SIAM J. ALG. DISC. METH.
Vol. 3, No. 3, September 1982

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0196-5212/82/0303-0010 $01.00/0

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THE COMPLEXITY OF THE PARTIAL ORDER DIMENSION PROBLEM*

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Abstract. The dimension of a partial order P is the minimum number of linear orders whose intersection is P. There are efficient algorithms to test if a partial order has dimension 1 or 2. We prove that it is NP-complete to determine if a partial order has dimension 3. As a consequence, several other related dimension-type problems are shown to be NP-complete.

Key words. partial order, dimension, NP-complete, interval dimension, threshold dimension, boxicity, cubicity

1. Introduction. A *partial order* P of a finite[1] set N is an irreflexive, transitive binary relation on N; i.e., $(x, x) \notin P$ for each $x \in N$, and if (x, y) and $(y, z) \in P$ then $(x, z) \in P$. A *linear order* L of N is a partial order which contains (x, y) or (y, x) for any two distinct elements x, y of N. The linear order L is a (*linear*) *extension* of a partial order P if $P \subseteq L$. A partial order P can be viewed as a transitive directed acyclic graph (DAG) with set of nodes N and arcs $x \to y$ for $(x, y) \in P$. A linear order L is then a complete DAG; it is a linear extension of P if P is a subgraph of L.

The intersection of any set of partial orders of a set N is obviously also a partial order. The *dimension* $d(P)$ of a partial order P of N is the minimum number of linear orders whose intersection is P [DM]. It is a well-defined parameter: every partial order is the intersection of some linear extensions of it. Moreover, $d(P) \leqq |N|/2$[H]. A geometric interpretation of the dimension (and justification of the term) is the following. Let π be a mapping from N to distinct points of the d-dimensional Euclidean space E^d. Let $P(\pi)$ be the partial order of N defined by: $(x, y) \in P$ if and only if each coordinate of $\pi(x)$ is less than the corresponding coordinate of $\pi(y)$. The dimension of a partial order P of N is the minimum d for which there exists such a mapping π from N to E^d with $P(\pi) = P$ [O].

Clearly, a partial order has dimension 1 if and only if it is a linear order. Duschnik and Miller [DM] proved a necessary and sufficient condition for a partial order P to have dimension 2. Two elements x and y of N are *comparable* if (x, y) or (y, x) belongs to P; otherwise they are *incomparable*. The *incomparability graph* of P is an undirected graph $I(P)$ with N as its set of nodes and edges connecting the pairs of incomparable elements. [DM] proved that P has dimension 2 if and only if $I(P)$ is *transitively orientable*; i.e., the edges of $I(P)$ can be oriented so that the resulting directed graph is transitive. (Such a graph is sometimes called a *comparability graph.*) This condition combined with an efficient algorithm for the recognition of transitively orientable graphs [PLE] gives a polynomial algorithm to test if a partial order has dimension at most 2. A complete set of forbidden subgraphs of such partial orders is given in [K], [TM].

In this paper we will prove that it is NP-complete to determine if the dimension of a partial order is at most 3, and consequently the same holds also for any fixed $k \geqq 3$. The complexity of the partial order dimension problem was unknown for any fixed $k \geqq 3$ and for an arbitrary k; it is one of the open problems in the list of Garey and Johnson [GJ]. For an exposition on NP-completeness see [GJ].

* Received by the editors July 8, 1981 and in revised form September 3, 1981.

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[1] All sets in this paper will be finite.

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2. Preliminaries. Before describing the reduction, let us get some insight into the problem. Let P be a partial order of N (transitive DAG on N) and suppose that N is partitioned into two sets S, S' so that there is no arc of P directed from a node of S' to a node of S. Let $B(P)$ be the bipartite graph with set of nodes N and set of edges $\{[x, y] | x \in S,\ y \in S',\ x$ and y incomparable$\}$. Consider now a linear extension L of P, and let $\bar{L}$ be the bipartite graph with set of nodes N and set of edges $\{[x, y] | x \in S, y \in S', (y, x) \in L\}$; i.e., $\bar{L}$ contains those arcs (without the direction) which are directed from nodes of S' to nodes of S. Since P has no such arcs and L is a linear extension of P, $\bar{L}$ must be a subgraph of $B(P)$.

