

On the parity conjecture for elliptic curves

Holly Green

University of Bristol

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Theorem (Dokchitser–G.–Konstantinou–Morgan)

Assuming III is finite, the Birch and Swinnerton-Dyer conjecture correctly predicts the parity of the rank of elliptic curves.

Theorem (Dokchitser–G.–Konstantinou–Morgan)

Assuming III is finite, then for all smooth, projective curves over number fields X/K

$$\text{rank}(\text{Jac}_X) \equiv \sum_{v \text{ place of } K} \Lambda_v(X) \pmod{2}$$

where $\Lambda_v \in \mathbb{Z}$ is an explicit invariant computed from curves over local fields.

Will assume III is finite throughout.

The Birch and Swinnerton-Dyer and parity conjectures

Let E be an elliptic curve over a number field K .

Birch–Swinnerton-Dyer conjecture (i)

$$\text{rank}(E) = \text{ord}_{s=1} L(E, s)$$

+

Conjectural functional equation

$$L^*(E, s) = w(E) L^*(E, 2 - s)$$

\Downarrow

The parity conjecture

$$(-1)^{\text{rank}(E)} = w(E) := \prod_{v \text{ place of } K} w_v(E)$$

When $v \mid \infty$, $w_v(E) = -1$. Otherwise,

$$w_v(E) = \begin{cases} +1 & E/K_v \text{ has good reduction,} \\ -1 & E/K_v \text{ has split multiplicative reduction,} \\ +1 & E/K_v \text{ has non-split multiplicative reduction,} \\ \dots & E/K_v \text{ has additive reduction.} \end{cases}$$

If E is semistable, the parity conjecture predicts that

$$(-1)^{\text{rank}(E)} = (-1)^{\#\{v|\infty\} + \#\{v\nmid\infty, E/K_v \text{ split multiplicative}\}}.$$

$E/\mathbb{Q} : y^2 = x^3 - \frac{1}{3}x + \frac{35}{108}$, $\Delta_E = -43$. E has non-split multiplicative reduction at 43

$\Rightarrow \text{rank}(E)$ is odd $\Rightarrow E$ has a \mathbb{Q} -point of infinite order.

If E/\mathbb{Q} is semistable with split multiplicative reduction at 2 then $\text{rank}(E/\mathbb{Q}(\zeta_8))$ is odd.

If K is imaginary quadratic and E/K has everywhere good reduction then $\text{rank}(E/K)$ is odd.

If L/K has even degree then $\text{rank}(E/L)$ is even and

$$\text{rank}(E/K) < \text{rank}(E/L).$$

Goal 1

Develop an arithmetic analogue of the parity conjecture:

$$(-1)^{\text{rank}(E)} = \prod_v (-1)^{\Lambda_v(E)}.$$

E.g., (Cassels) if $E \rightarrow E'$ is an isogeny of degree d , then $\Lambda_v(E) = \text{ord}_d(c_v(E)/c_v(E'))$.

Goal 2

Prove the parity conjecture:

$$(-1)^{\text{rank}(E)} = \prod_v w_v(E).$$

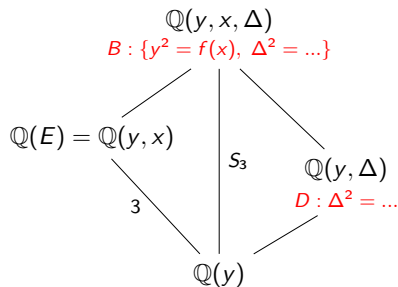
Relate $\Lambda_v(E)$ to $w_v(E)$, i.e. find $H_v \in \{\pm 1\}$ satisfying

$$(-1)^{\Lambda_v(E)} = H_v w_v(E) \quad \text{and} \quad \prod_v H_v = +1.$$

New idea: Use the arithmetic of covers of curves.

Taking covers of curves

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.



$$D : \Delta^2 = \text{Disc}_x(f(x) - y^2) = -27y^4 + 54by^2 - (4a^3 + 27b^2).$$

Theorem

Let Y/\mathbb{Q} be curve and $G \leq \text{Aut}_{\mathbb{Q}}(Y)$ finite.

