On the parity conjecture for elliptic curves

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Main results

Theorem (Dokchitser–G.–Konstantinou–Morgan)

Assuming Ш is finite, the Birch and Swinnerton-Dyer conjecture correctly predicts the parity of the rank of elliptic curves.

Theorem (Dokchitser–G.–Konstantinou–Morgan)

Assuming Ш is finite, then for all smooth, projective curves over number fields X/K

$$\mathrm{rank}(\mathrm{Jac}_X) \equiv \sum_{v \text{ place of } K} \Lambda_v(X) \mod 2$$

where $\Lambda_v \in \mathbb{Z}$ is an explicit invariant computed from curves over local fields.

Will assume Ш is finite throughout.

The Birch and Swinnerton-Dyer and parity conjectures

Let E be an elliptic curve over a number field K.

Birch–Swinnerton-Dyer conjecture (i)

$$\mathrm{rank}(E) = \mathrm{ord}_{s=1} L(E, s)$$

$+$

Conjectural functional equation

$$L^*(E, s) = w(E) L^*(E, 2-s)$$

$\Downarrow$

The parity conjecture

$$(-1)^{\mathrm{rank}(E)} \; = \; w(E) \; := \prod_{v \text{ place of } K} w_v(E)$$

When $v \mid \infty$, $w_v(E) = -1$. Otherwise,

$$w_v(E) = \begin{cases} +1 & E/K_v \text{ has good reduction,} \\ -1 & E/K_v \text{ has split multiplicative reduction,} \\ +1 & E/K_v \text{ has non-split multiplicative reduction,} \\ \ldots & E/K_v \text{ has additive reduction.} \end{cases}$$

Parity phenomena

If E is semistable, the parity conjecture predicts that

$$(-1)^{\mathrm{rank}(E)} = (-1)^{\#\{v|\infty\} \ + \ \#\{v\nmid\infty,\ E/K_v \text{ split multiplicative}\}}.$$

$E/\mathbb{Q} : y^2 = x^3 - \frac{1}{3}x + \frac{35}{108}$, $\Delta_E = -43$. E has non-split multiplicative reduction at 43

$$\Rightarrow \ \mathrm{rank}(E) \text{ is odd } \ \Rightarrow \ E \text{ has a } \mathbb{Q}\text{-point of infinite order.}$$

If $E/\mathbb{Q}$ is semistable with split multiplicative reduction at 2 then $\mathrm{rank}(E/\mathbb{Q}(\zeta_8))$ is odd.

If K is imaginary quadratic and E/K has everywhere good reduction then $\mathrm{rank}(E/K)$ is odd.
If L/K has even degree then $\mathrm{rank}(E/L)$ is even and

$$\mathrm{rank}(E/K) < \mathrm{rank}(E/L).$$

Strategy

Goal 1

Develop an arithmetic analogue of the parity conjecture:

$$(-1)^{\mathrm{rank}(E)} = \prod_v (-1)^{\Lambda_v(E)}.$$

E.g., (Cassels) if $E \to E'$ is an isogeny of degree d, then $\Lambda_v(E) = \mathrm{ord}_d(c_v(E)/c_v(E'))$.

Goal 2

Prove the parity conjecture:

$$(-1)^{\mathrm{rank}(E)} = \prod_v w_v(E).$$

Relate $\Lambda_v(E)$ to $w_v(E)$, i.e. find $H_v \in \{\pm 1\}$ satisfying

$$(-1)^{\Lambda_v(E)} = H_v w_v(E) \qquad \text{and} \qquad \prod_v H_v = +1.$$

New idea: Use the arithmetic of covers of curves.

Taking covers of curves

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.

$$
\begin{tikzcd}
 & \begin{array}{c} \mathbb{Q}(y,x,\Delta) \\ \color{red}{B : \{y^2 = f(x),\ \Delta^2 = \ldots\}} \end{array} \arrow[dl, dash] \arrow[dd, dash, "S_3"] \arrow[dr, dash] & \\
\mathbb{Q}(E) = \mathbb{Q}(y,x) \arrow[dr, dash, "3"'] & & \begin{array}{c} \mathbb{Q}(y,\Delta) \\ \color{red}{D : \Delta^2 = \ldots} \end{array} \arrow[dl, dash] \\
 & \mathbb{Q}(y) &
\end{tikzcd}
$$

$$
\begin{aligned}
D : \Delta^2 &= \mathrm{Disc}_x(f(x) - y^2) \\
&= -27y^4 + 54by^2 - (4a^3 + 27b^2).
\end{aligned}
$$

Theorem

Let $Y/\mathbb{Q}$ be curve and $G \leq \mathrm{Aut}_{\mathbb{Q}}(Y)$ finite.

