

Graph conductance, synchronization, and a new bottleneck measure

C. Tyler Diggans
Root Dynamix, Sewickley, PA 15143

As a quantification of the main bottleneck to flow over a graph, the network property of conductance plays an important role in the process of synchronization of network-coupled dynamical systems. Diffusive coupling terms serve not only to exchange information between nodes within a networked system, but ultimately to dissipate the entropy of the collective dynamic state down toward that which can be associated with a single dynamic node when the synchronization manifold is stable. While the graph conductance can characterize the coupling strength that is required to maintain widespread synchronization across a majority of the nodes in such a system, it offers no guarantee for a stable synchronization manifold, which involves all nodes in the system. We define a new measure called the *synchronization bottleneck* of a graph, which we denote by Ξ ; this new network property provides a quantification of the limiting bottleneck of the flow between any subset of nodes (regardless of its order) and the rest of the networked system. This quantity does control the coupling strength required for a stable synchronization manifold for a large class of dynamical systems. Solving for this quantity is combinatorial, as is the case with conductance, but heuristics based on this optimization problem can guide decentralized strategies for improving global synchronizability.

INTRODUCTION

Synchronization of coupled dynamical systems has been widely studied for more than half a century under various circumstances [1–6] and for a wide range of applications [7–10]. This system-level phenomenon increasingly appears relevant to the proper functioning of many emergent and distributed systems [11, 12], and a large body of literature has thus been devoted to the optimization of existing network structures for more robust synchronization [13–18]. When considered as a process of transferring and exchanging information, as was suggested in [19], the role that the coupling network topology, and specifically graph conductance, plays in determining the synchronizability of a system becomes evident [20]. This is supported by the observation that the ideal network structures for synchronization are so-called entangled networks as described in [21], which have both low average degree and small shortest path lengths.

The concept of a Master Stability Function (MSF), introduced in [22], enabled a separation of the analysis for the impact of the network topology from that of the particular dynamics and coupling functions on the stability of synchronization for a large class of systems. This resulted in an increased focus on spectral graph theory in the study of synchronization, especially in the case of chaotic dynamics, where positive topological entropy oscillators bring a more complex consideration of information generation, flow, and dissipation through diffusive coupling.

The definition of isoperimetric numbers for graphs and the generalization of the work of Jeff Cheeger [23] that provides bounds on these graph properties using the smallest non-zero eigenvalue of various Laplacian matrices is a central topic in spectral graph theory [24–26]. This along with the out-sized importance of the small-

est non-zero eigenvalue in the MSF formalism then indicates a potential role that isoperimetric numbers, like conductance, may play in the synchronizability of such systems [20–22, 27].

Here, it is important to differentiate between the traditional isoperimetric number [24, 26], which is usually referred to as the Cheeger constant of a graph, and a similarly defined property, which is usually referred to as the graph conductance [25]. While the latter is the more functionally useful as a measure of the bottleneck to flow over a graph, the former is the property that is associated with the smallest non-zero eigenvalue of the basic combinatorial Laplacian matrix, and so it is more common. We will spend some time on this distinction in Section for clarity.

For a majority of the nodes in a system to exhibit some form of synchronization, it is reasonable to expect that the conductance of the coupling network must meet some minimal value to allow enough bulk information flow to facilitate a sufficient dissipation of the entropy of the dynamics. However, from the definition of conductance (provided in Sec.) we will see that the division by the minimum volume of the two resulting subsets from the minimum cut indicates that this property is not well-suited to characterize stable global synchronization.

We define a new optimization problem whose solution quantifies the limiting bottleneck of flow within a graph. Similar to other isoperimetric numbers, this property is defined through the minimization of a fractional argument over non-empty subsets of the vertex set, however, here we require the cuts to separate the graph into two connected components, where the worst case for synchronization would be represented by the two subsets being fully synchronized independently. Here the measure then quantifies the information flow that can happen over the cut; and as such, this property does in fact control the

required coupling strength for a system to achieve global synchronization for a large class of dynamics.

We begin by providing context through an outline of the basic Master Stability Function (MSF) formalism and definitions of the traditional isoperimetric numbers used in spectral graph theory. Then, the new synchronization bottleneck measure is defined, for which we have chosen the symbol Ξ due to a combination of its limited use elsewhere in the graph literature and its symbolic connotation with a bottleneck. This property is investigated for a large set of representative networks through the consideration of a single weighted path graph.

