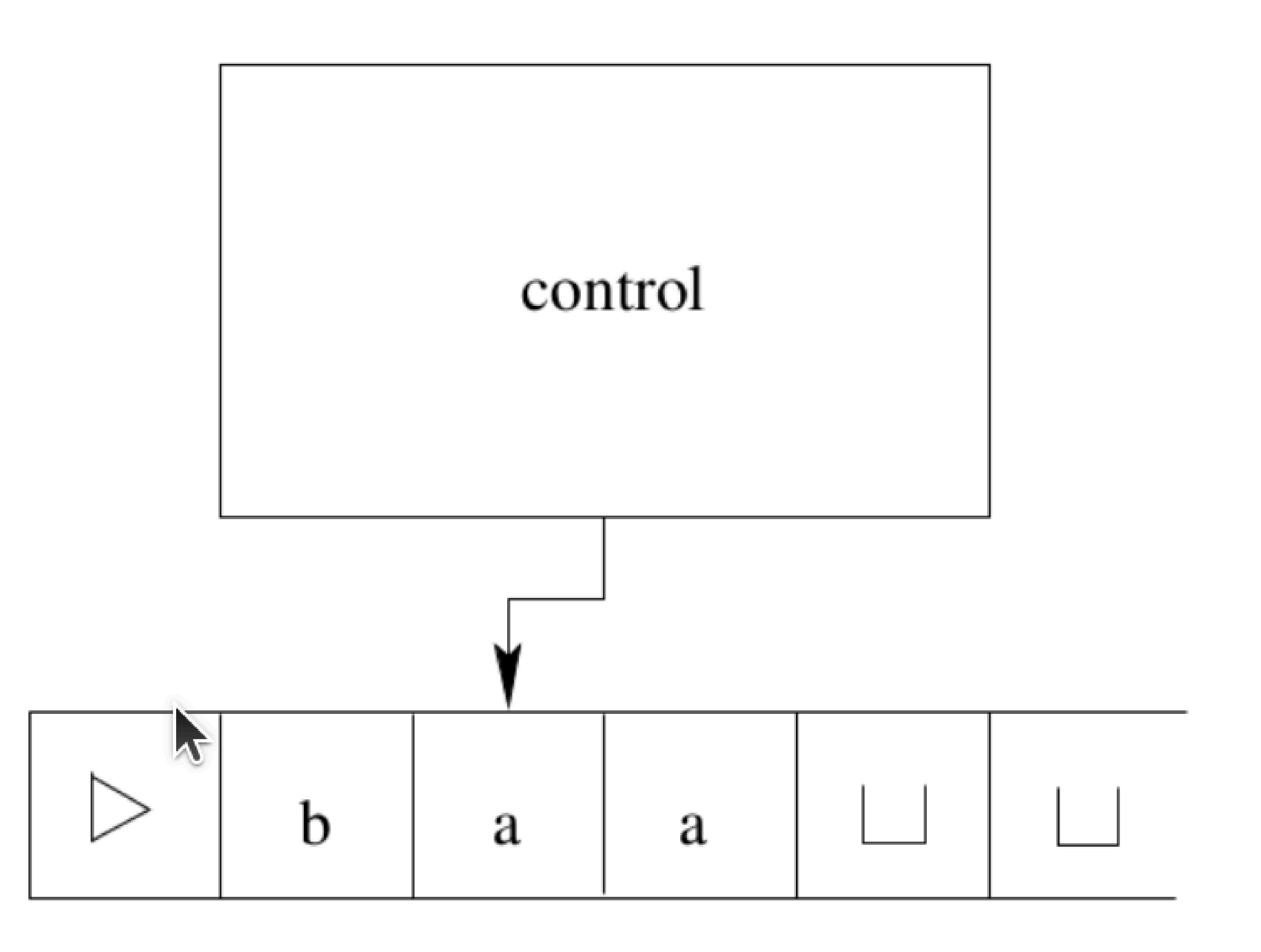
The Halting Problem

Will this ever end?

Objectives

- Give a mathematical model of an algorithm and a computer
- Present the Church-Turing Thesis.
- Understand the halting problem and find out whether it's possible to write a program that reasons about other programs

Turing Machines


control
▷
b
a
a
⊔
⊔

Components of a Turing Machine

- **Tape:** consists of an infinite number of cells, each of which could be empty or contains one of a fixed set of letters -- the "memory".
- **Working head:** reads and writes to the tape, and can be moved along the tape to access any cell.
- **Control module:** defines a set of instructions for the machine -- the program.

How it works

- The machine may be in one of a fixed set of *states*.
- Before it starts, we assume that the *input* is written on the tape.
- It begins in a given *initial state*, with the working head on the leftmost cell of the tape (at the $\triangleright$).
- The control module contains a set of instructions which determine what to do when the machine is in a given state and has just read a given letter.

