

# New perspectives in Kac-Moody algebras associated to higher dimensional manifolds

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## Abstract

In this review, we present a general framework for the construction of Kac-Moody (KM) algebras associated to higher-dimensional manifolds. Starting from the classical case of loop algebras on the circle  $S^1$ , we extend the approach to compact and non-compact group manifolds, coset spaces, and soft deformations thereof. After recalling the necessary geometric background on Riemannian manifolds, Hilbert bases and Killing vectors, we present the construction of generalized current algebras  $\mathfrak{g}(\mathcal{M})$ , their semidirect extensions with isometry algebras, and their central extensions. We show how the resulting algebras are controlled by the structure of the underlying manifold, and illustrate the framework through explicit realizations on  $SU(2)$ ,  $SU(2)/U(1)$ , and higher-dimensional spheres, highlighting their relation to Virasoro-like algebras. We also discuss the compatibility conditions for cocycles, the role of harmonic analysis, and some applications in higher-dimensional field theory and supergravity compactifications. This provides a unifying perspective on KM algebras beyond one-dimensional settings, paving the way for further exploration of their mathematical and physical implications.

**keyword:** Kac-Moody algebras; Virasoro algebra; central extensions; harmonic analysis; higher dimensional manifolds.

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Kac-Moody (KM) algebras on higher dimensional manifolds: Some generalities</b>	<b>6</b>
2.1	Relevant properties of manifolds . . . . .	7
2.2	A KM algebra associated to $\mathcal{M}$ . . . . .	9
<b>3</b>	<b>KM algebras on compact group manifolds</b>	<b>13</b>
3.1	Compact group manifolds . . . . .	14
3.2	Coset spaces of compact group manifolds . . . . .	16
3.3	KM algebras associated to compact Lie groups . . . . .	18
3.4	KM algebra associated to $SU(2)$ and $SU(2)/U(1)$ . . . . .	20
3.5	Virasoro algebra associated to the two- and three- sphere . . . . .	23
<b>4</b>	<b>KM algebras on non-compact Lie groups</b>	<b>28</b>
4.1	Some generalities . . . . .	28
4.2	A KM algebra associated to $SL(2, \mathbb{R})$ . . . . .	29
4.2.1	The group $SL(2, \mathbb{R})$ . . . . .	29
4.2.2	Matrix elements of $\mathfrak{sl}(2, \mathbb{R})$ and Plancherel Theorem . . . . .	31
4.2.3	Hilbert basis of $L^2(SL(2, \mathbb{R}))$ . . . . .	32
4.2.4	Clebsch-Gordan coefficients . . . . .	34
4.2.5	A KM algebra associated to $SL(2, \mathbb{R})$ . . . . .	35
<b>5</b>	<b>Soft manifolds</b>	<b>38</b>
5.1	KM algebras on soft manifolds . . . . .	38
<b>6</b>	<b>Roots systems and some elements of representation theory</b>	<b>46</b>
6.1	Roots system . . . . .	46
6.2	Some elements of representation theory . . . . .	49
6.3	Some results on toroidal algebras . . . . .	53
6.4	Unitary representations . . . . .	57
<b>7</b>	<b>Applications in physics</b>	<b>58</b>
7.1	Two-dimensional current algebra and CFT: WZW, Sugawara, Virasoro . . . . .	58
7.2	Compactifications and KK spectra . . . . .	59
7.3	Cosmological billiards and hidden symmetries of supergravity . . . . .	60
<b>8</b>	<b>Outlook: conjectures and novel applications</b>	<b>61</b>
8.1	Supergravity . . . . .	62
8.2	Superstrings and M-theory . . . . .	62
8.3	AdS/CFT and holography . . . . .	63

# 1 Introduction

## *From finite- to infinite-dimensional symmetry*

For quite a long time, finite-dimensional simple Lie algebras have provided the backbone of symmetry in mathematics and physics. The classification by Cartan and the subsequent development of representation theory turned these structures into essential tools throughout geometry, number theory and quantum theory. Beginning in the late 1960s, two independent lines of development—Moody’s and Kac’s—generalized this landscape to the infinite-dimensional realm, resulting in what are now called Kac–Moody (KM) algebras [1, 2, 3]. A particularly tractable and physically relevant subclass is formed by the *affine* algebras, realized as central extensions of loop algebras. Pressley and Segal’s monograph rigorously analyzed the functional-analytic and group-theoretic underpinnings in terms of loop groups and their projective unitary representations [4].

The importance of the affine KM symmetry in physics was then amplified by its tight relation to the Virasoro algebra, via the Sugawara construction and current algebra methods [5]. This connection underlies the solvability of two-dimensional conformal field theories (CFTs) and the Wess–Zumino–Witten (WZW) models, and it has been comprehensively discussed in the CFT literature [6, 7]. At the same time, the broader spectrum of KM algebras beyond the affine case (e.g. indefinite and hyperbolic generalizations) has been invoked in diverse corners of high-energy theoretical physics, notably in hidden symmetries of supergravity, cosmology and string/M-theory.

## *Beyond the circle : why ?*

The best-known laboratory for KM algebras is the loop algebra over the circle  $\mathbb{S}^1$ . Physically, the ubiquity of the circle stems from the role of closed strings and from the reduction of two-dimensional field theories on a spatial circle. Mathematically,  $\mathbb{S}^1$  is distinguished by its Fourier basis, as well as by the resulting algebraic simplifications that enable a complete representation theory for affine algebras. However, many problems of current interest in mathematical physics naturally involve *higher-dimensional* manifolds :

- Compactifications in Kaluza–Klein (KK) theory and string/M-theory crucially involve group manifolds and coset spaces; examples including spheres, tori, and symmetric spaces are e.g. discussed in [8, 9, 10, 11, 12, 13].
- Non-compact, Riemannian target spaces (e.g.  $SL(2, \mathbb{R})$  or  $SL(2, \mathbb{R})/U(1)$ ) arise in (ungauged and gauged) supergravity theories in diverse space-time dimensions, as well as in the theory of black hole attractors (see e.g. [14] and references therein).
- Deformations of group manifolds (*soft* or non-homogeneous manifolds) appear in the effective descriptions of flux compactifications and in group-geometric approaches to supergravity [15].

In all such settings, the natural notion of “currents” is no longer related to loops on  $\mathbb{S}^1$ , but rather it is connected to functions (or sections) on a manifold  $\mathcal{M}$ , with values in a (generally finite-dimensional) Lie algebra  $\mathfrak{g}$ . The resulting current algebra  $\mathfrak{g}(\mathcal{M})$  inherits both the algebraic structure from  $\mathfrak{g}$  and the geometric/analytic structure from  $\mathcal{M}$ . The central question we address in this review is: *to what extent do the characteristic features of affine KM algebras—such as central extensions, semidirect actions by diffeomorphisms or isometries, and rich representation theory—survive when  $\mathbb{S}^1$  is replaced by a higher-dimensional manifold  $\mathcal{M}$ ?*

## *A unifying viewpoint*

Our viewpoint is twofold : geometric and representation-theoretic. We begin and consider a Riemannian (or pseudo-Riemannian) manifold  $\mathcal{M}$ , equipped with a measure and a Hilbert space  $L^2(\mathcal{M})$  of square-integrable functions. A key point is the existence of a *Hilbert basis* adapted to the symmetries of  $\mathcal{M}$ . On the one hand, for compact Lie groups  $G_c$ , the Peter-Weyl theorem provides such a basis in terms of matrix elements of unitary irreducible representations [16]. On the other hand, for non-compact Lie groups  $G_{nc}$ , the Plancherel theorem organizes  $L^2(G_{nc})$  functions as a ‘mixture’ of discrete and continuous series contributions [17, 18]. This harmonic-analytic formalism allows us to expand  $\mathfrak{g}$ -valued fields on  $\mathcal{M}$  and thus to define current algebras  $\mathfrak{g}(\mathcal{M})$ , as well as their (semi-)direct extensions by symmetry algebras generated by Killing vectors of  $\mathcal{M}$  itself. In general, central extensions can be characterized cohomologically, in terms of closed currents on  $\mathcal{M}$ , and they can also be explicitly matched to local Schwinger terms in current commutators.

All this can be summarized into an algorithmic construction, which provides a coherent perspective encompassing spheres  $S^n$ , compact groups such as  $SU(2)$  and their homogeneous spaces, non-compact groups such as  $SL(2, \mathbb{R})$ , and generalizations with softened group structure. The steps of such an algorithmic construction list as follows :

1. Choose  $\mathcal{M}$  (compact or non-compact group manifold, Riemannian or pseudo-Riemannian coset thereof, soft deformations thereof, ...) and its isometry algebra.
2. Select an appropriate orthonormal basis on  $L^2(\mathcal{M})$  (as stated by the Peter-Weyl or Plancherel theorems), and then promote  $\mathfrak{g}$ -valued modes to generators.
3. Determine the semidirect actions by isometries/diffeomorphisms, and describe/classify all compatible 2-cocycles that yield central extensions.
4. Identify structural features (e.g. Witt/Virasoro analogues, area-preserving diffeomorphisms) related to the geometry of  $\mathcal{M}$ .

## *Applications in physics*

Historically, the amazingly fruitful cross-fertilization between infinite-dimensional algebras and physics has evolved along several lines of research. In two-dimensional CFT, the Virasoro and affine Kac–Moody algebras have provided the dynamical symmetry principles underlying the theory of exactly solvable models, modular covariance, and the classification of primary fields [7, 6, 5, 4]. In string theory, worldsheet reparameterization and current algebra symmetries determine crucial consistency conditions (termed as anomaly cancellation condition, possibly exploiting BRST cohomology), while WZW models provide exact backgrounds with affine symmetry. The celebrated and surprising Monstrous moonshine and generalized KM structures (Borcherds–Kac–Moody algebras) connect CFTs, sporadic groups, and automorphic forms, highlighting the depth of the algebraic structures which are emergent in quantum theories [19, 20, 21].

Another remarkable appearance of KM algebras pertains to the large hidden Lie symmetries arising in supergravity and in cosmological dynamics. For instance, cosmological billiards uncovered links between the asymptotic dynamics of gravity near spacelike singularities and hyperbolic KM symmetries [22, 23]. Moreover, extended supergravities and dimensional reductions have suggested hierarchies of very-extended algebras, as well as infinite towers of dual potentials; see, for instance, [24, 25, 26, 27, 28] and, for geometric and group-theoretic analyses, [29, 12, 13]. Within this context, the KK theory has provided a wealth of examples in which harmonic analysis (on compact  $\mathcal{M}$ ) describe the spectra of lower-dimensional fields [8, 9, 10, 11]. Finally, current algebras naturally arise on  $\mathcal{M}$ , being defined in terms of Noether charges integrated over the internal spaces, and a possible, subsequent mode expansion.

## *Scope and plan of this review*

Expectedly, the generalization from  $\mathbb{S}^1$  to higher-dimensional manifolds  $\mathcal{M}$  generates both opportunities and challenges. To list a few :

**Central extensions.** On  $\mathbb{S}^1$ , there is essentially a *unique* (up to normalization) central extension of the loop algebra, yielding an affine KM algebra. Typically, for higher-dimensional manifolds the space of admissible 2-cocycles is *infinite-dimensional*. Cohomological constructions tied to closed  $(n - 1)$ -currents on  $\mathcal{M}$  and divergence-free vector fields can be exploited, directly relating to *Schwinger terms* in the current algebra.

**Symmetry by isometries and diffeomorphisms.** The Lie algebra generated by the Killing vectors (or broader diffeomorphism algebras) acts naturally on  $\mathfrak{g}(\mathcal{M})$ . In favorable cases, one can identify subalgebras which are analogues to the Witt/Virasoro algebras (e.g. de Witt algebra on  $\mathbb{S}^n$ , or the area-preserving diffeomorphisms on  $\mathbb{S}^2$ ), yielding to semidirect products which hints to the affine–Virasoro interplay, familiar from two dimensions.

**Harmonic analysis and representation theory.** On compact groups and homogeneous spaces thereof, the Peter-Weyl theorem and the Clebsch–Gordan machinery allow to express products of basis functions in terms of representation-theoretic data. On the other hand, for non-compact groups such as  $SL(2, \mathbb{R})$ , a ‘mixture’ of discrete and continuous series appears via the Plancherel theory (implemented also through Bargmann’s classification), and new unitary structures emerge.

**Applications.** The above constructions provide an algebraic background for higher-dimensional current algebras in effective field theories, organize spectra in compactifications, and suggest new Virasoro-like structures that may control sectors of dynamics in higher-dimensional CFTs or holographic models.

This review aims to be pedagogical while remaining faithful to the breadth of the subject. We deliberately separate general principles from case studies:

1. We present a construction based on manifolds, that keeps track of metric, measure and symmetry data, and we adopt a formalism based on the Hilbert space, in order to emphasize unitarity and completeness of mode expansions.
2. We use group- and representation- theoretic tools (such as the Peter-Weyl and Plancherel theorems, or the Clebsch–Gordan coefficients) to highlight the algebraic structure, and to allow explicit computations of structure constants in current algebras on  $\mathcal{M}$ .
3. We discuss central extensions from two perspectives: cohomology of current algebras on  $\mathcal{M}$ , and anomalies/Schwinger terms in local commutators.
4. Throughout our treatment, we strive to highlight links to well-established areas in physics, such as two-dimensional CFT and string theory [4, 5, 6, 7], or higher-dimensional supergravity and compactifications [8, 9, 12, 29, 13, 10, 11, 24, 25, 26, 27, 28].

There exist several classic reference works on affine algebras and loop groups [4, 6, 5], on group manifolds and harmonic analysis [16, 17], and on supergravity/sigma-model applications [8, 9, 12, 29, 13]. In this respect, it is worth emphasizing that our focus is complementary: we aim at presenting a *single* construction, which applies to wide classes of manifolds, such as compact groups and cosets, non-compact groups and their homogeneous spaces, and (soft) deformations of group manifolds. As mentioned above, we also present a systematic analysis of the space of compatible central extensions. Various

examples are discussed and treated in some detail, in order to provide a glimpse of the wealth of the subject, rather than to exhaust all possibilities.

The plan of this review is as follows.

**Section 1 General framework on manifolds.** We recall the geometric background on (pseudo-)Riemannian manifolds, measures, and Killing vectors; we introduce  $L^2(\mathcal{M})$  and orthonormal bases adapted to symmetries (exploiting the Peter-Weyl or Plancherel theorems). We then define the current algebra  $\mathfrak{g}(\mathcal{M})$ , its semidirect extension by isometries, and set up the cohomological language for central extensions.

**Section 2 Compact group manifolds and cosets.** Invoking the Peter-Weyl theorem, we construct  $\mathfrak{g}(\mathcal{M})$  on  $G_c$  and on homogeneous spaces  $G_c/H$ . We explain how products of basis functions are controlled by Clebsch-Gordan coefficients and how central extensions are enumerated by closed currents. As examples, we highlight analogues of Witt/Virasoro subalgebras on spheres and their role in semidirect products.

**Section 3 Non-compact groups and Plancherel analysis.** We treat  $SL(2, \mathbb{R})$  and  $SL(2, \mathbb{R})/U(1)$  as canonical examples, reviewing Bargmann's discrete/continuous series and the Plancherel decomposition. We construct  $\mathfrak{g}(\mathcal{M})$  and compatible central extensions, discuss unitarity issues and constraints for the resulting representations.

**Section 4 Soft (deformed) group manifolds.** We consider the so-called *soft* deformations of group manifolds motivated by supergravity and string compactifications, discussing various aspects in some detail.

**Section 5 Root systems and representation theory.** We introduce a system of roots for the classes of Kac-Moody algebras introduced above. In a second part of this Section we introduce some elements of representations theory, stressing important differences with respect to affine Lie algebras. For simplicity's sake, we will here assume that  $\mathfrak{g}$  is a compact Lie algebra.

**Section 6 Applications to physics.** We discuss some applications of the above mathematical machinery to physics, most notably to : (i) two-dimensional current algebra/CFT (WZW, Sugawara, Virasoro); (ii) higher-dimensional compactifications and spectra in Kaluza–Klein theory; (iii) structures emerging within the theory of cosmological billiards, as well as underlying the hidden symmetries of supergravity.

**Section 7 Outlook and open problems.** We outline classification questions for central extensions on general  $\mathcal{M}$ , representation-theoretic challenges beyond compact groups, and possible applications to holography and integrability.

## 2 Kac-Moody (KM) algebras on higher dimensional manifolds: Some generalities

Let  $\mathfrak{g}$  be a semisimple (complex or real) Lie algebra. To the simplest (one-dimensional) manifold, namely the circle  $\mathbb{S}^1$ , one can associate two non-isomorphic infinite dimensional Lie algebras. The first is the affine extension of the loop algebra of smooth maps from  $\mathbb{S}^1$  to  $\mathfrak{g}$ , which allows to construct KM (or, better, affine) Lie algebras [2, 3, 1, 4, 5]. A second possibility is given by the Virasoro algebra, which is a central extension of the de Witt algebra (*i.e.*, of the algebra of polynomial vector fields on  $\mathbb{S}^1$ ), and is intimately connected to affine Kac-Moody algebras via the Sugawara construction (see e.g. [7, 30]).

A natural question concerns the possibility to define infinite-dimensional KM algebras related to higher dimensional manifolds, possibly maintaining some of the structural properties observed for the previous relevant cases. Along the years, a number

of infinite-dimensional KM algebras have been constructed. The first examples are associated to specific manifolds, such as the two-sphere, the three-sphere or the  $n$ -tori [31, 32, 33, 34, 13, 35, 36]. Then, a general study for more general manifolds were undertaken in [37, 38, 39, 40, 41, 42, 43, 44]. To this extent, various type of manifolds  $\mathcal{M}$  were considered. This systematic study started with  $\mathcal{M}$  being a compact  $n$ -dimensional manifold [37, 38, 39, 40, 41] (essentially, a compact Lie group  $G_c$  –not to be confused with  $G$ , the Lie group associated to  $\mathfrak{g}$ – or a coset space  $G_c/H$  with  $H$  a closed subgroup of  $G_c$ ). This construction is ultimately motivated by higher-dimensional physical theories, in which harmonic expansions *à la Kaluza-Klein* play a crucial role (see e.g. [8]-[11], with the latter reference being itself motivated by the supergravity context). The second series of extensions concerns non-compact manifolds, principally  $SL(2, \mathbb{R})$  or the coset  $SL(2, \mathbb{R})/U(1)$  [42, 43]. These constructions were motivated by the fact that the non-compact manifold  $SL(2, \mathbb{R})/U(1)$  appears as the target space of one complex scalar field in the bosonic sector of some Maxwell-Einstein supergravity theories in  $D = 3 + 1$  space-time dimensions. Finally, the last type of algebras [44] are associated to manifolds which are deformations of Lie groups, and called ‘soft’ group manifold (see e.g. [15]).

In this section, before considering the KM algebras associated to the specific types of manifolds presented above, we briefly recall some general properties of real manifolds and explain the various steps to construct the corresponding Kac-Moody algebra.

## 2.1 Relevant properties of manifolds

Let  $\mathcal{M}$  be an  $n$ -dimensional real Riemannian manifold. Let  $\mathcal{U} \subset \mathcal{M}$  be an open set of  $\mathcal{M}$ . We assume that the manifold is such that  $\mathcal{M} \setminus \mathcal{U}$  is of zero measure. Define an homomorphism from  $\mathcal{U}$  to an open set of  $\mathcal{O} \subset \mathbb{R}^n$  by

$$\begin{aligned} f : \mathcal{U} &\rightarrow \mathcal{O} \subset \mathbb{R}^n \\ m &\mapsto f(m) = (y^1, \dots, y^n), \end{aligned}$$

*i.e.*, we can associate a system of coordinates for points  $m$  in  $\mathcal{M}$  almost everywhere. Let  $g_{IJ}$  be the metric tensor of  $\mathcal{M}$  in the the system of coordinates  $y^I = (x^1, \dots, x^n)$ :

$$ds^2 = dx^I dx^J g_{IJ}$$

and define  $g = |\det g_{IJ}|$ . Consider now an infinitesimal diffeomorphism:

$$y^I \rightarrow y'^I + \epsilon \xi^I(y).$$

This transformation is an isometry of  $\mathcal{M}$ , *i.e.*, it preserves the tensor metric  $g_{IJ}$ , *iff* the  $\xi^I$  satisfy the Killing equation:

$$\nabla_I \xi_J(y) + \nabla_J \xi_I(y) = 0, \quad (2.1)$$

where  $\xi_I = g_{IJ} \xi^J$  and  $\nabla_I$  is the covariant derivative [45]. Suppose that the Killing equation has  $s$ -independent solutions  $\xi_A^I(y)$ ,  $A = 1, \dots, s$ , then the operators

$$\mathfrak{g}_\xi = \left\{ K_A = i \xi_A^I(y) \partial_I, A = 1, \dots, s \right\} \quad (2.2)$$

generate the Lie algebra with Lie brackets

$$[K_A, K_B] = i c_{AB}^C K_C, \quad (2.3)$$

since the composition of two isometries is an isometry, and where

$$-\xi_A^J \partial_J \xi_B^I + \xi_B^J \partial_J \xi_A^I = -c_{AB}^C \xi_C^I,$$

because the Killing vectors are associated with independent infinitesimal isometries.

We endow the manifold  $\mathcal{M}$  with an Hermitian scalar product:

$$(f, g) = \int_{\mathcal{O}} \sqrt{g} \, d^n y \, \overline{f(y)} g(y) \quad (2.4)$$

where  $\mathcal{O} = f(\mathcal{U}) \subset \mathbb{R}^n$ . Hermiticity of the operators  $K_A$  with respect to the scalar product (2.4) translates into two conditions:

$$\begin{aligned} 1: \quad & \partial_I (\sqrt{|g|} \xi_A^I) = 0, \\ 2: \quad & \xi_A^I = 0, \quad I = 0, \dots, n \end{aligned} \quad (2.5)$$

where  $\xi_A^I| = 0$  means that the boundary term associated to all directions vanishes. Note in particular that if  $\xi_A^I$  are periodic in a direction, condition 2 is automatically satisfied. Similarly, if a direction is unbounded, say  $y^{I_0} \in \mathbb{R}$ , condition 2 is also satisfied since  $\xi_a^{I_0} \rightarrow 0$  whenever  $y^{I_0} \rightarrow \pm\infty$ . Furthermore, as  $\xi_a^I$  is a Killing vector, we have a Levi-Civita connection satisfying  $\nabla_I g_{JK} = 0$ , hence the Killing condition (2.1) implies the condition 1 above<sup>1</sup>.

The last step of this general introduction is to consider the Hilbert space of square integrable functions on  $\mathcal{M}$ , namely  $L^2(\mathcal{M})$ , or more precisely  $L^2(\mathcal{O})$ . It is known that any Hilbert space admits a Hilbert basis [46, 47]. Let  $\mathcal{B} = \{b_M, M \in \mathbb{N}\}$  be a Hilbert basis of  $L^2(\mathcal{O})$ , i.e., a complete orthonormal set of vectors (for the scalar product (2.4)). In our case, the vectors  $b_N$  become functions on  $\mathcal{O}$ . We also assume that for any generators  $K_A$  of the Lie algebra generated by the Killing vectors (see (2.3)) and for any vector  $b_M$  in  $\mathcal{B}$ , the function  $K_A b_N$  is square integrable. Thus,

$$K_A b_M(y) = (M_A)_M^N b_N(y) \quad (2.6)$$

where the symbol of summation is omitted, where  $(M_A)_M^N$  is a matrix representing the action of  $\mathfrak{g}_\xi$  on  $b_M$ . Indeed, this equation in fact means that the vectors  $b_M$  transform according to a given representation of  $\mathfrak{g}_\xi$ . In general, this representation is fully reducible, but it may happen that a subset of vectors does not belong to an irreducible representation (see Sect. 4). Since the set of vectors  $b_M$  is orthonormal and constitutes a Hilbert basis of  $L^2(\mathcal{O})$ , these vectors also constitute a unitary representation of  $\mathfrak{g}_\xi$ , hence the operators  $K_A$  are Hermitian and thus satisfy (2.5) (and, in particular, the second condition).

In the following, two types of manifolds will be considered. The first type corresponds to group manifolds of compact Lie groups  $G_c$ , while the second corresponds to group manifolds of non-compact Lie groups  $G_{nc}$ . In both cases the decomposition of square integrable functions (or the Hilbert basis  $\mathcal{B}$ ) organize within the representation theory of  $G_c$  (resp.  $G_{nc}$ ), but it should be observed that these two cases are structurally quite different. If the group is compact, its unitary representations are finite dimensional and it turns out that, once correctly normalized, the set of all matrix elements constitutes an orthonormal Hilbert basis of  $L^2(G_c)$ . This corresponds to the Peter–Weyl Theorem [16]. If the group is non-compact, its unitary representations are infinite dimensional and square integrable functions decompose into a sum over the matrix elements of the discrete series and an integral over the matrix elements of the continuous series (see below). In this situation, the Plancherel Theorem has to be used [48].

Returning to the Hilbert basis  $\mathcal{B}$ , we assume further that for any elements  $b_N, b_M$  of  $\mathcal{B}$  the product  $b_N b_M$  is square integrable, thus in particular we have

$$b_M(y) b_N(y) = C_{MN}^P b_P(y) \quad (2.7)$$

---

<sup>1</sup>Recall that  $\nabla_I V^I = 1/\sqrt{g} \partial_I (\sqrt{g} V^I)$ .

where the symbol of summation is omitted. In general, the coefficients  $C_{MN}^P$  are difficult to compute explicitly, except (at least) for two types of manifolds: (i)  $\mathcal{M}$  is related to a compact Lie group  $G_c$ , or (ii)  $\mathcal{M}$  is related to a non-compact Lie group  $G_{nc}$ . In both cases, the coefficients  $C_{MN}^P$  can be expressed explicitly by means of the Clebsch-Gordan coefficients of  $G_c$  (resp.  $G_{nc}$ ).

Since  $\mathcal{M}$  is a real manifold, one can choose the set of functions  $b_M$  to be real functions. It is however useful for certain manifolds  $\mathcal{M}$  to consider complex-valued functions. Reality of  $\mathcal{M}$  imposes then that for any  $b_M$  in  $\mathcal{B}$  we have

$$\bar{b}^N(y) = \eta^{NM} b_M(y), \quad \text{or} \quad b_M(y) = \eta_{MN} \bar{b}^N(y), \quad (2.8)$$

where  $\eta_{MN} = \eta^{MN}$  and such that for a given  $M \in \mathbb{N}$ , there exists a unique  $N \in \mathbb{N}$  such that  $\eta_{MN}$  is non-vanishing. One can of course chose  $\eta_{MN} = 1$ , albeit for group manifolds we could also have  $\eta_{MN} = \pm 1$  (see Section 3). Finally, in this section the index of the functions  $b_M$  belongs to  $\mathbb{N}$ . In turn, other choices will be more relevant in the next section, in particular, negative integer values for  $M$ . In this case, conditions (2.8) will be obvious and  $\eta_{MN} \neq 0$  if  $N = -M$ .

Since the basis  $\mathcal{B}$  is complete, we have the completeness relation:

$$\sum_{N \in \mathbb{N}} \bar{b}^N(y) b_N(y') = \frac{\delta^n(y - y')}{\sqrt{g(y)}}. \quad (2.9)$$

## 2.2 A KM algebra associated to $\mathcal{M}$

Let  $\mathfrak{g} = \{T_a, a = 1, \dots, d\}$  be a  $d$ -dimensional simple Lie algebra with Lie brackets

$$[T_a, T_b] = i f_{ab}{}^c T_c.$$

The Lie algebra can be real or complex. We assume for simplicity that  $\mathfrak{g}$  is a compact Lie algebra with Killing form:

$$\langle T_a, T_b \rangle_0 = k_{ab} = \text{tr}(\text{ad}(T_a)\text{ad}(T_b)),$$

where  $\text{ad}$  denotes the adjoint representation of  $\mathfrak{g}$  and  $k_{ab}$  is definite positive.

In the next step of our construction we define the space of smooth mappings from  $\mathcal{M}$  into  $\mathfrak{g}$  as

$$\mathfrak{g}(\mathcal{M}) = \left\{ T_{aM} = T_a b_M(y), \quad a = 1, \dots, d, \quad M \in \mathbb{N} \right\}, \quad (2.10)$$

which inherits the structure of a Lie algebra

$$[T_{aM}, T_{bN}] = i f_{ab}{}^c C_{MN}{}^P T_{cP}, \quad (2.11)$$

where the coefficients  $C_{MN}{}^P$  are defined in (2.7). The Killing form in  $\mathfrak{g}(\mathcal{M})$  is given by

$$\langle X, Y \rangle_1 = \int_{\mathcal{O}} \sqrt{g} \, dx^n \langle X, Y \rangle_0,$$

for  $X, Y \in \mathfrak{g}(\mathcal{M})$ . From (2.8) it follows that

$$\langle T_{aM}, T_{bN} \rangle_1 = k_{ab} \eta_{MN}. \quad (2.12)$$

In fact, if we assume

$$[T_a(y), T_b(y')] = i f_{ab}{}^c T_c(y) \frac{\delta^n(y - y')}{\sqrt{g}}, \quad (2.13)$$

where

$$T_a(y) = \sum_{M \in \mathbb{N}} T_{aM} \bar{b}^M(y) \quad (2.14)$$

then integrating both sides of the equation above by  $\int \sqrt{g} d^n y b_P(y) \int \sqrt{g} d^n y' b_q(y')$ , (2.13) reproduce (2.11) using (2.9) and (2.7). Clearly the algebra  $\mathfrak{g}(\mathcal{M})$  is the generalization of the loop algebra corresponding to the  $n$ -dimensional manifold  $\mathcal{M}$  (the loop algebra is associated to  $\mathcal{M} = \mathbb{S}^1$ , the circle).

Continuing the construction, we recall that  $\mathcal{M}$  is such that the operators associated to the Killing vectors  $\mathfrak{g}_\xi = \{K_A, A = 1, \dots, s\}$  are Hermitian with respect to the scalar product (2.4) (see (2.5)). Let  $\mathfrak{g}(\mathcal{M}) \rtimes \mathfrak{g}_\xi$ . This algebra admits a semidirect structure:

$$\begin{aligned} [T_{aM}, T_{bN}] &= i f_{ab}{}^c C_{MN}{}^P T_{cP} , \\ [K_A, b_M(y)] &= (M_A)_M{}^N b_N(y) , \\ [K_A, K_B] &= i c_{AB}{}^C K_C , \end{aligned} \quad (2.15)$$

(see (2.6)). It may be interesting to identify the maximal set of commuting operators. Let  $r$  be the rank of  $\mathfrak{g}_\xi$ , and let  $\{D_1, \dots, D_r\}$  be a Cartan subalgebra. Without loss of generality, we can assume that the elements of the basis  $\mathcal{B}$  are eigenvectors of the  $D$ s:

$$[D_i, b_M(y)] = M(i) b_M(y) \quad (2.16)$$

with  $M(i) \in \mathbb{R}$ . It is important to emphasize that,

$$[D_i, b_M(y)] = M(i) b_M(y) \Leftrightarrow [D_i, \bar{b}^M(y)] = -M(i) \bar{b}^M(y) . \quad (2.17)$$

The algebra above can even be further extended. Introduce the Lie algebra of vector fields  $\mathfrak{X}(\mathcal{M}) = \{V_{IM} = -i b_M(y) \partial_I, I = 1, \dots, n, M \in \mathbb{N}\}$  on  $\mathcal{M}$ , where  $\partial_I = \frac{\partial}{\partial y^I}$ . The algebra (2.15) extends to  $\mathfrak{g}(\mathcal{M}) \rtimes \mathfrak{X}(\mathcal{M})$ :

$$\begin{aligned} [T_{aM}, T_{bN}] &= i f_{ab}{}^c c_{MN}{}^P T_{cP} , \\ [V_{IM}, V_{JN}] &= -i \left( (\partial_I b_N) V_{JM} - (\partial_J b_M) V_{IN} \right) , \\ [V_{IM}, T_{aN}] &= -i b_M \partial_I b_N T_a \equiv -i d_{IM,N}{}^P T_{aP} , \end{aligned} \quad (2.18)$$

and has also a semidirect product structure. This algebra admits (2.15) as a subalgebra, also some analogue of the de Witt algebras. Let

$$\text{Witt}_i(\mathcal{M}) = \left\{ \ell_{iM} = -b_M(y) D_i, \quad M \in \mathbb{N} \right\}, \quad i = 1, \dots, s \quad (2.19)$$

which are a subalgebras of the algebra of vector fields on  $\mathcal{M}$ . The algebra (2.15) extends to  $\mathfrak{g}(\mathcal{M}) \rtimes \text{Witt}_i(\mathcal{M})$

$$\begin{aligned} [T_{aM}, T_{bN}] &= i f_{ab}{}^c c_{MN}{}^P T_{cP} , \\ [\ell_{iM}, \ell_{jN}] &= (M(i) - N(j)) C_{MN}{}^P \ell_{iP} , \\ [\ell_{iM}, T_{aN}] &= -M(i) C_{MN}{}^P T_{aP} . \end{aligned} \quad (2.20)$$

The last step of the construction is to extend centrally the algebra  $\mathfrak{g}(\mathcal{M})$ . Central extensions of  $\mathfrak{g}(\mathcal{M})$  were identified by Pressley and Segal in [49] (see Proposition 4.28

therein). Given a one-chain  $C$  (*i.e.*, a closed one-dimensional piecewise smooth curve), the central extension is given by the two-cocycle

$$\omega_C(X, Y) = \oint_C \langle X, dY \rangle_0 , \quad (2.21)$$

where  $dY = \partial_I Y dy^I$  is the exterior derivative of  $Y$ . The two-cocycle can be written in alternative form. Indeed, we have [50]

$$\omega_C(X, Y) = \int_{\mathcal{O}} \langle X, dY \rangle_0 \wedge \gamma , \quad (2.22)$$

where  $\gamma$  is a closed  $(n - 1)$ -current (a distribution) associated to  $C$ . We now show that the cocycle (2.21) can also be associated to a vector field on  $L \in \mathfrak{X}(\mathcal{M})$ . Let  $L = f^I(y) \partial_I$  a vector field be such that

$$\partial_I(\sqrt{g}f^I) = 0 . \quad (2.23)$$

To the operator  $L$  one can associate the 1-form:

$$F(y) = f_I(y) dy^I \quad (2.24)$$

where  $f_I = g_{IJ}f^J$ , and the  $(n - 1)$ -form:

$$\gamma(y) = \sqrt{g} \sum_{I=1}^n (-)^{I-1} f^I(y) dy^1 \wedge \cdots \wedge dy^{I-1} \wedge dy^{I+1} \wedge \cdots \wedge dy^n . \quad (2.25)$$

Obviously, if  $\mathcal{M}$  is orientable,  $\gamma = {}^*F$  is the Hodge dual of  $F$ . Now the condition (2.23) implies that  $\gamma$  is closed, equivalently, that  $F$  is co-closed. Moreover, if the cohomology group  $H^{n-1}(\mathcal{M}) = 1$  is trivial, then  $F$  is co-exact or  $\gamma$  is exact and there exists a one-form  $G$  such that

$$F = {}^*d^*G , \quad \gamma = (-1)^{q+n-1} dh ,$$

where  $h = {}^*G$  and the metric  $g$  has signature  $(p, q)$  with  $q$  minus signs. Since the  $(n - 1)$ -form associated to the operator  $L$  is closed, the corresponding two-cocycle (2.22) reduces to

$$\omega(X, Y)_f = \int_{\mathcal{O}} \sqrt{g} d^n y \langle X, LY \rangle_0 = \int_{\mathcal{O}} \langle X, dY \rangle_0 \wedge \gamma . \quad (2.26)$$

We conclude that there is an infinite number of central extensions associated to any operator  $L$  satisfying (2.23). In [51], Feigin showed that the number of central extensions is equal to the dimension of  $H^2(\mathfrak{g}(\mathcal{M}))$ , and proved that when  $\dim \mathcal{M} > 1$  and  $\mathcal{M}$  is compact, we have an infinite number of central extensions. Central extensions were systematically computed in the case of the two- and three-sphere in [33, 35, 36].