BACKGROUND

Diffusively Coupled Dynamical Systems

Assume a collection of N identical dynamical systems of the form $\dot{\mathbf{x}} = f(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^d$, diffusively coupled through a (weighted and undirected) network defined by the graph $G = (V, E, W)$, where W is an $N \times N$ symmetric matrix consisting of positive real entries $W_{i,j} > 0$ for each edge $\{i, j\} \in E$ and zero otherwise. Generalizations will apply to more complex coupling through consideration of the linearization, but our focus will be on the linear diffusively coupled system for clarity, where we use the combinatorial Laplacian matrix, $L = D - W$, with D being the diagonal matrix of row sums of W . A large class of systems can then be represented by the equation

$$\dot{\vec{x}} = F(\vec{x}) + \sigma L \otimes H(\vec{x}), \quad (1)$$

where $\vec{x} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N]$ represents the set of state vectors for N nodes, $F(\vec{x}) = [f(\mathbf{x}_1), f(\mathbf{x}_2), \dots, f(\mathbf{x}_N)]$ is the uncoupled dynamics, σ is a global coupling strength, and $(H(\vec{x}))$ is a matrix that represents a coupling function used for multidimensional state vectors; however, we restrict our discussion to the case of $H = \mathbb{I}_d$ for simplicity.

This vectorized equation (1) with H being the identity then encodes N coupled differential equations of the form

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \sigma \sum_j W_{ij} (\mathbf{x}_i - \mathbf{x}_j), \quad (2)$$

where the global coupling strength, σ , allows for variation in proportion to L of the impact of coupling in relation to the uncoupled dynamics.

As argued before in [19], the process of synchronization can be viewed as an exchange between (and dissipation of) the information that is associated with the set of initial conditions of the set of dynamical systems. More specifically, one can consider the unique infinite binary representation of the continuous valued initial state of a node at $t = 0$. A symbolic dynamics for the system can map the trajectory of that initial state through the

uncoupled dynamics, revealing an infinite symbolic representation that would map to the unique binary string. When coupled, the information being revealed through the symbolic dynamics of the unfolding system will no longer map to the set of initial conditions, and if the system synchronizes, then the dynamics is only capable of describing a single shared trajectory having an initial state that may be some form of weighted average of the set of initial conditions. It follows that the process of exchanging information through diffusive coupling also results in a loss of information about the complex initial state of the system.

Master Stability Function

The Master Stability Function (MSF) approach to stability analysis proceeds by considering the eigen-decomposition of Equ. (1), which leads to a set of variational equations for each component of the N state vectors. Each of these equations are of dimension d , but collectively can be represented by

$$\dot{\xi} = [\mathbf{1} \otimes Df - \sigma L \otimes DH] \xi. \quad (3)$$

A subsequent change of coordinates into the eigenbasis of L results in the N uncoupled variational equations

$$\dot{\zeta}_i = [Df - KH] \zeta_i, \quad (4)$$

which now represent the variation in the direction of each eigenvector of L where K plays the role of a σ -boosted eigenvalue, i.e. $K = \sigma\lambda$. Assuming the coupled system is path connected, there is a single direction associated with the eigenvalue $\lambda_1 = 0$ of L , which can be associated with the synchronization manifold.

The MSF, denoted as $\Psi(K)$, is then defined as the Largest Lyapunov Exponent (LLE) of the generic variational equation (4) as a function of the parameter K . In practice, $\Psi(K)$ is estimated numerically for a range of parameter values and the regions for which $\Psi(K) < 0$ are estimated; see [27] for a detailed analysis for many common chaotic oscillator systems where the dynamics are classified into various types associated with the number of roots of $\Psi(K)$. Given this MSF and a network, if a single σ value can be chosen such that $\Psi(\sigma\lambda_k) < 0$ for all $k \in 2, \dots, N$, then the synchronization manifold for those dynamics coupled over that network is deemed stable.

For those dynamics having a so-called Type II MSF, where $\Psi(K)$ is negative on a single interval (K_α, K_β) [27], a synchronizability ratio is often defined to be $R = K_\beta/K_\alpha$; it then follows that for any network having $\lambda_N/\lambda_2 < R$ there exists a $\sigma > 0$ such that the system will have a stable synchronization manifold for those dynamics. While that class have particularly interesting features, we will consider the more common case for

a dynamical system, which have been termed Type I MSFs in [27], wherein $\Psi(K) < 0$ on (K_α, ∞) for some $K_\alpha > 0$. In this case (and the case of zero-entropy oscillators where $K_\alpha = 0$), any connected network will be able to synchronize given a large enough global coupling strength σ . And here, σ can be seen as a dial on the rate of information exchange and dissipation. As such, the value of K_α defined by the MSF does control the minimum rate of exchange required for a stable synchronization manifold, but this is not directly related to graph conductance.

