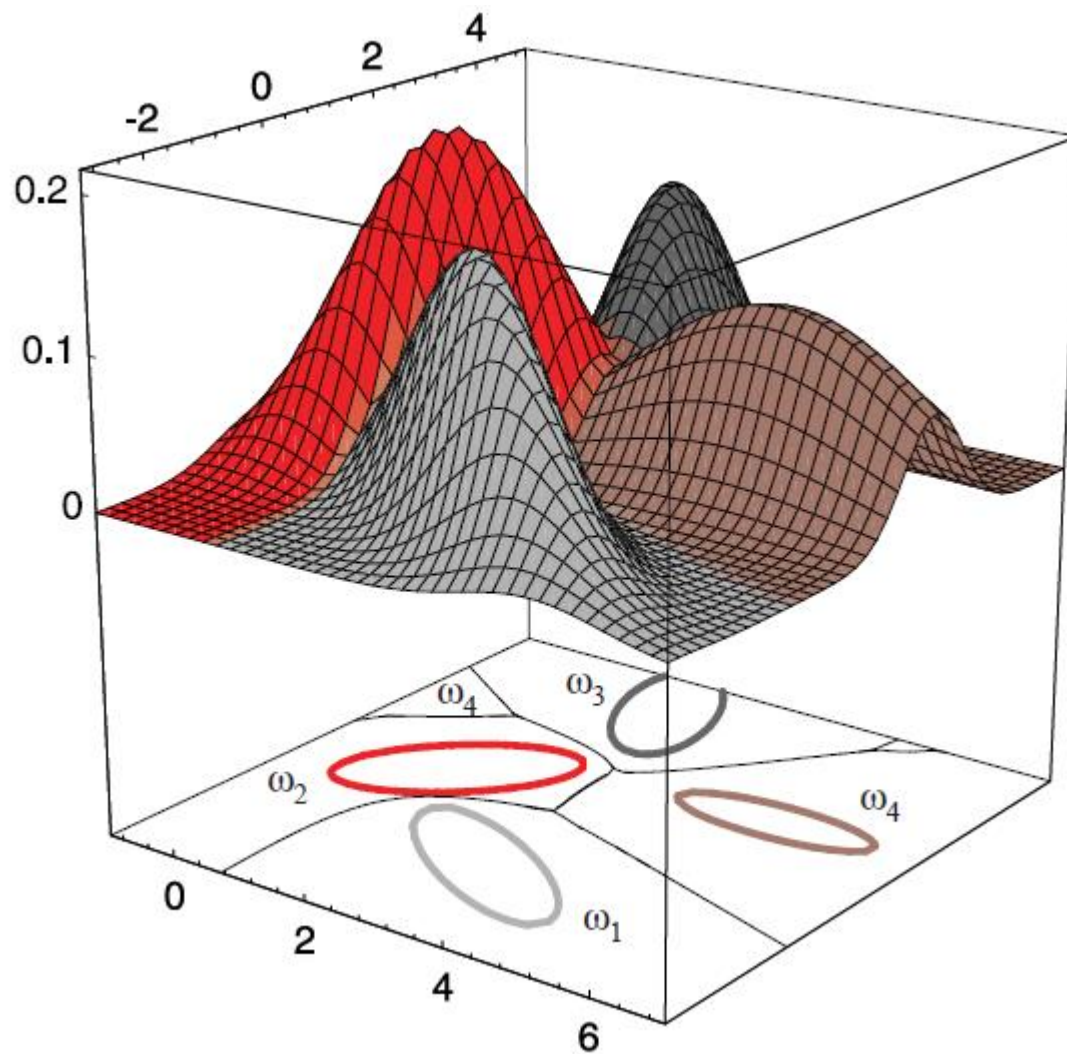
3D的例子



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2D, 4类的例子



- 一个具体例子

- 2类, 2D $P(\omega_1) = P(\omega_2) = 0.5$

$$\mu_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \quad \Sigma_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