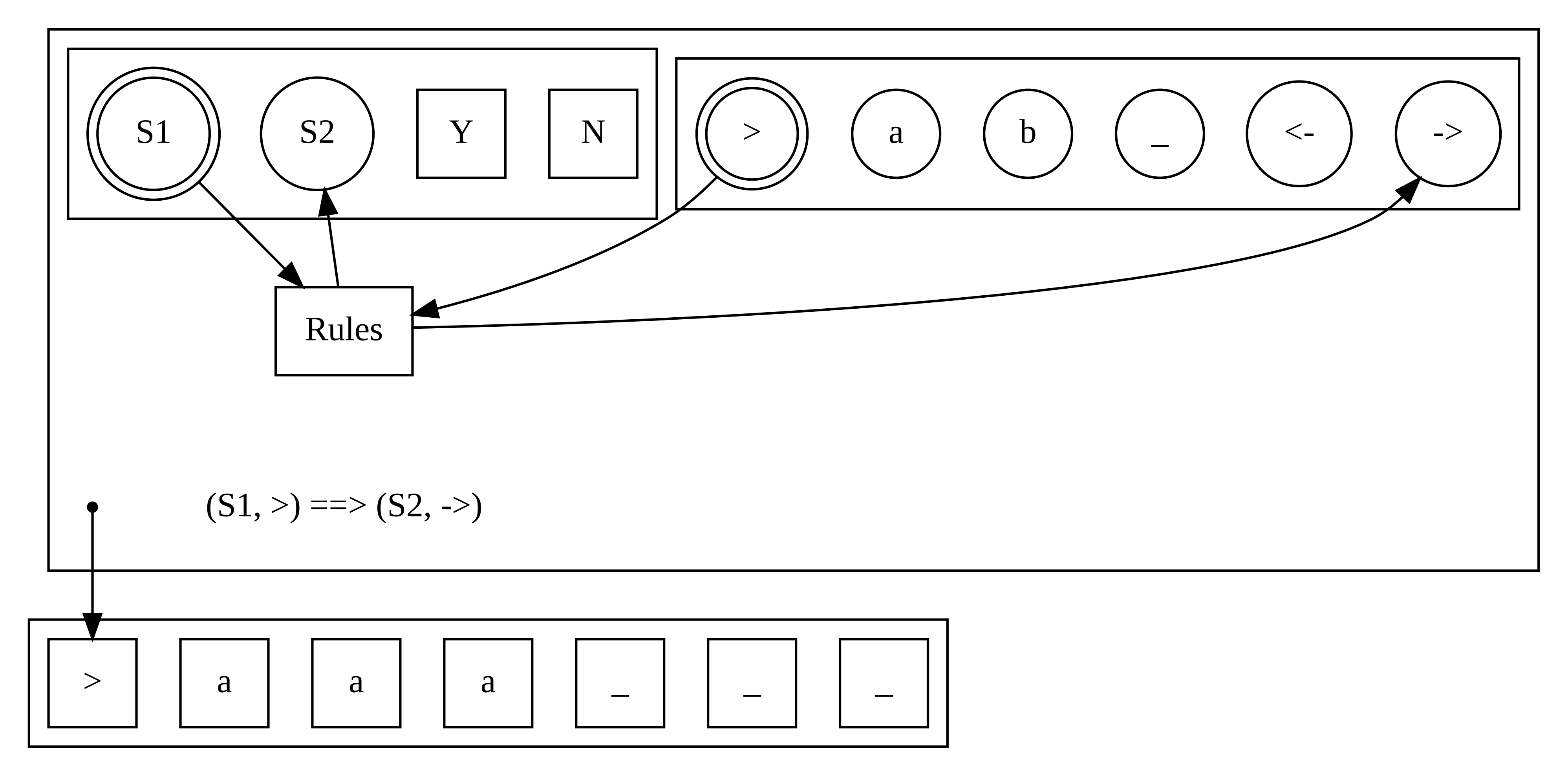
How it works

