

# Conformal bootstrap in Mellin space from GG systems

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## Abstract

A simple scheme to express the Mellin transform of  $D$ -dimensional Euclidean conformal bootstrap equation is presented by relating conformal blocks to a Gauss-Grassmann (GG) system due to Gelfand-Graev, associated to conformal integrals, which, in turn, are generalised hypergeometric functions. Usefulness of the expression for obtaining bounds on the spectrum of fields is demonstrated.

Specification of a conformal field theory (CFT) entails providing the spectrum of primary fields in terms of quantum numbers associated to the generators of the conformal algebra and the constants in the ring of fields, called the operator product expansion (OPE) coefficients. Correlation functions of fields of a CFT depend on these data. The conformal bootstrap program is a scheme which seeks to restrict these data [1–4]. Inequivalent permutations of a configuration of fields are called channels. Obtaining restrictions on OPE coefficients as well as the spectrum of fields by equating correlation functions of configurations of  $N$  points, called  $N$ -point functions, in different channels is the desideratum. The bootstrap program has been extensively studied from its inception to the present day [5]. Since the form of 2- and 3-point functions are completely fixed by conformal covariance of the fields, such a scheme starts from the 4-point functions. An  $N$ -point function for  $N > 3$  contains a piece, called the conformal block, which is a function of geometric conformal invariants of the spacetime coordinates, called cross ratios. Since any function of the invariants is an invariant itself, these introduce a certain arbitrariness of the correlation functions, which is fixed by demanding consistency with the representation of the conformal group. While the conformal block is generally expressed as an infinite series as (generalised) hypergeometric functions of cross ratios from the representation theory of the conformal group, certain details require going beyond [6]. Thus, the bootstrap equations are equations involving infinite series of cross ratios, rendering them difficult to solve, to say the least. Furthermore, the conformal weights, that is the quantum numbers associated with the scaling generator of the conformal group, may be valued in a countable or an uncountable set of real or complex numbers. Correlation functions are expressed as a sum

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or integral over these weights, posing another hurdle for the bootstrap programme. A variety of techniques, both analytic and numerical, along with their combinations, have been devised over the years to deal with these equations [7–15]. Most of these tackle the problem by integrating the bootstrap equation over the cross ratios, perhaps with a weight function, by considering a functional of conformal blocks. This is usually achieved by continuing the real cross ratios to the domain of complex numbers which requires settling extremely subtle issues of choosing integration contours in conformity with Fubini’s theorem [16]. A variant of this approach is to consider Mellin transform of the bootstrap equation with respect to the cross ratios, where the weight of integration is fixed, although the domain of convergence of the infinite series in the conformal block still poses subtleties [17–19].

In this article we set up the bootstrap equations for a conformal field theory of scalar fields alone. We consider the Mellin transform of the equations with respect to cross ratios. Issues pertaining to the convergence of infinite series in the conformal block is eschewed by taking cognizance of the fact that the conformal block is written in terms of a solution to the GG system of hypergeometric functions which has an integral presentation. The conformal correlation functions have been expressed earlier in terms of conformal integrals which in turn are written as generalised GKZ hypergeometric functions and known in all dimensions [20–22]. While the conformal integral is covariant under conformal transformations, it contains an invariant piece which is related to the conformal block. Writing the conformal integral in terms of the solution to a GG system allows expressing the conformal block as an integral, containing the cross ratios as parameters [23, 24]. It is then easy to take the Mellin transform of it avoiding the difficulties associated with the convergence of infinite series.

In the rest of the article we explain the identification of conformal blocks with solutions of GG systems, leading to the expression of bootstrap equations in a straightforward manner. In order to keep the discussion simple we demonstrate the idea for the 4-point correlation function of a scalar conformal field theory. Generalisation to higher point functions is obviously possible along the same line.

Let  $\phi_\Delta(x)$  denote a  $D$ -dimensional primary scalar field with conformal weight  $\Delta$ , where  $x \in \mathbf{R}^D$ , the  $D$ -dimensional Euclidean space. The Euclidean norm of  $x$  is denoted  $|x|$  and we write  $x_{ij} = x_i - x_j$ . The  $N$ -point correlation function is denoted as

$$G_{\Delta}^{(N)}(\mathbf{x}) = \langle \phi_{\Delta_1}(x_1) \cdots \phi_{\Delta_N}(x_N) \rangle, \quad (1)$$