We say that two edges $[x, y]$, $[z, w]$, of a graph are *independent* if the nodes x, y, z, w are distinct and the subgraph induced by them consists of exactly these two edges. Suppose that $\bar{L}$ had two independent edges $[x, y]$, $[z, w]$ with $x, z \in S$ and $y, w \in S'$. From the definition of $\bar{L}$ then, L would contain (y, x), (w, z), (z, y) and (x, w); i.e., L would contain a cycle $y \to x \to w \to z \to y$, contradicting the fact that L is a linear order (complete DAG). Thus, $\bar{L}$ has no pair of independent edges. We call a bipartite graph with this property, a *chain graph*. It is also characterized by the property that the neighborhoods Γ_x (sets of nodes adjacent to x) of nodes x in S are totally ordered by set inclusion; i.e. for every x, y in S, either $\Gamma_x \subseteq \Gamma_y$ or $\Gamma_y \subseteq \Gamma_x$ [Y].

Suppose now that P has dimension d, and let $L_1, \cdots, L_d$ be linear extensions of P with intersection P. Let $\bar{L}_1, \cdots, \bar{L}_d$ be the bipartite graphs that we defined above. Each $\bar{L}_i$ is a chain subgraph of $B(P)$. Since the intersection of the L_i's is P, for every edge $[x, y]$ of $B(P)$ with $x \in S$, $y \in S'$, the arc (y, x) must appear in at least one of the L_i's; thus $[x, y]$ is covered by (appears in) at least one of the $\bar{L}_i$'s. For a bipartite graph G, let $\mathrm{ch}(G)$ be the minimum number of chain subgraphs of G that cover all the edges of G. We have shown:

LEMMA 1. $d(P) \geqq \mathrm{ch}(B(P))$.

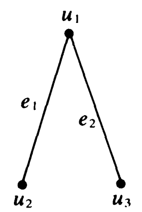
Thus, for example, if P is the *crown* on nodes $\{v_1, \cdots, v_k, v'_1, \cdots, v'_k\}$ with arcs $v_i \to v'_j$ for $i \neq j$, and we take $S = \{v_1, \cdots, v_k\}$, $S' = \{v'_1, \cdots, v'_k\}$, then $B(P)$ consists of k pairwise independent edges $[v_1, v'_1], \cdots, [v_k, v'_k]$, and $d(P) \geqq \mathrm{ch}(B(P)) = k$.

3. The reduction. The reduction is from the chromatic number 3 problem; i.e., given a graph G determine if the nodes of G can be colored with 3 colors so that adjacent nodes receive different colors. This problem was shown NP-complete in [GJS]. From G we will construct a partial order P so that G can be colored with 3 colors if and only if $d(P) \leqq 3$.

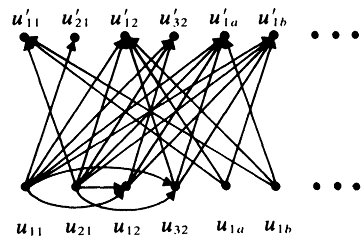
Let $V = \{u_1, \cdots, u_n\}$ be the nodes of G, and $E = \{e_1, \cdots, e_m\}$ its edges. P is a partial order on the union N of two disjoint sets S and S'. S contains two nodes u_{ia}, u_{ib} for every node u_i of V, and two nodes u_{ik}, u_{jk} for every edge $e_k = [u_i, u_j]$ in E. S' contains the primed versions of the nodes in S; thus, altogether N has $4n + 4m$ elements. The partial order P is defined as follows.