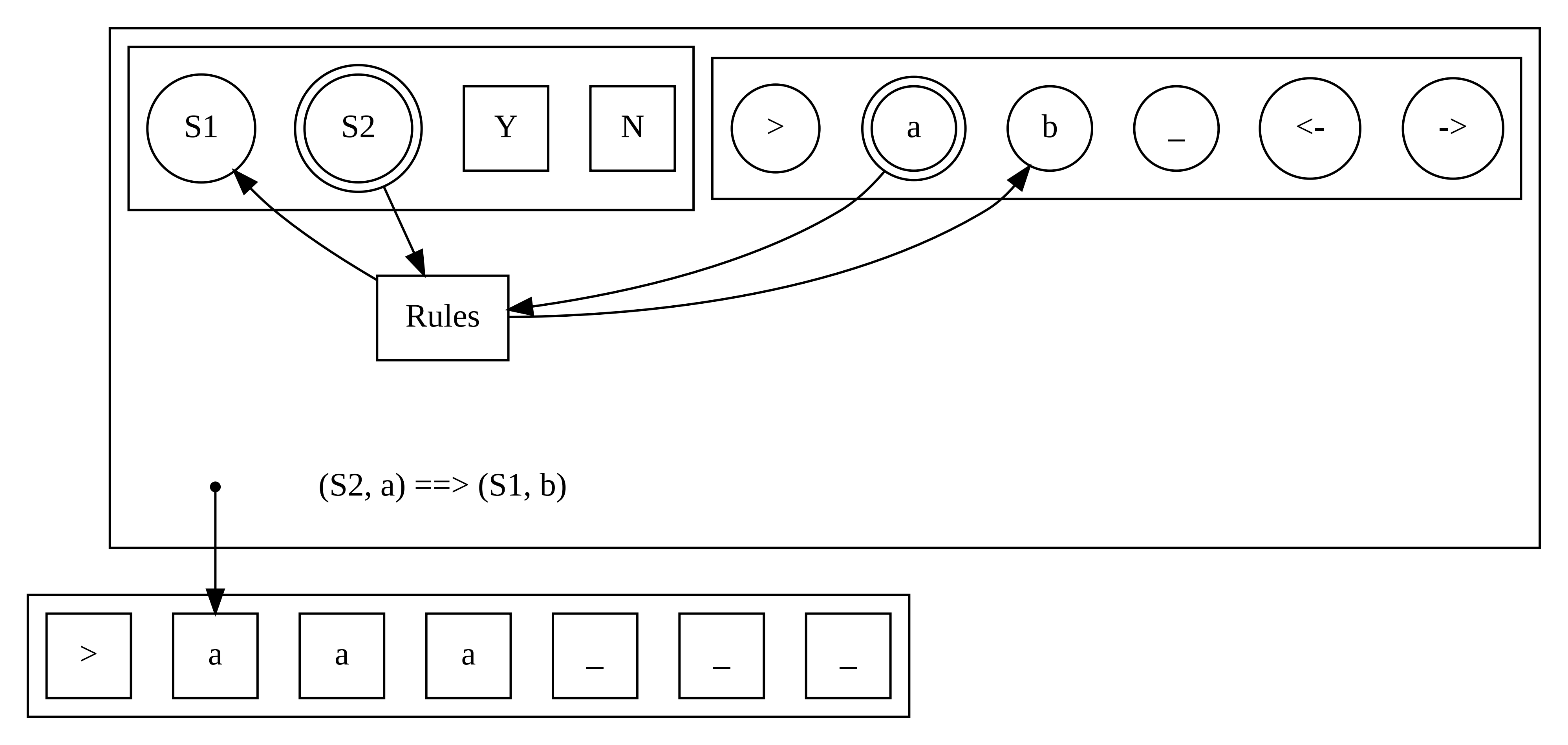
Each instruction does the following thing:

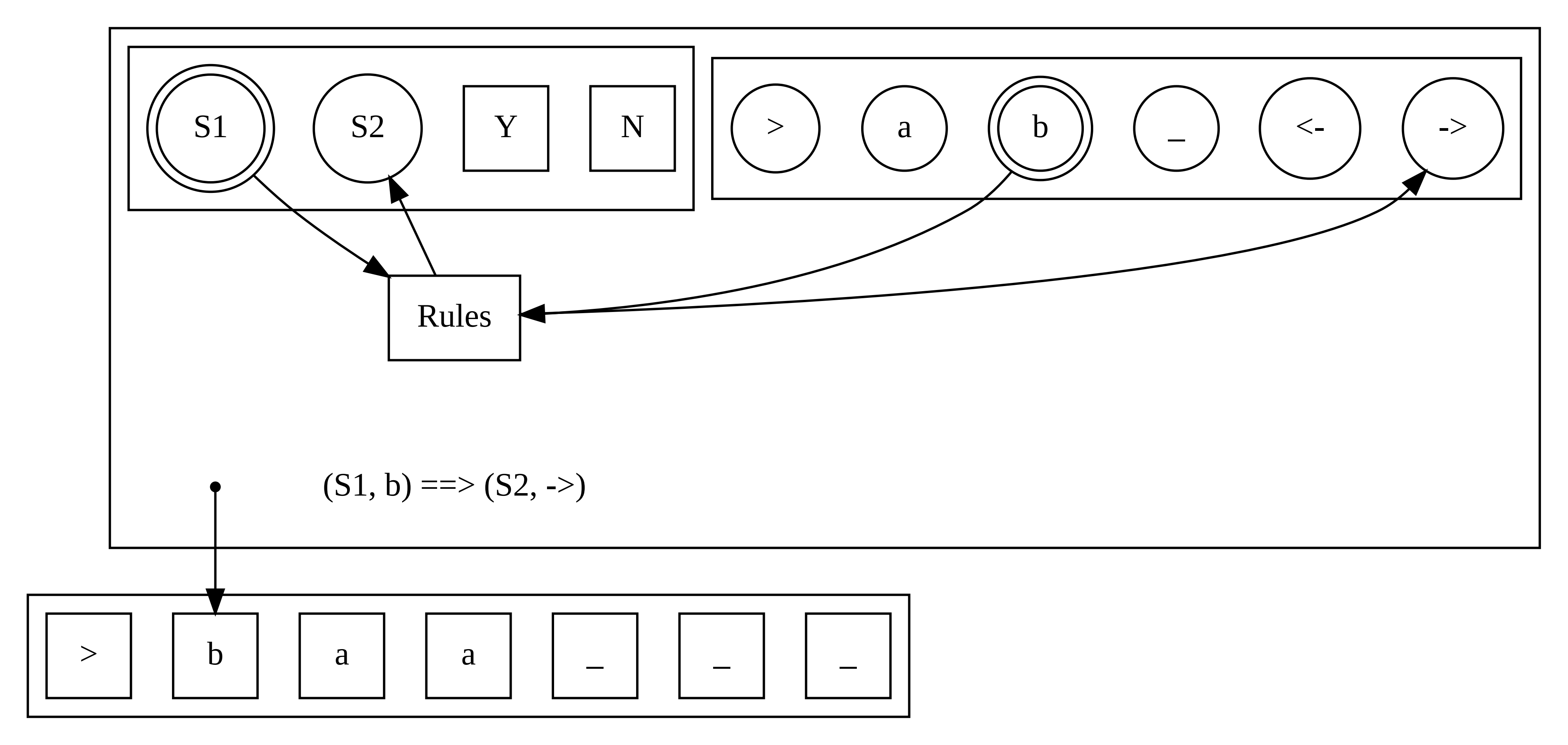
- Change state,

and

- write a letter on the current cell of the tape, or
- move working head left $\leftarrow$, or
- move working head right $\rightarrow$.

