Practical Optimization Algorithms and Applications

Chapter I: Introduction

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General Information about the Course

- Prerequisites:
  - Some knowledge of linear algebra;
  - The standard sequence of calculus courses;
- Books:
  - J. Nocedal and S. Wright. *Numerical Optimization* SECOND EDITION. Springer, New York, USA., 2006.
  - W-Y. Sun and Y-X. Yuan. *Optimization Theory and Methods: Nonlinear Programming*. Springer, New York, USA., 2006.
- Additional Online Recourse:
  - `http://www.mcs.anl.gov/otc/Guide/`
  - `http://www.mcs.anl.gov/otc/Guide/SoftwareGuide`
  - `http://neos-server.org/neos/`

What is Optimization?

Optimization is the minimization or maximization of a function subject to constraints on its variables.

- It traces the roots to the calculus of variations and the work of Euler and Lagrange.
- It is often called **mathematical programming**, a somewhat confusing term coined in the 1940s, before the word "programming" became inextricably linked with computer software.

Notation

We will use the following notation in this course:

- x is the vector of **variables**, also called *unknowns* or *parameters*;
- f is the **objective function**, a (scalar) function of x that we want to maximize or minimize;
- c_i are **constraint** functions, which are scalar functions of x that define certain equations and inequations that the unknown vector x must satisfy.

Mathematics Formulation

Using the notations, the optimization problem can be written as the following **Standardized Formulation**:

$$
\begin{aligned}
\min \quad & f(x) && && \text{(1a)}\\
\text{s.t.} \quad & c_i(x) = 0, && i \in \mathcal{E} = \{1, \dots, m_e\} && \text{(1b)}\\
& c_i(x) \geq 0, && i \in \mathcal{I} = \{m_e + 1, \dots, m\} && \text{(1c)}
\end{aligned}
$$

Here "s.t." means "subject to", $\mathcal{E}$ and $\mathcal{I}$ are the sets of indices for equality and inequality constraints, respectively.

- If $\mathcal{E} = \mathcal{I} = \emptyset$, (1) is called **unconstrained optimization**;
- If $\mathcal{E} \neq \emptyset$ or $\mathcal{I} \neq \emptyset$, (1) is called **constrained optimization**.

Example: A Transportation Problem

A chemical company has 2 factories F_1 and F_2 and a dozen retail outlets $R_1, R_2, \cdots, R_{12}$.

- Each factory F_i can produce a_i tons of a certain chemical product each week; a_i is called the *capacity* of the plant.
- Each retail outlet R_j has a known weekly *demand* of b_j tons of the product.
- The cost of shipping one tone of the product from factory F_i to retail outlet R_j is c_{ij}

The problem is to determine how much of the product to ship from each factory to each outlet so as to satisfy all the requirements and minimize cost.

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\min \quad & \sum_{i,j=1}^{i=2,j=12} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{12} x_{ij} \le a_i, \qquad i = 1, 2, \\
& \sum_{i=1}^{2} x_{ij} \ge b_j, \qquad j = 1, \cdots, 12, \\
& x_{ij} \ge 0, \qquad i = 1, 2, j = 1, \cdots, 12.
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Because the objective and constraints are all linear function. The above problem is called **linear programming**.

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Suppose there are p factories instead of 2, i.e. $F_1 \cdots F_p$, and each factory has q outlets $R_1, R_2, \cdots, R_q$, p and q are positive integer. Then the model becomes

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\begin{aligned}
\min \quad & \sum_{i,j=1}^{i=p,j=q} c_{ij}x_{ij} && \text{(2a)}\\
\text{s.t.} \quad & \sum_{j=1}^{q} x_{ij} \le a_i, \qquad i = 1, \cdots, p, && \text{(2b)}\\
& \sum_{i=1}^{p} x_{ij} \ge b_j, \qquad j = 1, \cdots, q, && \text{(2c)}\\
& x_{ij} \ge 0, \qquad i = 1, \cdots, p, j = 1, \cdots, q. && \text{(2d)}
\end{aligned}
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Example: A Transportation Problem

Suppose there are volume discounts for shipping the product. For example the cost (2c) could be represented by $c_{ij}\sqrt{\delta + x_{ij}}$, where $\delta > 0$ is a small subscription fee. Then the model can be formulaes as

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Because the objective is described as a nonlinear function, the above problem is called a **nonlinear programming**.

Thanks for your attention!