- $\mathbb{Q}(Y)^G = \mathbb{Q}(Y/G)$,
- $\Omega^1(Y)^G = \Omega^1(Y/G)$,
- $(\mathrm{Jac}_Y(\mathbb{Q}) \otimes \mathbb{Q})^G = \mathrm{Jac}_{Y/G}(\mathbb{Q}) \otimes \mathbb{Q}$.

Example: B has genus 3

$\Omega^1(B) = \mathbb{1}^{\oplus s} \oplus \epsilon^{\oplus t} \oplus \rho^{\oplus u} \Rightarrow B$ has genus $s + t + 2u$.

$$s = \dim \Omega^1(B)^{S_3} = \dim \Omega^1(\mathbb{P}^1) = 0, \qquad s + t = \dim \Omega^1(B)^{C_3} = \dim \Omega^1(D) = 1,$$

$$s + u = \dim \Omega^1(B)^{C_2} = \dim \Omega^1(E) = 1.$$

Exhibiting isogenies

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.

$$\begin{tikzcd}[column sep=small]
 & \underset{\color{red}{B : \{y^2 = f(x),\ \Delta^2 = \mathrm{Disc}_x(f(x)-y^2)\}}}{\mathbb{Q}(y,x,\Delta)} \arrow[dl, dash] \arrow[dd, dash, "S_3"] \arrow[dr, dash] & \\
\mathbb{Q}(E) = \mathbb{Q}(y,x) \arrow[dr, dash] & & \underset{\color{red}{D : \Delta^2 = \mathrm{Disc}_x(f(x)-y^2)}}{\mathbb{Q}(y,\Delta)} \arrow[dl, dash] \\
 & \mathbb{Q}(y) &
\end{tikzcd}$$

Theorem (Kani–Rosen)

Let $Y/\mathbb{Q}$ be a curve and $G \leq \mathrm{Aut}_{\mathbb{Q}}(Y)$ finite. Suppose that $\bigoplus_i \mathbb{C}[G/H_i] \cong \bigoplus_j \mathbb{C}[G/H'_j]$ for some $H_i, H'_j \leq G$. Then there's an isogeny

$$\prod_i \mathrm{Jac}_{Y/H_i} \longrightarrow \prod_j \mathrm{Jac}_{Y/H'_j}.$$

Example: there's an isogeny $E \times E \times \mathrm{Jac}_D \to \mathrm{Jac}_B$

$$\mathbb{C}[S_3/1] = \mathbb{1} \oplus \epsilon \oplus \rho^{\oplus 2}, \quad \mathbb{C}[S_3/C_2] = \mathbb{1} \oplus \rho, \quad \mathbb{C}[S_3/C_3] = \mathbb{1} \oplus \epsilon, \quad \mathbb{C}[S_3/S_3] = \mathbb{1}.$$

$\implies$ there's an isogeny
$$\underset{\underset{E}{\|}}{\mathrm{Jac}_{B/C_2}} \times \underset{\underset{E}{\|}}{\mathrm{Jac}_{B/C_2}} \times \underset{\underset{\mathrm{Jac}_D}{\|}}{\mathrm{Jac}_{B/C_3}} \to \underset{\underset{\mathrm{Jac}_B}{\|}}{\mathrm{Jac}_{B/1}} \times \underset{\underset{0}{\|}}{\mathrm{Jac}_{B/S_3}} \times \underset{\underset{0}{\|}}{\mathrm{Jac}_{B/S_3}}.$$

Isogeny invariance of BSD

Birch–Swinnerton-Dyer conjecture (ii)

$$\frac{L^{(r_E)}(E,1)}{r_E!} = \mathrm{BSD}_E := \frac{\#\text{Ш}_E \cdot \mathrm{Reg}_E \cdot C_E}{\#E(\mathbb{Q})^2_{\mathrm{tors}}}$$

Theorem (Cassels–Tate)

The BSD coefficient is invariant under isogeny.

Apply to the isogeny $E \times E \times \mathrm{Jac}_D \to \mathrm{Jac}_B$.

$$\square \cdot 3^{\mathrm{rank}(E)+\mathrm{rank}(\mathrm{Jac}_D)} = \frac{\mathrm{Reg}_{\mathrm{Jac}_B}}{\mathrm{Reg}_E^2\ \mathrm{Reg}_{\mathrm{Jac}_D}} = \frac{\#\mathrm{Jac}_B(\mathbb{Q})^2_{\mathrm{tors}}}{\#E(\mathbb{Q})^4_{\mathrm{tors}}\ \#\mathrm{Jac}_D(\mathbb{Q})^2_{\mathrm{tors}}} \cdot \frac{\#\text{Ш}_E^2\ \#\text{Ш}_{\mathrm{Jac}_D}}{\#\text{Ш}_{\mathrm{Jac}_B}} \cdot \frac{C_E^2\ C_{\mathrm{Jac}_D}}{C_{\mathrm{Jac}_B}} = \square \cdot \frac{C_E^2\ C_{\mathrm{Jac}_D}}{C_{\mathrm{Jac}_B}}$$