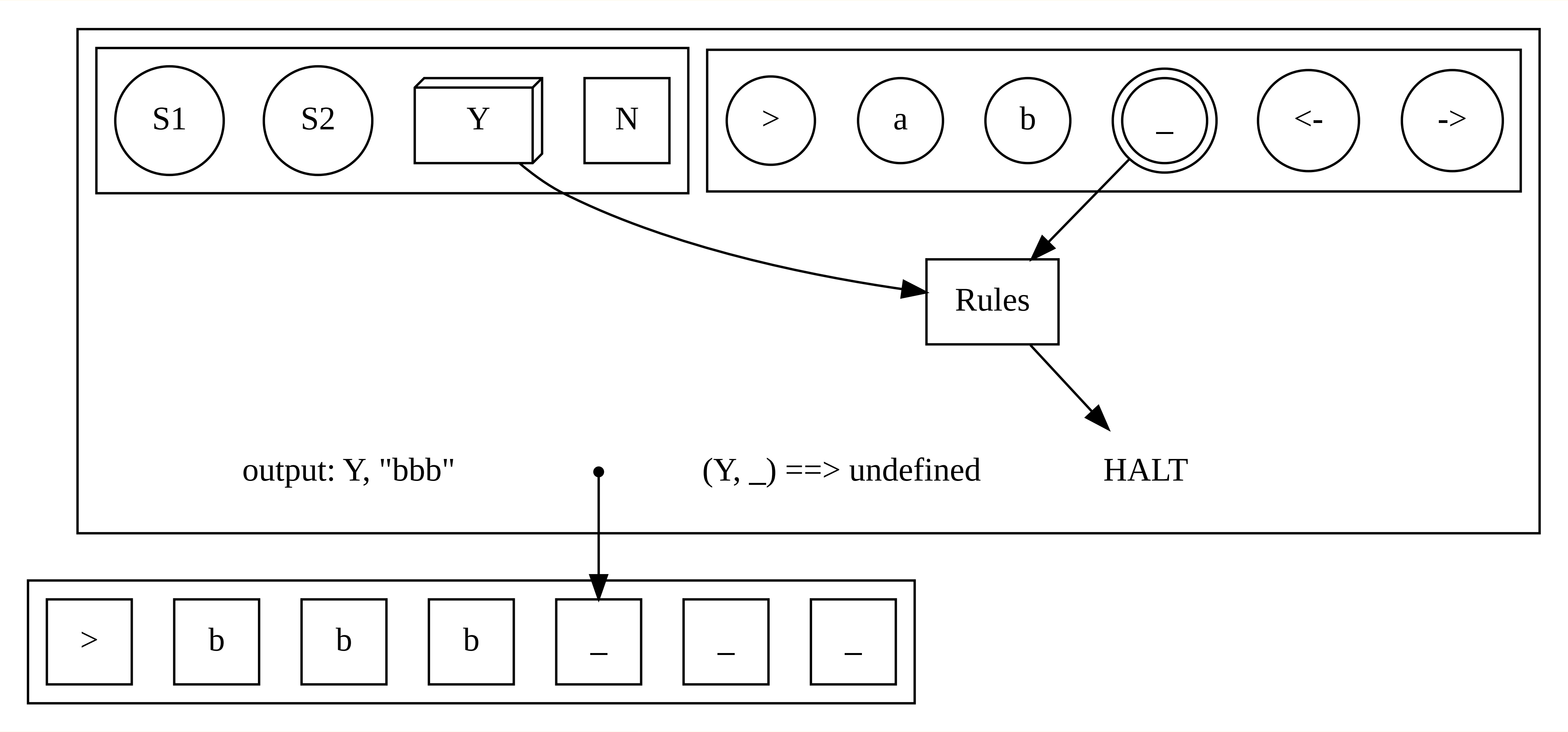

S1
S2
Y
N
>
a
b
_
<-
->
Rules
(S1, >) ==> (S2, ->)
>
a
a
a
_
_
_


S1
S2
Y
N
>
a
b
_
<-
->
Rules
(S2, a) ==> (S1, b)
>
a
a
a
_
_
_


S1
S2
Y
N
>
a
b
_
<-
->
Rules
(S1, b) ==> (S2, ->)
>
b
a
a
_
_
_

Stopping and output

- The set of states contains a number of *halting states*.
- When the machine is in one of these *halting states*, it stops running.
- Its "output" is the state it stopped in and the letters written on the tape when it stopped.


S1
S2
Y
N
>
a
b
_
<-
->
Rules
output: Y, "bbb"
(Y, _) ==> undefined
HALT
>
b
b
b
_
_
_

Programming with a Turing Machine

The set of instructions which determine what the turing machine will do is called the transition function.

For example:

- If I read an a and I'm in state S_2, then write a b and move to state S_1,
- If I read a b and I'm in state S_1, then move to state S_2 and move right.

Partial function

Mathematically, the transition function is a partial function. I.e.,

- Given some letter x and some (non-halting) state Y, the transition function defines exactly one action.
- Actions for every combination of letter and non-halting state are defined.

Example

Even or odd length

Given a word of all a's, decide whether its length is even or odd.

The word is written on the tape as follows

$$\triangleright \mid a \mid a \mid \cdots \mid a \mid \sqcup \mid \sqcup \mid \cdots$$

and the working head is initially at the start of the tape (reading $\triangleright$).

Pseudo-code

- Read word one letter at a time.
- Flip-flop between two states: one for even one for odd.
- When every letter in the word is read, halt with either **YES** or **NO**.

Coding the transition function

Turing machines are very low-level. We don't have nice constructs like loops and conditionals.

Instead we'll have to code it up using states to represent odd and even length, and read the word by moving right.

Defining the machine

- States: $\{E, O, Y, N\}$,
- Halting states: $\{Y, N\}$ (Y for yes it's even, N for no it's not)
- Initial state: E.

Defining the transition function

State	Letter	New State	Operation
O	$\triangleright$	E	$\rightarrow$
E	$\triangleright$	E	$\rightarrow$
O	a	E	$\rightarrow$
E	a	O	$\rightarrow$
O	$\sqcup$	N	$\sqcup$
E	$\sqcup$	Y	$\sqcup$

Exercises

Build Turing Machines to solve the following problems

1. - **Input:** a string of only a's
   - **Output:** deletes every letter and HALT.
2. - **Input**: A string which may contain a's, b's, or a mixture of both.
   - **Output:** YES if the input string contains only a's, NO otherwise.
3. - *How might you modify the TM you just built to output YES if the input consists of only a's or only b's?*
4. - **Input**: A string which may contain a's, b's, or a mixture of both.
   - **Output:** YES if the string contains the same number of a's and b's, NO otherwise.

The Universal Turing Machine

The Universal Turing Machine

A special Turing machine which can run other Turing Machines on a given input.