Recall, there are two main isoperimetric numbers defined for graph structures, and each has a spectral relationship to a different Laplacian matrix. We now review these relationships in greater detail before moving forward.

Isoperimetric Numbers, Graph Conductance, and the Cheeger Inequalities

There is a well-known association of the smallest non-zero eigenvalue of the graph Laplacian, $L = D - W$, with a measure of the main bottleneck for a graph [24]. This measure, often referred to as the Cheeger constant of the graph, is the discrete analog of the famous isoperimetric number for compact Riemannian manifolds defined by Jeff Cheeger [23]. However, this measure is distinct from the conductance of the graph, which is the more useful property when considering flows on a network. The discrepancy stems from the use of either the volume of a subset $S \subset V$ or the order of that set, $|S|$, in the denominator of the optimization argument. This difference has meaningful implications that lead to the bottleneck described being associated with either the normalized symmetric Laplacian, defined as $\mathcal{L} = \mathbb{I}_N - D^{-1/2}WD^{-1/2}$, or the combinatorial Laplacian, $L = D - W$, respectively.

The common convention (though not always followed, e.g., [25]) is to use the term Cheeger constant for the measure associated with L , while the term graph conductance is used for the measure associated with \mathcal{L} . We seek to avoid further confusion on this point by defining these quantities directly using some of the notation from [25].

Given a symmetric weighted graph $G = (V, E, W)$, we define the *edge boundary* of a set $S \subset V$ to be

$$\partial S = \{\{u, v\} \in E : u \in S, v \notin S\} \quad (5)$$

Similarly, we define the (outside) *vertex boundary* of a set $S \subset V$ to be

$$\delta S = \{v \notin S : \{u, v\} \in E, u \in S\} \quad (6)$$

The *Cheeger constant* of a graph, G , which we denote by h_G , can then be simply defined as

$$h_G = \min_{\{S \subseteq V : |S| \leq |V|/2\}} \frac{|\partial S|}{|S|}. \quad (7)$$

where $|\cdot|$ represents different operations depending on whether the argument is a set of vertices or edges. If the argument is an edge set, i.e. $F \subseteq E$, then $|F|$ represents the sum of edge weights in the edge set, $\sum_{f \in F} W_f$; on the other hand, if it is a set of vertices, $S \subset V$, then $|S|$ is simply the order of the vertex set, i.e. the number of vertices in the subset.

Solving the optimization problem (7) is computationally expensive, but for a connected graph, its value can be bounded using the value of λ_2 , i.e. the first non-zero eigenvalue of the Laplacian matrix L , i.e.

$$\lambda_2/2 \leq h_G \leq \sqrt{2\Delta\lambda_2}, \quad (8)$$

where Δ is the maximum degree in G [24, 26].

In contrast, the *graph conductance* of G , denoted by Φ_G , focuses on edge weights in the denominator by using the volume of the set S (instead of the order), i.e.

$$\Phi_G = \min_{\{S \subseteq V : vol(S) \leq vol(V)/2\}} \frac{|\partial S|}{vol(S)}, \quad (9)$$

where $vol(A) = \sum_{v \in A} deg(v) = \sum_{v \in A} \sum_{j=1}^N W_{ij}$ is the sum of the weights of edges with an endpoint in the set A .

Similar to the Cheeger constant, it is related to the eigenvalues of the normalized symmetric Laplacian, which we denote by μ_k for $k = 1, \dots, N$, by a Cheeger-like inequality [25],

$$\mu_2/2 \leq \Phi_G \leq \sqrt{2\mu_2}. \quad (10)$$

These bounds are not generally tight in either case, but it has been argued that Φ_G is more fundamental than h_G as the resulting inequalities are not scaled by the maximum degree. This may be the reason that some references use the term Cheeger constant for Φ_G , and may even use h_G instead of Φ_G [25]. Regardless of naming conventions, both of these properties indicate some ability of the majority of the system to synchronize, i.e. something akin to a “giant synchronizable component” of the system. However, in practice, neither can be used to effectively quantify the coupling strength required to achieve even that level of synchronization due to the looseness of the Cheeger bounds.

