
Part II

Local Structures

The continuity of a real-valued function on a topological space is a local property: If one knows the function only in a small neighborhood of each point of the domain, one can decide whether the function is continuous or not. On the other hand, the integrability of a real-valued function on a topological space equipped with a measure is not a local property. For example, for the constant function 1 on \mathbb{R} , the restriction to any bounded open set is integrable, but the entire function is not.

For open subsets of \mathbb{R}^n , the differentiability of real-valued functions is also a local property. Given the usefulness of differentiable functions, the question arises whether the concept of differentiability can be extended to other topological spaces. What requirement must one impose on a topological space for this? One answer to this question is: “The topological space must locally look like \mathbb{R}^n .” Thus, it is required that the topological space carries an additional structure that is of a local nature.

In this part of the book, the idea of a local structure is clarified and a number of significant local structures are introduced. We begin in Chap. 5 with the abstract concept of a sheaf, which on the one hand clarifies the idea of a local property and on the other hand is a very versatile and powerful conceptual tool that is indispensable in modern mathematics.

As a first concrete local structure, we consider the concept of a manifold, whose local model is a vector space with a suitable sheaf of functions. We restrict ourselves to finite-dimensional vector spaces over \mathbb{R} or \mathbb{C} , which then leads to finite-dimensional real or complex manifolds. For the associated sheaves of functions, there are still various choices. For continuous functions, one obtains the concept of a topological manifold. If at least one continuous derivative is required, one obtains differentiable manifolds of various degrees of regularity. At the end of the scale, there are the analytic manifolds, for which the functions of the local model are the power series with positive radius of convergence. For real manifolds, a good part of the local theory is covered in advanced calculus courses, in which differential and integral calculus in one and several variables is treated. Therefore, the focus in Chap. 6 is on the construction of derived local structures such as the tangent bundle and the sheaf of differential forms. With the help of differential forms, one

can develop an integration theory for manifolds. The approach here is exemplary for local structures: One knows integrals in the local models and knows (from the transformation formula) how they behave under diffeomorphisms. This knowledge is used to define a global integral of differential forms of maximum degree by gluing. Also exemplary for this context is Stokes' theorem, which is obtained by gluing from the fundamental theorem of calculus.

The local theory of complex manifolds, i.e., the differential and integral calculus of complex-valued functions in one or more complex variables, is usually not covered in calculus courses. Since a central result for this, the Cauchy integral theorem, can be easily derived from Stokes' theorem, we prove some basic local results and thus show as an exemplary global result that every complex-differentiable function on a compact complex manifold is constant.

In Chap. 7 we turn to the second family of local structures to be discussed in more detail in this part: algebraic varieties and schemes. Here, the local theory requires further algebraic preparations. In particular, we show Hilbert's Nullstellensatz, which generalizes the fundamental theorem of algebra and allows to describe the zero sets of polynomials with coefficients in an algebraically closed field by ideals in polynomial rings. These zero sets are the local models for algebraic varieties, whose points can then be identified with maximal ideals of a ring. The local models for schemes are even more diverse and serve the purpose of also being able to make statements about zero sets of polynomials with coefficients in arbitrary fields (or even rings).