- $\mathbb{Q}(Y)^G = \mathbb{Q}(Y/G),$
- $\Omega^1(Y)^G = \Omega^1(Y/G),$
- $(\text{Jac}_Y(\mathbb{Q}) \otimes \mathbb{Q})^G = \text{Jac}_{Y/G}(\mathbb{Q}) \otimes \mathbb{Q}.$

Example: B has genus 3

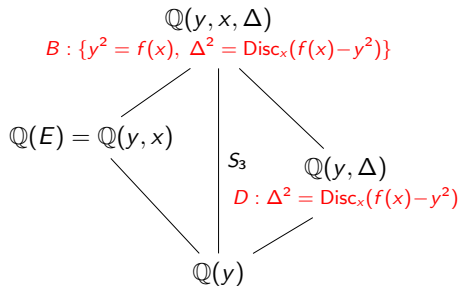
$$\Omega^1(B) = \mathbb{1}^{\oplus s} \oplus \epsilon^{\oplus t} \oplus \rho^{\oplus u} \Rightarrow B \text{ has genus } s + t + 2u.$$

$$s = \dim \Omega^1(B)^{S_3} = \dim \Omega^1(\mathbb{P}^1) = 0, \quad s + t = \dim \Omega^1(B)^{C_3} = \dim \Omega^1(D) = 1,$$

$$s + u = \dim \Omega^1(B)^{C_2} = \dim \Omega^1(E) = 1.$$

Exhibiting isogenies

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.



Theorem (Kani–Rosen)

Let Y/\mathbb{Q} be a curve and $G \leq \text{Aut}_{\mathbb{Q}}(Y)$ finite. Suppose that $\bigoplus_i \mathbb{C}[G/H_i] \cong \bigoplus_j \mathbb{C}[G/H'_j]$ for some $H_i, H'_j \leq G$. Then there's an isogeny

$$\prod_i \text{Jac}_{Y/H_i} \rightarrow \prod_j \text{Jac}_{Y/H'_j}.$$

Example: there's an isogeny $E \times E \times \text{Jac}_D \rightarrow \text{Jac}_B$

$$\mathbb{C}[S_3/1] = 1 \oplus \epsilon \oplus \rho^{\oplus 2}, \quad \mathbb{C}[S_3/C_2] = 1 \oplus \rho, \quad \mathbb{C}[S_3/C_3] = 1 \oplus \epsilon, \quad \mathbb{C}[S_3/S_3] = 1.$$

\implies there's an isogeny $\text{Jac}_{B/C_2} \times \text{Jac}_{B/C_2} \times \text{Jac}_{B/C_3} \rightarrow \text{Jac}_{B/1} \times \text{Jac}_{B/S_3} \times \text{Jac}_{B/S_3}$.

$$\begin{array}{cccccc} \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\ E & E & \text{Jac}_D & \text{Jac}_B & 0 & 0 \end{array}$$

Isogeny invariance of BSD

Birch–Swinnerton-Dyer conjecture (ii)

$$\frac{L^{(r_E)}(E, 1)}{r_E!} = \text{BSD}_E := \frac{\# \text{III}_E \cdot \text{Reg}_E \cdot C_E}{\# E(\mathbb{Q})_{\text{tors}}^2}$$

Theorem (Cassels–Tate)

The BSD coefficient is invariant under isogeny.

Apply to the isogeny $E \times E \times \text{Jac}_D \rightarrow \text{Jac}_B$.

$$\square \cdot \text{rank}(E) + \text{rank}(\text{Jac}_D) = \frac{\text{Reg}_{\text{Jac}_B}}{\text{Reg}_E^2 \text{Reg}_{\text{Jac}_D}} = \frac{\# \text{Jac}_B(\mathbb{Q})_{\text{tors}}^2}{\# E(\mathbb{Q})_{\text{tors}}^4 \# \text{Jac}_D(\mathbb{Q})_{\text{tors}}^2} \cdot \frac{\# \text{III}_E^2 \# \text{III}_{\text{Jac}_D}}{\# \text{III}_{\text{Jac}_B}} \cdot \frac{C_E^2 C_{\text{Jac}_D}}{C_{\text{Jac}_B}} = \square \cdot \frac{C_E^2 C_{\text{Jac}_D}}{C_{\text{Jac}_B}}$$