Theorem

Assuming that $\text{Ш}_E[3^\infty]$ *and* $\text{Ш}_{\mathrm{Jac}_D}[3^\infty]$ *are finite,*

$$\mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) \equiv \sum_v \mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right) \mod 2.$$

Let $E : y^2 = x^3 - \frac{1}{3}x + \frac{35}{108} \Rightarrow \mathrm{Jac}_D : y^2 = x^3 - \frac{35}{4}x^2 + x \qquad (\Delta_E = -43,\ \Delta_{\mathrm{Jac}_D} = 3^3 \cdot 43)$

$$\mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) \equiv \underbrace{\mathrm{ord}_3\left(\frac{1^2\cdot 3}{3}\right)}_{v=3} + \underbrace{\mathrm{ord}_3\left(\frac{1^2\cdot 1}{1}\right)}_{v=43} + \underbrace{\mathrm{ord}_3\left(\frac{5.46\ldots^2 \cdot 2.14\ldots}{21.26\ldots}\right)}_{v=\infty} \equiv 1 \mod 2.$$

An arithmetic analogue of the parity conjecture

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.

$$\begin{tikzcd}
& \begin{array}{c}\mathbb{Q}(y,x,\Delta) \\ B : \{y^2 = f(x),\ \Delta^2 = \mathrm{Disc}_x(f(x)-y^2)\}\end{array} \arrow[dl, dash] \arrow[dd, dash, "S_3"] \arrow[dr, dash] & \\
\mathbb{Q}(E) = \mathbb{Q}(y,x) \arrow[dr, dash] & & \begin{array}{c}\mathbb{Q}(y,\Delta) \\ D : \Delta^2 = \mathrm{Disc}_x(f(x)-y^2)\end{array} \arrow[dl, dash] \\
& \mathbb{Q}(y) &
\end{tikzcd}$$

$$D : \Delta^2 = -27y^4 + 54by^2 - (4a^3 + 27b^2) =: g(y^2)$$

Theorem

$$\mathrm{rank}(E) + \mathrm{rank}(\mathrm{Jac}_D) \equiv \sum_v \mathrm{ord}_3 \left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)} \right)$$

$$\begin{tikzcd}
& \begin{array}{c}\mathbb{Q}(D) = \mathbb{Q}(\sqrt{x}, \sqrt{g(x)}) \\ D : \{y^2 = x,\ \Delta^2 = g(x)\}\end{array} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{x}) \arrow[dr, dash] & \begin{array}{c}\mathbb{Q}(x, \sqrt{xg(x)}) \\ E' : z^2 = xg(x)\end{array} \arrow[d, dash] & \mathbb{Q}(x, \sqrt{g(x)}) \arrow[dl, dash] \\
& \mathbb{Q}(x) &
\end{tikzcd}$$

Suppose $g(x)$ is quadratic.

There's an isogeny $\mathrm{Jac}_D \to E' \Rightarrow \mathrm{BSD}_{\mathrm{Jac}_D} = \mathrm{BSD}_{E'}$

$$\Rightarrow \square \cdot 2^{\mathrm{rank}(\mathrm{Jac}_D)} = \frac{\mathrm{Reg}_{E'}}{\mathrm{Reg}_{\mathrm{Jac}_D}} = \square \cdot \frac{C_{\mathrm{Jac}_D}}{C_{E'}}$$

$$\Rightarrow \mathrm{rank}(\mathrm{Jac}_D) \equiv \sum_v \mathrm{ord}_2 \left(\frac{c_v(\mathrm{Jac}_D)}{c_v(E')} \right) \mod 2.$$

An arithmetic analogue of the parity conjecture

Theorem (Dokchitser–G.–Konstantinou–Morgan)

Assume Ш *is finite. Let $X/\mathbb{Q}$ be a smooth, projective curve. There is an explicit invariant $\Lambda \in \mathbb{Z}$ computed from curves over local fields such that*

$$\mathrm{rank}(\mathrm{Jac}_X) \equiv \sum_v \Lambda_v(X) \mod 2.$$

E.g., When $E : y^2 = x^3 + ax + b$,

$$\Lambda_v(E) = \mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right) + \mathrm{ord}_2\left(\frac{c_v(\mathrm{Jac}_D)}{c_v(E')}\right).$$

The parity conjecture

$$(-1)^{\mathrm{rank}(\mathrm{Jac}_X)} = \prod_v w_v(\mathrm{Jac}_X).$$

Summary

Let $E/\mathbb{Q} : y^2 = f(x) := x^3 + ax + b$ be an elliptic curve, $a \neq 0$.