$$
\begin{aligned}
P = \{(u_{ia}, u'_{it}) | 1 \leqq i \leqq n, t \neq a\} &\cup \{(u_{ib}, u'_{it}) | 1 \leqq i \leqq n, t \neq b\} \\
&\cup \{(u_{ik}, u'_{jl}) | 1 \leqq i, j \leqq n, 1 \leqq k \leqq m, l > k \text{ or } l = a \text{ or } b\} \\
&\cup \{(u_{ik}, u'_{jk}) 1 \leqq k \leqq m, e_k = [u_i, u_j]\} \\
&\cup \{(u_{ik}, u_{jl}) | 1 \leqq i, j \leqq n, 1 \leqq k < l \leqq m\}.
\end{aligned}
$$

Notice that all arcs between S and S' are directed from S to S'. Let $B(P)$ be the bipartite graph defined in the previous section. Let Q be the nodes u_{ik} in S that correspond to edges of G (i.e., with $1 \leqq k \leqq m$), R the rest of the nodes in S (i.e. with $k = a, b$), and let Q', R' be the analogous subsets of S'. In simple words, $B(P)$ has the following structure: a node in Q is connected to its primed version and to all nodes of Q' with a strictly smaller second index; a node in R is connected to its primed version and all nodes of S' with a different first index. In Fig. 1 we give an example of the construction; only part of the partial order is shown for clarity.



(a) A graph G.



(b) Part of the partial order P.

FIG. 1

LEMMA 2. *If* ch $(B(P)) \leqq 3$ *then* G *can be colored with* $\leqq 3$ *colors.*

Proof. Suppose that ch $(B(P)) \leqq 3$, and let B_1, B_2, B_3 be three chain subgraphs of $B(P)$ that cover it. Consider the subgraph H_i of $B(P)$ induced by all nodes u_{it}, u'_{it} with first index i. It has three connected components: the edge $[u_{ia}, u'_{ia}]$, the edge $[u_{ib}, u'_{ib}]$ and the subgraph induced by the u_{ik}, u'_{ik} with $1 \leqq k \leqq m$. Since a chain graph cannot contain two independent edges, none of the B_j's can contain two edges from different components of H_i. Since H_i has three components, all edges of the third component are in the same B_j; color node u_i with the index of this B_j.

We must show now that this is a legal coloring of G. Suppose that there are two adjacent nodes u_i, u_j with the same color, say color 1. Let $e_k = [u_i, u_j]$. From the definition of the coloring we have $[u_{ik}, u'_{ik}]$, $[u_{jk}, u'_{jk}] \in B_1$. But these two edges are independent in $B(P)$, hence also in B_1, contradicting the assumption that B_1 is a chain graph. $\square$

LEMMA 3. *If* G *can be colored with* $\leqq 3$ *colors then* $d(P) \leqq 3$.

Proof. Suppose that G can be colored with 3 colors, and let C_1, C_2, C_3 be the sets of nodes that receive color 1, 2, 3 respectively in a legal coloring of G. We will construct three linear orders L_1, L_2, L_3 whose intersection is P. We will show only L_1 corresponding to the color class C_1; the two other linear orders are analogous.

We need some notation for describing linear orders. We shall write a linear order as a string where every element is less than the elements to its right. If X is a set then X will stand also for an arbitrary linear order of X. If F_1, F_2 are linear orders of disjoint sets X_1, X_2, then F_1F_2 is the concatenation of the two strings. If $I = \{i_1, \cdots, i_k\}$ is an index set with $i_1 < i_2 < \cdots < i_k$, and $F_{i_1}, \cdots, F_{i_k}$ are linear orders of disjoint sets $X_{i_1}, \cdots, X_{i_k}$, then we will denote $F_{i_1}F_{i_2} \cdots F_{i_k}$ by $\langle F_i \uparrow i \in I \rangle$, and $F_{i_k}F_{i_{k-1}} \cdots F_{i_1}$ by $\langle F_i \downarrow i \in I \rangle$.