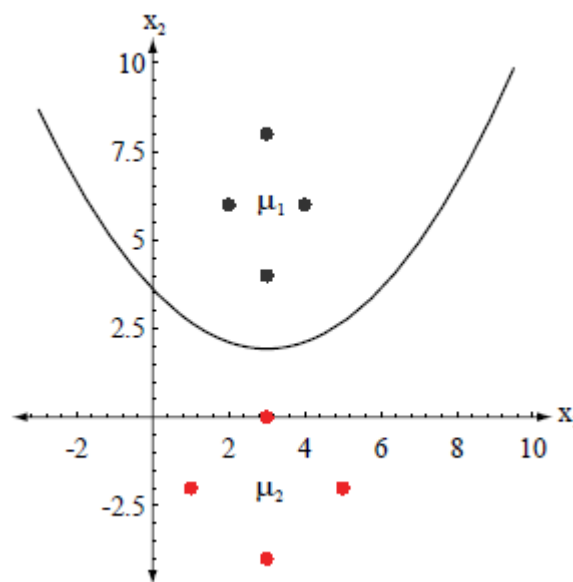
$$\Sigma_1^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\mu_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \quad \Sigma_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Sigma_2^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

- 决策面 $g_1(\mathbf{x}) = g_2(\mathbf{x})$

$$x_2 = 3.514 - 1.125x_1 + 0.1875x_1^2$$



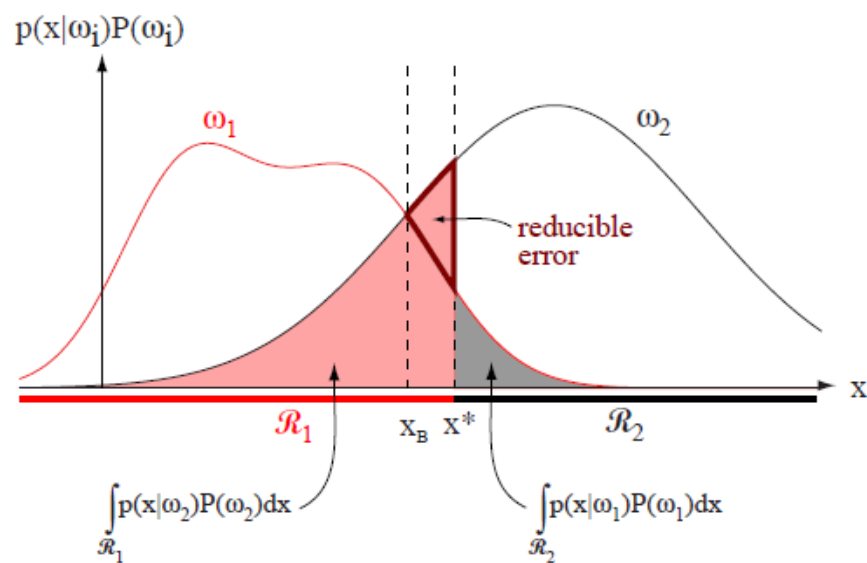
分类错误率

- 2类的情況

$$\begin{aligned}P(error) &= P(x \in \mathcal{R}_2, \omega_1) + P(x \in \mathcal{R}_1, \omega_2) \\&= P(x \in \mathcal{R}_2 | \omega_1)P(\omega_1) + P(x \in \mathcal{R}_1 | \omega_2)P(\omega_2) \\&= \int_{\mathcal{R}_2} p(x|\omega_1)P(\omega_1) dx + \int_{\mathcal{R}_1} p(x|\omega_2)P(\omega_2) dx.\end{aligned}$$

- 一般情况

$$\begin{aligned}P(correct) &= \sum_{i=1}^c P(x \in \mathcal{R}_i, \omega_i) \\&= \sum_{i=1}^c P(x \in \mathcal{R}_i | \omega_i)P(\omega_i) \\&= \sum_{i=1}^c \int_{\mathcal{R}_i} p(x|\omega_i)P(\omega_i) dx\end{aligned}$$