The result above can also be obtained differently. Indeed, one can centrally extend the ‘loop’ algebra  $\mathfrak{g}(\mathcal{M})$ , using the current algebra and introducing an anomaly or a Schwinger term:

$$[T_{a_1}(y_1), T_{a_2}(y_2)] = \left( i f_{a_1 a_2}{}^b T_b(y_2) + k_{a_1 a_2} f^I(y_2) \partial_I^{(2)} \right) \frac{\delta^n(y_1 - y_2)}{\sqrt{g(y_2)}} , \quad (2.27)$$

where  $\partial_I^{(2)}$  means that we take the derivative with respect to the second variable  $y_2^I$ . Thus,

$$\begin{aligned} [[T_{a_1}(y_1), T_{a_2}(y_2)], T_{a_3}(y_3)] &= \left[ \left( i f_{a_1 a_2}{}^b T_b(y_2) + k_{a_1 a_2} f^I(y_2) \partial_I^{(2)} \right) \frac{\delta^n(y_1 - y_2)}{\sqrt{g(y_2)}}, T_{a_3}(y_3) \right] \\ &= \cdots + i f_{a_1 a_2 a_3} f^I(y_3) \frac{\delta^n(y_1 - y_2)}{\sqrt{g(y_2)}} \partial_I^{(3)} \left( \frac{\delta^n(y_2 - y_3)}{\sqrt{g(y_3)}} \right) \end{aligned}$$

where  $\dots$  represents terms not involving the anomaly and  $f_{a_1 a_2 a_3} = f_{a_1 a_2}{}^b k_{ba_3}$ . Integration by parts, using the symmetry property of the Dirac  $\delta$ -distribution and the antisymmetry of the structure constants  $f_{a_1 a_2 a_3}$  in the computation  $\left[ [T_{a_1}(y_1), T_{a_2}(y_2)], T_{a_3}(y_3) \right] + \text{cyclic terms}$ , the Jacobi identity leads to (2.23), which is the two-cocycle condition that we have obtained using the Pressley-Segal two-cocycle. Furthermore, performing integration  $\int \sqrt{g(y)} d^n y b_M(y) \int \sqrt{g(y')} d^n y' b_N(y')$  for the anomaly term in (2.27) leads, due to the relation (2.23) above, to the two-cocycle (2.26). This means that the Pressley-Segal results on central extensions can equivalently be given in terms of an appropriate Schwinger term.

We now address the compatibility of central extensions and the differential operators of  $\mathfrak{X}(\mathcal{M})$ . Assume that we have one central extension associated to the operator  $L$  above or the two-cocycle  $\omega_L$ . The new Lie brackets read

$$[X, Y]_L = [X, Y]_0 + \omega_L(X, Y) , \quad (2.28)$$

where  $[ , ]_0$  are the Lie brackets without central extension, *i.e.*, the Lie brackets of  $\mathfrak{g}(\mathcal{M})$ . Let  $L'$  be an operator in  $\mathfrak{X}(\mathcal{M})$ . The compatibility condition, *i.e.*, the Jacobi identities, between the KM algebra with two-cocycle  $\omega_L$  and  $L'$  turn out to be

$$\omega_L(L'X, Y) + \omega_L(X, L'Y) = 0 .$$

Because of (2.23), after integration by parts, we get that  $L' = L$  is an obvious solution.

Let now  $(L, \omega_L)$  be a doublet differential operator/two-cocycle. On the one hand, the two-cocycle  $\omega_L$  and the differential operator  $L$  are in duality (see (2.24) and (2.25)) and on the other hand, they are compatible. One may wonder if for a given manifold  $\mathcal{M}$  one can have more than one couple  $(\omega_L, L)$  in duality and compatible with each other. In fact, as will be seen in Section 3.1, in general this is the case. In particular, if we consider the commuting Hermitian operators  $D_i$  (see (2.16)) and the corresponding two-cocycle  $\omega_i$ :

$$\omega_i(T_{aN}, T_{bM}) = M(i)k_{ab}\eta_{MN} , \quad (2.29)$$

the compatibility condition above between  $D_j$  and  $\omega_i$  reduces to

$$\omega_i(D_j T_{aN}, T_{bM}) + \omega_i(T_{aN}, D_j T_{bM}) = k_{ab}M(i)(N(j) + M(j))\eta_{MN} = 0 .$$

The condition of compatibility translates into

$$(N(j) + M(j))\eta_{MN} = 0 , \quad \forall M, N . \quad (2.30)$$

Let now  $\mathfrak{g}_D = \left\{ D_i, i = 1, \dots, r' \leq r : (2.30) \text{ is satisfied} \right\}$ . We thus define the KM algebra  $\tilde{\mathfrak{g}}(\mathcal{M}) = \left\{ \mathcal{T}_{aM}, a = 1, \dots, d, M \in \mathbb{N}, k^i, i = 1, \dots, r' \right\} \rtimes \left\{ D_i, i = 1, \dots, r' \right\}$ . From (2.11), (2.16) and (2.29) the non-vanishing Lie brackets are:

$$\begin{aligned} [\mathcal{T}_{aM}, \mathcal{T}_{bN}] &= i f_{ab}{}^c C_{MN}{}^P \mathcal{T}_{cP} + k_{ab} \eta_{MN} \sum_{i=1}^{r'} k^i M(i) , \\ [D_i, \mathcal{T}_{aM}] &= M(i) \mathcal{T}_{aM} . \end{aligned} \quad (2.31)$$

Note that  $k^i, i = 1, \dots, r'$  are the central charges associated to the two-cocycle  $\omega_i$ .

In the construction above, we determined the maximal set of compatible cocycles and differentials operators  $(\omega_L, L)$ . We consider here another possibility which will be relevant for the construction of the analogue of the Virasoro algebra (see Sec. 3.4 for  $\mathcal{M} = \text{SU}(2)$ ). Let  $\omega_L$  be a cocycle associated to the differential operator  $L$  and let  $T$  be any function in  $L^2(\mathcal{M})$ . The compatibility condition of the cocycle  $\omega_L$  with the operator  $L$  reads

$$\omega_L(T(m)LX, Y) + \omega_L(X, T(m)LY) = 0 , \quad (2.32)$$

whose second term on the LHS is

$$\omega_L(X, T(m)LY) = \int_{\mathcal{M}} \langle X, L((T(m)LY)) \rangle_0 \sqrt{g} d^n m ,$$

whilst the first one, after an integration by parts, reduces to

$$\begin{aligned} \omega_L(T(m)LX, Y) &= \int_{\mathcal{M}} \langle T(m)LX, LY \rangle_0 \sqrt{g} d^n m \\ &= - \int_{\mathcal{M}} \left( \langle X, L((T(m)LY)) \rangle_0 \sqrt{g} + \langle X, (T(m)LY) \rangle_0 \partial_m(\xi^m \sqrt{g}) \right) d^n m \\ &\quad + \int_{\mathcal{M}} \partial_m \left( \xi^m \langle X, (T(m)LY) \rangle_0 \sqrt{g} \right) d^n m , \end{aligned}$$

where we have used  $L = \xi^m \partial_m$ . The second term in the second equality vanishes because of (2.23). On the other hand, the boundary term vanishes along any compact direction of  $\mathcal{M}$ , but also vanishes along non-compact directions of  $\mathcal{M}$ , since  $X$  and  $Y$  go to zero at infinity. Finally, Eq.[2.32] is satisfied and the set of differential operators

$$D_L = \left\{ L^T := T(m)L , T(m) \in L^2(\mathcal{M}) \right\} \quad (2.33)$$

is compatible with the cocycle  $\omega_L$ . Furthermore, we have the following commutation relations

$$[L^{T_1}, L^{T_2}] = L^{T_{12}} , \quad T_{12}(m) = T_1(m)(LT_2(m)) - T_2(m)(LT_1(m)) .$$

Thus  $D_L \subset \mathfrak{X}(\mathcal{M})$  is a subalgebra of the algebra of vector fields on  $\mathcal{M}$ , which should be regarded as the analogue of the de Witt algebra in the case of  $\mathcal{M} = \text{U}(1)$ . This algebra was found in [33] for  $\mathcal{M} = \text{SU}(2)$ . We shall see in Section 3.4 that, in the case of  $\text{SU}(2)$ , a convenient choice of  $L$  enables us to introduce an analogue of the Virasoro algebra on  $\text{SU}(2)$ , *i.e.*, to central extend the algebra  $D_L$ . This construction certainly extends to other manifolds, as, for instance, any compact group manifold  $\mathcal{M} = G_c$ . In particular, for a given cocycle associated to  $L$ , further compatible differential operators can be added. For instance, if  $L = D_i$ , the set  $D_{D_i}$  is the de Witt algebra (see (2.19) and the second equation of (2.20)). Next, the algebra  $D_{D_i}$  can certainly be centrally extended along the lines of Sec. 3.4, yielding to an analogue of the semidirect product of the Virasoro algebra with the Kac-Moody algebra.

Before ending this section, it should be noted that the loop algebra  $\mathfrak{g}(\mathcal{M})$  admits also other central extensions. See *e.g.* [52, 53] for the three-dimensional case and [54] (and references therein) for higher-dimensional cases.

### 3 KM algebras on compact group manifolds

In the previous section we have given a general strategy to construct a generalized KM algebra associated to a manifold  $\mathcal{M}$ . We have assumed throughout (2.6) and (2.7). Let  $G_c$  be a compact Lie group manifold and let  $H \subset G_c$  be a closed subgroup of  $G_c$ . In this section we are considering the case where  $\mathcal{M}$  is either the compact group manifold  $\mathcal{M} = G_c$  or the coset space  $\mathcal{M} = G_c/H$ . In both cases, the conditions (2.6) and (2.7) (together with (2.5)(2)) are natural.

### 3.1 Compact group manifolds

Let  $G_c$  be an  $n$ -dimensional compact Lie group. Let  $m^A$  with  $i = A, \dots, n$  be a parameterization of  $G_c$ . Then an element of  $G_c$  connected to the identity takes the form:

$$g(m) = e^{im^A J_A}, \quad (3.1)$$

where  $J_A, A = 1, \dots, n$  are the generators of  $\mathfrak{g}_c$ , the Lie algebra of  $G_c$  (not to be confused with the generators of  $\mathfrak{g}$ ). Then, the coordinates of a group element (in a local coordinate chart) are

$$g(m)^M \equiv m^M, \quad M = 1, \dots, n. \quad (3.2)$$

The indices  $A, B, \dots$  are tangent space indices, *i.e.*, flat indices, whilst the indices  $M, N, \dots$  are world indices, *i.e.*, curved indices. It should be mentioned that for specific parameterizations, the variables  $m^A$  decompose into  $m^A = (\varphi^i, \theta^r), i = 1, \dots, p, r = 1, \dots, q, p+q = n$  where the matrix elements of  $m^M$  are periodic in  $\varphi$  and non-periodic in  $\theta$  (see Section 3).

The Vielbein one-form associated to the parameterization (3.1) is thus

$$e(m) = g(m)^{-1} \mathrm{d}g(m), \quad (3.3)$$

where  $\mathrm{d}g(m)$  is the exterior derivative of  $g$ . The Vielbein satisfies the Maurer-Cartan equation

$$\mathrm{d}e + e \wedge e = 0. \quad (3.4)$$

Expanding the Vielbein in the basis  $J_A$ :  $e = ie^A J_A$  we have for the one-forms  $e^A$ :

$$e^A(m) = e_M{}^A(m) \mathrm{d}m^M.$$

The one-forms are left-invariant by construction, *i.e.*, invariant by the left multiplication  $L_h(g) = hg$  with  $h \in G_c$ . Note that defining the alternative Vielbein  $e'(m) = \mathrm{d}g(m)g(m)^{-1}$ , we obtain a right-invariant one-forms, *i.e.*, invariant by the right multiplication  $R_h(g) = gh$ . The metric tensor on  $G_c$  can be defined by the left or right-invariant one-forms

$$g_{MN} = e_M{}^A(m)e_N{}^B(m)\delta_{AB} = e'_M{}^A(m)e'_N{}^B(m)\delta_{AB}. \quad (3.5)$$

We endow the manifold  $G_c$  with the scalar product

$$(f, g) = \frac{1}{V} \int_{G_c} \sqrt{g} \mathrm{d}\varphi^p \mathrm{d}\theta^q \overline{f(\varphi, \theta)} f(\varphi, \theta), \quad (3.6)$$

where  $V$  is the volume of  $G_c$ .

To identify a Hilbert basis of  $L^2(G_c)$ , we first recall that since  $G_c$  is a compact Lie group, its unitary irreducible representations are finite dimensional. Let  $\hat{\mathcal{R}} = \{\mathcal{R}_k, k \in \hat{G}_c\}$  be the set of all irreducible unitary representations of  $G_c$ , and let  $\hat{G}_c$  be the set of labels of such representations (see below). Let  $d_k$  be the dimension of the representation  $\mathcal{R}_k$  and let  $D_{(k)}{}^i{}_j(g)$  be its matrix elements. With these notations, have the following theorem:

**Theorem 3.1 (Peter-Weyl [16])** *Let  $\hat{\mathcal{R}} = \{\mathcal{R}_k, k \in \hat{G}_c\}$  be the set of all unitary irreducible representations of  $G_c$ , and let  $D_{(k)}(g) \in \mathcal{R}_k$  for  $g \in G_c$ . Then the set of functions on  $G_c$ ,*

$$\psi_{(k)}{}^i{}_j(g) = \sqrt{d_k} D_{(k)}{}^i{}_j(g), \quad k \in \hat{G}_c, i, j = 1, \dots, d_k, g \in G_c, \quad (3.7)$$

forms a complete Hilbert basis of  $L^2(G_c)$  with inner product

$$\int_{G_c} \sqrt{g} d\varphi^p d\theta^q \bar{\psi}^{(k)}{}_i{}^j(g) \psi_{(k')}{}^{i'}{}_{j'}(g) = \delta_{k'}^k \delta_i^{i'} \delta_{j'}^j .$$

Since the notations of Theorem 3.1 are not appropriate for our purpose, we introduce more convenient notations. Assume that the Lie algebra  $\mathfrak{g}_c$  is of rank  $\ell$ . Then  $\mathfrak{g}_c$  admits  $\ell$  independent Casimir operator  $\{C_1, \dots, C_\ell\}$ . Let  $\mathcal{R}$  be a unitary representation of  $G_c$ ,  $\mathcal{R}$  is uniquely specified by the eigenvalues of the  $\ell$  Casimir operators (alternatively the representation can also be specified by the highest weight with respect to a given Cartan subalgebra  $\mathfrak{h} \in \mathfrak{g}_c$ ). Denote  $Q = (c_1, \dots, c_\ell)$  the eigenvalues  $\{C_1, \dots, C_\ell\}$ . Racah showed in [55] that a number of  $1/2(\dim g - \ell)$  internal labels are required to separate the states within the multiplet  $\mathcal{R}$ . Given a Cartan subalgebra, its eigenvalues constitute an appropriate set of  $\ell$  labels. The choice of the remaining additional internal labels is far from being unique, and usually depends on a specific chain of proper subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}_c$$

such that, in each step, the Casimir operators of the subalgebra are used to separate states [56, 57]. Let  $D_{(k)}$  and  $D'_{(k)}$  be two matrix elements of the representation  $\mathcal{R}_k$ . Since the product  $D_{(k)} D'_{(k)}$  is still a matrix element of  $\mathcal{R}_k$ , the lines (resp. the column) of  $D_{(k)}$  are in the representation  $\mathcal{R}_k$  (lines are in left representations and column in right representations). Consequently, we denote now any matrix element appearing in Theorem 3.1 by  $\Psi_{LQR}$ , where  $Q$  denotes the  $\ell$  eigenvalues of the Casimir operators that identify the representation, and  $L$  (resp.  $R$ ) the  $1/2(\dim g - \ell)$  labels to separate the states within the left (resp. right) action of  $G_c$ . Let  $\mathcal{I}$  be the range for the variables  $L, Q$  and  $R$ ; with these notations, the Hilbert basis of  $L^2(G_c)$  associated to the Peter-Weyl theorem is given by:

$$\mathcal{B}_{G_c} = \left\{ \Psi_{LQR} , \quad (L, Q, R) \in \mathcal{I} \right\} , \quad (3.8)$$

and the matrix elements are normalized:

$$(\Psi_{LQR}, \Psi_{L'Q'R'}) = \delta_Q^Q \delta_L^L \delta_R^R .$$

The parameterisation  $m^A = (\varphi^i, \theta^r)$  leads naturally to a differential realization of the Lie algebra  $\mathfrak{g}_c$  for the generators of the left and right action. Indeed, since the manifold  $G_c$  has a natural left and right action, the Killing equation (2.1) automatically admits solutions, namely the generators of the left (resp. right) action and  $\mathfrak{g}_\xi = (\mathfrak{g}_c)_L \oplus (\mathfrak{g}_c)_R$ . Let  $L_A$  (resp.  $R_A$ )  $A = 1, \dots, n$ , be the generators of the left (resp. right) action. We have

$$[L_A, L_B] = i c_{AB}{}^C L_C , \quad [R_A, R_B] = i c_{AB}{}^C R_C , \quad [L_A, R_B] = 0 , \quad (3.9)$$

where  $c_{AB}{}^C$  are the structure constants of  $\mathfrak{g}_c$ . Furthermore, the operators  $L_A$  and  $R_A$  are Hermitian with respect to the scalar product (3.6) and act naturally on the matrix elements  $\Psi_{LQR}$ :

$$\begin{aligned} L_A \Psi_{LQR}(\varphi, \theta) &= (M_A^Q)_L{}^{L'} \Psi_{L'QR}(\varphi, \theta) \\ R_A \Psi_{LQR}(\varphi, \theta) &= (M_A^Q)_R{}^{R'} \Psi_{LQR'}(\varphi, \theta) \end{aligned}$$

where  $(M_A^Q)_L{}^{L'}, (M_A^Q)_R{}^{R'}$  are the matrix elements of the left or right action for the representation specified by  $Q$ .

Let  $\mathcal{D}_Q, \mathcal{D}_{Q'}$  be two representations of  $G_c$  specified by the eigenvalues of the Casimir operators. Consider the tensor product:

$$\mathcal{D}_Q \otimes \mathcal{D}_{Q'} = \bigoplus_{Q''} \mathcal{D}_{Q''} .$$

Introducing the Clebsch-Gordan coefficient  $\begin{pmatrix} Q & Q' & Q'' \\ L & L' & L'' \end{pmatrix}$ , we have

$$|Q'', L''\rangle = \sum \begin{pmatrix} Q & Q' & Q'' \\ L & L' & L'' \end{pmatrix} |Q, L\rangle \otimes |Q', L'\rangle \quad (3.10)$$

Thus, observing that

$$\Psi_{LQR}(0) = \sqrt{d} \delta_{LR} ,$$

where  $d$  is the dimension of the representation  $\mathcal{D}_Q$ , we have

$$\Psi_{LQR}(m) \Psi_{L'Q'R'}(m) = C_{LQR; L'Q'R'}^{L''Q''R''} \Psi_{L''Q''R''}(m) , \quad (3.11)$$

where the summation is implicit, and with

$$C_{LQR; L'Q'R'}^{L''Q''R''} = \sqrt{\frac{dd'}{d''}} \begin{pmatrix} Q & Q' & Q'' \\ L & L' & L'' \end{pmatrix} \overline{\begin{pmatrix} Q & Q' & Q'' \\ R & R' & R'' \end{pmatrix}} , \quad (3.12)$$

where  $d$  (resp.  $d', d''$ ) are the dimension of  $\mathcal{D}_Q$  (resp.  $\mathcal{D}_{Q'}, \mathcal{D}_{Q''}$ ).

### 3.2 Coset spaces of compact group manifolds

Now let  $H$  be a closed subgroup of  $G_c$ ,  $\pi : G_c \rightarrow G_c/H$  be the quotient map and let  $n = \dim G_c, s = \dim H$ . Then we can always find a chart  $(U, \varphi = (y^1, \dots, y^n))$  around the identity in  $G_c$  such that

1.  $\varphi(U) = \{(\xi^1, \dots, \xi^n) \mid |\xi^p| < \varepsilon, p = 1, \dots, n\}$  for some  $\varepsilon > 0$ .
2. Each slice with  $y^{s+1} = \xi^{s+1}, \dots, y^n = \xi^n$  is a relatively open set in some coset  $gH$ , and these cosets are all distinct.
3. The restriction of  $\pi$  to the slice  $y^q = 0, q = 1, \dots, s$  is an open homeomorphism, and hence determines a chart around the identity element in  $G_c/H$ .

Using these charts, the coset  $G_c/H$  can be endowed with the structure of a differential manifold, such that the  $\pi$  and the action of  $G_c$  are differentiable. If in addition  $H$  is a normal subgroup, then  $G_c/H$  inherits the structure of a Lie group, and the differential of the  $G_c \rightarrow G_c/H$  corresponds to the to the quotient map  $\mathfrak{g}_c \rightarrow \mathfrak{g}_c/\mathfrak{h}$ , where  $\mathfrak{h}$  is the Lie algebra of  $H$ .

In the general case (i.e., there is some  $g \in G_c$  with  $g^{-1}Hg \neq H$ ), the coset space  $G_c/H$  is not a Lie group, and hence the factor space  $\mathfrak{g}_c/\mathfrak{h}$  is not a Lie algebra. We write the generators of  $\mathfrak{g}_c$ , namely  $T_A$  (with  $A = 1, \dots, \dim \mathfrak{g}_c$ ), as follows:  $U_i$  with  $i = 1, \dots, \dim \mathfrak{h}$ , and  $V_p$  with  $p = 1, \dots, \dim \mathfrak{g}_c - \dim \mathfrak{h}$ , where  $V_p$  denotes the complementary space of  $\mathfrak{h}$  in  $\mathfrak{g}_c$ . The commutations relations take the generic form

$$\begin{aligned} (a) \quad & [U_j, U_k] = i g_{jk}{}^\ell U_\ell , \\ (b) \quad & [U_j, V_p] = i N_{jp}{}^\ell U_\ell + i (R_j)_p{}^q V_q , \\ (c) \quad & [V_p, V_q] = i g_{pq}{}^r V_r . \end{aligned}$$

The relations (a) are trivially satisfied, as  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{g}_c$ , whereas the relations (b) imply that  $\mathfrak{g}_c/\mathfrak{h}$  is a representation of  $\mathfrak{g}_c$ , whenever the condition  $N_{jp}^\ell = 0$  is satisfied.

In particular, the coset space  $G_c/H$  is called reductive if there exists an  $Ad(H)$ -invariant subspace  $\mathfrak{m}$  of  $\mathfrak{g}_c$  that is complementary to  $\mathfrak{h}$  in  $\mathfrak{g}_c$ .<sup>2</sup> It follows from the invariance that  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ . A sufficient condition for the coset space to be reductive is that either  $H$  or  $Ad(H)$  is compact [17]. If, in addition  $g_{pq}{}^r = 0$  holds in (c), then the manifold  $G_c/H$  is related to a specially relevant class of homogeneous spaces (endowed with a Riemannian structure), the so-called symmetric spaces [58].

In the following, we shortly summarize the construction steps of the previous section, adapted to (generic) homogeneous spaces. A generic element of  $G_c/H$  is of the form:

$$L = e^{im^p V_p} ,$$

and the Vielbein one-form given by

$$e = L^{-1} dL , \quad (3.13)$$

which decomposes as

$$e(m) = ie^p U_p + ie^i V_i . \quad (3.14)$$

Denoting

$$e^p(m) = e^p{}_{\tilde{p}}(m) dm^{\tilde{p}} ,$$

where  $p, q, \dots$  represent the indices in the flat tangent space and  $\tilde{p}, \tilde{q}, \dots$  the curved indices of the manifold  $G_c/H$ , the metric tensor on the coset space is given by

$$g_{\tilde{p}\tilde{q}} = e^p{}_{\tilde{p}}(m) e^q{}_{\tilde{q}}(m) \delta_{pq} . \quad (3.15)$$

We identify the Killing vectors and the corresponding symmetry algebra of the manifold  $G_c/H$

$$\mathfrak{g}_\xi = \left\{ K_A = i\xi_A^p \partial_p : \xi_A^p \text{ Killing vector of } g_{\tilde{p}\tilde{q}} \right\} \subset (\mathfrak{g}_c)_L \oplus (\mathfrak{g}_c)_R . \quad (3.16)$$

We endow the manifold  $G_c/H$  with the scalar product

$$(f, g) = \frac{1}{V'} \int_{G_c/H} \sqrt{g'} dm^N \overline{f(m)} g(m) , \quad (3.17)$$

where  $V'$  is the volume of  $G_c/H$ , which is the restriction of (3.6) to  $G_c/H$  and  $N = \dim \mathfrak{g}_c - \dim \mathfrak{h}$ .

We now identify a Hilbert basis of  $G_c/H$ . Given a representation  $\mathcal{R}_k$  of  $G_c$ , which is such that  $\mathcal{R}_k$  admits a scalar representation with respect to the embedding  $H \subset G_c$ . Let  $\mathcal{R}_k|_H$  be the set of representations with this property. The set of matrix elements  $\Psi_{LQR}$  of a representation  $\mathcal{R}_k|_H$  decomposes into

$$\left\{ \Psi_{LQR} , (LQR) \in \mathcal{I} \right\} = \left\{ \Psi_{LQR_0} , (LQR) \in \mathcal{I} : \Psi_{LQR_0} \text{ trivial under the right action of } H \right\} \oplus \dots$$

Stated differently,  $R_0$  corresponds to the set of indices associated to the trivial representation of the right  $H$ -action. The matrix elements  $\Psi_{LQR_0}$  contribute to the harmonic analysis in  $G_c/H$ . With the notation of Section 3.1 we denote the corresponding normalized matrix elements

$$\phi_{LQ} = \psi_{LQR_0} , \quad (3.18)$$

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<sup>2</sup>The space  $\mathfrak{m}$  is sometimes called the Lie subspace for  $G_c/H$ .

where the index  $R_0$  means that that  $\phi_{LQR_0}$  transform trivially under the right action of  $H$  [8, 37]. The number of internal labels is  $\ell$  (the rank of  $\mathfrak{g}_c$ ) to separate the representations, and  $1/2(\dim \mathfrak{g}_c - \ell)$  internal labels to distinguish states within representations [37]. Solving the Killing equation for the coset leads to the generators of the left action  $G_c$ , and possibly additional generators associated to the right action which survive the coset process. Finally, computing the product  $\phi_{LQ}\phi_{L'Q'}$ , we obtain coefficients that can be deduced from (3.12) (see 3.3).

### 3.3 KM algebras associated to compact Lie groups

Consider first  $\mathcal{M} = G_c$ . The first step to associate a KM algebra to  $G_c$ , is to define the Lie algebra

$$\mathfrak{g}(G_c) = \left\{ T_{aLQR}(m) = T_a \psi_{LQR}(m) , \quad A = 1, \dots, d, \quad (L, Q, R) \in \mathcal{I} \right\}$$

(see (3.8)) with Lie brackets

$$[T_{aLQR}(m), T_{a'L'Q'R'}(m)] = i f_{aa'}^{a''} C_{LQR; L'Q'R'}^{L''Q''R''} T_{a''L''Q''R''}(m). \quad (3.19)$$

This algebra can be obtained in a different way. Let

$$T_a(m) = \sum_{(L, Q, R) \in \mathcal{I}} T_{aLQR} \overline{\psi^{LQR}}(m),$$

then

$$[T_a(m), T_b(m')] = i f_{ab}^c T_c(m') \frac{\delta(m - m')}{\sqrt{g}} \quad (3.20)$$

lead to (3.19) upon integration by  $\int d^n m \sqrt{g} \Psi_{LQR}(m) \int d^n m' \sqrt{g} \Psi_{L'Q'R'}(m')$ .

This algebra can indeed be obtained easily in the context of Kaluza-Klein theories. Consider the spacetime  $K = \mathbb{R}^{1,D-1} \times G_c$ , *i.e.*, a  $D$ -dimensional Minkowski spacetime with a compact manifold  $G_c$  of dimension  $\dim \mathfrak{g}_c = n$  as internal space. Denote  $X^I = (x^\mu, m^M)$  the components in  $K$  where  $\mu = 0, \dots, D-1$  and  $M = 1, \dots, n$ . Assume further the metric on  $K$  (in this simple analysis we don't endow  $\mathbb{R}^{1,d-1}$  with a Riemannian structure, thus the metric is Minkowskian)

$$ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} - dm^M dm^N g_{MN} := g_{IJ} dX^I dX^J,$$

where  $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$  is the Minkowski metric on  $\mathbb{R}^{1,d-1}$  and  $g_{MN}$  is the metric (3.5) on  $G_c$ .

Consider a massless free complex scalar field, with action

$$\mathcal{S} = \int_{\mathbb{R}^{1,D-1}} dx^D \int_{G_c} dm^n \sqrt{g} g^{IJ} \partial_I \Phi^\dagger(x, m) \partial_J \Phi(x, m).$$

The equations of the motion read

$$(\square - \nabla^2) \Phi(x, m) = 0,$$

where  $\square = \partial_\mu \partial_\nu \eta^{\mu\nu}$  and  $\nabla^2 = \frac{1}{\sqrt{g}} \partial_M (g^{MN} \partial_N)$  are respectively the  $D$ -dimensional d'Alembertian and the Laplace-Beltrami operator of  $G_c$ . Next, decomposing the scalar field in the compact manifold  $G_c$  as

$$\Phi(x, m) = \sum_{(L, Q, R) \in \mathcal{I}} \phi_{LQR}(x) \overline{\psi^{LQR}}(m),$$

and observing that the action of the Laplace-Beltrami operator on the scalar functions is related to the quadratic Casimir operator [17, 59],

$$\nabla^2 = -kC_2 , \quad k > 0 ,$$

(where  $C_2$  is the quadratic Casimir operator of  $\mathfrak{g}_c$  with eigenvalue  $c_2$ ) we obtain

$$\nabla^2 \bar{\psi}^{LQR}(m) = -kc_2 \bar{\psi}^{LQR}(m) , \quad c_2 \geq 0 ,$$

with the mass of the field  $\phi_{LQR}$  given by  $\sqrt{kc_2}$ . Thus the solution of the relativistic wave equation takes the form:

$$\Phi(x, m) = \sum_{(L, Q, R) \in \mathcal{I}} \int \frac{d^{D-1}x}{\sqrt{2E_{c_2}}} \left( a_{\vec{p}, LQR} e^{-i\mathbf{p} \cdot \mathbf{x}} + b_{\vec{p}, LQR}^\dagger e^{i\mathbf{p} \cdot \mathbf{x}} \right) \bar{\psi}^{LQR}(m) ,$$

where  $\mathbf{p} \cdot \mathbf{x}$  is the scalar product in the Minkowski spacetime, and  $\mathbf{p} \cdot \mathbf{p} = E^2 - \vec{p} \cdot \vec{p} = kc_2$ , or  $E_{c_2} \equiv \sqrt{\vec{p} \cdot \vec{p} + kc_2}$ .

Let  $\Pi(x, m) = \Phi^\dagger(x, m)$  be the conjugate momentum of  $\Phi$ . If we now quantize the scalar field by the usual equal-time commutation relations, an elementary computation leads to

$$\begin{aligned} [\Phi(t, \vec{x}, m), \Pi(t, \vec{x}', m')] &= i\delta^{D-1}(\vec{x} - \vec{x}') \frac{\delta^n(m - m')}{\sqrt{g}} \\ \Leftrightarrow \begin{cases} [a_{\vec{p}, LQR}, a_{\vec{p}', L'Q'R'}^\dagger] &= (2\pi)^{D-1} \delta^{D-1}(\vec{p} - \vec{p}') \delta_{LL'} \delta_{QQ'} \delta_{RR'} \\ [b_{\vec{p}, LQR}, b_{\vec{p}', L'Q'R'}^\dagger] &= (2\pi)^{D-1} \delta^{D-1}(\vec{p} - \vec{p}') \delta_{LL'} \delta_{QQ'} \delta_{RR'} . \end{cases} \end{aligned}$$

Consider now, that the scalar field is in a representation  $\mathcal{R}$  of the compact Lie algebra  $\mathfrak{g}$ . Let  $M_a$  be the corresponding matrix representation. Then the Noether theorem leads to the conserved current

$$j_a^\mu(t, \vec{x}, m) = i(\Pi(t, \vec{x}, m)M_a\Phi(t, \vec{x}, m) - \Phi^\dagger(t, \vec{x}, m)M_a\Pi^\dagger(t, \vec{x}, m))$$

and the equal-time commutation relations imply

$$[j_a^0(t, \vec{x}, m), j_b^0(t', \vec{x}', m')] = if_{ab}{}^c j_c^0(t, \vec{x}, m) \delta^{D-1}(\vec{x} - \vec{x}') \frac{\delta^n(m - m')}{\sqrt{g}} .$$

Upon space integration  $\int dx^{D-1}$  we reproduce (3.20). A second integration  $\int \sqrt{g} dm^d$  leads to (3.19). Thus the conserved charges  $Q_a = \int dx^{D-1} \int \sqrt{g} dm^d j_a^0(t, \vec{x}, m)$  generate the Lie algebra (3.19), which appears naturally upon compactification. This fact was already observed in [60]. for the case  $G_c = U(1)$ .