where  $\Delta = (\Delta_1, \dots, \Delta_N)$  is the  $N$ -tuple of weights of the fields and  $\mathbf{x} = (x_1, \dots, x_N) \in (\mathbf{R}^D)^N$ . Covariance under conformal transformations fixes the 2-point and 3-point correlation functions up to numerical constants as

$$G_{(\Delta, \Delta')}^{(2)}(x_1, x_2) = \frac{C_{\Delta} \delta_{\Delta \Delta'}}{|x_{12}|^{2\Delta}}, \quad (2)$$

$$G_{(\Delta_1, \Delta_2, \Delta_3)}^{(3)}(x_1, x_2, x_3) = \frac{C_{\Delta_1 \Delta_2 \Delta_3}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}}, \quad (3)$$

where  $C_{\Delta}$  is a constant, and  $C_{\Delta_1 \Delta_2 \Delta_3}$  are the OPE coefficients. Defining the conformal integral

$$I_N^{\mu}(\mathbf{x}) = \int \frac{d^D x}{|x - x_1|^{2\mu_1} |x - x_2|^{2\mu_2} \cdots |x - x_N|^{2\mu_N}}, \quad (4)$$

where  $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)$  is an  $N$ -tuple of weights with  $|\boldsymbol{\mu}| := \sum_{i=1}^N \mu_i = D$ , and redefining the constant  $C_\Delta$  appropriately, the 4-point correlation function  $G_{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}^{(4)}(x_1, x_2, x_3, x_4)$  is written as [20]

$$G_{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}^{(4 \succleftarrow)}(x_1, x_2, x_3, x_4) = \sum_{\Delta} C_{\Delta_1 \Delta_2 \Delta} C_{\Delta \Delta_3 \Delta_4} \frac{I_4\left(\frac{\Delta_1 - \Delta_2 + \Delta}{2}, \frac{\Delta_2 - \Delta_1 + \Delta}{2}, \frac{\Delta_3 - \Delta_4 + D - \Delta}{2}, \frac{\Delta_4 - \Delta_3 + D - \Delta}{2}\right)(x_1, x_2, x_3, x_4)}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta} |x_{34}|^{\Delta_3 + \Delta_4 + \Delta - D}} \quad (5)$$

in the  $s$ -channel. In the  $t$ -channel the 4-point correlation function assumes the form

$$G_{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}^{(4 \searrow)}(x_1, x_2, x_3, x_4) = \sum_{\Delta} C_{\Delta_1 \Delta_4 \Delta} C_{\Delta \Delta_3 \Delta_2} \frac{I_4\left(\frac{\Delta_1 - \Delta_4 + \Delta}{2}, \frac{\Delta_2 - \Delta_3 + D - \Delta}{2}, \frac{\Delta_3 - \Delta_2 + D - \Delta}{2}, \frac{\Delta_4 - \Delta_1 + \Delta}{2}\right)(x_1, x_2, x_3, x_4)}{|x_{14}|^{\Delta_1 + \Delta_4 - \Delta} |x_{23}|^{\Delta_2 + \Delta_3 + \Delta - D}}, \quad (6)$$

obtained from (5) by simultaneous exchange of  $(x_2, \Delta_2)$  and  $(x_4, \Delta_4)$ . The bootstrap equation is obtained by equating the correlation functions in the two channels,

$$G_{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}^{(4 \succleftarrow)}(x_1, x_2, x_3, x_4) = G_{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}^{(4 \searrow)}(x_1, x_2, x_3, x_4), \quad (7)$$

or, equivalently, presented diagrammatically, as,

$$\sum_{\Delta} \begin{array}{c} \Delta_2 \\ \diagdown \\ \Delta_1 \end{array} \begin{array}{c} \Delta \\ \diagup \\ \Delta_3 \end{array} \begin{array}{c} \Delta_4 \\ \diagdown \\ \Delta \end{array} \begin{array}{c} \Delta_3 \\ \diagup \\ \Delta \end{array} = \sum_{\Delta} \begin{array}{c} \Delta_2 \\ \diagdown \\ \Delta \end{array} \begin{array}{c} \Delta_3 \\ \diagup \\ \Delta \end{array} \begin{array}{c} \Delta_4 \\ \diagdown \\ \Delta \end{array} \begin{array}{c} \Delta_1 \\ \diagup \\ \Delta \end{array}. \quad (8)$$