$$\begin{tikzcd}
& \begin{array}{c}\mathbb{Q}(y,x,\Delta)\\ B : \{y^2 = f(x),\ \Delta^2 = \mathrm{Disc}_x(f(x)-y^2)\}\end{array} \arrow[dl, dash] \arrow[dd, dash, "S_3"] \arrow[dr, dash] & \\
\mathbb{Q}(E) \arrow[dr, dash] & & \begin{array}{c}\mathbb{Q}(y,\Delta)\\ D : \Delta^2 = \mathrm{Disc}_x(f(x)-y^2)\end{array} \arrow[dl, dash] \\
& \mathbb{Q}(\mathbb{P}^1) &
\end{tikzcd}$$

There's an isogeny $E \times E \times \mathrm{Jac}_D \to \mathrm{Jac}_B$.

Theorem

Assuming $\text{Ш}_E[3^\infty]$, $\text{Ш}_{\mathrm{Jac}_D}[3^\infty]$ are finite,

$$(-1)^{\mathrm{rank}(E)+\mathrm{rank}(\mathrm{Jac}_D)} = \prod_v (-1)^{\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)}.$$

The parity conjecture for $E \times \mathrm{Jac}_D$

$$(-1)^{\mathrm{rank}(E)+\mathrm{rank}(\mathrm{Jac}_D)} = \prod_v w_v(E) w_v(\mathrm{Jac}_D).$$

Goal: Relate $\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)$ to $w_v(E)w_v(\mathrm{Jac}_D)$.

E.g., Let $E : y^2 = x^3 - \frac{1}{3}x + \frac{35}{108} \Rightarrow \mathrm{Jac}_D : y^2 = x^3 - \frac{35}{4}x^2 + x \quad (\Delta_E = -43,\ \Delta_{\mathrm{Jac}_D} = 3^3 \cdot 43)$

v	$\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}$	$(-1)^{\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)}$	$w_v(E)$	$w_v(\mathrm{Jac}_D)$
3	$\frac{1^2 \cdot 3}{3} = 1$	$+1$	-1	$+1$
43	$\frac{1^2 \cdot 1}{1} = 1$	$+1$	$+1$	$+1$
∞	$\frac{5.46\ldots^2 \cdot 2.14\ldots}{21.26\ldots} = 3$	-1	-1	-1
$p \neq 3, 43$	1	$+1$	$+1$	$+1$

Theorem

Let v be a place of $\mathbb{Q}$. Then,

$$(-1)^{\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)} = \begin{cases} -w_v(E)w_v(\mathrm{Jac}_D) & v = 3 \text{ or } \infty, \\ w_v(E)w_v(\mathrm{Jac}_D) & \text{otherwise.} \end{cases}$$

Proving the parity conjecture for E

$$(-1)^{\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)} = \begin{cases} -w_v(E)w_v(\mathrm{Jac}_D) & v = 3 \text{ or } \infty, \\ w_v(E)w_v(\mathrm{Jac}_D) & \text{otherwise.} \end{cases}$$

Theorem

Let $E/\mathbb{Q}$ be an elliptic curve. Assuming $Ш_E[3^\infty]$, $Ш_{\mathrm{Jac}_D}[3^\infty]$, $Ш_{\mathrm{Jac}_D}[2^\infty]$ are finite, the parity conjecture holds for E.

Proof.

Write $E : y^2 = x^3 + ax + b$ with $a \neq 0$. Then,

$$(-1)^{\mathrm{rank}(E)+\mathrm{rank}(\mathrm{Jac}_D)} = \prod_v (-1)^{\mathrm{ord}_3\left(\frac{c_v(E)^2 c_v(\mathrm{Jac}_D)}{c_v(\mathrm{Jac}_B)}\right)} = (-1)^2 \prod_v w_v(E)w_v(\mathrm{Jac}_D).$$

Additionally,

$$(-1)^{\mathrm{rank}(\mathrm{Jac}_D)} = \prod_v (-1)^{\mathrm{ord}_2\left(\frac{c_v(\mathrm{Jac}_D)}{c_v(E')}\right)} = \prod_v (3a, -3b)_v (6b, 3\Delta_E)_v w_v(\mathrm{Jac}_D) = \prod_v w_v(\mathrm{Jac}_D).$$

$\square$

Further applications to the parity conjecture

Theorem (G.–Maistret)

The 2-parity conjecture holds for Jac_C *where* $C : y^2 = f(x^2)$ *has genus 2.*

Theorem (Nekovář, Dokchitser^2, G.–Maistret)

The p-parity conjecture holds for elliptic curves over totally real fields.

Work in progress (Dokchitser–G.–Morgan)

Assume Ш *is finite. The parity conjecture holds for Jacobians of semistable* hyperelliptic curves.*

Thank you for your attention!