Let $R_1 = \{u_{ia}, u_{ib} | u_i \in C_1\}$ and similarly for R'_1, R_2 etc. Let $e_k = [u_i, u_j]$ be an edge of G. If none of u_i, u_j has color 1 then $E_k = \{u_{ik}, u_{jk}\}$. If one of e_k's nodes, say u_i, has color 1 then E_k is the order $u_{jk}u'_{ik}u_{ik}$.

Let u_i be a node that receives color 2 or 3. Define a linear order K_i on $\{u_{ia}, u_{ib}, u'_{ia}, u'_{ib}\} \cup \{u'_{ik} | u_i \in e_k\}$ as follows. If u_i has color 2 then K_i is $u_{ib}u'_{ia}u_{ia}u'_{ib}$ followed by the elements of the second set in decreasing order of the second index (k). If u_i has color 3 then K_i is $u_{ia}u'_{ib}u_{ib}u'_{ia}$ followed by the elements of the second set again in decreasing order of k.

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The linear order L_1 now is $R_1 < E_k \uparrow k \in \{1, \cdots, m\} > R'_1 \langle K_i \uparrow u_i \in C_2 \rangle \langle K_i \downarrow u_i \in C_3 \rangle$. The two other orders are defined in a cyclically symmetric fashion.

It is straightforward to verify that each L_i is a linear extension of P. We will show now that for every pair of incomparable elements x, y of P, (x, y) is in one of the L_i's.

Case 1. $x \in S'$, $y \in S$. This case amounts to showing that the chain subgraphs $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$ of $B(P)$ cover all of its edges. Let $a = u'_{ik}$, $y = u_{jt}$, and suppose u_i has color 1, and u_j color c. If $c \neq 1$, or $t \in \{1, \cdots, m\}$ then $(x, y) \in L_1$. Assume then $c = 1$ and $t \in \{a, b\}$. If $i > j$ then $(x, y) \in L_2$; if $i < j$ then $(x, y) \in L_3$. If $i = j$ we must have $k = t$ since x and y are incomparable and $t = a$ or b; then $(x, y) \in L_2$ or L_3.

Case 2. $x, y \in S'$. Let $x = u'_{ik}$, $y = u'_{jt}$. If $i = j$ and $1 \leqq t < k \leqq m$, then, $(x, y) \in L_c$ where c is not the color of u_i. In all other cases there is a node z of S incomparable with x and such that $(z, y) \in P$: If $i \neq j$ or $[i = j$ and $k \in \{a, b\}$, say $k = a]$ then u_{ja} is such a node z. *If* $i = j$ and $k \in \{1, \cdots, m\}$ then we must have $t \in \{a, b\}$ or $t > k$, and u_{ik} is such a node z. From Case 1, (x, z) is in one of the linear extensions L_r of P and therefore $(x, y) \in L_r$ by transitivity.

Case 3. $x \in S$, $y \in S'$. Clearly, there is a $z \in S'$ such that $z \neq y$, $(x, z) \in P$. Since z and y are incomparable, (z, y) is in some L_r from Case 2 and by transitivity $(x, y) \in L_r$.

Case 4. $x, y \in S$. If there is a z in S' incomparable to y with $(x, z) \in P$, then (x, y) is in some L_r as before. If there is no such z, then it is easy to see that $x = u_{ik}$ for some $k \in \{a, b\}$ and $y = u_{jt}$ for some $t \in \{1, \cdots, m\}$. Then $(x, y) \in L_c$ where c is the color of u_i. □

THEOREM 1. *It is* NP-*complete to determine if the dimension of a given partial order is at most* 3.