The construction of the KM algebras associated to  $G_c$  follows naturally from the description given in Section 2.2. We now introduce the generators of the Cartan subalgebra of  $\mathfrak{g}_c$  for the left (resp. right action)  $D_i^L, i = 1, \dots, \ell' \leq \ell$  (resp.  $D_i^R, i = 1, \dots, \ell'$ ) satisfying (2.30), together with their associated two-cocycles

$$\begin{aligned} \omega_i^L(T_{aLQR}, T_{a'L'Q'R'}) &= \int dm^d \sqrt{g} \left\langle T_{aLQR}, D_i^L T_{a'L'Q'R'} \right\rangle_0 = k_{aa'} L'(i) \eta_{LQR, L'Q'R'} , \\ \omega_i^R(T_{aLQR}, T_{a'L'Q'R'}) &= \int dm^d \sqrt{g} \left\langle T_{aLQR}, D_i^R T_{a'L'Q'R'} \right\rangle_0 = k_{aa'} R'(i) \eta_{LQR, L'Q'R'} , \end{aligned} \quad (3.21)$$

(see (2.25)). As the detailed steps where given in Sect. 2.2, we merely indicate the results without further details. The KM algebra associated to  $G_c$  can be defined (with the

notations of Sect. 2.2) by  $\tilde{\mathfrak{g}}(G_c) = \{\mathcal{T}_{aLQR}, a = 1 \cdots, d, (LQR) \in \mathcal{I}, D_i^L, D_i^R, k_L^i, k_R^j, i = 1, \cdots, \ell'\}$ , with non-vanishing Lie brackets:

$$\begin{aligned} [\mathcal{T}_{aLQR}, \mathcal{T}_{a'L'Q'R'}] &= i f_{aa'}^{a''} C_{LQR; L'Q'R'}^{L''Q''R''} \mathcal{T}_{a''L''Q''R''} \\ &\quad + k_{aa'} \eta_{LQR, L'Q'R'} \sum_{i=1}^{\ell'} (k_L^i L(i) + k_R^i R(i)) , \\ [D_i^L, \mathcal{T}_{aLQR}] &= L(i) \mathcal{T}_{aLQM} , \\ [D_i^R, \mathcal{T}_{aLQR}] &= R(i) \mathcal{T}_{aLQP} . \end{aligned}$$

Note that here,  $L(i), R(i)$  are the components of the weight vectors of the corresponding representation.

Consider now briefly the construction of a KM algebra associated to a coset  $G_c/H$  where  $H \subset G_c$  is a closed subgroup of  $G_c$ . The construction of a KM algebra associated to the manifold  $G_c/H$  is entirely analogous to the construction of the KM algebra associated to  $G_c$ , *mutas mutandis* the following replacements:

1. the metric on  $G/H$  is given by (3.15);
2. the scalar product reduces to (3.17);
3. the Hilbert basis of  $L^2(G_c/H)$  is given by the set of  $H$ -invariant vectors of the right-action (3.18);
4. the Killing vectors and the corresponding symmetry algebra of  $G/H$  are given in (3.16). Denote  $D_i, i = 1, \cdots, \ell' \leq \ell$  the generators of the Cartan subalgebra satisfying (2.30).
5. the two-cocycles are obtained through (2.29) (with the commuting Hermitian operators satisfying (2.30)  $D_i, i = 1, \cdots, \ell' \leq \ell$ ).

The Lie brackets of the algebra  $\tilde{\mathfrak{g}}(G_c/H)$  are given by the Lie brackets (3.22) *modulo* the substitutions above.

### 3.4 KM algebra associated to $SU(2)$ and $SU(2)/U(1)$

In this section we give explicit examples of KM algebras associated to compact group manifolds or cosets. A detailed account on their construction was given in [37], for this reason we merely present two examples here.

We first consider  $G_c = SU(2)$ . We give here the principal steps, the details of which can be found also in [37, 44]. We have

$$SU(2) = \left\{ z_1, z_2 \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \right\} \cong \mathbb{S}^3 ,$$

where  $\mathbb{S}^3$  is the three-sphere. A parameterisation of  $\mathbb{S}^3$  is given by

$$z_1 = \cos \frac{\theta}{2} e^{i \frac{\varphi+\psi}{2}} , \quad z_2 = \sin \frac{\theta}{2} e^{i \frac{\varphi-\psi}{2}} , \quad 0 \leq \theta \leq \pi , \quad 0 \leq \varphi < 2\pi , \quad 0 \leq \psi < 4\pi . \quad (3.22)$$

From

$$m = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in SU(2) \quad (3.23)$$

equation (3.3) leads to the left-invariant one-forms

$$\begin{aligned} \lambda_1 &= \sin \varphi \, d\theta - \cos \varphi \, \sin \theta \, d\psi \\ \lambda_2 &= \cos \varphi \, d\theta + \sin \varphi \, \sin \theta \, d\psi \\ \lambda_3 &= \cos \theta \, d\psi + d\varphi \end{aligned} \quad (3.24)$$

and to the metric tensor

$$ds^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta d\phi d\psi . \quad (3.25)$$

The left/right invariant vectors fields obtained by solving the Killing equation (2.1) are

$$\begin{aligned} L_{\pm} &= e^{\pm i\psi} \left( -\frac{i}{\sin\theta} \partial_{\varphi} + i \cot\theta \partial_{\psi} \mp \partial_{\theta} \right), & L_0 &= -i\partial_{\psi} \\ R_{\pm} &= e^{\pm i\varphi} \left( \frac{i}{\sin\theta} \partial_{\psi} - i \cot\theta \partial_{\varphi} \mp \partial_{\theta} \right), & R_0 &= -i\partial_{\varphi} \end{aligned}$$

and satisfy the commutation relations

$$\begin{aligned} [L_0, L_{\pm}] &= \pm L_{\pm}, & [L_+, L_-] &= 2L_0, & [L_a, R_b] &= 0 . \\ [R_0, R_{\pm}] &= \pm R_{\pm}, & [R_+, R_-] &= 2R_0, \end{aligned}$$

The quadratic Casimir operator reduces to:

$$C_2 = -\partial_{\theta}^2 - \cot\theta \partial_{\theta} - \frac{1}{\sin^2\theta} (\partial_{\varphi}^2 + \partial_{\psi}^2) + 2\frac{\cos\theta}{\sin^2\theta} \partial_{\varphi} \partial_{\psi} .$$

Finally, the SU(2)-scalar product is given by

$$(f, g) = \frac{1}{16\pi^2} \int_{\mathbb{S}^3} \sin\theta d\theta d\psi d\phi \overline{f(\theta, \varphi, \psi)} g(\theta, \varphi, \psi) .$$

We now identify the matrix elements. Since  $\dim \mathfrak{su}(2) = 3$  and  $\text{rk } \mathfrak{su}(2) = 1$ , we need two labels to identify the matrix elements of  $\mathfrak{su}(2)$ , one external corresponding to the eigenvalue of the quadratic Casimir operator, and one internal label determined by the Cartan subalgebra eigenvalues of the left/right action of  $\mathfrak{su}(2)$ . We thus obtain [44] ( $s \in \frac{1}{2}\mathbb{N}, -s \leq n, m \leq s$ ):

$$\psi_{nsm}(\theta, \varphi, \psi) = \begin{cases} \frac{(-1)^{m-n}\sqrt{(2s+1)}}{(n-m)!} \sqrt{\frac{(s+n)!}{(s-n)!} \frac{(s-m)!}{(s+m)!}} e^{im\varphi+in\psi} \cos^{-n-m}\frac{\theta}{2} \sin^{n-m}\frac{\theta}{2} & n \geq m \\ {}_2F_1(-m-s, -m+s+1; 1-m+n; \sin^2\frac{\theta}{2}) & -n-m \geq 0 \\ \frac{\sqrt{(2s+1)}}{(m-n)!} \sqrt{\frac{(s+m)!}{(s-m)!} \frac{(s-n)!}{(s+n)!}} e^{im\varphi+in\psi} \cos^{-n-m}\frac{\theta}{2} \sin^{m-n}\frac{\theta}{2} & m \geq n \\ {}_2F_1(-n-s, -n+s+1; 1-n+m; \sin^2\frac{\theta}{2}) & -n-m \geq 0 \\ \frac{(-1)^{m-n}\sqrt{2s+1}}{(n-m)!} \sqrt{\frac{(s+n)!}{(s-n)!} \frac{(s-m)!}{(s+m)!}} e^{im\varphi+in\psi} \cos^{n+m}\frac{\theta}{2} \sin^{n-m}\frac{\theta}{2} & n \geq m \\ {}_2F_1(n-s, n+s+1; 1+n-m; \sin^2\frac{\theta}{2}) & n+m \geq 0 \\ \frac{\sqrt{2s+1}}{(m-n)!} \sqrt{\frac{(s+m)!}{(s-m)!} \frac{(s-n)!}{(s+n)!}} e^{im\varphi+in\psi} \cos^{n+m}\frac{\theta}{2} \sin^{m-n}\frac{\theta}{2} & m \geq n \\ {}_2F_1(m-s, m+s+1; 1+m-n; \sin^2\frac{\theta}{2}) & n+m \geq 0 \end{cases} \quad (3.26)$$

where  ${}_2F_1$  denotes the Euler hypergeometric polynomial (see e.g. [42] for definitions). A direct inspection gives rise to

$$\overline{\psi^{nsm}}(\theta, \varphi, \psi) = (-1)^{n-m} \psi_{-ns-m}(\theta, \varphi, \psi) ,$$

leading to  $\eta_{nm,n'm'} = (-1)^{n-m} \delta_{n,-n'} \delta_{m,-m'}$  (see Sect. 2.1). These matrix elements were obtained solving the differential equations

$$\begin{aligned} L_0 \psi_{nsm}(\theta, \varphi, \psi) &= n \psi_{nsm}(\theta, \varphi, \psi) , \\ R_0 \psi_{nsm}(\theta, \varphi, \psi) &= m \psi_{nsm}(\theta, \varphi, \psi) , \\ C_2 \psi_{nsm}(\theta, \varphi, \psi) &= s(s+1) \psi_{nsm}(\theta, \varphi, \psi) \end{aligned}$$

where  $C_2$  is the quadratic Casimir operator [44], and we have

$$\begin{aligned} L_{\pm}\psi_{nsm}(\theta, \varphi, \psi) &= \sqrt{(s \mp n)(s \pm n + 1)}\psi_{n\pm 1sm}, \\ L_0\psi_{nsm}(\theta, \varphi, \psi) &= n\psi_{nsm}(\theta, \varphi, \psi) \\ R_{\pm}\psi_{nsm}(\theta, \varphi, \psi) &= \sqrt{(s \mp m)(s \pm m + 1)}\psi_{nsm\pm 1}, \\ R_0\psi_{nsm}(\theta, \varphi, \psi) &= m\psi_{nsm}(\theta, \varphi, \psi). \end{aligned}$$

The set

$$\mathcal{B} = \left\{ \psi_{nsm}, s \in \frac{1}{2}\mathbb{N}, -s \leq n, m \leq s \right\} \quad (3.27)$$

constitutes a Hilbert basis of  $L^2(\mathrm{SU}(2))$ , and the following is satisfied :

$$(\psi_{nsm}, \psi_{n',s',m'}) = \delta_{ss'}\delta_{nn'}\delta_{mm'} . \quad (3.28)$$

We have now all the ingredients to define the KM algebra associated to  $\mathrm{SU}(2)$ . First observe that the differential operators  $L_0, R_0$  satisfy both (2.30). Let  $\omega_L, \omega_R$  be the corresponding two-cocycles. One may now wonder whether there exists additional differential operators compatible with the two cocycles  $\omega_L, \omega_R$ . This will be studied in the next section. Thus in turn we are able to define the possible extensions of the algebra  $\mathfrak{g}(\mathcal{M}) = \left\{ T_a \psi_{nsm}(\theta, \varphi, \psi), a = 1, \dots, d, \ell \in \frac{1}{2}\mathbb{N}, -\ell \leq n, m \leq \ell \right\}$  (see (3.19)) by

$$\tilde{\mathfrak{g}}(\mathrm{SU}(2)) \rtimes \{L_0, R_0\} = \left\{ \mathcal{T}_{ansm}, k_L, k_R, a = 1, \dots, d, \ell \in \frac{1}{2}\mathbb{N}, -\ell \leq n, m \leq \ell \right\} \rtimes \{L_0, R_0\}$$

with Lie brackets (see [37])

$$\begin{aligned} [\mathcal{T}_{ansm}, \mathcal{T}_{a'n's'm'}] &= i f_{aa'}^{a''} C_{ss'nn'mm'}^{s''} \mathcal{T}_{a''n+n's''m+m'} \\ &\quad + k_{ab}(-1)^{m-n} \delta_{ss'} \delta_{n,-n'} \delta_{m,-m'} (k_L n' + k_R m') \\ [L_0, \mathcal{T}_{ansm}] &= n \mathcal{T}_{ansm} \\ [R_0, \mathcal{T}_{ansm}] &= m \mathcal{T}_{ansm}. \end{aligned} \quad (3.29)$$

Here we have

$$C_{s_1 s_2 n_1 n_2 m_1 m_2}^S = \sqrt{\frac{(2s_1 + 1)(2s_2 + 1)}{2S + 1}} \begin{pmatrix} s_1 & s_2 & S \\ n_1 & n_2 & n_1 + n_2 \end{pmatrix} \overline{\begin{pmatrix} s_1 & s_2 & S \\ m_1 & m_2 & m_1 + m_2 \end{pmatrix}} \quad (3.30)$$

with  $\begin{pmatrix} s_1 & s_2 & S \\ n_1 & n_2 & n_1 + n_2 \end{pmatrix}$  the Clebsch-Gordan coefficients. Note that we recover the algebra obtained in [37, 44]. There exists a second algebra associated to  $\mathfrak{g}(\mathrm{SU}(2))$ , that will be studied in the next section.

The KM algebra associated to  $\mathrm{SU}(2)/\mathrm{U}(1) \cong \mathbb{S}^2$  follows directly from the construction above. A point in the coset is parameterized by

$$m = \begin{pmatrix} e^{i\varphi} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & e^{-i\varphi} \cos \frac{\theta}{2} \end{pmatrix}$$

obtained by the substitution  $\psi = -\varphi$  in (3.22). From (3.14), we obtain the left-invariant vector fields

$$\omega_1 = \sin \theta \sin \varphi d\varphi - \sin \varphi d\theta, \quad \omega_2 = \sin \theta \cos \varphi d\varphi + \cos \varphi d\theta ,$$

such that the metric on the two-sphere reduces to

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 .$$

The invariant vector fields obtained by solving the Killing equation (2.1) are

$$L_{\pm} = e^{\pm i\varphi} \left( i \cot \theta \partial_{\varphi} \pm \partial_{\theta} \right) , \quad L_0 = -i\partial_{\varphi} ,$$

and generate the  $\mathfrak{so}(3)$ -Lie algebra. We finally define the scalar product

$$(f, g) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \sin \theta \, d\theta \, d\varphi \, \overline{f(\theta, \varphi)} \, g(\theta, \varphi) .$$

Following Sec. 3.2, the only representations to be considered for harmonic analysis on the two-sphere are those having matrix elements with eigenvalues  $R_0$  equal to zero (see (3.18)). This is possible *iff*  $s \in \mathbb{N}$  and for  $m = 0$ . Substituting  $m = 0$  in (3.26), we obtain ( $\ell \in \mathbb{N}$ ):

$$Y_{\ell n}(\theta, \varphi) = \frac{(-1)^{\ell + \frac{|n|+n}{2}} \sqrt{2\ell+1}}{|n|!} \sqrt{\frac{(\ell+|n|)!}{(\ell-|n|)!}} e^{in\varphi} \cos^{|n|} \frac{\theta}{2} \sin^{|n|} \frac{\theta}{2} {}_2F_1(|n|-\ell, |n|+\ell+1; 1+|n|; \sin^2 \frac{\theta}{2}) ,$$

where

$$\overline{Y^{\ell n}}(\theta, \varphi) = (-1)^n Y^{\ell-n}(\theta, \varphi) .$$

The KM algebras associated to  $\mathbb{S}^2$  follows immediately:

$$\tilde{\mathfrak{g}}(\mathrm{SU}(2)/\mathrm{U}(1)) = \left\{ \mathcal{T}_{a\ell m}, k, \quad a = 1, \dots, d, \ell \in \mathbb{N}, -\ell \leq m \leq \ell \right\} \rtimes \{L_0\}$$

with Lie brackets (see [37])

$$\begin{aligned} [\mathcal{T}_{a\ell m}, \mathcal{T}_{a'\ell' m'}] &= i f_{aa'}{}^a{}'' C_{\ell\ell' mm'}^{\ell''} \mathcal{T}_{a''m+m'\ell''} + kk_{ab}(-1)^m \delta_{\ell\ell'} \delta_{m,-m'} \\ [L_0, \mathcal{T}_{a\ell m}] &= m \mathcal{T}_{a\ell m} \end{aligned}$$

Since in the coset  $\mathrm{SU}(2)/\mathrm{U}(1)$  the matrix elements of the left action are obtained from the matrix elements of  $\mathrm{SU}(2)$ ,  $\psi_{n\ell m}$  for  $m = 0$ , the coefficients (3.30) reduce to

$$C_{\ell_1 \ell_2 n_1 n_2}^L = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{2L+1}} \begin{pmatrix} \ell_1 & \ell_2 & L \\ n_1 & n_2 & n_1+n_2 \end{pmatrix} \overline{\begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix}}$$

This algebra was obtained for the first time in [31], see also [37, 44].

### 3.5 Virasoro algebra associated to the two- and three- sphere

In this paragraph we show that, for two- and the three-spheres, an analogue of the Virasoro algebra can be constructed.

Consider first the case of  $\mathbb{S}^2$ . Let  $L$  be a vector field on  $\mathbb{S}^2$ :

$$L = A_{\varphi}(\theta, \varphi) \partial_{\varphi} + A_{\theta}(\theta, \varphi) \partial_{\theta}$$

As  $H^1(\mathbb{S}^2) = 1$ , every closed one-form on the sphere is exact. Let  $\gamma = dh$  be a closed one-form associated to the cocycle

$$\omega_h(X, Y) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \langle X, dY \rangle \wedge dh .$$

The compatibility condition between the KM algebra with two-cocycle  $\omega_h$

$$[X, Y]_h = [X, Y]_0 + \omega_h(X, Y) ,$$

where  $[ , ]_0$  are the Lie brackets without central extension, *i.e.*, the Lie brackets of  $\mathfrak{g}(\mathbb{S}^2)$ , turns out to be

$$Lh = c ,$$

with  $c$  a constant whose solutions are given by (see [33])

$$\begin{aligned} L_h^T(\theta, \varphi) &= T(\theta, \varphi)(\partial_\theta h \partial_\varphi - \partial_\varphi h \partial_\theta) , \\ \Lambda(\theta, \varphi) &= \frac{\partial_\theta h \partial_\varphi + \partial_\varphi h \partial_\theta}{\partial_\theta h \partial_\varphi h} . \end{aligned} \quad (3.31)$$

The generators  $L_h^T$  generate the area preserving diffeomorphism algebra [61, 62]. The second solution is possible if the denominator does not vanish. In particular, if we define the cocycle associated to  $L_0$  the generators of the de Witt algebra

$$\ell_{\ell m} = -Y_{\ell m}(\theta, \varphi)L_0 , \quad (3.32)$$

are compatible with the two-cocycle  $\omega_0$ . We are thus able to define the algebra

$$\begin{aligned} &\tilde{\mathfrak{g}}'(\mathrm{SU}(2)/\mathrm{U}(1)) \rtimes \mathrm{Witt} \\ &= \left\{ \mathcal{T}_{a\ell m}, k \mid a = 1, \dots, d, \ell \in \mathbb{N}, -\ell \leq m \leq \ell \right\} \rtimes \left\{ \ell_{\ell m} \mid \ell \in \mathbb{N}, -\ell \leq m \leq \ell \right\} \end{aligned}$$

with the following Lie brackets:

$$\begin{aligned} [\mathcal{T}_{a\ell m}, \mathcal{T}_{a'\ell' m'}] &= i f_{aa'}{}^{a''} C_{\ell\ell'mm'}^{\ell''} \mathcal{T}_{a''\ell''m+m'} + k_0 m' k_{ab} (-1)^m \delta_{\ell\ell'} \delta_{m,-m'} \\ [L_0, \mathcal{T}_{a\ell m}] &= m \mathcal{T}_{a\ell m} \end{aligned} \quad (3.33)$$

A similar analysis holds for  $\mathbb{S}^3$ . As  $H^2(\mathbb{S}^3) = 1$ , a closed one-form is exact. Let  $(u_1, u_2, u_3) = (\varphi, \psi, \cos \theta)$  and let  $h(u) = h_i(u)du^i$ . The corresponding two-cocycle associated to the closed form  $dh$  is given by

$$\omega_h(X, Y) = \int_{\mathbb{S}^3} \langle X, dY \rangle \wedge dh .$$

From  $d(XdY \wedge h) = dX \wedge dY \wedge h - XdY \wedge dh$  and since  $\mathbb{S}^3$  has no boundary, we have

$$\begin{aligned} \omega_h(X, Y) &= \int_{\mathbb{S}^3} \langle X \wedge dY \rangle \wedge dh = \int_{\mathbb{S}^3} \langle dX \wedge dY \rangle \wedge h \\ &= \int_{\mathbb{S}^3} \left( h_1 \{X, Y\}_{23} + h_2 \{X, Y\}_{31} + h_3 \{X, Y\}_{12} \right) du_1 du_2 du_3 , \end{aligned}$$

where

$$\{X, Y\}_{ij} = \partial_i X \partial_j Y - \partial_j X \partial_i Y .$$

It can be shown that the vector field (see also Eq.[2.33])

$$L_h = A(u_1, u_2, u_3) \left( (\partial_1 h_2 \partial_3 - \partial_3 h_2 \partial_1) + (\partial_2 h_3 \partial_1 - \partial_1 h_3 \partial_2) + (\partial_3 h_1 \partial_2 - \partial_2 h_1 \partial_3) \right) ,$$

is compatible with the two-cocycle  $\omega_h$  [33]. The algebra generated by the operators of the form  $L_h$  above are called area preserving diffeomorphisms in [61, 62]. We are thus able to define (pay attention to the fact that, in this case, we only have one central charge, say  $k_L = k$ )

$$\tilde{\mathfrak{g}}'(\mathrm{SU}(2)) \rtimes \mathrm{Witt}_{L_0} = \\ \left\{ \mathcal{T}_{ansm}, k, \quad a = 1, \dots, d, s \in \frac{1}{2}\mathbb{N}, -s \leq n, m \leq s \right\} \rtimes \left\{ \ell_{msn}, s \in \frac{1}{2}\mathbb{N}, -s \leq n, m \leq s \right\}$$

where

$$\ell_{msn} = -\psi_{msn}(\varphi, \psi, \theta)L_0$$

and with Lie brackets:

$$\begin{aligned} [\mathcal{T}_{ansm}, \mathcal{T}_{a'n's'm'}] &= if_{aa'}^{a''} C_{ss'nn'mm'}^{s''} \mathcal{T}_{a''n+n's''m+m'} + n'k k_{aa'} (-1)^{m+n} \delta_{ss'} \delta_{n,-n'} \delta_{m,-m'} \\ [\ell_{nsn}, \mathcal{T}_{a'n's'm'}] &= -n' C_{ss'nn'mm'}^{s''} \mathcal{T}_{an+n's''m+m'} \\ [\ell_{nsn}, \ell_{n's'm'}] &= (n-n') C_{ss'nn'mm'}^{s''} \ell_{n+n's''m+m'} \end{aligned} \quad (3.34)$$

We now show that the de Witt algebra of the two- and three-sphere can be centrally extended. Before analysing possible central extensions, let us make several observations. A basis of the algebra of vector fields  $\mathfrak{X}(\mathbb{S}^2)$  is given by  $\{X_{\ell m}^\varphi = Y_{\ell m} \partial_\varphi, X_{\ell m}^\theta = Y_{\ell m} \partial_\theta, \ell \in \mathbb{N}, -\ell \leq m \leq \ell\}$ . Several subalgebras of  $\mathfrak{X}(\mathbb{S}^2)$  with different properties may be considered. For instance, Floratos and Iliopoulos studied the area preserving diffeomorphism algebra which appears in the analysis of bosonic membranes. If the membrane has the topology of the two-sphere, this algebra is generated by the vector fields of the form [61]  $L_h = \partial_{\cos\theta} h(\theta, \varphi) \partial_\varphi - \partial_\varphi h(\theta, \varphi) \partial_{\cos\theta}$  and satisfy

$$[V_{h_1}, V_{h_2}] = V_{\{h_2, h_1\}}, \quad (3.35)$$

where  $\{h_1, h_2\} = \partial_{\cos\theta} h_1(\theta, \varphi) \partial_\varphi h_2(\theta, \varphi) - \partial_{\cos\theta} h_2(\theta, \varphi) \partial_\varphi h_1(\theta, \varphi)$  is the Poisson bracket. In particular, we have for  $V_{\ell m} = \partial_{\cos\theta} Y_{\ell m}(\theta, \varphi) \partial_\varphi - \partial_\varphi Y_{\ell m}(\theta, \varphi) \partial_{\cos\theta}$

$$[V_{\ell_1 m_1}, V_{\ell_2 m_2}] = -g_{\ell_1 m_1 \ell_2 m_2}^{\ell_3 m_3} V_{\ell_3 m_3}, \quad (3.36)$$

where the structure constants

$$\{Y_{\ell_1 m_1}, Y_{\ell_2 m_2}\} = g_{\ell_1 m_1 \ell_2 m_2}^{\ell_3 m_3} Y_{\ell_3 m_3},$$

are given in [63]. The second relevant algebra is that associated to the compatibility of the cocycle  $\omega_h$  [33]. This algebra is generated by the vector fields of the form  $L_h^T = T(\theta, \varphi) (\partial_{\cos\theta} h(\theta, \varphi) \partial_\varphi - \partial_\varphi h(\theta, \varphi) \partial_{\cos\theta})$ :

$$[L_h^{T_1}, L_h^{T_2}] = L_h^{T_3}, \quad T_3 = T_1 \{T_2, h\} - T_2 \{T_1, h\}. \quad (3.37)$$

The difference between the algebras given in (3.35) and (3.37) is that, for the former, the function  $h$  in  $V_h$  varies, whilst in the latter, the function  $h$  in  $V_h^T$  is fixed. It has been shown in [62] that the algebra (3.35), or the area preserving diffeomorphism algebra, does not admit central extensions. Differently, we now show that the algebra (3.37) for  $h(\theta, \varphi) = Y_{\ell m}(\theta, \varphi)$  admits a central extension. Introduce the spherical harmonics in the form:

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos\theta) e^{im\varphi} = Q_{\ell m}(\cos\theta) e^{im\varphi}, \quad (3.38)$$

where  $P_{\ell m}$  are the associated Legendre functions, satisfying the orthonormality property for  $\ell_1, \ell_2 \geq |m|$ :

$$(Q_{\ell_1 m}, Q_{\ell_2 m}) = \frac{1}{2} \int_0^\pi \sin \theta d\theta Q_{\ell_2 m}(\cos \theta) Q_{\ell_1 m}(\cos \theta) = \delta_{\ell_1 \ell_2}. \quad (3.39)$$

We also have

$$Q_{\ell_1 m_1}(\cos \theta) Q_{\ell_2 m_2}(\cos \theta) = C_{\ell_1 \ell_2 m_1 m_2}^\ell Q_{\ell m_1 + m_2}(\cos \theta), \quad (3.40)$$

and

$$Q_{\ell -m}(\cos \theta) = (-1)^m Q_{\ell m}(\cos \theta), \quad (3.41)$$

which are a direct consequence of analogous relations for  $Y_{\ell m}$  and from the definition of  $Q_{\ell m}$  (see (3.38)). Let

$$L(\theta, \varphi) = \sum_{m=-\infty}^{+\infty} \left( \sum_{\ell=|m|}^{+\infty} L_{\ell m} Q_{\ell m}(\cos \theta) \right) e^{im\varphi} = \sum_{m=-\infty}^{+\infty} L_m(\theta) e^{im\varphi}.$$

We further assume that the  $L_m(\theta)$  satisfy the relations

$$[L_m(\theta), L_n(\theta')] = \left( (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \right) \delta(\cos \theta - \cos \theta') \quad (3.42)$$

Now, using the orthonormality relation (3.39), (3.40) and (3.41), integration by

$$\frac{1}{2} \int du Q_{\ell m}(u) \frac{1}{2} \int du' Q_{\ell' m'}(u'), \quad u = \cos \theta, u' = \cos \theta'$$

leads to:

$$[L_{\ell_1 m_1}, L_{\ell_2 m_2}] = (m-n)C_{\ell_1 \ell_2 m_1 m_2}^{\ell_3} L_{\ell_3 m_1 + m_2} + (-1)^{m_1} \frac{c}{12} m_1 (m_1^2 - 1) \delta_{m_1, -m_2} \delta_{\ell_1 \ell_2}$$

We now check that the Jacobi identities are satisfied using (3.42). Indeed,

$$\begin{aligned} & [[L_{m_1}(\theta_1), L_{m_2}(\theta_2)], L_{m_3}(\theta_3)] \\ & + [[L_{m_2}(\theta_2), L_{m_3}(\theta_3)], L_{m_1}(\theta_1)] \\ & + [[L_{m_3}(\theta_3), L_{m_1}(\theta_1)], L_{m_2}(\theta_2)] \\ & = -\frac{c}{12} (m_2 - m_3)(m_1 - m_3)(m_1 - m_2)(m_1 + m_3 + m_2) \\ & \quad \delta_{m_1 + m_2 + m_3, 0} \delta(\cos \theta_1 - \cos \theta_2) \delta(\cos \theta_2 - \cos \theta_3) = 0 \end{aligned}$$

where we have used  $(m_1 - m_2)(m_3^3 - m_3) + (m_2 - m_3)(m_1^3 - m_1) + (m_3 - m_3)(m_1^3 - m_1) = (m_2 - m_3)(m_1 - m_3)(m_1 - m_2)(m_1 + m_3 + m_2)$ .

Thus altogether, the algebra

$$\begin{aligned} & \tilde{\mathfrak{g}}'(\mathbb{S}^2) \rtimes \text{Vir}(\mathbb{S}^2) = \\ & \left\{ \mathcal{T}_{a\ell m}, k, \quad a = 1, \dots, d, \ell \in \mathbb{N}, -\ell \leq m \leq \ell \right\} \rtimes \left\{ L_{\ell m}, \ell \in \mathbb{N}, -\ell \leq m \leq \ell \right\} \end{aligned}$$

has Lie brackets:

$$\begin{aligned} [\mathcal{T}_{a\ell m}, \mathcal{T}_{a'\ell' m'}] &= i f_{aa'}^{a''} C_{\ell\ell' mm'}^{\ell''} \mathcal{T}_{a''\ell'' m+m'} + k_0 m' k_{ab} (-1)^m \delta_{\ell\ell'} \delta_{m,-m'} \\ [\mathcal{L}_{\ell_1 m_1}, \mathcal{L}_{\ell_2 m_2}] &= (m-n)C_{\ell_1 \ell_2 m_1 m_2}^{\ell_3} L_{\ell_3 m_1 + m_2} + (-1)^{m_1} \frac{c}{12} m_1 (m_1^2 - 1) \delta_{m_1 + m_2} \delta_{\ell_1 \ell_2} \\ [\mathcal{L}_{\ell m}, \mathcal{T}_{a_1 \ell_1 m_1}] &= -m_1 C_{\ell\ell_1 mm_1}^{\ell'} \mathcal{T}_{a_1 \ell' m+m_1}, \end{aligned} \quad (3.43)$$

where now  $L_0 = -L_{00}$ .

It remains to be shown that the algebra above is the *only* centrally extended Lie algebra obtained from  $\mathfrak{g}(\mathbb{S}^2)$  and  $\mathfrak{X}(\mathbb{S}^2)$  that one can consistently define. Recall that  $\mathfrak{g}(\mathbb{S}^2) \rtimes \mathfrak{X}(\mathbb{S}^2) = \{T_{alm}, a = 1, \dots, \dim \mathfrak{g}, \ell \in N, -\ell \leq m \leq \ell\} \rtimes \{X_{\ell m}^\theta = Y_{\ell m}(\theta, \varphi)\partial_\theta, X_{\ell m}^\varphi = Y_{\ell m}(\theta, \varphi)\partial_\varphi, \ell \in \mathbb{N}, -\ell \leq m \leq \ell\}$ , which contains as a subalgebra  $\mathfrak{g}(\mathbb{S}^2) \rtimes \{L_0, L_+, L_-\}$  (where  $\{L_0, L_+, L_-\}$  are the algebra associated with the isometry of  $\mathbb{S}^2$ ), has Lie brackets:

$$\begin{aligned} [T_{a_1 \ell_1 m_1}, T_{a_2 \ell_2 m_2}] &= i f_{a_1 a_2}{}^{a_3} C_{\ell_1 \ell_2 m_1 m_2}^{\ell_3} T_{a_3 \ell_3 m_1 + m_2} \\ [L_0, T_{a_1 \ell_1 m_1}] &= m_1 T_{a_1 \ell_1 m_1} \\ [L_\pm, T_{a_1 \ell_1 m_1}] &= \sqrt{(\ell_1 \mp m_1)(\ell_1 \pm m_1 + 1)} T_{a_1 \ell_1 m_1 \pm \pm 1} \end{aligned}$$

The algebra  $\mathfrak{g}(\mathbb{S}^2)$  admits a infinite number of central extensions [49]. Explicit expressions can be found in [33, 35, 36]. Now the compatibility of the algebra which centrally extends  $\mathfrak{g}(\mathbb{S}^2)$  and  $\mathfrak{X}(\mathbb{S}^2)$  is two-fold: on the one hand only one cocycle can be considered, say  $\omega_h$ . On the other hand, only the vector fields (3.31) are compatible with  $\omega_h$ . If we now centrally extend  $\mathfrak{g}(\mathbb{S}^2)$  introducing the two-cocyle  $\omega_0$ , the only generators of the first algebra which are compatible with  $\omega_0$  are the operators  $\ell_{\ell m}$  (see (3.32)) and the only operator of the second algebra which is compatible with  $\omega_0$  is  $L_0$ <sup>3</sup>:

$$\begin{aligned} &\{\mathcal{T}_{a \ell m}, a = 1, \dots, \dim \mathfrak{g}, \ell \in N, -\ell \leq m, k_0 \leq \ell\} \rtimes \{L_0\} \\ &\subset \{\mathcal{T}_{a \ell m}, a = 1, \dots, \dim \mathfrak{g}, \ell \in N, -\ell \leq m, k_0 \leq \ell\} \rtimes \{\ell_{\ell m}, \ell \in N, -\ell \leq m, k_0 \leq \ell\}. \end{aligned}$$

Finally if we centrally extend the de Witt algebra  $\{\ell_{\ell m}, \ell \in N, -\ell \leq m \leq \ell\}$ , we obtain the Virasoro algebra of the two-sphere. Thus, in conclusion, the algebra  $\tilde{\mathfrak{g}}'(\mathbb{S}^2) \rtimes \text{Vir}(\mathbb{S}^2)$  with brackets given in (3.43) is the only centrally extended algebra that one can define along these lines.

