CERN-TH-2025-188

arXiv:2509.20449v1 [hep-th] 24 Sep 2025

Geometric Building Blocks of Effective Field Theory Amplitudes

Timothy Cohen,[a,b,c] **Xu-Xiang Li,**[d] **and Zhengkang Zhang**[d]

[a] *Theoretical Physics Department, CERN, 1211 Geneva, Switzerland*

[b] *Theoretical Particle Physics Laboratory, EPFL, 1015 Lausanne, Switzerland*

[c] *Institute for Fundamental Science, University of Oregon, Eugene, OR 97403, USA*

[d] *Department of Physics & Astronomy, University of Utah, Salt Lake City, UT 84112, USA*

E-mail: tim.cohen@cern.ch, xuxiang.li@utah.edu, z.k.zhang@utah.edu

Abstract: On-shell amplitudes are invariant under field redefinitions. Nonderivative field redefinitions have a natural interpretation as coordinate transformations on the target manifold. General field redefinitions, which may involve derivatives, can be viewed as coordinate transformations on the field configuration manifold. We present a unified perspective for the geometry of both the target manifold and the field configuration manifold for scalar effective field theories. In both cases, we identify vertices that can be used to build the tree-level amplitudes, with the property that they transform covariantly in the vacuum and on-shell limits. We identify a choice of metric on the field configuration manifold, for which amplitude expressions on the target manifold can be easily reproduced from their counterparts on the field configuration manifold. This clarifies the relation between the well-established framework of field space geometry and recent proposals for functional geometry.

Contents

1 Introduction

The laws of physics do not depend on the coordinate system chosen by physicists. In an Effective Field Theory (EFT), the dynamical degrees of freedom are fields, and the coordinate system corresponds to a field basis. Given an EFT action $S[\phi]$, we are free to redefine the fields:

$$\phi = f\big(\widetilde{\phi}\,,\, \partial_\mu\widetilde{\phi}\,,\, \partial_\mu\partial_\nu\widetilde{\phi}\,,\, \ldots\big) \tag{1.1}$$

for some function f, and express the same theory using a redefined action $\widetilde{S}[\widetilde{\phi}]$ in terms of the new set of fields. The two actions before and after the field redefinition, $S[\phi]$ and $\widetilde{S}[\widetilde{\phi}]$, must contain the same physics (e.g., they must predict the same on-shell amplitudes) because all we have done is a change of basis.

This seemingly innocuous statement has been a source of complication and confusion across the EFT literature. For example, the Standard Model EFT (SMEFT) features thousands of effective operators already at the dimension-six level [1]. Many field bases have been proposed for SMEFT [1–5] and, because of the sheer number of operators, it is typically not straightforward to compare different analyses if they do not use the same basis. Another crucial question is what theories of new physics have their low-energy imprints captured by SMEFT, and what theories require the more general framework of the Higgs EFT (HEFT). Delineating the boundary between different classes of UV completions of the Standard Model requires going beyond the classification in terms of linearly vs. nonlinearly realized electroweak symmetry, in part because this distinction becomes ambiguous under basis change.

To appreciate the problem in a concrete setting, consider the standard algorithm for computing scattering amplitudes in an EFT. Starting from the action, we derive the Feynman rules and draw diagrams to connect vertices with propagators. These conventional building blocks of EFT amplitudes are field basis dependent. For example, a set of irrelevant operators that appear to modify the propagators in one basis might have their effects encoded in vertex corrections in a different basis. Only when the building blocks are assembled into on-shell amplitudes do we see that the final results are the same across all bases.

In this paper, we develop an approach where the individual off-shell building blocks of EFT amplitudes have well-defined transformation properties under field redefinitions at every step of the calculation. This makes the ultimate field redefinition invariance of the on-shell amplitudes manifest. This approach has the benefit that the physical implications of a given theory are not obscured by change of basis.

Realizing this goal naturally evokes ideas from differential geometry, which physicists are usually introduced to in the context of general relativity. Indeed, a central goal of general relativity is to express physical laws in terms of tensors which transform covariantly under coordinate changes on the spacetime manifold. In order to adopt the same strategy for EFTs, we would like to interpret fields as charting some manifold. This enables the introduction of geometric quantities, which serve as the elemental objects with which we can construct observables such as scattering amplitudes. This is the underlying logic of the EFT geometry program, which has a long history [6–17] and has undergone significant new developments in recent years: SMEFT vs. HEFT classification of Standard Model extensions [18–22], all order in $\frac{v^2}{\Lambda^2}$ expressions for electroweak and Higgs observables [23–26], geometric soft theorems [27–29], EFT matching and renormalization group evolution equations [30–38], and incorporation of fermions [33, 38–42], to name a few.

In the simplest version of EFT geometry, the manifold of interest is the target manifold of the field map (also commonly referred to as the field space manifold). To be concrete, consider N real scalar fields $\phi^i(x)$, with $i = 1, \ldots, N$. We say that the values of $\phi^i(x)$ at each spacetime point x provide a coordinate chart on an N-dimensional manifold, which is the target manifold of the map ϕ. The associated geometric framework is known as *field space geometry*. It is built on two key observations:

- field redefinitions that *do not involve derivatives*, $\phi = f(\widetilde{\phi})$, can be viewed as coordinate transformations on the target manifold, and

- for an EFT *truncated at* $\mathcal{O}(\partial^2)$, defined by $\mathcal{L} = \frac{1}{2}\, g_{ij}(\phi)\, (\partial_\mu \phi^i)(\partial^\mu \phi^j) - U(\phi)$, the coefficient function $g_{ij}(\phi)$ of the two-derivative term naturally defines a metric on the target manifold.

With a metric in hand, it becomes straightforward to apply the machinery familiar from general relativity to introduce other geometric notions like the Levi-Civita connection, covariant derivative, Riemann curvature tensor, etc. One can then derive on-shell amplitudes in terms of the geometric objects, which results in expressions that are manifestly invariant under *nonderivative* field redefinitions [21, 27, 39–41, 43].

Accommodating *derivative* field redefinitions, on the other hand, is more challenging. When derivatives of ϕ are present, Eq. (1.1) is not a coordinate transformation on the target manifold. Moreover, it is often necessary in phenomenological applications to go beyond $\mathcal{O}(\partial^2)$ in the EFT Lagrangian. Progress toward a more general geometric construction has been made in two complementary directions, by including derivatives of fields as additional coordinates [44–47] or by defining geometric objects on the manifold of field configurations [48–51]. The former approach is naturally formulated in the jet bundle formalism, while the latter approach draws on the language of functional methods in field theory. Our present work is in line with the second direction and, following Ref. [48], we refer to this approach as *functional geometry*. Two key observations underlying the initial proposal of functional geometry are:

- general field redefinitions of the form Eq. (1.1), which replace ϕ by a functional of $\widetilde{\phi}$, can be viewed as coordinate transformations on the field configuration manifold (referred to as the functional manifold in Refs. [50, 51]), and
- for a general EFT, amplitudes can be constructed recursively by taking functional derivatives of correlation functions.

The second point follows from the standard path integral formulation of field theory: the second functional derivative of $S[\phi]$ gives the inverse propagator, and higher functional derivatives give the standard Feynman vertices. Furthermore, replacing the classical action $S[\phi]$ by the one-particle-irreducible (1PI) effective action, one obtains the loop-corrected propagator and vertices.

Importantly, although on-shell amplitudes can only be measured at the physical vacuum, in order to take functional derivatives we must work with objects defined on the entire field configuration manifold. The focal point of the analyses in Refs. [48, 50, 51] is a set of objects $\mathcal{M}_{1\ldots n}$, which are a natural generalization of on-shell amplitudes away from the vacuum and continued to off-shell momenta. Away from the vacuum and on-shell limits, however, $\mathcal{M}_{1\ldots n}$ generally do not transform as tensors. This leads to the notion of *on-shell covariance*, meaning the inhomogeneous pieces that appear upon performing a field redefinition vanish at the vacuum for on-shell external momenta [48, 50]. Ref. [51] further elaborated this notion by showing that not only are $\mathcal{M}_{1\ldots n}$ on-shell covariant, for general scalar EFTs at tree level, they can also be recursively constructed from on-shell covariant building blocks. The main finding of Ref. [51] can be summarized as:

$$\mathcal{M}_{1\ldots n} = \mathcal{F}_{1\ldots n}\big(\Delta^{ab}, \{V_{a_1\ldots a_m}\}\big) = \mathcal{F}_{1\ldots n}\big(\Delta^{ab}, \{\mathcal{V}_{a_1\ldots a_m}\}\big) \,, \tag{1.2}$$

where Δ^{ab} and $V_{a_1\ldots a_m}$ are the standard Feynman propagator and vertices, and $\mathcal{V}_{a_1\ldots a_m}$ represents a new set of vertices that are on-shell covariant.[1] The statement of Eq. (1.2) is that one can take the standard tree-level expression of $\mathcal{M}_{1\ldots n}$, and "upgrade" $V_{a_1\ldots a_m}$ to $\mathcal{V}_{a_1\ldots a_m}$, in order to obtain a manifestly on-shell covariant expression for the off-shell, off-vacuum generalization of the amplitude. Note that Δ^{ab} is already on-shell covariant, so nothing needs to be done to the propagators. See Fig. 1 for a graphic illustration.

A primary goal of the present paper is to elucidate the relation between recent developments in functional geometry [48, 50, 51] and existing results in field space geometry, focusing on scalar EFTs. In particular, we will show that Eq. (1.2) is not unique to functional geometry; there is also a field space geometry version of this equation which we derive for arbitrary-point tree-level amplitudes in scalar EFTs truncated at $\mathcal{O}(\partial^2)$. This will result in a unified perspective on the notion of on-shell

[1] As a technical note, the $\mathcal{V}_{a_1\ldots a_m}$ in this work are the fully symmetric components of the $\mathcal{V}_{a_1\ldots a_m}$ introduced in Ref. [51]. Eq. (1.2) holds for both versions of $\mathcal{V}_{a_1\ldots a_m}$.

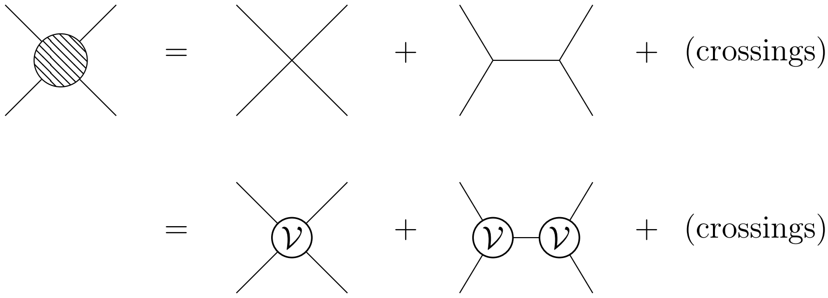


Figure 1. Graphic illustration of Eq. (1.2) for $n = 4$. $\mathcal{V}$ represents a new type of vertices built from geometric objects on the field configuration manifold, which transform covariantly in the vacuum and on-shell limits. The geometric expression in the second line makes on-shell covariance of the amplitude manifest.

covariance of EFT amplitudes and their building blocks, both on the target manifold and on the field configuration manifold. Reconciling previous formulations of field space geometry and functional geometry, which appear to have adopted different starting points as we can see from the bullet points above, is an essential step toward refining many of the recent results in field space geometry, e.g., classification of EFTs and soft theorems, to accommodate the more general set of field redefinitions that include derivatives. This paper also serves as a jumping-off point to discover the generalizations of these results to more generic EFTs.

This paper is organized as follows. In Sec. 2, we formulate EFT amplitude calculations in field space geometry in a way that can be easily generalized to functional geometry. Many of the results in this section are already known from previous works, but an important novelty here is that we consider correlation functions defined on the entire target manifold (not just at the vacuum for on-shell momenta), and show that they can be constructed from on-shell covariant building blocks. In Sec. 3, we follow a similar strategy to derive EFT amplitudes in functional geometry. In Ref. [51], the main result, Eq. (1.2), was obtained from an off-shell recursion relation which does not seem to have a field space geometry counterpart. We reproduce this result (more precisely, a slight variant of the result in Ref. [51] which involves symmetric vertices) using a different approach based on normal coordinates, as a natural generalization of the discussion in Sec. 2. The notion of on-shell covariance which unites the developments in both Secs. 2 and 3 relies on the existence of a connection but not necessarily a metric, so we stay agnostic about the definition of metric throughout Sec. 3. In Sec. 4, we discuss proposals to establish a metric—hence a Riemannian geometry and its associated Levi-Civita connection—on the field configuration manifold. A new development here is that we outline a procedure to define an unambiguous metric from a general EFT action, which allows us to easily reproduce amplitude results in field space geometry from the more general framework of functional geometry. We conclude in Sec. 5 and provide additional technical details in the appendices. We consistently work at tree level throughout this paper, and leave a generalization of our results to loop-level amplitudes to future work.

2 EFT amplitudes on the target manifold

In this section, we revisit the well-established framework of field space geometry. We provide a new perspective on the on-shell covariance of EFT amplitudes, as defined on the target manifold. This sets the stage for a natural generalization to functional geometry as defined on the field configuration manifold.

2.1 Target manifold

Consider an EFT of N real scalars ϕ^i, with $i = 1, \dots, N$. The action truncated at $\mathcal{O}(\partial^2)$ can be written as:

$$S[\phi] = \int_x \left[\frac{1}{2}\, g_{ij}(\phi)(\partial_\mu \phi^i)(\partial^\mu \phi^j) - U(\phi) \right], \tag{2.1}$$

where

$$\int_x \;\equiv\; \int \mathrm{d}^d x\,, \tag{2.2}$$

and $g_{ij}(\phi) = g_{ji}(\phi)$ is symmetric and positive-definite at the vacuum, defined to be the minimum of $U(\phi)$. The starting point of field space geometry is that $g_{ij}(\phi)$ is a metric on the target manifold, while $U(\phi)$ is a scalar function [18, 19]. Indeed, under a coordinate transformation on the target manifold, i.e., a nonderivative field redefinition, $\phi^i = f^i(\widetilde{\phi})$ where f^i are a set of analytic functions:

$$\begin{aligned} S[\phi] &= \int_x \left[\frac{1}{2}\, g_{ij}\big(f(\widetilde{\phi})\big)\, \frac{\partial \phi^i}{\partial \widetilde{\phi}^k} \frac{\partial \phi^j}{\partial \widetilde{\phi}^l}\, (\partial_\mu \widetilde{\phi}^k)(\partial^\mu \widetilde{\phi}^l) - U\big(f(\widetilde{\phi})\big) \right] \\ &\equiv \int_x \left[\frac{1}{2}\, \widetilde{g}_{kl}(\widetilde{\phi})(\partial_\mu \widetilde{\phi}^k)(\partial^\mu \widetilde{\phi}^l) - \widetilde{U}(\widetilde{\phi}) \right] = \widetilde{S}[\widetilde{\phi}]\,, \end{aligned} \tag{2.3}$$

where

$$\widetilde{g}_{kl}(\widetilde{\phi}) = \frac{\partial \phi^i}{\partial \widetilde{\phi}^k} \frac{\partial \phi^j}{\partial \widetilde{\phi}^l}\, g_{ij}\big(f(\widetilde{\phi})\big) \qquad \text{and} \qquad \widetilde{U}\big(\widetilde{\phi}\big) = U\big(f(\widetilde{\phi})\big)\,, \tag{2.4}$$

in agreement with the transformations of a $(0, 2)$ tensor and a scalar, respectively.[2]

[2] As usual, we refer to "tensor functions" simply as "tensors." It should be clear from the context whether we are talking about a tensor function on the entire manifold or a tensor at a given point on the manifold.

The identification of $g_{ij}(\phi)$ as a metric endows the target manifold with a Riemannian structure. From the metric, we can obtain the Levi-Civita connection and Riemann curvature tensor in the usual way:

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right), \tag{2.5a}$$

$$R^i{}_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{kl}\,, \tag{2.5b}$$

$$\begin{aligned} R_{ijkl} = g_{im}R^m{}_{jkl} &= \frac{1}{2}\left(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}\right) \\ &\quad + g_{mn}\Gamma^m_{il}\Gamma^n_{jk} - g_{mn}\Gamma^m_{ik}\Gamma^n_{jl}\,, \end{aligned} \tag{2.5c}$$

where comma denotes a partial derivative on the target manifold, e.g., $g_{ij,k} = \frac{\partial g_{ij}}{\partial \phi^k}$. Covariant derivatives are denoted by semicolons, e.g., for a tensor $T^{i\dots}{}_{j\dots}$,

$$T^{i\dots}{}_{j\dots;k} = T^{i\dots}{}_{j\dots,k} + \left(\Gamma^i_{kl}\, T^{l\dots}{}_{j\dots} + \cdots\right) - \left(\Gamma^l_{kj}\, T^{i\dots}{}_{l\dots} + \cdots\right). \tag{2.6}$$

Of particular interest are covariant derivatives of the scalar potential $U(\phi)$:

$$U_{;i_1} = U_{,i_1}\,, \tag{2.7a}$$

$$U_{;i_1i_2} = U_{,i_1i_2} - \Gamma^j_{i_1i_2}\, U_{,j}\,, \tag{2.7b}$$

$$U_{;i_1i_2i_3} = U_{,i_1i_2i_3} - (\Gamma^j_{i_1i_2}\, U_{,ji_3})_{3\text{ terms}} - \Gamma^j_{i_1i_2i_3}\, U_{,j}\,, \tag{2.7c}$$

$$\begin{aligned} U_{;i_1i_2i_3i_4} = U_{,i_1i_2i_3i_4} &- (\Gamma^j_{i_1i_2}\, U_{,ji_3i_4})_{6\text{ terms}} + (\Gamma^j_{i_1i_2}\,\Gamma^k_{i_3i_4}\, U_{,jk})_{3\text{ terms}} \\ &- (\Gamma^j_{i_1i_2i_3}\, U_{,ji_4})_{4\text{ terms}} - \Gamma^j_{i_1i_2i_3i_4}\, U_{,j}\,, \end{aligned} \tag{2.7d}$$

where we have introduced the generalized Christoffel symbols $\Gamma^j_{i_1\dots i_n}$. They are defined recursively via:

$$\Gamma^j_{i_1\dots i_n i_{n+1}} \equiv \Gamma^j_{i_1\dots i_n,i_{n+1}} - \sum_{a=1}^{n} \Gamma^k_{i_{n+1}i_a}\, \Gamma^j_{i_1\dots \not{i}_a k\dots i_n}\,, \tag{2.8}$$

where $\not{i}_a k$ means the index i_a is replaced by k. We use $(\dots)_{n\text{ terms}}$ as a shorthand for a sum of inequivalent n terms of the form of the expression in the parentheses. For example, $(\Gamma^j_{i_1i_2}\, U_{,ji_3})_{3\text{ terms}} = \Gamma^j_{i_1i_2}\, U_{,ji_3} + \Gamma^j_{i_1i_3}\, U_{,ji_2} + \Gamma^j_{i_2i_3}\, U_{,ji_1}$ is obtained by summing over permutations of open indices, while $(\Gamma^j_{i_1i_2i_3}\, U_{,ji_4})_{4\text{ terms}} = \Gamma^j_{i_1i_2i_3}\, U_{,ji_4} + \Gamma^j_{i_1i_2i_4}\, U_{,ji_3} + \Gamma^j_{i_1i_3i_4}\, U_{,ji_2} + \Gamma^j_{i_2i_3i_4}\, U_{,ji_1}$ is obtained by summing over terms where each of $i_1, \dots, i_4$ plays the role of i_4 in $\Gamma^j_{i_1i_2i_3}\, U_{,ji_4}$, and the ordering of the remaining three indices is preserved.

2.2 Amplitudes and correlation functions

To derive scattering amplitudes from the geometric quantities introduced above, we need to first find the particle states. At tree level, this amounts to diagonalizing the quadratic part of Eq. (2.1) around the vacuum $\overline{\phi}$. We expand $\phi^i(x) = \overline{\phi}^i + \eta^i(x)$ with η the quantum fluctuation, and obtain:

$$S[\overline{\phi}+\eta] = \int_x \bigg[\frac{1}{2}\Big(\overline{g}_{ij} + \overline{g}_{ij,k}\,\eta^k + \frac{1}{2}\,\overline{g}_{ij,kl}\,\eta^k\eta^l + \cdots \Big)(\partial_\mu\eta^i)(\partial^\mu\eta^j) \\ - \overline{U} - \frac{1}{2}\,\overline{U}_{,ij}\,\eta^i\eta^j - \frac{1}{3!}\,\overline{U}_{,ijk}\,\eta^i\eta^j\eta^k - \cdots \bigg], \tag{2.9}$$

where bar means evaluated at $\phi = \overline{\phi}$, e.g., $\overline{g}_{ij} \equiv g_{ij}(\overline{\phi})$. The metric can be diagonalized by a vielbein $e^\alpha{}_i$ at any point on the manifold.[3] At the vacuum point $\overline{\phi}$, we can choose the vielbein such that $\overline{U}_{,ij}$ is also diagonalized:

$$\overline{g}_{ij} = \overline{e}^\alpha{}_i\,\overline{e}^\beta{}_j\,\delta_{\alpha\beta}\,, \qquad \text{and} \qquad \overline{U}_{,ij} = \overline{e}^\alpha{}_i\,\overline{e}^\beta{}_j\,m^2_{\alpha\beta}\,, \tag{2.10}$$

where $m^2_{\alpha\beta} = m^2_\alpha\,\delta_{\alpha\beta}$ is the diagonal mass matrix. We use Greek letters $\alpha, \beta, \ldots$ to label mass eigenstates.

It is more convenient to work with momentum space fields, which are related to position space fields via:

$$\eta^i(p) = \int_x e^{ip\cdot x}\,\eta^i(x) \qquad \text{and} \qquad \eta^i(x) = \int_p e^{-ip\cdot x}\,\eta^i(p)\,, \tag{2.11}$$

where

$$\int_p \;\equiv\; \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,. \tag{2.12}$$

In terms of $\eta^i(p)$, the action reads:

$$S[\overline{\phi}+\eta] = S[\overline{\phi}] + \frac{1}{2}\int_p \eta^i(-p)\,\overline{\Delta}^{-1}_{ij}(p)\,\eta^j(p) + \mathcal{O}(\eta^3) \\ = S[\overline{\phi}] + \frac{1}{2}\int_p \big(\overline{e}^\alpha{}_i\,\eta^i(-p)\big)\,\overline{\Delta}^{-1}_{\alpha\beta}(p)\,\big(\overline{e}^\beta{}_j\,\eta^j(p)\big) + \mathcal{O}(\eta^3)\,, \tag{2.13}$$

where

$$\overline{\Delta}^{-1}_{ij}(p) \equiv \overline{g}_{ij}\,p^2 - \overline{U}_{,ij} = \overline{e}^\alpha{}_i\,\overline{e}^\beta{}_j\,\overline{\Delta}^{-1}_{\alpha\beta}(p)\,, \tag{2.14a}$$

$$\overline{\Delta}^{-1}_{\alpha\beta}(p) \equiv \delta_{\alpha\beta}(p^2 - m^2_\alpha)\,. \tag{2.14b}$$

The inverse of these expressions, $\overline{\Delta}^{ij}(p)$, $\overline{\Delta}^{\alpha\beta}(p)$, which satisfy $\overline{\Delta}^{ij}(p)\,\overline{\Delta}^{-1}_{jk}(p) = \delta^i_k$, $\overline{\Delta}^{\alpha\beta}(p)\,\overline{\Delta}^{-1}_{\beta\gamma}(p) = \delta^\alpha_\gamma$, give propagators in the flavor and mass bases, respectively.