And with respect to global synchronization, it is entirely possible for a small number of nodes, or even a single node (a pendant vertex with a small weight edge connecting it to the rest of the graph) to be the limiting bottleneck in a system’s ability to maintain global synchronization. This detail leads us to a new measure.

THE SYNCHRONIZATION BOTTLENECK

We define the *synchronization bottleneck* for a weighted symmetric graph $G = (V, E, W)$ as

$$\Xi = \Xi(G) = \min_{S \subseteq V} \frac{|\partial S|^2}{\text{vol}(\delta \bar{S})}, \quad (11)$$

where \bar{A} is the compliment of A , i.e. $V \setminus A$. And, since $\text{vol}(\delta \bar{S})$, by definition, includes the edge weights in the sum $|\partial S|$, the synchronization bottleneck associated with a particular set S satisfies $0 < \Xi_G(S) \leq |\partial S|$. Further, whenever the set S consists of some collection of pendant vertices, then $\text{vol}(\delta \bar{S}) = |\partial S|$ and so $\Xi_G(S) = |\partial S|$. But, since we are minimizing this argument, we need only consider the set of *minimal cuts* where the resulting sets S and $V \setminus S$ are both connected components. Finally, whenever the set S has edges that are internal to S , then we have $\Xi_G(S) < |\partial S|$.

While this new graph property is defined in the same spirit as the two isoperimetric numbers, h_G and Φ_G , due to the potential for S to be a very small subset, we do not initially expect any straight forward relation to the graph spectra through Cheeger-like inequalities. Regardless, this property does capture an important feature of the graph for the stability of global synchronization, especially in the case of dynamical systems with Type I MSF, having many applications across domains, e.g., electricity transmission grids.

RESULTS

Because the measure Ξ is entirely defined in terms of edge weights, a simple model of the weighted path graph on five vertices suffices to explore a large set of interesting cases. Figure 1 shows an example of one such graph where one of the links maintains a weight of one and plays the role of the cut that defines the graph conductance Φ_G by design, while the three other links take on general values of w_l , w_c and w_r , where we assume $w_l > w_c \gg 1 > w_r$ in order to create a case where the two cuts that define Ξ_G and Φ_G are not equal.

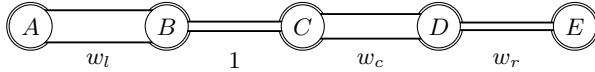


FIG. 1. A weighted path graph on five vertices, which can represent the bottlenecks for dynamics coupled on many graphs when $w_l > w_c \gg 1 > w_r$.

If we further assume $w_c + w_r < w_l$ and that $w_r > 1/2$, then it is straight forward to show that $h_G = 1/2$ and $\Phi_G = 1/w_l$, both relying on the minimizer set $S = \{A, B\}$ with the minimum cut being the single edge $\{B, C\}$. It may be instructive to note that if $w_r < 1/2$,

h_G would then be defined in terms of the cut $\{D, E\}$, while Φ_G would remain defined by the cut $\{B, C\}$; this being an indication of why Φ is generally preferable as a measure of the main bottleneck to flow.

If we now consider the value of the argument $|\partial S|^2/\text{vol}(\delta \bar{S})$ over all subsets $S \subseteq V$ where both S and $V \setminus S$ are connected components, we find the values in Table I. For the example values of $w_l = 5$ and $w_c = 3$, the minimal value of $\Xi = w_r^2/(w_c + w_r)$ is achieved by the minimizer set $S = \{A, B, C, D\}$ with the graph cut being the single edge $\{D, E\}$ with weight w_r . This is true for all potential values of $0 < w_r < 1$, meaning this graph property is robust and identifies the true limiting bottleneck to global flow between a subset of the nodes and the rest of the graph. It is instructive to also consider the same graph as the weight w_l increases for a fixed value of w_r ; eventually, the minimizer would become $S = \{A, B\}$ as the strength of coupling between A and B becomes so strong that the flow to C becomes the limiting bottleneck to global synchronization. Of course, for systems with a Type I MSF, this just means that the coupling strength must be increased to overcome this bottleneck, but this case indicates the added challenge that arises with Type II MSF systems where the diameter of the subsets may play an important role still to be determined.