The analysis for the three-sphere is similar. Starting from

$$\begin{aligned} L(\theta, \varphi, \psi) &= \sum_{\epsilon=0,1/2} \sum_{\ell \in \mathbb{N}+\epsilon} \sum_{m,n=-\ell}^{\ell} L_{n \ell m} \psi_{n \ell m}(\theta, \varphi, \psi) \\ &= \sum_{\epsilon=0,1/2} \sum_{n \in \mathbb{Z}+\epsilon} \left( \sum_{\ell \geq |n|} \sum_{m=-\ell}^{\ell} L_{n \ell m} F_{\ell m}(\theta, \varphi) \right) e^{i n \psi} \\ &\equiv \sum_{\epsilon=0,1/2} \sum_{n \in \mathbb{Z}+\epsilon} L_n(\theta, \varphi) e^{i n \psi} \end{aligned}$$

We have introduced the functions  $F_{n \ell m}$  defined by  $\psi_{nsm}(\theta, \varphi, \psi) = F_{nsm}(\theta, \varphi) e^{i n \psi}$ . Thus, if we assume:

$$[L_n(\theta, \varphi), L_{n'}(\theta', \varphi')] = \left( (n - n') L_{n+n'}(\theta, \varphi) + \frac{c}{12} (n^3 - n) \delta_{n,-n'} \right) \frac{\delta(\theta - \theta') \delta(\varphi - \varphi')}{\sin \theta'}$$

as for the two-sphere, we obtain the algebra

$$\begin{aligned} \tilde{\mathfrak{g}}'(\text{SU}(2)) \rtimes \text{Vir}(\text{SU}(2)) &= \\ \left\{ \mathcal{T}_{ansm}, k, \quad a = 1, \dots, d, s \in \frac{1}{2}\mathbb{N}, -s \leq n, m \leq s \right\} &\rtimes \left\{ L_{msn}, s \in \frac{1}{2}\mathbb{N}, -s \leq n, m \leq s \right\} \end{aligned}$$

where

$$\ell_{msn} = -\psi_{msn}(\varphi, \psi, \theta) L_0$$

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<sup>3</sup>Strictly speaking the operator  $R_0$  is also compatible with  $\omega_0$ .

and with Lie brackets:

$$\begin{aligned} [\mathcal{T}_{ansm}, \mathcal{T}_{a'n's'm'}] &= i f_{aa'}^{a''} C_{ss'nn'mm'}^{s''} \mathcal{T}_{a''n+n's''m+m'} + n' k k_{aa'} (-1)^{m+n} \delta_{ss'} \delta_{n,-n'} \delta_{m,-m'} \\ [L_{nsm}, \mathcal{T}_{a'n's'm'}] &= -n' C_{ss'nn'mm'}^{s''} \mathcal{T}_{an+n's''m+m'} \\ [L_{nsm}, L_{n's'm'}] &= (n - n') C_{ss'nn'mm'}^{s''} \mathbb{L}_{n+n's''m+m'} + \frac{c}{12} (n^3 - n) \delta_{ss'} \delta_{n,-n'} \delta_{m,-m'} . \end{aligned}$$

We now have  $L_0 = -L_{000}$ . This algebra has been defined in [41].

## 4 KM algebras on non-compact Lie groups

The purpose of this section is to extend the results of the previous section to the case of non-compact manifolds.

### 4.1 Some generalities

Let  $G_{nc}$  be a non-compact Lie group and let  $\mathfrak{g}_{nc} = \{J_A, A = 1, \dots, n\}$  be its corresponding Lie algebra. In contrast to compact Lie algebras, where the Killing form is positive definite<sup>4</sup>, for non-compact Lie algebras  $\mathfrak{g}_{nc}$ , the Killing form has signature  $(n_+, n_-)$  with  $n = n_+ + n_-$ . In spite of this difference, the metric  $g_{MN}$  on  $G_{nc}$  is obtained in a similar way than the metric on  $G_c$  (see Eqs.[3.1-3.5]). Since in this case the manifold  $G_{nc}$  is non-compact, its volume is infinite (the volume is finite for a compact Lie group  $G_c$ ). Thus we define the scalar product for  $f, g \in L^2(G_{nc})$  by:

$$(f, g) = \int_{G_{nc}} \sqrt{g} \, dm^n \overline{f(m)} g(m) \quad (4.1)$$

for the variables  $m$ , with the same notations as in Sect. 3.1. As it holds for compact manifolds, when solving the Killing equations, the generators of the left (resp. right) action and  $\mathfrak{g}_\xi = (\mathfrak{g}_{nc})_L \oplus (\mathfrak{g}_{nc})_R$  corresponding of the left/right action of  $G_{nc}$  automatically appear as solutions. For the corresponding generators, we adopt here the same notation used for  $G_c$  (see (3.9)).

The next step in the construction of a KM algebra associated to  $G_{nc}$  is to decompose square integrable functions on  $L^2(G_{nc})$ . However, the situation for non-compact Lie groups is very different than the situation for compact Lie groups. The first difference resides in the unitary representations of  $G_{nc}$ . Due to non-compactness, unitary representations are infinite dimensional (recall that for a compact Lie group, all unitary representations are finite dimensional). Next, there exist two types of unitary representations: the discrete series and the continuous series [64]. A non-compact Lie group admits always continuous series as unitary representation, but the former exist *iff* the rank of the non-compact group is equal to the rank of its maximal compact subgroup. The discrete series are characterized by discrete eigenvalues of the Casimir operators, whereas the continuous series have continuous eigenvalues of the Casimir operators. This in particular means that if we realize the Lie algebra on the manifold  $G_{nc}$ , the continuous series is non normalizable, whilst the discrete series is normalizable. Hence, the harmonic analysis on  $L^2(G_{nc})$  is more involved in this case, and is summarized in the Plancherel theorem [48]. This theorem basically states that any square integrable functions on  $G_{nc}$  decomposes as a sum over the matrix elements of the discrete series and an integral over the matrix

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<sup>4</sup>We use the physicists' notation, in which the Lie brackets have an additional  $i$ -factor. Thus, for unitary representations the generators of the Lie algebra  $\mathfrak{g}$  are Hermitian, and the Killing form is definite positive for a compact semisimple Lie algebra.

elements of the continuous series. However, since we are considering Hermitian operators with continuous spectra their eigenfunctions are not normalizable. This means that some care must be taken. We thus consider a Gel'fand triple of a Hilbert space  $\mathcal{H}$  (see for instance [65]). Let  $\mathcal{S}$  be a dense subspace of smooth functions of  $\mathcal{H}$  and its dual  $\mathcal{S}'$

$$\mathcal{S} \overset{J}{\subset} \mathcal{H} \overset{K}{\subset} \mathcal{S}' , \quad (4.2)$$

where  $J : \mathcal{S} \rightarrow \mathcal{H}$  is an injective bounded operator with dense image and  $K$  is the composition of the canonical isomorphism  $\mathcal{H} \simeq \mathcal{H}'$  determined by the inner product (given by the Riesz theorem) and the dual  $J' : \mathcal{H}' \rightarrow \mathcal{S}'$  of  $J$ . The set  $\mathcal{S}$  we are considering here is the space of rapidly decreasing functions in any non-compact directions of  $G_{nc}$  or, for short, Schwartz functions. Schwartz functions (which generalize the well-known set of Schwartz functions for  $\mathbb{R}$ ) were defined for any semisimple Lie group by Harish-Chandra (see also [66] and references therein). With these definitions, it turns out that the matrix elements of the continuous series belong to  $\mathcal{S}'$ , and a closed expression of the Plancherel Theorem can be given (see for instance [65], Theorem 1, p. 426, and Eqs.[34-37] p. 429, or [66] Chap. 8).

We are considering also a second way to expand any square integrable functions on  $G_{nc}$ , by identifying a Hilbert basis of  $L^2(G_{nc})$ . This is always possible, since any Hilbert space admits a Hilbert basis (*i.e.* a countable, complete set of orthonormal vectors) [46].

Then, following the steps of section 3.3, one can naturally associate to  $G_{nc}$  a corresponding KM algebra. In this review we don't consider a generic non-compact manifold  $G_{nc}$ , but only focus on specific  $G_{nc}$ , namely  $\mathrm{SL}(2, \mathbb{R})$ . There are two isomorphic presentations of the KM algebra associated to  $\mathrm{SL}(2, \mathbb{R})$ . The first one is based on the Plancherel theorem and involves integrals and Dirac  $\delta$ -distributions in its Lie brackets. The second is based on the identification of a Hilbert basis of  $L^2(G_{nc})$ , and involves only sums and Kronecker symbols in its Lie brackets. Since we are only considering Schwartz functions throughout, both presentations are isomorphic.

## 4.2 A KM algebra associated to $\mathrm{SL}(2, \mathbb{R})$

In this section we give with some details about the construction of the KM algebra associated to  $\mathrm{SL}(2, \mathbb{R})$ . At the end of the section we briefly comment the construction associated to  $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ .

### 4.2.1 The group $\mathrm{SL}(2, \mathbb{R})$

The group  $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$  is defined by the set of  $2 \times 2$  complex matrices

$$\begin{aligned} \mathrm{SL}(2, \mathbb{R}) &= \left\{ U = \begin{pmatrix} z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad z_1, z_2 \in \mathbb{C}: \quad |z_1|^2 - |z_2|^2 = 1 \right\} \\ &\cong \left\{ z_1, z_2 \in \mathbb{C}^2 : |z_1|^2 - |z_2|^2 = 1 \right\} \equiv \mathbb{H}_{2,2} , \end{aligned} \quad (4.3)$$

where  $\mathbb{H}_{2,2}$  is the hyperboloid which can be parameterized as follows:

$$z_1 = \cosh \rho e^{i\varphi_1}, \quad z_2 = \sinh \rho e^{i\varphi_2}, \quad \rho \geq 0, 0 \leq \varphi_1, \varphi_2 < 2\pi .$$

From this parameterization we obtain the left-invariant one-forms (see (3.3))

$$\begin{aligned} \lambda_0 &= \cosh^2 \rho d\varphi_1 - \sinh^2 \rho d\varphi_2 \\ \lambda_1 &= \sin(\varphi_1 + \varphi_2) d\rho - \sinh \rho \cosh \rho \cos(\varphi_1 + \varphi_2) (d\varphi_1 - d\varphi_2) \\ \lambda_2 &= -\cos(\varphi_1 + \varphi_2) d\rho - \sinh \rho \cosh \rho \sin(\varphi_1 + \varphi_2) (d\varphi_1 - d\varphi_2) . \end{aligned}$$

Thus the metric tensor is:

$$ds^2 = \lambda_0^2 - \lambda_2^2 - \lambda_3^2 = -d\rho^2 + \cosh^2 \rho d\varphi_1^2 - \sinh^2 \rho d\varphi_2^2.$$

The generators of left/right action of  $\mathfrak{sl}(2, \mathbb{R})$ , obtained by solving the Killing equation (2.1), are

$$\begin{aligned} L_{\pm} &= \frac{1}{2} e^{i(\varphi_1 \mp \varphi_2)} \left[ i \tanh \rho \partial_1 \pm \partial_\rho - i \coth \rho \partial_2 \right], & L_0 &= \frac{i}{2} (\partial_2 - \partial_1), \\ R_{\pm} &= \frac{1}{2} e^{\pm i(\varphi_1 + \varphi_2)} \left[ -i \tanh \rho \partial_1 \mp \partial_\rho - i \coth \rho \partial_2 \right], & R_0 &= -\frac{i}{2} (\partial_2 + \partial_1), \end{aligned}$$

and satisfy the commutation relations

$$\begin{aligned} [L_0, L_{\pm}] &= \pm L_{\pm}, & [L_+, L_-] &= -2L_0, & [L_a, R_b] &= 0, \\ [R_0, R_{\pm}] &= \pm R_{\pm}, & [R_+, R_-] &= -2R_0, \end{aligned}$$

The Casimir operator is given by

$$C_2 = \frac{1}{2} \coth(2\rho) \partial_\rho + \frac{1}{4} \partial_\rho^2 - \frac{1 - \tanh^2 \rho}{4} \partial_1^2 + \frac{\coth^2 \rho - 1}{4} \partial_2^2.$$

and the scalar product on  $\mathbb{H}_{2,2}$  reduces to:

$$(f, g) = \frac{1}{4\pi^2} \int_{\mathbb{H}_{2,2}} \cosh \rho \sinh \rho d\rho d\varphi_1 d\varphi_2 \overline{f(\rho, \varphi_1, \varphi_2)} g(\rho, \varphi_1, \varphi_2). \quad (4.4)$$

We recall that unitary representations  $\mathcal{D}_\Lambda = \{|\Lambda, n\rangle, n \in I_\Lambda\}$  of  $\mathfrak{sl}(2, \mathbb{R})$  were classified by Bargmann [18] (see also [42]):

$$\begin{aligned} L_0 |\Lambda, n\rangle &= n |\Lambda, n\rangle, \\ L_+ |\Lambda, n\rangle &= \vartheta \sqrt{(n + \Lambda)(n - \Lambda + 1)} |\Lambda, n + 1\rangle, \\ L_- |\Lambda, n\rangle &= \vartheta \sqrt{(n - \Lambda)(n + \Lambda - 1)} |\Lambda, n - 1\rangle, \\ C_2 |\Lambda, n\rangle &= \Lambda(\Lambda + 1) |L, n\rangle, \end{aligned} \quad (4.5)$$

with the sign  $\vartheta$  being conveniently chosen for each representation (see Proposition 4.1). Unitarity restricts  $\Lambda$  and  $I_\Lambda$ :

**Proposition 4.1** *Unitary representations are:*

1.  $\mathcal{D}_\lambda^+$ : discrete series bounded from below  $\Lambda = \lambda, \lambda \in \mathbb{N} \setminus \{0\}$  or  $\frac{1}{2} + \mathbb{N}$  and  $I_\lambda = \{n \geq \lambda\}$ ;
2.  $\mathcal{D}_\lambda^-$ : discrete series bounded from above  $\Lambda = \lambda, \lambda \in \mathbb{N} \setminus \{0\}$  or  $\frac{1}{2} + \mathbb{N}$  and  $I_\lambda = \{n \leq -\lambda\}$ ;
3.  $\mathcal{C}_{i\sigma}^\epsilon$ : principal continuous series  $\Lambda = \frac{1}{2} + \frac{i}{2}\sigma, \sigma > 0$  and  $I_{i\sigma} = \mathbb{Z}$  or  $\frac{1}{2} + \mathbb{Z}$  ( $n \in \frac{1}{2} + \mathbb{Z}$  or  $n \in \mathbb{Z}$ );
4.  $\mathcal{C}_\sigma$ : supplementary continuous series  $\Lambda = \frac{1}{2} + \frac{\sigma}{2}, 0 < \sigma^2 < 1$  and  $I_\sigma = \mathbb{Z}$  ( $n \in \mathbb{Z}$ ).

The sign  $\vartheta$  can be taken equal to 1, however, conveniently we take  $\vartheta = 1$  for the discrete series bounded from below, as well as for the (principal and supplementary) continuous series, whilst  $\vartheta = -1$  for the discrete series bounded from above.

The discrete series  $\mathcal{D}_\lambda^+$  is bounded from below whilst the discrete series  $\mathcal{D}_\lambda^-$  is bounded from above. The continuous series  $\mathcal{C}_{i\sigma}^\epsilon$  and  $\mathcal{C}_\sigma$  are unbounded. The eigenvalue of  $C_2$  is discrete for the two-discrete series, and continuous for the two continuous series. Note that  $\mathfrak{sl}(2, \mathbb{R})$  admits discrete series because  $\mathfrak{u}(1) \subset \mathfrak{sl}(2, \mathbb{R})$  and  $\text{rk } \mathfrak{u}(1) = \text{rk } \mathfrak{sl}(2, \mathbb{R})$ .

### 4.2.2 Matrix elements of $\mathfrak{sl}(2, \mathbb{R})$ and Plancherel Theorem

The Plancherel Theorem only involves the two discrete series and the principal continuous series. Since the supplementary continuous series plays no role (see [48, 67, 68]), we hereafter only consider discrete and continuous principal series. The corresponding matrix matrix elements are denoted  $\psi_{m\bar{\Lambda}n}$  or, more specifically: for the discrete series  $\psi_{m\lambda n}^\eta$  where  $\bar{\Lambda} = (\lambda, \eta)$ ,  $\eta = \pm$ ,  $\lambda > 1/2$ ,  $\eta m, \eta n \geq \lambda$ , for the discrete series bounded from below ( $\eta = 1$ ) above ( $\eta = -1$ <sup>5</sup>), and for the continuous principal series  $\psi_{mi\sigma n}^\epsilon$  where  $\bar{\Lambda} = (\frac{1}{2} + \frac{i}{2}\sigma, \epsilon)$ ,  $\sigma > 0$ ,  $\epsilon = 0, 1/2$ ,  $n, m \in \mathbb{Z} + \epsilon$ , for the continuous principal bosonic (resp. fermionic)  $\epsilon = 0$  (resp.  $\epsilon = 1/2$ ). These matrix elements are obtained solving the differential equations

$$\begin{aligned} L_0 \psi_{m\Lambda n}(\rho, \varphi_1, \varphi_2) &= m \psi_{m\Lambda n}(\rho, \varphi_1, \varphi_2) \\ R_0 \psi_{m\Lambda n}(\rho, \varphi_1, \varphi_2) &= n \psi_{m\Lambda n}(\rho, \varphi_1, \varphi_2) \\ C_2 \psi_{m\Lambda n}(\rho, \varphi_1, \varphi_2) &= \Lambda(\Lambda + 1) \psi_{m\Lambda n}(\rho, \varphi_1, \varphi_2). \end{aligned}$$

We can unify all matrix elements in the following form (for  $m \geq n$ )

$$\begin{aligned} \psi_{n\bar{\Lambda}m}(\rho, \varphi_1, \varphi_2) &= \frac{\mathcal{N}}{(m-n)!} \sqrt{\frac{\Gamma(\vartheta m + 1 - \Lambda)}{\Gamma(\vartheta n + 1 - \Lambda)}} e^{i(m+n)\varphi_1 + i(m-n)\varphi_2} \\ &\quad \cosh^{-\vartheta(m+n)} \rho \sinh^{\vartheta(m-n)} \rho {}_2F_1(-\vartheta n + \Lambda, -\vartheta n - \lambda + 1; 1 + \vartheta(m-n); -\sinh^2 \rho) \end{aligned}$$

where  $\bar{\Lambda} = (\Lambda = \lambda, +)$ ,  $(\Lambda = \lambda, -)$  with  $\lambda > 1/2$  for the discrete series and  $\bar{\Lambda} = (\Lambda = \frac{1}{2} + \frac{i}{2}\sigma, 0)$ ,  $(\Lambda = \frac{1}{2} + \frac{i}{2}\sigma, \frac{1}{2})$  with  $\sigma > 0$  for the continuous series. We have defined  $\vartheta = 1$  for the discrete series bounded from below and the continuous series and  $\vartheta = -1$  for the discrete series bounded from above, and  $\mathcal{N} = \sqrt{2(2\lambda - 1)}$  (resp.  $\mathcal{N} = 1$ ) for the discrete (resp. continuous series). The matrix elements for  $n \geq m$  are obtained with the substitution  $m \leftrightarrow n$  everywhere, except in the exponential factor, which is unaffected, and are multiplied by an overall factor  $(-1)^{n-m}$ . Recall that  ${}_2F_1$  is a hypergeometric function (see [42] for precise definition). The matrix elements of discrete series are thus expressed in terms of hypergeometric polynomials, whereas the matrix elements of continuous series are expressed in terms of hypergeometric functions [42]. They are normalized such that

$$\begin{aligned} (\psi_{m_1\lambda_1 n_1}^{\eta_1}, \psi_{m_2\lambda_2 n_2}^{\eta_2}) &= \delta^{\eta_1\eta_2} \delta_{\lambda_1\lambda_2} \delta_{m_1m_2} \delta_{n_1n_2} \quad \text{discrete series} \\ \psi_{mi\sigma n}^\epsilon(0) &= \delta_{mn} \quad \text{principal continuous series} \end{aligned} \tag{4.6}$$

and satisfy (4.5) for the left and right action. As stated previously, the matrix elements of the discrete series are normalizable (see (4.6)), but satisfy

$$\psi_{m\lambda n}^\eta(0) = \sqrt{2(2\lambda - 1)} \delta_{mn}, \tag{4.7}$$

whilst the matrix elements of continuous principal series are not normalizable and satisfy

$$(\psi_{ni\sigma m}^\epsilon, \psi_{n'i\sigma' m'}^{\epsilon'}) = \frac{1}{\sigma \tanh \pi(\sigma + i\epsilon)} \delta_{\epsilon\epsilon'} \delta_{mm'} \delta_{nn'} \delta(\sigma - \sigma'), \tag{4.8}$$

(see [42]). Finally, we have

$$\begin{aligned} \overline{\psi_{\eta}^{m\lambda n}}(\rho, \varphi_1, \varphi_2) &= \psi_{-n\lambda-m}^{-\eta}(\rho, \varphi_1, \varphi_2), \\ \overline{\psi_{\epsilon}^{mi\sigma n}}(\rho, \varphi_1, \varphi_2) &= \psi_{-m\sigma-n}^{\epsilon}(\rho, \varphi_1, \varphi_2). \end{aligned} \tag{4.9}$$

Let  $\mathcal{S}$  be the set of Schwartz (or rapidly decreasing) functions in the  $\rho$ -direction and let  $\mathcal{S}'$  be its dual (see (4.2)). The asymptotic (when  $\rho \rightarrow +\infty$ ) expansion of the matrix

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<sup>5</sup>Unitarity of  $\mathfrak{sl}(2, \mathbb{R})$ -representations implies  $\lambda > 0$ , but in order to have normalizable matrix elements we now have  $\lambda > 1/2$  [18, 42].

elements of the discrete series was studied in [18, 42], and it turns out that the matrix elements  $\psi_{m\lambda n}^\eta$  belong to  $\mathcal{S}$ . Moreover, the matrix elements of the continuous principal series  $\psi_{m i \sigma n}^\epsilon$  belong to  $\mathcal{S}'$ ; see for instance [65], Theorem 1, p. 426, and Eqs.[34-37] p. 429, or [66] Chap. 8. Then, considering a function  $f \in \mathcal{S}$ , the Plancherel Theorem enables us to expand  $f$  as follows (see [68] and [67], pp. 336-337):

$$\begin{aligned} f(\rho, \varphi_1, \varphi_2) &= \sum_{\lambda > \frac{1}{2}} \sum_{m,n \geq \lambda} f_+^{n\lambda m} \psi_{n\lambda m}^+(\rho, \varphi_1, \varphi_2) + \sum_{\lambda > \frac{1}{2}} \sum_{m,n \leq -\lambda} f_-^{n\lambda m} \psi_{n\lambda m}^-(\rho, \varphi_1, \varphi_2) \\ &\quad + \int_0^{+\infty} d\sigma \sigma \tanh \pi \sigma \sum_{m,n \in \mathbb{Z}} f_0^{nm}(\sigma) \psi_{ni\sigma m}^0(\rho, \varphi_1, \varphi_2) \\ &\quad + \int_0^{+\infty} d\sigma \sigma \coth \pi \sigma \sum_{m,n \in \mathbb{Z} + \frac{1}{2}} f_{\frac{1}{2}}^{nm}(\sigma) \psi_{ni\sigma m}^{\frac{1}{2}}(\rho, \varphi_1, \varphi_2) \end{aligned} \quad (4.10)$$

i.e., as a sum over the matrix elements of the discrete series and an integral over the matrix elements of the principal continuous series. This is the Plancherel Theorem for  $\mathrm{SL}(2, \mathbb{R})$ . The components of  $f$  are given by the scalar products (4.4)

$$\begin{aligned} f_\pm^{n\lambda m} &= (\psi_{n\lambda m}^\pm, f) \\ f_{nm}^\epsilon(\sigma) &= (\psi_{ni\sigma m}^\epsilon, f). \end{aligned} \quad (4.11)$$

Introducing the symbol  $\sum$  to denote the summation over all discrete and continuous series, we can rewrite (4.10) as

$$f(\rho, \varphi_1, \varphi_2) = \sum_{\Lambda, m, n} f^{n\Lambda m} \psi_{n\Lambda, m}(\rho, \varphi_1, \varphi_2) \quad (4.12)$$

with  $\Lambda = (+, \lambda), (-, \lambda), (0, i\sigma), (1/2, i\sigma)$ .

### 4.2.3 Hilbert basis of $L^2(\mathrm{SL}(2, \mathbb{R}))$

The Plancherel Theorem gives rise to an expansion of Schwartz functions as a sum over the matrix elements of the discrete series and an integral over the matrix elements of the continuous series (see (4.10)). As any Hilbert space is known to admit a Hilbert basis [46], we would like now to identify a Hilbert basis of  $L^2(\mathrm{SL}(2, \mathbb{R}))$ . Let  $L^2(\mathrm{SL}(2, \mathbb{R})) = L^2(\mathrm{SL}(2, \mathbb{R}))^d \oplus L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$ , where  $L^2(\mathrm{SL}(2, \mathbb{R}))^d$  constitutes the set of square-integrable functions expanded within the discrete series (first line of (4.10)). By definition, the set of matrix elements of discrete series is a Hilbert basis of  $L^2(\mathrm{SL}(2, \mathbb{R}))^d$ . Differently, since the matrix elements of the continuous series are not normalizable, they don't constitute a Hilbert basis of  $L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$ . V. Losert identified for us a Hilbert basis for  $L^2(\mathrm{SL}(2, \mathbb{R}))$  (see [42]). Let  $W_{mn}$  be the eigenspaces of the operators  $L_0$  and  $R_0$

$$W_{nm} = \left\{ F \in L^2(\mathrm{SL}(2, \mathbb{R})) , \quad F(\rho, \varphi_1, \varphi_2) = e^{i(m+n)\varphi_1 + i(m-n)\varphi_2} f(\rho) \right\}.$$

For  $F \in W_{nm}$  we get

$$\begin{aligned} L_0 F(\rho, \varphi_1, \varphi_2) &= nF(\rho, \varphi_1, \varphi_2), \\ R_0 F(\rho, \varphi_1, \varphi_2) &= mF(\rho, \varphi_1, \varphi_2). \end{aligned} \quad (4.13)$$

The main idea of V. Losert is to identify a Hilbert basis of  $W_{nm}$  for each  $n$  and  $m$ . Let

$$\mathcal{B}_{nm} = \left\{ \Phi_{nnk}(\rho, \varphi_1, \varphi_2) = e^{i(m+n)\varphi_1 + i(m-n)\varphi_2} e_{nmk}(\cosh 2\rho), \quad k \in \mathbb{N} \right\}$$

be a Hilbert basis of  $W_{nm}$ . Observing that for the discrete series bounded from below (resp. above) we have  $nm > 0, n, m \geq \lambda > 1/2$  (resp.  $mn > 0, n, m \leq -\lambda < -1/2$ ) (see Proposition 4.1), three cases have to be considered for  $W_{nm}$  [42]:

1.  $mn > 0, m, n > 1/2$ :  $\mathcal{B}_{nm}$  contains matrix elements of the discrete series bounded from below, together with elements of  $L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$ .
2.  $mn > 0, m, n < -1/2$ :  $\mathcal{B}_{nm}$  contains matrix elements of the discrete series bounded from above, together with elements of  $L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$ .
3.  $mn < 0$  or  $m = 0, 1/2$  or  $n = 0, 1/2$ :  $\mathcal{B}_{nm}$  contains only elements of  $L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$ .

Let  $\epsilon = 0$  if  $m, n \in \mathbb{Z}$  and  $\epsilon = 1/2$  if  $m, n \in \mathbb{Z} + \frac{1}{2}$ , and define the condition

$$\mathcal{C} : mn > 0, |m|, |n| > 1/2 \quad (4.14)$$

which ensures that  $W_{mn} \cap L^2(\mathrm{SL}(2, \mathbb{R}))^d \neq \emptyset$ . Then set

$$k_{\min} = \begin{cases} \min(|m|, |n|) - \epsilon & \text{if } \mathcal{C} \text{ is satisfied} \\ 0 & \text{if } \mathcal{C} \text{ is not satisfied} \end{cases} \quad (4.15)$$

such that if

$$\begin{aligned} k < k_{\min} &\Rightarrow \Phi_{nmk} \in L^2(\mathrm{SL}(2, \mathbb{R}))^d \\ k \geq k_{\min} &\Rightarrow \Phi_{nmk} \in L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}, \end{aligned} \quad (4.16)$$

showing that  $W_{mn} \cap L^2(\mathrm{SL}(2, \mathbb{R}))^d = \emptyset$  when the condition (4.14) does not hold. For  $x = \cosh 2\rho$ , we have for  $n \geq m$  [42]

$$e_{nmk}(x) = \sqrt{2^{2m-1} \frac{(2m-2k-1)k!(m+n-k-1)!}{(2m-k-1)!(n-m+k)!}} \times \quad (4.17)$$

$$(x-1)^{\frac{n-m}{2}} (x+1)^{-\frac{m+n}{2}} P_k^{(n-m, -n-m)}(x) \in L^2(\mathrm{SL}(2, \mathbb{R}))^d, \quad 0 \leq k < k_{\min},$$

$$e_{nmk}(x) = \sqrt{2^{2k+2\epsilon+1} \frac{(2k+n-m+2\epsilon+1)(k+n-m+2\epsilon)!k!}{(k+2\epsilon)!(n-m+k)!}} \times \quad$$

$$(x-1)^{\frac{n-m}{2}} (x+1)^{\frac{m-n}{2}-k-\epsilon-1} P_k^{(n-m, m-n-2k-2\epsilon-1)}(x) \in L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}, \quad k \geq k_{\min},$$

where  $P_k^{(a,b)}$  are Jacobi polynomials<sup>6</sup>. Recall that  $\deg P_k^{(a,b)} = k$ . Thus,  $L^2(\mathrm{SL}(2, \mathbb{R}))^d$  involves Jacobi polynomials of degree  $< k_{\min}$  and  $L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$  involves Jacobi polynomials of degree  $\geq k_{\min}$  [42]. We have similar expressions for  $n \geq m$  and the conjugation relations:

$$\overline{\Phi_{nmk}}(\rho, \varphi_1, \varphi_2) = \Phi_{-n-mk}(\rho, \varphi_1, \varphi_2). \quad (4.18)$$

The set  $\cup_{n,m \in \mathbb{Z}} \mathcal{B}_{nm} \cup_{n,m \in \mathbb{Z} + 1/2} \mathcal{B}_{nm}$  is a complete orthonormal Hilbert basis of  $L^2(\mathrm{SL}(2, \mathbb{R}))$  called the Losert basis:

$$(\Phi_{nmk}, \Phi_{n'm'k'}) = \delta_{nn'} \delta_{mm'} \delta_{kk'} . \quad (4.19)$$

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<sup>6</sup>The matrix elements of the discrete series are expressed in terms of hypergeometric polynomials. However, one may show that these hypergeometric polynomials can be easily related to Jacobi polynomials [42].

Therefore, if  $f \in L^2(\mathrm{SL}(2, \mathbb{R}))$ , we have

$$\begin{aligned} f(\rho, \varphi_1, \varphi_2) &= \sum_{\epsilon=0,1/2} \sum_{n,m \in \mathbb{Z} + \epsilon} \sum_{k=0}^{+\infty} f^{nmk} \Phi_{nmk}(\rho, \varphi_1, \varphi_2) , \\ f_{nmk} &= (\Phi_{nmk}, f) . \end{aligned} \quad (4.20)$$

Since the  $\mathfrak{sl}(2, \mathbb{R})$  generators act on  $L^2(\mathrm{SL}(2, \mathbb{R}))$ , and since  $L^2(\mathrm{SL}(2, \mathbb{R}))^\mathrm{d} \subset L^2(\mathrm{SL}(2, \mathbb{R}))$  is a sub-representation, *i.e.*, an invariant subspace of  $L^2(\mathrm{SL}(2, \mathbb{R}))$ , it follows that  $L^2(\mathrm{SL}(2, \mathbb{R}))^\mathrm{d}^\perp$  is also a representation of  $\mathrm{SL}(2, \mathbb{R})$ . The action of  $L_\pm, R_\pm$  on  $\Phi_{nmk} \in L^2(\mathrm{SL}(2, \mathbb{R}))^\mathrm{d}^\perp$  is given by

$$\begin{aligned} L_+(\Phi_{nmk}) &= \alpha_{nmk}^{+L} \Phi_{n+1mk+1} + \beta_{nmk}^{+L} \Phi_{n+1mk} + \gamma_{nmk}^{+L} \Phi_{n+1nk-1} & \left\{ \begin{array}{ll} \gamma_{nmk}^{+L} = 0 & m > n \\ \alpha_{nmk}^{+L} = 0 & n \geq m \end{array} \right. \\ L_-(\Phi_{nmk}) &= \alpha_{nmk}^{-L} \Phi_{n-1mk+1} + \beta_{nmk}^{-L} \Phi_{n-1mk} + \gamma_{nmk}^{-L} \Phi_{n-1mk-1} & \left\{ \begin{array}{ll} \alpha_{nmk}^{-L} = 0 & m \geq n \\ \gamma_{nmk}^{-L} = 0 & n > m \end{array} \right. \\ R_+(\Phi_{nmk}) &= \alpha_{nmk}^{+R} \Phi_{nm+1k+1} + \beta_{nmk}^{+R} \Phi_{nm+1k} + \gamma_{nmk}^{+R} \Phi_{nm+1k-1} & \left\{ \begin{array}{ll} \alpha_{nmk}^{+R} = 0 & m \geq n \\ \gamma_{nmk}^{+R} = 0 & n > m \end{array} \right. \\ R_-(\Phi_{nmk}) &= \alpha_{nmk}^{-R} \Phi_{nm-1k+1} + \beta_{nmk}^{-R} \Phi_{nm-1k} + \gamma_{nmk}^{-R} \Phi_{nm-1k-1} & \left\{ \begin{array}{ll} \gamma_{nmk}^{-R} = 0 & m > n \\ \alpha_{nmk}^{-R} = 0 & n \geq m \end{array} \right. \end{aligned} \quad (4.21)$$

The action of the Casimir operator reduces to

$$C_2 \Phi_{nmk}(\rho, \varphi_1, \varphi_2) = a_{nmk} \Phi_{nmk-1}(\rho, \varphi_1, \varphi_2) + b_{nmk} \Phi_{nmk}(\rho, \varphi_1, \varphi_2) + c_{nmk} \Phi_{nmk+1}(\rho, \varphi_1, \varphi_2) .$$

This action clearly shows that this representation is unitary but *is not irreducible*. Explicit expressions of the coefficients  $\alpha, \beta$  and  $a, b, c$  are given in [42].