By the Lehmann-Symanzik-Zimmermann (LSZ) formula, scattering amplitudes are given by residues on the poles of time-ordered correlation functions of mass

[3] Since the dimension of the manifold is arbitrary, we use the term "vielbein" as opposed to "tetrad" (which is often used in general relativity since the spacetime manifold is four dimensional).

eigenstate fields $\overline{e}^{\alpha}{}_{i}\,\eta^{i}$. At tree level (neglecting loop contributions to wave function renormalization factors), we have:

$$\begin{aligned}
&\left\langle \overline{e}^{\alpha_1}{}_{i_1}\,\eta^{i_1}(p_1)\ldots\,\overline{e}^{\alpha_n}{}_{i_n}\,\eta^{i_n}(p_n)\right\rangle_{\mathrm{c}} \\
&\quad = \frac{i}{p_1^2-m_{\alpha_1}^2}\,\cdots\,\frac{i}{p_n^2-m_{\alpha_n}^2}\,(2\pi)^d\,\delta^d(p_1+\cdots+p_n)\,i\mathcal{A}_{\alpha_1\ldots\alpha_n}(-p_1,\ldots,-p_n) \\
&\qquad + (\text{non-singular}) \\
&\quad = i^{n+1}\,\overline{\Delta}^{\alpha_1\beta_1}(p_1)\ldots\,\overline{\Delta}^{\alpha_n\beta_n}(p_n)\,(2\pi)^d\,\delta^d(p_1+\cdots+p_n)\,\mathcal{A}_{\beta_1\ldots\beta_n}(-p_1,\ldots,-p_n) \\
&\qquad + (\text{non-singular})\,,
\end{aligned}\tag{2.15}$$

where "c" means "connected,"[4] and all momenta in the argument of $\mathcal{A}$ are incoming. Our convention here is such that upper (lower) indices are consistently associated with outgoing (incoming) momenta. This explains the awkward minus signs in Eq. (2.15) but will turn out convenient later. Multiplying both sides of Eq. (2.15) by inverse propagators and taking the on-shell limit, we obtain:

$$\begin{aligned}
&(2\pi)^d\,\delta^d(p_1+\cdots+p_n)\,\mathcal{A}_{\alpha_1\ldots\alpha_n}(-p_1,\ldots,-p_n) \\
&\quad = \frac{1}{i^{n+1}}\lim_{p_a^2\to m_{\alpha_a}^2}\,\overline{e}_{\alpha_1}{}^{i_1}\ldots\,\overline{e}_{\alpha_n}{}^{i_n}\,\overline{\Delta}^{-1}_{i_1j_1}(p_1)\,\ldots\,\overline{\Delta}^{-1}_{i_nj_n}(p_n)\left\langle\eta^{j_1}(p_1)\ldots\,\eta^{j_n}(p_n)\right\rangle_{\mathrm{c}}\,,
\end{aligned}\tag{2.16}$$

where $\overline{e}_{\alpha}{}^{i}$ is the inverse of $\overline{e}^{\alpha}{}_{i}$ satisfying $\overline{e}_{\alpha}{}^{i}\,\overline{e}^{\beta}{}_{i}=\delta^{\beta}_{\alpha}$, and we have used the relation between $\overline{\Delta}^{-1}_{ij}(p)$ and $\overline{\Delta}^{-1}_{\alpha\beta}(p)$ from Eq. (2.14a).

The object $\left\langle\eta^{i_1}(p_1)\ldots\,\eta^{i_n}(p_n)\right\rangle_{\mathrm{c}}$ in Eq. (2.16) represents a connected correlation function among the fluctuations of the fields around the vacuum point $\overline{\phi}$ on the target manifold (we leave time ordering implicit throughout the paper). It can be computed from the path integral by taking functional derivatives with respect to a source that couples to η:

$$\begin{aligned}
&\left\langle\eta^{i_1}(p_1)\ldots\,\eta^{i_n}(p_n)\right\rangle_{\mathrm{c}} \\
&\quad = \left[\frac{(2\pi)^d}{i}\right]^n\frac{\delta^n}{\delta J_{i_1}(p_1)\ldots\delta J_{i_n}(p_n)}\log\int\mathcal{D}\eta\;e^{i\left(S[\overline{\phi}+\eta]+\int_x J_i\,\eta^i\right)}\bigg|_{J_i(x)=0}\,,
\end{aligned}\tag{2.17}$$

where $J_i(p)=\int_x e^{-ip\cdot x}\,J_i(x)$, consistent with the sign convention discussed below Eq. (2.15). Substituting Eq. (2.9) into Eq. (2.17) and perturbatively evaluating the path integral, one can derive the standard Feynman rules, where the propagator is given by $\overline{\Delta}^{ij}(p)$ and vertices are given by functions of $\overline{g}_{ij,k\ldots}$, $\overline{U}_{,ijk\ldots}$ and momenta.

As mentioned in the Introduction, in order to connect to recent developments in functional geometry, we must study correlation functions defined away from the

[4]Technically, the LSZ formula applies for full correlation functions. Here we are extracting the connected piece, which is proportional to the overall momentum-conserving δ-function, to define $\mathcal{A}$.

vacuum. Expanding the action around an arbitrary point ϕ on the target manifold, we have:

$$S[\phi+\eta] = \int_x \bigg[\frac{1}{2}\Big(g_{ij}(\phi) + g_{ij,k}(\phi)\,\eta^k + \frac{1}{2}\, g_{ij,kl}(\phi)\,\eta^k\eta^l + \cdots \Big) (\partial_\mu \eta^i)(\partial^\mu \eta^j) \\ - U(\phi) - U_{,i}(\phi)\,\eta^i - \frac{1}{2}\, U_{,ij}(\phi)\,\eta^i\eta^j - \frac{1}{3!}\, U_{,ijk}(\phi)\,\eta^i\eta^j\eta^k - \cdots \bigg] . \tag{2.18}$$

Eq. (2.18) takes the same form as Eq. (2.9), but with derivatives of U and g_{ij} evaluated at ϕ instead of $\overline{\phi}$. If we include a constant source $J_i(x) = U_{,i}(\phi)$ to cancel the linear term, and define:

$$\big\langle \eta^{i_1}(p_1) \dots \eta^{i_n}(p_n) \big\rangle_{\mathrm{c},\,\phi} \\ \equiv \left[\frac{(2\pi)^d}{i}\right]^n \frac{\delta^n}{\delta J_{i_1}(p_1)\dots\delta J_{i_n}(p_n)} \log \int \mathcal{D}\eta\, e^{i\left(S[\phi+\eta]+\int_x J_i\,\eta^i\right)} \bigg|_{J_i(x)=U_{,i}(\phi)} , \tag{2.19}$$

we would obtain the same set of Feynman rules as we would from Eq. (2.17), but with $\overline{\Delta}^{ij}(p)$, $\overline{g}_{ij,k\dots}$, $\overline{U}_{,ijk\dots}$ replaced by $\Delta^{ij}(\phi\,;p)$, $g_{ij,k\dots}(\phi)$, $U_{,ijk\dots}(\phi)$, where $\Delta^{ij}(\phi\,;p)$ is the inverse of

$$\Delta^{-1}_{ij}(\phi\,;p) = g_{ij}(\phi)\,p^2 - U_{,ij}(\phi)\,. \tag{2.20}$$

A more useful set of quantities to consider are *amputated correlation functions*, which we denote by $\mathcal{M}_{i_1\dots i_n}$. They are defined at any point ϕ on the target manifold by factoring out the external propagators and the overall momentum-conserving δ-function from connected correlation functions:

$$\big\langle \eta^{i_1}(p_1) \dots \eta^{i_n}(p_n) \big\rangle_{\mathrm{c},\,\phi} \\ \equiv i^n\, \Delta^{i_1 j_1}(\phi\,;p_1)\dots\Delta^{i_n j_n}(\phi\,;p_n)\,(2\pi)^d\,\delta^d(p_1+\dots+p_n)\, i\mathcal{M}_{j_1\dots j_n}(\phi\,;-p_1,\dots,-p_n)\,. \tag{2.21}$$

Combining Eqs. (2.16) and (2.21), we obtain:

$$\mathcal{A}_{\alpha_1\dots\alpha_n}(p_1,\dots,p_n) = \lim_{p_a^2\to m_{\alpha_a}^2} \overline{e}_{\alpha_1}{}^{i_1}\dots\,\overline{e}_{\alpha_n}{}^{i_n}\, \overline{\mathcal{M}}_{i_1\dots i_n}(p_1,\dots,p_n)\,, \tag{2.22}$$

where $\overline{\mathcal{M}}_{i_1\dots i_n}(p_1,\dots,p_n)$ denotes $\mathcal{M}_{i_1\dots i_n}(\phi\,;p_1,\dots,p_n)$ evaluated at $\phi=\overline{\phi}$ (corresponding to $J_i(\phi)=0$).

2.3 On-shell covariance

From Eq. (2.22), we see that on-shell amplitudes $\mathcal{A}_{\alpha_1\dots\alpha_n}(p_1,\dots,p_n)$ can be obtained from amputated correlation functions $\mathcal{M}_{i_1\dots i_n}(\phi\,;p_1,\dots,p_n)$ on the target manifold by

1) taking the vacuum limit $\phi\to\overline{\phi}$,

2) contracting with vielbein factors $\overline{e}_{\alpha_a}{}^{i_a}$ for all external particles, and

3) taking the on-shell limit $p_a^2 \to m_{\alpha_a}^2$.

For a fixed set of particles labeled by $\alpha_1 \ldots \alpha_n$, the vielbein factors transform as vectors under coordinate changes on the target manifold. Since $\mathcal{A}_{\alpha_1\ldots\alpha_n}(p_1, \ldots, p_n)$ are physical observables, Eq. (2.22) implies that in the vacuum and on-shell limits, amputated correlation functions $\mathcal{M}_{i_1\ldots i_n}(\phi\,; p_1, \ldots, p_n)$ must transform as tensors. More precisely, under a nonderivative field redefinition $\phi^i = f^i(\widetilde{\phi})$,

$$\widetilde{\mathcal{M}}_{i_1\ldots i_n}(\widetilde{\phi}) = \frac{\partial\phi^{j_1}}{\partial\widetilde{\phi}^{i_1}} \cdots \frac{\partial\phi^{j_n}}{\partial\widetilde{\phi}^{i_n}}\, \mathcal{M}_{j_1\ldots j_n}\big(f(\widetilde{\phi})\big) + X_{i_1\ldots i_n}(\widetilde{\phi})\,, \tag{2.23}$$

where $X_{i_1\ldots i_n}$ satisfies

$$\lim_{p_a^2\to m_{\alpha_a}^2} \overline{e}_{\alpha_1}{}^{j_1}\, \frac{\partial\widetilde{\phi}^{i_1}}{\partial\phi^{j_1}} \,\cdots\, \overline{e}_{\alpha_n}{}^{j_n}\, \frac{\partial\widetilde{\phi}^{i_n}}{\partial\phi^{j_n}}\, \overline{X}_{i_1\ldots i_n} = 0\,, \tag{2.24}$$

and momentum arguments $p_1, \ldots, p_n$ are implicit. We will abbreviate Eq. (2.24) as:

$$X_{i_1\ldots i_n} \overset{\text{OS}}{=} 0\,, \tag{2.25}$$

where "$\overset{\text{OS}}{=}$" means equal upon performing the three operations listed at the beginning of this subsection. We call this relation between objects on the target manifold *on-shell equivalence* to emphasize the "taking the on-shell limit" operation, though it is understood that the vacuum limit is also taken and the open indices are contracted with vielbein factors. Eq. (2.25) says $X_{i_1\ldots i_n}$ is on-shell equivalent to zero.[5] Two other examples of objects that are on-shell equivalent to zero are:

$$U_{,j} \overset{\text{OS}}{=} 0 \qquad \text{and} \qquad \Delta^{-1}_{ji_a}(p_a) \overset{\text{OS}}{=} 0\,, \tag{2.26}$$

where p_a is an external momentum. The first derivative of the potential vanishes as soon as the vacuum limit is taken: $\overline{U}_{,j} = 0$. The inverse propagator vanishes after all three operations listed above are performed:

$$\lim_{p_a^2\to m_{\alpha_a}^2} \overline{e}_{\alpha_a}{}^{i_a}\, \overline{\Delta}^{-1}_{ji_a}(p_a) = \lim_{p_a^2\to m_{\alpha_a}^2} \overline{\Delta}^{-1}_{\alpha_a\beta_a}(p_a)\, \overline{e}^{\beta_a}{}_j = 0\,. \tag{2.27}$$

We say an object on the target manifold is *on-shell covariant* if it transforms as a tensor up to inhomogeneous terms that are on-shell equivalent to zero. On-shell covariance of amputated correlation functions was discussed previously in the context of functional geometry [48, 50, 51]. Here we see its counterpart in field space geometry: $\mathcal{M}_{i_1\ldots i_n}$ are generally not tensors on the target manifold, but must be on-shell covariant. Two properties of on-shell covariant objects immediately follow from this definition: sums and products of on-shell covariant objects are on-shell covariant; if an object is on-shell equivalent to a tensor, it must be on-shell covariant.

[5] Objects that are on-shell equivalent to zero are referred to "evanescent" in Ref. [50]. We choose not to use "evanescent" here to avoid confusion with the more common use of the term as referring to operators that are present in dimensional regularization but vanish in four-dimensional spacetime.

2.4 On-shell covariant building blocks

We can make the on-shell covariance of $\mathcal{M}_{i_1\dots i_n}(\phi\,;p_1,\dots,p_n)$ manifest by expressing them in terms of on-shell covariant building blocks. Let us first illustrate this explicitly for $n=3$ and 4, before generalizing to arbitrary n. From the standard Feynman rules, we obtain:

$$\mathcal{M}_{i_1i_2i_3}(p_1,p_2,p_3) = V_{i_1i_2i_3}(p_1,p_2,p_3)\,, \tag{2.28a}$$

$$\begin{aligned}\mathcal{M}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4) &= V_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4)\\ &\quad - \Big[\Delta^{jk}(p_{12})\,V_{ji_1i_2}(-p_{12},p_1,p_2)\,V_{ki_3i_4}(p_{12},p_3,p_4)\Big]_{3\text{ terms}}\,,\end{aligned} \tag{2.28b}$$

where $p_{ab}\equiv p_a+p_b$, and $V_{ijk\dots}$ are the vertex functions with all momenta incoming:

$$\begin{aligned}V_{i_1i_2i_3}(p_1,p_2,p_3) &= -\Big[(p_1\cdot p_2)\,g_{i_1i_2,i_3}\Big]_{3\text{ terms}} - U_{,i_1i_2i_3}\\ &= \frac{1}{2}\Big[p_1^2\,(g_{i_1i_2,i_3}+g_{i_1i_3,i_2}-g_{i_2i_3,i_1})\Big]_{3\text{ terms}} - U_{,i_1i_2i_3}\,,\end{aligned} \tag{2.29a}$$

$$\begin{aligned}V_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4) &= -\Big[(p_1\cdot p_2)\,g_{i_1i_2,i_3i_4}\Big]_{6\text{ terms}} - U_{,i_1i_2i_3i_4}\\ &= -\frac{1}{2}\Big[p_{12}^2\,(g_{i_1i_2,i_3i_4}+g_{i_3i_4,i_1i_2})\Big]_{3\text{ terms}}\\ &\quad+\frac{1}{2}\Big[p_1^2\,(g_{i_1i_2,i_3i_4}+g_{i_1i_3,i_2i_4}+g_{i_1i_4,i_2i_3})\Big]_{4\text{ terms}} - U_{,i_1i_2i_3i_4}\,.\end{aligned} \tag{2.29b}$$

Here and in what follows, we leave the ϕ arguments implicit to reduce notational clutter. It is understood that all quantities are evaluated at a general point ϕ on the target manifold, unless specified otherwise.

We now consider how the objects entering Eq. (2.29) transform under nonderivative field redefinitions. The internal propagator is the inverse of $\Delta^{-1}_{ij}(p)$ defined in Eq. (2.20), which transforms as

$$\widetilde{\Delta}^{-1}_{ij}(p) = \frac{\partial\phi^k}{\partial\widetilde{\phi}^i}\,\frac{\partial\phi^l}{\partial\widetilde{\phi}^j}\,\Delta^{-1}_{kl}(p) - \frac{\partial^2\phi^k}{\partial\widetilde{\phi}^i\partial\widetilde{\phi}^j}\,U_{,k}\,. \tag{2.30}$$

This is not a tensorial transformation. However, the inhomogeneous term on the right-hand side of Eq. (2.30) is on-shell equivalent to zero because $U_{,k}\overset{\text{OS}}{=}0$. Therefore, $\Delta^{-1}_{ij}(p)$ is on-shell covariant, and so is $\Delta^{ij}(p)$. A quicker way to reach the same conclusion is to note that $\Delta^{ij}(p)$ is on-shell equivalent to a tensor:

$$\Delta^{ij}(p)\overset{\text{OS}}{=}\boldsymbol{\Delta}^{ij}(p)\,, \tag{2.31}$$

where $\boldsymbol{\Delta}^{ij}(p)$ (note boldface) is the inverse of

$$\boldsymbol{\Delta}^{-1}_{ij}(p) \equiv g_{ij}\, p^2 - U_{;ij}\,. \tag{2.32}$$

Note that $U_{;ij} = U_{,ij} - \Gamma^k_{ij}\, U_{,k}$ is symmetric since the Levi-Civita connection satisfies $\Gamma^k_{ij} = \Gamma^k_{ji}$.

The vertex functions in Eq. (2.29), on the other hand, do not have simple transformation properties under field redefinitions. Only when they are assembled into $\mathcal{M}_{i_1 i_2 i_3}$ and $\mathcal{M}_{i_1 i_2 i_3 i_4}$ according to Eq. (2.28) do we know that the results must be on-shell covariant. However, we can regroup the terms in Eq. (2.28) such that the individual building blocks are on-shell covariant. Using Eqs. (2.5) and (2.7), we find, after some algebra:

$$\mathcal{M}_{i_1 i_2 i_3}(p_1,p_2,p_3) = \mathcal{V}_{i_1 i_2 i_3}(p_1,p_2,p_3)\,, \tag{2.33a}$$

$$\begin{aligned}\mathcal{M}_{i_1 i_2 i_3 i_4}(p_1,p_2,p_3,p_4) &= \mathcal{V}_{i_1 i_2 i_3 i_4}(p_1,p_2,p_3,p_4)\\ &\quad - \Big[\boldsymbol{\Delta}^{jk}(p_{12})\, \mathcal{V}_{j i_1 i_2}(-p_{12},p_1,p_2)\, \mathcal{V}_{k i_3 i_4}(p_{12},p_3,p_4)\Big]_{3\text{ terms}}\,,\end{aligned} \tag{2.33b}$$

where

$$\begin{aligned}\mathcal{V}_{i_1 i_2 i_3}(p_1,p_2,p_3) &= -U_{;(i_1 i_2 i_3)} + \Gamma^j_{(i_1 i_2 i_3)} U_{,j}\\ &\quad + \Gamma^j_{i_2 i_3}\, \boldsymbol{\Delta}^{-1}_{j i_1}(p_1) + \Gamma^j_{i_1 i_3}\, \boldsymbol{\Delta}^{-1}_{j i_2}(p_2) + \Gamma^j_{i_1 i_2}\, \boldsymbol{\Delta}^{-1}_{j i_3}(p_3)\,,\end{aligned} \tag{2.34a}$$

$$\begin{aligned}\mathcal{V}_{j i_1 i_2}(p_j,p_1,p_2) &= -U_{;(j i_1 i_2)} + \Gamma^l_{(j i_1 i_2)} U_{,l}\\ &\quad + \Gamma^l_{j i_2}\, \boldsymbol{\Delta}^{-1}_{l i_1}(p_1) + \Gamma^l_{j i_1}\, \boldsymbol{\Delta}^{-1}_{l i_2}(p_2)\,,\end{aligned} \tag{2.34b}$$

$$\begin{aligned}\mathcal{V}_{i_1 i_2 i_3 i_4}(p_1,p_2,p_3,p_4) &= -\frac{2}{3}\Big[p_{12}^2\, R_{i_1 (i_3 i_4) i_2}\Big]_{3\text{ terms}} - U_{;(i_1 i_2 i_3 i_4)} + \Gamma^j_{(i_1 i_2 i_3 i_4)} U_{,j}\\ &\quad + \Big[\Gamma^j_{(i_2 i_3 i_4)}\, \boldsymbol{\Delta}^{-1}_{j i_1}(p_1)\Big]_{4\text{ terms}}\,.\end{aligned} \tag{2.34c}$$

Here and in what follows, parentheses denote full symmetrization of indices, e.g., $R_{i_1(i_3 i_4) i_2} = \frac{1}{2}\left(R_{i_1 i_3 i_4 i_2} + R_{i_1 i_4 i_3 i_2}\right)$, and $U_{;(i_1 i_2 i_3)} = \frac{1}{3!}\left(U_{;i_1 i_2 i_3}\right)_{6\text{ terms}}$. The first two equations in Eqs. (2.34) can be expressed in the general form:

$$\mathcal{V}_{i_1 i_2 i_3}(p_1,p_2,p_3) = -U_{;(i_1 i_2 i_3)} + \Gamma^j_{(i_1 i_2 i_3)} U_{,j} + \sum_{a\in\text{ext}} \Gamma^j_{i_1 \ldots \not{i_a} \ldots i_3}\, \boldsymbol{\Delta}^{-1}_{j i_a}(p_a)\,, \tag{2.35}$$

where "$a \in \text{ext}$" means a runs over the subset of $\{1,2,3\}$ for which p_a is an external momentum.

Note that Eq. (2.33) takes exactly the same form as the standard expressions Eq. (2.28) for the 3- and 4-point amputated correlation functions, but with the

Feynman vertices $V_{i_1 \dots i_m}$ replaced by a new type of vertices $\mathcal{V}_{i_1 \dots i_m}$. In other words, we have just seen that:

$$\boxed{\mathcal{M}_{i_1 \dots i_n}(p_1, \dots, p_n) = \mathcal{F}_{i_1 \dots i_n}\big(\Delta^{ij}, \{V_{j_1 \dots j_m}\}\big) = \mathcal{F}_{i_1 \dots i_n}\big(\Delta^{ij}, \{\mathcal{V}_{j_1 \dots j_m}\}\big)\,,} \tag{2.36}$$

for $n = 3, 4$. We emphasize that the ϕ arguments are implicit, i.e., this equation is satisfied at any point on the target manifold. In Sec. 2.5, we will show that Eq. (2.36) holds for all $n \geq 3$, with properly defined $\mathcal{V}_{i_1 \dots i_m}$ (see Eq. (2.51)). The new vertices $\mathcal{V}_{i_1 \dots i_m}$ are not tensors on the entire target manifold because they contain nontensorial terms of the form:

$$\Gamma^j_{(i_1 \dots i_m)}\, U_{,j} \qquad \text{and} \qquad \Gamma^j_{(i_1 \dots \not{i_a} \dots i_m)}\, \Delta^{-1}_{j i_a}(p_a) \quad (a \in \text{ext})\,. \tag{2.37}$$

Importantly, these nontensorial terms are on-shell equivalent to zero because of Eq. (2.26). As a result, the $\mathcal{V}_{i_1 \dots i_m}$ vertices are on-shell equivalent to a set of tensors (which have appeared previously in on-shell amplitudes expressions [21, 27]):

$$\mathcal{V}_{i_1 i_2 i_3}(p_1, p_2, p_3) \overset{\text{OS}}{=} -U_{;(i_1 i_2 i_3)}\,, \tag{2.38a}$$

$$\mathcal{V}_{i_1 i_2 i_3 i_4}(p_1, p_2, p_3, p_4) \overset{\text{OS}}{=} -\frac{2}{3}\left[p_{12}^2\, R_{i_1 (i_3 i_4) i_2}\right]_{3\text{ terms}} - U_{;(i_1 i_2 i_3 i_4)}\,, \tag{2.38b}$$

and similarly for higher-point $\mathcal{V}_{i_1 \dots i_m}$ vertices as we will see below. In other words, the $\mathcal{V}_{i_1 \dots i_m}$ vertices are on-shell covariant. Eq. (2.36) therefore gives a manifestly on-shell covariant expression for the n-point amputated correlation function for an EFT of the form Eq. (2.1) (the special cases of $n = 3, 4$ are given in Eq. (2.33)).

2.5 Origin of nontensorial terms

There is one special case where the nontensorial terms Eq. (2.37) vanish even away from the vacuum and on-shell limits: if we are in a field basis that corresponds to a set of Riemann normal coordinates at ϕ, (generalized) Christoffel symbols evaluated at ϕ would vanish upon symmetrization of its lower indices. In this subsection, we will show explicitly how the presence of nontensorial terms in the expressions of $\mathcal{M}_{i_1 \dots i_n}$ can be understood as coming from a mismatch between a general field basis and Riemann normal coordinates at ϕ. We will also provide a proof of Eq. (2.36) for general n-point amputated correlation functions.