Let us recall some properties of the conformal integral which we use to write the bootstrap equation in a convenient form. The conformal integral transforms covariantly under the conformal transformations of  $\mathbf{R}^D$ . It is, therefore, a function of  $|x_{ij}|$ . It is a generalised hypergeometric function satisfying the GKZ equations associated to the toric matrix

$$\mathcal{A}_{i, (jk)} = \delta_{ij} + \delta_{ik}, \quad (9)$$

obtained from the weights of conformal transformations [21, 22]. It is an  $N \times N(N-1)/2$  matrix with  $N(N-3)/2$  linearly independent null eigenvectors  $\ell$ , that is,

$$\sum_{\substack{j,k=1 \\ j < k}}^N \mathcal{A}_{i, (jk)} \ell_{jk}^A = 0, \quad A = 1, 2, \dots, N(N-3)/2. \quad (10)$$

The GKZ equations associated to this data are

$$\left( \prod_{\ell_{ij}^A > 0} \partial_{ij}^{\ell_{ij}^A} - \prod_{\ell_{ij}^A < 0} \partial_{ij}^{\ell_{ij}^A} \right) I_N^\mu(\mathbf{x}) = 0, \quad (11)$$

$$\sum_{j,k=1}^N \mathcal{A}_{i,(jk)} |x_{jk}|^2 \partial_{jk} I_N^\mu(\mathbf{x}) + \mu_i I_N^\mu(\mathbf{x}) = 0, \quad \forall i = 1, 2, \dots, N, \quad (12)$$

where we defined  $\partial_{ij} = \frac{\partial}{\partial |x_{ij}|^2}$ . As a solution of the GKZ equations the conformal integral (4) is written as [22]

$$I_N^\mu(\mathbf{x}) = \prod_{\substack{i,j=1 \\ i < j}}^N |x_{ij}|^{2\beta_{ij}} I_N^0(\boldsymbol{\xi}), \quad (13)$$

where  $\boldsymbol{\xi} = \{\xi^A; A = 1, 2, \dots, N(N-3)/2\}$  are real-valued conformal invariant cross ratios, with

$$\xi^A = \prod_{\substack{i,j=1 \\ i < j}}^N |x_{ij}|^{2\ell_{ij}^A}, \quad (14)$$

and  $\beta$ 's are parameters related to  $\boldsymbol{\mu}$ , satisfying

$$\begin{aligned} \beta_{ii} &= 0, \quad \beta_{ji} = \beta_{ij}, \\ \sum_{\substack{i,j=1 \\ i < j}}^N \mathcal{A}_{i,(jk)} \beta_{jk} &= -\mu_i, \end{aligned} \quad (15)$$

$i, j, k = 1, 2, \dots, N$ . By virtue of (10), the conformal integral (13) remains unaltered under a shift of  $\beta$ 's by arbitrary multiples of the null vectors, *viz.*

$$\beta_{ij} \longrightarrow \beta_{ij} + \sum_A n_A \ell_{ij}^A. \quad (16)$$

The invariant part  $I_N^0(\boldsymbol{\xi})$  of the conformal integral solves the GG system and is given by [23]

$$I_N^0(\boldsymbol{\xi}) = \int \prod_{\substack{i,j=1 \\ i < j}}^N \frac{dt_{ij}}{t_{ij}} t_{ij}^{-\beta_{ij}} e^{-t_{ij}} \prod_{A=1}^{N(N-3)/2} \delta \left( \frac{\prod_{\substack{i,j=1 \\ i < j}}^N t_{ij}^{\ell_{ij}^A}}{\xi^A} - 1 \right). \quad (17)$$

Although there are  $N(N-1)/2$   $\beta$ 's satisfying  $N$  equations (15), the conformal integral (4) written as (13) is independent of the choice of  $\beta$ 's [22, 23]. By the same token, the invariant  $I_N^0(\boldsymbol{\xi})$  depends on the choice of  $\beta$ 's. Upon choosing appropriate paths of integration the integrals evaluate to infinite series whose convergence depend on  $\xi^A$  [21, 22]. On the other hand, we can integrate over the cross ratios over the real line within the integral. Let us perform a Mellin

transform of  $I_N^0(\boldsymbol{\xi})$  with respect to the cross ratios. For  $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_{N(N-3)/2})$ , the Mellin transform of (17) is

$$\mathcal{I}_N(\boldsymbol{\nu}, \beta, L) = \prod_{A=1}^{N(N-3)/2} \int_0^\infty d\xi^A (\xi^A)^{\nu_A-1} I_N^0(\boldsymbol{\xi}). \quad (18)$$

