Saddle Surfaces

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1 Differential Geometry of Saddles

A saddle is a generalization of a surface of negative curvature. A surface of negative curvature is a two-dimensional surface in three-dimensional Euclidean space that has negative Gaussian Curvature $K<0$ at every point [3]. The concept of a surface of negative curvature can be generalized with respect to the dimension of the surface itself or the dimension and structure of the ambient space. A surface is called a saddle surface if it is impossible to cut off a crust by any plane. Examples of a saddle surface are a one-sheet hyperboloid, a hyperbolic paraboloid, and some ruled surfaces.

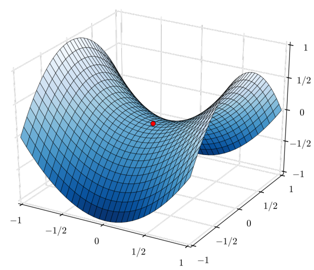

Figure 1: A saddle surface [1]

The basic parametrization of a saddle surface is $\vec{x}(u,v)=(u,v,uv)$ [2]. A saddle surface has geodesics that consist of the cross section in the middle of the surface passing through the saddle point where the surface lies on both sides of the tangent plane. These geodesics are intrinsically straight so that the geodesic curvature, $\vec{k}_g$, does not exist in the tangent plane. This also means that the curvature vector, $\vec{k}_\alpha$, and the normal curvature, $\vec{k}_n$, are displayed as the same in the tangent plane. This shows the projection of the curvature onto the normal.

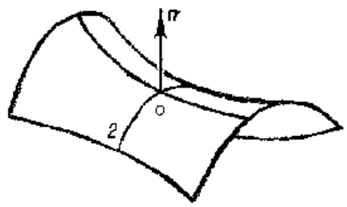

Figure 2: Geodesics on a saddle surface [4]

It makes sense that the geodesics must nicely bisect through the main saddle point at the origin because at a saddle point the surface is saddle shaped. This leads to the fact that the saddle point is the most interesting point on a saddle. This point is typically at the origin of the surface which would be $\vec{x}(u,v)$=(0,0,0). As said before, there is negative Gaussian curvature everywhere, which includes the saddle point. Gaussian curvature, K, is intrinsic and determines the deviance of a surface from being a plane at each point. This curvature for a saddle is of the form $K=\frac{-1}{(1+u^2+v^2)^2}$. This is because one of the extrinsic principal curvatures, $\vec{k}_1$, is negative and the other, $\vec{k}_2$, is positive. When they are multiplied they yield a negative Gaussian curvature. Gauss curvature can be found by using the formula $K=\frac{ln-m^2}{EG-F^2}$. How to find E, G, F, l, m, and n will be explained in the next paragraph. The main thing to know about having negative Gaussian curvature in relation to the saddle is that one piece of the tangent plane goes up while the other is directed down. Therefore, the surface lies on both sides of the tangent plane. On top of Gauss curvature, there is also the mean curvature, H, which is just what it sounds like. It is the mean or average of the two principal curvatures where $H=\frac{k_1+k_2}{2}=\frac{lG-2mF+nE}{2(EG-F^2)}$.

The first fundamental form is special to surfaces because it helps determine a surface up to rigid motion intrinsically rather than extrinsically. In other words it can measure on the surface without knowledge of the embedding. The first fundamental form is made up of $E = \vec{x}_u \cdot \vec{x}_u$, $F = \vec{x}_u \cdot \vec{x}_v$, and $G = \vec{x}_v \cdot \vec{x}_v$ which all help with finding length, angles, and area. However, the metric form focuses on only length. After obtaining E,F, and G from the saddle parametrization, the metric form comes out to be $(\frac{ds}{dt})^2=(v^2+1)(\frac{du}{dt})^2+(2uv)(\frac{du}{dt})(\frac{dv}{dt})+(u^2+1)(\frac{dv}{dt})^2$, where $E=(v^2+1)$, $F=uv$, and $G=(u^2+1)$. This metric form shows that the Pythagorean theorem does not hold because there is a nonzero F, and E, F do not equate to a coefficient of 1. The only way for F to be zero would be if u∩v=0. Then there would not be a middle term in the metric form because $2uv$ would cancel out, and E, F would be left as coefficients of 1. This cannot happen because then the parametrization would not be a saddle. The parametrization would be $\vec{x}(u,v)$=(0,0,0) which is just a point at the origin of the plane as stated above. The first fundamental form helps describe the surface area of the surface as well. This surface area is of the form $\int\int\sqrt{EF-F^2}dudv$. For a saddle this surface area integral would look like $\int\int\sqrt{(v^2+1)(u^2+1)-(uv)^2}dudv$. The second fundamental form helps describe the surface embedded in space as U changes where U=$\frac{\vec{x}_u\times\vec{x}_v}{|\vec{x}_u\times\vec{x}_v|}$. The second fundamental form includes coefficients of l=$\vec{x}_{uu}\cdot U$, m=$\vec{x}_{uv}\cdot U$, and n=$\vec{x}_{vv}\cdot U$.