A set of Riemann normal coordinates at ϕ can be constructed as follows. Denote the geodesic from ϕ to any point $\phi + \eta$ in the vicinity of ϕ by $\gamma(\lambda)$, which satisfies

$$\frac{\mathrm{d}^2 \gamma^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk}\, \frac{\mathrm{d}\gamma^j}{\mathrm{d}\lambda}\, \frac{\mathrm{d}\gamma^k}{\mathrm{d}\lambda} = 0\,, \tag{2.39}$$

with $\gamma(0) = \phi$, $\gamma(1) = \phi + \eta$. The Riemann normal coordinates for the point $\phi + \eta$ are components of the tangent vector to the geodesic at the origin $\xi^i = \frac{\mathrm{d}\gamma^i}{\mathrm{d}\lambda}(0)$. The relation between η^i and ξ^i can be derived by solving Eq. (2.39) order by order:

$$\eta^i = \xi^i - \sum_{m=2}^{\infty} \frac{1}{m!}\, \Gamma^i_{(i_1\dots i_m)}\, \xi^{i_1}\dots\xi^{i_m}\,, \tag{2.40}$$

where $\Gamma^i_{(i_1\dots i_m)}$ are all evaluated at ϕ. It follows that $\Gamma^i_{(i_1\dots i_m)}$ must vanish if we are already in a basis where η^i are a set of Riemann normal coordinates at ϕ.

On the other hand, if we start from a general basis, we can use Eq. (2.40) to rewrite the action in Eq. (2.18) in terms of the Riemann normal coordinates ξ^i:

$$\begin{aligned} S[\phi+\eta] = \int_x \bigg[&\frac{1}{2}\Big(g_{ij} + \frac{1}{3}\, R_{i(kl)j}\, \xi^k\xi^l + \cdots\Big)\, (\partial_\mu\xi^i)(\partial^\mu\xi^j) \\ &- U - U_{;i}\,\xi^i - \frac{1}{2}\, U_{;ij}\,\xi^i\xi^j - \frac{1}{3!}\, U_{;(ijk)}\,\xi^i\xi^j\xi^k - \cdots \bigg]. \end{aligned} \tag{2.41}$$

The coefficient functions in Eq. (2.41) are all tensors, which is expected since ξ^i are components of the tangent vector to the geodesic. As a result, Feynman rules for the ξ fields as derived from $S[\phi+\eta]$ are written entirely in terms of tensors. The propagator is given by $\boldsymbol{\Delta}^{ij}(p)$, and the vertices are given by:

$$V^{(\xi)}_{i_1i_2i_3}(p_1,p_2,p_3) = -U_{;(i_1i_2i_3)}\,, \tag{2.42a}$$

$$\begin{aligned} V^{(\xi)}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4) &= -\frac{2}{3}\Big[(p_1\cdot p_2)\, R_{i_1(i_3i_4)i_2}\Big]_{6\text{ terms}} - U_{;(i_1i_2i_3i_4)} \\ &= -\frac{2}{3}\Big[p_{12}^2\, R_{i_1(i_3i_4)i_2}\Big]_{3\text{ terms}} - U_{;(i_1i_2i_3i_4)}\,, \end{aligned} \tag{2.42b}$$

etc., where we have used $R_{i_1i_2i_3i_4} + R_{i_1i_3i_4i_2} + R_{i_1i_4i_2i_3} = 0$ to arrive at the last line. Expressions for higher-point vertices can be derived using formulae for the derivatives of the metric and potential in Riemann normal coordinates in App. A of Ref. [21] (see also Refs. [52, 53]). We will present another algorithm that allows us to easily obtain higher-point vertices in Sec. 4.4. The result takes the following form:

$$V^{(\xi)}_{i_1\dots i_m}(p_1,\dots,p_m) = \mathcal{R}_{i_1\dots i_m}(p_1,\dots,p_m) - U_{;(i_1\dots i_m)}\,, \tag{2.43}$$

where $\mathcal{R}_{i_1\dots i_m}$ is a symmetric tensor built from the Riemann curvature tensor and its covariant derivatives (see Eq. (4.41)).

To derive full (as opposed to connected) correlation functions of η, we can relate them to correlation functions of ξ using Eq. (2.40):

$$\begin{aligned} \langle\eta^{i_1}\dots\eta^{i_n}\rangle_\phi &= \langle\xi^{i_1}\dots\xi^{i_n}\rangle_\phi \\ &\quad - \sum_{a=1}^{n}\sum_{m=2}^{\infty}\frac{1}{m!}\,\Gamma^{i_a}_{(j_1\dots j_m)}\,\big\langle\xi^{i_1}\dots\cancel{\xi^{i_a}}(\xi^{j_1}\dots\xi^{j_m})\dots\xi^{i_n}\big\rangle_\phi + \cdots\,, \end{aligned} \tag{2.44}$$

where $\not{\xi}^{i_a}(\xi^{j_1}\dots\xi^{j_m})$ means ξ^{i_a} is replaced by $\xi^{j_1}\dots\xi^{j_m}$, and the ellipses at the end of the equation represent terms where more than one ξ's in the correlation function are replaced by products of ξ's. The correlation functions in Eq. (2.44) are all evaluated with respect to the following action plus source term:[6]

$$
\begin{aligned}
&S[\phi+\eta] + \int_x U_{,i}\,\eta^i \\
&\qquad = S[\phi+\eta] + \int_x U_{,i}\,\xi^i - \int_x \sum_{m=2}^{\infty} \frac{1}{m!}\left(\Gamma^i_{(i_1\dots i_m)}\,U_{,i}\right)\xi^{i_1}\dots\xi^{i_m}\,.
\end{aligned}
\tag{2.45}
$$

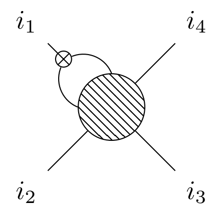
We see that when computing the ξ correlation functions appearing on the right-hand side of Eq. (2.44), we must include the extra terms $-\int_x \sum\limits_{m=2}^{\infty} \frac{1}{m!}\left(\Gamma^i_{(i_1\dots i_m)}\,U_{,i}\right)\xi^{i_1}\dots\xi^{i_m}$ in addition to the original action. In particular, the quadratic term restores the propagator to its standard nontensorial form:

$$
\begin{aligned}
S[\phi+\eta]\Big|_{\mathcal{O}(\xi^2)} - \frac{1}{2}\int_x \Gamma^i_{jk}\,U_{,i}\,\xi^j\xi^k &= \frac{1}{2}\int_p \xi^j(-p)\left[\boldsymbol{\Delta}^{-1}_{jk}(p) - \Gamma^i_{jk}\,U_{,i}\right]\xi^k(p) \\
&= \frac{1}{2}\int_p \xi^j(-p)\,\Delta^{-1}_{jk}(p)\,\xi^k(p)\,.
\end{aligned}
\tag{2.46}
$$

Meanwhile, the cubic and higher terms in Eq. (2.45) amount to an extra contribution to the vertices:

$$
\delta V^{(\xi)}_{i_1\dots i_m} = -\Gamma^i_{(i_1\dots i_m)}\,U_{,i}\,. \tag{2.47}
$$

Now let us take a closer look at the additional terms in the second line of Eq. (2.44), which give the difference between $\langle \eta^{i_1}\dots\eta^{i_n}\rangle$ and $\langle \xi^{i_1}\dots\xi^{i_n}\rangle$. These terms involve correlation functions between fields and local operators. For example, consider the case of $n=4$. The $a=1, m=2$ term on the right-hand side of Eq. (2.44) can be represented as:

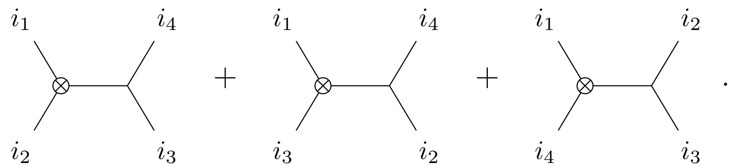


(2.48)

where $\otimes$ denotes an insertion of local operator $-\frac{1}{2}\,\Gamma^{i_1}_{j_1 j_2}\,\xi^{j_1}\xi^{j_2}$. The blob represents a 5-point full (as opposed to connected) correlation function of ξ, which can be computed by inserting vertices and performing Wick contractions as usual. If we

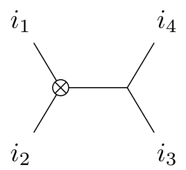
[6] The path integral measures $\mathcal{D}\eta$ and $\mathcal{D}\xi$ differ by a Jacobian, which is a functional determinant and can be neglected for tree-level calculations. Even at loop level, the Jacobian factor for a nonderivative field redefinition like Eq. (2.40) only gives rise to scaleless integrals which vanish in dimensional regularization (see e.g., Refs. [54–57]).

require the full diagram to be a tree diagram, then the only possibility is to insert a 3-point vertex and obtain:



(2.49)

The diagrams in Eq. (2.49) have the same topology as the standard set of s, t, u-channel Feynman diagrams contributing to the 4-point correlation function, but with one of the 3-point vertices replaced by the local operator insertion discussed above. Concretely, let us work through the first diagram for example:



$$\begin{aligned} &= -\Gamma^{i_1}_{j_1 j_2}\, i\Big[V^{(\xi)}_{kj_3j_4}(p_{12},p_3,p_4) + \delta V^{(\xi)}_{kj_3j_4}\Big] \\ &\qquad i\Delta^{j_1k}(p_{12})\, i\Delta^{i_2j_2}(p_2)\, i\Delta^{i_3j_3}(p_3)\, i\Delta^{i_4j_4}(p_4) \\ &= i\Big[\Gamma^m_{lj_2}\,\Delta^{-1}_{mj_1}(p_1)\Big]\, i\Big[V^{(\xi)}_{kj_3j_4}(p_{12},p_3,p_4) + \delta V^{(\xi)}_{kj_3j_4}\Big] \\ &\qquad i\Delta^{kl}(p_{12})\, i\Delta^{i_1j_1}(p_1)\, i\Delta^{i_2j_2}(p_2)\, i\Delta^{i_3j_3}(p_3)\, i\Delta^{i_4j_4}(p_4)\,, \end{aligned} \tag{2.50}$$

where symmetry factors from the vertices are canceled by combinatoric factors from Wick contractions. From Eq. (2.50) it is clear that the contribution from this diagram to the 4-point η correlation function can be reproduced by including an extra term $\Gamma^m_{lj_2}\,\Delta^{-1}_{mj_1}(p_1)$ in the 3-point vertex.

Following the same strategy, we see that each additional term on the right-hand side of Eq. (2.44) yields a similar set of diagrams, which have the same topology as the standard set of tree-level Feynman diagrams contributing to the n-point correlation function, but with one or more of the vertices replaced by $\Gamma^j_{(i_1\dots \not{i_a}\dots i_m)}\,\Delta^{-1}_{ji_a}(p_a)$, where p_a represents an external momentum. The same contributions can be reproduced if we stipulate that for each m-point vertex in a Feynman diagram, we must include an extra term $\Gamma^j_{(i_1\dots \not{i_a}\dots i_m)}\,\Delta^{-1}_{ji_a}(p_a)$ for each external particle (if any) that is directly connected to the vertex.

Combining the analysis above and the discussion around Eqs. (2.46) and (2.47), we see that correlation functions of η can be obtained from the standard set of Feynman diagrams, with propagators given by $\Delta^{ij}(p)$ and vertices given by:

$$\begin{aligned} \mathcal{V}_{i_1\dots i_m}(p_1,\dots,p_m) &= V^{(\xi)}_{i_1\dots i_m}(p_1,\dots,p_m) + \delta V^{(\xi)}_{i_1\dots i_m} + \sum_{a\in\text{ext}} \Gamma^j_{(i_1\dots \not{i_a}\dots i_m)}\,\Delta^{-1}_{ji_a}(p_a) \\ &= \mathcal{R}_{i_1\dots i_m}(p_1,\dots,p_m) - U_{;(i_1\dots i_m)} \\ &\qquad - \Gamma^j_{(i_1\dots i_m)}\,U_{,j} + \sum_{a\in\text{ext}} \Gamma^j_{(i_1\dots \not{i_a}\dots i_m)}\,\Delta^{-1}_{ji_a}(p_a)\,, \end{aligned} \tag{2.51}$$

where $\mathcal{R}_{i_1i_2i_3} = 0$, $\mathcal{R}_{i_1i_2i_3i_4} = -\frac{2}{3}\big[p_{12}^2\, R_{i_1(i_3i_4)i_2}\big]_{3\text{ terms}}$, and $\mathcal{R}_{i_1\dots i_m}$ with higher m can be constructed recursively in terms of the Riemann curvature tensor and its covariant derivatives, as we will discuss in detail in Sec. 4.4. We have therefore proved that Eq. (2.36), which expresses $\mathcal{M}_{i_1\dots i_n}$ in terms of on-shell covariant building blocks, holds for general n-point amputated correlation functions. For $m = 3$ and 4, Eq. (2.51) reproduces our previous results in Eqs. (2.34) and (2.35) obtained by manually regrouping terms in $\mathcal{M}_{i_1i_2i_3}$ and $\mathcal{M}_{i_1i_2i_3i_4}$. As discussed below Eq. (2.37), the nontensorial terms in the last line of Eq. (2.51) are on-shell equivalent to zero, so they do not contribute to on-shell amplitudes. This is consistent with the fact that η and ξ interpolate the same one-particle states.

We end this section by offering a slightly more general perspective on the appearance of nontensorial terms. At a given point ϕ on the target manifold, we can define a set of normal coordinates with respect to any connection, not just the Levi-Civita connection derived from the metric (in which case the normal coordinates are Riemann normal coordinates). The relation between a general basis η and normal coordinates ξ is given by the same Eq. (2.40), with $\Gamma^i_{(i_1\dots i_m)}$ defined from the chosen connection by recursively applying Eq. (2.8). Expanding the action in the ξ basis, one again obtains tensorial coefficients, and hence tensorial propagator and vertices. However, for a general connection, the expansion coefficients contain additional terms that involve the torsion and covariant derivatives of the metric.[7] On the other hand, the fact that Eq. (2.40) holds regardless of the choice of connection implies that the difference between the η and ξ bases can be accommodated by exactly the same nontensorial terms in the vertices as shown in the last line of Eq. (2.51), with $\Gamma^i_{(i_1\dots i_m)}$ computed from the chosen connection. Therefore, the presence of nontensorial terms in $\mathcal{M}_{i_1\dots i_n}$ can be attributed to a mismatch between a general field basis and normal coordinates at ϕ with respect to any connection.

It is useful to keep in mind that the notion of (on-shell) covariance relies on a connection, but not necessarily a metric. In field space geometry, we start out by defining a metric, and then simply stick to the unique Levi-Civita connection associated with the metric. There is no obstacle to working with a different connection, which would define a different set of on-shell covariant vertices. But there does not

[7]For example, if one chooses a metric-compatible connection that has torsion, Eq. (2.41) would be modified to [58]:

$$
\begin{aligned}
S[\phi+\eta] = \int_x \bigg\{ \frac{1}{2}\bigg[& g_{ij} - T_{(ij)k}\,\xi^k \\
& + \left(\frac{1}{3}\,R_{i(kl)j} - \frac{2}{3}\,T_{(ij)(k;l)} + \frac{1}{4}\,T^m{}_{ik}T_{mjl} - \frac{1}{3}\,T^m{}_{ik}T_{jlm}\right)\xi^k\xi^l + \cdots \bigg](\partial_\mu\xi^i)(\partial^\mu\xi^j) \\
& - U - U_{;i}\,\xi^i - \frac{1}{2}\,U_{;ij}\,\xi^i\xi^j - \frac{1}{3!}\,U_{;(ijk)}\,\xi^i\xi^j\xi^k - \cdots \bigg\}, \qquad (2.52)
\end{aligned}
$$

where $T^i{}_{jk} \equiv \Gamma^i_{jk} - \Gamma^i_{kj}$ and $T_{ijk} = g_{il}\,T^l{}_{jk}$.

seem to be any advantage either. However, the conceptual separation between a metric and a connection is useful when we generalize the discussion to functional geometry, where the definition of a metric is less obvious, as we will see in the next two sections.

3 EFT amplitudes on the field configuration manifold

The field space geometry framework discussed in the previous section has two limitations: it concerns EFT actions truncated at $\mathcal{O}(\partial^2)$, and manifests on-shell covariance under nonderivative field redefinitions only. As mentioned in the Introduction, general field redefinitions (with or without derivatives) can be viewed as coordinate transformations on the field configuration manifold. In this section, we generalize the discussion of Sec. 2 to find on-shell covariant building blocks of amplitudes on the field configuration manifold for general scalar EFT actions.

3.1 Field configuration manifold

The field configuration manifold is charted by $\phi^i(x)$ or, more conveniently $\phi^i(p)$. Unlike in field space geometry, positions or momenta are not external quantities, but must be considered as part of the coordinates of the field configuration manifold; in other words, we should think of x in $\phi^i(x)$ as an additional index, which is on the same footing as the flavor index i, and similarly for p in $\phi^i(p)$. To emphasize this point, we collect both flavor and position/momentum indices and write:

$$\phi^{(ix)} \equiv \phi^i(x) \qquad \text{and} \qquad \phi^{(ip)} \equiv \phi^i(p)\,, \tag{3.1}$$

and similarly for other objects on the field configuration manifold introduced below. The target manifold in field space geometry can be viewed as a submanifold of the field configuration manifold, which corresponds to $\phi^{(ix)} = \phi^i =$ constant, or equivalently, $\phi^{(ip)} = \phi^i\,(2\pi)^d\delta^d(p)$. We assume that the vacuum preserves spacetime translation symmetry, so it corresponds to a point $\overline{\phi}$ on the constant field submanifold.

In what follows, we will encounter expressions with many flavor and momentum indices. To reduce notational clutter, we will often use $(i_1p_1), (i_2p_2), \ldots$ for these indices, and abbreviate $(i_ap_a) \to a$ as in DeWitt notation, e.g.,

$$\phi^a \equiv \phi^{(i_ap_a)}\,. \tag{3.2}$$

In this notation, derivatives on the field configuration manifold are normalized as

$$_{,a} \equiv {}_{,(i_ap_a)} \equiv \frac{\partial}{\partial\phi^a} \equiv \frac{\partial}{\partial\phi^{(i_ap_a)}} \equiv (2\pi)^d \frac{\delta}{\delta\phi^{i_a}(p_a)}\,, \tag{3.3}$$

where δ denotes the standard functional derivative, and we use ∂ for its normalized version in momentum space. The identity on the field configuration manifold is normalized as:

$$\delta^a_b \equiv \delta^{i_a}_{i_b}\, \delta^{p_a}_{p_b} \equiv \frac{\partial \phi^a}{\partial \phi^b} = \delta^{i_a}_{i_b}\, (2\pi)^d \delta^d(p_a - p_b)\,. \tag{3.4}$$

We will also encounter more general momentum-conserving δ-functions, which we abbreviate as:

$$\delta^{q_1\dots q_m}_{p_1\dots p_n} \equiv (2\pi)^d\, \delta^d(q_1 + \cdots + q_m - p_1 - \cdots - p_n)\,. \tag{3.5}$$

When we contract indices on the field configuration manifold, an integral over momentum is implied, in addition to a sum over flavor indices:

$$^a{}_a \equiv {}^{(i_a p_a)}{}_{(i_a p_a)} \equiv \sum_{i_a} \int_{p_a} = \sum_{i_a} \int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\,. \tag{3.6}$$

Let us demonstrate all this notation by considering the expansion of a general EFT action, which is a scalar on the field configuration manifold, around a point ϕ. The result can be compactly written as:

$$\begin{aligned} S[\phi+\eta] &= \sum_{n=0}^{\infty} \frac{1}{n!}\, S_{,1\dots n}[\phi]\, \eta^1 \dots \eta^n \\ &\equiv S[\phi] + S_{,a}[\phi]\, \eta^a + \frac{1}{2}\, \Delta^{-1}_{ab}[\phi]\, \eta^a \eta^b + \sum_{n=3}^{\infty} \frac{1}{n!}\, V_{1\dots n}[\phi]\, \eta^1 \dots \eta^n\,, \end{aligned} \tag{3.7}$$

where we have defined:

$$\Delta^{-1}_{ab}[\phi] \equiv S_{,ab}[\phi] \qquad \text{and} \qquad V_{1\dots n}[\phi] \equiv S_{,1\dots n}[\phi]\,. \tag{3.8}$$

These are functionals of ϕ. For EFT actions of the form Eq. (2.1) and on the constant field submanifold, Δ^{-1}_{ab} and $V_{1\dots n}$ reduce to quantities defined in Sec. 2:

$$\Delta^{-1}_{ab}[\phi]\big|_{\phi(x)=\phi} = \Delta^{-1}_{i_a i_b}(\phi\,; p_a)\, \delta_{p_a p_b}\,, \tag{3.9a}$$

$$V_{1\dots n}[\phi]\big|_{\phi(x)=\phi} = V_{i_1\dots i_n}(\phi\,; p_1, \dots, p_n)\, \delta_{p_1\dots p_n}\,. \tag{3.9b}$$

We use the same symbols for the propagator and vertices on the field configuration manifold and the constant field submanifold, keeping in mind that they differ by momentum-conserving δ-functions on the constant field submanifold. It should be clear from the index notation whether we are referring to a quantity on the target manifold or the full field configuration manifold. For nonconstant field configurations $\phi(x)$, $\Delta^{-1}_{ab}[\phi]$ and $V_{1\dots n}[\phi]$ are generally not proportional to momentum-conserving δ-functions because a nonconstant background field breaks spacetime translation symmetry.

Note that at this point, we have not introduced any Riemannian structure. All we have assumed so far is that the field configuration manifold is a differentiable manifold, where differentiation is implemented by the familiar functional derivative. We have not defined a metric on this manifold. As discussed at the end of Sec. 2, what we really need to construct on-shell covariant vertices is a connection. In field space geometry, it is natural to use the Levi-Civita connection derived from the metric, which is naturally defined from the two-derivative terms in the action. In functional geometry, the identification of a metric is more subtle, as we will see in Sec. 4. In this section, we will focus on obtaining on-shell covariant building blocks of EFT amplitudes on the field configuration manifold, and will stay agnostic about the existence of a metric on this manifold.

3.2 Amplitudes and correlation functions

In Sec. 2, we studied correlation functions as quantities on the target manifold: at any point ϕ on the target manifold, we expanded the action as in Eq. (2.18), and defined correlation functions in the presence of a constant source $J_i(x) = -U_{,i}(\phi)$ to cancel the linear term in the expansion; see Eq. (2.19). The result was equivalent to defining correlation functions via the standard Feynman rules. A natural generalization of this prescription would be to expand the action around a point ϕ on the field configuration manifold as in Eq. (3.7), and define correlation functions in the presence of a nonconstant source that cancels the linear term, $J_i(x) = -\frac{\delta S}{\delta\phi^i(x)}$, or equivalently, $J_a = -S_{,a}[\phi]$:

$$\left\langle \eta^1 \dots \eta^n \right\rangle_{\mathrm{c},\,\phi} \equiv \frac{1}{i^n} \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_n} \log \int \mathcal{D}\eta \, e^{i\left(S[\phi+\eta]+J_a\,\eta^a\right)} \bigg|_{J_a=-S_{,a}[\phi]} . \tag{3.10}$$

The result is again equivalent to defining correlation functions via the standard Feynman rules, with propagator given by $\Delta^{ab}[\phi]$, i.e., the inverse of $\Delta^{-1}_{ab}[\phi]$ (which should exist in a finite neighborhood of $\overline{\phi}$ for any healthy theory):

$$\Delta^{ab}[\phi]\,\Delta^{-1}_{bc}[\phi] = \delta^a_c\,, \tag{3.11}$$

and vertices given by $V_{1\dots n}$. The only difference compared to the target manifold case in Sec. 2 is that momentum conservation at each vertex is replaced by index contraction between the vertex and the propagators connected to it—see e.g., Eq. (3.13) below. As in Sec. 2, we will often leave the ϕ arguments implicit in what follows to reduce notational clutter. It is understood that all quantities on the field configuration manifold are evaluated at a general point ϕ unless specified otherwise.

Generalizing Eq. (2.21), we can define amputated correlation functions $\mathcal{M}_{1\dots n}$ on the field configuration manifold by factoring out the functional version of external propagators:

$$\left\langle \eta^1 \cdots \eta^n \right\rangle_{\mathrm{c},\,\phi} \equiv i^n \Delta^{11'}[\phi] \cdots \Delta^{nn'}[\phi]\, i\mathcal{M}_{1'\dots n'}[\phi]\,. \tag{3.12}$$

For example, the 3- and 4-point amputated correlation functions are given by:

$$\mathcal{M}_{123} = V_{123}\,, \tag{3.13a}$$

$$\mathcal{M}_{1234} = V_{1234} - \big(\Delta^{ab}\, V_{a12}\, V_{b34}\big)_{3\text{ terms}}\,. \tag{3.13b}$$

In the constant field limit and for actions of the form Eq. (2.1),

$$\Delta^{aa'}[\phi]\big|_{\phi(x)=\phi} = \Delta^{i_a i_{a'}}(\phi\,;p_a)\,\delta^{p_a p_{a'}}\,, \tag{3.14a}$$

$$\mathcal{M}_{1'\dots n'}[\phi]\big|_{\phi(x)=\phi} = \mathcal{M}_{i_{1'}\dots i_{n'}}(\phi\,;p_{1'},\dots,p_{n'})\,\delta_{p_{1'}\dots p_{n'}}\,, \tag{3.14b}$$

so we reproduce Eq. (2.21) from Eq. (3.12).