Somewhat like an interpreter, the Universal Turing Machine U takes as input

- M: an encoding of a Turing Machine (source code)
- w: the input to run M on.

Then:

- U writes the instructions for the machine M on its tape, followed by the input w.
- It then runs the instructions for M on the input w and returns its output (via states and the tape).

Turing Completeness

A programming language is called **turing complete** if it can be used to build the Universal Turing Machine.

Not all computer languages are Turing complete.

For example

Turing Complete

- Most general-purpose programming languages.
- E.g., C / C++, Java Python.
- More surprisingly, Emacs (lisp), Vim and even, one could argue, LaTex!

Not Turing Complete

- Many markup languages.
- E.g., HTML, CSS, Markdown.
- Also some querying languages, e.g., ANSI SQL

Brainf*ck

- An esoteric programming language.
- It *is* turing complete...
- Although there are only 8 commands!
- See https://en.wikipedia.org/wiki/Brainfuck

Hardcore Exercise

- In a (Turing complete) language of your choice, build a Brainf*ck interpreter!
- Extension: build a C++ compiler in Brainf*ck!

The Church-Turing Thesis

Statement

One useful way of stating the church-Turing Thesis is:

> “ **Any algorithm can be implemented using a Turing Machine.** ”

Evidence in favour

- Every attempt to program algorithms on Turing machines worked so far
- All versions of the Turing machine have been proved to be equivalent
- All other models of algorithm have been proved to be equivalent to Turing Machines (see, e.g., the lambda calculus)

The Halting Problem

Please tell me whether it's going to end!

Infinite loops

Some programs run forever.

For example:

```
let num = 0;

while (true) {
    num = num + 1;
}
```

It's easy [for a human] to see that this won't halt.

Semi-decidable sets

- If we code this up as a Turing Machine (Church-Turing says we can), it will never enter a halting state.
- Some Turing Machines halt and give us an answer, while others will run forever!
- The set of all Turing Machines is called *semi-decidable*.

Exercise

1. Design a Turing Machine which never halts.

Exercise

1. Design a Turing Machine which never halts.
2. What if we're only allowed to use finitely many cells of the tape?

A program and its input

Does this program always halt?

```
let n = user_input();
let fac = 1;

while (n != 0) {
    fac = fac * n;

    n = n - 1;
}

return fac;
```

The Halting Problem

Input

- M: a Turing machine (e.g., some program code)
- w: an input for M.

Output:

- **YES** if M halts on w, **NO** otherwise.

Is it possible to write a program to solve this problem?

Russell's Paradox and Self-Reference

Computers are stupid

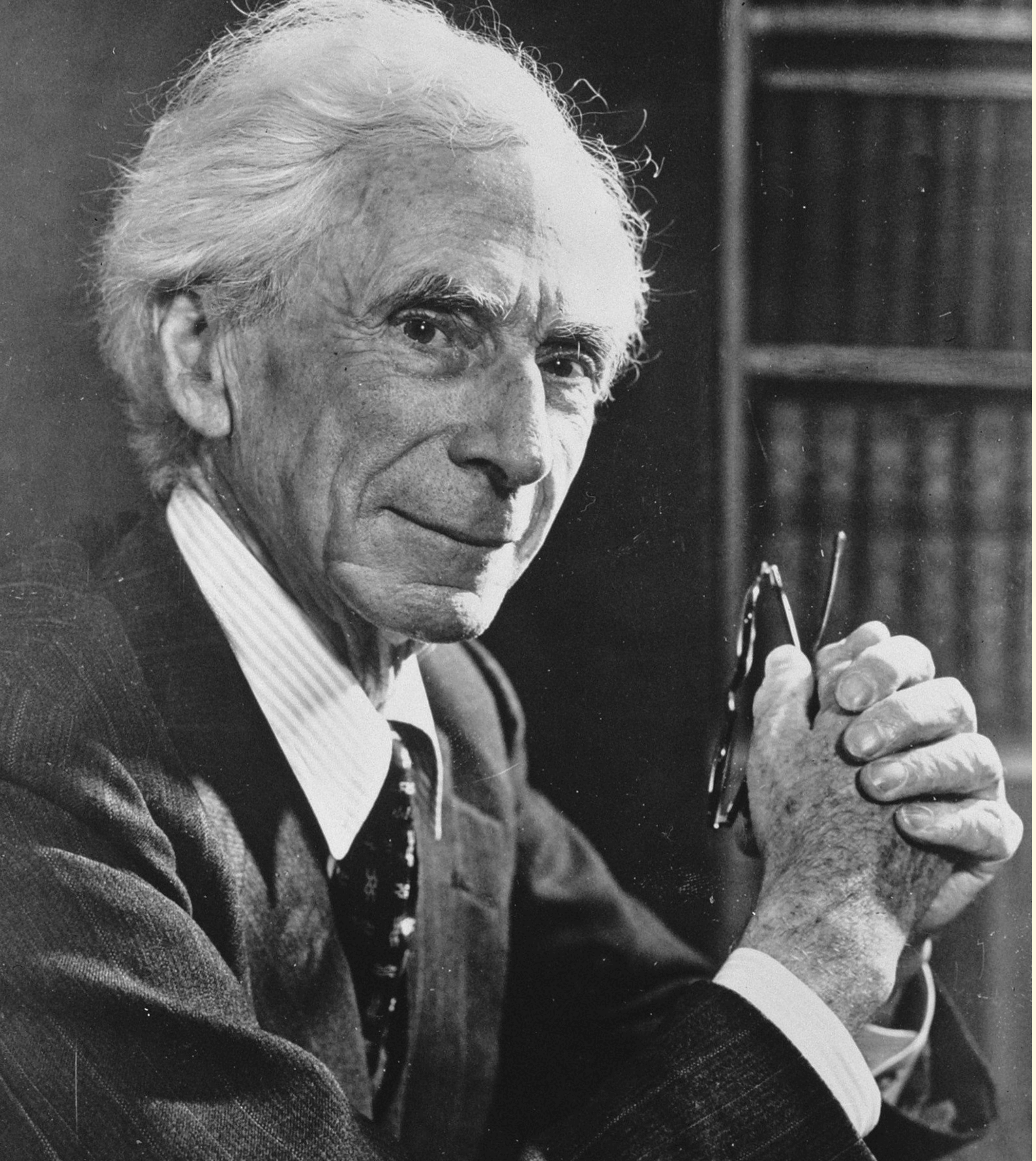
Computers can perform a sequence of specific instructions with speed and accuracy.