TABLE I. The computation of $\Xi = |\partial S|^2/\text{vol}(\delta \bar{S})$ for all subsets S resulting from a minimal cut of the vertex set $V = \{A, B, C, D, E\}$ into two connected components, where the minimizer is the set $S = \{A, B, C, D\}$ leading to the synchronization bottleneck value of $\Xi = w_r^2/(w_r + w_c)$. Considering the specific values of $w_l = 5$ and $w_c = 3$ with $w_r < 1$, we compare results for $w_r = 0.75$ with those for $w_r = 0.25$ to provide an illustrative case where h_G and Φ_G disagree.

S	$ \partial S $	$\delta \bar{S}$	$\text{vol}(\delta \bar{S})$	$\frac{ \partial S ^2}{\text{vol}(\delta \bar{S})}$	$w_r = 0.75$	$w_r = 0.25$
$\{A\}$	w_l	$\{A\}$	w_l	w_l	5	5
$\{E\}$	w_r	$\{E\}$	w_r	w_r	0.75	0.25
$\{A, B\}$	1	$\{B\}$	$w_l + 1$	$\frac{1}{w_l + 1}$	0.167	0.167
$\{D, E\}$	w_c	$\{D\}$	$w_c + w_r$	$\frac{w_c^2}{w_c + w_r}$	2.4	2.769
$\{A, B, C\}$	w_c	$\{C\}$	$1 + w_c$	$\frac{w_c^2}{1 + w_c}$	2.25	2.25
$\{C, D, E\}$	1	$\{C\}$	$1 + w_c$	$\frac{1}{1 + w_c}$	0.25	0.25
$\{A, B, C, D\}$	w_r	$\{D\}$	$w_c + w_r$	$\frac{w_r^2}{w_c + w_r}$	0.15	0.0192
$\{B, C, D, E\}$	w_l	$\{B\}$	$1 + w_l$	$\frac{w_l^2}{w_l + 1}$	4.167	4.167

Having obtained the graph cut that defines this synchronization bottleneck, we are not better informed about the stability of the synchronization manifold, since we do not have any Cheeger-like bounds connecting this feature to the MSF formalism. However, we are informed about the most important links where increasing edge weights would improve the stability in this graph at least for any dynamics having a Type I MSF for which the synchronization manifold is already stable. The MSF formalism does not help us in this respect either, and

furthermore, the serious challenges in quantifying information flows become paramount to making additional progress on this point. But, for many practical applications, the simple suggestion to increase the weights of the edges of this cut to improve the synchronizability of any dynamics on this graph structure can be helpful.

CONCLUSIONS

We have described the reasons behind why both the graph conductance and the more traditional Cheeger constant are not sufficient to characterize global synchronization of all nodes in a graph. We have then defined a new graph property that successfully characterizes a limiting bottleneck for global synchronization in a network. However, it is important to note that this property is not sufficient to characterize synchronizability in the cases of dynamical systems with MSFs of Type II and higher, where $\Psi(K) < 0$ on either bounded intervals or multiple intervals, as these systems introduce an added issue with synchronization over long diameter graphs. In these cases, it is not merely a bottleneck to flow that is the problem, but likely the speed at which information is exchanged with respect to the dynamic's Lyapunov exponent that ultimately control global synchronization.

Despite this more general failing, the graph property Ξ does in fact control the synchronizability of a graph for any dynamics having a Type I MSF (and is informative of at least a bound on coupling strengths that ensure synchronization for other odd type MSF), meaning that if a graph G has a stable synchronization manifold for a given system with that type and a second graph H is such that $\Xi(H) \geq \Xi(G)$, then H will also exhibit the same kind of stability of the synchronization manifold for that same system of dynamics.

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- [1] Yoshiki Kuramoto. Self-entrainment of a population of coupled non-linear oscillators. In Araki H., editor, *Lecture Notes in Physics*, volume 39. Springer, Berlin, Heidelberg, 1975.
 - [2] Louis M. Pecora and Thomas L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64:821–824, Feb 1990.
 - [3] Michael G Rosenblum, Arkady S Pikovsky, and Jürgen Kurths. Phase synchronization of chaotic oscillators. *Physical review letters*, 76(11):1804, 1996.
 - [4] Takashi Nishikawa, Adilson E. Motter, Ying-Cheng Lai, and Frank C. Hoppensteadt. Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize? *Phys. Rev. Lett.*, 91:014101, Jul 2003.
 - [5] Igor Belykh, Martin Hasler, Menno Lauret, and Henk Nijmeijer. Synchronization and graph topology. *International Journal of Bifurcation and Chaos*, 15(11):3423–3433, 2005.