The rate of change of the surface normal, U, is not a multiple of $\vec{x}_u$ or $\vec{x}_v$ individually but can be a multiple by combining the two. The reason individual multiples of $\vec{x}_u$ and $\vec{x}_v$ can not be obtained is because they both have zero components where $\vec{x}_u$=(1,0,v) and $\vec{x}_v$=(0,1,u). Therefore, there is an algebraic congruence but not an isometric one. So the covariant derivatives and shape operator, which is used to define a type of extrinsic curvature, would include:

$$\mathrm{S}(\vec{x}_u)=\mathrm{aU}_u+\mathrm{bU}_v,$$
$$\mathrm{S}(\vec{x}_v)=\mathrm{cU}_u+\mathrm{dU}_v$$

where a,b,c, and d are multiples.

When one thinks of a surface with negative curvature, usually a saddle is the first surface that comes to mind. This leads to other surfaces with negative curvature being isometric to a saddle in different ways. For example, Enneper's looks as if it might be the same surface as a saddle when viewed at certain parameters such as $u = -\frac{1}{2}..\frac{1}{2}$ and $v = -\frac{1}{2}..\frac{1}{2}$. However, after comparing their metric forms it becomes clear that they are not isometric. The first fundamental form of Enneper's surface is

$$I = \begin{bmatrix} (u^2+v^2+1)^2 & 0 \\ 0 & (u^2+v^2+1)^2 \end{bmatrix}$$

which, as you can see, is much different than the metric form of the saddle.

After being able to understand the differential geometry behind saddles, finding ways to relate the basic parametrization and properties of a 'normal' saddle to other types of saddles was made easier. Below is a classroom worksheet on a monkey saddle. After reading the differential geometry behind a saddle, one should be able to apply this knowledge to find the corresponding information of a similar saddle like the monkey saddle.

2 Monkey Saddle Worksheet

The parametrization of a monkey saddle is $\vec{x}$=(u,v,u^3-3uv^2) [5].

1. Use this parametrization to show that the coefficients of the first fundamental form are E = 1 + 9(u^2-v^2)^2, F = -18uv(u^2-v^2), and G = 1 + 36u^2v^2.

2. Use these coefficient values to show that the metric form is $(\frac{ds}{dt})^2$=[1 + (3u^2 - 3v^2)^2]$(\frac{du}{dt})^2$-2[18uv(u^2-v^2)]$(\frac{du}{dt})(\frac{dv}{dt})$+(1+36$u^2v^2$)$(\frac{dv}{dt})^2$. Does the Pythagorean theorem hold? (no)

3. Use the coefficients to set up (don't evaluate) the surface area element integral. The integral should look like $\int\int\sqrt{1+9(u^2+v^2)^2}dudv$ in its final form.

4. Show that the coefficients of the second fundamental form are l=$\frac{6u}{\sqrt{1+9(u^2+v^2)^2}}$, m=-$\frac{6v}{\sqrt{1+9(u^2+v^2)^2}}$, and n=-$\frac{6u}{1+9(u^2+v^2)^2}$.

5. Use all of these coefficient values to show that the mean curvature is H=$\frac{27u(-u^4+2u^2v^2+3v^4)}{[1+9(u^2+v^2)^2]^{3/2}}$ and the Gauss curvature is K=-$\frac{36(u^2+v^2)}{[1+9(u^2+v^2)^2]^2}$. Does this imply that every point on the monkey saddle has negative Gaussian curvature? (yes)

3 Acknowledgements and References

References

[1] Sokolov, D.D. Saddle surface. *Encyclopedia of Mathematics*. February 7, 2011. `http://www.encyclopediaofmath.org/index.php?title=Saddle_surface&oldid=17506`

This source gave me a more detailed description of what properties a saddle surface has. This includes how saddles have negative Gauss curvature and what that means. This source is also where I found the picture of a saddle that I included in this document labeled Figure 1.

[2] Oprea, J. Differential Geometry and Its Applications. *Mathematical Association of America*. 2007

This source is that textbook we used all throughout the semester. It gave me the parametrization of a saddle and exampled of saddle surfaces. It also gave me all the definition I used in the first half of tis project document. I was then able to apply these definitions to find out the values of each topic for both a basic saddle and a monkey saddle.

[3] A'Campo, N; Papadopoulos, A. On Klein So-called Non-Euclidean geometry. 10, 91-136, Jun. 27, 2015

This source gave me a greater insight on what Euclidean geometry is so I was able to talk about how a saddle is related to this in the beginning of this document.

[4] Efimov, N.V. Negative curvature, surface of. *Encyclopedia of Mathematics*. April 15, 2012. `https://www.encyclopediaofmath.org/index.php/Negative_curvature,_surface_of`

This source allowed me to apply the properties that a saddle surface has. The main thing this source talked about is what negative curvature is and what it means for a surface to have this at every point. It also described the geodesics on a saddle, and it provided me with the picture I included in this document labeled Figure 2.

[5] Weisstein, E. Monkey Saddle. *Wolfram MathWorld*. May 3, 2018. `mathworld.wolfram.com/MonkeySaddle.html`

This source gave me the parametrization of a monkey saddle. It also provided me with results to the questions I asked on the classroom worksheet that I made. This allowed me to be able to check my work to make sure it was right before I put it on the worksheet.

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