Proof. Follows immediately from Lemmas 1, 2, 3. □

COROLLARY 1. *It is* NP-*complete to determine if a given bipartite graph can be covered by* 3 *chain subgraphs.*

Proof. Same as for Theorem 1. □

We should mention here that it is possible to determine in polynomial time if a bipartite graph can be covered by 2 chain subgraphs; this follows from the results of Ibaraki and Peled in [IP] and Lemma 7 in the next section.

The *height* of a partial order P is the length of the longest path in (the DAG) P. [Ki] showed that a partial order P can be transformed efficiently into another partial order P' of height 1 with $d(P) \leqq d(P') \leqq d(P) + 1$.[2] Therefore, an efficient algorithm for computing the dimension of partial orders of height 1 would give a good approximation of the dimension of an arbitrary partial order.

COROLLARY 2. *It is* NP-*complete to determine if the dimension of a partial order of height* 1 *is at most* 4.

Proof. Let B_1 be a bipartite graph with S_1, S'_1 a bipartition of its nodes. Let B be obtained from B_1 by adding two new nodes u, u' and an edge $[u, u']$. Let $S = S_1 \cup \{u\}$ and $S' = S'_1 \cup \{u'\}$. Let P be the partial order of height one with $(x, y) \in P$ if and only if $x \in S$, $y \in S'$ and $[x, y] \notin B$. Notice that B is the graph $B(P)$ that we defined in § 2. We claim that $\mathrm{ch}\,(B_1) \leqq 3$ if and only if $d(P) \leqq 4$ (if and only if $\mathrm{ch}\,(B) \leqq 4$).

(*if*) Suppose that $d(P) \leqq 4$. From Lemma 1 then $\mathrm{ch}\,(B) \leqq 4$. Let G_1, G_2, G_3, G_4 be chain subgraphs of B that cover its edges and suppose without loss of generality that $[u, u'] \in G_1$. Since $[u, u']$ is independent from all the other edges of B, the rest of the edges must all appear in the other three chain subgraphs.

[2] The partial order P' of height 1 has the property $d(P) = \mathrm{ch}\,(B(P'))$; that is, the construction of [Ki] is actually a reduction of the dimension problem to the chain covering problem. It is easy to see also that for any order P' of height 1, $d(P') \leqq \mathrm{ch}\,(B(P')) + 1$—a proof is essentially contained in Corollary 2.

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(*only if*) Let G_1, G_2, G_3 be three chain subgraphs of B_1 that cover its edges. From any chain graph G_i we can get a linear order L_i on $S_1 \cup S_1'$ such that $\bar{L}_i = G_i$. To see this, recall from § 2 that the neighborhoods in G_i of all nodes of S_1 (and S_1') are totally ordered by set inclusion. From this it follows easily that the nodes in S_1 can be partitioned into sets $R_1, R_2, \cdots, R_k$ and the nodes in S_1' into sets $R_1', R_2', \cdots, R_k'$ so that the neighborhood of each node in R_j is $\bigcup_{t>j} R_t'$. (R_1' and/or R_k may be empty.) The linear order $L_i = R_k R_k' R_{k-1} R_{k-1}', \cdots, R_1 R_1'$ satisfies then $\bar{L}_i = G_i$ (see Fig. 2).

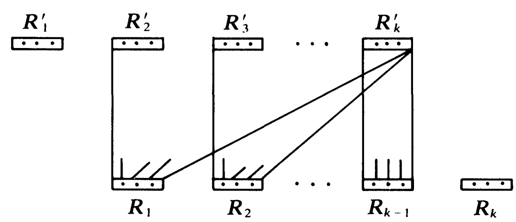

FIG. 2

Let L_1, L_2, L_3 therefore be linear orders with $\bar{L}_i = G_i$. Let F_i be the intersection of the restrictions of L_1, L_2, L_3 on S_1, and F_1' the intersection of the restrictions on S_1'. Let F, F' be any linear orders of S_1, S_1' respectively with $F \cap F_1 = \emptyset, F' \cap F_1' = \emptyset$, e.g., F can be the inverse of any topological sort (linear extension) of the DAG F_1.