The asymptotic behavior of the functions  $e_{nmk} \in W_{nm}$  were studied in [42], and it turns out that the functions  $\Phi_{nmk}$  are actually Schwartz functions. Thus, for  $\Phi_{nmk} \in W_{nm} \cap L^2(\mathrm{SL}(2, \mathbb{R}))^\mathrm{d}^\perp$ , one can use the Plancherel Theorem and write (4.10)

$$\Phi_{nmk}(\rho, \varphi_1, \varphi_2) = \int_0^{+\infty} d\sigma \sigma \tanh \pi(\sigma + i\epsilon) f_{nmk}(\sigma) \psi_{ni\sigma\epsilon m}(\rho) , \quad (4.22)$$

where  $\epsilon = 0$  if  $n, m$  are integers and  $\epsilon = 1/2$  if  $n, m$  are half-integers. It should be observed that there is no summation on  $n$  and  $m$ . Furthermore, using (4.11), we have (with the scalar product (4.4))

$$f^{nmk}(\sigma) = (\psi_{ni\sigma\epsilon m}, e_{nmk}) .$$

This implies that the relation (4.22) can be inverted

$$\psi_{ni\sigma\epsilon m}(\rho, \varphi_1, \varphi_2) = \sum_{k \geq k_{\min}} \overline{f^{mnk}}(\sigma) \Phi_{nmk}(\rho, \varphi_1, \varphi_2) , \quad (4.23)$$

with  $k_{\min}$  defined in (4.16). We are thus able to express the matrix elements of the (principal) continuous series in terms of the Losert basis, and conversely. This observation is important for the construction of the algebra below.

#### 4.2.4 Clebsch-Gordan coefficients

The next step in the construction of a KM algebra associated to  $\mathrm{SL}(2, \mathbb{R})$  is the computation of the Clebsch-Gordan coefficients corresponding to the decomposition  $\mathcal{D}_\Lambda \otimes \mathcal{D}_{\Lambda'}$ .

This has been studied in [69, 70]. The coupling of two discrete series was studied by means of a bosonic realization of the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ ; for the case in which at least one continuous series is involved, the result was obtained by a cumbersome analytic continuation. An heuristic decomposition can be deduced from a back and forth. More precisely let  $\mathcal{D}_\Lambda, \mathcal{D}_{\Lambda'}$  be two representations of  $\text{SL}(2, \mathbb{R})$  in the Plancherel basis, expand the matrix elements of  $\mathcal{D}_\Lambda$  and  $\mathcal{D}_{\Lambda'}$  in the Losert basis using (4.23), subsequently decompose the product  $\Psi_{m\Lambda n} \Psi_{m'\Lambda' n'}$  in the Losert basis and then express the results back in the Plancherel basis using (4.22). For instance, consider two discrete series bounded from above:  $\mathcal{D}_\lambda^+ \otimes \mathcal{D}_{\lambda'}^+$ . The product of two matrix elements  $\psi_{n\lambda m}^+ \psi_{n'\lambda' m'}^+$  is such that  $nm > 0, n'm' > 0, n, m, n', m' > 1/2$  (see Condition 4.14). So we have  $(n+n')(m+m') > 0, (n+n'), (m+m') > 1/2$ . Thus the product  $\mathcal{D}_\lambda^+ \otimes \mathcal{D}_{\lambda'}^+$  decomposes only in the discrete series bounded from above. The situation is very different for the product  $\mathcal{D}_{\lambda_+}^+ \otimes \mathcal{D}_{\lambda_-}^-$  which involves both matrix elements of discrete series and continuous series. Indeed, the product of two matrix elements  $\psi_{m+\lambda+n_+}^+ \psi_{m-\lambda-n_-}^-$  is such that  $m_\pm n_\pm > 0$  and  $\pm m_\pm, \pm n_\pm \geq \lambda_\pm$ . So it may happen that  $(m_+ + m_-)(n_+ + n_-) < 0$  (depending of the respective value of  $m_\pm$  and  $n_\pm$ ), and thus  $\psi_{m+\lambda+n_+}^+ \psi_{m-\lambda-n_-}^-$  belongs to  $L^2(\text{SL}(2, \mathbb{R}))^{d^\perp}$ . So, in this case, by (4.22)  $\psi_{m+\lambda+n_+}^+ \psi_{m-\lambda-n_-}^-$  decomposes as an integral over matrix elements of continuous representations.

We thus generically write:

$$\psi_{m_1 \bar{\Lambda}_1 m'_1}(\rho, \varphi_1, \varphi_2) \psi_{m_2 \bar{\Lambda}_2 m'_2}(\rho, \varphi_1, \varphi_2) = \sum_{\bar{\Lambda}} C_{\bar{\Lambda}_1, \bar{\Lambda}_2 m_1, m_2, m'_1, m'_2}^{\bar{\Lambda}} \psi_{m_1 + m_2 \bar{\Lambda} m'_1 + m'_2}(\rho, \varphi_1, \varphi_2) \quad (4.24)$$

with the notations (4.12), and where  $\bar{\Lambda}_1, \bar{\Lambda}_2 = (\lambda, +), (\lambda, -), (i\sigma, 0)$  or  $(i\sigma, 1/2)$  and  $\bar{\Lambda}$  takes one of the allowed values occurring in the tensor product decomposition (see [69, 70] and [42] for a case by case treatment with the same notations). The coefficients are given by the corresponding product of Clebsch-Gordan coefficients (see [42, 43]).

Similarly, if we proceed with the Losert basis, as the product  $e_{nmk}(x) e_{n'm'k'}(x)$  is square integrable (even better, it is a Schwartz function) [42] we have

$$\Phi_{nmk}(\rho, \varphi_1, \varphi_2) \Phi_{n'm'k'}(\rho, \varphi_1, \varphi_2) = C_{kk'nn'mm'}^{k''} \Phi_{n+n'm+m'k''}(\rho, \varphi_1, \varphi_2). \quad (4.25)$$

From now on, we call the Plancherel basis the set of matrix elements of the discrete and principal continuous series, *i.e.*, the  $\psi$ s and the Losert basis the  $\Phi$ s.

#### 4.2.5 A KM algebra associated to $\text{SL}(2, \mathbb{R})$

The KM algebra associated to  $\text{SL}(2, \mathbb{R})$  is obtained in an analogous way to the construction of the KM algebra associated to a compact Lie group  $G_c$ . It is a central extension of the algebra  $\mathfrak{g}(\text{SL}(2, \mathbb{R}))$ . In the Plancherel basis (PB), we have

$$\begin{aligned} \mathfrak{g}(\text{SL}(2, \mathbb{R})) &= \left\{ T^a \psi_{n\lambda m}^+(\rho, \varphi_1, \varphi_2), T^a \psi_{n\lambda m}^-(\rho, \varphi_1, \varphi_2), \lambda > \frac{1}{2}, mn > 0, |m|, |n| > \lambda, \right. \\ &\quad \left. T^a \psi_{ni\sigma m}^\epsilon(\rho, \varphi_1, \varphi_2), \epsilon = 0, \frac{1}{2}, \sigma > 0, m, n \in \mathbb{Z} + \epsilon, a = 1, \dots, \dim \mathfrak{g} \right\} \\ &= \left\{ T_{m\bar{\Lambda}n}^a = T^a \psi_{m\bar{\Lambda}n}, \bar{\Lambda} = (\lambda, +), (\lambda, -), (i\sigma, 0), (i\sigma, \frac{1}{2}), a = 1, \dots, \dim \mathfrak{g} \right\} \end{aligned}$$

and in the Losert basis (LB) we get

$$\mathfrak{g}(\text{SL}(2, \mathbb{R})) = \left\{ T_{mnk}^a = T^a \phi_{m,n,k}(\rho, \varphi_1, \varphi_2), m, n \in \mathbb{Z} + \epsilon, k \in \mathbb{N}, \epsilon = 0, \frac{1}{2} \right\}.$$

The Lie brackets take the form

$$\begin{aligned} [T_{m\bar{\Lambda}n}^a, T_{m'\bar{\Lambda}'n'}^{a'}] &= if^{aa'} \sum_{\bar{\Lambda}''} C_{\bar{\Lambda}, \bar{\Lambda}'}^{\bar{\Lambda}''} m, m', n, n' T_{m+m'\bar{\Lambda}''n+n'}^{a''} \quad (\text{PB}) \\ [T_{mnk}^a, T_{m'n'k'}^{a'}] &= if^{aa'} \sum_{k''} C_{kk'm, m', n, n'}^{k''} T_{m+m'n+n'k''}^{a''} \quad (\text{LB}) \end{aligned} \quad (4.26)$$

by (4.24) and (4.25). Of course, since we can express the  $T_{mnk}^a$  in terms of the  $T_{m\bar{\Lambda}n}^a$  and vice versa, because of (4.23) and (4.22), the two presentations of the algebra are isomorphic. In (4.26) we explicitly write the symbol  $\sum$  in the LB (*i.e.*, we do not use the Einstein summation convention) in order to emphasize the different presentation of the algebra in the PB and in the LB bases. Indeed, in the former basis the algebra involves an integral, while in the latter, it involves sum.

The next step in the construction is to introduce Hermitian operators and compatible two-cocycles. The construction is similar to the SU(2) case. We first introduce the two commuting operators  $L_0, R_0$ . In the Losert basis. The action of the commuting Hermitian operators is given in (4.13), and in the Plancherel basis the action of Hermitian operators is given in (4.5) with the corresponding value of  $\Lambda$ . We then associate to  $L_0, R_0$  compatible two-cocycles  $\omega_L, \omega_R$ :

$$\begin{aligned} \omega_L(X, Y) &= -\frac{k_L}{4\pi^2} \int_{\mathbb{H}_{2,2}} d\rho \sinh \rho \cosh \rho d\varphi_1 d\varphi_2 \langle X, L_0 Y \rangle_0 \\ \omega_R(X, Y) &= -\frac{k_R}{4\pi^2} \int_{\mathbb{H}_{2,2}} d\rho \sinh \rho \cosh \rho d\varphi_1 d\varphi_2 \langle X, R_0 Y \rangle_0. \end{aligned}$$

Thus, using the orthogonality conditions (4.6) and (4.8) (resp. (4.19)) and the conjugation property (4.9) (resp. (4.18)) in the Plancherel (resp. the Losert) basis, we obtain

$$\begin{aligned} \omega_L(T_{n\lambda\eta m}^a, T_{n'\lambda'\eta' m'}^{a'}) &= n k_L k^{aa'} \delta_{\lambda, \lambda'} \delta_{\eta, -\eta'} \delta_{m, -m'} \delta_{n, -n'} \\ \omega_L(T_{ni\sigma\epsilon m}^a, T_{n'i\sigma'\epsilon' m'}^{a'}) &= n k_L k^{aa'} \frac{\delta(\sigma - \sigma')}{\sigma \tanh \pi(\sigma + i\epsilon)} \delta_{\epsilon, \epsilon'} \delta_{m, -m'} \delta_{n, -n'} \end{aligned} \quad (4.27)$$

in the PB basis, and

$$\omega_L(T_{nmk}^a, T_{n'm'k'}^{a'}) = nk_L k^{aa'} \delta_{kk'} \delta_{m, -m'} \delta_{n, -n'} \quad (4.28)$$

in the LB basis. Defining

$$\delta(\bar{\Lambda}, \bar{\Lambda}') = \begin{cases} \delta_{\lambda, \lambda'} \delta_{\eta, -\eta'} & \bar{\Lambda} = (\lambda, \eta), \bar{\Lambda}' = (\lambda', \eta') \\ \frac{\delta(\sigma - \sigma')}{\sigma \tanh \pi(\sigma + i\epsilon)} \delta_{\epsilon, \epsilon'} & \bar{\Lambda} = (i\sigma, \epsilon), \bar{\Lambda}' = (i\sigma', \epsilon') \\ 0 & \text{elsewhere} \end{cases}$$

we obtain

$$\omega_L(T_{n\bar{\Lambda}m}^a, T_{n'\bar{\Lambda}'m'}^{a'}) = nk_L k^{aa'} \delta(\bar{\Lambda}, \bar{\Lambda}') \delta_{m, -m'} \delta_{n, -n'}$$

in the PB basis. We have a similar expression for  $\omega_R$ .

With the same notations as before, we define the KM algebra associated to  $\text{SL}(2, \mathbb{R})$ :

$$\tilde{\mathfrak{g}}(\text{SL}(2, \mathbb{R})) = \begin{cases} \left\{ \mathcal{T}_{m\bar{\Lambda}n}^a, L_0, R_0, k_L, k_L \right\} & (\text{PB}) \\ \left\{ \mathcal{T}_{mnk}^a, L_0, R_0, k_L, k_L \right\} & (\text{LB}) \end{cases} \quad (4.29)$$

From (4.26). (4.27) and (4.6) and (4.8), the Lie brackets are

$$\begin{aligned}
[\mathcal{T}_{am\bar{\Lambda}n}, \mathcal{T}_{a'm'\bar{\Lambda}'n'}] &= if_{aa'}{}^{a''} \sum_{\bar{\Lambda}''} C_{\bar{\Lambda}, \bar{\Lambda}'m, m', n, n'}^{\bar{\Lambda}''} \mathcal{T}_{a''m+m'\bar{\Lambda}''n+n'} \\
&\quad + (mk_L + nk_R) h_{aa'} \delta(\bar{\Lambda}, \bar{\Lambda}') \delta_{m, -m'} \delta_{n, -n'}, \\
[L_0, \mathcal{T}_{am\bar{\Lambda}n}] &= m \mathcal{T}_{am\bar{\Lambda}n}, \\
[R_0, \mathcal{T}_{am\bar{\Lambda}n}] &= n \mathcal{T}_{am\bar{\Lambda}n},
\end{aligned} \tag{4.30}$$

in the Plancherel basis, and from (4.26). (4.27) and (4.8) the Lie brackets are

$$\begin{aligned}
[\mathcal{T}_{amnk}, \mathcal{T}_{a'm'n'k'}] &= if_{aa'}{}^{a''} \sum_{k''} C_{kk'm, m', n, n'}^{k''} \mathcal{T}_{a''m+m'n+n'k''} \\
&\quad + (mk_L + nk_R) h_{aa'} \delta_{kk'} \delta_{m, -m'} \delta_{n, -n'}, \\
[L_0, \mathcal{T}_{amnk}] &= m \mathcal{T}_{amnk}, \\
[R_0, \mathcal{T}_{amnk}] &= n \mathcal{T}_{amnk},
\end{aligned} \tag{4.31}$$

in the Losert basis. This algebra was obtained in [42].

Since we can expand the Losert basis in the Plancherel basis (4.22) and the Plancherel basis in the Losert basis (4.23), the two presentations of the algebra  $\tilde{\mathfrak{g}}(\mathrm{SL}(2, \mathbb{R}))$  (4.30) and (4.31) are equivalent. Each presentation has its own advantages. In the former case, elements of  $\tilde{\mathfrak{g}}(\mathrm{SL}(2, \mathbb{R}))$  are expressed by means of unitary irreducible representations of  $\mathfrak{sl}(2, \mathbb{R})$ , but the Lie brackets involve integrals and Dirac  $\delta$ -distributions, as a consequence of the continuous basis. In the latter case, the Lie brackets involve only sums and Kronecker symbols, but the elements of  $\tilde{\mathfrak{g}}(\mathrm{SL}(2, \mathbb{R}))$  are expressed by means of a reducible representation of  $\mathfrak{sl}(2, \mathbb{R})$  associated to the Hilbert basis of  $L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$ .

This algebra is very different to its analogue of KM algebra of compact Lie groups. Indeed, for a compact Lie group we have  $\mathfrak{g} \subset \tilde{\mathfrak{g}}(G_c)$ , because the trivial representation is a unitary representation of  $G_c$ . However, in the non-compact case  $G_{nc}$ ,  $\mathfrak{g}$  is not included in  $\tilde{\mathfrak{g}}(G_{nc})$ , because the trivial representation is non-normalizable on the  $\mathrm{SL}(2, \mathbb{R})$ -manifold.

We conclude this section mentioning that one can associate a Virasoro algebra to  $\mathrm{SL}(2, \mathbb{R})$ . This construction follows the same lines as the corresponding construction in Section 3.5. We introduce

$$\begin{aligned}
\ell_{m\bar{\Lambda}n} &= -\psi_{m\bar{\Lambda}n}(\theta, \varphi_1, \varphi_2) L_0 \quad (\text{PB}) \\
\ell_{mnk} &= -\Phi_{mnk}(\theta, \varphi_1, \varphi_2) L_0 \quad (\text{LB})
\end{aligned}$$

which are compatible with  $\omega_L$ . The commutation relations read:

$$\begin{aligned}
[\ell_{m\bar{\Lambda}n}, \ell_{m'\bar{\Lambda}'n'}] &= (m - m') \sum_{\bar{\Lambda}''} C_{\bar{\Lambda}, \bar{\Lambda}'m, m', n, n'}^{\bar{\Lambda}''} \ell_{m+m'\bar{\Lambda}''n+n'} \quad (\text{PB}) \\
[\ell_{mnk}, \ell_{m'n'k'}] &= (m - m') \sum_{k''} C_{kk'm, m', n, n'}^{k''} \ell_{m+m'n+n'k''} \quad (\text{LB})
\end{aligned}$$

This algebra admits a central extension, exactly along the lines of (3.4), and with the same notations, we introduce the generators:

$$\mathrm{Vir}(\mathrm{SL}(2, \mathbb{R})) = \begin{cases} \{L_{m\bar{\Lambda}n}, c\} & (\text{PB}) \\ \{L_{mnk}, c\} & (\text{LB}) \end{cases}$$

with Lie brackets in the PB basis:

$$\begin{aligned} [L_{m\bar{\Lambda}n}, L_{m'\bar{\Lambda}'n'}] &= (m-m') \sum_{\bar{\Lambda}''} C_{\bar{\Lambda}, \bar{\Lambda}' m, m', n, n'}^{\bar{\Lambda}''} L_{m+m'\bar{\Lambda}''n+n'} \\ &\quad + \frac{c}{12}(m^3 - m) \delta_{m,-m'} \delta_{n,-n'} \delta(\bar{\Lambda}, \bar{\Lambda}') \end{aligned} \quad (4.32)$$

and in the LB basis:

$$[L_{mnk}, \ell_{m'n'k'}] = (m-m') \sum_{k''} C_{kk'm, m', n, n'}^{k''} L_{m+m'n+n'k''} + \frac{c}{12}(m^3 - m) \delta_{m,-m'} \delta_{n,-n'} \delta_{kk'} . \quad (4.33)$$

Thus,  $\tilde{\mathfrak{g}}'(\mathrm{SL}(2, \mathbb{R})) \rtimes \mathrm{Vir}(\mathrm{SL}(2, \mathbb{R}))$  (similarly to  $\mathrm{SU}(2)$ , we only introduce one central charge, say  $k_L$ ) has a semidirect structure with the action of the Virasoro algebra on the KM algebra, the remaining part being:

$$\begin{aligned} [L_{m\bar{\Lambda}n}, \mathcal{T}_{m'\bar{\Lambda}'n'}^a] &= -m' \sum_{\bar{\Lambda}''} C_{\bar{\Lambda}, \bar{\Lambda}' m, m', n, n'}^{\bar{\Lambda}''} \mathcal{T}_{m+m'\bar{\Lambda}''n+n'}^a \quad (\text{PB}) \\ [L_{mnk}, \mathcal{T}_{m'n'k'}^a] &= -m' \sum_{k''} C_{kk'm, m', n, n'}^{k''} \mathcal{T}_{m+m'n+n'k''}^a \quad (\text{LB}) \end{aligned} .$$

The algebra  $\tilde{\mathfrak{g}}'(\mathrm{SL}(2, \mathbb{R})) \rtimes \mathrm{Vir}(\mathrm{SL}(2, \mathbb{R}))$  is thus defined by (4.30)/(4.31) (with the central charge  $k_R = 0$ ), (4.32)/(4.33) and (4.34). Again, the Virasoro algebra of  $\mathrm{SL}(2, \mathbb{R})$  is very different to the Virasoro algebra of  $\mathrm{SU}(2)$ . Recall that for the Virasoro algebra of  $\mathrm{SU}(2)$  we have  $L_0 = -L_{000}$  (see (3.44)), but here, as the trivial representation is not normalizable, we cannot relate  $L_0$  to a specific element of  $\mathrm{Vir}(\mathrm{SL}(2, \mathbb{R}))$ . However, one can extend the Virasoro algebra to  $\mathrm{Vir}'(\mathrm{SL}(2, \mathbb{R})) = \mathrm{Vir}(\mathrm{SL}(2, \mathbb{R})) \times \{L_0\}$ , where the action of  $L_0$  is given by

$$[L_0, L_{nsm}] = nL_{nsm} .$$

We finish this section mentioning that a KM algebra on  $\mathrm{SL}(2, \mathbb{R})/U(1)$  can be easily deduced from the KM algebra on  $\mathrm{SL}(2, \mathbb{R})$ . The only representations which appear in the harmonic analysis on  $\mathrm{SL}(2, \mathbb{R})/U(1)$  are the representations, which are chargeless with respect to  $R_0$ . The only representations that survive this condition in the PB basis are the matrix elements of the bosonic principal continuous series  $\psi_{ni\sigma 0}^0$ , and in the LB basis, the elements of  $W_{m0} : \Phi_{m0k}$  (LB). More details and the explicit brackets can be found in [42].

## 5 Soft manifolds

This section is devoted to the construction of KM algebras on soft group manifolds, that is, on group manifolds with a soft deformation.

### 5.1 KM algebras on soft manifolds

In Sections 3 and 4 we have considered KM algebras associated to (compact and non-compact) group manifolds. These manifolds have a large isometry group (associated to the left and right action of the group itself). Furthermore, for these manifolds the Vielbein

satisfies the Maurer-Cartan equation (3.4) or is a left invariant one-form. Moreover, for these manifolds the metric tensor is naturally deduced from the Vielbein (3.5).

Let  $G_m$  be a group (compact or non-compact), we now consider a smooth, ‘softening’ deformation  $G_m^\mu$  of the Lie group  $G_m$ , locally diffeomorphic to  $G_m$  itself (see e.g. [15] and references therein). We further assume that the manifold  $G_m^\mu$  has the same parameterisation  $m^M$  (see Section 3 for  $G_m = G_c$  a compact manifold and Section 4 for  $G_m = G_{nc}$  a non-compact manifold), the only difference between  $G_m$  and  $G_m^\mu$  being at the level of the metric tensor. We thus assume that the Vielbein  $\mu$  is an intrinsic one-form (valued in the Lie algebra  $\mathfrak{g}_m$  of  $G_m$ )

$$\mu^A(m) = \mu_M{}^A(m) \, dm^M$$

i.e., it is not a Maurer-Cartan one-form (it does not satisfy equation (3.4)):

$$d\mu + \mu \wedge \mu = R, \quad (5.1)$$

where  $R$  is the curvature two-form of  $\mu$ . In other words,  $\mu$  is *not* left-invariant (i.e., it is a ‘soft’, intrinsic one-form). It is in this sense that we consider that the ‘soft’ group manifold  $G_m^\mu$  is a *deformation* of  $G_m$ . The metric tensor is now defined by

$$g_{MN}^\mu(m) = \mu_M{}^A(m) \mu_N{}^B(m) \eta_{AB}, \quad (5.2)$$

where  $\eta_{AB}$  is the Killing form of the Lie algebra  $\mathfrak{g}_m$  (positive definite for compact manifolds and indefinite for non-compact manifolds). Taking the exterior derivative of both sides of Eq. (5.1), one obtains the Bianchi identity for the curvature of  $\mu$ ,

$$dR + 2R \wedge \mu = 0 \Leftrightarrow \nabla R = 0,$$

where the covariant derivative operator  $\nabla$  on  $G_m^\mu$  has been introduced. For instance, if  $G_m = \text{ISO}(1, d-1)/\text{SO}(1, d-1)$ , where  $\text{SO}(1, d-1)$  (resp.  $\text{ISO}(1, d-1)$ ) corresponds to the Lorentz (resp. Poincaré) transformations in  $D$ -dimensions,  $G_m$  is the  $D$ -dimensional Minkowski spacetime of Special Relativity, and its deformation  $G_m^\mu$  the  $D$ -dimensional Riemann spacetime of General Relativity [12].

The fact that the metric tensors differ for the manifold  $G_m$  (see (3.5) or its analogue for a non-compact manifold) and its deformation  $G_m^\mu$  (see (5.2)) is not the only difference between  $G_m$  and  $G_m^\mu$ . Indeed, as seen previously, the manifold  $G_m$  has a large isometry group, namely  $(G_m)_L \times (G_m)_R$ , but in general, the isometry group of  $G_m^\mu$  is reduced with respect to its undeformed analogue. Stated differently, solving the Killing equation (2.1) leads to less invariant vector fields. Consequently, if we follow the construction of KM algebras in Section 2.1, we will obtain a less richer structure. For this reason, we associate a KM algebra to  $G_m$  following a different strategy. We endow  $G_m^\mu$  with the scalar product  $G_m^\mu$

$$(f, g)_\mu = \frac{1}{C} \int_{G_m} \sqrt{g^\mu} \, d^n m \, \overline{f(m)} \, g(m), \quad (5.3)$$

where  $g^\mu = |\det(g_{MN}^\mu)|$  and  $n$  is the dimension of  $G_m$ . The coefficient  $C$  depends of the manifold. For a compact manifold  $G_c$  we take  $C = V$ , the volume of  $G_c$ . For a non-compact manifold, see (4.4) for  $G_m = \text{SL}(2, \mathbb{R})$ . Notice that, since the parameterisation of  $G_m^\mu$  and  $G_m$  is the same, the limits of integration are again  $G_m$  in this case. As a final hypothesis concerning the deformed manifold, we assume that  $G_m^\mu$  is non-singular and well defined at any point of  $G_m^\mu$  (see (5.5) below).

Let  $L^2(G_m)$  be the set of square integrable functions on  $G_m$ . We would like to identify a Hilbert basis of  $L^2(G_m^\mu)$  from a Hilbert basis of  $L^2(G_m)$ <sup>7</sup>. The results presented here

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<sup>7</sup>For  $G_m = G_{nc}$  a non compact manifold, the Plancherel theorem involves normalizable and non-normalizable functions (see Section 4). However the technique presented here extends to non-normalizable functions. For  $G_m = G_{nc}$  we will consider two explicit examples, with an obvious generalization to any  $G_{nc}$ .

are due to Mackey [71] (see also [42]). Let  $\mathcal{M}$  be a manifold (not necessarily a group manifold) and let  $L^2(\mathcal{M})$  be the set of square integrable functions defined on  $\mathcal{M}$ . Assume that the manifold  $\mathcal{M}$  is endowed with two different scalar products with measure  $d\alpha$  and  $d\beta$  respectively:

$$\begin{aligned} (\mathcal{M}, d\alpha) : \quad (f, g)_\alpha &= \int_{\mathcal{M}} d\alpha \overline{f(m)} g(m) , \\ (\mathcal{M}, d\beta) : \quad (f, g)_\beta &= \int_{\mathcal{M}} d\beta \overline{f(m)} g(m) . \end{aligned}$$

We assume further that there exists a mapping  $T_{\beta\alpha}$ :

$$T_{\beta\alpha} : L^2(\mathcal{M}, d\beta) \rightarrow L^2(\mathcal{M}, d\alpha) ,$$

such that

$$\int_{\mathcal{M}} d\alpha = \int_{\mathcal{M}} d\beta T_{\beta\alpha} .$$

For an  $n$ -dimensional Riemannian manifold  $\mathcal{M}$  parameterized by  $m_1, \dots, m_n$  with metric  $g_\alpha$  (resp.  $g_\beta$ ), we have  $d\alpha = \sqrt{|\det g_\alpha|} d^n m$  (resp.  $d\beta = \sqrt{|\det g_\beta|} d^n m$ ) and thus  $T_{\beta\alpha} = \sqrt{|\det g_\alpha| / |\det g_\beta|}$ . This means that if  $\{f_i^\beta, i \in \mathbb{N}\}$  is a Hilbert basis of  $L^2(\mathcal{M}, d\beta)$ , then  $\{f_i^\alpha = \frac{f_i^\beta}{\sqrt{T_{\beta\alpha}}}, i \in \mathbb{N}\}$  is a Hilbert basis for  $L^2(\mathcal{M}, d\alpha)$ , and we obviously have

$$(f_i^\beta, f_j^\beta)_\beta = \delta_{ij} \iff (f_i^\alpha, f_j^\alpha)_\alpha = \delta_{ij} \quad (5.4)$$

and the map  $T_{\beta\alpha}$  is unitary.

Returning to our softly deformed manifold, let  $g_{MN}$  be the metric of  $G_m$ , let  $g_{MN}^\mu$  be the metric on  $G_m^\mu$  and let  $g = |\det(g_{MN})|$ ,  $g^\mu = |\det(g_{MN}^\mu)|$ . Let  $\mathcal{B} = \{\rho_I(m), I \in \mathcal{I}\}$  be a Hilbert basis of  $L^2(G_m)$ ; by (5.4)

$$\mathcal{B}_\mu = \left\{ \rho_I^\mu(m) = \sqrt{T^\mu} \rho_I(m), I \in \mathcal{I} \right\} ,$$

where  $T = \sqrt{\frac{g}{g^\mu}}$  is a Hilbert basis of  $L^2(G_m^\mu)$  and we have

$$(\rho_I^\mu, \rho_J^\mu)_\mu = (\rho_I, \rho_J) = \delta_{IJ} ,$$

where in the second equality we have used the scalar product over the manifold  $G_m$ . We assume that the transition function is non-degenerate, namely that

$$t_{\min} \leq T^\mu(m) \leq t_{\max}, \quad \forall m \in G_m^\mu \quad (5.5)$$

with  $t_{\min}, t_{\max} \in \mathbb{R}_+ \setminus \{0\}$ .

This construction can be adapted easily for non-normalizable functions. As an illustration, consider the Minkowski spacetime in  $D$  dimensions. By the Fourier theorem, the set of plane waves

$$\mathcal{B} = \left\{ \Phi_{\mathbf{p}}(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}} e^{i\mathbf{p}\cdot\mathbf{x}}, \mathbf{p} \in \mathbb{R}^{1,D-1} \right\} \quad (5.6)$$

where  $\mathbf{p}\cdot\mathbf{x}$  is the usual Minkowski scalar product, enables to expand any Schwartz function. Indeed, in this case, by Fourier transformation we have:

$$\begin{aligned} \Psi(\mathbf{x}) &= \int_{\mathbb{R}^{1,D-1}} d\mathbf{p} \tilde{\Psi}(\mathbf{p}) \Phi_{\mathbf{p}}(\mathbf{x}) , \\ \tilde{\Psi}(\mathbf{p}) &= (\Phi_{\mathbf{p}}, \Psi) = \int_{\mathbb{R}^{1,D-1}} d\mathbf{x} \overline{\Phi_{\mathbf{p}}(\mathbf{x})} \Psi(\mathbf{x}) , \end{aligned}$$

and

$$(\Phi_{\mathbf{p}}, \Phi_{\mathbf{q}}) = \int_{\mathbb{R}^{1,D-1}} d\mathbf{p} \overline{\Phi_{\mathbf{p}}(\mathbf{x})} \Phi_{\mathbf{p}}(\mathbf{x}) = \delta(\mathbf{p} - \mathbf{q}) .$$

The procedure of Mackey can be extended easily in this case and

$$\mathcal{B}_\mu = \left\{ \Phi_{\mathbf{p}}^\mu(\mathbf{x}) = \frac{1}{(g^\mu)^{1/4}} \Phi_{\mathbf{p}}(\mathbf{x}) , \quad \mathbf{p} \in \mathbb{R}^{1,D-1} \right\}$$

with

$$(\Phi_{\mathbf{p}}^\mu, \Phi_{\mathbf{q}}^\mu)_\mu = (\Phi_{\mathbf{p}}, \Phi_{\mathbf{q}}) = \delta(\mathbf{p} - \mathbf{q}) .$$

Finally, the closure relation of the wave functions  $\Phi_{\vec{p}}$  implies

$$\int d\mathbf{p} \overline{\Phi_{\mathbf{p}}^\mu(\mathbf{x})} \Phi_{\mathbf{p}}^\mu(\mathbf{y}) = \frac{\delta(\mathbf{x} - \mathbf{y})}{\sqrt{g^\mu}} ,$$

thus the set  $\mathcal{B}_\mu$  is complete, and these elements enable to expand any Schwartz function of the Riemannian spacetime:

$$\Psi(\mathbf{x}) = \int_{\mathbb{R}^{1,D-1}} d\mathbf{p} \tilde{\Psi}(\mathbf{p}) \Phi_{\mathbf{p}}^\mu(\mathbf{x}) , \quad \tilde{\Psi}(\mathbf{p}) = (\Phi_{\mathbf{p}}^\mu, \Psi)_\mu .$$

Note however that the functions  $\Phi_{\mathbf{p}}^\mu$  in  $\mathcal{B}_\mu$  are defined by means of the scalar product in the Minkowski spacetime, and not the scalar product in the Riemann spacetime. As a second example, we now consider a deformation of  $\mathrm{SL}(2, \mathbb{R})$ . In this case, the Mackey procedure leads to

$$\begin{aligned} \psi_{n\lambda m}^\eta(\rho, \varphi_1, \varphi_2) &\rightarrow \psi_{n\lambda m}^{\mu\eta}(\rho, \varphi_1, \varphi_2) = \left( \frac{\cosh \rho \sinh \rho}{g^\mu} \right)^{1/4} \psi_{n\lambda m}^\eta(\rho, \varphi_1, \varphi_2) \quad (5.7) \\ \psi_{ni\sigma m}^\epsilon(\rho, \varphi_1, \varphi_2) &\rightarrow \psi_{ni\sigma m}^{\mu\epsilon}(\rho, \varphi_1, \varphi_2) = \left( \frac{\cosh \rho \sinh \rho}{g^\mu} \right)^{1/4} \psi_{ni\sigma m}^\epsilon(\rho, \varphi_1, \varphi_2) , \end{aligned}$$

for respectively the matrix elements of the discrete and principal continuous series. These matrix elements satisfy the first equation in (4.6) and Eq.[4.8] with the scalar product (5.3), instead of the scalar product (4.4). In a similar manner, as seen for the previous example, the set of functions in (5.7) is a complete set. Of course, only the former functions are normalized, while the latter functions are not normalized. Even because of hypothesis (5.5), the functions  $\psi_{n\lambda m}^{\mu\eta}$  are Schwartzian. This set of functions enables us to have a ‘Plancherel’ decomposition on  $G_{nc}^\mu$ . Actually, if  $f$  is a Schwartz function, we have

$$\begin{aligned} f(\rho, \varphi_1, \varphi_2) &= \sum_{\Lambda, m, n} f_{\pm}^{n\lambda m} \psi_{n\Lambda, m}^\mu(\rho, \varphi_1, \varphi_2) \\ f_{\pm}^{n\lambda m} &= (\psi_{n\lambda m}^{\mu\pm}, f)_\mu , \quad f_{nm}^\epsilon(\sigma) = (\psi_{ni\sigma m}^{\mu\epsilon}, f)_\mu . \end{aligned}$$

with the notations of (4.12) and (4.11).

