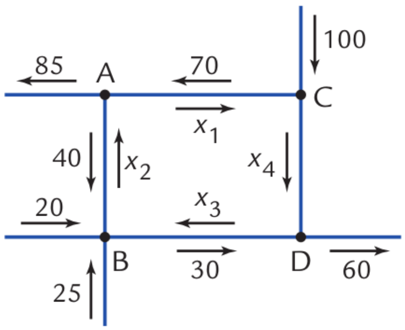
Problem 2, page 43

The volume of traffic for a collection of intersections is show in the figure below. Find all possible values for x_1, x_2, x_3 and x_4. What is the minimum volume of traffic from C to D?



Solution

The number of cars entering each intersection must equal the number of cars leaving. There are four intersections, so we can get the following system of equations:

$$\begin{aligned} 70 + x_2 &= 85 + 40 + x_1 \\ 40 + 20 + 25 + x_3 &= 30 + x_2 \\ 100 + x_1 &= 70 + x_4 \\ 30 + x_4 &= x_3 + 60 \end{aligned}$$

We can rewrite these equations into the standard form of a system of linear equations:

$$\begin{aligned} -x_1 + \quad x_2 \qquad\qquad\qquad &= 55 \\ -x_2 + x_3 \qquad\quad &= -55 \\ x_1 \qquad\qquad\qquad\quad -x_4 &= -30 \\ x_3 + \quad x_4 &= 30 \end{aligned}$$

which can be converted into an augmented matrix and converted to echelon form:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 55 \\ 0 & -1 & 1 & 0 & -55 \\ 1 & 0 & 0 & -1 & -30 \\ 0 & 0 & -1 & 1 & 30 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 55 \\ 0 & -1 & 1 & 0 & -55 \\ 0 & 0 & 1 & -1 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we can get the following set of equations:

$$\begin{aligned} -x_1 + \quad x_2 \qquad\qquad &= 55 \\ -x_2 + x_3 \qquad &= -55 \\ x_3 + x_4 &= 30 \end{aligned}$$

Using back substitution and letting $x_4 = s_4$ as a free parameter, the set of solutions is:

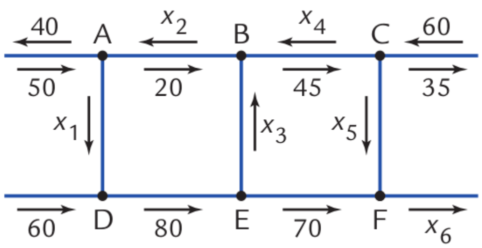
$$x_1 = s_4 - 30,000, \quad x_2 = s_4 + 25,000, \quad x_3 = s_4 - 30,000, \quad x_4 = s_4$$

Thus there are infinitely many possible distributions of cars for the four intersections. It is not possible for a volume of cars to be negative, so $x_4 \geq 30,000$.

The minimum volume of cars from C to D is **30,000 cars.**

Problem 4, page 43

The volume of traffic for a collection of intersections shown in the figure below. Find all possible values for x_1, x_2, x_3, x_4, x_5 and x_6.



Solution

There are six intersections, so we will have six equations to represent the volume of traffic:

$$\begin{aligned} x_2 + 50 &= 20 + 40 + x_2 \\ x_3 + x_4 + 20 &= x_2 + 45 \\ 45 + 60 &= x_4 + x_5 + 35 \\ x_1 + 60 &= 80 \\ 80 &= 70 + x_3 \\ x_5 + 70 &= x_6 \end{aligned}$$

We can rewrite these equations into the standard form of a system of linear equations:

$$\begin{aligned} -x_1 + x_2 \qquad\qquad\qquad\qquad &= 10 \\ - x_2 + x_3 + x_4 \qquad\quad &= 25 \\ x_4 + x_5 \qquad &= 70 \\ x_1 \qquad\qquad\qquad\qquad\qquad\qquad &= 20 \\ x_3 \qquad\qquad\qquad\quad &= 10 \\ x_5 - x_6 &= -70 \end{aligned}$$

which can be converted into an augmented matrix and converted to echelon form:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 55 \\ 0 & -1 & 1 & 1 & 0 & 0 & -55 \\ 0 & 0 & 0 & 1 & 1 & 0 & 70 \\ 1 & 0 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & -1 & -70 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 10 \\ 0 & -1 & 1 & 1 & 0 & 0 & 25 \\ 0 & 0 & 1 & 1 & 0 & 0 & 55 \\ 0 & 0 & 0 & 1 & 1 & 0 & 70 \\ 0 & 0 & 0 & 0 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 0 & -1 & -95 \end{bmatrix}$$