Changing the order of integrations to perform the integrations over the cross ratios first, followed by integrating over the  $t$ 's we obtain

$$\mathcal{I}_N(\boldsymbol{\nu}, \beta, L) = (-1)^{\frac{N(N-3)}{2}} \prod_{\substack{i,j=1 \\ i < j}}^N \Gamma \left( \sum_A \nu_A \ell_{ij}^A - \beta_{ij} \right). \quad (19)$$

Let us now return to the bootstrap equation. Plugging the conformal integral (13) in (5) and (6), the bootstrap equation (7) can be written as a product of  $I_N^0(\boldsymbol{\xi})$  and powers of the cross ratios. The bootstrap equation, thereby, is conformal invariant. We can perform Mellin transformation of all terms of the equation with respect to the cross ratios to arrive at an equation depending on the OPE coefficients, the  $\beta$ 's and the parameters  $\boldsymbol{\nu}$ . This results in a particularly simple form of the bootstrap equation, because, instead of infinite series on both sides of (7), only the product of finite number of terms, as in (19) appear. Let us point out that the sum over  $\Delta$ , remains.

Let us now demonstrate this for the 4-point bootstrap equation. The toric matrix (9) in the case of  $N = 4$  is given by [21, 22]

$$\mathcal{A} = \begin{array}{c} i \ (jk) \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \begin{pmatrix} (12) & (13) & (14) & (23) & (24) & (34) \\ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} & \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \end{array} \end{pmatrix}. \quad (20)$$

In order to define the cross ratios (14), we choose a basis of the null vectors, arranged into a matrix. In the  $s$ -channel the null vectors are chosen as

$$L = \begin{pmatrix} \ell^1 & 0 & 1 & -1 & -1 & 1 & 0 \\ \ell^2 & 1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \quad (21)$$

As indicated on the right, the rows of the matrix define two cross ratios,

$$\xi^1 = \frac{|x_{13}|^2 |x_{24}|^2}{|x_{14}|^2 |x_{23}|^2}, \quad \xi^2 = \frac{|x_{12}|^2 |x_{34}|^2}{|x_{14}|^2 |x_{23}|^2}. \quad (22)$$

In the  $t$ -channel, the null vectors and cross ratios are

$$L' = \begin{pmatrix} \ell'^1 & -1 & 1 & 0 & 0 & 1 & -1 \\ \ell'^2 & -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \xi'^1 \\ \xi'^2 \end{pmatrix} = \begin{pmatrix} \xi^1 / \xi^2 \\ 1 / \xi^2 \end{pmatrix}, \quad (23)$$

obtained by exchanging  $x_2$  and  $x_4$ .

Channel	$\mu$	$\beta$
s-channel $\bowtie$	$\mu_1 = \frac{\Delta_1 - \Delta_2 + \Delta}{2}$	$\beta_{12} = 0$
	$\mu_2 = \frac{\Delta_2 - \Delta_1 + \Delta}{2}$	$\beta_{14} = \frac{\Delta_2 - \Delta_1 - \Delta}{2} - \beta_{13}$
	$\mu_3 = \frac{\Delta_3 - \Delta_4 + D - \Delta}{2}$	$\beta_{23} = \frac{\Delta_4 - \Delta_3 - \Delta}{2} - \beta_{13}$
	$\mu_4 = \frac{\Delta_4 - \Delta_3 + D - \Delta}{2}$	$\beta_{24} = \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{2} + \beta_{13}$
		$\beta_{34} = \Delta - \frac{D}{2}$
t-channel $\bowtie$	$\mu_1 = \frac{\Delta_1 - \Delta_4 + \Delta}{2}$	$\beta'_{12} = \frac{\Delta_4 - \Delta_1 - \Delta}{2} - \beta'_{13}$
	$\mu_2 = \frac{\Delta_2 - \Delta_3 + D - \Delta}{2}$	$\beta'_{14} = 0$
	$\mu_3 = \frac{\Delta_3 - \Delta_2 + D - \Delta}{2}$	$\beta'_{23} = \Delta - \frac{D}{2}$
	$\mu_4 = \frac{\Delta_4 - \Delta_1 + \Delta}{2}$	$\beta'_{24} = \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{2} + \beta'_{13}$
		$\beta'_{34} = \frac{\Delta_2 - \Delta_3 - \Delta}{2} - \beta'_{13}$