Furthermore, if  $G_m = G_{nc}$  is non-compact, it is always possible to identify a Hilbert basis, *i.e.*, a complete set of orthonormal functions. The Mackey procedure follows easily in this case. Again for  $G_{nc} = \mathrm{SL}(2, \mathbb{R})$ , this leads to

$$\Phi_{nmk}(\rho, \varphi_1, \varphi_2) \rightarrow \Phi_{nmk}^\mu(\rho, \varphi_1, \varphi_2) = \left( \frac{\cosh \rho \sinh \rho}{g^\mu} \right)^{1/4} \Phi_{nmk}(\rho, \varphi_1, \varphi_2) , \quad (5.8)$$

which form a Hilbert basis of  $L^2(\mathrm{SL}(2, \mathbb{R})^\mu)$  and are also Schwartzian.

Following the notations of Section 3.1,<sup>8</sup> and in particular results due to Racah [55], we denote  $\Psi_{LQR}$  (see (3.8), for  $G_m = G_c$ ) the set of matrix elements which allows us to expand any Schwartz functions on  $G_m$ , and define

$$\mathcal{B}_\mu = \left\{ \Psi_{LQR}^\mu(m) = \sqrt{T^\mu} \Psi_{LQR}(m) , (LQR) \in \mathcal{I} \right\} , \quad (5.9)$$

the corresponding set in  $G_m^\mu$  through the Mackey procedure.

The next step in our construction is to decompose the product  $\Psi_{LQR}^\mu(m) \Psi_{L'Q'R'}^\mu(m)$  in the basis (5.9). We analyze first the case when  $G_m = G_c$  is a compact Lie group. As the metric tensor of the soft manifold satisfies (5.5), this in particular means that any function  $\Psi_{LQR}^\mu$  can be expanded in the Hilbert basis of  $L^2(G_c)$  (3.8). Conversely, any function  $\Psi_{LQR}$  can be expanded in the Hilbert basis of  $L^2(G_c^\mu)$ :

$$\Psi_{LQR}^\mu(m) = P_{LQR}^{L'Q'R'} \Psi_{L'Q'R'}(m) , \quad \Psi_{LQR}(m) = (P^{-1})_{LQR}^{L'Q'R'} \Psi_{L'Q'R'}^\mu(m) . \quad (5.10)$$

Therefore the product  $\Psi_{L_1 Q_1 R_1}^\mu(m) \Psi_{L_2 Q_2 R_2}^\mu(m)$  decomposes as:

$$\Psi_{L_1 Q_1 R_1}^\mu(m) \Psi_{L_2 Q_2 R_2}^\mu(m) = C_{L_1 Q_1 R_1; L_2 Q_2 R_2}^{\mu L_3 Q_3 R_3} \Psi_{L_3 Q_3 R_3}^\mu(m) , \quad (5.11)$$

where

$$C_{L_1 Q_1 R_1; L_2 Q_2 R_2}^{\mu L_3 Q_3 R_3} = P_{L_1 Q_1 R_1}^{L'_1 Q'_1 R'_1} P_{L_2 Q_2 R_2}^{L'_2 Q'_2 R'_2} (P^{-1})_{L'_3 Q'_3 R'_3}^{L_3 Q_3 R_3} C_{L'_1 Q'_1 R'_1; L'_2 Q'_2 R'_2}^{L'_3 Q'_3 R'_3} \quad (5.12)$$

If  $G_{nc}$  is a non-compact manifold, this procedure extends naturally to any Hilbert basis such as (5.7) for  $\text{SL}(2, \mathbb{R})$ . In [44] we proceeded in a different but equivalent way to obtain the decomposition of the product  $\Psi_{L_1 Q_1 R_1}^\mu(m) \Psi_{L_2 Q_2 R_2}^\mu(m)$ . However, the proof given in [44] does not extend to the case when  $G_m = G_{nc}$  is a non-compact manifold. We now consider non compact-manifolds, and consider  $\text{SL}(2, \mathbb{R})_\mu$  as an illustration. Since the matrix element  $\psi_{n\lambda m}^\eta$  are Schwartz functions and the metric of  $\text{SL}(2, \mathbb{R})_\mu$  satisfies (5.5), the functions  $\psi_{n\lambda m}^{\mu\eta}$  (see (5.7)) are also Schwartzian, and can thus be decomposed in the Hilbert basis of  $L^2(\text{SL}(2, \mathbb{R}))^\perp$ <sup>9</sup> (see Section 4.2.3):

$$\begin{aligned} \psi_{n\lambda m}^{\mu\eta}(\rho, \varphi_1, \varphi_2) &= P_{n\lambda m}^{n'\lambda'm'} \psi_{n'\lambda'm'}^\eta(\rho, \varphi_1, \varphi_2) , \\ \psi_{n\lambda m}^\eta(\rho, \varphi_1, \varphi_2) &= (P^{-1})_{n\lambda m}^{n'\lambda'm'} \psi_{n'\lambda'm'}^{\mu\eta}(\rho, \varphi_1, \varphi_2) . \end{aligned} \quad (5.13)$$

We now turn on the decomposition of  $\psi_{ni\sigma m}^{\mu\epsilon}$ . We recall that, in the case of  $\text{SL}(2, \mathbb{R})$ , the Plancherel theorem enables us to express the matrix elements of the continuous series in terms of the Losert basis of  $L^2(\text{SL}(2, \mathbb{R}))^{d^\perp}$  and *vice versa* [42] (see (4.22) and (4.23)). Thus, in the case of  $\text{SL}(2, \mathbb{R})_\mu$  because of (5.7) and (5.8), we have exactly the same expansion, and (5.7), (5.8) reduce to:

$$\begin{aligned} \Phi_{nmk}^\mu(\rho, \varphi_1, \varphi_2) &= \int_0^{+\infty} d\sigma \sigma \tanh \pi(\sigma + i\epsilon) f_{nmk}(\sigma) \psi_{ni\sigma m}^{\mu\epsilon}(\rho, \varphi_1, \varphi_2) , \quad k \geq k_{\min} \\ \psi_{ni\sigma m}^{\mu\epsilon}(\rho, \varphi_1, \varphi_2) &= \sum_{k \geq k_{\min}} \overline{f^{nmk}}(\sigma) \Phi_{nmk}^\mu(\rho, \varphi_1, \varphi_2) . \end{aligned}$$

Observe that we have only a sum over  $k$  in the second equality and an integral over  $\sigma$  in the first equality. We also have<sup>10</sup>

$$\begin{aligned} \Phi_{nmk}^\mu(\rho, \varphi_1, \varphi_2) &= P_{nmk}^{n'm'k'} \Phi_{n'm'k'}^\mu(\rho, \varphi_1, \varphi_2) , \\ \Phi_{nmk}^\mu(\rho, \varphi_1, \varphi_2) &= (P^{-1})_{nmk}^{n'm'k'} \Phi_{n'm'k'}^\mu(\rho, \varphi_1, \varphi_2) \end{aligned}$$

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<sup>8</sup>Similar results hold to label states of non-compact manifolds, so we take the same notations in both cases.

<sup>9</sup>In this analysis, in order to simplify the presentation we assume that  $\psi_{n\lambda m}^{\mu\eta}$  decompose in the set of functions  $\psi_{n\lambda m}^\eta$ , more general situations can be encountered.

<sup>10</sup>As in Footnote 9, we assume for simplicity that if  $\Phi_{nmk} \in L^2(\text{SL}(2, \mathbb{R}))^{d^\perp}$ , then  $\Phi_{nmk}^\mu \in L^2(\text{SL}(2, \mathbb{R}))^{d^\perp}$ .

where for the sum over  $k'$  we have  $k' \geq k_{\min}$ , and the sum over  $n, m$  is unconstrained (note that if  $\Phi_{nmk}^\mu \notin L^2(\mathrm{SL}(2, \mathbb{R}))^{d^\perp}$  we have no restriction on the summation over  $k'$ ). Thus

$$\begin{aligned}\psi_{ni\sigma m}^{\mu\epsilon}(\rho, \varphi_1, \varphi_2) &= \overline{f^{nmk}}(\sigma) P_{nmk}{}^{n'm'k'} \Phi_{n'm'k'}(\rho, \varphi_1, \varphi_2) \\ &= \int_0^{+\infty} d\sigma' \sigma' \tanh \pi(\sigma' + i\epsilon) \overline{f^{nmk}}(\sigma) P_{nmk}{}^{n'm'k'} f_{n'm'k'}(\sigma') \psi_{n'i\sigma'm'}^{\epsilon}(\rho, \varphi_1, \varphi_2) \\ &\equiv \int d\sigma' P_{nm}^\epsilon(\sigma, \sigma') \psi_{ni\sigma'm'}^{\epsilon}(\rho, \varphi_1, \varphi_2),\end{aligned}\tag{5.14}$$

where

$$P_{nm}^\epsilon(\sigma, \sigma') = \sigma' \tanh \pi(\sigma' + i\epsilon) \overline{f^{nmk}}(\sigma) P_{nmk}{}^{n'm'k'} f_{n'm'k'}(\sigma').$$

Note that there is no summation over  $n$  and  $m$ . This relation can be inverted

$$\psi_{ni\sigma m}^{\epsilon}(\rho, \varphi_1, \varphi_2) = \int d\sigma' (P^{-1})_{nm}^\epsilon(\sigma, \sigma') \psi_{ni\sigma'm'}^{\epsilon}(\rho, \varphi_1, \varphi_2),\tag{5.15}$$

with

$$(P^{-1})_{nm}^\epsilon(\sigma, \sigma') = \sigma' \tanh \pi(\sigma' + i\epsilon) \overline{f^{nmk}}(\sigma) (P^{-1})_{nmk}{}^{n'm'k'} f_{n'm'k'}(\sigma').$$

Then, the product  $\psi_{n_1\bar{\Lambda}_1m_1}^\mu \psi_{n_2\bar{\Lambda}_2m_2}^\mu$  decomposes as

$$\psi_{n_1\bar{\Lambda}_1m_1}^\mu(\rho, \varphi_1, \varphi_2) \psi_{n_2\bar{\Lambda}_2m_2}^\mu(\rho, \varphi_1, \varphi_2) = \sum_{\bar{\Lambda}} C^{\mu\bar{\Lambda}}_{\bar{\Lambda}_1, \bar{\Lambda}_2 m_1, m_2, m'_1, m'_2} \psi_{m_1+m_2\bar{\Lambda}m'_1+m'_2}^\mu(\rho, \varphi_1, \varphi_2)\tag{5.16}$$

where the coefficients  $C^{\mu\bar{\Lambda}}_{\bar{\Lambda}_1, \bar{\Lambda}_2 m_1, m_2, m'_1, m'_2}$  can be deduced either from the transformations properties (5.13), (5.16), (5.15) or from the relations (4.24) involving the Clebsch-Gordan coefficients of  $\mathrm{SL}(2, \mathbb{R})$ .

To summarize, we have for any soft manifold  $G_m^\mu$

$$\Psi_{LQR}^\mu(m) = \sum_{L'Q'R'} P_{LQR}{}^{L'Q'R'} \Psi_{L'Q'R'}^\mu(m), \quad \Psi_{LQR}(m) = \sum_{L'Q'R'} (P^{-1})_{LQR}{}^{L'Q'R'} \Psi_{L'Q'R'}^\mu(m)\tag{5.17}$$

The decomposition of the product  $\Psi_{LQR}^\mu \Psi_{L'Q'R'}^\mu$  follows at once by (5.17) and (3.11). The Lie algebra

$$\mathfrak{g}(G_m^\mu) = \left\{ T_{aLQR}^\mu(m) = T_a \psi_{LQR}^\mu(m), \quad a = 1, \dots, d, \quad (L, Q, R) \in \mathcal{I} \right\}$$

has Lie brackets

$$[T_{a_1L_1Q_1R_1}^\mu(m), T_{a_2L_2Q_2R_2}^\mu(m)] = i f_{a_1a_2}{}^{a_3} \sum_{L_3Q_3R_3} C_{L_1Q_1R_1; L_2Q_2R_2}^{\mu L_3Q_3R_3} T_{a_3L_3Q_3R_3}^\mu(m),$$

by (3.19) and (5.17). Note that, due to relations (5.17), the Lie algebras  $\mathfrak{g}(G_m^\mu)$  and  $\mathfrak{g}(G_m)$  are isomorphic.

We now turn our attention to the identification of Hermitian operators. As mentioned previously, the isometry group of the soft manifold  $G_m^\mu$  is smaller than its undeformed analogue. Thus, if we solve the Killing equation (2.1) for  $G_m^\mu$ , we have less Killing vectors

than for  $G_m$ . In order to have a number of generators larger than the dimension of the isometry group of  $G_m^\mu$ , define [44] (see also [72] for an analogous definition)

$$L_A^\mu = \sqrt{T^\mu} L_A \frac{1}{\sqrt{T^\mu}}, \quad R_A^\mu = \sqrt{T^\mu} R_A \frac{1}{\sqrt{T^\mu}}. \quad (5.18)$$

In general, these generators do not generate the isometry group of  $G_m^\mu$ , but it follows at once that

$$[L_A^\mu, L_B^\mu] = i c_{AB}{}^C L_C^\mu, \quad [R_A^\mu, R_B^\mu] = i c_{AB}{}^C R_C^\mu, \quad [L_A^\mu, R_B^\mu] = 0.$$

and

$$\begin{aligned} L_A^\mu \Psi_{LQR}^\mu(m) &= (M_A^Q)_L{}^{L'} \Psi_{L'QR}^\mu(m) \\ R_A^\mu \Psi_{LQR}(m) &= (M_A^Q)_R{}^{R'} \Psi_{LQR'}^\mu(m) \end{aligned} \quad (5.19)$$

with  $(M_A^Q)_L{}^{L'}$  the matrix elements of the representation of  $G_m$  associated to the eigenvalues of the Casimir operators  $Q$ . Thus  $\{L_A^\mu, A = 1, \dots, n\}$  and  $\{R_A^\mu, A = 1, \dots, n\}$  generate the Lie algebra  $\mathfrak{g}_m$  and  $\Psi_{LQR}^\mu$  are the corresponding matrix elements of  $G_m$  (but not of  $G_m^\mu$ , which is not a group). This means that, locally,  $G_m^\mu$  and  $G_m$  are diffeomorphic. Moreover, since the operators  $L_A, R_A$  are Hermitian with respect to the scalar product on  $G_m$ , the operators  $L_A^\mu, R_A^\mu$  will be Hermitian with respect to the scalar product on  $G_m^\mu$ . Note however, that the generators  $L_A^\mu, R_A^\mu$  are *certainly not* linear combinations of the generators  $L_A, R_A$ , simply because, in general, the soft manifold  $G_m^\mu$  has less Killing vectors than the manifold  $G_m$ .

One observation is required. As we have seen,  $\psi_{LQR}^\mu$  are in the left and right representation of  $G_m$  (see (5.19)). However, since the metric tensor is deformed by the parameter  $T^\mu$ , we have to take into account this deformation parameter when considering tensor products of representations. In particular, if we define

$$\begin{aligned} \Psi_{LQR}^\mu(m) \otimes_\mu \Psi_{L'Q'R'}^\mu(m) &\equiv \frac{1}{\sqrt{T^\mu}} \Psi_{LQR}^\mu(m) \Psi_{L'Q'R'}^\mu(m) = \sqrt{T^\mu} \Psi_{LQR}(m) \Psi_{L'Q'R'}(m) \\ &= \sum_{L''Q''R''} C_{LQRL;L'Q'R'}^{L''Q''R''} \Psi_{L''Q''R''}^\mu(m), \end{aligned}$$

thus recovering the usual results instead of (5.11) (for a compact manifold).

The last step in our construction is to identify relevant central extensions. First observe that two-cocycles can be defined equivalently to the case  $G_m$  by (2.21) or (2.22). However, because of (5.18), this construction is more involved. We identify relevant central extensions following a different strategy. Indeed, we have already obtained central extensions for the algebra  $\mathfrak{g}(G_m)$  denoted  $k_L^i, k_R^i$  associated and to the two-cocycles  $\omega_i^L, \omega_i^R$  (see (3.21) for  $G_m = G_c$  a compact Lie group, and with a similar relation for  $G_m = G_{nc}$ , a non-compact Lie group).

Before going further, recall some well-known properties about two-cocycles. Let  $(\{X_a, a = 1, \dots, \}, [\ , \ ])$  be a Lie algebra with Lie brackets  $[X_a, X_b] = ih_{ab}{}^c X_c$ . Then, if we define a new algebra  $(\{X_a, a = 1, \dots, \}, [\ , \ ]')$  with a new bracket  $[X_a, X_b]' = [X_a, X_b] + \omega(X_a, X_b)$  ( $\omega(X_a, X_b)$  belongs to  $\mathbb{R}$  or  $\mathbb{C}$  depending if we consider real or complex Lie algebras) this algebra is endowed with the structure of a Lie algebra if  $\omega$  is a two-cocycle. This is equivalent to say that the Jacobi identity is satisfied, *i.e.*, that we have

$$h_{bc}{}^d \omega(X_a, X_d) + h_{ca}{}^d \omega(X_b, X_d) + h_{ab}{}^d \omega(X_c, X_d) = 0.$$

Obviously, if we perform a change of basis:  $X'_a = P_a^b X_b$ , the two-cocycle is given by

$$\omega(X'_a, X'_b) = P_a^c P_b^c \omega(X_a, X_b) , \quad (5.20)$$

and, the Jacoby identity above is satisfied with the new two-cocycle.

Returning to  $G_m^\mu$ , relations (5.17) allow to define two-cocycles of  $G_m^\mu$  from two-cocycles of  $G_m$ , because of (5.20). In the case of a compact Lie group  $G_m = G_c$ ,<sup>11</sup> the two-cocycles take the form:

$$\omega_i^{\mu L}(T_{a_1 L_1 Q_1 R_1}^\mu, T_{a_2 L_2 Q_2 R_2}^\mu) = k_{a_1 a_2} P_{L_1 Q_1 R_1}^{L'_1 Q'_1 R'_1} P_{L_2 Q_2 R_2}^{L'_2 Q'_2 R'_2} L'_2(i) \eta_{L'_1 Q'_1 R'_1, L'_2 Q'_2 R'_2} \quad (5.21)$$

(with a similar relation for  $\omega_i^R$ ) because of (5.10). It is important to emphasize that the two-cocycles above are defined with integration on the manifold  $G_m$  and *not* on  $G_m^\mu$ . This observation is very important when we identify the series of operators which are compatible with these cocycles. It turns out that the operators compatible with  $\omega_i^{\mu L}, \omega_i^{\mu R}$  are the operators  $D_i^L, D_i^R$  satisfying (2.30) and *not* the operators  $D_i^{\mu L}, D_i^{\mu R}$ .

The KM algebra associated to  $G_c^\mu$  (with  $G_c$  a compact Lie group) is then defined by

$$\tilde{\mathfrak{g}}(G_c^\mu) = \{\mathcal{T}_{aLQR}^\mu, a = 1 \cdots, d, (LQR) \in \mathcal{I}, k_L^i, k_R^i, i = 1, \cdots, \ell'\} \rtimes \{D_i^L, D_i^R, i = 1, \cdots, \ell'\}$$

with Lie brackets

$$\begin{aligned} [\mathcal{T}_{aL_1 Q_1 R_1}^\mu, \mathcal{T}_{a_2 L_2 Q_2 R_2}^\mu] &= i f_{a_1 a_2} {}^{a_3} C_{L_1 Q_1 R_1; L_2 Q_2 R_2}^{\mu L_3 Q_3 R_3} \mathcal{T}_{a_3 L_3 Q_3 R_3}^\mu \\ &\quad + k_{a_1 a_2} P_{L_1 Q_1 R_1}^{L'_1 Q'_1 R'_1} P_{L_2 Q_2 R_2}^{L'_2 Q'_2 R'_2} \eta_{L'_1 Q'_1 R'_1, L'_2 Q'_2 R'_2} (k_L^i L'_2(i) + k_R^i R'_2(i)) , \\ [D_i^L, \mathcal{T}_{aLQR}] &= P_{LQR}^{L'Q'R'} L'(i) \mathcal{T}_{aL'Q'R'}^\mu , \\ [D_i^R, \mathcal{T}_{aLQR}] &= P_{LQR}^{L'Q'R'} R'(i) \mathcal{T}_{aLQR'}^\mu , \end{aligned} \quad (5.22)$$

where  $C_{L_1 Q_1 R_1; L_2 Q_2 R_2}^{\mu L_3 Q_3 R_3}$  is defined in (5.12). Analogous expressions hold for  $G_m = G_{nc}$  a non-compact Lie group.

Because the algebras  $\mathfrak{g}(G_c)$  and  $\mathfrak{g}(G_c^\mu)$  are isomorphic, and because of (5.21), it seems at a first glance that the algebras  $\tilde{\mathfrak{g}}(G_c)$  and  $\tilde{\mathfrak{g}}(G_c^\mu)$  are also isomorphic. However, for the algebra associated to  $G_c$ , the differential operator  $D_i^{L,R}$  are associated to a Killing vector of  $G_c$ , while for the algebra  $G_c^\mu$ , we consider the same differential operators, but they *are not* associated to any Killing vector. This observation simply means that the algebras  $\tilde{\mathfrak{g}}(G_c)$  and  $\tilde{\mathfrak{g}}(G_c^\mu)$  are only diffeomorphic locally, as  $G_m$  and  $G_m^\mu$  are locally diffeomorphic. The case where  $G_m = U(1)$  is very specific. Indeed, if we consider a soft manifold  $U(1)^\mu$  with scalar product

$$(f, g)_F = \int_0^{2\pi} d\theta F(\theta) \bar{f}(\theta) g(\theta) ,$$

a change of variables  $F(\theta) d\theta = d\varphi$  leads to the isomorphism  $\tilde{g}(U(1)^\mu) \cong \tilde{g}(U(1))$ , and there do not exist non-trivial soft deformations of affine Lie algebras [44]. This is a consequence of the property that any one-dimensional manifold without boundary is homeomorphic to the one-dimensional sphere  $\mathbb{S}^1$ .

It is also possible in some cases to associate a Virasoro algebra of  $G_m^\mu$ . We briefly comment this point. Let  $D_i^L, D_i^R$  be the set of generators of the Cartan subalgebra of

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<sup>11</sup>The case of a non compact Lie group is easily obtained with the substitution  $\sum \rightarrow \sum$ .

$(\mathfrak{g}_m)_L \oplus (\mathfrak{g}_m)_R$  satisfying condition (2.30). The de Witt algebra (with obvious relations for  $L \rightarrow R$ ) reads:

$$\text{Witt}_i(G_m^\mu) = \left\{ \ell_{iLQR}^\mu = -\psi_{LQR}^\mu D_i^L, (LQR) \in \mathcal{I} \right\} \quad (5.23)$$

and has Lie brackets

$$[\ell_{iL_1Q_1R_1}^\mu, \ell_{iL_2Q_2R_2}^\mu] = P_{L_1Q_1R_1}^{L'_1Q'_1R'_1} P_{L_2Q_2R_2}^{L'_2Q'_2R'_2} (L'_1(i) - L'_2(i)) C_{L'_1Q'_1R'_1; L'_2Q'_2R'_2}^{L''Q''R''} \ell_{iL''Q''R''}^\mu$$

because of (2.20) and (5.17). On the one hand, if the corresponding operators  $\ell_{iLQR}$  in  $\text{Witt}_i(G_m)$  are compatible with the cocycle  $\omega_i^L$ , the operators  $\ell_{iLQR}^\mu$  are compatible with the cocycle  $\omega_i^{\mu L}$ . On the other hand, if  $\text{Witt}_i(G_m)$  admits a central extension, then  $\text{Witt}_i(G_m^\mu)$  also admits a central extension, because (5.20) holds.

In this review, we will not consider any example of KM algebras associated to soft deformations of (compact)Lie groups, since their construction goes essentially along the same lines of the compact group manifolds. The interested reader may consult [44] for more details.

## 6 Roots systems and some elements of representation theory

In this section we would like to introduce a system of roots for the Kac-Moody algebras considered in previous sections. In a second part we shall introduce some elements of representations theory and in particular we shall see that the situation is very different to the corresponding situation for affine Lie algebras. Notwithstanding that the results of the previous section, (*i.e.*, the construction of the KM algebra  $\tilde{\mathfrak{g}}(\mathcal{M})$ ) can easily be extended to any complex or real Lie algebra  $\mathfrak{g}$ , in this section we *are assuming that  $\mathfrak{g}$  is a compact Lie algebra*.

### 6.1 Roots system

For simplicity we now consider Kac-Moody algebras associated to compact Lie groups that is  $\mathcal{M} = G_c$  or  $G_c/H$  with  $H$  a subgroup of  $G_c$  [37]. The other cases can also be considered, but with more technical difficulties irrelevant for the analyses below –see [42] for the case where  $\mathcal{M}$  is a non-compact Lie group.

Let  $\mathfrak{g}$  be a compact Lie algebra (not to be confused with the compact Lie algebra  $\mathfrak{g}_c$  of Section 3.1) and assume that  $\mathfrak{g}$  is of rank  $r$ . Let  $\{H^i, i = 1, \dots, r\} \subset \mathfrak{g}$  be a Cartan subalgebra of  $\mathfrak{g}$ , let  $\Sigma$  be the root system of  $\mathfrak{g}$  and let  $E_\alpha, \alpha \in \Sigma$  be the corresponding root-vector. To ease the notations, we denote  $I = (LQR)$  with the notations of Section 3.1 then in the Cartan-Weyl basis the Kac-Moody algebra is generated by

$$\tilde{\mathfrak{g}}(\mathcal{M}) = \left\{ H_I^i, E_{\alpha I}, i = 1, \dots, r, \alpha \in \Sigma, I \in \mathcal{I}, D_i, k_i, i = 1 \dots, \ell' \right\}$$

and the non-vanishing Lie brackets are:

$$\begin{aligned}
[H_I^i, H_{I'}^{i'}] &= \eta_{II'} h^{ii'} \sum_{i=1}^{\ell'} I'(i) k_i , \\
[H_I^i, E_{\alpha J}] &= c_{IJ}{}^K \alpha^i E_{\alpha K} , \\
[E_{\alpha I}, E_{\beta J}] &= \begin{cases} \epsilon(\alpha, \beta) c_{IJ}{}^K E_{\alpha+\beta K} , & \alpha + \beta \in \Sigma, \\ c_{IJ}{}^K \alpha \cdot H_K + \eta_{IJ} \sum_{i=1}^{\ell'} J(i) k_i & \alpha + \beta = 0, \\ 0, & \begin{cases} \alpha + \beta \neq 0 , \\ \alpha + \beta \notin \Sigma, \end{cases} \end{cases} \\
[D_i, E_{\alpha J}] &= J(i) E_{\alpha J} , \\
[D_i, H_J^j] &= J(i) H_J^j ,
\end{aligned} \tag{6.1}$$

where

$$h^{ij} = \langle H^i, H^j \rangle_0 , \quad h_{ij} = (h^{-1})_{ij} , \quad \alpha \cdot H_I = h_{ij} \alpha^i H_I^j$$

with the Killing form  $\langle \cdot, \cdot \rangle_0$  defined in Section 3.1, and the operators associated to roots of  $\mathfrak{g}$  normalized as

$$\langle E_\alpha, E_\beta \rangle_0 = \delta_{\alpha, -\beta} .$$

The coefficients  $c(\alpha, \beta)$  depends on  $\mathfrak{g}$  and are equal to  $\pm 1$  if  $\mathfrak{g}$  is simply laced. Note the hermiticity relations:

$$(H_I^i)^\dagger = H_{I^c}^i , \quad (E_{\alpha I})^\dagger = E_{-\alpha I^c} , \quad (k_i)^\dagger = k_i ,$$

where  $I^c$  means  $I^c(i) = -I(i), i = 1, \dots, \ell'$ .

In a way entirely analogous to the usual affine Lie algebra (see e.g. [5], p. 343-344) [37], we can extend the Killing form to  $\tilde{\mathfrak{g}}(\mathcal{M})$  by:

$$\begin{aligned}
\langle \mathcal{T}_{aI}, \mathcal{T}_{bJ} \rangle_1 &= \eta_{IJ} k_{ab} , \\
\langle D_j, \mathcal{T}_{aI} \rangle_1 &= \langle k_j, \mathcal{T}_{aI} \rangle_1 = 0 , \\
\langle k_i, k_j \rangle_1 &= \langle D_i, D_j \rangle_1 = 0 , \\
\langle D_i, k_j \rangle_1 &= \delta_{ij} .
\end{aligned} \tag{6.2}$$

Let  $\mathcal{Q}$  be the possible eigenvalues of the Casimir operators, let  $q \in \mathcal{Q}$  and let  $\mathcal{D}^q$  be the corresponding representation. Define  $\mathcal{D}_0^q = \{\Psi_{q,1}, \dots, \Psi_{q,n_q}\} \subset \mathcal{D}^q$  to be the set of vectors with zero weight. Let  $\{\mathcal{H}_{q,1}^i, \dots, \mathcal{H}_{q,n_q}^i, i = 1, \dots, r\} \subset \{H_I^i, I, i = 1, \dots, r \in \mathcal{I}\}$  be the corresponding elements in  $\tilde{\mathfrak{g}}(\mathcal{M})$ . These elements are obviously commuting. <sup>12</sup> Thus from (6.1), the maximal set of commuting operators is given by

$$\tilde{\mathfrak{h}} = \left\{ \mathcal{H}_{n,q}^i, i = 1, \dots, r, q \in \mathcal{Q}, n = 1, \dots, n_q, D_i, k_i, i = 1, \dots, \ell' \right\} .$$

This means that the Kac-Moody algebra  $\tilde{\mathfrak{g}}(\mathcal{M})$  is an infinite rank Lie algebra. Due to the difficulty to deal with infinitely many commuting generators to be diagonalised simultaneously, we define the root-space of  $\tilde{\mathfrak{g}}(\mathcal{M})$  considering the finite-dimensional subalgebra  $\tilde{\mathfrak{h}}_0 \subset \tilde{\mathfrak{h}}$  defined by:

$$\tilde{\mathfrak{h}}_0 = \left\{ H_0^i, i = 1, \dots, r, D_i, k_i, i = 1, \dots, \ell' \right\} ,$$

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<sup>12</sup>For instance, if  $\mathfrak{g} = \mathfrak{su}(2)$ , all bosonic, or integer spin, representations have exactly one zero-weight vector, and in this case  $\tilde{\mathfrak{g}}(\mathrm{SU}(2))$  is an infinite rank-Lie algebra.

(with the notations of Sect. 3.1 for the definition of  $\ell'$ ) where  $H_0^i$  correspond to the generators of the ‘loop’ algebra  $\mathfrak{g}(\mathcal{M})$  associated to the trivial representation  $\mathcal{D}^0$  of  $\mathfrak{g}$ . The corresponding root spaces are given by

$$\begin{aligned}\mathfrak{g}_{(\alpha, n_1, \dots, n_{\ell'})} &= \left\{ E_{\alpha I} \text{ with } I(1) = n_1, \dots, I(\ell') = n_{\ell'} \right\}, \alpha \in \Sigma, n_1, \dots, n_{\ell'} \in \mathbb{Z}, \\ \mathfrak{g}_{(0, n_1, \dots, n_{\ell'})} &= \left\{ H_I^i \text{ with } I(1) = n_1, \dots, I(\ell') = n_{\ell'} \right\}, n_1, \dots, n_{\ell'} \in \mathbb{Z}.\end{aligned}\quad (6.3)$$

Since, in general, the number  $\ell'$  of compatible commuting operator  $D_I$  is such that  $\ell' < n = \dim \mathcal{M}$ , unlike the usual Kac-Moody algebras, the root spaces associated to roots are infinite dimensional and we have

$$\begin{aligned}[\mathfrak{g}_{(0, \mathbf{n})}, \mathfrak{g}_{(\alpha, \mathbf{m})}] &\subset \mathfrak{g}_{(\alpha, \mathbf{m+n})}, \\ [\mathfrak{g}_{(\alpha, \mathbf{m})}, \mathfrak{g}_{(\beta, \mathbf{n})}] &\subset \mathfrak{g}_{(\alpha+\beta, \mathbf{m+n})}, \quad \alpha + \beta \in \Sigma\end{aligned}$$

with  $\mathbf{n} = (n_1, \dots, n_{\ell'})$ . There is one notable exception. The Kac-Moody algebra associated to  $U(1)^n$  admits exactly  $\ell' = n$  compatible commuting operators [37] (these algebra are called toroidal algebras in [34] and quasi-simple Lie algebras in [32]). Thus, in this cases the root space  $\mathfrak{g}_{(\alpha, n_1, \dots, n_n)}$  is one-dimensional. Finally, it is important to observe that the Lie brackets between two elements involve not only the root structure, but also the representation theory of  $G_c$ , in the form of the Clebsch-Gordan coefficients  $c_{IJ}^K$  (see (6.1)).

