Definition of a Subspace: A subset S of $\mathbf{R^n}$ is a subspace if S satisfies the following three conditions:

(i) S contains $\mathbf{0}$, the zero vector.

(ii) If $\mathbf{u}, \mathbf{v}$ are in S, then $\mathbf{u} + \mathbf{v}$ is in S. This is known as *closure under addition.*

(iii) If $\mathbf{u}$ is in S, then $r\mathbf{u}$ is in S for every real number r. This is known as *closure under scalar multiplication.*

We can determine if this subset is a subspace by checking if it satisfies all three of these conditions.

We can also use **Theorem 4.2**, which states: Let $S = \text{span}(\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_m}\})$ be a subset of R^n. Then S is a subspace of R^n.

Proofs

63. Prove that if $\mathbf{b} \neq 0$, then the set of solutions to $A\mathbf{x} = \mathbf{b}$ is not a subspace.

Proof

We will prove this by assuming the contrapositive.
Assume that the set of solutions to $A\mathbf{x} = \mathbf{b}$ is in fact a subspace. If this is the case, then the set of solutions must contain the zero vector $\mathbf{0}$.
This also means that the zero vector $\mathbf{0}$ must satisfy the matrix equation $A\mathbf{x} = \mathbf{b}$, such that $A\mathbf{0} = \mathbf{x}$.
We know this isn't true, because $A\mathbf{0} = \mathbf{0}$, which means that b must be $\mathbf{0}$, but it is given that $b \neq \mathbf{0}$.
Thus, since the only way for the set of solutions to be a subspace is for it to contain the zero vector is if $b = \mathbf{0}$, which is not true, the set of solutions cannot be a subspace.

■

69. Let A be a matrix and $T(\mathbf{x}) = A\mathbf{x}$ a linear transformation. Show that $ker(T) = \{\mathbf{0}\}$ if and only if the columns of A are linearly independent.

Proof

The transformation $T(\mathbf{x}) = A\mathbf{x}$ is linear, so we know that the following conditions must be satisfied:

Definition of a Linear Transformation

a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

b) $T(c\mathbf{u}) = cT(\mathbf{u})$

Recall that the kernel of T, $\ker(T)$ is the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.
Also recall that in order for a set of vectors to be linearly independent, the only solution to the vector equation

$$\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_n} = \mathbf{0}$$

must be the trivial solution.
Since $\ker(T)$ is the set of vectors that satisfies $\mathrm{T}(\mathbf{x}) = \mathbf{0}$ and the matrix equation $A\mathbf{x} = 0$ corresponds with this matrix, the only vector in $\ker(T)$ must be the trivial solution.
So $\ker(T)$ must be equal to $\mathbf{0}$ - in other words, the trivial solution - if and only if the columns of A are linearly independent.

■

70. If T is a linear transformation, show that $\mathbf{0}$ is always in $\ker(T)$.

Proof

The transformation T is linear, so we know that the following conditions must be satisfied:

Definition of a Linear Transformation

a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

b) $T(c\mathbf{u}) = cT(\mathbf{u})$

Recall that the kernel of T, $\ker(T)$ is the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.
Since $T(c\mathbf{u}) = cT(\mathbf{u})$ by the definition of a linear transformation, we know that $T(0\mathbf{u}) = 0T(\mathbf{u})$ This must equal the zero vector $\mathbf{0}$.
Since $T(0) = \mathbf{0}$, the zero vector is part of the set of vectors where $T(\mathbf{x}) = \mathbf{0}$, so the zero vector $\mathbf{0}$ must be in $\ker(T)$.

■

71. Prove that if u and v are in a subspace S, then so is $\mathbf{u} - \mathbf{v}$.

Proof

If $\mathbf{u}$ and $\mathbf{v}$ are in the subspace S, this means that all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ are also in the subspace, since the definition of a subspace requires that the closure properties - closure under addition and closure under scalar multiplication - be satisfied.
Particularly, since S is closed under addition, $\mathbf{u} + -\mathbf{v} = \mathbf{u} + \mathbf{v}$ must be in S.
The subtraction of vectors can be understood as the addition of negative scalar multiples, which is satisfied by the closure properties.

■

72. Prove Theorem 4.6: If T is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Proof

Theorem 4.6
Given a linear transformation $T : \mathbf{R^m} \to \mathbf{R^n}$, T must be one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Prove that T being one-to-one implies that $\ker(T) = \{\mathbf{0}\}$.

Assume that T is one-to-one. This means that there is at most one solution to $T(\mathbf{x})$. We know this because of Theorem 3.5.

Theorem 3.5
Let T be a linear transformation. T is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Since T is a linear transformation, we know that $T(\mathbf{0}) = 0$ since $T(c\mathbf{0}) = cT(\mathbf{0}) = \mathbf{0}$. Thus, $\ker(T) = \{\mathbf{0}\}$

Prove that implies that $\ker(T) = \{\mathbf{0}\}$ implies that T is one-to-one.

Assume that $\ker(T)$ only contains the trivial solution in its set of vectors.
If $T(\mathbf{u}) = T(\mathbf{v})$, then $T(u) - T(v) = 0$. This implies that $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. We know this is true since T is a linear transformation, and $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$. Since $T(\mathbf{x}) = \mathbf{0}$) only has the trivial solution, if $\ker(T) = \{\mathbf{0}\}$, it follows that $\mathbf{u} - \mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{v}$. Thus, T is one-to-one.

$\blacksquare$