To derive scattering amplitudes from correlation functions, we again need to find the particle states. Let us label the one-particle states by $|\alpha, P\rangle$, where P is an on-shell 4-momentum, i.e., $P^2 = m_\alpha^2$. Here and in what follows, we use capital letters for on-shell momenta. For EFT actions of the form Eq. (2.1), we were able to use a vielbein obtained from the target manifold metric to find the mass basis. At tree level, the vielbein can also be extracted from the overlap between states created by the fluctuation fields η around the vacuum point $\overline{\phi}$ and one-particle states [27]:

$$\langle 0|\eta^i(x)|\alpha, P\rangle = \overline{e}_\alpha{}^i\, e^{-iP\cdot x}\,. \tag{3.15}$$

For more general EFT actions, we can use the same Eq. (3.15) to define a set of objects $\overline{e}_\alpha{}^i$, although they may not represent a vielbein. Then just as in field space geometry, $\overline{e}^\alpha{}_i\,\eta^i$ (where $\overline{e}^\alpha{}_i$ is the inverse of $\overline{e}_\alpha{}^i$) are the fields that create mass eigenstates from the vacuum, whose wave functions are plane waves with unit normalization:

$$\langle 0|\,\overline{e}^\alpha{}_i\,\eta^i(x)|\alpha, P\rangle = e^{-iP\cdot x}\,. \tag{3.16}$$

The propagator in the mass basis is given by:

$$\begin{aligned}\overline{\Delta}^{(\alpha p)(\beta q)} &\equiv \overline{e}^\alpha{}_i\,\overline{e}^\beta{}_j\,\overline{\Delta}^{(ip)(jq)} = \overline{e}^\alpha{}_i\,\overline{e}^\beta{}_j\,\langle \eta^{(ip)}\,\eta^{(jq)}\rangle_{\overline{\phi}} \\ &= \delta^{\alpha\beta}\delta^{pq}\,\frac{i}{p^2 - m_\alpha^2} + (\text{non-singular})\,.\end{aligned} \tag{3.17}$$

As in Sec. 2, we use bar to denote quantities evaluated at $\phi = \overline{\phi}$.

To write on-shell amplitudes in terms of quantities defined on the field configuration manifold, let us define:

$$\mathcal{A}_{(\alpha_1 P_1)\dots(\alpha_n P_n)} \equiv \delta_{P_1\dots P_n}\,\mathcal{A}_{\alpha_1\dots\alpha_n}(P_1,\dots,P_n)\,. \tag{3.18}$$

The intuition is that $\mathcal{A}_{(\alpha_1 P_1)\dots(\alpha_n P_n)}$ should be related to $\overline{\mathcal{M}}_{1\dots n} \equiv \overline{\mathcal{M}}_{(i_1 p_1)\dots(i_n p_n)}$ via a set of objects connecting the fields labeled by $(i_a p_a)$ to the particle states labeled by $(\alpha_a P_a)$, with P_a a set of on-shell momenta. To see that this is indeed the case, we apply the LSZ formula to extract residues of correlation functions among mass eigenstate fields as follows:

$$\begin{aligned}
\mathcal{A}_{(\alpha_1 P_1)\dots(\alpha_n P_n)} &= \frac{1}{i^{n+1}} \lim_{p_a \to P_a} \overline{\Delta}^{-1}_{(\alpha_1 p_1)(\beta_1 q_1)} \dots \overline{\Delta}^{-1}_{(\alpha_n p_n)(\beta_n q_n)} \big\langle \overline{e}^{\beta_1}{}_{j_1}\, \eta^{(j_1 q_1)} \dots \overline{e}^{\beta_n}{}_{j_n}\, \eta^{(j_n q_n)} \big\rangle_{\mathrm{c},\overline{\phi}} \\
&= \frac{1}{i^{n+1}} \lim_{p_a \to P_a} \overline{e}_{\alpha_1}{}^{i_1}\, \overline{\Delta}^{-1}_{(i_1 p_1)(j_1 q_1)} \dots \overline{e}_{\alpha_n}{}^{i_n}\, \overline{\Delta}^{-1}_{(i_n p_n)(j_n q_n)} \big\langle \eta^{(j_1 q_1)} \dots \eta^{(j_n q_n)} \big\rangle_{\mathrm{c},\overline{\phi}} \\
&= \lim_{p_a \to P_a} \overline{e}_{\alpha_1}{}^{i_1} \dots \overline{e}_{\alpha_n}{}^{i_n}\, \overline{\mathcal{M}}_{(i_1 p_1)\dots(i_n p_n)} \\
&= \overline{e}_{\alpha_1}{}^{i_1}\, \delta^{p_1}_{P_1} \dots \overline{e}_{\alpha_n}{}^{i_n}\, \delta^{p_n}_{P_n}\, \overline{\mathcal{M}}_{(i_1 p_1)\dots(i_n p_n)} \, .
\end{aligned} \tag{3.19}$$

We have used the relation between flavor- and mass-basis propagators in Eq. (3.17) to arrive at the second line, and used Eq. (3.12) to arrive at the third line. Finally, we have rewritten $\lim\limits_{p_a \to P_a}$ as integrals over momentum space δ-functions (recall from Eq. (3.6) that contracted momentum indices imply integration). The objects connecting $\mathcal{A}_{(\alpha_1 P_1)\dots(\alpha_n P_n)}$ and $\overline{\mathcal{M}}_{(i_1 p_1)\dots(i_n p_n)}$ in Eq. (3.19) are nothing but the wave function factors in momentum space:

$$\overline{e}_{\alpha}{}^{i}\, \delta^{p}_{P} = \langle 0 | \eta^{(ip)} | \alpha, P \rangle \equiv \Psi^{(ip)}_{(\alpha P)} \, , \tag{3.20}$$

We can now write Eq. (3.19) in a compact form, applying our shorthand notation $(i_a p_a) \to a$:

$$\mathcal{A}_{(\alpha_1 P_1)\dots(\alpha_n P_n)} = \Psi^{1}_{(\alpha_1 P_1)} \dots \Psi^{n}_{(\alpha_n P_n)}\, \overline{\mathcal{M}}_{1\dots n} \, . \tag{3.21}$$

This is the functional version of Eq. (2.22).

3.3 On-shell covariance

From Eq. (3.21) we see that on-shell amplitudes $\mathcal{A}_{(\alpha_1 P_1)\dots(\alpha_n P_n)}$ can be obtained from amputated correlation functions $\mathcal{M}_{1\dots n}[\phi]$ on the field configuration manifold by

1) taking the vacuum limit $\phi \to \overline{\phi}$, and
2) contracting with wave function factors $\Psi^{(ip)}_{(\alpha P)}$ defined in Eq. (3.20), which implements both the on-shell limit and projection onto mass eigenstates.

Under a general field redefinition, $\phi^{(ip)} = F^{(ip)}[\widetilde{\phi}]$ with $F^{(ip)}$ a set of analytic functionals, which maps $\widetilde{\phi}$ to ϕ and $\widetilde{\phi} + \widetilde{\eta}$ to $\phi + \eta$, we have:

$$\eta^{(ip)} = F^{(ip)}[\widetilde{\phi} + \widetilde{\eta}] - F^{(ip)}[\widetilde{\phi}] = \frac{\partial \phi^{(ip)}}{\partial \widetilde{\phi}^{(jq)}}\, \widetilde{\eta}^{(jq)} + \mathcal{O}(\widetilde{\eta}^2) \, . \tag{3.22}$$

The wave function factor therefore transforms as:

$$\widetilde{\Psi}^{(ip)}_{(\alpha P)} = \left. \frac{\partial \widetilde{\phi}^{(ip)}}{\partial \phi^{(jq)}} \right|_{\phi = \overline{\phi}} \Psi^{(jq)}_{(\alpha P)} + \langle 0 | \mathcal{O}(\eta^2) | \alpha, P \,\rangle \, . \tag{3.23}$$

The $\mathcal{O}(\eta^2)$ terms can have a nontrivial overlap with one-particle states, but only at loop level. Therefore, as far as tree-level calculations are concerned, the wave function factors on the right-hand side of Eq. (3.21) transform as vectors at the vacuum point $\overline{\phi}$, for a fixed set of particle states labeled by $(\alpha_1 P_1)\dots(\alpha_n P_n)$.

Following the same logic as in Sec. 2.3, we conclude that amputated correlation functions $\mathcal{M}_{1\dots n}[\phi]$ defined on the field configuration manifold must transform as tensors in the vacuum and on-shell limits. More precisely, under a general field redefinition $\phi^{(ip)} = F^{(ip)}[\widetilde{\phi}]$ which can involve derivatives, we have

$$\widetilde{\mathcal{M}}_{1\dots n}[\widetilde{\phi}] = \frac{\partial \phi^{1'}}{\partial \widetilde{\phi}^1} \cdots \frac{\partial \phi^{n'}}{\partial \widetilde{\phi}^n} \mathcal{M}_{1'\dots n'}\big[F[\widetilde{\phi}]\big] + X_{1\dots n}[\widetilde{\phi}]\,, \tag{3.24}$$

where $X_{1\dots n}$ satisfies

$$\widetilde{\Psi}^1_{(\alpha_1 P_1)} \dots \widetilde{\Psi}^n_{(\alpha_n P_n)}\, \overline{X}_{1\dots n} = 0\,. \tag{3.25}$$

This implies that the *on-shell equivalence* relation "$\overset{\text{OS}}{=}$" that we introduced in Sec. 2.3 can be naturally generalized to the field configuration manifold: we say that two objects on the field configuration manifold are on-shell equivalent if they are equal upon performing the two operations listed at the beginning of this subsection. With this definition, we can abbreviate Eq. (3.25) as:

$$X_{1\dots n} \overset{\text{OS}}{=} 0\,. \tag{3.26}$$

In addition to $X_{1\dots n}$, two other objects that are on-shell equivalent to zero are:

$$S_{,c} \overset{\text{OS}}{=} 0 \qquad \text{and} \qquad \Delta^{-1}_{ca} \overset{\text{OS}}{=} 0\,, \tag{3.27}$$

where p_a is an external momentum.

We say an object on the field configuration manifold is *on-shell covariant* if it transforms as a tensor up to inhomogeneous terms that are on-shell equivalent to zero. $\mathcal{M}_{1\dots n}$ are generally not tensors, but must be on-shell covariant [48, 50]. Just like on the target manifold, sums and products of on-shell covariant objects on the field configuration manifold are on-shell covariant. If an object is on-shell equivalent to a tensor, it must be on-shell covariant.

3.4 On-shell covariant building blocks

In Sec. 2, we saw that for an EFT action truncated at $\mathcal{O}(\partial^2)$, amputated correlation functions on the target manifold $\mathcal{M}_{i_1\dots i_n}$ can be expressed in terms of elementary building blocks Δ^{ij} and $\{\mathcal{V}_{j_1\dots j_m}\}$, each of which is on-shell covariant under nonderivative field redefinitions $\phi^i = f^i(\widetilde{\phi})$. Here we have seen that for a general EFT action, amputated correlation functions on the field configuration manifold $\mathcal{M}_{1\dots n} \equiv \mathcal{M}_{(i_1 p_1)\dots(i_n p_n)}$ must be on-shell covariant under general field redefinitions $\phi^{(ip)} = F^{(ip)}[\widetilde{\phi}]$. The next step is to show that we can express $\mathcal{M}_{1\dots n}$ in terms of on-shell covariant building blocks on the field configuration manifold.

Up to this point, our discussion in this section has only relied on the existence of the field configuration manifold, upon which we can define functions (which are functionals of ϕ) and take their derivatives (which are the ordinary functional derivatives, up to normalization). In particular, the EFT action $S[\phi]$ is a scalar on the field configuration manifold, and its derivatives give the standard Feynman propagator and vertices; see Eq. (3.8). We would like to construct on-shell covariant versions of the vertices just as in Sec. 2. For this we need to introduce some additional structures on the field configuration manifold, namely a connection with which we can covariantize derivatives.

Suppose we have a connection Γ^a_{bc} on the field configuration manifold. For a tensor $T^{a\cdots}{}_{b\ldots}$, we can take its covariant derivative with respect to this connection following the standard formula:

$$T^{a\ldots}{}_{b\ldots;c} \equiv T^{a\ldots}{}_{b\ldots,c} + \left(\Gamma^a_{cd}\, T^{d\ldots}{}_{b\ldots} + \cdots\right) - \left(\Gamma^d_{cb}\, T^{a\ldots}{}_{d\ldots} + \cdots\right). \tag{3.28}$$

Note that we do not assume the connection Γ^a_{bc} is torsion-free and we must therefore be careful about the ordering of its lower indices. Under a coordinate transformation on the field configuration manifold, the connection transforms as:

$$\widetilde{\Gamma}^a_{bc} = \frac{\partial\widetilde{\phi}^a}{\partial\phi^d}\,\frac{\partial\phi^e}{\partial\widetilde{\phi}^b}\,\frac{\partial\phi^f}{\partial\widetilde{\phi}^c}\,\Gamma^d_{ef} + \frac{\partial\widetilde{\phi}^a}{\partial\phi^d}\,\frac{\partial^2\phi^d}{\partial\widetilde{\phi}^b\partial\widetilde{\phi}^c}\,. \tag{3.29}$$

We leave the explicit expression of the connection unspecified at this point. In principle, we could define many different connections on any manifold, each of which leads to a distinct notion of covariant differentiation. Our discussion in this section is valid for any connection on the field configuration manifold. As we will see shortly, just like in field space geometry, any connection defines a set of on-shell covariant building blocks for $\mathcal{M}_{1\ldots n}$.

Given a connection Γ^a_{bc}, we can now repeat the analysis in Sec. 2.5 on the field configuration manifold. Expanding the fields around a general point ϕ on the field configuration manifold, we can solve for the geodesic from ϕ to $\phi+\eta$ and obtain:

$$\eta^a = \xi^a - \sum_{m=2}^{\infty}\frac{1}{m!}\,\Gamma^a_{(1\ldots m)}\,\xi^1\ldots\xi^m\,, \tag{3.30}$$

where ξ are the normal coordinates at ϕ with respect to the connection Γ^a_{bc}, and $\Gamma^a_{1\ldots m}$ are defined recursively via

$$\Gamma^a_{1\ldots m(m+1)} \equiv \Gamma^a_{1\ldots m,m+1} - \sum_{b=1}^{m}\Gamma^c_{(m+1)b}\,\Gamma^a_{1\ldots \not{b}c\ldots m}\,. \tag{3.31}$$

Similarly to Eq. (2.8), the notation $\not{b}c$ means the index b is replaced by c. In terms of ξ, the expansion of the action has a manifestly covariant expression:

$$S[\phi+\eta] = \sum_{m=0}^{\infty}\frac{1}{m!}\,S_{;(1\ldots m)}\,\xi^1\ldots\xi^m\,. \tag{3.32}$$

That the expansion coefficients of a scalar function in normal coordinates are the symmetrized covariant derivatives can be proved following the same steps as in App. A of Ref. [21]; they give rise to covariant Feynman rules for the correlation functions in the ξ basis [9, 39, 43]. In the original η basis, including the source term with $J_a = -S_{,a}$, we have

$$S[\phi+\eta] + J_a\eta^a = S + \frac{1}{2}\,\Delta^{-1}_{ab}\,\xi^a\xi^b + \sum_{m=3}^{\infty}\frac{1}{m!}\left[S_{;(1\dots m)} + \Gamma^a_{(1\dots m)}\,S_{,a}\right]\xi^1\dots\xi^m\,, \tag{3.33}$$

where we have used $S_{;(12)} = S_{,12} - \Gamma^a_{(12)}\,S_{,a} = \Delta^{-1}_{12} - \Gamma^a_{(12)}\,S_{,a}$ to simplify the quadratic term. In complete analogy to Eq. (2.44), we can compute correlation functions of η in a general background by relating them to correlation functions of ξ:

$$\begin{aligned}\langle\eta^1\dots\eta^n\rangle_\phi &= \langle\xi^1\dots\xi^n\rangle_\phi \\ &\quad - \sum_{a=1}^{n}\sum_{m=2}^{\infty}\frac{1}{m!}\,\Gamma^a_{(1'\dots m')}\,\big\langle\xi^1\dots\cancel{\xi}^a(\xi^{1'}\dots\xi^{m'})\dots\xi^n\big\rangle_\phi + \cdots\,,\end{aligned} \tag{3.34}$$

where $\cancel{\xi}^a(\xi^{1'}\dots\xi^{m'})$ means ξ^a is replaced by $\xi^{1'}\dots\xi^{m'}$, and the ellipses at the end of the equation represent terms where more than one ξ's in the correlation function are replaced by products of ξ's. All correlation functions in Eq. (3.34) are evaluated with respect to the action plus source term shown in Eq. (3.33).[8] Following the same logic as in Sec. 2.5, we conclude that correlation functions of η can be obtained from the standard set of Feynman diagrams, with propagators given by Δ^{ab}, which is the inverse of

$$\Delta^{-1}_{ab} = S_{,ab} = S_{;(ab)} + \Gamma^c_{(ab)}\,S_{,c}\,, \tag{3.35}$$

and vertices given by:

$$\mathcal{V}_{1\dots m} = S_{;(1\dots m)} + \Gamma^c_{(1\dots m)}\,S_{,c} + \sum_{a\in\mathrm{ext}}\Gamma^c_{(1\dots\cancel{a}\dots m)}\,\Delta^{-1}_{ca}\,. \tag{3.36}$$

In other words, we now have the functional version of Eq. (2.36):

$$\boxed{\mathcal{M}_{1\dots n} = \mathcal{F}_{1\dots n}\big(\Delta^{ab},\{V_{a_1\dots a_m}\}\big) = \mathcal{F}_{1\dots n}\big(\Delta^{ab},\{\mathcal{V}_{a_1\dots a_m}\}\big)\,.} \tag{3.37}$$

For example, the 3- and 4-point amputated correlation functions are given by (compare Eq. (3.13)):

$$\mathcal{M}_{123} = \mathcal{V}_{123}\,, \tag{3.38a}$$

$$\mathcal{M}_{1234} = \mathcal{V}_{1234} - \big(\Delta^{ab}\,\mathcal{V}_{a12}\,\mathcal{V}_{b34}\big)_{3\text{ terms}}\,, \tag{3.38b}$$

[8] As in Sec. 2.5, we can neglect the difference between the path integral measures $\mathcal{D}\eta$ and $\mathcal{D}\xi$ for tree-level calculations. Care must be taken if one wishes to go beyond tree level because, depending on the connection chosen, the Jacobian factor from Eq. (3.30) may have nontrivial momentum dependence and give rise to loop integrals that do not vanish even in dimensional regularization; see e.g., Ref. [57].

where

$$\mathcal{V}_{123} = S_{;(123)} + \Gamma^c_{(123)}\, S_{,c} + \Gamma^c_{(23)}\, \Delta^{-1}_{c1} + \Gamma^c_{(13)}\, \Delta^{-1}_{c2} + \Gamma^c_{(12)}\, \Delta^{-1}_{c3}\,, \tag{3.39a}$$

$$\mathcal{V}_{a12} = S_{;(a12)} + \Gamma^c_{(a12)}\, S_{,c} + \Gamma^c_{(a2)}\, \Delta^{-1}_{c1} + \Gamma^c_{(a1)}\, \Delta^{-1}_{c2}\,, \tag{3.39b}$$

$$\mathcal{V}_{b34} = S_{;(b34)} + \Gamma^c_{(b34)}\, S_{,c} + \Gamma^c_{(b4)}\, \Delta^{-1}_{c3} + \Gamma^c_{(b3)}\, \Delta^{-1}_{c4}\,, \tag{3.39c}$$

$$\mathcal{V}_{1234} = S_{;(1234)} + \Gamma^c_{(1234)}\, S_{,c} + \left[\Gamma^c_{(234)}\, \Delta^{-1}_{c1}\right]_{4\text{ terms}}. \tag{3.39d}$$

The propagator Δ^{ab} and vertices $\mathcal{V}_{1\dots m}$ defined in Eqs. (3.35) and (3.36) are not tensors, but they are all on-shell covariant. The nontensorial terms, which again come from the mismatch between a general field basis η and normal coordinates ξ, are of the form:

$$\Gamma^c_{(1\dots m)}\, S_{,c} \qquad \text{and} \qquad \Gamma^c_{(1\dots \not{a} \dots m)}\, \Delta^{-1}_{ca} \quad (a \in \text{ext})\,. \tag{3.40}$$

So, as long as $\Gamma^c_{(1\dots m)}$ and $\Gamma^c_{(1\dots \not{a} \dots m)}$ are not singular, the nontensorial terms are on-shell equivalent to zero because of Eq. (3.27). As a result, Δ^{ab} and $\mathcal{V}_{1\dots m}$ are on-shell equivalent to a set of tensors:

$$\Delta^{ab} \overset{\text{OS}}{=} \boldsymbol{\Delta}^{ab}\,, \tag{3.41a}$$

$$\mathcal{V}_{1\dots m} \overset{\text{OS}}{=} S_{;(1\dots m)}\,, \tag{3.41b}$$

where $\boldsymbol{\Delta}^{ab}$ (note boldface) is the inverse of

$$\boldsymbol{\Delta}^{-1}_{ab} \equiv S_{;(ab)}\,. \tag{3.41c}$$

A subtlety is that some choices of the functional connection $\Gamma^c_{(1\dots \not{a} \dots m)}$ may have a pole in the momentum p_c, in which case $\Gamma^c_{(1\dots \not{a} \dots m)}\, \Delta^{-1}_{ca}$ can be nonzero in the on-shell limit, meaning $\mathcal{V}_{1\dots m}$ is not on-shell equivalent to $S_{;(1\dots m)}$. This happens, for example, in any massive theory for the set of connection choices discussed in Ref. [51]; more on this in Sec. 4.2. However, we show in Sec. A that even in this case, the $\mathcal{V}_{1\dots m}$ are still on-shell covariant under field redefinitions that do not artificially introduce single-particle poles. Therefore, Eq. (3.37) gives a manifestly on-shell covariant expression for the n-point amputated correlation function in a general scalar EFT; the special cases of $n = 3, 4$ are given in Eq. (3.38).

Having understood how EFT amplitudes can be constructed from on-shell covariant building blocks on the field configuration manifold, we now turn to the question of defining a Riemannian geometry on this manifold.

4 Geometry of the field configuration manifold

In the previous section, we showed that EFT amplitudes can be expressed in terms of on-shell covariant propagators and vertices on the field configuration manifold; see

Eq. (3.37). Throughout that discussion, we stayed agnostic about the existence of a metric on the field configuration manifold. The point we wish to emphasize is that Eq. (3.37) is valid—and the propagator Δ^{ab} and vertices $\mathcal{V}_{1\ldots m}$ are on-shell covariant under nonsingular field redefinitions—for *any* Γ^a_{bc} that transforms as a connection, which may or may not be associated with a metric. Nevertheless, given the similarity between on-shell covariant amplitude expressions on both the target manifold and the field configuration manifold—see Eqs. (2.36) and (3.37)—one would naturally wonder whether we can identify a metric on the field configuration manifold that is a natural generalization of the field space geometry metric. This is the question we aim to address in this section. We will first review the idea introduced in Ref. [49] and further discussed in Ref. [51], and discuss its features and limitations. We then advocate a variation on this proposal that gives an algorithm to unambiguously define a metric on the field configuration manifold, which renders the reduction from functional geometry to field space geometry transparent.