Introduce  $\mathbf{0} = (0, \dots, 0)$ , then a root of  $\tilde{\mathfrak{g}}(\mathcal{M})$  is defined by  $\tilde{\alpha} = (\alpha, \mathbf{0}, \mathbf{n})$  were  $\alpha \in \Sigma$  corresponds to the root of the simple compact Lie algebra  $\mathfrak{g}$ , the following  $\ell'$  entries correspond to the vanishing eigenvalues of the central charges  $k_1, \dots, k_{\ell'}$  and  $\mathbf{n} \in \mathbb{Z}^{\ell'}$  are the eigenvalues of  $D_1, \dots, D_{\ell'}$ .

A root  $(\alpha, \mathbf{0}, n_1, \dots, n_{\ell'})$  is said to be positive if it satisfy:

$$(\alpha, \mathbf{0}, n_1, \dots, n_{\ell'}) > 0 \quad \text{if} \quad \begin{cases} \text{either} & \begin{cases} \exists k \in \{1, \dots, \ell'\} \text{ s.t.} \\ n_{\ell'} = \dots = n_{k+1} = 0 \text{ and } n_k > 0 \end{cases} \\ \text{or} & n_{\ell'} = \dots = n_1 = 0 \text{ and } \alpha > 0. \end{cases} \quad (6.4)$$

By (6.2), we can endow the weight space with the scalar product:

$$(\alpha, c_1, \dots, c_{\ell'}, n_1, \dots, n_{\ell'}) \cdot (\alpha', c'_1, \dots, c'_{\ell'}, n'_1, \dots, n'_{\ell'}) = \alpha \cdot \alpha' + \sum_{j=1}^{\ell'} (n_j c'_j + n'_j c_j),$$

where  $\alpha \cdot \alpha'$  is the usual scalar product in  $\Sigma$ .

As happens for usual Kac-Moody algebras [3], we have two types of roots. The set of roots  $(\alpha, \mathbf{0}, \mathbf{n})$  of  $\mathfrak{g}_{(\alpha, \mathbf{n})}$  with  $\alpha \in \Sigma, \mathbf{n} \in \mathbb{Z}^{\ell'}$  satisfy

$$(\alpha, \mathbf{0}, \mathbf{n}) \cdot (\alpha, \mathbf{0}, \mathbf{n}) = \alpha \cdot \alpha > 0,$$

and are called *real roots*, whilst the set  $(0, \mathbf{0}, \mathbf{n})$  of  $\mathfrak{g}_{(0, \mathbf{n})}$  with  $\mathbf{n} \in \mathbb{Z}^{\ell'}$  and satisfying

$$(0, \mathbf{0}, \mathbf{n}) \cdot (0, \mathbf{0}, \mathbf{n}') = 0,$$

is called the set of *imaginary roots*. The imaginary root-space is  $\ell'$ -dimensional.

Recall that  $\ell'$  denotes the number of central charges, that we call the order of centrality. We now show that unless  $\ell' = 1$ , we cannot find a system of simple roots for  $\tilde{\mathfrak{g}}(\mathcal{M})$ . To this extent, introduce  $\alpha_i, i = 1, \dots, r$  the simple roots of  $\mathfrak{g}$ . If  $\ell' = 1$ , and we denote by  $\psi$  the highest root of  $\mathfrak{g}$ , it is easy to see that

$$\hat{\alpha}_i = (\alpha_i, 0, 0), \quad i = 1, \dots, r, \quad \hat{\alpha}_{r+1} = (-\psi, 0, 1) \quad (6.5)$$

is a system of simple roots of  $\tilde{\mathfrak{g}}(\mathcal{M})$ . Now, if we suppose that  $\ell' = 2$ , as the positive roots are given by (i)  $(\alpha, 0, 0, 0, 0)$  with  $\alpha > 0$ , or (ii)  $(\alpha, 0, 0, n_1, 0)$  with  $\alpha \in \Sigma, n_1 > 0$ , or (iii)  $(\alpha, 0, 0, n_1, n_2), \alpha \in \Sigma, n_1 \in \mathbb{Z}, n_2 > 0$  and since the roots  $(\alpha, 0, 0, n_1, 0)$  are neither bounded from below nor from above because  $n_1 \in \mathbb{Z}$ , we cannot define a simple root of the form  $(-\psi, 0, 0, -n_{\max}, 1)$ , where  $n_{\max}$  corresponds to the highest possible value of  $n_1$  (or  $-n_{\max}$  the lowest possible value of  $n_1$ ). This means that for  $\ell' \geq 2$  we can't construct a system of simple roots. In other words, the only generalised Kac-Moody algebras that admit simple roots are (obviously) the usual affine algebras, but also the Kac-Moody algebras associated to  $SU(2)/U(1)$  studied in [37].

Note that there exists an alternative generalization of affine Lie algebras called also Kac-Moody algebras and introduced independently by Kac and Moody. These algebras admit a system of simple roots and are defined by a generalized Cartan matrix [3, 1, 73]. However both generalizations of affine Lie algebras, *i.e.*, Kac-Moody algebras associated to a Manifold  $\mathcal{M}$  and Kac-Moody algebras associated to a generalized Cartan matrix exhibit fundamentally different features. Indeed, the former is of infinite-rank –except for the algebra  $\tilde{\mathfrak{g}}(U(1)^n)$ , see Section 6.3–, but all its roots and the corresponding generators are explicitly known, whereas the latter has finite rank, but its generators are only iteratively known in terms of the Chevalley-Serre relations.

## 6.2 Some elements of representation theory

Let  $G_c$  be a compact Lie group and let  $H \subset G_c$  be a subgroup. Let  $\mathcal{M} = G_c$  or  $\mathcal{M} = G_c/H$  and let  $\tilde{\mathfrak{g}}(\mathcal{M})$  be the corresponding Kac-Moody algebra. Representation theory of  $\tilde{\mathfrak{g}}(\mathcal{M})$  is very different than representation theory of usual affine Lie algebras when  $\dim \mathcal{M} > 1$ . In fact we will show that firstly unitary representations of  $\tilde{\mathfrak{g}}(\mathcal{M})$  exist *iff* there is *only one non-vanishing* central charge. Further unitary representations forbid highest weigh representations.

To begin, consider the Kac-Moody algebra  $\tilde{\mathfrak{g}}(U(1)^n)$ . In the Cartan-Weyl basis we have

$$\tilde{\mathfrak{g}}(U(1)^n) = \{H_{\mathbf{m}}^i, E_{\alpha, \mathbf{m}}, \alpha \in \Sigma, \mathbf{m} \in \mathbb{Z}^n, d_i, k_i, i = 1, \dots, n\},$$

and the Lie brackets are given by ( $\mathbf{m} = (m_1, \dots, m_n)$ ) <sup>13</sup>

$$\begin{aligned} [H_{\mathbf{m}}^i, H_{\mathbf{m}'}^{i'}] &= h^{ii'} \sum_{i=1}^n m_i k_i \delta_{\mathbf{m}, -\mathbf{m}'}, \\ [H_{\mathbf{m}}^i, E_{\alpha \mathbf{m}}] &= \alpha^i E_{\alpha \mathbf{m} + \mathbf{n}}, \\ [E_{\alpha \mathbf{m}}, E_{\beta \mathbf{n}}] &= \begin{cases} \epsilon(\alpha, \beta) E_{\alpha + \beta \mathbf{m} + \mathbf{n}}, & \alpha + \beta \in \Sigma, \\ \alpha \cdot H_{\mathbf{m} + \mathbf{n}} + \sum_{i=1}^n m_i k_i \delta_{\mathbf{m}, -\mathbf{n}}, & \alpha + \beta = 0, \\ 0, & \alpha + \beta \neq 0, \\ & \alpha + \beta \notin \Sigma, \end{cases} \\ [d_i, E_{\alpha \mathbf{m}}] &= m_i E_{\alpha \mathbf{m}}, \\ [d_i, H_{\mathbf{m}}^j] &= m_i H_{\mathbf{m}}^j. \end{aligned} \tag{6.6}$$

Stress again that in this case the root-space  $\mathfrak{g}_{(\alpha, m_1, \dots, m_n)}, \alpha \in \Sigma, m_1, \dots, m_n \in \mathbb{Z}$  are one-dimensional. Moreover, because of this last property, the Cartan subalgebra, differently than for the generic cases ( $G_c \neq U(1)^n$ ), *is not* infinite dimensional, but its dimension is equal to  $\text{rk } \mathfrak{g} + 2n$ .

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<sup>13</sup>To be in accordance with the literature on affine Lie algebras, please note the change of sign in the central charges  $k_i$  compared to (6.1).

Recall that a root  $(\alpha, \mathbf{0}, \mathbf{m})$  is said to be positive if it satisfies (6.4). Assume that  $\text{rk } \mathfrak{g} = r$  and let  $\alpha_i, i = 1, \dots, r$  be the simple roots of  $\mathfrak{g}$ . Introduce further  $\mu^i, i = 1, \dots, r$  the fundamental weights of  $\mathfrak{g}$  satisfying

$$2 \frac{\alpha_i \cdot \mu^j}{\alpha_i \cdot \alpha_i} = \delta_i^j .$$

Since  $\mathfrak{g} \subset \tilde{\mathfrak{g}}(U(1)^n)$ , if  $\mathcal{D}$  is a unitary representation of  $\tilde{\mathfrak{g}}(U(1)^n)$ , it decomposes into  $\mathcal{D} = \bigoplus \mathcal{D}_i$  where  $\mathcal{D}_i$  are unitary representations of  $\mathfrak{g}$ . However, unitary representations of  $\mathfrak{g}$  are in one-to-one correspondence with highest weight vectors  $\mu_0 = \sum_{i=1}^r p_i \mu^i, p_i \in \mathbb{N}$ .

**Proposition 6.1 ([37])** *Let  $n > 1$  and let  $\tilde{\mathfrak{g}}(U(1)^n)$  be a generalized Kac-Moody algebra. Let  $\mu_0 = p_i \mu^i, p_i \in \mathbb{N}$  and suppose that  $|\mu_0, \mathbf{c}, \mathbf{m}_0\rangle$  is such that*

$$\begin{aligned} E_{\alpha, \mathbf{m}} |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle &= 0 \quad \text{if } (\alpha, \mathbf{0}, \mathbf{m}) > 0 , \\ H_{\mathbf{m}}^i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle &= 0 \quad \text{if } (0, \mathbf{0}, \mathbf{m}) > 0 , \\ H_{\mathbf{0}}^i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle &= \mu_0^i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle , \\ k_i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle &= c_i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle , \\ d_i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle &= (m_0)_i |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle . \end{aligned} \tag{6.7}$$

Define  $\mathcal{D}_{\mu_0, \mathbf{c}, \mathbf{m}_0}$  to be the highest weight representation obtained by the action of  $E_{\alpha, \mathbf{m}}$  with  $(\alpha, \mathbf{0}, \mathbf{m}) < 0$  on the vacuum state  $|\mu_0, \mathbf{c}, \mathbf{m}_0\rangle$ .

- (i) If  $\mathcal{D}_{\mu_0, \mathbf{c}, \mathbf{m}_0}$  is a unitary representation all central charges vanishes, but one.
- (ii) If  $|\mu_0, \mathbf{c}, \mathbf{m}_0\rangle$  is a highest weight satisfying (6.7) then  $\mathcal{D}_{\mu_0, \mathbf{c}, \mathbf{m}_0}$  is not a unitary representation.

*Proof.* (i) Let  $(-\alpha, \mathbf{0}, \mathbf{m})$  be a root of  $\tilde{\mathfrak{g}}(U(1)^n)$ . The generators

$$X_{\alpha, \mathbf{m}}^{\pm} = \sqrt{\frac{2}{\alpha \cdot \alpha}} E_{\mp \alpha, \pm \mathbf{m}}, \quad h_{\alpha} = \frac{2}{\alpha \cdot \alpha} \left( -\alpha \cdot H_{\mathbf{0}} + \sum_{i=1}^n m_i k_i \right),$$

span an  $\mathfrak{su}(2)$ -subalgebra. Assume  $(\alpha, \mathbf{0}, \mathbf{m}) > 0$  thus  $X_{\alpha, \mathbf{m}}^+ |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle = 0$ . Unitarity condition  $((X_{\alpha, \mathbf{m}}^+)^{\dagger} = X_{\alpha, \mathbf{m}}^-)$

$$\begin{aligned} \|X_{\alpha, \mathbf{m}}^- |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle\|^2 &= \langle \mu_0, \mathbf{c}, \mathbf{m}_0 | [X_{\alpha, \mathbf{m}}^+, X_{\alpha, \mathbf{m}}^-] |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle = \\ &= \langle \mu_0, \mathbf{c}, \mathbf{m}_0 | h_{\alpha} | \mu_0, \mathbf{c}, \mathbf{m}_0\rangle = \frac{2}{\alpha \cdot \alpha} \left( -\alpha \cdot \mu_0 + \sum_{i=1}^n c_i m_i \right) \geq 0 \end{aligned}$$

implies

$$\sum_{i=1}^n c_i m_i \geq \alpha \cdot \mu_0 . \tag{6.8}$$

Suppose  $\alpha > 0$  then  $\alpha \cdot \mu_0 > 0$ . In this case (6.8) is very strong. Indeed since  $(-\alpha, \mathbf{0}, \mathbf{m}) > 0$ , this means that  $\mathbf{m} = (m_1, \dots, m_{k-1}, m_k, 0, \dots, 0)$  with  $0 < k \leq n, m_k > 0$  and  $m_1, \dots, m_{k-1} \in \mathbb{Z}$ . The condition of (6.8), which must be satisfied for any  $m_i \in \mathbb{Z}, i = 1, \dots, k-1$ , is equivalent to impose that only one central charge is non-vanishing. This proves (i).

(ii): Since only one central charge is non-vanishing, without loss of generality we can suppose  $c_n = c \neq 0$  and  $c_i = 0, i = 1, \dots, n-1$ . Assume again  $\alpha > 0$  and let

$(-\alpha, \mathbf{0}, m_1, \dots, m_k, 0, \dots, 0) > 0$  with  $0 < k < n$ ,  $m_k > 0$  and  $(m_1, \dots, m_{k-1}) \in \mathbb{Z}^{k-1}$ . The operators (with  $\mathbf{m} = (m_1, \dots, m_k, 0, \dots, 0)$ )

$$Y_{\alpha, \mathbf{m}}^\pm = \sqrt{\frac{2}{\alpha \cdot \alpha}} E_{\mp \alpha, \pm \mathbf{m}}, \quad h'_\alpha = -\frac{2}{\alpha \cdot \alpha} \alpha \cdot H_{\mathbf{0}}, \quad (6.9)$$

also generate an  $\mathfrak{su}(2)$ -subalgebra. As before, the condition  $\|Y_{\alpha, \mathbf{n}}^- |\mu_0, \mathbf{c}, \mathbf{m}_0\rangle\|^2 \geq 0$  holds, from which we deduce that

$$\sum_{i=1}^k m_i c_i \geq \alpha \cdot \mu_0. \quad (6.10)$$

However, as  $c_1 = \dots = c_{n-1} = 0$ , this cannot be satisfied if  $\alpha > 0$ . Thus the representation  $\mathcal{D}_{\mu_0, \mathbf{c}}$  is not unitary. Which prove (ii). QED

**Corollary 6.2** *Let  $\mathfrak{g}$  be a simple compact Lie algebra, then the Kac-Moody Lie algebra  $\tilde{\mathfrak{g}}(U(1)^n)$ ,  $n > 1$  doesn't have any unitary highest weight representation.*

These results can be extended to the case of a more general Kac-Moody algebra as  $\tilde{\mathfrak{g}}(G_c)$  or  $\tilde{\mathfrak{g}}(G_c/H)$ . Some elements of the proof are given in [37]. It is important to observe that, in absence of symmetries between the generators  $D_i$ , we can have  $\ell'$  different possibilities given by (eventually reordering the eigenvalues of the operators  $D_i$  to define positive roots, see equation (6.4))

$$\mathbf{c} = (0, \dots, 0, c_p, 0, \dots, 0), \quad p \in \{1, \dots, \ell'\}.$$

These results have been obtained in a different manner in [74]. Let  $G_c$  be a compact Lie group and let  $\mathfrak{g}(G_c)$  and  $\tilde{\mathfrak{g}}(G_c)$  be respectively the ‘loop algebra’ (see (3.19)) and the Kac-Moody algebra (see Section 3.3). On purpose, the authors introduced spinors of  $G_c$  ( $G_c$  is assumed to be a spin manifold). Then, they observed that it is easy to obtain a unitary representation of the ‘loop’-algebra  $\mathfrak{g}(G_c)$ , however this representation is *not* a highest weight representation. In other words we get a ‘first quantized’ representation with no vacuum states annihilated by all fermion annihilation operators. They then showed that if we try to define a ‘second quantized’ or a highest weight representation, some divergences do appear (see p. 380, [74]) and operators are ill defined. However, considering bosonic condensate fermions, they were able to construct a highest weight representation, but no invariant inner product was identified.

To illustrate the situation, let us consider an explicit example. Let  $\mathfrak{g}$  be a compact Lie algebra and let  $\mathcal{D}$  be a real unitary  $d$ -dimensional representation. Denote the generators of  $\mathfrak{g}$  in the representation  $\mathcal{D}$  by the Hermitian matrices  $M_a$ ,  $a = 1 \dots, \dim \mathfrak{g}$ . Introduce now  $H^i$ ,  $i = 1, \dots, d$  real fermions in the representation  $\mathcal{D}$ .

Consider first the case where  $\mathcal{M} = \mathbb{S}^1$ . We have the following decomposition (to ease the presentation we only assume here Neveu-Schwarz (NS) –and not Ramond– fermions), *i.e.*, with anti-periodic boundary conditions:

$$H^i(\theta) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n^i e^{-in\theta}.$$

We further assume the quantization relations (with  $\{a, b\} = ab + ba$ )

$$\{b_n^i, b_m^j\} = \delta^{ij} \delta_{m, -n},$$

and the reality condition

$$(b_m^i)^\dagger = b_{-m}^i .$$

The vacuum of the representation is defined by

$$b_n^i |0\rangle = 0 , \quad n > 0 ,$$

in other words  $b_n^i, n < 0/n > 0$  are creation/annihilation operators. Consider now the current

$$T_a(\theta) = \frac{1}{2} H^i(\theta) (M_a)_{ij} H^j(\theta) .$$

Since  $T_a$  is bilinear in the fermion fields, in order to have well defined quantities, a normal ordering prescription must be defined. As usual we put to the right the creation operators:

$$\stackrel{\circ}{b}_n^i b_m^j \stackrel{\circ}{=} \begin{cases} -b_m^i b_n^j & \text{if } n > 0 \\ b_m^i b_n^j & \text{if } n < 0 \end{cases}$$

If we decompose  $T_a(\theta) = \sum_{M \in \mathbb{Z}} T_{aM} e^{-iM\theta}$  we obtain

$$T_{aM} = \frac{1}{2} (M_a)_{ij} \sum_{m \in \mathbb{Z}} \stackrel{\circ}{b}_m^i b_{M-m}^j \stackrel{\circ}{=} -\frac{1}{2} (M_a)_{ij} \sum_{m > M} b_{M-m}^j b_m^i + \frac{1}{2} (M_a)_{ij} \sum_{m < M} b_m^i b_{M-m}^j \quad (6.11)$$

This normal ordering prescription has two advantages: (i) the vacuum is well defined  $\langle 0 | T_{aM} | 0 \rangle = 0$  and (ii)  $\|T_{aM}|0\rangle\|$  is finite. For the second condition (ii) observe that in (6.11), in the second sum the term  $b_{M-m}^j$  is an annihilation operator and thus annihilate the vacuum. In the first sum the second term  $b_m^i$  is a creation operator if  $M < m < 0$ . Thus if  $M < 0$ , there are  $-M - 2$  possible values of  $m$  where  $b_{M-m}^j b_m^i$  acts non-trivially on  $|0\rangle$ , and consequently  $\|T_{aM}|0\rangle\|$  is finite. So the operators  $T_{aM}$  are always well defined. Furthermore, the normal ordering prescription is also of crucial importance. Indeed, using Wick theorem one can show that in fact the operators  $T_{aM}$  generate the affine Lie algebra  $\tilde{\mathfrak{g}}(\mathbb{S}^1)$  with a well defined central extension expressed in terms of the quadratic Casimir operator in the representation  $\mathcal{D}$  [5] and thus we obtain a unitary highest weight representation of  $\tilde{\mathfrak{g}}(\mathbb{S}^1)$ .

We now analyze briefly the case where  $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^1$ . Again considering (NS,NS) modes we get

$$H^i(\theta_1, \theta_2) = \sum_{m_1, m_2 \in \mathbb{Z} + \frac{1}{2}} b_{m_1 m_2}^i e^{-i(m_1 \theta_1 + m_2 \theta_2)} . \quad (6.12)$$

As for the circle, we assume the quantization relations

$$\{b_{m_1 m_2}^i, b_{n_1 n_2}^j\} = \delta^{ij} \delta_{m_1, -n_1} \delta_{m_2, -n_2} ,$$

and the reality condition

$$(b_{m_1 m_2}^i)^\dagger = b_{-m_1 -m_2}^i . \quad (6.13)$$

To define annihilation and creation operators we introduce the following order relation <sup>14</sup>:

$$(m, p) > 0 \iff m > 0 .$$

---

<sup>14</sup>Since we are considering only (NS,NS) fermions,  $m$  and  $p$  can't be equal to zero. For R-fermions we have to account on the possibility that  $m$  or  $n$  equal to zero to define our order relation.

The vacuum is defined by:

$$b_{m_1 m_2}^i |0\rangle = 0 , \quad m_1 > 0 , \quad \forall m_2 \in \mathbb{Z} + \frac{1}{2} ,$$

and the normal ordering prescription by:

$$\circ b_{mp}^i b_{nq}^j \circ = \begin{cases} -b_{nq}^j b_{mp}^i & \text{if } m > 0 \quad \forall p \in \mathbb{Z} + \frac{1}{2} \\ b_{mp}^i b_{nq}^j & \text{if } m < 0 \quad \forall p \in \mathbb{Z} + \frac{1}{2} \end{cases}$$

Decomposing as for the circle

$$T_a(\theta_1, \theta_2) = \frac{1}{2} (M_a)_{ij} \circ H^i(\theta_1, \theta_2) H^j(\theta_1, \theta_2) \circ$$

we get

$$\begin{aligned} T_{aMP} &= \frac{1}{2} (M_a)_{ij} \sum_{m,p \in \mathbb{Z} + \frac{1}{2}} \circ b_{mp}^i b_{M-mP-p}^j \circ \\ &= -\frac{1}{2} (M_a)_{ij} \sum_{m > M} \sum_{p \in \mathbb{Z} + (\frac{1}{2})} b_{M-mP-p}^j b_{mp}^i + \frac{1}{2} (M_a)_{ij} \sum_{m < M} \sum_{p \in \mathbb{Z} + (\frac{1}{2})} b_{mp}^i b_{M-mP-p}^j \end{aligned}$$

As for the circle the vacuum is well defined, but now  $\|T_{aMP}|0\rangle\|$  diverges because when  $M < 0$  the first sum above involves infinitely many terms (because of the sum over the second indice which belongs to  $\mathbb{Z}$ ). This situation has been analyzed explicitly in [40, 39] where in order to have well defined generators, beyond the usual normal ordering prescription, a regulator was introduced, and infinite sums were regularized by means of Riemann  $\zeta$ -function. Next, the normal ordering prescription also leads to a representation of  $\tilde{\mathfrak{g}}(\mathbb{S}^1 \times \mathbb{S}^1)$  but with only one central extension, and again the central charge can be expressed in terms of the quadratic Casimir operator of  $\mathfrak{g}$  in the representation  $\mathcal{D}$  [40]. Even if it was possible to construct fermions [40] or bosons [39] representations of  $\tilde{\mathfrak{g}}(\mathbb{S}^1 \times \mathbb{S}^1)$  this procedure is not fully satisfactory because some cut-off is needed in order to have well defined quantities. This construction is *not* in contradiction with Corollary 6.2 because we didn't consider any cut-off or regularization prescription in Proposition 6.1.

### 6.3 Some results on toroidal algebras

In this section we would like to study more into the details the Kac-Moody algebra  $\tilde{\mathfrak{g}}(U(1)^n)$ . We first consider the case  $n = 2$ , having in mind that the general case  $n > 2$  follows easily. Starting from the ‘loop’-algebra  $\mathfrak{g}(U(1)^2) = \{T_{am_1 m_2} = T_a e^{im_1 \theta_1 + im_2 \theta_2}, a = 1, \dots, \dim \mathfrak{g}, m_1, m_2 \in \mathbb{Z}\}$ , its central extensions are characterized by closed one-forms. Since  $H^1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \oplus \mathbb{Z}$  a general closed one-form reads:

$$\gamma = ik_1 d\theta_2 - ik_2 d\theta_1 - idh := k_1 \gamma_1 + k_2 \gamma_2 + \gamma_h ,$$

where  $k_1, k_2 \in \mathbb{R}$  and  $h$  is an arbitrary periodic function on the two-torus. Let  $h_{m_1, m_2}(\theta_1, \theta_2) = e^{im_1 \theta_1 + im_2 \theta_2}$ ,  $(m_1, m_2) \neq (0, 0)$  and let  $\gamma_{m_1, m_2} = -k_{m_1, m_2} dh_{m_1, m_2}$  be the corresponding exact one-form. The differential operators associated to the one-forms  $\gamma_1, \gamma_2, \gamma_{m_1, m_2}$  are respectively:

$$\begin{aligned} d_1 &= -i\partial_1 , \\ d_2 &= -i\partial_2 , \\ d_{n_1, n_2} &= e^{im_1 \theta_1 + im_2 \theta_2} (-im_2 \partial_1 + im_1 \partial_2) . \end{aligned}$$

A generic two-cocycle is thus given by

$$\begin{aligned}\omega(T_{an_1n_2}, T_{a'm'_1n'_2}) &= -k_1 \omega_1(T_{an_1n_2}, T_{a'm'_1n'_2}) - k_2 \omega_2(T_{an_1n_2}, T_{a'm'_1n'_2}) \\ &\quad - \sum_{(m_1, m_2) \in \mathbb{Z} \setminus \{0, 0\}} k_{m_1, m_2} \omega_{m_1, m_2}(T_{an_1n_2}, T_{a'm'_1n'_2})\end{aligned}$$

where (see Eq.[2.26])

$$\begin{aligned}\omega_1(T_{an_1n_2}, T_{a'm'_1n'_2}) &= n'_1 k_{aa'} \delta_{n_1, -n'_1} \delta_{n_2, -n'_2}, \\ \omega_2(T_{an_1n_2}, T_{a'm'_1n'_2}) &= n'_2 k_{aa'} \delta_{n_1, -n'_1} \delta_{n_2, -n'_2}, \\ \omega_{m_1, m_2}(T_{an_1n_2}, T_{a'm'_1n'_2}) &= k_{aa'} (m_2 n'_1 - m_1 n'_2) \delta_{n_1 + n'_1, -m_1} \delta_{n_2 + n'_2, -m_2} \\ &= k_{aa'} (n_1 n'_2 - n_2 n'_1) \delta_{n_1 + n'_1, -m_1} \delta_{n_2 + n'_2, -m_2}.\end{aligned}\tag{6.14}$$

These cocycles were also obtained in [33]. It is direct to observe by (2.28) that the differential operators  $d_1$  and  $d_2$  are compatible with the cocycles  $\omega_1, \omega_2, \omega_{m_1, m_2}$ .

In [34] Moody and his collaborators obtained, using Kähler differentials [75, 76], the universal central extension of the ‘loop’ algebra they called toroidal algebra. It turns out, that in fact the central extensions in [34] coincide with the two-cocycle above (6.14). Indeed, the first cases (resp. second cases) in Eq.[3] of [76] correspond to the cocycle  $\omega_{m_1, m_2}$  –after an appropriate rescalling– (resp.  $\omega_1, \omega_2$ ). Moreover, in [76] the authors also proved that their formulæ coincide with the formulæ obtained in [75] used by Moody and collaborators. This means that to centrally extend the ‘loop’-algebra  $\mathfrak{g}(U(1)^2)$  we can equivalently use the cocycle (6.14) or Kähler differentials as in [34].

For completeness and for further use, we briefly recall the main feature of the Moody *et al* construction [34]. Let  $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$  be the Laurent polynomial ring in two variables and let  $\mathfrak{g} \otimes \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$  be the ‘loop’ algebra (see (3.19) in our notations). To define the universal central extension of the ‘loop’ algebra introduce  $\Omega_1$  the set of one-forms of  $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ . Finally let  $\bar{\phantom{x}} : \Omega_1 \rightarrow \Omega_1 / \text{Imd}$ . Thus for any  $F \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$  we have  $\overline{dF} = 0$ . The universal central extension of the ‘loop’ algebra is given by [34]

$$[F \otimes x, G \otimes y]' = FG \otimes [x, y] + \overline{(dF)G} \langle x, y \rangle_0,$$

for any  $x, y \in \mathfrak{g}, F, G \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$  with  $[, ]$  (resp.  $\langle , \rangle_0$ ) the Lie brackets (the Killing form) of  $\mathfrak{g}$ . Since

$$\overline{d(z_1^{k_1} z_2^{k_2})} = k_1 \overline{z_1^{k_1} z_2^{k_2} z_1^{-1} dz_1} + k_2 \overline{z_1^{k_1} z_2^{k_2} z_2^{-1} dz_2} = 0,$$

a basis of  $\Omega_1 / \text{Imd}$  is given by

$$\begin{aligned}C_{k_1 k_2} &= \overline{z_1^{k_1} z_2^{k_2} z_2^{-1} dz_2} && \text{if } k_1 \neq 0 \\ C_{0 k_2} &= \overline{z_2^{k_2} z_1^{-1} dz_1} && \text{if } k_1 = 0, k_2 \neq 0 \\ c_1 &= \overline{z_1^{-1} dz_1}, \quad c_2 = \overline{z_2^{-1} dz_2} && \text{if } k_1 = k_2 = 0\end{aligned}$$

and

$$\overline{d(z_1^{n_1} z_2^{n_2}) z_1^{n'_1} z_2^{n'_2}} = \begin{cases} (n_2 n'_1 - n_1 n'_2) \frac{C_{n_1 + n'_1 n_2 + n'_2}}{n_1 + n'_1} & \text{if } n_1 + n'_1 \neq 0 \\ (n_2 n'_1 - n_1 n'_2) \frac{C_{0 n_2 + n'_2}}{n_2 + n'_2} & \text{if } n_1 + n'_1 = 0, n_2 + n'_2 \neq 0 \\ n_1 c_1 + n_2 c_2 & \text{if } n_1 + n'_1 = n_2 + n'_2 = 0 \end{cases}$$

which corresponds exactly to the cocycles given in (6.14). Thus the two constructions coincide.

Coming back to our notations, the toroidal algebra is  $T_2(\mathfrak{g}) = \{\mathcal{T}_{an_1n_2}, d_1, d_2, k_1, k_2, k_{m_1m_2}, a = 1, \dots, \dim \mathfrak{g}, n_1, n_2 \in \mathbb{Z}, (m_1, m_2) \in \mathbb{Z}^2 \setminus \{0, 0\}\}$  with Lie brackets (in our notations):

$$\begin{aligned} [\mathcal{T}_{an_1n_2}, \mathcal{T}_{a'n'_1n'_2}] &= if_{aa'}{}^b \mathcal{T}_{bn_1+n'_1n_2+n'_2} + (k_1n_1 + k_2n_2) k_{aa'} \delta_{n_1, -n'_1} \delta_{n_2, -n'_2} \\ &\quad + \sum_{(m_1, m_2) \in \mathbb{Z}^2 \setminus \{0, 0\}} k_{m_1m_2} (n_2n'_1 - n_1n'_2) k_{aa'} \delta_{n_1+n'_1, -m_1} \delta_{n_2+n'_2, -m_2}, \\ &= if_{aa'}{}^b \mathcal{T}_{bn_1+n'_1n_2+n'_2} + (k_1n_1 + k_2n_2) k_{aa'} \delta_{n_1, -n'_1} \delta_{n_2, -n'_2} \quad (6.15) \\ &\quad + k_{n_1+n'_1, n_2+n'_2} (n_2n'_1 - n_1n'_2) k_{aa'}, \\ [d_1, \mathcal{T}_{an_1n_2}] &= n_1 \mathcal{T}_{an_1n_2}, \\ [d_2, \mathcal{T}_{an_1n_2}] &= n_2 \mathcal{T}_{an_1n_2}. \end{aligned}$$

This algebra admits an interesting subalgebra where only two-central charges are non-vanishing, say  $k_1, k_2$  and corresponds to the Kac-Moody algebra considered throughout this review:  $\tilde{\mathfrak{g}}(U(1)^2) = \{\mathcal{T}_{am_1m_2}, d_1, d_2, k_1, k_2, a = 1, \dots, \dim \mathfrak{g}, m_1, m_2 \in \mathbb{Z}\}$ . This algebra is also named the double affine algebra in [77]. All the results of the two-toroidal and the double affine algebras extend naturally for  $n > 2$  [78], *i.e.*, to the  $n$ -toroidal and  $n$ -affine algebras.

Now we turn to representation theory of the double affine Lie algebra (or Kac-Moody algebra associated to  $\mathcal{M} = (U(1)^2)$ ). In Section 6.2 the results were given, considering the root system  $\Sigma$  of  $\mathfrak{g}$ . In [34] non-unitary representations of  $\tilde{\mathfrak{g}}(U(1)^2)$  were explicitly obtained considering the roots of  $\widehat{\mathfrak{g}}$ , the affine extensions of  $\mathfrak{g}$ . The key observation is the following: even if  $\tilde{\mathfrak{g}}(U(1)^2)$  doesn't admit a system of simple roots, it is possible to have a Chevalley-Serre presentation of  $\tilde{\mathfrak{g}}(U(1)^2)$  (and of course also of the toroidal algebra  $T_2(\mathfrak{g})$ ) in two different but equivalent manners—(1) with the simple roots of  $\mathfrak{g}$ , or (2) with the simple roots of  $\widehat{\mathfrak{g}}$ . Consider now the double-affine algebra or the Kac-Moody algebra  $\tilde{\mathfrak{g}}(U(1)^2)$  with two central extensions  $k_1, k_2$  whose Lie brackets are given by (6.6) with  $n = 2$  (note that all results presented below extend easily for the toroidal algebra with non-vanishing central charge  $k_{m_1m_2}$ ).