Table 1:  $\beta$ 's in the two channels

The weights  $\mu$  of the conformal integral appearing in (5) and (6) in the form (13) are used to solve for the  $\beta$ 's, as shown in Table 1. OPE consistency [6] of (5) and (6) fixes  $\beta_{12} = 0$  in the  $s$ -channel and  $\beta_{14} = 0$  in the  $t$ -channel. Plugging in the values of parameters in (5) and (6) and equating them we obtain an equation involving solely the cross ratios,

$$(\xi^1)^{\beta_{13}} \sum_{\Delta} C_{\Delta_1 \Delta_2 \Delta} C_{\Delta \Delta_3 \Delta_4} (\xi^2)^{\frac{\Delta - \Delta_2}{2}} I_4^0(\xi^1, \xi^2)^{\bowtie} = (\xi'^1)^{\beta'_{13}} \sum_{\Delta} C_{\Delta_1 \Delta_4 \Delta} C_{\Delta \Delta_3 \Delta_2} (\xi'^2)^{\frac{\Delta - \Delta_4}{2}} I_4^0(\xi'^1, \xi'^2)^{\bowtie}, \quad (24)$$

as mentioned above. Let us point out that taking into account the definition of cross ratios in the two channels, the LHS of the bootstrap equation goes over to the RHS as  $(x_2, \Delta_2)$  and  $(x_4, \Delta_4)$  are swapped at once. The GG integral  $I_4^0(\xi)$  appears as the summand in the conformal block. Multiplying both sides with  $(\xi^1)^{\nu_1 - 1} (\xi^2)^{\nu_2 - 1}$ , and integrating over both cross ratios as in (18) with appropriate variable change on the RHS, and using (19), we obtain

$$\sum_{\Delta} C_{\Delta_1 \Delta_2 \Delta} C_{\Delta \Delta_3 \Delta_4} \mathcal{I}_4((\beta_{13} + \nu_1, \frac{\Delta - \Delta_2}{2} + \nu_2), \beta, L) = \sum_{\Delta} C_{\Delta_1 \Delta_4 \Delta} C_{\Delta \Delta_3 \Delta_2} \mathcal{I}_4((\beta'_{13} + \nu_1, \frac{\Delta - \Delta_4}{2} - \nu_1 - \nu_2), \beta', L'), \quad (25)$$

where

$$\mathcal{I}_4((\beta_{13} + \nu_1, \frac{\Delta - \Delta_2}{2} + \nu_2), \beta, L) = \Gamma(\nu_1) \Gamma(\nu_1 - \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{2}) \Gamma(\frac{\Delta - \Delta_2}{2} + \nu_2) \Gamma(\frac{\Delta_1}{2} - \nu_1 - \nu_2) \Gamma(\frac{\Delta_2 + \Delta_3 - \Delta_4}{2} - \nu_1 - \nu_2) \Gamma(\frac{D - \Delta - \Delta_2}{2} + \nu_2), \quad (26)$$

$$\mathcal{I}_4((\beta'_{13} + \nu_1, \frac{\Delta - \Delta_4}{2} - \nu_1 - \nu_2), \beta', L') = \Gamma(\nu_1) \Gamma(\nu_1 - \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{2}) \Gamma(\frac{\Delta - \Delta_4}{2} - \nu_1 - \nu_2) \Gamma(\frac{\Delta_1}{2} + \nu_2) \Gamma(\frac{\Delta_4 + \Delta_3 - \Delta_2}{2} + \nu_2) \Gamma(\frac{D - \Delta - \Delta_4}{2} - \nu_1 - \nu_2). \quad (27)$$

Interestingly, the bootstrap equation is independent of the parameters  $\beta$ , only the external weights appear in it. This is the general form of the Mellin transform of the bootstrap equation depending only on the CFT data.

To demonstrate the usefulness of this form of the bootstrap equation, let us consider the special case of external fields of equal weight,  $\Delta_i = \Delta_0$ , for  $i = 1, 2, 3, 4$ . The bootstrap equation (25) simplifies to

$$\sum_{\Delta} C_{\Delta_0 \Delta_0 \Delta}^2 (L_{\Delta} - R_{\Delta}) = 0, \quad (28)$$

where

$$L_{\Delta} = \Gamma\left(\frac{\Delta - \Delta_0}{2} + \nu_2\right) \Gamma\left(\frac{D - \Delta - \Delta_0}{2} + \nu_2\right) \Gamma\left(\frac{\Delta_0}{2} - \nu_1 - \nu_2\right)^2, \quad (29)$$

$$R_{\Delta} = \Gamma\left(\frac{\Delta - \Delta_0}{2} - \nu_1 - \nu_2\right) \Gamma\left(\frac{D - \Delta - \Delta_0}{2} - \nu_1 - \nu_2\right) \Gamma\left(\frac{\Delta_0}{2} + \nu_2\right)^2. \quad (30)$$