Theorem

Assuming that $\text{III}_E[3^\infty]$ and $\text{III}_{\text{Jac}_D}[3^\infty]$ are finite,

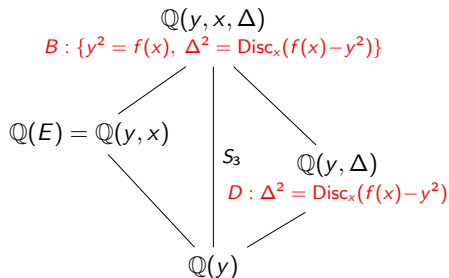
$$\text{rank}(E) + \text{rank}(\text{Jac}_D) \equiv \sum_v \text{ord}_3 \left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)} \right) \pmod{2}.$$

$$\text{Let } E : y^2 = x^3 - \frac{1}{3}x + \frac{35}{108} \Rightarrow \text{Jac}_D : y^2 = x^3 - \frac{35}{4}x^2 + x \quad (\Delta_E = -43, \Delta_{\text{Jac}_D} = 3^3 \cdot 43)$$

$$\text{rank}(E) + \text{rank}(\text{Jac}_D) \equiv \text{ord}_3 \left(\frac{12 \cdot 3}{3} \right) + \text{ord}_3 \left(\frac{12 \cdot 1}{1} \right) + \text{ord}_3 \left(\frac{5 \cdot 46 \dots^2 \cdot 2 \cdot 14 \dots}{21 \cdot 26 \dots} \right) \equiv 1 \pmod{2}.$$

An arithmetic analogue of the parity conjecture

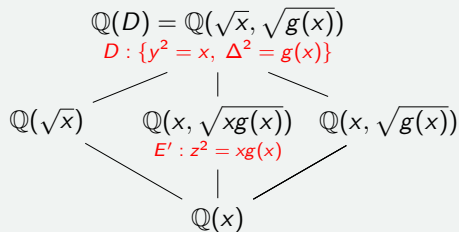
Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.



$$D : \Delta^2 = -27y^4 + 54by^2 - (4a^3 + 27b^2) =: g(y^2)$$

Theorem

$$\text{rank}(E) + \text{rank}(\text{Jac}_D) \equiv \sum_v \text{ord}_3 \left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)} \right)$$



Suppose $g(x)$ is quadratic.

There's an isogeny $\text{Jac}_D \rightarrow E' \Rightarrow \text{BSD}_{\text{Jac}_D} = \text{BSD}_{E'}$

$$\Rightarrow \square \cdot 2^{\text{rank}(\text{Jac}_D)} = \frac{\text{Reg}_{E'}}{\text{Reg}_{\text{Jac}_D}} = \square \cdot \frac{C_{\text{Jac}_D}}{C_{E'}}$$

$$\Rightarrow \text{rank}(\text{Jac}_D) \equiv \sum_v \text{ord}_2 \left(\frac{c_v(\text{Jac}_D)}{c_v(E')} \right) \pmod{2}.$$

An arithmetic analogue of the parity conjecture

Theorem (Dokchitser–G.–Konstantinou–Morgan)

Assume III is finite. Let X/\mathbb{Q} be a smooth, projective curve. There is an explicit invariant $\Lambda \in \mathbb{Z}$ computed from curves over local fields such that

$$\text{rank}(\text{Jac}_X) \equiv \sum_v \Lambda_v(X) \pmod{2}.$$

E.g., When $E : y^2 = x^3 + ax + b$,

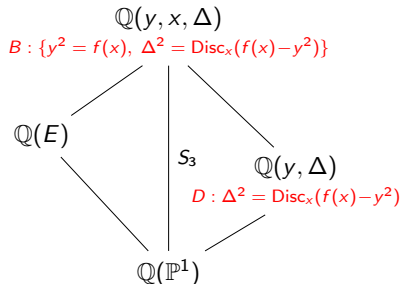
$$\Lambda_v(E) = \text{ord}_3 \left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)} \right) + \text{ord}_2 \left(\frac{c_v(\text{Jac}_D)}{c_v(E')} \right).$$

The parity conjecture

$$(-1)^{\text{rank}(\text{Jac}_X)} = \prod_v w_v(\text{Jac}_X).$$

Summary

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.