Let $L_1' = uL_1u'$, $L_2' = uL_2u'$, $L_3' = uL_3u'$, $L_4' = Fu'uF'$, and let P^* be the intersection of the L_i''s. Clearly, all L_i''s are linear extensions of P; thus $P \subseteq P^*$. From our choice of F and F', the restrictions of P^* on S and S' are empty. For $x \in S$, $y \in S'$ with x, y incomparable we have $(x, y) \in L_4'$ or $x = u$, $y = u'$ and $(x, y) \in L_1'$. For $x \in S_1'$, $y \in S_1$ incomparable, (x, y) is in the L_i' that corresponds to the chain subgraph G_i that covers $[x, y]$. Finally, (u', u) is in L_4'. Thus, $P = P^*$. □

COROLLARY 3. *For any $k \geqq 3$ it is* NP-*complete to determine if the dimension of a partial order is at most k.*

Proof. Apply the reduction of Corollary 2, $k-3$ times. □

Note. E. Lawler and O. Vornberger showed recently (and independently) the result in the case of arbitrary dimension; i.e., given partial order P and integer k it is NP-complete to determine if $d(P) \leqq k$ [L].

4. Related problems. In this section we will show that several related dimension-type problems are NP-complete, using Corollary 1.

Interval dimension. Let X be a set of closed intervals on the real line. We can define a partial order P on X, where for x, y in X we have $(x, y) \in P$ if and only if the right endpoint of interval x is to the left of the left endpoint of interval y. A partial order that can be constructed in this way from a set of intervals is called an *interval order.* Clearly, every linear order is an interval order; in this case the intervals can be taken to be distinct points.

The *interval dimension* of a partial order P, denoted id (P), is the minimum number of interval orders whose intersection is P [TB]. Since every linear order is also an interval order, we have id $(P) \leqq d(P)$.

Interval orders of height 1 are closely related to chain graphs. A necessary and sufficient condition for a partial order P to be an interval order is that P does not contain a pair of independent arcs, i.e., two arcs (x, y) and (u, v) with x, y, u, v distinct elements and such that the subgraph of P induced by them consists of exactly these

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two arcs [F]. Thus, a partial order of height 1 is an interval order if and only if its underlying graph (its comparability graph) is a chain graph.

Let P be a partial order of height 1 and let S be the set of elements of height 1 and S' the set of elements of height 0. Clearly all arcs of P are directed from S to S'. Let $G(P)$ be the underlying graph of P, and $B(P)$ the graph that we defined in § 2; i.e., $B(P)$ is the bipartite graph with the set of edges $\{[x, y] | x \in S, y \in S', (x, y) \notin P\}$. It follows easily from the definitions that $G(P)$ is a chain graph if and only if $B(P)$ is a chain graph.

LEMMA 4. *Let P be a partial order of height* 1. *Then* $\mathrm{id}(P) = \mathrm{ch}(B(P))$.

Proof.

(1) $\mathrm{id}(P) \leqq \mathrm{ch}(B(P))$. Let $B_1, \cdots, B_k$ be chain subgraphs of $B(P)$ that cover its edges. For each B_i define the partial order $P_i = \{(x, y) | x \in S, y \in S', [x, y] \notin B_i\}$. Since B_i is a chain subgraph of $B(P)$, P_i is an interval order that contains P. Since the B_i's cover the edges of $B(P)$, the intersection of the P_i's is P.