This can be used to find bounds on the conformal weights of fields. Since  $C_{\Delta_0\Delta\Delta}^2$  is positive, a bound on  $\Delta$  is obtained as the ratio  $L_{\Delta}/R_{\Delta}$  crosses unity. There are four parameters in the expression, namely,  $D, \Delta_0, \nu_1$ , and  $\nu_2$ . The ratio  $L_{\Delta}/R_{\Delta}$  is plotted against  $\Delta$  in Figure 1. The weight  $\Delta$  and  $\Delta_0$  are taken to obey the unitarity bound  $\Delta > (D - 2)/2$ . In Figure 1a the ratio is plotted for different values of  $\Delta_0$ , showing that it crosses unity for some value of  $\Delta$ . In Figure 1b the ratio is plotted for different dimensions, the graphs crossing unity at a finite value of  $\Delta$  in each case, thereby providing a lower bound on  $\Delta$ .

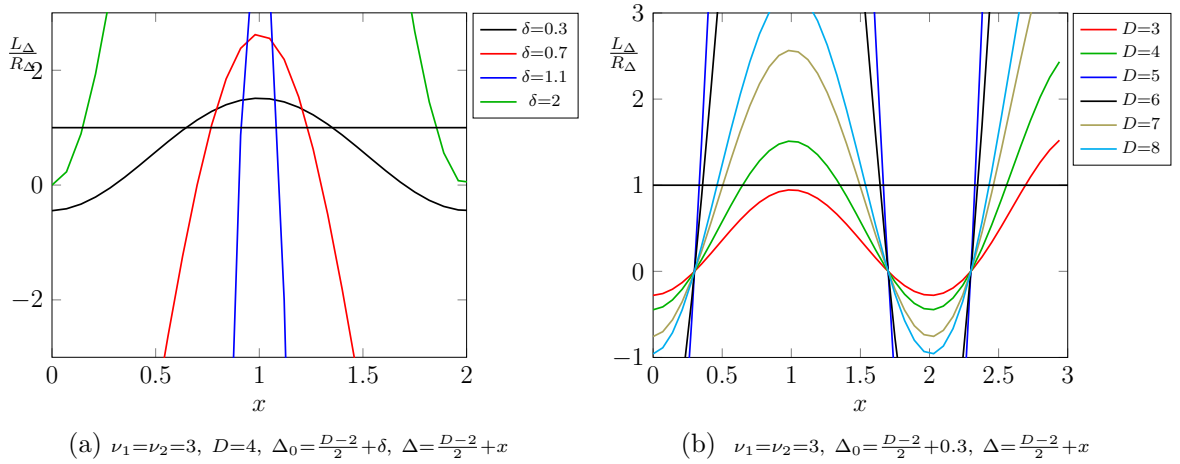


Figure 1: Bounds on conformal weights

To summarise, we have obtained a simple expression for the Mellin transform of the 4-point bootstrap equation for scalar conformal fields in the  $D$ -dimensional Euclidean space. First, we use the expression of the correlation functions in terms of conformal integrals in the  $s$ - and  $t$ -channel, (5) and (6), respectively. Then, writing the conformal integral in the form (13), the invariant summand of the conformal blocks depending only on the cross ratios is identified as the solution of a GG system corresponding to a toric GKZ system furnished by the conformal transformations. Writing the bootstrap equation solely in terms of cross ratios and using the integral form (18), we write down the Mellin transform of the bootstrap equation (25). The expression is a sum over the conformal weights of the “internal” fields,  $\Delta$  and has a product of a finite number,  $N(N - 3)/2$ , of gamma functions only. This formulation avoids dealing with subtleties related to the convergence of infinite series, as well as analytic continuation, in the cross ratios. Let us note that equation (25) is in reality an infinitude of equations depending on the Mellin parameters  $\nu$ . While we fixed these arbitrarily in Figure 1 to exhibit the usefulness of the present formulation, the best bound may depend on particular values.

Dealing with a conformal field theory with scalar fields alone is clearly an over-simplification. However, the computations presented here can be generalised to incorporate spin as well as to

bootstrap higher point correlation functions [25,26], since the higher point correlation functions can be expressed in terms of conformal integrals, which, in turn, are solutions of GG systems. Such generalisations will be reported in future.

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