But they don't "understand" what they are doing.

Self-reference

In order to determine whether our input program will halt, the computer will neeed to understand what the program does.

In other words, it would need to understand itself!



“ **Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.** -- Bertrend Russell ”

Quick aside on set theory

- A set is a "bag of things".
- The things could be anything -- numbers; people; zombies, ghosts and skeletons; even sets.
- Sets may be finite, e.g., the set of all people in the world,
- Infinite but countable, e.g., the set of natural (counting) numbers,
- or infinite and uncountable, e.g., the real (decimal) numbers.

Defining Sets

We can define a set using properties of its elements.

E.g.,

- $\{p \in P \mid \text{Person } p \text{ likes dogs.}\}$
- $\{x \in \mathbb{Z} \mid x \text{ is even}\}$

Sets of sets

We can also define sets of sets in this way. Let $\mathcal{U}$ be the set of all possible sets, then

$$\{ A \in \mathcal{U} \mid A \text{ has exactly two elements } \}.$$

Question: is this set finite or infinite?

The Russell Set

Now let's define the Russell Set

$$R := \{A \in \mathcal{U} \mid A \notin A\}.$$

In words

> “ The set of all sets that do not contain themselves. ”

Question: is R a member of itself?

Russell's Paradox

Yes!

- Suppose that $R \in R$,
- By the definition, R cannot contain itself,
- So $R \notin R$.

Contradiction! An element cannot be both *in* and *not in* a set at the same time.

Russell's Paradox

Then the answer must be no!

- Suppose $R \notin R$.
- So far, so good, it satisfies the property $R \notin R$ from the definition.
- But that means $R \in R$, and we get another contradiction!

Paradox

- Either $R \in R$ or $R \notin R$,
- but both situations lead to a contradiction!

The Barber Paradox

There is a remote village where only one barber, a lovely fella called Bill, lives.

Bill has some strict morals about who he will shave.

Bill shaves every resident, and only those residents, who do not shave themselves.

Question: Does Bill shave himself?

Back to the Halting Problem

Using self-reference to explain why there can be no solution

Proof by contradiction

Assume we have a program `halts`.

Input:

- M: the source code of a program
- w: some input

Output: YES if M halts on w, **NO** if M does not halt on w.