(2) $\mathrm{ch}(B(P)) \leqq \mathrm{id}(P)$. Let $P_1, \cdots, P_k$ be interval orders whose intersection is P. For each i, let P'_i be the subgraph of P_i that consists of those arcs of P_i that are directed from S to S'. Let (x, y), (u, v) be two arcs of P'_i with the nodes $x, u \in S$ and $y, v \in S'$ distinct. Since P_i is an interval order, these two arcs cannot be independent in P_i. Suppose that P_i contains an arc from one of $\{x, y\}$ to one of $\{u, v\}$ (the other case is symmetric). Then, by transitivity, (x, v) is in P_i, and therefore also in P'_i. Thus, P'_i is an interval order of height 1, and consequently $B(P'_i)$ is a chain graph. Since the intersection of the P_i's is P and all arcs of P are directed from S to S', the intersection of the P'_i's is also P. Therefore, the $B(P'_i)$'s are chain subgraphs of $B(P)$ that cover its edges. □

COROLLARY 4. *It is* NP-*complete to determine if the interval dimension of a partial order of height* 1 *is at most* 3.

Proof. Follows from Corollary 1 and Lemma 4. □

[TM] presents a characterization of partial orders of height 1 that have interval dimension 2, in terms of forbidden subgraphs. However, the interval dimension 2 problem for general partial orders is open.

Boxicity. Let X be a set of closed intervals on the real line. We can construct a graph G with the intervals as nodes, and an edge between any two intervals with a nonempty intersection. A graph that can be constructed in this way from a set of intervals is called an *interval graph.* Thus, an interval graph is the incomparability graph of an interval order.

If $G_1, \cdots, G_k$ are graphs with the same set of nodes, their intersection is a graph with the same nodes and with those edges that are contained in all the G_i's. The *boxicity* of a graph G, denoted $b(G)$, is the minimum number of interval graphs whose intersection is G. A geometric interpretation (and justification of the term) is the following. Let X be a set of boxes in the k-dimensional space with sides parallel to the coordinate axis. Their intersection graph has set of nodes X and an edge between any two boxes with a nonempty intersection. The boxicity of a graph G is the minimum k such that G is the intersection graph of a set of such boxes in the k-dimensional space [R]. Thus, G has boxicity 1 if and only if it is an interval graph, boxicity 2 if and only if it is the intersection graph of rectangles in the plane with sides parallel to the axis, etc.

LEMMA 5. *Let* $\bar{B}$ *be the complement of a bipartite graph B. Then,* $b(\bar{B}) = \mathrm{ch}(B)$.

Proof. At first let us show that the complement $\bar{G}$ of a bipartite graph G is an interval graph if and only if G is a chain graph. If G is a chain graph with S, S' a

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bipartition of its nodes, then the partial order P obtained from G by directing all its edges from S to S' is an interval order. Therefore, $\bar{G}$, the incomparability graph of P, is an interval graph. Conversely, if $\bar{G}$ is an interval graph then it is the incomparability graph of an interval order P. Therefore G, the underlying graph of P, does not contain a pair of independent edges. Since G is also bipartite, it is a chain graph.

(1) $b(\bar{B}) \leqq \mathrm{ch}\,(B)$. Let $B_1, \cdots, B_k$ be chain subgraphs of B that cover its edges. Their complements $\bar{B}_1, \cdots, \bar{B}_k$ are interval graphs whose intersection is $\bar{B}$.

(2) $\mathrm{ch}\,(B) \leqq b(\bar{B})$. Let $\bar{B}_1, \cdots, \bar{B}_k$ be interval graphs whose intersection is $\bar{B}$. The complements $B_1, \cdots, B_k$ of the $\bar{B}_i$'s are subgraphs of B and therefore are bipartite. Thus, the B_i's are chain subgraphs of B that cover its edges. □

COROLLARY 5. *It is* NP-*complete to determine if the boxicity of a graph is at most* 3.

Cozzens showed recently the NP-completeness of the boxicity problem for arbitrary k, i.e., that given graph G and number k it is NP-complete to tell if $b(G) \leqq k$ [C]. The boxicity 2 case remains open.