From this, we can get the following set of equations:

$$\begin{aligned} -x_1 + x_2 \qquad\qquad\qquad\qquad &= 10 \\ - x_2 + x_3 + x_4 \qquad\quad &= 25 \\ x_3 + x_4 \qquad\quad &= 55 \\ x_4 + x_5 \qquad &= 70 \\ x_5 \qquad &= 25 \\ - x_6 &= -95 \end{aligned}$$

Using back substitution, the set of solutions is:

$$\mathbf{x_1 = 20}, \quad \mathbf{x_2 = 30}, \quad \mathbf{x_3 = 10}, \quad \mathbf{x_4 = 45}, \quad \mathbf{x_5 = 25}, \quad \mathbf{x_6 = 95}$$

Use Example 5 as a guide to find the subspace of values that balances the given chemical equation.

Problem 74, page 160

Methane burns in oxygen to form carbon dioxide and steam.

$$x_1CH_4 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O$$

Solution

In order to balance the chemical equation, we need to find values of x_1, x_2, x_3, x_4, such that the number of atoms for each element in the equation is the same for both sides.
We can create the linear system:

$$\begin{aligned} -x_1 \quad - \quad x_3 \quad &= 0 \\ 4x_1 \qquad\qquad - 2x_4 &= 0 \\ 2x_2 - 2x_3 - \quad x_4 &= 0 \end{aligned}$$

This can be converted into the following augmented matrix, which can be reduced into echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

From this we can get the following set of solutions:

$$x_1 = \frac{1}{2}s_4, \quad x_2 = s_4, \quad x_3 = \frac{1}{2}s_4, \quad x_4 = s_4$$

where s_4 is a free parameter.
We can use this to get a set of values that will satisfy the chemical equation. This is the subspace of values that balances the equation:

$$\text{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

Problem 76, page 160

Ethyl alcohol reacts with oxygen to form vinegar and water.

$$x_1C_2H_5OH + x_2O_2 \to x_3HC_2H_3O_2 + x_4H_2O$$

Solution

In order to balance the chemical equation, we need to find values of x_1, x_2, x_3, x_4, such that the number of atoms for each element in the equation is the same for both sides.
We can create the linear system:

$$\begin{aligned} 2x_1 \qquad\quad - 2x_3 \qquad\quad &= 0 \\ 5x_1 \qquad\quad - 4x_3 - 2x_4 &= 0 \\ x_1 + 2x_2 - 2x_3 - x_4 &= 0 \\ x_1 \qquad\quad - x_3 \qquad\quad &= 0 \end{aligned}$$

This can be converted into the following augmented matrix, which can be reduced into echelon form:

$$\begin{bmatrix} 2 & 0 & -2 & 0 & 0 \\ 5 & -4 & 0 & 0 & 0 \\ 1 & 2 & -2 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we can get the following set of solutions:

$$x_1 = \frac{2}{3}s_4, \quad x_2 = \frac{5}{6}s_4, \quad x_3 = \frac{2}{3}s_4, \quad x_4 = s_4$$

where s_4 is a free parameter.
We can use this to get a set of values that will satisfy the chemical equation. This is the subspace of values that balances the equation:

$$\text{span}\left\{ \begin{bmatrix} \frac{2}{3} \\ \frac{5}{6} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right\}$$

Problem 28, page 44

Find the values of the coefficients a, b, and c so the given conditions for the function f and its derivatives are met.

$$f(x) = ae^x + be^{2x} + ce^{-3x}; f(0) = 2, f'(0) = 1, \text{ and } f''(0) = 19.$$

Solution

Compute the first and second derivatives of the function $f(x)$:

$$f'(x) = ae^x + 2be^{2x} - 2ce^{-3x}$$
$$f''(x) = ae^x + 4be^{2x} + 9ce^{-3x}$$

Plug in $x = 0$ to cancel out the e^x terms since $e^0 = 1$:

$$f(0) = a + b + c$$
$$f'(0) = a + 2b - 3c$$
$$f''(0) = a + 4b + 9c$$

Set these equations equal to the given initial conditions to form a system of equations:

$$a + b + c = 2$$
$$a + 2b - 3c = 1$$
$$a + 4b + 9c = 19$$

This can be converted to the following augmented matrix, which can be reduced to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & -3 & 1 \\ 1 & 4 & 9 & 19 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

From this we can obtain values for the coefficents of the equation:

$$a = -2, \quad b = 3, \quad c = 1$$