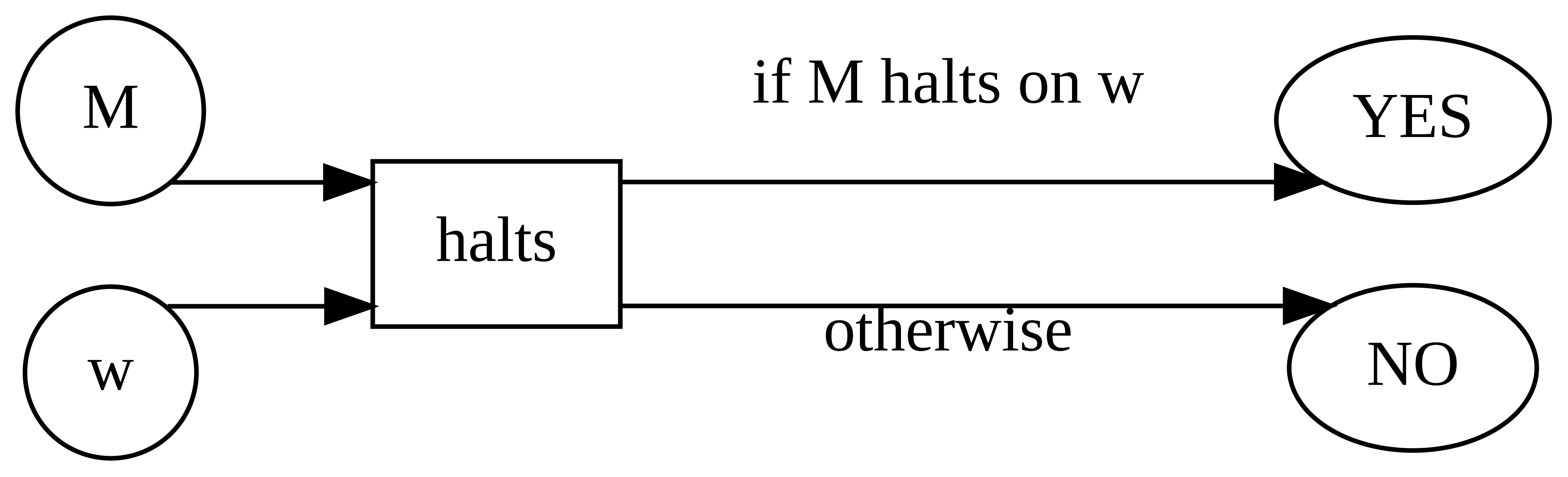


Diagram of `halts` machine.

I.e., assume we have solved the halting problem.

Self-reference

Now we define another program `contra` as follows:

Input:

- M: program source code

Output:

- Halt with **YES** if M **does not** halt on itself,
- Otherwise, **do not halt**

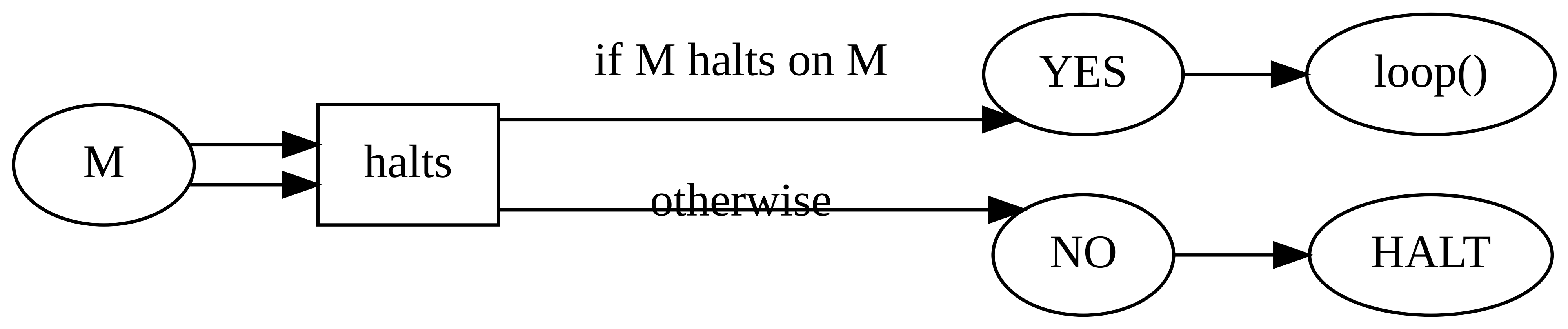


Diagram of `contra` machine, which assumes `halts` always returns YES or NO.

Self-reference

`contra` is a program, and it takes a program as input...

Run `contra` on itself!

Contradiction

- If `contra' halts on 'contra',
- Our 'halts' program returns YES.
- Then 'contra' enters the forever `loop()`.
- Thus `contra` **does not** halt on `contra`.

Contradiction

- If `contra` does not halt on `contra`,
- `halts` outputs NO, and the `contra` program halts.
- Another contradiction!

Consequence

Programming is hard!

It is not possible to automatically check if any program we write is correct

Large companies (Meta, Microsoft, Google etc) are developing tools to automatically check *some* types of programs, but it will never be possible to check every program.

In some safety-critical applications, restrictions are made on the types of programs that can be written, and these can be verified.

Thank-you

I hope you enjoyed it!

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