Let  $\alpha_i, i = 1, \dots, r$  be the simple roots of  $\mathfrak{g}$  and let  $A_{ij}$  be the corresponding Cartan matrix:

$$A_{ij} = 2 \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}, \quad 1 \leq i, j \leq r.$$

The matrix  $A$  is non-singular, *i.e.*,  $\det A \neq 0$ . Associated to any simple roots of the semisimple Lie algebra  $\mathfrak{g}$ , we define the three operators for  $(m_1, m_2) \in \mathbb{Z}^2$ :

$$h_{im_1m_2} = \frac{2}{\alpha_i \cdot \alpha_i} \alpha_i \cdot H_{m_1m_2}, \quad e_{im_1m_2}^\pm = \sqrt{\frac{2}{\alpha_i \cdot \alpha_i}} E_{\pm \alpha_i m_1, m_2}, \quad 1 \leq i \leq r.$$

The Chevalley-Serre relations are <sup>15</sup>

$$\begin{aligned} [k_1, h_{im_1m_2}] &= [k_1, e_{im_1m_2}^\pm] = [k_2, h_{im_1m_2}] = [k_2, e_{im_1m_2}^\pm] = 0 \\ [h_{im_1m_2}, h_{jn_1n_2}] &= (k_1m_1 + k_2m_2) \alpha_i^\vee \cdot \alpha_j^\vee \delta_{m_1, -n_1} \delta_{m_2, -n_2} \\ [h_{im_1m_2}, e_{jn_1n_2}^\pm] &= \pm A_{ij} e_{jm_1+n_1m_2+n_2}^\pm \\ [e_{im_1m_2}^\pm, e_{i,n_1n_2}^\pm] &= 0 \\ [e_{im_1m_2}^+, e_{jn_1n_2}^-] &= \delta_{ij} \left( h_{im_1+n_1m_2+n_2} + \frac{2}{\alpha_i \cdot \alpha_i} (k_1m_1 + k_2m_2) \delta_{m_1, -n_1} \delta_{m_2, -n_2} \right) \\ \text{ad}^{1-A_{ij}}(e_{imn}^\pm) \cdot e_{jpq}^\pm &= 0, \quad i \neq j, \end{aligned}$$

---

<sup>15</sup>In [34] the authors were given a presentation of the toroidal (and not the double-affine Lie) algebra.

$(\text{ad}(x) \cdot y = [y, x], \text{ad}^2(x) \cdot y = [[y, x], x], \text{ etc. and } \alpha^\vee = 2\alpha/\alpha \cdot \alpha \text{ is the co-root})$  enable to reproduce the whole algebra (see (6.6) with  $n = 2$ ) using the Serre relation, *i.e.*, the last equation.

The algebra  $\tilde{\mathfrak{g}}(U(1)^2)$  can be equivalently presented using the simple roots of the affine Lie algebra  $\hat{\mathfrak{g}}$  associated to  $\mathfrak{g}$ . Let  $\hat{\alpha}_{(i)}, i = 0, \dots, r$  be the simple roots of  $\hat{\mathfrak{g}}$ :

$$\begin{aligned}\hat{\alpha}_{(i)} &= (\alpha_i, 0, 0), \quad i = 1, \dots, r, \\ \hat{\alpha}_0 &= (-\psi, 0, 1),\end{aligned}\tag{6.16}$$

where  $\alpha_i, \psi$  are respectively the simple roots and the highest root of  $\mathfrak{g}$ . Let  $\hat{A}_{ij}$  be the Cartan matrix of  $\hat{\mathfrak{g}}$ :

$$\hat{A}_{ij} = 2 \frac{\hat{\alpha}_i \cdot \hat{\alpha}_j}{\hat{\alpha}_i \cdot \hat{\alpha}_i}, \quad 0 \leq i, j \leq r$$

where the scalar product (see for example [5]) is

$$\hat{\alpha} \cdot \hat{\alpha}' = (\alpha, k, m) \cdot (\alpha', k', m') = \alpha \cdot \alpha' + km' + k'm.$$

Note that now the Cartan matrix is singular and  $\dim \text{Ker} A = 1$ , *i.e.*,  $A$  is of corank one. Associated to any simple roots of  $\hat{\mathfrak{g}}$  we define for  $m \in \mathbb{Z}$ :

$$\begin{aligned}\hat{e}_{im}^\pm &= e_{i0m}^\pm, & \hat{h}_{im} &= h_{i0m}, & i &= 1, \dots, r \\ \hat{e}_{0m}^\pm &= \sqrt{\frac{2}{\psi \cdot \psi}} E_{\mp \psi \pm 1m}, & \hat{h}_{0m} &= \frac{2}{\psi \cdot \psi} (-\psi \cdot H_{1m} + k_1).\end{aligned}\tag{6.17}$$

This determines a presentation of the algebra as

$$\begin{aligned}[k_2, \hat{h}_{im}] &= [k_2, \hat{e}_{im}^\pm] = 0, \\ [\hat{h}_{im}, \hat{h}_{jn}] &= k_2 m \hat{\alpha}_i^\vee \cdot \hat{\alpha}_j^\vee \delta_{m,-n}, \\ [\hat{h}_{im}, \hat{e}_{jn}^\pm] &= \pm \hat{A}_{ij} \hat{e}_{jm+n}^\pm, \\ [\hat{e}_{im}^+, \hat{e}_{jn}^-] &= \delta_{ij} (\hat{h}_{im+n} + \frac{2}{\hat{\alpha}_i \cdot \hat{\alpha}_i} k_2 m \delta_{m,-n}), \\ [\hat{e}_{im}^\pm, \hat{e}_{in}^\pm] &= 0, \\ \text{ad}^{1-\hat{A}_{ij}}(\hat{e}_{im}^\pm) \cdot \hat{e}_{jn}^\pm &= 0, \quad i \neq j.\end{aligned}$$

Again the last relation, *i.e.*, the Serre relation enables to reproduce the whole algebra (6.6). Indeed it has been shown in [41] that the two presentations (6.16) and (6.18) leads to isomorphic algebras (6.6). Similar presentations (in terms of the roots of  $\mathfrak{g}$  or  $\hat{\mathfrak{g}}$ ) hold for the toroidal Lie algebra  $T_n(\mathfrak{g})$  and for the Kac-Moody algebra  $\tilde{\mathfrak{g}}(U(1)^n)$  (or  $n$ -affine Lie algebra) when  $n > 2$  [78].

Even if the two presentations are isomorphic they present some structural differences. Indeed, within the presentation of  $\tilde{\mathfrak{g}}(U(1)^n)$  with the roots of the semisimple Lie algebra  $\mathfrak{g}$  the Cartan matrix is non-singular, whereas with the presentation with the roots of the affine Lie algebra  $\hat{\mathfrak{g}}$  the Cartan matrix is singular. The presentation of torodial Lie algebras in terms of roots of affine Lie algebras has been used by several authors in order to construct explicit representations of  $T_2(\mathfrak{g})$  and  $T_n(\mathfrak{g}), n > 2$ . In [34, 78] a Vertex representation of the Toroidal Lie algebra  $T_n(\mathfrak{g})$  (when  $\mathfrak{g}$  is simply laced) was obtained. In [77] a fermion realization of  $\tilde{\mathfrak{g}}(U(1)^2)$ , *i.e.*, of the double affine algebra in the terminolgy of [77], when  $\mathfrak{g}$  is a classical Lie algebra was explicitly constructed. Note that representation theory of toroidal Lie algebras have been studied extensively (see e.g. [77] and references therein). However, since the Cartan matrix of  $\hat{\mathfrak{g}}$  is singular, and since all these articles are based on the Chevalley-Serre presentation of the algebra in terms of the roots of  $\hat{\mathfrak{g}}$ , all these representations are non-unitary.

## 6.4 Unitary representations

Now we turn to construct explicit unitary representations. On purpose we assume that the only non-vanishing central charges are  $k_1$  and  $c_m = -mk_{0,m}, m \neq 0$ . Let  $\bar{\mathfrak{g}} = \{\mathcal{T}_{anm}, k_1, c_k, a = 1, \dots, \dim \mathfrak{g}, m, n \in \mathbb{Z}, k \in \mathbb{Z} \setminus \{0\}\}$ . With these notations the non-vanishing Lie brackets (6.15) reduce to

$$[\mathcal{T}_{an_1 n_2}, \mathcal{T}_{a' n'_1 n'_2}] = i f_{aa'}{}^b \mathcal{T}_{bn_1 + n'_1 n_2 + n'_2} + n_1 \left( k_1 \delta_{n_2, -n'_2} + c_{n_2 + n'_2} \right) k_{aa'} \delta_{n_1, -n'_1} \quad (6.18)$$

Consider now the mapping

$$\begin{aligned} \mathcal{T}_{an_1 n_2} &\mapsto e^{in_2 \theta_2} \mathcal{T}_{an_1} \\ k_1 &\mapsto k_1 \\ c_m &\mapsto e^{im \theta_2} k_1 \end{aligned}$$

where  $\mathcal{T}_{am}$  are the generators of the affine Lie algebra  $\widehat{\mathfrak{g}}$ . Through this mapping (6.18) reduces to

$$[e^{in_2 \theta_2} \mathcal{T}_{an_1}, e^{in'_2 \theta_2} \mathcal{T}_{a' n'_1}] = i f_{aa'}{}^b e^{i(n_2 + n'_2)\theta_2} \mathcal{T}_{bn_1 + n'_1} + n_1 k_1 e^{i(n_2 + n'_2)\theta_2} h_{aa'} \delta_{n_1, -n'_1}, \quad (6.19)$$

and the two algebras (6.18) and (6.19) are isomorphic (see [34] Proposition 2.8, or [79] Theorem 3.3). It is immediate to observe that the latter algebra (6.19) is the loop algebra of the affine algebra and thus we have the following isomorphism  $\bar{\mathfrak{g}} \cong \widehat{\mathfrak{g}}(\mathbb{S}^1) \cong \widehat{\mathfrak{g}} \otimes \mathbb{S}^1$ . Please note that  $[d_2, c_m] = 0$ , but  $[d_2, e^{im \theta_2} k_1] = m e^{im \theta_2} k_1$ .

A certain class of unitary representations follows at once and are directly obtained from unitary representations of the affine Lie algebra  $\widehat{\mathfrak{g}}$ . Unitary highest weight representations of  $\widehat{\mathfrak{g}}$  are well known. The simple roots of  $\widehat{\mathfrak{g}}$  are given in (6.16) and the highest weight read (recall that  $\text{rk } \mathfrak{g} = r$ )

$$\frac{\psi}{\psi \cdot \psi} = \sum_{i=1}^r q^i \frac{\alpha_i}{\alpha_i \cdot \alpha_i}, \quad q^i \in \mathbb{N}$$

The corresponding fundamental weights [5] are

$$\hat{\mu}^0 = (0, \frac{1}{2} q^0 \psi \cdot \psi, 0) \quad \text{with} \quad q^0 = 1, \quad \hat{\mu}^i = (\mu^i, \frac{1}{2} q^i \psi \cdot \psi, 0), \quad i = 1, \dots, r.$$

Let  $\hat{\mu}_0 = \sum_{i=0}^r p_i \hat{\mu}^i, p_i \in \mathbb{N}$  and let  $|\hat{\mu}_0\rangle$  be a highest weight defined by (with the notations of Section 6.3, Eq.[6.17])

$$\begin{aligned} \hat{e}_i^+ |\hat{\mu}_0\rangle &= 0, \\ \hat{h}_i |\hat{\mu}_0\rangle &= p_i |\hat{\mu}_0\rangle. \end{aligned}$$

Because of the expression of  $h_0$  in term of the central charge  $k$  (we denote  $c$  its eigenvalue) a highest weight is equivalently specified by  $\mu_0 = \sum_{i=1}^r p_i \mu^i$  (the highest weight of the semisimple Lie algebra  $\mathfrak{g}$ , with its associated fundamental weight  $\mu^i$ ) and the eigenvalue of the central charge  $c$ <sup>16</sup>. Let  $x = 2 \frac{c}{\psi \cdot \psi}$  be the level of the representation. The representation  $\mathcal{D}_{\hat{\mu}_0}$  is unitary if (see e.g. [5]):

$$\left( x \in \mathbb{Z} \text{ and } c \geq \psi \cdot \mu_0 \geq 0 \right) \Leftrightarrow \left( x = \sum_{i=0}^r p_i q^i \in \mathbb{Z} \text{ and } x \geq \sum_{i=1}^r p_i q^i \right)$$

---

<sup>16</sup>The Cartan subalgebra of  $\widehat{\mathfrak{g}}$  is  $\{h_i, i = 0, \dots, r, k, d\}$ , but the eigenvalue of  $d$  for the highest state  $|\hat{\mu}_0\rangle$  is irrelevant. Indeed if  $d|\hat{\mu}_0\rangle = n_0|\hat{\mu}_0\rangle$ , redefining  $d \rightarrow d - \frac{n_0}{c}k$  we have  $d|\hat{\mu}_0\rangle = 0$  and we can take  $n_0 = 0$  [80].

Consider now

$$\mathcal{R} = \{|m\rangle, m \in \mathbb{Z}\}, \quad (6.20)$$

the set of unitary representations of  $U(1)$ :

$$d_2|m\rangle = m|m\rangle.$$

Using the harmonic expansion on  $U(1)$

$$\langle \theta_2 | m \rangle = e^{im\theta_2},$$

unitary representations of  $\bar{\mathfrak{g}}$  are given by the tensor product

$$\bar{\mathcal{D}}_{\hat{\mu}_0} = \mathcal{D}_{\hat{\mu}_0} \otimes \mathcal{R} \quad (6.21)$$

and correspond to a harmonic expansion of the unitary representation  $\mathcal{D}_{\hat{\mu}_0}$  of  $\hat{\mathfrak{g}}$  on the manifold  $U(1)$ . These results were anticipated in [37]. Note that whilst  $\mathcal{D}_{\hat{\mu}_0}$  is a highest weight representation of  $\hat{\mathfrak{g}}$ ,  $\bar{\mathcal{D}}_{\hat{\mu}_0}$  is not a highest weight representation of  $\bar{\mathfrak{g}}$ . A similar analysis holds for the algebra  $\hat{\mathfrak{g}} \otimes \mathbb{T}_{n-1}$ ,  $n > 1$  where  $T_{n-1}$  is the  $(n-1)$ -dimensional torus [37].

## 7 Applications in physics

The generalized KM current algebra  $\mathfrak{g}(\mathcal{M})$ , its semidirect symmetry actions, and its cohomological central extensions constitute a versatile toolkit, which can be used to investigate various subfields of high-energy theoretical physics. This section surveys three arenas in which the mathematical framework developed in this review finds concrete applications in physics. Our emphasis will be on : (i) two-dimensional current algebra and CFT, including WZW models, Sugawara stress tensors and the Virasoro algebra; (ii) higher-dimensional compactifications and spectra in KK theory, and (iii) structures emerging in cosmological billiards and in the hidden symmetries of supergravity. We will attempt at keeping the presentation self-contained and pedagogical, referring to the relevant literature for technical details and a broader background. Across all such three contexts, the crucial inputs have a threefold nature : geometric (e.g., choice of  $\mathcal{M}$  and its symmetry), analytic (e.g., harmonic analysis on  $\mathcal{M}$ ), and algebraic (e.g., compatibility of cocycles and representations).

### 7.1 Two-dimensional current algebra and CFT: WZW, Sugawara, Virasoro

#### *Affine symmetry from loops and its generalization*

On a two-dimensional worldsheet, currents  $J^a(z)$  valued in a finite-dimensional Lie algebra  $\mathfrak{g}$  satisfy the operator product expansions (OPEs) that encode an *affine* KM algebra at a certain level  $k$ . This structure emerges canonically in WZW models, where the basic field  $g(z, \bar{z}) \in G$  is a group-valued map and the action includes a Wess–Zumino topological term; see e.g. [4, 5, 6]. The holomorphic currents realize a centrally extended loop algebra of maps  $\mathbb{S}^1 \rightarrow \mathfrak{g}$ , and the stress tensor  $T(z)$  is obtained by the Sugawara construction, which expresses the Virasoro generators as quadratic combinations of KM modes, producing a central charge  $c = \frac{k \dim \mathfrak{g}}{k + \mathfrak{g}^\vee}$ , where  $\mathfrak{g}^\vee$  is the dual Coxeter number of  $\mathfrak{g}$  [5, 6].

Within the general framework presented in this review, the circle  $\mathbb{S}^1$  is replaced by a higher-dimensional manifold  $\mathcal{M}$ , thus yielding to Lie algebra  $\mathfrak{g}(\mathcal{M})$  of  $\mathfrak{g}$ -valued functions

on  $\mathcal{M}$ , organized through a Hilbert basis adapted to the symmetries of  $\mathcal{M}$  (by exploiting the Peter-Weyl theorem on compact groups/cosets, or the Plancherel theorem on non-compact groups/cosets). Semidirect actions by Killing vector fields or diffeomorphism algebras may enrich this symmetry, and compatible two-cocycles yield central extensions that generalize the affine case. From the two-dimensional worldsheet perspective, this construction can be viewed as a *mode expansion of currents along internal manifolds*: the worldsheet current algebra fibers over  $\mathcal{M}$ , so that KM modes carry, in addition, the harmonic labels on  $\mathcal{M}$  itself. This is particularly manifest when  $\mathcal{M}$  is a *compact* group manifold or a homogeneous space thereof, because in this case the harmonic analysis is representation-theoretically well-known and studied [4, 17, 16]. Possible issues related to normal ordering prescriptions in manifolds of dimension  $> 1$  have been discussed in [39, 40]; we also briefly recalled them in Sec. 6.

### **Group manifolds and coset CFTs**

Let  $\mathcal{M} = G_c$  be a compact Lie group manifold. The Peter-Weyl theorem provides an orthonormal basis  $\{\Psi_{LQR}\}$  in  $L^2(G_c)$ , built from matrix elements of irreducible unitary representations, and the product of two basis elements decomposes in terms of Clebsch–Gordan coefficients. The machinery developed in this review lifts the  $\mathfrak{g}$ -valued modes  $T_{a;LQR}$  to generators of a current algebra  $\mathfrak{g}(G_c)$  with structure constants directly expressed in terms of the representation theory of  $G_c$ . In the CFT context, this gives a natural language for coset models  $G/H$ , in which the surviving harmonics are the  $H$ -invariant components on the right (*or* left), and the operator content obeys the selection rules of the  $G \rightarrow H$  branching. This perspective clarifies how the coset primaries and their fusion rules relate to the representation-theoretic decomposition on  $G/H$ , as well as to the geometry of the coset [6, 13, 17].

### **Diffeomorphism algebras and Virasoro analogues**

For  $\mathcal{M} = \mathbb{S}^1$  the semidirect ‘partner’ of the affine algebra is the Witt algebra (and its central extension, the Virasoro algebra). On higher spheres  $\mathbb{S}^n$  or other higher-dimensional manifolds  $\mathcal{M}$ , families of vector fields play an analogous role. For instance, for  $\mathbb{S}^2$  the algebra of area-preserving diffeomorphisms is relevant, while, more generally, one can construct subalgebras of vector fields (generated by Killing vectors or selected modes) acting on  $\mathfrak{g}(\mathcal{M})$ . This leads to semidirect products that generalize to  $\mathcal{M}$  the affine–Virasoro ‘interplay’ on  $\mathbb{S}^1$ . Actually, this idea can be traced back to the early literature on generalized KM algebras related to diffeomorphism groups of closed surfaces [33, 32]. It should also here be recalled that the appearance of generalized (Borcherds–Kac–Moody) algebras in CFT and moonshine phenomena may be regarded as a hint to the depth of algebraic structures that can emerge beyond the affine case [19, 20, 21]..

## **7.2 Compactifications and KK spectra**

### **Mode expansions and mass towers**

Consider a  $d$ -dimensional field theory on a spacetime of the form  $\mathbb{R}^{1,d-1} \times \mathcal{M}$ , where  $\mathcal{M}$  is a compact internal manifold (group manifold, coset, or deformation thereof); the expansion of fields in an orthonormal basis of  $L^2(\mathcal{M})$  produces KK towers whose masses are determined by eigenvalues of Laplace-type operators on  $\mathcal{M}$ . This finds extensive application and crucial relevance in general KK theories, as well as in dimensional compactifications in (super)string theory or M-theory [8, 9, 10, 11, 12].

Within this framework, the generalization from  $\mathbb{S}^1$  to  $\mathcal{M}$  yields the following consequences: (i) the Noether charges associated with a compact gauge algebra  $\mathfrak{g}$  become

families of charges indexed by the harmonic labels on  $\mathcal{M}$ , thus realizing the Lie algebra  $\mathfrak{g}(\mathcal{M})$ ; (ii) the semidirect action of the isometries of  $\mathcal{M}$  provides an interesting geometric perspective on the selection rules and spectral degeneracies; (iii) the central extensions encode anomalies that can appear in the commutators of KK currents when integrating over  $\mathcal{M}$ , with the relevant 2-cocycles in one-to-one correspondence with closed  $(\dim \mathcal{M} - 1)$ -currents on  $\mathcal{M}$  (as developed in this review).

### ***Compact group manifolds and cosets***

For  $\mathcal{M} = G_c$ , the Peter-Weyl theory organizes the KK modes of fields into representations of  $G_c$ . The product of modes is controlled by Clebsch-Gordan coefficients, and the associated KM algebra  $\mathfrak{g}(G_c)$  provides underlies the interactions and selection rules among KK modes. For  $\mathcal{M} = G_c/H$ , the  $H$ -invariant modes in the right (or left) action are retained, thus reproducing the well-known truncations of coset compactifications and their spectra [17, 12, 13]. Here we confine ourselves to adding that this intriguingly relates to generalized Scherk–Schwarz reductions, where group-theoretic data constrain and determine consistent truncations of physical theories.

### ***Toroidal and deformed (soft) manifolds***

When the internal, higher-dimensional manifold is a torus, one recovers familiar Abelian current algebras and their higher-rank generalizations, while deformations of group manifolds (which go under the name of *soft* manifolds), as used in group-geometric approaches to supergravity, naturally fit into the same formalism [15, 12]. The central extension analysis identifies which deformations admit non-trivial 2-cocycles compatible with symmetry actions, and thus when anomalous terms can affect the reduced dynamics. Related algebraic structures, such as toroidal Lie algebras, also arise in the analysis of currents on higher-dimensional tori, and they have been studied in the mathematical literature since quite a long time [32, 34, 78].

### ***Kac–Moody symmetries in KK theories and beyond***

KM-like enhancements in the symmetry algebras of KK reductions and related stringy settings have been quite extensively investigated [60, 12, 29]. In this respect, we should stress that the present framework clarifies when and how such enhancements occur : the algebra  $\mathfrak{g}(\mathcal{M})$  organizes the towers of KK currents, thus allowing for compatible central extensions to emerge precisely when the geometry of  $\mathcal{M}$  admits closed  $(\dim \mathcal{M} - 1)$ -currents, in turn giving rise to the cohomological 2-cocycles. Therefore, this framework naturally highlights the dependence on the topology and metric of  $\mathcal{M}$ , thereby providing a controlled setting to explore consistent truncations, dualities and anomaly structures of higher-dimensional (super) gravity theories.

## **7.3 Cosmological billiards and hidden symmetries of supergravity**

### ***BKL dynamics and billiards***

Near spacelike singularities, the Belinsky-Khalatnikov-Lifshitz (BKL) analysis reveals chaotic, piecewise Kasner regimes interrupted by curvature-induced reflections - the so-called ‘mixmaster’ behavior [22, 81]. Remarkably, this dynamics can be geometrized as a billiard motion in a region of hyperbolic space bounded by walls associated to dominant gravitational and  $p$ -form contributions. Somewhat surprisingly, in many supergravity theories the billiard domain coincides with the fundamental Weyl chamber of a hyperbolic

KM algebra, thus suggesting that an underlying infinite-dimensional symmetry controls the near-singularity regime [23, 29].

### ***Hidden symmetries and very-extended KM algebras***

Dimensional reduction of supergravities give rise to large, non-compact global symmetries, controlling the electric-magnetic duality of the resulting theory; in string/M-theory, such a symmetry gets defined over discrete fields, and it is named *U*-duality. In general, the target space of scalar fields can be regarded as a manifold coordinatized by the scalars themselves. In a wide class of cases, the target space can be modeled as a sigma-model on cosets  $G/H$ , where  $G$  is a non-compact Lie group and  $H$  a maximal compact subgroup. The emergence of indefinite or hyperbolic extensions in further dimensional reductions as well as in conjectural uplifts ( $E_{10}/E_{11}$ -type structures) has been widely discussed, as it may be shown to encode ‘hidden’ symmetries of M-theory and cosmological dynamics (see e.g. [24, 25, 26, 27, 28], and references therein). In this respect, the framework presented in this review exhibits two remarkable features : (i) the sigma-model structure on non-compact target spaces (e.g.  $SL(2, \mathbb{R})$  and  $SL(2, \mathbb{R})/U(1)$ ) is inherently consistent with the non-compact harmonic analysis used to build  $\mathfrak{g}(\mathcal{M})$ ; (ii) the billiard/wall identifications hint at the possibility to describe the asymptotic dynamics through KM root systems, again consistently with the appearance of infinite-dimensional algebras in reductions and duality webs [23, 29].

### ***Cocycles, anomalies, and constraints***

The analysis of the possible central extensions of the current algebras on  $\mathcal{M}$  provides crucial insights on the existence of anomaly-like terms in the effective dynamics, as well as on the relevant constraints associated with conserved charges. In the aforementioned cosmological settings, integrating currents over compact slices (or along suitable cycles) in  $\mathcal{M}$  yields to Schwinger terms controlled by the cohomology of  $L^2(\mathcal{M})$ ; this essentially extends the Pressley-Segal cocycles in the affine case to the richer  $(\dim \mathcal{M} - 1)$ -current data. Thus, while many open questions remain (e.g. precise matching between very-extended algebras and full dynamics), the formal machinery and tools presented in this review may allow to establish when *bona fide* infinite-dimensional symmetries are actually compatible with the underlying geometry of the internal manifold and with the employed boundary conditions; at the same time, potential topological and/or flux obstructions may be detected.

## **8 Outlook: conjectures and novel applications**

The framework developed in this review - generalized KM current algebras  $\mathfrak{g}(\mathcal{M})$  on a manifold  $\mathcal{M}$ , semidirect actions by isometries/diffeomorphisms, and cohomological classification of central extensions - suggests several avenues for new applications in high-energy theoretical and mathematical physics. In this final section, we will briefly mention some potential, partly conjectural, developments in supergravity theories, superstring/M-theory, AdS/CFT and holography. The proposals below should be regarded as possible lines of investigation rather than established research venues; in fact, we will stress their *conjectural* nature, but also indicate how they could in principle be tested and checked. We leave all this for further, future works.

## 8.1 Supergravity

### *KM-structured charge algebras in consistent truncations*

Consistent truncations on group manifolds and cosets are organized in terms of geometric data (left-invariant frames, structure constants) and by harmonic analysis on  $\mathcal{M}$  [12, 15]. In light of the topics discussed in the present review, it may be conjectured that in any truncation admitting a global orthonormal frame (e.g. generalized parallelisations), the full tower of Noether charges of the truncated theory closes into a centrally-extended current algebra of the form  $\mathfrak{g}(\mathcal{M})$ , with 2-cocycles determined by closed  $(\dim \mathcal{M} - 1)$ -currents on  $\mathcal{M}$  itself. This conjecture is nothing but an extension of the standard, affine construction on the circle  $S^1$ , and it may potentially detect possible anomalies through the study of the cohomology on  $\mathcal{M}$ . Recent progress on consistent truncations in deformed/generalized settings provides us with a promising testing ground [13, 82]; in this respect, a concrete test may consist in computing Schwinger terms for KK-reduced currents in gauged supergravities, and match them against the cohomological classification of 2-cocycles outlined in this review.

### *Soft manifolds, fluxes, and anomaly constraints*

Group-geometric approaches embed supergravity in a *soft* deformation of group manifolds, encoding fluxes and torsion as geometric data [15, 12]. In this framework, it may be conjectured that the existence of non-trivial central extensions of  $\mathfrak{g}(\mathcal{M})$  compatible with isometries imposes integrability conditions on the flux backgrounds; such integrability conditions should turn out to be equivalent to a subset of the Bianchi identities and tadpole/anomaly-cancellation conditions in the dimensionally reduced, resulting physical theory. In practice, these conditions would translate into the co-closedness of certain  $(\dim \mathcal{M} - 1)$ -currents, as well as into quantization constraints for the Wess–Zumino-type terms. Remarkably, this would provide a powerful, algebraic criterion for the consistency of flux compactifications, as well as of their gauged supergravity limits [12, 13].

### *Hidden symmetries beyond billiards*

The appearance of hyperbolic KM structures in cosmological billiards suggests a more prominent role of infinite-dimensional symmetries in supergravity dynamics [23]. One may be led to naturally conjecture that, away from the near-singularity regime, a subset of these symmetries survives as an *algebra of generalized currents* on suitable slices (e.g., homogeneous spatial sections) of  $\mathcal{M}$ , with central terms fixed by boundary conditions and topological data. In principle, this conjecture should be testable/falsifiable within numerical relativity setups, or in analytic families of Bianchi cosmologies, by constructing conserved charges associated to divergence-free vector fields on  $\mathcal{M}$  and checking their commutators against the 2-cocycles discussed in this review. As another conjectural remark, we should not forget to mention potential connections to duality orbits in extended supergravities, which may further constrain the admissible cocycles [29].

## 8.2 Superstrings and M-theory

### *Worldvolume current algebras with internal labels*

In the WZW models and related worldsheet theories, the affine symmetry controls the spectrum and dynamics [5, 6]. Then, it may be conjectured a *lift* of such a framework, in which the currents acquire harmonic labels on an internal manifold  $\mathcal{M}$ , thus effectively realizing  $\mathfrak{g}(\mathcal{M})$  on the worldsheet. When  $\mathcal{M}$  is a group manifold or coset, Peter-Weyl theorem allows for the explicit construction of operator bases; for non-compact  $\mathcal{M}$ , Plancherel

theorem should provide the relevant distributions. All in all, this approach seemingly combines target-space geometry and worldsheet current algebras, in a way which is compatible with string field theory, as well as with group-geometric formulations [12, 15, 83]. A possible test of this conjecture would concern heterotic compactifications on group manifolds/cosets, in which one should compute OPE coefficients across harmonic sectors.

### ***Brane boundaries and defect algebras***

M2/M5 branes admit boundary/defect descriptions with current algebras on lower-dimensional intersections. One may thus put forward the conjecture that, in backgrounds with isometries along an internal manifold  $\mathcal{M}$ , the boundary algebra could be identified with a central extension of the generalized KM current  $\mathfrak{g}(\mathcal{M})$ , with the 2-cocycle determined by the pullback of fluxes to  $(\dim \mathcal{M} - 1)$ -cycles. Interestingly, this would generalize level quantization in WZW models to higher-dimensional defects. Hints of such structures have already appeared in early works on membranes [63], as well within the systematic investigation of WZW-like terms in string/M-theory [83].

### ***Non-geometric backgrounds and doubled/exceptional geometry***

In doubled and exceptional field theories, gauge symmetries generally mix diffeomorphisms and  $p$ -form transformations. In this context, we may conjecture that the algebra of generalized diffeomorphisms and gauge transformations on a compact manifold  $\mathcal{M}$  can be organized as a centrally extended  $\mathfrak{g}(\mathcal{M})$  built from the relevant finite-dimensional algebra  $\mathfrak{g}$  (e.g.  $E_{d(d)}$ ) and a  $L^2(\mathcal{M})$  basis. However, we should stress that the compatibility of the central extensions with section/closure constraints would become a sharp algebraic condition, possibly hard to meet/check. However, recent progress on deformed generalized parallelisations and consistent truncations [82] seemingly supports the idea that constructing such algebras on explicit  $\mathcal{M}$  may be feasible; in this context, flux quantization would thus arise out as the quantization of the central charge in the extended algebra. Among other aspects, this perspective could clarify the representation content of KK towers in exceptional field theory, as well as their coupling to the so-called ‘dual’ graviton sectors [29, 11].

## **8.3 AdS/CFT and holography**

### ***Asymptotic symmetries and boundary current algebras***

In  $AdS_3$ , the Virasoro symmetry arises as an asymptotic symmetry; in higher dimensions, boundary symmetries are more subtle. In this framework, it could be conjectured that, whenever the bulk includes an internal compact manifold  $\mathcal{M}$  with a certain isometry algebra, the boundary CFT exhibits a tower of conserved currents organized by  $\mathfrak{g}(\mathcal{M})$ , whose central terms should also be sensitive to holographic counterterms and global anomalies [84, 85, 86]. In practice, such a holographic renormalization supplies the bilinear form used to compute 2-cocycles (in terms of boundary two-point functions of currents), while the harmonic analysis on  $\mathcal{M}$  would determine the multiplet structure. This is in principle testable in truncations whose consistent embeddings are known, as well as in top-down approaches to holography, in which the manifold  $\mathcal{M}$  is explicitly specified [82].

### ***Coset holography and harmonic selection rules***

For compact  $\mathcal{M} = G_c/H$ , we expect a tight match between boundary operator algebras and the representation-theoretic decomposition on  $\mathcal{M}$ , governed by Clebsch-Gordan coefficients and  $H$ -invariance. We also expect the correlators to generally obey some selection

rules, which should in turn mirror the product structure of  $L^2(\mathcal{M})$ ; in this way, central terms should therefore be predicted by the cohomology classes of  $(\dim \mathcal{M} - 1)$ -currents. This seems to be potentially explorable in the holographic duals built from consistent truncations on  $G_c/H$ , and comparable with CFT data obtained via bootstrap or integrability methods [84, 86].

### *Holographic anomalies from cocycles*

One may also put forward the conjecture that certain boundary anomalies (e.g., of mixed flavor-gravitational type) could be captured by the cohomological 2-cocycles of  $\mathfrak{g}(\mathcal{M})$  computed from the bulk : in this sense, harmonic analysis and de Rham cohomology on  $L^2(\mathcal{M})$  [85, 84] would become crucial. This would be especially remarkable, since it would provide an algebraic holographic dictionary between fluxes on  $\mathcal{M}$  and the central extensions in the boundary current algebra.

All in all, one conceptual consequence of the programme reviewed in this work is the determination of a uniform algebraic language for the description and investigation of charges, anomalies and selection rules in string/M-theory, supergravity and holography. Even partial validations of the several conjectures stated above would shed light onto long-standing structural questions about hidden symmetries and consistent truncations; on the other hand, failures would be interesting too, since they would help establishing the true scope of current-algebraic methods beyond the circle, at least for what concerns the aforementioned physical applications.

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