4.1 Preliminaries

A key observation that allowed us to identify a metric in field space geometry is that $\partial_\mu \phi^i$ transforms like a vector under a nonderivative field redefinition $\phi^i = f^i(\widetilde{\phi})$:

$$\partial_\mu \phi^i = \frac{\partial \phi^i}{\partial \widetilde{\phi}^j}\, \partial_\mu \widetilde{\phi}^j \,. \tag{4.1}$$

Now consider a local field redefinition that involves derivatives:

$$\phi^i = f^i\big(\widetilde{\phi}\,,\, \partial_\mu \widetilde{\phi}\,,\, \partial_\mu \partial_\nu \widetilde{\phi}\,,\, \ldots\big)\,. \tag{4.2}$$

Here "local" means $\phi^i(x)$ is a polynomial function of $\widetilde{\phi}^j(x)$, $\partial_\mu \widetilde{\phi}^j(x)$, $\partial_\mu \partial_\nu \widetilde{\phi}^j(x)$, etc., all evaluated at the same spacetime point x. Taking a spacetime derivative, we find:

$$\partial_\mu \phi^i = \frac{\partial f^i}{\partial \widetilde{\phi}^j}\, \partial_\mu \widetilde{\phi}^j + \frac{\partial f^i}{\partial(\partial_\nu \widetilde{\phi}^j)}\, \partial_\mu \partial_\nu \widetilde{\phi}^j + \frac{\partial f^i}{\partial(\partial_\nu \partial_\rho \widetilde{\phi}^j)}\, \partial_\mu \partial_\nu \partial_\rho \widetilde{\phi}^j + \cdots\,, \tag{4.3}$$

where all terms are evaluated at x. Comparing with

$$\frac{\delta \phi^i(x)}{\delta \widetilde{\phi}^j(y)} = \frac{\partial f^i}{\partial \widetilde{\phi}^j}\bigg|_x \delta^d(x-y) + \frac{\partial f^i}{\partial(\partial_\nu \widetilde{\phi}^j)}\bigg|_x \partial^x_\nu\, \delta^d(x-y) + \frac{\partial f^i}{\partial(\partial_\nu \partial_\rho \widetilde{\phi}^j)}\bigg|_x \partial^x_\nu\, \partial^x_\rho\, \delta^d(x-y) + \cdots\,, \tag{4.4}$$

where the superscripts x on ∂ denote that partial derivatives are taken with respect to x, we see that Eq. (4.3) is equivalent to a vector transformation law on the field configuration manifold:

$$\partial_\mu \phi^i(x) = \int_y \frac{\delta \phi^i(x)}{\delta \widetilde{\phi}^j(y)}\, \partial_\mu \widetilde{\phi}^j(y)\,, \tag{4.5}$$

thereby generalizing Eq. (4.1).

We can in fact relax the requirement that the field redefinition be local. The essential property of the field redefinitions we consider is that they are translation-invariant. Denoting the infinitesimal spacetime translation operator by $\mathcal{T}_\epsilon$, we have

$$\mathcal{T}_\epsilon\, \phi^i(x) = \phi^i(x+\epsilon) = \phi^i(x) + \epsilon^\mu\, \partial_\mu \phi^i(x) + \mathcal{O}(\epsilon^2)\,. \tag{4.6}$$

The left-hand side of this equation should be understood as the function $\mathcal{T}_\epsilon\, \phi^i$, which is obtained by acting $\mathcal{T}_\epsilon$ on the function ϕ^i, evaluated at x. We say a field redefinition $\phi^i = F^i[\widetilde{\phi}]$ is *translation-invariant* if it commutes with spacetime translation:

$$\mathcal{T}_\epsilon\, \phi^i = \mathcal{T}_\epsilon\, F^i[\widetilde{\phi}] = F^i[\mathcal{T}_\epsilon\, \widetilde{\phi}]\,. \tag{4.7}$$

Any local field redefinitions of the form Eq. (4.2) trivially commute with spacetime translation, but the set of translation-invariant field redefinitions is larger than the set of local field redefinitions.

To prove the vector transformation of $\partial_\mu \phi^i$ on the field configuration manifold under translation-invariant field redefinitions, we can expand both sides of Eq. (4.7) in ϵ. The left-hand side, evaluated at x, is already given by Eq. (4.6). The right-hand side is:

$$F^i[\mathcal{T}_\epsilon\, \widetilde{\phi}] = F^i[\widetilde{\phi}] + \int_y \frac{\delta F^i}{\delta \widetilde{\phi}^j(y)}\, \epsilon^\mu\, \partial_\mu \widetilde{\phi}^j(y) + \mathcal{O}(\epsilon^2)\,. \tag{4.8}$$

Evaluating this expression at x, we see that Eq. (4.7) implies that

$$\mathcal{T}_\epsilon\, \phi^i(x) = \phi^i(x) + \epsilon^\mu \int_y \frac{\delta \phi^i(x)}{\delta \widetilde{\phi}^j(y)}\, \partial_\mu \widetilde{\phi}^j(y) + \mathcal{O}(\epsilon^2)\,. \tag{4.9}$$

Comparing Eqs. (4.6) and (4.9), we immediately reproduce Eq. (4.5).

Let us introduce some notation for later convenience. We denote:

$$(\partial_\mu \phi)^{(ix)} \equiv \partial_\mu \phi^i(x)\,, \tag{4.10}$$

to be read as the (ix) component of the vector function $(\partial_\mu \phi)$ on the field configuration manifold (which is a functional of ϕ). Upon Fourier transformation:

$$(\partial_\mu \phi)^a = (\partial_\mu \phi)^{(i_a p_a)} = -i p_{a\mu} \phi^{(i_a p_a)} = -i p_{a\mu} \phi^a\,. \tag{4.11}$$

Note that there is no sum over a when $p_{a\mu}$ appears as a multiplicative factor in an expression. The vector transformation law Eq. (4.5) can then be written concisely as

$$(\partial_\mu \phi)^a = \frac{\partial \phi^a}{\partial \widetilde{\phi}^b}\, (\partial_\mu \widetilde{\phi})^b\,. \tag{4.12}$$

It is a useful exercise to check Eq. (4.12) by considering field redefinitions in momentum space [51]. Assuming $\phi^i = F^i[\widetilde{\phi}]$ is nonsingular at $\widetilde{\phi} = 0$, we can expand:

$$\phi^a = (c_0)^a + (c_1)^a_1\, \widetilde{\phi}^1 + (c_2)^a_{12}\, \widetilde{\phi}^1 \widetilde{\phi}^2 + \cdots = \sum_{n=0}^{\infty} (c_n)^a_{1\ldots n}\, \widetilde{\phi}^1 \cdots \widetilde{\phi}^n\,, \tag{4.13}$$

where, consistent with our notation, $(c_n)^a_{1\ldots n}$ carries flavor indices $i_a, i_1, \ldots, i_n$ and is also a function of momenta $p_a, p_1, \ldots, p_n$. For translation-invariant field redefinitions, these coefficients must satisfy momentum conservation:

$$(c_n)^a_{1\ldots n} \propto \delta^{p_a}_{p_1\ldots p_n} \,. \tag{4.14}$$

As a result,

$$(\partial_\mu \phi)^a = -i \sum_{n=0}^{\infty} (c_n)^a_{1\ldots n}\, p_{a\mu}\, \widetilde{\phi}^1 \cdots \widetilde{\phi}^n = -i \sum_{n=0}^{\infty} (c_n)^a_{1\ldots n}\, (p_1 + \cdots + p_n)_\mu\, \widetilde{\phi}^1 \cdots \widetilde{\phi}^n \,, \tag{4.15}$$

which is equal to:

$$\frac{\partial \phi^a}{\partial \widetilde{\phi}^b}\, (\partial_\mu \widetilde{\phi})^b = -i \sum_{n=0}^{\infty} \sum_{b=1}^{n} (c_n)^a_{1\ldots n}\, \widetilde{\phi}^1 \cdots p_{b\mu}\, \widetilde{\phi}^b \cdots \widetilde{\phi}^n \,, \tag{4.16}$$

where the expression on the right-hand side means we are replacing $\widetilde{\phi}^b$ in $\widetilde{\phi}^1 \cdots \widetilde{\phi}^n$ by $p_{b\mu}\, \widetilde{\phi}^b$. This verifies Eq. (4.12). If we further restrict ourselves to local field redefinitions, $(c_n)^a_{1\ldots n}$ would be polynomial functions of momenta. From the discussion above it is clear that the vector transformation of $(\partial_\mu \phi)^a$ relies only on translation invariance but does not require the field redefinition to be local. On the other hand, it is worth noting that there exist many field redefinitions that are not translation-invariant, under which $(\partial_\mu \phi)^a$ does not transform like a vector. A simple example is $\phi^{i_a}(x) = \varphi^{i_a}(x) + \widetilde{\phi}^{i_a}(x)$, with $\varphi^{i_a}(x)$ a spacetime-dependent background field. In this case, $(c_0)^a = \int_x e^{ip_a \cdot x}\, \varphi^{i_a}(x)$ is not proportional to δ^{p_a} for $\varphi^{i_a}(x) \neq$ constant, and indeed, $\frac{\partial \phi^a}{\partial \widetilde{\phi}^b}\, (\partial_\mu \widetilde{\phi})^b = (\partial_\mu \widetilde{\phi})^a \neq (\partial_\mu \phi)^a$.

4.2 Toward defining a metric

Now that we have a vector $(\partial_\mu \phi)^a$ on the field configuration manifold, any metric $g_{ab}[\phi]$ must render the combination $\frac{1}{2}\, g_{ab}[\phi]\, (\partial_\mu \phi)^a (\partial^\mu \phi)^b$ a scalar under translation-invariant field redefinitions. Meanwhile, we know of a scalar on the field configuration manifold—the action $S[\phi]$. It is therefore tempting to identify:

$$S[\phi] = \frac{1}{2}\, g_{ab}[\phi]\, (\partial_\mu \phi)^a (\partial^\mu \phi)^b \,, \tag{4.17}$$

or, more explicitly:

$$\begin{aligned} S[\phi] &= \frac{1}{2} \int_{x_a, x_b} g_{(i_a x_a)(i_b x_b)}[\phi]\, \big(\partial_\mu \phi^{i_a}(x_a)\big)\, \big(\partial^\mu \phi^{i_b}(x_b)\big) \\ &= -\frac{1}{2} \int_{p_a, p_b} g_{(i_a p_a)(i_b p_b)}[\phi]\, (p_a \cdot p_b)\, \phi^{i_a}(p_a)\, \phi^{i_b}(p_b) \,. \end{aligned} \tag{4.18}$$

This is the approach taken in Refs. [49, 51]. As a technical note, since $(\partial_\mu \phi)^a$ transforms like a vector only under translation-invariant field redefinitions, whereas $S[\phi]$

is a scalar under all field redefinitions, the identification in Eq. (4.17) is consistent only across field bases that are related by translation-invariant field redefinitions. Practically this does not cause any issue because we almost always work with field bases where translation invariance is manifest (e.g., there are no spacetime-dependent couplings),[9] and any two such bases are related by a translation-invariant field redefinition.

As a simple example, consider a one-flavor ϕ^3 theory:

$$\begin{aligned} S[\phi] &= \int_x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{6}\mu\phi^3 \right] \\ &= -\frac{1}{2}\int_{p_a,p_b} \delta_{p_a p_b}(p_a\cdot p_b + m^2)\,\phi(p_a)\,\phi(p_b) - \frac{1}{6}\,\mu\int_{p_a,p_b,p_c} \delta_{p_a p_b p_c}\,\phi(p_a)\phi(p_b)\phi(p_c)\,. \end{aligned} \tag{4.19}$$

One can readily identify:

$$\begin{aligned} g_{ab}[\phi] &= \frac{1}{p_a\cdot p_b}\left[\delta_{p_a p_b}\big(p_a\cdot p_b + m^2\big) + \frac{1}{3}\,\mu\int_{p_c} \delta_{p_a p_b p_c}\phi(p_c) \right] \\ &= \delta_{p_a p_b}\left(1 - \frac{m^2}{p_a^2}\right) + \frac{1}{3}\,\mu\int_{p_c}\delta_{p_a p_b p_c}\frac{1}{p_a\cdot p_b}\,\phi(p_c)\,. \end{aligned} \tag{4.20}$$

This example demonstrates a general property of metrics satisfying Eq. (4.17): for massive theories, g_{ab} vanishes in the vacuum and on-shell limits [51]. In Eq. (4.20), the (metastable) vacuum limit corresponds to $\overline{\phi} = 0$, so $\overline{g}_{ab}$ has a zero at $p_a^2 = m^2$ if $m \neq 0$. As a result, the Levi-Civita connection $\overline{\Gamma}^a_{bc}$, which involves the inverse metric, becomes singular when p_a goes on-shell. This is not necessarily a problem: as mentioned at the end of Sec. 3.4 and elaborated in Sec. A, a simple pole in the connection does not ruin the on-shell covariance of $\mathcal{V}_{1\ldots m}$ under nonsingular field redefinitions.

Another notable feature of Eq. (4.17) is that, while a metric uniquely defines an action, an action does not uniquely define a metric [49, 51]. Given an EFT action, if one finds a $g_{ab}[\phi]$ that satisfies Eq. (4.17), an equally good metric choice would be $g_{ab}[\phi]+h_{ab}[\phi]$, where $h_{ab}[\phi]$ is an arbitrary functional that satisfies $h_{ab}[\phi]\,(\partial_\mu\phi)^a\,(\partial^\mu\phi)^b = 0$. For example, in the ϕ^3 theory above, an alternative choice of metric that satisfies Eq. (4.17) would be:

$$g'_{ab}[\phi] = \delta_{p_a p_b}\left(1 - \frac{m^2}{p_a^2}\right) + \mu\int_{p_c}\delta_{p_a p_b p_c}\frac{1}{p_a\cdot p_b + p_b\cdot p_c + p_c\cdot p_a}\,\phi(p_c)\,. \tag{4.21}$$

[9] An exception is when we expand the fields around a spacetime-dependent background, as discussed at the end of Sec. 4.1, in which case one obtains couplings that depend on the nonconstant background fields. So for example, normal coordinates around a point ϕ that is not on the constant field submanifold would not be a basis where we can identify $S[\phi]$ with $\frac{1}{2}\,g_{ab}[\phi]\,(\partial_\mu\phi)^a(\partial^\mu\phi)^b$. However, once we have obtained the metric via Eq. (4.17) in a basis where translation invariance is manifest, it is straightforward to obtain the metric in any other bases (including normal coordinates) by simply performing the appropriate field redefinition.

All we can say is that, if we start from an EFT action written in some basis and identify *a metric* that satisfies Eq. (4.17), its tensor transformation would give *a metric* in the new basis, in the sense that $\widetilde{S}[\widetilde{\phi}] = \frac{1}{2}\,\widetilde{g}_{ab}[\widetilde{\phi}]\,(\partial_\mu\widetilde{\phi})^a\,(\partial^\mu\widetilde{\phi})^b$. However, it might not be easy to identify the same metric $\widetilde{g}_{ab}[\widetilde{\phi}]$ starting from $\widetilde{S}[\widetilde{\phi}]$. In other words, we do not have a prescription to extract a unique metric from the action that is consistent across all bases. Different metric choices giving the same EFT action define different Riemannian geometries of the field configuration manifold—for example, they give different curvatures. Therefore, the lack of a prescription to unambiguously fix a metric poses a challenge if one wishes to assign physical meaning to geometric objects.

To further illustrate this challenge, let us consider an extreme example. Starting from a general scalar EFT action,

$$S[\phi] = \frac{1}{2}\,\overline{\Delta}_{ab}^{-1}[\phi]\,\phi^a\phi^b + \sum_{n=3}^{\infty}\frac{1}{n!}\,\overline{V}_{1\dots n}[\phi]\,\phi^1\dots\phi^n\,, \tag{4.22}$$

where we assume the vacuum is at $\overline{\phi}=0$, we can make the following perturbative field redefinition:

$$\widehat{\phi}^a = \phi^a + (c_2)^a_{12}\,\phi^1\phi^2 + (c_3)^a_{123}\,\phi^1\phi^2\phi^3 + \cdots\,. \tag{4.23}$$

By making judicious choices of $(c_2)^a_{12}$, $(c_3)^a_{123}$, etc., we can actually arrive at a free theory in the $\widehat{\phi}$ basis:

$$(c_2)^a_{12} = \frac{1}{3!}\,\overline{\Delta}^{ab}\,\overline{V}_{b12}\,,\quad (c_3)^a_{123} = \frac{1}{4!}\left[\overline{\Delta}^{ab}\,\overline{V}_{b123} - \frac{1}{9}\left(\overline{\Delta}^{ab}\,\overline{\Delta}^{cd}\,\overline{V}_{bc1}\,\overline{V}_{d23}\right)_{3\text{ terms}}\right],\quad \dots$$
$$\Rightarrow\qquad \widehat{S}[\widehat{\phi}] = \frac{1}{2}\,\overline{\Delta}_{ab}^{-1}\,\widehat{\phi}^a\widehat{\phi}^b\,. \tag{4.24}$$

While turning an interacting action into a free action might appear unusual, there is really no contradiction here. We have made a nonlocal field redefinition (due to the $\overline{\Delta}^{ab}$ factors) for which modifications to the source term cannot be neglected [57]; in fact, all interactions are encoded in the coupling to the source in the $\widehat{\phi}$ basis. Making the obvious choice of metric in the $\widehat{\phi}$ basis,

$$\widehat{g}_{ab}[\widehat{\phi}] = -\frac{1}{p_a\cdot p_b}\,\overline{\Delta}_{ab}^{-1}\,, \tag{4.25}$$

we then obtain a metric in the original ϕ basis via a tensor transformation:

$$\begin{aligned}
g_{ab}[\phi] &= \frac{\partial\widehat{\phi}^c}{\partial\phi^a}\,\frac{\partial\widehat{\phi}^d}{\partial\phi^b}\,\widehat{g}_{cd}[\widehat{\phi}] \\
&= -\frac{1}{p_a\cdot p_b}\,\overline{\Delta}^{-1}_{ab} + \frac{1}{3}\left(\frac{1}{p_a^2}+\frac{1}{p_b^2}\right)\overline{V}_{abc}\,\phi^c \\
&\quad + \left\{\left(\frac{1}{p_a^2}+\frac{1}{p_b^2}\right)\left[\frac{1}{8}\,\overline{V}_{abcd} - \frac{1}{72}\,\overline{\Delta}^{ef}(\overline{V}_{eab}\,\overline{V}_{fcd} + 2\,\overline{V}_{eac}\,\overline{V}_{fbd})\right]\right. \\
&\qquad \left. - \frac{1}{9}\,\frac{1}{(p_a+p_c)\cdot(p_b+p_d)}\,\overline{\Delta}^{ef}\,\overline{V}_{eac}\,\overline{V}_{fbd}\right\}\phi^c\,\phi^d + \cdots\,.
\end{aligned} \tag{4.26}$$

Once we have Eq. (4.26), we can forget about the $\widehat{\phi}$ basis because it is straightforward to verify that Eq. (4.26) is a valid metric choice for the action in Eq. (4.22), in the sense that Eq. (4.17) is satisfied. This metric is flat on the entire field configuration manifold because it is secretly the tensor transformation of a constant metric. One can also check explicitly, order by order in ϕ, that the Riemann curvature tensor computed from Eq. (4.26) vanishes identically. Therefore, we have shown that for any action of the form Eq. (4.22), there exists a metric consistent with Eq. (4.17) for which the curvature of the field configuration manifold is identically zero.[10]

4.3 An unambiguous metric

In light of the challenge discussed in the previous subsection, we would like to consider an alternative approach to identify a metric on the field configuration manifold. Our starting point is again that $(\partial_\mu\phi)^a$ is a vector, so any metric $g_{ab}[\phi]$ must render the expression $\frac{1}{2}\,g_{ab}[\phi]\,(\partial_\mu\phi)^a(\partial^\mu\phi)^b$ a scalar under translation-invariant field redefinitions. However, rather than identifying this scalar with the full action $S[\phi]$, we identify it with the two-derivative part of $S[\phi]$ in a special set of bases which we call *Warsaw-like bases*:

$$S_{(\partial^2)}[\phi] = \frac{1}{2}\,g_{ab}[\phi]\,(\partial_\mu\phi)^a(\partial^\mu\phi)^b \qquad\qquad \text{(Warsaw-like bases)}\,. \tag{4.27}$$

Concretely, up to any fixed derivative order, we can enumerate the operators and then follow the same procedure that leads to the Warsaw basis for SMEFT dimension-six operators [1] to minimize the number of derivatives acting on each field. Up to $\mathcal{O}(\partial^4)$,

[10]Note that there is no contradiction between a flat metric and nonvanishing amplitudes, because amplitude expressions contain nontensorial terms which do not vanish in the vacuum and on-shell limits in the present case (cf. the discussion at the end of Sec. 3.4). There is also no contradiction between nonvanishing $\mathcal{V}_{1\dots m}$ vertices in the ϕ basis and vanishing vertices in the $\widehat{\phi}$ basis: the two bases are related by a singular field redefinition that introduces single-particle poles, so the proof of on-shell covariance of $\mathcal{V}_{1\dots m}$ in Sec. A does not go through.

this procedure yields:

$$S[\phi] = \int_x \bigg[-U(\phi) + \frac{1}{2}\, g_{ij}(\phi)(\partial_\mu\phi^i)(\partial^\mu\phi^j) + \frac{1}{8}\, \lambda_{ijkl}(\phi)(\partial_\mu\phi^i)(\partial^\mu\phi^j)(\partial_\nu\phi^k)(\partial^\nu\phi^l) + \mathcal{O}(\partial^6)\bigg] \qquad \text{(Warsaw-like bases)}\,, \tag{4.28}$$

where U, $g_{ij} = g_{(ij)}$ and $\lambda_{ijkl} = \lambda_{(ij)(kl)} = \lambda_{(kl)(ij)}$ are real analytic functions of ϕ; see Sec. B for details. Note that Eq. (4.28) specifies a set of bases rather than a single basis, because there is still the freedom to make nonderivative field redefinitions, which preserve the form of Eq. (4.28). In other words, a nonderivative field redefinition takes us from one Warsaw-like basis to another Warsaw-like basis.

We can rewrite Eq. (4.28) as:

$$S[\phi] = -U[\phi] + \frac{1}{2}\, g_{ab}[\phi](\partial_\mu\phi)^a(\partial^\mu\phi)^b + \frac{1}{8}\, \lambda_{abcd}[\phi](\partial_\mu\phi)^a(\partial^\mu\phi)^b(\partial_\nu\phi)^c(\partial^\nu\phi)^d + \mathcal{O}(\partial^6) \qquad \text{(Warsaw-like bases)}\,, \tag{4.29}$$

where

$$U[\phi] \equiv \int_x U\big(\phi(x)\big)\,, \tag{4.30a}$$

$$g_{ab}[\phi] \equiv \int_x e^{-i(p_a+p_b)\cdot x}\, g_{i_a i_b}\big(\phi(x)\big)\,, \tag{4.30b}$$

$$\lambda_{abcd}[\phi] \equiv \int_x e^{-i(p_a+p_b+p_c+p_d)\cdot x}\, \lambda_{i_a i_b i_c i_d}\big(\phi(x)\big)\,. \tag{4.30c}$$

With $g_{ab}[\phi]$ identified as the metric on the field configuration manifold in a Warsaw-like basis, we can follow the standard tensor transformation rule to obtain the metric in any other basis that is related to our Warsaw-like basis by a field redefinition. Similarly, we can demand that $U[\phi]$ and $\lambda_{abcd}[\phi]$ transform as a scalar and a $(0,4)$ tensor, respectively, and obtain their expressions in any other basis. See Sec. B for details. Note that if we have worked with a different Warsaw-like basis, we would obtain the same metric, in the sense that its components are related to $g_{ab}[\phi]$ by the correct tensor transformation that corresponds to the nonderivative field redefinition connecting the two bases; we expand on this point in Sec. C. The metric defined in this way is nonsingular and free from the ambiguity of the prescription discussed in Sec. 4.2.[11]

[11] We note that our metric choice here coincides with that in Ref. [43] when the action is truncated at $\mathcal{O}(\partial^2)$, and was also discussed in Ref. [51] in the special case of nonlinear sigma models. Since many phenomenologically interesting EFTs (e.g., SMEFT and HEFT) contain an infinite series of higher-derivative operators, the prescription introduced here is needed to define a metric that has the correct tensorial transformation under derivative field redefinitions.