Cubicity. A *unit-interval graph* is the intersection graph of unit intervals (closed intervals of length 1) on the real line. The *cubicity* $c(G)$ of a graph G is the minimum number of unit-interval graphs whose intersection is G. Geometrically, the cubicity of G is the minimum number k such that G is the intersection graph of unit cubes with sides parallel to the coordinate axes in the k-dimensional space [R]. Clearly, $b(G) \leqq c(G)$.

LEMMA 6. *Let $\bar{B}$ be the complement of a bipartite graph B. Then*, $c(\bar{B}) = \mathrm{ch}\,(B)$.

Proof. In view of Lemma 5 it suffices to show that the complement of a chain graph G is a unit interval graph. Let G be a chain graph that has the form of Fig. 1. We shall construct a unit-interval model for G. Associate with every node of R_i $(i = 1, \cdots, k)$ the (closed) unit interval $[i/k,\ 1 + i/k]$, and with every node of R_i' the interval $[1 + i/k,\ 2 + i/k]$. It is easy to see then that the intersection graph of these intervals is $\bar{G}$, the complement of G. □

COROLLARY 6. *It is* NP-*complete to determine if the cubicity of a graph is at most* 3. □

Threshold dimension. Let G be a graph with nodes $v_1, \cdots, v_n$. With every subset X of nodes we can associate its characteristic vector $\mathbf{x} = \langle x_1, \cdots, x_n \rangle$, where x_i is 1 or 0 depending on whether the node v_i is in X or not. The *threshold dimension* $\theta(G)$ of G is the minimum number of linear inequalities in the variables $x_1, \cdots\ x_n$ such that a set of nodes X is independent (i.e., does not induce any edge) if and only if its characteristic vector satisfies the inequalities [CH1]. A graph G with $\theta(G) \leqq 1$ is called a *threshold graph*. The threshold dimension of a graph G can be defined in an equivalent way as the minimum number of threshold subgraphs of G that cover its edges.

A threshold graph has the following structure. Its nodes can be partitioned into an independent set of nodes P and a clique Q so that the subgraph of G consisting of the edges of G between P and Q is a chain graph. Equivalently, G is a threshold graph if and only if it does not contain as an induced subgraph a pair of independent edges, a path of length 3, or a cycle of length 4 (see [CH1], [G] for more details).

LEMMA 7. *Let B be a bipartite graph with P, Q a bipartition of its nodes. Let B' be obtained from B by including all edges between nodes in Q (i.e., making Q a clique). Then* $\mathrm{ch}\,(B) = \theta(B')$.

Proof. (1) $\mathrm{ch}\,(B) \geqq \theta(B')$. Let $B_1, \cdots, B_k$ be chain subgraphs of B that cover its edges, and let $B_1', \cdots, B_k'$ respectively be obtained from them by turning Q into a clique. Then the B_i's are threshold subgraphs of B' that cover its edges.

(2) $\mathrm{ch}(B) \leqq \theta(B')$. Let $B'_1, \cdots, B'_k$ be threshold subgraphs of B' that cover its edges. For each i, let B_i consist of the edges of B'_i between P and Q. Then the B_i's cover the edges of B. We claim that they are also chain graphs. For, suppose that B_i has a pair of independent edges $[x, y]$, $[u, v]$ with $x, u \in P$ and $y, v \in Q$. Then the subgraph of B'_i induced by these four nodes is either a path of length 3 (if it contains the edge $[y, v]$) or a pair of independent edges (if it does not contain $[y, v]$). In either case B'_i is not a threshold graph. □

COROLLARY 7. *It is* NP-*complete to determine if a given graph has threshold dimension at most* 3.

Chvatal and Hammer [CH2] had shown the NP-completeness of the threshold dimension problem in the case of arbitrary dimension. The case of dimension 2 remains open.

Acknowledgment. I wish to thank Martin Golumbic for pointing out the implication for the threshold dimension problem (Corollary 7) and for helpful discussions on the problems of § 4.

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