- [6] Jie Sun, Erik M Bollt, and Takashi Nishikawa. Synchronization stability of coupled near-identical oscillator network. In *Complex Sciences: First International Conference, Complex 2009, Shanghai, China, February 23–25, 2009. Revised Papers, Part 1*, pages 900–911. Springer, 2009.
- [7] Arthur T. Winfree. Biological rhythms and the behavior of populations of coupled oscillators. *Journal of Theoretical Biology*, 16(1):15–42, 1967.
- [8] Bhargava Ravoori, Adam B Cohen, Jie Sun, Adilson E Motter, Thomas E Murphy, and Rajarshi Roy. Robustness of optimal synchronization in real networks. *Physical review letters*, 107(3):034102, 2011.
- [9] Adilson E Motter, Seth A Myers, Marian Anghel, and Takashi Nishikawa. Spontaneous synchrony in power-grid networks. *Nature Physics*, 9(3):191–197, 2013.
- [10] Rafael Gallego, Ernest Montbrió, and Diego Pazó. Synchronization scenarios in the winfree model of coupled oscillators. *Phys. Rev. E*, 96:042208, Oct 2017.
- [11] Stefano Boccaletti, Vito Latora, Yamir Moreno, Martin Chavez, and D-U Hwang. Complex networks: Structure and dynamics. *Physics reports*, 424(4–5):175–308, 2006.
- [12] Florian Dörfler and Francesco Bullo. Synchronization in complex networks of phase oscillators: A survey. *Automatica*, 50(6):1539–1564, 2014.
- [13] Juan G Restrepo, Edward Ott, and Brian R Hunt. Characterizing the dynamical importance of network nodes and links. *Physical review letters*, 97(9):094102, 2006.
- [14] Louis M Pecora. Synchronization of oscillators in complex networks. *Pramana*, 70:1175–1198, 2008.
- [15] Aric Hagberg and Daniel Schult. Rewiring networks for synchronization. *Chaos*, 18:037105, 10 2008.
- [16] Mahdi Jalili. Enhancing synchronizability of diffusively coupled dynamical networks: a survey. *IEEE transactions on neural networks and learning systems*, 24(7):1009–1022, 2013.
- [17] Per Sebastian Skardal, Dane Taylor, and Jie Sun. Optimal synchronization of complex networks. *Physical review letters*, 113(14):144101, 2014.
- [18] Mahyar Fazlyab, Florian Dörfler, and Victor M Preciado. Optimal network design for synchronization of coupled oscillators. *Automatica*, 84:181–189, 2017.
- [19] Erik M Bollt. Synchronization as a process of sharing and transferring information. *International Journal of Bifurcation and Chaos*, 22(11):1250261, 2012.
- [20] C Tyler Diggans, Jeremie Fish, Abd AlRahman R AlMomanii, and Erik M Bollt. The essential synchronization backbone problem. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 31(11), 2021.
- [21] Luca Donetti, Pablo I Hurtado, and Miguel A Munoz. Entangled networks, synchronization, and optimal network topology. *Physical Review Letters*, 95(18):188701, 2005.
- [22] Louis M. Pecora and Thomas L. Carroll. Master stability functions for synchronized coupled systems. *Phys. Rev. Lett.*, 80:2109–2112, Mar 1998.
- [23] Jeff Cheeger. “a lower bound for the smallest eigenvalue of the laplacian”. In R. C. Gunning, editor, *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, page 195–199. Princeton University Press, Princeton, 1970.
- [24] Bojan Mohar. Isoperimetric numbers of graphs. *Journal of combinatorial theory, Series B*, 47(3):274–291, 1989.
- [25] F.R.K. Chung. *Spectral Graph Theory*. Number 92 in

- CBMS Regional Conference Series. American Mathematical Society, 1997.
- [26] Bogdan Nica. A brief introduction to spectral graph theory. In *EMS Textbooks in Mathematics*. European Mathematical Society, 2018.
- [27] Liang Huang, Qingfei Chen, Ying-Cheng Lai, and Louis M Pecora. Generic behavior of master-stability functions in coupled nonlinear dynamical systems. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, 80(3):036204, 2009.