A key advantage of the metric definition in this subsection is that the relation to field space geometry becomes totally transparent. If we neglect four- and higher-derivative terms in the action, $g_{i_a i_b}(\phi)$ is nothing but the metric in field space geometry; the metric in functional geometry $g_{ab}[\phi]$ is then obtained via a Fourier transform. Similarly, the (generalized) Christoffel symbols and Riemann curvature are given by:

$$\Gamma^a_{1\dots m}[\phi] = \int_x e^{i(p_a - p_1 - \dots - p_m)\cdot x}\, \Gamma^{i_a}_{i_1\dots i_m}\big(\phi(x)\big)\,, \tag{4.31a}$$

$$R^a{}_{bcd}[\phi] = \int_x e^{i(p_a - p_b - p_c - p_d)\cdot x}\, R^{i_a}{}_{i_b i_c i_d}\big(\phi(x)\big)\,. \tag{4.31b}$$

where $\Gamma^{i_a}_{i_1\dots i_m}$, $R^{i_a}{}_{i_b i_c i_d}$ are the corresponding quantities in field space geometry, as one can verify using:

$$\frac{\partial}{\partial\phi^{(ip)}} = (2\pi)^d \int_x \frac{\delta\phi^j(x)}{\delta\phi^i(p)}\, \frac{\delta}{\delta\phi^j(x)} = \int_x e^{-ip\cdot x}\, \frac{\delta}{\delta\phi^i(x)}\,. \tag{4.32}$$

Covariant derivatives of the potential, curvature and coefficient functional λ_{abcd} on the field configuration manifold are similarly related to those on the target manifold via a Fourier transform:

$$U_{;1\dots m}[\phi] = \int_x e^{-i(p_1 + \dots + p_m)\cdot x}\, U_{;i_1\dots i_m}\big(\phi(x)\big)\,, \tag{4.33a}$$

$$R^a{}_{bcd;1\dots m}[\phi] = \int_x e^{i(p_a - p_b - p_c - p_d - p_1 - \dots - p_m)\cdot x}\, R^{i_a}{}_{i_b i_c i_d;\, i_1\dots i_m}\big(\phi(x)\big)\,, \tag{4.33b}$$

$$\lambda_{abcd;1\dots m}[\phi] = \int_x e^{-i(p_a + p_b + p_c + p_d + p_1 + \dots + p_m)\cdot x}\, \lambda_{i_a i_b i_c i_d;\, i_1\dots i_m}\big(\phi(x)\big)\,. \tag{4.33c}$$

A useful feature of the Riemannian geometry on the field configuration manifold defined in this subsection is that $(\partial_\mu\phi)^a$ is a set of Killing vectors associated with spacetime translation symmetry. Indeed, from Eq. (4.6) we see that $(\partial_\mu\phi)^a$ generates a diffeomorphism on the field configuration manifold that is equivalent to a spacetime translation. The metric $g_{ab}[\phi]$ is translation-invariant because it is an integral of a local function (see Eq. (4.30b)), so the diffeomorphism generated by $(\partial_\mu\phi)^a$ is an isometry. We can also explicitly verify that $(\partial_\mu\phi)^a = -ip_{a\mu}\phi^a$ satisfies Killing's equation:

$$\begin{aligned}
2\,(\partial_\mu\phi)_{(a;b)} &= g_{ac}\,(\partial_\mu\phi)^c{}_{;b} + g_{bc}\,(\partial_\mu\phi)^c{}_{;a} \\
&= g_{ac}\left(-ip_{b\mu}\,\delta^c_b + \Gamma^c_{bd}\,(\partial_\mu\phi)^d\right) + g_{bc}\left(-ip_{a\mu}\,\delta^c_a + \Gamma^c_{ad}\,(\partial_\mu\phi)^d\right) \\
&= -i(p_a+p_b)_\mu\, g_{ab} + g_{ab,d}\,(\partial_\mu\phi)^d \\
&= -i(p_a+p_b)_\mu\, g_{ab} + \int_{p_d,x_d} e^{-ip_d\cdot x_d}\,\frac{\delta g_{ab}}{\delta\phi^{i_d}(x_d)}\left(-ip_{d\mu}\,\phi^{i_d}(p_d)\right) \\
&= \int_x \left[\partial_\mu\left(e^{-i(p_a+p_b)\cdot x}\right) g_{i_a i_b}\big(\phi(x)\big) + e^{-i(p_a+p_b)\cdot x}\,\partial_\mu g_{i_a i_b}\big(\phi(x)\big)\right] \\
&= \int_x \partial_\mu\left[e^{-i(p_a+p_b)\cdot x}\, g_{i_a i_b}\big(\phi(x)\big)\right] = 0\,,
\end{aligned} \tag{4.34}$$

where we have used Eq. (4.32) and $g_{ab;d} = g_{ab,d} - g_{bc}\,\Gamma^c_{da} - g_{ac}\,\Gamma^c_{db} = 0$. An especially useful identity for Killing vectors is

$$(\partial_\mu\phi)^a{}_{;\,bc} = R^a{}_{bcd}\,(\partial_\mu\phi)^d\,, \tag{4.35}$$

which follows directly from Killing's equation and the definition of Riemann curvature.[12]

4.4 Reduction to field space geometry

With the discussion in the previous subsection, we now have a very simple procedure to derive on-shell covariant amplitude expressions in field space geometry starting from the more general framework of functional geometry. Neglecting four- and higher-derivative terms, we can write the action as:

$$S[\phi] = \frac{1}{2}\, g_{ab}[\phi]\,(\partial_\mu\phi)^a(\partial^\mu\phi)^b - U[\phi]\,. \tag{4.36}$$

To obtain the on-shell covariant vertex functions $\mathcal{V}_{1\dots m}$ defined in Eq. (3.36), we need to take covariant derivatives of the action on the field configuration manifold. Covariant derivatives of $U[\phi]$ are given by Eq. (4.33a), while covariant derivatives of the metric with respect to the Levi-Civita connection vanish. Using Eq. (4.35), we can express all covariant derivatives of $(\partial_\mu\phi)^a$ in terms of $(\partial_\mu\phi)^a$, $(\partial_\mu\phi)^a{}_{;b}$ and (covariant derivatives of) the Riemann curvature tensor:

$$(\partial_\mu\phi)^a{}_{;12} = R^a{}_{12b}\,(\partial_\mu\phi)^b\,, \tag{4.37a}$$

$$(\partial_\mu\phi)^a{}_{;123} = R^a{}_{12b}\,(\partial_\mu\phi)^b{}_{;3} + R^a{}_{12b;3}\,(\partial_\mu\phi)^b\,, \tag{4.37b}$$

$$(\partial_\mu\phi)^a{}_{;1234} = R^a{}_{12b;4}\,(\partial_\mu\phi)^b{}_{;3} + R^a{}_{12b;3}\,(\partial_\mu\phi)^b{}_{;4} + R^a{}_{12b}\,R^b{}_{34c}\,(\partial_\mu\phi)^c + R^a{}_{12b;34}\,(\partial_\mu\phi)^b\,, \tag{4.37c}$$

[12]To the best of our knowledge, Eq. (4.35) first appeared in Ref. [59] in the context of chiral perturbation theory.

and so on. To reproduce results in field space geometry, we are interested in the constant field submanifold, $\phi^i(x) = \phi^i = \text{constant}$. In the constant field limit, we have

$$g_{ab}[\phi]\big|_{\phi(x)=\phi} = g_{i_a i_b}(\phi)\,\delta_{p_a p_b}\,, \tag{4.38a}$$

$$\Gamma^a_{1\ldots m}[\phi]\big|_{\phi(x)=\phi} = \Gamma^{i_a}_{i_1\ldots i_m}(\phi)\,\delta^{p_a}_{p_1\ldots p_m}\,, \tag{4.38b}$$

$$U_{;1\ldots n}[\phi]\big|_{\phi(x)=\phi} = U_{;i_1\ldots i_n}(\phi)\,\delta_{p_1\ldots p_n}\,, \tag{4.38c}$$

$$R^a{}_{12b;3\ldots}[\phi]\big|_{\phi(x)=\phi} = R^{i_a}{}_{i_1 i_2 i_b;i_3\ldots}(\phi)\,\delta^{p_a}_{p_1 p_2 p_b p_3\ldots}\,, \tag{4.38d}$$

$$(\partial_\mu\phi)^a\big|_{\phi(x)=\phi} = 0\,, \tag{4.38e}$$

$$(\partial_\mu\phi)^a{}_{;1}\big|_{\phi(x)=\phi} = -ip_{1\mu}\delta^a_1 = -ip_{1\mu}\delta^{i_a}_{i_1}\,\delta^{p_a}_{p_1}\,, \tag{4.38f}$$

$$(\partial_\mu\phi)^a{}_{;12}\big|_{\phi(x)=\phi} = 0\,, \tag{4.38g}$$

$$(\partial_\mu\phi)^a{}_{;123}\big|_{\phi(x)=\phi} = -ip_{3\mu}\,R^{i_a}{}_{i_1 i_2 i_3}(\phi)\,\delta^{p_a}_{p_1 p_2 p_3}\,, \tag{4.38h}$$

$$(\partial_\mu\phi)^a{}_{;1234}\big|_{\phi(x)=\phi} = -i\big(p_{3\mu}\,R^{i_a}{}_{i_1 i_2 i_3;i_4}(\phi) + p_{4\mu}\,R^{i_a}{}_{i_1 i_2 i_4;i_3}(\phi)\big)\,\delta^{p_a}_{p_1 p_2 p_3 p_4}\,, \tag{4.38i}$$

and one can similarly obtain higher covariant derivatives of $(\partial_\mu\phi)^a$. Meanwhile, for the (inverse) propagator, we know from Eq. (3.9a) that:

$$\Delta^{-1}_{ab}[\phi]\big|_{\phi(x)=\phi} = \Delta^{-1}_{i_a i_b}(\phi\,;p_a)\,\delta_{p_a p_b}\,, \tag{4.39a}$$

$$\Delta^{ab}[\phi]\big|_{\phi(x)=\phi} = \Delta^{i_a i_b}(\phi\,;p_a)\,\delta^{p_a p_b}\,. \tag{4.39b}$$

Combining all the results above, we find that in the constant field limit, the on-shell covariant vertices on the field configuration manifold defined in Eq. (3.36) reduce to their counterparts in field space geometry:

$$\begin{aligned}\mathcal{V}_{1\ldots m}\big|_{\phi(x)=\phi} &= \Big[\mathcal{R}_{i_1\ldots i_m}(p_1,\ldots,p_m) - U_{;(i_1\ldots i_m)} \\ &\qquad - \Gamma^{i_c}_{i_1\ldots i_m}\,U_{,i_c} + \sum_{a\in\text{ext}}\Gamma^{i_c}_{i_1\ldots \cancel{i_a}\ldots i_m}\,\Delta^{-1}_{i_c i_a}(p_a)\Big]\,\delta_{p_1\ldots p_m} \\ &= \mathcal{V}_{i_1\ldots i_m}(p_1,\ldots,p_m)\,\delta_{p_1\ldots p_m}\,,\end{aligned} \tag{4.40}$$

where

$$\delta_{p_1\ldots p_m}\,\mathcal{R}_{i_1\ldots i_m}(p_1,\ldots,p_m) = \frac{1}{2}\,g_{ab}[\phi]\big[(\partial_\mu\phi)^a\,(\partial^\mu\phi)^b\big]_{;(1\ldots m)}\Big|_{\phi(x)=\phi} \tag{4.41}$$

is a function of the curvature and its covariant derivatives that can be derived following Eq. (4.37) and using the constant field limit expressions in Eq. (4.38). For

$n = 3$ and 4, we find:

$$\mathcal{R}_{i_1 i_2 i_3}(p_1, p_2, p_3) = 0\,, \tag{4.42a}$$

$$\begin{aligned}\mathcal{R}_{i_1 i_2 i_3 i_4}(p_1, p_2, p_3, p_4) &= -\frac{2}{3}\Big[(p_1 \cdot p_2)\, R_{i_1 (i_3 i_4) i_2}\Big]_{6\text{ terms}} \\ &= -\frac{2}{3}\Big[p_{12}^2\, R_{i_1 (i_3 i_4) i_2}\Big]_{3\text{ terms}}\,,\end{aligned} \tag{4.42b}$$

in agreement with our results in Sec. 2. Computing higher-point on-shell covariant vertices is straightforward.

We end this section by outlining how our discussion can be generalized to scalar EFT actions containing four- and higher-derivative terms. We have seen that, up to $\mathcal{O}(\partial^2)$ in a Warsaw-like basis, the action is specified by a scalar $U[\phi]$ and a symmetric $(0,2)$ tensor $g_{ab}[\phi]$ which we identify as the metric. At the next order $\mathcal{O}(\partial^4)$, we have an additional coefficient functional $\lambda_{abcd}[\phi]$, which is a $(0,4)$ tensor; see Eq. (4.29). When we take covariant derivatives of $S[\phi]$ to derive the on-shell covariant vertices, we would obtain additional contributions that depend on (covariant derivatives of) λ_{abcd} and the Riemann curvature tensor. Going to higher derivative orders, we will encounter additional operators which can be written in terms of:

$$(\partial_\mu \phi)^a\,, \qquad (\mathcal{D}_\nu \partial_\mu \phi)^a\,, \qquad (\mathcal{D}_\rho \mathcal{D}_\nu \partial_\mu \phi)^a\,, \qquad \ldots \tag{4.43}$$

These are all vectors defined recursively via [18, 19]

$$\begin{aligned}(\mathcal{D}_{\mu_m} \mathcal{D}_{\mu_{m-1}} \ldots \mathcal{D}_{\mu_2} \partial_{\mu_1} \phi)^a \equiv \int_x e^{ip_a \cdot x} \Big[&\partial_{\mu_m} (\mathcal{D}_{\mu_{m-1}} \ldots \mathcal{D}_{\mu_2} \partial_{\mu_1} \phi)^{(i_a x)} \\ &+ \Gamma^{i_a}_{i_b i_c}\big(\phi(x)\big)\, \big(\partial_{\mu_m} \phi^{i_b}(x)\big)\, \big(\mathcal{D}_{\mu_{m-1}} \ldots \mathcal{D}_{\mu_2} \partial_{\mu_1} \phi)^{(i_c x)}\big)\Big]\,,\end{aligned} \tag{4.44}$$

and they satisfy

$$\begin{aligned}(\mathcal{D}_{\mu_m} \mathcal{D}_{\mu_{m-1}} \ldots \mathcal{D}_{\mu_2} \partial_{\mu_1} \phi)^a &= (\partial_{\mu_m} \phi)^a{}_{;b}\, (\mathcal{D}_{\mu_{m-1}} \ldots \mathcal{D}_{\mu_2} \partial_{\mu_1} \phi)^b \\ &= (\partial_{\mu_m} \phi)^{a_m}{}_{;a_{m-1}}\, (\partial_{\mu_{m-1}} \phi)^{a_{m-1}}{}_{;a_{m-2}} \cdots (\partial_{\mu_1} \phi)^{a_1}\,,\end{aligned} \tag{4.45}$$

as one can show using the torsion-free property of the Levi-Civita connection. As in the $\mathcal{O}(\partial^4)$ case, we can demand that the coefficient functional of each operator transforms as a tensor. As a result, scattering amplitudes can be expressed in a manifestly on-shell covariant manner in terms of a finite number of tensors and their covariant derivatives up to any finite order in the derivative expansion. We leave a detailed exploration of the explicit construction of Warsaw-like bases with higher-derivative tensorial operators and applications of functional geometry to EFTs containing higher-derivative operators to future work.

As a final remark, we would like note a subtle difference in philosophy between our approach and some of the recent attempts to extend field space geometry [46, 49, 51]. Extending the construction from the target manifold to a larger manifold (either the field configuration manifold or a jet bundle) makes it possible to encode everything about an EFT into a metric, in the sense that specifying a metric on the larger manifold is sufficient to define an EFT action. While the idea of encoding all the physics of an EFT into the intrinsic geometry of a Riemannian manifold is appealing, one must be careful about assigning physical meaning to geometric objects because of the freedom to define a multitude of metrics from the same EFT action. In this paper, we have taken a different approach which centers around establishing a direct connection to the amplitude expressions in field space geometry. In particular, we have emphasized that to achieve a manifestly on-shell covariant construction of EFT amplitudes, all we need are *a*) an action, which is a scalar on the field configuration manifold and invariantly defines the EFT, and *b*) a connection, which defines a notion of covariant differentiation. The role of a metric is secondary, in that it provides a natural definition of a connection, but is not central to the construction. With our choice of metric in Sec. 4.3, the physical content of an EFT is captured by not just the metric, but a tower of tensors $\{U\,,\, g_{ab}\,,\, \lambda_{abcd}\,,\, \dots\}$ which all descend from the action and are simply related to their field space geometry counterparts by a Fourier transform.

5 Conclusions

In this work, we have presented a unified perspective on both the geometry of the target manifold (commonly known as "field space geometry") and its generalization to the field configuration manifold (which we call "functional geometry") that accommodates derivative field redefinitions in scalar EFTs. In both cases, we focused on the amputated correlation functions $\mathcal{M}$, whose vacuum and on-shell limits give physical scattering amplitudes. Working at tree level, we found expressions for $\mathcal{M}$ in terms of building blocks that transform covariantly in the vacuum and on-shell limits under coordinate changes on the respective manifolds. Our main findings are summarized as follows.

- In field space geometry, $\mathcal{M}$ carry flavor indices $i_1 \dots i_n$ and are functions of momenta $p_1, \dots, p_n$. They satisfy:

  $$\mathcal{M}_{i_1\dots i_n}(p_1,\dots,p_n) = \mathcal{F}_{i_1\dots i_n}\big(\Delta^{ij}, \{V_{j_1\dots j_m}\}\big) = \mathcal{F}_{i_1\dots i_n}\big(\Delta^{ij}, \{\mathcal{V}_{j_1\dots j_m}\}\big)\,, \tag{5.1}$$

  where the middle expression represents the function of propagator Δ^{ij} and vertices $V_{j_1\dots j_m}$ obtained from standard Feynman rules, while the last expression represents the same function, but with the standard vertices replaced by a new set of vertices $\mathcal{V}_{j_1\dots j_m}$ given in Eq. (2.51) for a scalar EFT action truncated at $\mathcal{O}(\partial^2)$. Under coordinate changes on the target manifold, which are nonderivative field redefinitions, Δ^{ij} and $\mathcal{V}_{j_1\ldots j_m}$ transform as tensors in the vacuum and on-shell limits. Away from the vacuum and on-shell limits, Δ^{ij} and $\mathcal{V}_{j_1\ldots j_m}$ generally depend on nontensorial terms, and so do $\mathcal{M}_{i_1\ldots i_n}$. The nontensorial terms do not contribute to on-shell amplitudes, and, as we have shown here for the first time, can be understood as originating from a mismatch between a generic field basis and normal coordinates at any given point on the target manifold.

- In functional geometry, $\mathcal{M}$ carry both flavor and momentum indices, collectively denoted by $1\ldots n$. They satisfy:

$$\mathcal{M}_{1\ldots n} = \mathcal{F}_{1\ldots n}\big(\Delta^{ab}, \{V_{a_1\ldots a_m}\}\big) = \mathcal{F}_{1\ldots n}\big(\Delta^{ab}, \{\mathcal{V}_{a_1\ldots a_m}\}\big)\,, \tag{5.2}$$

  where the middle expression represents the function of propagator Δ^{ab} and vertices $V_{a_1\ldots a_m}$ obtained from standard Feynman rules, while the last expression represents the same function, but with the standard vertices replaced by a new set of vertices $\mathcal{V}_{a_1\ldots a_m}$ given in Eq. (3.36) for a general scalar EFT. Under coordinate changes on the field configuration manifold, which are general field redefinitions that may involve derivatives, Δ^{ab} and $\mathcal{V}_{a_1\ldots a_m}$ transform as tensors in the vacuum and on-shell limits. Away from the vacuum and on-shell limits, Δ^{ab} and $\mathcal{V}_{a_1\ldots a_m}$ generally contain nontensorial terms, and so do $\mathcal{M}_{1\ldots n}$. The nontensorial terms can be understood as originating from a mismatch between a generic field basis and normal coordinates at any given point on the field configuration manifold. They do not contribute to on-shell amplitudes for an appropriately chosen nonsingular connection.

- In both field space geometry and functional geometry, the on-shell covariant building blocks discussed above can be defined for any connection, independent of the existence of a metric. On the target manifold, it is natural to identify the coefficient function of the two-derivative terms in the action as the metric. On the field configuration manifold, there appeared to be an ambiguity in the definition of metric [49, 51]. We proposed to sidestep this ambiguity by identifying the coefficient functional of the two-derivative part of the action in Warsaw-like bases as the metric. With this choice of metric, the relation between geometric objects in field space geometry and functional geometry becomes transparent. We then showed that amplitude expressions in field space geometry can be easily obtained from the constant field limit of the corresponding expressions in functional geometry.

By formulating field space geometry and functional geometry in a unified manner, our work sets the stage for revisiting many results that have so far only been established using field space geometry—e.g., classification of EFTs (free vs. interacting, renormalizable vs. nonrenormalizable, SMEFT vs. HEFT), soft theorems, EFT matching and renormalization group evolution equations—through the lens of functional geometry, thereby achieving conclusions that are robust under both nonderivative and derivative field redefinitions. In particular, while four- and higher-derivative terms have been rarely considered in field space geometry, they are straightforward to include in functional geometry, following the procedure outlined at the end of Sec. 4.4. A systematic study of their effects in the Higgs sector of the Standard Model from the geometric perspective will shed new light on the interpretation of precision measurements at the LHC and future colliders. Another future direction is to extend functional geometry to accommodate fermion and vector fields, thus generalizing recent efforts to include nonzero-spin fields in field space geometry [33, 38–43, 60]. Finally, it would be interesting to see if similar on-shell covariant building blocks can be identified for loop amplitudes and 1PI effective actions. We are optimistic that we are only beginning to uncover applications to the theory and phenomenology of EFTs that can be understood using the geometry of the field configuration manifold.

Acknowledgments

We are grateful to Xiaochuan Lu for collaboration in the early stages of the project. We thank Andreas Helset and Yu-Tse (Alan) Lee for helpful discussions and feedback on a preliminary draft. T.C. is supported by the U.S. Department of Energy under grant DE-SC0011640. X.-X.L. and Z.Z. are supported in part by the U.S. National Science Foundation under grant PHY-2412880. This work was performed in part at the Aspen Center for Physics, which is supported by National Science Foundation grant PHY-1607611.

Note added: We thank Mohammad Alminawi, Ilaria Brivio and Joe Davighi for sharing their new preprint [61], where they present covariant Feynman rules in the complementary formulation of EFT geometry using jet bundles. They incorporate both the potential and two-derivative interactions into a metric on the 0-jet bundle and derive on-shell amplitude expressions that are manifestly covariant under nonderivative field redefinitions.

Appendices

A On-shell covariance of $\mathcal{V}_{1\ldots m}$

In this appendix, we show that even for a connection Γ^c_{ab} on the field configuration manifold that has a simple pole in p_c, the $\mathcal{V}_{1\ldots m}$ vertices given in Eq. (3.36) are still on-shell covariant. We first note that, in our definition of on-shell equivalence and on-shell covariance in Sec. 3.3, the vacuum limit should be taken before contracting with wave function factors because the wave function factors are only defined at the vacuum. Among the two types of nontensorial terms in Eq. (3.36), $\Gamma^c_{(1\ldots m)}\, S_{,c}$ vanishes as soon as the vacuum limit is taken because $\overline{S}_{,c}=0$, provided the connection is not singular at the vacuum, which we assume. Therefore,

$$\Gamma^c_{(1\ldots m)}\, S_{,c} \overset{\text{OS}}{=} 0\,. \tag{A.1}$$

All that remains to prove the on-shell covariance of $\mathcal{V}_{1\ldots m}$ is to show that $\Gamma^c_{(1\ldots\cancel{a}\ldots m)}\, \Delta^{-1}_{ca}$ ($a\in$ ext) is on-shell covariant, even if it is not on-shell equivalent to zero. It turns out that symmetrization of the indices is not important, and we will show that $\Gamma^c_{1\ldots\cancel{a}\ldots m}\, \Delta^{-1}_{ca}$ is on-shell covariant; in other words, we will show that the inhomogeneous terms in the transformation of $\Gamma^c_{1\ldots\cancel{a}\ldots m}\, \Delta^{-1}_{ca}$ are on-shell equivalent to zero.

To simplify notation, let us rewrite m as $m+1$ and, without loss of generality, pick the term with $a=m+1$, assuming $p_a=p_{m+1}$ is an external momentum. Under a coordinate transformation $\phi=F[\widetilde{\phi}]$ on the field configuration manifold, the inverse propagator becomes:

$$\widetilde{\Delta}^{-1}_{ca} = \frac{\partial\phi^e}{\partial\widetilde{\phi}^c}\,\frac{\partial\phi^b}{\partial\widetilde{\phi}^a}\,\Delta^{-1}_{eb} + \frac{\partial^2\phi^b}{\partial\widetilde{\phi}^c\,\partial\widetilde{\phi}^a}\, S_{,b}\,. \tag{A.2}$$

This is the functional analog of Eq. (2.30). The transformation of $\Gamma^c_{1\ldots m}$ takes the form:

$$\widetilde{\Gamma}^c_{1\ldots m} = \frac{\partial\widetilde{\phi}^c}{\partial\phi^d}\,\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^d_{1'\ldots m'} + \Theta^c_{1\ldots m}\,, \tag{A.3}$$

where $\Theta^c_{1\ldots m}$ collects the inhomogeneous terms. Combining the two equations above, we have:

$$\begin{aligned}\widetilde{\Gamma}^c_{1\ldots m}\,\widetilde{\Delta}^{-1}_{ca} &= \frac{\partial\phi^b}{\partial\widetilde{\phi}^a}\,\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^c_{1'\ldots m'}\,\Delta^{-1}_{cb}\\ &\quad + \Theta^c_{1\ldots m}\,\widetilde{\Delta}^{-1}_{ca} + \frac{\partial^2\phi^b}{\partial\widetilde{\phi}^c\,\partial\widetilde{\phi}^a}\,\frac{\partial\widetilde{\phi}^c}{\partial\phi^e}\,\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^e_{1'\ldots m'}\, S_{,b}\,.\end{aligned} \tag{A.4}$$

Among the inhomogeneous terms in the second line, the one proportional to $S_{,b}$ is obviously on-shell equivalent to zero because $\overline{S}_{,b}=0$. To show that $\Theta^c_{1\ldots m}\,\widetilde{\Delta}^{-1}_{ca}$ is also on-shell equivalent to zero, we need to show that $\Theta^c_{1\ldots m}$ does not contain poles in p_c, since $\widetilde{\Delta}^{-1}_{ca} \overset{\text{OS}}{=} 0$.