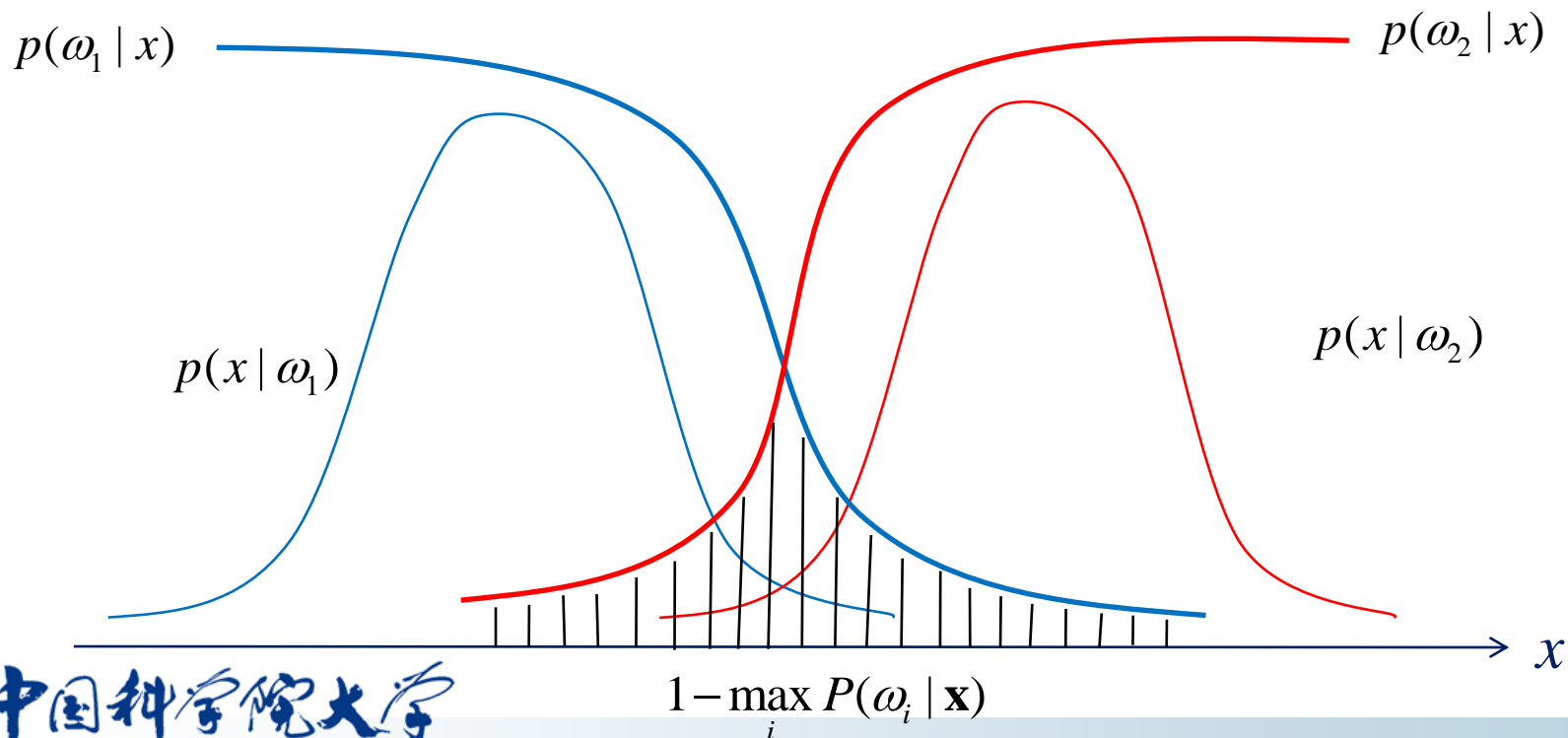
决策面为 x_B 时为最小错误率分类

- 最大后验概率决策(0-1 loss)的情况

$$P(\text{correct}) = \int_{\mathbf{x}} \max_i P(\mathbf{x} | \omega_i) P(\omega_i) d\mathbf{x}$$

$$= \int_{\mathbf{x}} \max_i P(\omega_i | \mathbf{x}) P(\mathbf{x}) d\mathbf{x}$$

$$P(\text{error}) = \int_{\mathbf{x}} \left[1 - \max_i P(\omega_i | \mathbf{x}) \right] P(\mathbf{x}) d\mathbf{x}$$



讨论

- 贝叶斯分类器(基于贝叶斯决策的分类器)是最优的吗？
 - 最小风险、最大后验概率决策
 - 最优的条件：概率密度、风险能准确估计
 - 具体的参数法、非参数法是贝叶斯分类器的近似，实际中难以达到最优
- 判别模型：回避了概率密度估计，以较小复杂度估计后验概率或判别函数
- 什么方法能胜过贝叶斯分类器：在不同的特征空间才有可能！

下次课内容

- 第2章
 - 离散变量的贝叶斯决策
 - 复合模式分类
- 第3章
 - 最大似然参数估计
 - 贝叶斯估计