There's an isogeny $E \times E \times \text{Jac}_D \rightarrow \text{Jac}_B$.

Theorem

Assuming $\text{III}_E[3^\infty], \text{III}_{\text{Jac}_D}[3^\infty]$ are finite,

$$(-1)^{\text{rank}(E) + \text{rank}(\text{Jac}_D)} = \prod_v (-1)^{\text{ord}_3 \left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)} \right)}.$$

The parity conjecture for $E \times \text{Jac}_D$

$$(-1)^{\text{rank}(E) + \text{rank}(\text{Jac}_D)} = \prod_v w_v(E) w_v(\text{Jac}_D).$$

Goal: Relate $\text{ord}_3 \left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)} \right)$ to $w_v(E) w_v(\text{Jac}_D)$.

Local comparison

E.g., Let $E : y^2 = x^3 - \frac{1}{3}x + \frac{35}{108} \Rightarrow \text{Jac}_D : y^2 = x^3 - \frac{35}{4}x^2 + x$ ($\Delta_E = -43$, $\Delta_{\text{Jac}_D} = 3^3 \cdot 43$)

v	$\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)}$	$(-1)^{\text{ord}_3\left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)}\right)}$	$w_v(E)$	$w_v(\text{Jac}_D)$
3	$\frac{1^2 \cdot 3}{3} = 1$	+1	-1	+1
43	$\frac{1^2 \cdot 1}{1} = 1$	+1	+1	+1
∞	$\frac{5.46 \dots^2 \cdot 2.14 \dots}{21.26 \dots} = 3$	-1	-1	-1
$p \neq 3, 43$	1	+1	+1	+1

Theorem

Let v be a place of \mathbb{Q} . Then,

$$(-1)^{\text{ord}_3\left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)}\right)} = \begin{cases} -w_v(E)w_v(\text{Jac}_D) & v = 3 \text{ or } \infty, \\ w_v(E)w_v(\text{Jac}_D) & \text{otherwise.} \end{cases}$$

Proving the parity conjecture for E

$$(-1)^{\text{ord}_3\left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)}\right)} = \begin{cases} -w_v(E)w_v(\text{Jac}_D) & v = 3 \text{ or } \infty, \\ w_v(E)w_v(\text{Jac}_D) & \text{otherwise.} \end{cases}$$

Theorem

Let E/\mathbb{Q} be an elliptic curve. Assuming $\text{III}_E[3^\infty]$, $\text{III}_{\text{Jac}_D}[3^\infty]$, $\text{III}_{\text{Jac}_D}[2^\infty]$ are finite, the parity conjecture holds for E .

Proof.

Write $E : y^2 = x^3 + ax + b$ with $a \neq 0$. Then,

$$(-1)^{\text{rank}(E)+\text{rank}(\text{Jac}_D)} = \prod_v (-1)^{\text{ord}_3\left(\frac{c_v(E)^2 c_v(\text{Jac}_D)}{c_v(\text{Jac}_B)}\right)} = (-1)^2 \prod_v w_v(E)w_v(\text{Jac}_D).$$

Additionally,

$$(-1)^{\text{rank}(\text{Jac}_D)} = \prod_v (-1)^{\text{ord}_2\left(\frac{c_v(\text{Jac}_D)}{c_v(E')}\right)} = \prod_v (3a, -3b)_v (6b, 3\Delta_E)_v w_v(\text{Jac}_D) = \prod_v w_v(\text{Jac}_D).$$

□

Further applications to the parity conjecture

Theorem (G.–Maistret)

The 2-parity conjecture holds for Jac_C where $C : y^2 = f(x^2)$ has genus 2.

Theorem (Nekovář, Dokchitser², G.–Maistret)

The p -parity conjecture holds for elliptic curves over totally real fields.

Work in progress (Dokchitser–G.–Morgan)

Assume III is finite. The parity conjecture holds for Jacobians of semistable hyperelliptic curves.*

Thank you for your attention!