The proof proceeds by induction. For $m=2$, we have:

$$\Theta^c_{12} = \frac{\partial\widetilde{\phi}^c}{\partial\phi^b}\,\frac{\partial^2\phi^b}{\partial\widetilde{\phi}^1\,\partial\widetilde{\phi}^2}\,, \tag{A.5}$$

which obviously does not have a pole as long as we consider nonsingular field redefinitions, meaning field redefinitions that do not artificially introduce single-particle poles.[13] To complete the induction step, we assume that $\Theta^c_{1\ldots m}$ does not have poles in p_c. Using Eqs. (A.3) and (3.31) to derive $\Theta^c_{1\ldots m+1}$, we find:

$$\begin{aligned}
\widetilde{\Gamma}^c_{1\ldots m+1} &= \widetilde{\Gamma}^c_{1\ldots m,m+1} - \sum_{b=1}^{m} \widetilde{\Gamma}^d_{(m+1)b}\, \widetilde{\Gamma}^c_{1\ldots \not{b} d\ldots m} \\
&= \frac{\partial}{\partial\widetilde{\phi}^{m+1}}\left(\frac{\partial\widetilde{\phi}^c}{\partial\phi^d}\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^d_{1'\ldots m'} + \Theta^c_{1\ldots m}\right) \\
&\qquad - \sum_{b=1}^{m}\left(\frac{\partial\widetilde{\phi}^d}{\partial\phi^e}\frac{\partial\phi^g}{\partial\widetilde{\phi}^b}\frac{\partial\phi^f}{\partial\widetilde{\phi}^{m+1}}\,\Gamma^e_{fg} + \frac{\partial\widetilde{\phi}^d}{\partial\phi^e}\frac{\partial^2\phi^e}{\partial\widetilde{\phi}^b\,\partial\widetilde{\phi}^{m+1}}\right) \\
&\qquad\qquad \left(\frac{\partial\widetilde{\phi}^c}{\partial\phi^h}\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{b'}}{\partial\widetilde{\phi}^d}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^h_{1'\ldots b'\ldots m'} + \Theta^c_{1\ldots\not{b}d\ldots m}\right).
\end{aligned} \tag{A.6}$$

Collecting the inhomogeneous terms, we find:

$$\begin{aligned}
\Theta^c_{1\ldots m+1} &= \frac{\partial}{\partial\widetilde{\phi}^{m+1}}\left(\frac{\partial\widetilde{\phi}^c}{\partial\phi^d}\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\right)\Gamma^d_{1'\ldots m'} + \Theta^c_{1\ldots m,m+1} \\
&\qquad - \sum_{b=1}^{m}\left(\widetilde{\Gamma}^d_{(m+1)b}\,\Theta^c_{1\ldots\not{b}d\ldots m} + \frac{\partial\widetilde{\phi}^c}{\partial\phi^d}\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial^2\phi^{b'}}{\partial\widetilde{\phi}^b\,\partial\widetilde{\phi}^{m+1}}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^d_{1'\ldots b'\ldots m'}\right) \\
&= \Theta^c_{1\ldots m,m+1} - \sum_{b=1}^{m}\widetilde{\Gamma}^d_{(m+1)b}\,\Theta^c_{1\ldots\not{b}d\ldots m} + \frac{\partial^2\widetilde{\phi}^c}{\partial\phi^d\,\partial\phi^e}\frac{\partial\phi^e}{\partial\widetilde{\phi}^{m+1}}\frac{\partial\phi^{1'}}{\partial\widetilde{\phi}^1}\cdots\frac{\partial\phi^{m'}}{\partial\widetilde{\phi}^m}\,\Gamma^d_{1'\ldots m'}\,.
\end{aligned} \tag{A.7}$$

If $\Theta^c_{1\ldots m}$ does not have poles in p_c, differentiation will not generate a pole, so the first term in the last line of Eq. (A.7) is safe. The remaining terms depend on $\widetilde{\Gamma}^d_{(m+1)b}$ and $\Gamma^d_{1'\ldots m'}$, which may have poles when p_d goes on-shell. However, at the vacuum point, each object appearing in Eq. (A.7) must be proportional to a momentum-conserving δ-function due to translation invariance, e.g. $\frac{\partial^2\widetilde{\phi}^c}{\partial\phi^d\,\partial\phi^e} \propto \delta^d(p_c - p_d - p_e)$. The momentum p_c is never set equal to p_d by momentum conservation (it is interesting to note that terms proportional to $\frac{\partial\widetilde{\phi}^c}{\partial\phi^d} \propto \delta^d(p_c - p_d)$ cancel out). Therefore, $\Theta^c_{1\ldots m+1}$ does not have poles in p_c, so we have completed the proof.

B Functional geometry up to $\mathcal{O}(\partial^4)$ in a general basis

In this appendix, we explicitly demonstrate the procedure introduced in Sec. 4.3 to derive the potential $U[\phi]$, metric $g_{ab}[\phi]$, and the coefficient functional $\lambda_{abcd}[\phi]$ of the

[13]Note, for example, that the nonlocal field redefinition that turns an interacting action into a free action discussed around Eq. (4.23) is a singular field redefinition. So the discussion below does not apply in that case.

four-derivative operator starting from an EFT action written in a general basis. Up to $\mathcal{O}(\partial^4)$ and neglecting topological terms (which depend on the spacetime dimension and whose incorporation presents no additional difficulty), the most general form of the action is:

$$\begin{aligned}
S[\phi] = \int_x \Big[&- U'(\phi) + \frac{1}{2}\, g''_{ij}(\phi)(\partial_\mu\phi^i)(\partial^\mu\phi^j) + h_i(\phi)(\partial^2\phi^i) \\
&+ \frac{1}{8}\,\lambda''_{ijkl}(\phi)\,\mathcal{O}_1^{ijkl}(\phi) + \frac{1}{2}\,\lambda''_{ijk}(\phi)\,\mathcal{O}_2^{ijk}(\phi) + \frac{1}{2}\,\lambda''_{ij}(\phi)\,\mathcal{O}_3^{ij}(\phi) \\
&+ \frac{1}{2}\,\kappa_{ijk}(\phi)\,\mathcal{O}_4^{ijk}(\phi) + \kappa_{ij}(\phi)\,\mathcal{O}_5^{ij}(\phi) + \frac{1}{2}\,\kappa'_{ij}(\phi)\,\mathcal{O}_6^{ij}(\phi) \\
&+ \kappa_i(\phi)\,\mathcal{O}_7^i(\phi) + \mathcal{O}(\partial^6)\Big]\,,
\end{aligned} \tag{B.1}$$

where

$$\mathcal{O}_1^{ijkl}(\phi) = (\partial_\mu\phi^i)(\partial^\mu\phi^j)(\partial_\nu\phi^k)(\partial^\nu\phi^l)\,, \tag{B.2a}$$

$$\mathcal{O}_2^{ijk}(\phi) = (\partial_\mu\phi^i)(\partial^\mu\phi^j)(\partial^2\phi^k)\,, \tag{B.2b}$$

$$\mathcal{O}_3^{ij}(\phi) = (\partial^2\phi^i)(\partial^2\phi^j)\,, \tag{B.2c}$$

$$\mathcal{O}_4^{ijk}(\phi) = (\partial_\mu\phi^i)(\partial_\nu\phi^j)(\partial^\mu\partial^\nu\phi^k)\,, \tag{B.2d}$$

$$\mathcal{O}_5^{ij}(\phi) = (\partial_\mu\phi^i)(\partial^\mu\partial^2\phi^j)\,, \tag{B.2e}$$

$$\mathcal{O}_6^{ij}(\phi) = (\partial_\mu\partial_\nu\phi^i)(\partial^\mu\partial^\nu\phi^j)\,, \tag{B.2f}$$

$$\mathcal{O}_7^{i}(\phi) = (\partial^4\phi^i)\,, \tag{B.2g}$$

and the coefficient functions have the following symmetry properties:

$$\begin{aligned}
&g''_{ij} = g''_{(ij)}\,, \qquad \lambda''_{ijkl} = \lambda''_{(ij)(kl)} = \lambda''_{(kl)(ij)}\,, \qquad \lambda''_{ijk} = \lambda''_{(ij)k}\,, \\
&\lambda''_{ij} = \lambda''_{(ij)}\,, \qquad \kappa_{ijk} = \kappa_{(ij)k}\,, \qquad \kappa'_{ij} = \kappa'_{(ij)}\,.
\end{aligned} \tag{B.3}$$

To go to a Warsaw-like basis, we first note that five of the operators in Eq. (B.1) can be rewritten as a linear combination of the remaining operators via integration by parts:

$$\int_x h_i(\phi)(\partial^2\phi^i) = -\int_x h_{i,j}(\phi)\,(\partial_\mu\phi^i)(\partial^\mu\phi^j)\,, \tag{B.4a}$$

$$\begin{aligned}\int_x \frac{1}{2}\,\kappa_{ijk}(\phi)\,\mathcal{O}_4^{ijk}(\phi) = \int_x \bigg[\bigg(\frac{1}{4}\,\kappa_{ijk,l}(\phi) - \frac{1}{2}\,\kappa_{ikj,l}(\phi)\bigg)\,\mathcal{O}_1^{ijkl}(\phi)&\\ + \frac{1}{4}\big(\kappa_{ijk}(\phi) - \kappa_{kij}(\phi) - \kappa_{kji}(\phi)\big)\,\mathcal{O}_2^{ijk}(\phi)\bigg]&\,,\end{aligned} \tag{B.4b}$$

$$\int_x \kappa_{ij}(\phi)\,\mathcal{O}_5^{ij}(\phi) = -\int_x \big[\kappa_{ij,k}(\phi)\,\mathcal{O}_2^{ikj}(\phi) + \kappa_{ij}(\phi)\,\mathcal{O}_3^{ij}(\phi)\big]\,, \tag{B.4c}$$

$$\int_x \frac{1}{2}\,\kappa'_{ij}(\phi)\,\mathcal{O}_6^{ij}(\phi) = -\frac{1}{2}\int_x \big[\kappa'_{ij,k}(\phi)\,\mathcal{O}_4^{ikj}(\phi) + \kappa'_{ij}(\phi)\,\mathcal{O}_5^{ij}(\phi)\big]\,, \tag{B.4d}$$

$$\int_x \kappa_i(\phi)\,\mathcal{O}_7^{i}(\phi) = -\int_x \kappa_{i,j}(\phi)\,\mathcal{O}_5^{ji}(\phi)\,. \tag{B.4e}$$

Showing Eqs. (B.4) is mostly straightforward. The only relation that requires a little bit of work to show is Eq. (B.4b):

$$\begin{aligned}\int_x \frac{1}{2}\,\kappa_{ijk}(\phi)\,\mathcal{O}_4^{ijk}(\phi) &= \int_x \frac{1}{4}\,\kappa_{ijk}(\phi)\big((\partial_\mu\phi^i)(\partial_\nu\phi^j) + (\partial_\nu\phi^i)(\partial_\mu\phi^j)\big)(\partial^\mu\partial^\nu\phi^k)\\ &= -\frac{1}{4}\int_x \bigg[\kappa_{ijk,l}(\phi)\big((\partial_\mu\phi^i)(\partial_\nu\phi^j) + (\partial_\nu\phi^i)(\partial_\mu\phi^j)\big)(\partial^\mu\phi^k)(\partial^\nu\phi^l)\\ &\qquad + \kappa_{ijk}(\phi)\big((\partial_\mu\phi^i)(\partial^2\phi^j) + (\partial^2\phi^i)(\partial_\mu\phi^j)\big)(\partial^\mu\phi^k)\\ &\qquad + \kappa_{ijk}(\phi)\,\partial_\mu\big((\partial_\nu\phi^i)(\partial^\nu\phi^j)\big)(\partial^\mu\phi^k)\bigg]\\ &= \int_x \bigg[\bigg(\frac{1}{4}\,\kappa_{ijk,l}(\phi) - \frac{1}{2}\,\kappa_{ikj,l}(\phi)\bigg)\,\mathcal{O}_1^{ijkl}(\phi)\\ &\qquad + \frac{1}{4}\big(\kappa_{ijk}(\phi) - \kappa_{kij}(\phi) - \kappa_{kji}(\phi)\big)\,\mathcal{O}_2^{ijk}(\phi)\bigg]\,.\end{aligned} \tag{B.5}$$

As a result of the integration by parts relation in Eqs. (B.4), we have:

$$\begin{aligned}S[\phi] = \int_x \bigg[&-U'(\phi) + \frac{1}{2}\,g'_{ij}(\phi)(\partial_\mu\phi^i)(\partial^\mu\phi^j)\\ &+ \frac{1}{8}\,\lambda'_{ijkl}(\phi)\,\mathcal{O}_1^{ijkl}(\phi) + \frac{1}{2}\,\lambda'_{ijk}(\phi)\,\mathcal{O}_2^{ijk}(\phi) + \frac{1}{2}\,\lambda'_{ij}(\phi)\,\mathcal{O}_3^{ij}(\phi) + \mathcal{O}(\partial^6)\bigg]\,,\end{aligned} \tag{B.6}$$

where

$$g'_{ij} = g''_{ij} - h_{i,j} - h_{j,i}\,, \tag{B.7a}$$

$$\begin{aligned}\lambda'_{ijkl} = \lambda''_{ijkl} &+ \frac{1}{2}\big(\kappa_{ijk,l} + \kappa_{ijl,k} + \kappa_{kli,j} + \kappa_{klj,i} \\ &\qquad - \kappa_{ikj,l} - \kappa_{ikl,j} - \kappa_{ilj,k} - \kappa_{ilk,j} - \kappa_{jki,l} - \kappa_{jkl,i} - \kappa_{jli,k} - \kappa_{jlk,i}\big) \\ &- \kappa'_{ij,kl} - \kappa'_{kl,ij} + \kappa'_{ik,jl} + \kappa'_{jl,ik} + \kappa'_{il,jk} + \kappa'_{jk,il}\,,\end{aligned} \tag{B.7b}$$

$$\lambda'_{ijk} = \lambda''_{ijk} + \frac{1}{2}\,\kappa_{ijk} - \kappa_{k(ij)} - \kappa_{ik,j} - \kappa_{jk,i} + \frac{1}{2}\big(\kappa'_{ik,j} + \kappa'_{jk,i}\big) + \kappa_{i,jk} + \kappa_{j,ik}\,, \tag{B.7c}$$

$$\lambda'_{ij} = \lambda''_{ij} - 2\,\kappa_{(ij)} + \kappa'_{ij} + \kappa_{i,j} + \kappa_{j,i}\,. \tag{B.7d}$$

Next, we can eliminate $\mathcal{O}_2^{ijk}$ and $\mathcal{O}_3^{ij}$ via a derivative field redefinition:

$$\phi^i = \widetilde{\phi}^i + c^i_j(\widetilde{\phi})\,(\partial^2\widetilde{\phi}^j) + \frac{1}{2}\,c^i_{jk}(\widetilde{\phi})\,(\partial_\mu\widetilde{\phi}^j)(\partial^\mu\widetilde{\phi}^k)\,, \tag{B.8}$$

where $c^i_{jk} = c^i_{kj}$. In the $\widetilde{\phi}$ basis, the action becomes:

$$\begin{aligned}\widetilde{S}[\widetilde{\phi}] = \int_x \bigg[&- \widetilde{U}(\widetilde{\phi}) + \frac{1}{2}\,\widetilde{g}_{ij}(\phi)(\partial_\mu\widetilde{\phi}^i)(\partial^\mu\widetilde{\phi}^j) \\ &+ \frac{1}{8}\,\widetilde{\lambda}_{ijkl}(\widetilde{\phi})\,\mathcal{O}_1^{ijkl}(\widetilde{\phi}) + \frac{1}{2}\,\widetilde{\lambda}_{ijk}(\widetilde{\phi})\,\mathcal{O}_2^{ijk}(\widetilde{\phi}) + \frac{1}{2}\,\widetilde{\lambda}_{ij}(\widetilde{\phi})\,\mathcal{O}_3^{ij}(\widetilde{\phi}) + \mathcal{O}(\partial^6)\bigg]\,,\end{aligned} \tag{B.9}$$

where

$$\widetilde{U} = U'\,, \tag{B.10a}$$

$$\widetilde{g}_{ij} = g'_{ij} + U'_{,k}\big(c^k_{i,j} + c^k_{j,i} - c^k_{ij}\big) + U'_{,ik}\,c^k_j + U'_{,jk}\,c^k_i\,, \tag{B.10b}$$

$$\widetilde{\lambda}_{ijkl} = \lambda'_{ijkl} - 2\,g'_{mn}\big(\Gamma'^m_{ij}\,c^n_{kl} + \Gamma'^m_{kl}\,c^n_{ij}\big) - U'_{,mn}\,c^m_{ij}\,c^n_{kl}\,, \tag{B.10c}$$

$$\widetilde{\lambda}_{ijk} = \lambda'_{ijk} - g'_{kl}\,c^l_{ij} - 2\,g'_{lm}\,\Gamma'^l_{ij}\,c^m_k - U'_{,lm}\,c^l_{ij}\,c^m_k\,, \tag{B.10d}$$

$$\widetilde{\lambda}_{ij} = \lambda'_{ij} - g'_{ik}\,c^k_j - g'_{jk}\,c^k_i - U'_{,kl}\,c^k_i\,c^l_j\,. \tag{B.10e}$$

In these equations, $\Gamma'^k_{ij} \equiv \frac{1}{2}\,g'^{kl}(g'_{li,j} + g'_{lj,i} - g'_{ij,l})$ represents the Levi-Civita connection computed as if g'_{ij} is the metric.

If we choose c^i_j and c^i_{jk} such that

$$\widetilde{\lambda}_{ijk} = 0 \qquad \text{and} \qquad \widetilde{\lambda}_{ij} = 0\,, \tag{B.11}$$

then the $\widetilde{\phi}$ basis is a Warsaw-like basis. The condition Eq. (B.11) can be solved perturbatively if we assume a power counting $\lambda'_{ijk} \sim \frac{1}{\Lambda^3}$, $\lambda'_{ij} \sim \frac{1}{\Lambda^2}$, while $U'_{ij} \sim m^2 \ll$

Λ^2, where Λ is the EFT cutoff, in which case we obtain:

$$c^i_j = \frac{1}{2}\, g'^{ik}\, \lambda'_{kj} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right), \tag{B.12a}$$

$$c^i_{jk} = g'^{il}\left(\lambda'_{jkl} - \Gamma'^m_{jk}\, \lambda'_{lm}\right) + \mathcal{O}\left(\frac{1}{\Lambda^5}\right). \tag{B.12b}$$

In the Warsaw-like $\widetilde{\phi}$ basis, we have, according to Eq. (4.30):

$$\widetilde{U}[\widetilde{\phi}] \equiv \int_x \widetilde{U}\big(\widetilde{\phi}(x)\big)\,, \tag{B.13a}$$

$$\widetilde{g}_{ab}[\widetilde{\phi}] \equiv \int_x e^{-i(p_a+p_b)\cdot x}\, \widetilde{g}_{i_a i_b}\big(\widetilde{\phi}(x)\big)\,, \tag{B.13b}$$

$$\widetilde{\lambda}_{abcd}[\widetilde{\phi}] \equiv \int_x e^{-i(p_a+p_b+p_c+p_d)\cdot x}\, \widetilde{\lambda}_{i_a i_b i_c i_d}\big(\widetilde{\phi}(x)\big)\,, \tag{B.13c}$$

where the functions in the integrand on the right-hand side of these equations are given by Eqs. (B.10).

Now that we have the Warsaw-like basis expressions, we can obtain $U[\phi]$, $g_{ab}[\phi]$ and $\lambda_{abcd}[\phi]$ in the original basis following the standard tensor transformation rules. For the scalar potential, we have:

$$U[\phi] = \widetilde{U}\big[F[\phi]\big] = \int_x \widetilde{U}\big(F[\phi](x)\big) = \int_x U'\big(F[\phi](x)\big)\,, \tag{B.14}$$

where $\widetilde{\phi}^i = F^i[\phi]$ is the inverse of the field redefinition in Eq. (B.8), obtained iteratively as:

$$\begin{aligned}
F^i[\phi] = \phi^i - c^i_j(\phi)(\partial^2\phi^j) - \frac{1}{2}\, c^i_{jk}(\phi)(\partial_\mu\phi^j)(\partial^\mu\phi^k) \\
+ \Big[c^i_{j,l}(\phi)(\partial^2\phi^j) + \frac{1}{2}\, c^i_{jk,l}(\phi)(\partial_\mu\phi^j)(\partial^\mu\phi^k) + c^i_l(\phi)\,\partial^2 + c^i_{jl}(\phi)(\partial_\mu\phi^j)\,\partial^\mu\Big] \\
\Big[c^l_m(\phi)(\partial^2\phi^m) + \frac{1}{2}\, f^l_{mn}(\phi)(\partial_\nu\phi^m)(\partial^\nu\phi^n)\Big] + \mathcal{O}(\partial^6)\,.
\end{aligned} \tag{B.15}$$

Substituting Eq. (B.15) into Eq. (B.14) and expanding around ϕ, we obtain:

$$
\begin{aligned}
U[\phi] = \int_x \Big\{ & U'(\phi) + \frac{1}{2}\left(g'_{ij}(\phi) - \widetilde{g}_{ij}(\phi)\right)(\partial_\mu \phi^i)(\partial^\mu \phi^j) \\
& + U'_{,i}(\phi)\left[c^i_{j,l}(\phi)(\partial^2\phi^j) + \frac{1}{2}\,c^i_{jk,l}(\phi)(\partial_\mu\phi^j)(\partial^\mu\phi^k) + c^i_l(\phi)\,\partial^2 + c^i_{jl}(\phi)(\partial_\mu\phi^j)\,\partial^\mu\right] \\
& \qquad \left[c^l_m(\phi)(\partial^2\phi^m) + \frac{1}{2}\,f^l_{mn}(\phi)(\partial_\nu\phi^m)(\partial^\nu\phi^n)\right] \\
& + \frac{1}{2}\,U'_{,ij}(\phi)\left[c^i_k(\phi)(\partial^2\phi^k) + \frac{1}{2}\,f^i_{kl}(\phi)(\partial_\mu\phi^k)(\partial^\mu\phi^l)\right] \\
& \qquad \left[c^j_m(\phi)(\partial^2\phi^m) + \frac{1}{2}\,f^j_{mn}(\phi)(\partial_\nu\phi^m)(\partial^\nu\phi^n)\right] + \mathcal{O}(\partial^6)\Big\}\,.
\end{aligned}
\tag{B.16}
$$

With some algebra, one can also rewrite the $\mathcal{O}(\partial^4)$ terms as a linear combination of $\mathcal{O}_1^{ijkl}(\phi)$, $\mathcal{O}_2^{ijk}(\phi)$ and $\mathcal{O}_3^{ij}(\phi)$. We see from Eq. (B.16) that the potential functional contains not only the spacetime integral of the potential function $U'(\phi)$ in the original basis, but also part of the two- and four-derivative terms.

