1. a) $\displaystyle\int_1^\infty x \frac{\beta}{x^{\beta+1}}\,dx$ ① part marks

$= \displaystyle\int_1^\infty \beta x^{-\beta}\,dx$

$= \left.\dfrac{\beta x^{-\beta+1}}{-\beta+1}\right|_1^\infty$

$= \dfrac{\beta}{-\beta+1}\left[x^{-\infty} - 1\right]$

$= \dfrac{-\beta}{-\beta+1} = \boxed{\dfrac{\beta}{\beta-1}}$ ← full marks

b) $E[X] = \bar{X}$

$\dfrac{\beta}{\beta-1} = \bar{X}$ ① part marks

$\beta = (\beta-1)\bar{X}$

$\beta = \beta\bar{X} - \bar{X}$

$\beta - \beta\bar{X} = -\bar{X}$

$\beta(1-\bar{X}) = -\bar{X}$

$\beta = \dfrac{-\bar{X}}{1-\bar{X}}$

$\boxed{\hat{\beta} = \dfrac{\bar{X}}{\bar{X}-1}}$ ← full marks

c) $$f(x_1, \dots, x_n; \beta) = \prod_{i=1}^{n} \frac{\beta}{x_i^{\beta+1}}$$

$$= \beta^n \left(\prod_{i=1}^{n} x_i\right)^{-\beta-1} \quad ①$$

$$\ln f(\cdot) = n \ln \beta - (\beta+1) \ln\left(\prod_{i=1}^{n} x_i\right) \quad ①$$

$$\frac{\partial}{\partial \beta} \ln f(\cdot) = \frac{n}{\beta} - \sum_{i=1}^{n} \ln x_i = 0$$

$$\frac{n}{\beta} = \sum \ln x_i$$

$$\frac{\beta}{n} = \frac{1}{\sum \ln x_i}$$

$$\boxed{\hat{\beta} = \frac{n}{\sum \ln x_i}} \quad ①$$

d) $$f(x;\beta) = \frac{\beta}{x^{\beta+1}}$$

$$\ln f(\cdot) = \ln\beta - (\beta+1)\ln x \quad 1$$

$$\frac{\partial}{\partial\beta}\ln f(\cdot) = \frac{1}{\beta} - \ln x \quad 1$$

$$\frac{\partial^2}{\partial\beta^2}\ln f(\cdot) = \frac{-1}{\beta^2}$$

$$I(\beta) = -E\left[\frac{-1}{\beta^2}\right] = \frac{1}{\beta^2} \quad 1$$

so $$\boxed{\beta \to N\left(\beta, \frac{1}{n\beta^2}\right)}$$

1 0.5 0.5

2. a) $$\begin{aligned} E[2\bar{X}] &= 2E[\bar{X}] \\ &= 2E[X] \\ &= 2\left[\frac{1}{2}(0+\theta)\right] \\ &= \theta \quad \blacksquare \end{aligned}$$

1 must show work

Yes, it is unbiased 1

b) $MSE(\hat{\theta}) = Var(\hat{\theta}) - (bias)^2$

$$= Var\,\hat{\theta}$$

$$Var[\hat{\theta}] = Var[2\bar{X}]$$

$$= 4\,Var[\bar{X}]$$

$$= 4\,\frac{Var[X]}{n}$$

$$= \frac{4}{n}\left[\frac{1}{12}(\theta - 0)^2\right]$$

$$= \frac{1}{3n}\theta^2 \quad 1$$

$$\boxed{MSE = \frac{\theta^2}{3n}} \quad 1$$

c) $f(x_1, \dots, x_n; \theta) = \frac{1}{\theta} \qquad 0 < x_i < \theta$

$$= \frac{1}{\theta} I(0 < \min(x_i), \max(x_i) < \theta) \quad 1$$

$\therefore$ not sufficient for θ. 1