Next, to derive the metric in the original basis, it is easier to work in position space, where

$$
g_{(i_a x_a)(i_b x_b)}[\phi] = \int_{y_a, y_b} \frac{\delta\widetilde{\phi}^{(j_a y_a)}}{\delta\phi^{(i_a x_a)}}\,\frac{\delta\widetilde{\phi}^{(j_b y_b)}}{\delta\phi^{(i_b x_b)}}\,\widetilde{g}_{(j_a y_a)(j_b y_b)}\big[F[\phi]\big]\,,
\tag{B.17}
$$

with

$$
\begin{aligned}
\widetilde{g}_{(j_a y_a)(j_b y_b)}\big[F[\phi]\big] &= \int_{p_a, p_b} e^{i(p_a\cdot y_a + p_b\cdot y_b)}\,\widetilde{g}_{(j_a p_a)(j_b p_b)}\big[F[\phi]\big] \\
&= \int_x \delta^d(x - y_a)\,\delta^d(x - y_b)\,\widetilde{g}_{j_a j_b}\big(F[\phi](x)\big) \\
&= \int_x \delta^d(x - y_a)\,\delta^d(x - y_b)\,\Big[\widetilde{g}_{j_a j_b}\big(\phi(x)\big) \\
&\qquad - \widetilde{g}_{j_a j_b, i}(\phi)\left(c^i_j(\phi)(\partial^2\phi^j) + \frac{1}{2}\,c^i_{jk}(\phi)(\partial_\mu\phi^j)(\partial^\mu\phi^k)\right) + \mathcal{O}(\partial^4)\Big]\,.
\end{aligned}
\tag{B.18}
$$

The functional derivatives entering Eq. (B.17) can be obtained as:

$$
\begin{aligned}
\frac{\delta\widetilde{\phi}^{(jy)}}{\delta\phi^{(ix)}} = \Big[&\delta^j_i - c^j_{k,i}(\phi)(\partial^2\phi^k) - \frac{1}{2}\,c^j_{kl,i}(\phi)(\partial_\mu\phi^k)(\partial^\mu\phi^l) \\
& - c^j_i(\phi)\,\partial^2 - c^j_{ik}(\phi)(\partial_\mu\phi^k)\,\partial^\mu + \mathcal{O}(\partial^4)\Big]_y\,\delta^d(y - x)\,,
\end{aligned}
\tag{B.19}
$$

where $[\dots]_y$ means the fields and their derivatives are evaluated at y, and open derivatives are taken with respect to y. Substituting Eqs. (B.18) and (B.19) into Eq. (B.17), and Fourier transforming back to momentum space, we obtain

$$\begin{aligned}
g_{ab}[\phi] &= \int_{x_a, x_b} e^{-i(p_a\cdot x_a + p_b\cdot x_b)}\, g_{(i_a x_a)(i_b x_b)}[\phi] \\
&= \int_x e^{-i(p_a+p_b)\cdot x} \Big\{ \widetilde{g}_{i_a i_b}(\phi) \\
&\qquad - \Big[\widetilde{g}_{i_a i_b, i}(\phi)\, c^i_j(\phi) + \widetilde{g}_{i_a i}(\phi)\, c^i_{j,i_b}(\phi) + \widetilde{g}_{i_b i}(\phi)\, c^i_{j,i_a}(\phi) \Big] (\partial^2 \phi^j) \\
&\qquad - \frac{1}{2} \Big[\widetilde{g}_{i_a i_b, i}(\phi)\, c^i_{jk}(\phi) + \widetilde{g}_{i_a i}(\phi)\, c^i_{jk,i_b}(\phi) + \widetilde{g}_{i_b i}(\phi)\, c^i_{jk,i_a}(\phi) \Big] (\partial_\mu \phi^j)(\partial^\mu \phi^k) \\
&\qquad + \widetilde{g}_{i_a i}(\phi) \Big[p_b^2\, c^i_{i_b}(\phi) + i p_b^\mu\, c^i_{i_b j}(\phi)\, (\partial_\mu \phi^j) \Big] \\
&\qquad + \widetilde{g}_{i_b i}(\phi) \Big[p_a^2\, c^i_{i_a}(\phi) + i p_a^\mu\, c^i_{i_a j}(\phi)\, (\partial_\mu \phi^j) \Big] + \mathcal{O}(\partial^4) \Big\} ,
\end{aligned} \tag{B.20}$$

where we have counted $p_a, p_b \sim \mathcal{O}(\partial)$. Noting that $\widetilde{g}_{ij}$ is given by Eq. (B.10b) and c^i_j, c^i_{jk} solve Eq. (B.11), we see that the metric $g_{ab}[\phi]$ contains information about zero- and four-derivative terms as well as two-derivative terms in the original basis.

Finally, for $\lambda_{abcd}[\phi]$, since we are only working up to $\mathcal{O}(\partial^4)$ in the action, it is sufficient to keep the leading order term in Eq. (B.19), or equivalently,

$$\frac{\partial \widetilde{\phi}^a}{\partial \phi^b} = \delta^a_b + \mathcal{O}(\partial^2)\,. \tag{B.21}$$

We obtain:

$$\begin{aligned}
\lambda_{abcd}[\phi] &= \frac{\partial \widetilde{\phi}^e}{\partial \phi^a} \frac{\partial \widetilde{\phi}^f}{\partial \phi^b} \frac{\partial \widetilde{\phi}^g}{\partial \phi^c} \frac{\partial \widetilde{\phi}^h}{\partial \phi^d}\, \widetilde{\lambda}_{efgh}\big[F[\phi]\big] \\
&= \int_x e^{-i(p_a+p_b+p_c+p_d)\cdot x} \Big[\widetilde{\lambda}_{i_a i_b i_c i_d}(\phi) + \mathcal{O}(\partial^2) \Big] ,
\end{aligned} \tag{B.22}$$

with $\widetilde{\lambda}_{ijkl}$ given in Eq. (B.10).

C Nonderivative field redefinitions on the field configuration manifold

In this appendix, we discuss in more detail nonderivative field redefinitions as a special class of coordinate transformations on the field configuration manifold. Consider $\phi^i = f^i(\widetilde{\phi})$, where f^i is a set of real analytic functions. As a coordinate transformation on the field configuration manifold, it can be written as:

$$\phi^{(ix)} = F^{(ix)}[\widetilde{\phi}] \equiv \int_y \delta^d(x-y)\, f^i\big(\widetilde{\phi}(y)\big)\,. \tag{C.1}$$

Therefore,

$$\frac{\delta\phi^{(ix)}}{\delta\widetilde{\phi}^{(jy)}} = \frac{\partial\phi^i}{\partial\widetilde{\phi}^j}\,\delta^d(x-y)\,, \qquad \frac{\delta^2\phi^{(ix)}}{\delta\widetilde{\phi}^{(jy)}\delta\widetilde{\phi}^{(kz)}} = \frac{\partial^2\phi^i}{\partial\widetilde{\phi}^j\partial\widetilde{\phi}^k}\,\delta^d(x-y)\,\delta^d(x-z)\,, \tag{C.2}$$

etc., where $\frac{\partial\phi^i}{\partial\widetilde{\phi}^j}$ and $\frac{\partial^2\phi^i}{\partial\widetilde{\phi}^j\partial\widetilde{\phi}^k}$ represent partial derivatives of $\phi^i = f^i(\widetilde{\phi})$ with respect to $\widetilde{\phi}$ evaluated at x. We can also use Eq. (4.32) to take functional derivatives in momentum space:

$$\frac{\partial\phi^{(ip)}}{\partial\widetilde{\phi}^{(jq)}} = \int_y e^{-iq\cdot y}\,\frac{\delta}{\delta\widetilde{\phi}^{(jy)}}\int_x e^{ip\cdot x}\,\phi^{(ix)} = \int_x e^{i(p-q)\cdot x}\,\frac{\partial\phi^i}{\partial\widetilde{\phi}^j}\,, \tag{C.3a}$$

$$\frac{\partial^2\phi^{(ip)}}{\partial\widetilde{\phi}^{(jq)}\partial\widetilde{\phi}^{(kr)}} = \int_z e^{-ir\cdot z}\,\frac{\delta}{\delta\widetilde{\phi}^{(kz)}}\int_x e^{i(p-q)\cdot x}\,\frac{\partial\phi^i}{\partial\widetilde{\phi}^j} = \int_x e^{i(p-q-r)\cdot x}\,\frac{\partial^2\phi^i}{\partial\widetilde{\phi}^j\partial\widetilde{\phi}^k}\,, \tag{C.3b}$$

etc.

A direct consequence of these equations is that, under nonderivative field redefinitions, any object on the field configuration manifold that is defined as the Fourier transform of an object on the target manifold transforms in the same way as its target manifold counterpart. As a nontrivial example, consider the Levi-Civita connection obtained from the metric defined in Eq. (4.30b), which is the Fourier transform of the Levi-Civita connection in field space geometry (see Eq. (4.31a)):

$$\Gamma^a_{bc}[\phi] = \Gamma^{(i_a p_a)}_{(i_b p_b)(i_c p_c)}[\phi] \equiv \int_x e^{i(p_a-p_b-p_c)\cdot x}\,\Gamma^{i_a}_{i_b i_c}\big(\phi(x)\big)\,. \tag{C.4}$$

Under a nonderivative field redefinition, $\Gamma^{i_a}_{i_b i_c}$ transforms as a connection on the target manifold. So if we compute Γ^a_{bc} from the transformed target manifold connection, we would obtain:

$$\begin{aligned}\widetilde{\Gamma}^a_{bc}[\widetilde{\phi}] &= \int_x e^{i(p_a-p_b-p_c)\cdot x}\,\widetilde{\Gamma}^{i_a}_{i_b i_c}\big(\widetilde{\phi}(x)\big)\\ &= \int_x e^{i(p_a-p_b-p_c)\cdot x}\left[\frac{\partial\widetilde{\phi}^{i_a}}{\partial\phi^{i_d}}\,\frac{\partial\phi^{i_e}}{\partial\widetilde{\phi}^{i_b}}\,\frac{\partial\phi^{i_f}}{\partial\widetilde{\phi}^{i_c}}\,\Gamma^{i_d}_{i_e i_f}\big(f(\widetilde{\phi})\big) + \frac{\partial\widetilde{\phi}^{i_a}}{\partial\phi^{i_d}}\,\frac{\partial^2\phi^{i_d}}{\partial\widetilde{\phi}^{i_b}\partial\widetilde{\phi}^{i_c}}\right].\end{aligned} \tag{C.5}$$

Using Eq. (C.3), it is straightforward to check that this is equivalent to

$$\widetilde{\Gamma}^a_{bc}[\widetilde{\phi}] = \frac{\partial\widetilde{\phi}^a}{\partial\phi^d}\,\frac{\partial\phi^e}{\partial\widetilde{\phi}^b}\,\frac{\partial\phi^f}{\partial\widetilde{\phi}^c}\,\Gamma^d_{ef}\big[F(\widetilde{\phi})\big] + \frac{\partial\widetilde{\phi}^a}{\partial\phi^d}\,\frac{\partial^2\phi^d}{\partial\widetilde{\phi}^b\partial\widetilde{\phi}^c}\,, \tag{C.6}$$

as expected for a connection on the field configuration manifold. We can see the same in position space, where the connection on the field configuration manifold is given by:

$$\begin{aligned}\Gamma^{(i_a x_a)}_{(i_b x_b)(i_c x_c)}[\phi] &= \int_{p_a,p_b,p_c} e^{-i(p_a\cdot x_a - p_b\cdot x_b - p_c\cdot x_c)}\,\Gamma^{(i_a p_a)}_{(i_b p_b)(i_c p_c)}[\phi]\\ &= \int_x \Gamma^{i_a}_{i_b i_c}\big(\phi(x)\big)\,\delta^d(x-x_a)\,\delta^d(x-x_b)\,\delta^d(x-x_c)\,.\end{aligned} \tag{C.7}$$

Under a nonderivative field redefinition,

$$
\widetilde{\Gamma}^{(i_a x_a)}_{(i_b x_b)(i_c x_c)}[\widetilde{\phi}] = \int_x \widetilde{\Gamma}^{i_a}_{i_b i_c}\big(\widetilde{\phi}(x)\big)\, \delta^d(x - x_a)\, \delta^d(x - x_b)\, \delta^d(x - x_c)
$$

$$
= \int_x \left[\frac{\partial \widetilde{\phi}^{i_a}}{\partial \phi^{i_d}} \frac{\partial \phi^{i_e}}{\partial \widetilde{\phi}^{i_b}} \frac{\partial \phi^{i_f}}{\partial \widetilde{\phi}^{i_c}} \Gamma^{i_d}_{i_e i_f}\big(f(\widetilde{\phi})\big) + \frac{\partial \widetilde{\phi}^{i_a}}{\partial \phi^{i_d}} \frac{\partial^2 \phi^{i_d}}{\partial \widetilde{\phi}^{i_b} \partial \widetilde{\phi}^{i_c}} \right] \delta^d(x - x_a)\, \delta^d(x - x_b)\, \delta^d(x - x_c)
$$

$$
= \int_{x_d, x_e, x_f} \frac{\delta \widetilde{\phi}^{(i_a x_a)}}{\delta \phi^{(i_d x_d)}} \frac{\delta \phi^{(i_e x_e)}}{\delta \widetilde{\phi}^{(i_b x_b)}} \frac{\delta \phi^{(i_f x_f)}}{\delta \widetilde{\phi}^{(i_c x_c)}} \Gamma^{(i_d x_d)}_{(i_e x_e)(i_f x_f)}\big[f(\widetilde{\phi})\big] + \int_{x_d} \frac{\delta \widetilde{\phi}^{(i_a x_a)}}{\delta \phi^{(i_d x_d)}} \frac{\delta^2 \phi^{(i_e x_d)}}{\delta \widetilde{\phi}^{(i_b x_b)} \delta \widetilde{\phi}^{(i_c x_c)}}\,, \tag{C.8}
$$

which is again the desired transformation of a connection on the field configuration manifold.

References

[1] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, *Dimension-Six Terms in the Standard Model Lagrangian*, *JHEP* **10** (2010) 085, [arXiv:1008.4884].

[2] K. Hagiwara, S. Ishihara, R. Szalapski, and D. Zeppenfeld, *Low-energy effects of new interactions in the electroweak boson sector*, *Phys. Rev. D* **48** (1993) 2182–2203.

[3] G. F. Giudice, C. Grojean, A. Pomarol, and R. Rattazzi, *The Strongly-Interacting Light Higgs*, *JHEP* **06** (2007) 045, [hep-ph/0703164].

[4] J. Elias-Miro, J. R. Espinosa, E. Masso, and A. Pomarol, *Higgs windows to new physics through d=6 operators: constraints and one-loop anomalous dimensions*, *JHEP* **11** (2013) 066, [arXiv:1308.1879].

[5] J. Elias-Miró, C. Grojean, R. S. Gupta, and D. Marzocca, *Scaling and tuning of EW and Higgs observables*, *JHEP* **05** (2014) 019, [arXiv:1312.2928].

[6] S. R. Coleman, J. Wess, and B. Zumino, *Structure of phenomenological Lagrangians. 1.*, *Phys. Rev.* **177** (1969) 2239–2247.

[7] C. G. Callan, Jr., S. R. Coleman, J. Wess, and B. Zumino, *Structure of phenomenological Lagrangians. 2.*, *Phys. Rev.* **177** (1969) 2247–2250.

[8] J. Honerkamp, *Chiral multiloops*, *Nucl. Phys. B* **36** (1972) 130–140.

[9] D. V. Volkov, *Phenomenological Lagrangians*, *Fiz. Elem. Chast. Atom. Yadra* **4** (1973) 3–41.

[10] L. Tataru, *One Loop Divergences of the Nonlinear Chiral Theory*, *Phys. Rev. D* **12** (1975) 3351–3352.

[11] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, *The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model*, *Annals Phys.* **134** (1981) 85.

[12] L. Alvarez-Gaume and D. Z. Freedman, *Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model*, *Commun. Math. Phys.* **80** (1981) 443.

[13] G. A. Vilkovisky, *The Unique Effective Action in Quantum Field Theory*, *Nucl. Phys. B* **234** (1984) 125–137.

[14] B. S. DeWitt, *The spacetime approach to quantum field theory*, in *Les Houches Summer School on Theoretical Physics: Relativity, Groups and Topology*, pp. 381–738, 1984.

[15] M. K. Gaillard, *The Effective One Loop Lagrangian With Derivative Couplings*, *Nucl. Phys. B* **268** (1986) 669–692.

[16] B. S. DeWitt, *The Effective Action*, in *Les Houches School of Theoretical Physics: Architecture of Fundamental Interactions at Short Distances*, pp. 1023–1058, 1987.

[17] H. Georgi, *On-shell effective field theory*, *Nucl. Phys. B* **361** (1991) 339–350.

[18] R. Alonso, E. E. Jenkins, and A. V. Manohar, *A Geometric Formulation of Higgs Effective Field Theory: Measuring the Curvature of Scalar Field Space*, *Phys. Lett. B* **754** (2016) 335–342, [arXiv:1511.00724].

[19] R. Alonso, E. E. Jenkins, and A. V. Manohar, *Geometry of the Scalar Sector*, *JHEP* **08** (2016) 101, [arXiv:1605.03602].

[20] T. Cohen, N. Craig, X. Lu, and D. Sutherland, *Is SMEFT Enough?*, *JHEP* **03** (2021) 237, [arXiv:2008.08597].

[21] T. Cohen, N. Craig, X. Lu, and D. Sutherland, *Unitarity violation and the geometry of Higgs EFTs*, *JHEP* **12** (2021) 003, [arXiv:2108.03240].

[22] R. Alonso and M. West, *Roads to the Standard Model*, *Phys. Rev. D* **105** (2022), no. 9 096028, [arXiv:2109.13290].

[23] A. Helset, A. Martin, and M. Trott, *The Geometric Standard Model Effective Field Theory*, *JHEP* **03** (2020) 163, [arXiv:2001.01453].

[24] C. Hays, A. Helset, A. Martin, and M. Trott, *Exact SMEFT formulation and expansion to $\mathcal{O}(v^4/\Lambda^4)$*, *JHEP* **11** (2020) 087, [arXiv:2007.00565].

[25] T. Corbett, A. Helset, A. Martin, and M. Trott, *EWPD in the SMEFT to dimension eight*, *JHEP* **06** (2021) 076, [arXiv:2102.02819].

[26] A. Martin and M. Trott, *More accurate $\sigma\left(\mathcal{GG} \to h\right), \Gamma\left(h \to \mathcal{GG}, \mathcal{AA}, \overline{\Psi}\Psi\right)$ and Higgs width results via the geoSMEFT*, *JHEP* **01** (2024) 170, [arXiv:2305.05879].

[27] C. Cheung, A. Helset, and J. Parra-Martinez, *Geometric soft theorems*, *JHEP* **04** (2022) 011, [arXiv:2111.03045].

[28] M. Derda, A. Helset, and J. Parra-Martinez, *Soft scalars in effective field theory*, *JHEP* **06** (2024) 133, [arXiv:2403.12142].

[29] T. Cohen, I. Fadakar, A. Helset, and F. Nardi, *Geometry of soft scalars at one loop*, *JHEP* **08** (2025) 140, [arXiv:2504.12371].

[30] R. Alonso, K. Kanshin, and S. Saa, *Renormalization group evolution of Higgs effective field theory*, *Phys. Rev. D* **97** (2018), no. 3 035010, [arXiv:1710.06848].

[31] R. Alonso and M. West, *On the effective action for scalars in a general manifold to any loop order*, *Phys. Lett. B* **841** (2023) 137937, [arXiv:2207.02050].

[32] A. Helset, E. E. Jenkins, and A. V. Manohar, *Renormalization of the Standard Model Effective Field Theory from geometry*, *JHEP* **02** (2023) 063, [arXiv:2212.03253].

[33] B. Assi, A. Helset, A. V. Manohar, J. Pagès, and C.-H. Shen, *Fermion geometry and the renormalization of the Standard Model Effective Field Theory*, *JHEP* **11** (2023) 201, [arXiv:2307.03187].

[34] E. E. Jenkins, A. V. Manohar, L. Naterop, and J. Pagès, *An algebraic formula for two loop renormalization of scalar quantum field theory*, *JHEP* **12** (2023) 165, [arXiv:2308.06315].

[35] E. E. Jenkins, A. V. Manohar, L. Naterop, and J. Pagès, *Two loop renormalization of scalar theories using a geometric approach*, *JHEP* **02** (2024) 131, [arXiv:2310.19883].

[36] X.-X. Li, X. Lu, and Z. Zhang, *The geometric universal one-loop effective action*, *JHEP* **08** (2025) 102, [arXiv:2411.04173].

[37] P. Aigner, L. Bellafronte, E. Gendy, D. Haslehner, and A. Weiler, *Renormalising the field-space geometry*, *JHEP* **07** (2025) 167, [arXiv:2503.09785].

[38] B. Assi, A. Helset, J. Pagès, and C.-H. Shen, *Renormalizing Two-Fermion Operators in the SMEFT via Supergeometry*, arXiv:2504.18537.

[39] K. Finn, S. Karamitsos, and A. Pilaftsis, *Frame covariant formalism for fermionic theories*, *Eur. Phys. J. C* **81** (2021), no. 7 572, [arXiv:2006.05831].

[40] V. Gattus and A. Pilaftsis, *Minimal supergeometric quantum field theories*, *Phys. Lett. B* **846** (2023) 138234, [arXiv:2307.01126].

[41] V. Gattus and A. Pilaftsis, *Supergeometric quantum effective action*, *Phys. Rev. D* **110** (2024), no. 10 105006, [arXiv:2406.13594].

[42] N. Craig, I.-K. Lee, and Y.-T. Lee, *Fermi Geometry of the Higgs Sector*, arXiv:2509.07101.

[43] K. Finn, S. Karamitsos, and A. Pilaftsis, *Frame Covariance in Quantum Gravity*, *Phys. Rev. D* **102** (2020), no. 4 045014, [arXiv:1910.06661].

[44] N. Craig, Y.-T. Lee, X. Lu, and D. Sutherland, *Effective field theories as Lagrange spaces*, *JHEP* **11** (2023) 069, [arXiv:2305.09722].

[45] N. Craig and Y.-T. Lee, *Effective Field Theories on the Jet Bundle*, *Phys. Rev. Lett.* **132** (2024), no. 6 061602, [arXiv:2307.15742].

[46] M. Alminawi, I. Brivio, and J. Davighi, *Jet Bundle Geometry of Scalar Field Theories*, *J. Phys. A* **57** (2024) 435401, [arXiv:2308.00017].

[47] Y.-T. Lee, *Field space geometry and nonlinear supersymmetry*, *Phys. Rev. D* **111** (2025), no. 10 105004, [arXiv:2410.21395].

[48] T. Cohen, N. Craig, X. Lu, and D. Sutherland, *On-Shell Covariance of Quantum Field Theory Amplitudes*, *Phys. Rev. Lett.* **130** (2023), no. 4 041603, [arXiv:2202.06965].

[49] C. Cheung, A. Helset, and J. Parra-Martinez, *Geometry-kinematics duality*, *Phys. Rev. D* **106** (2022), no. 4 045016, [arXiv:2202.06972].

[50] T. Cohen, X. Lu, and D. Sutherland, *On amplitudes and field redefinitions*, *JHEP* **06** (2024) 149, [arXiv:2312.06748].

[51] T. Cohen, X. Lu, and Z. Zhang, *What is the geometry of effective field theories?*, *Phys. Rev. D* **111** (2025), no. 8 085012, [arXiv:2410.21378].

[52] U. Muller, C. Schubert, and A. M. E. van de Ven, *A Closed formula for the Riemann normal coordinate expansion*, *Gen. Rel. Grav.* **31** (1999) 1759–1768, [gr-qc/9712092].

[53] A. Hatzinikitas, *A Note on Riemann normal coordinates*, hep-th/0001078.

[54] G. 't Hooft and M. J. G. Veltman, *DIAGRAMMAR*, *NATO Sci. Ser. B* **4** (1974) 177–322.

[55] C. Arzt, *Reduced effective Lagrangians*, *Phys. Lett. B* **342** (1995) 189–195, [hep-ph/9304230].

[56] J. C. Criado and M. Pérez-Victoria, *Field redefinitions in effective theories at higher orders*, *JHEP* **03** (2019) 038, [arXiv:1811.09413].

[57] T. Cohen, M. Forslund, and A. Helset, *Field Redefinitions Can Be Nonlocal*, arXiv:2412.12247.

[58] M. O. Katanaev, *Normal coordinates in affine geometry*, *Lobachevskii Journal of Mathematics* **39** (2018), no. 3 464–476.

[59] G. Ecker and J. Honerkamp, *Covariant perturbation theory and chiral superpropagators*, *Phys. Lett. B* **42** (1972) 253–256.

[60] A. Helset, E. E. Jenkins, and A. V. Manohar, *Geometry in scattering amplitudes*, *Phys. Rev. D* **106** (2022), no. 11 116018, [arXiv:2210.08000].

[61] M. Alminawi, I. Brivio, and J. Davighi, *Scalar Amplitudes from Fibre Bundle Geometry*.