
Part III

Outlook

From the structures that we have discussed in Parts I and II, a large number of new structures could easily be obtained by combination. For example, one could consider variants of each algebraic structure in which the underlying sets are topological spaces, manifolds, or varieties, and the operations are morphisms in the respective category. However, algebraic and local structures can also be mixed with other structures that are neither algebraic nor local. For example, one could require that there is a measure on a manifold that is defined on the σ -algebra generated by the open subsets. Or one might want to presuppose the existence of an orientation. However, most conceivable combined structures are not considered let alone studied in detail in practice.

There are different motivations to study new structures. One motivation can be that examples of such structures appear in already studied mathematical problems. For example, linear algebra originated as a structural abstraction of the study of (linear) differential equations. Another motivation is that additional structural elements are suggested by the mathematical modeling of physical or other phenomena. For many applications, the existence of certain functions is necessary for modeling the relevant situations. In relativity theory, for example, some kind of distance measurement in space-time is needed. In this section, we describe various classes of such specialized structures. We provide motivations and precise definitions, but largely refrain from detailed results. The structures described are each the subject of their own mature theory, and we limit ourselves to providing standard literature on the respective theory, which may serve the reader as a starting point for a search for more detailed information.

One could call the entire field of functional analysis, whose development is closely linked to the mathematical modeling of quantum physics, an example class for the study of specialized structures. It combines vector spaces with norms or topologies, that is, functional analysis deals with special vector spaces. However, the focus is on the fact that functional analysis generalizes methods and results of linear algebra to certain infinite-dimensional vector spaces. This draws attention to generalizations of structures and theories that are developed to solve given problems that could not be tackled with conventional methods. For example, it happens that a

certain combination of properties is needed to be able to use a known technique. The introduction of complex numbers in the study of polynomial equations can be read in this way, as can the introduction of distributions to solve differential equations. In both cases, these are extensions of the mathematical framework to be able to describe solutions to given problems in this extended framework, which can then be further investigated. The extension of \mathbb{C} -varieties to schemes to study rationality questions can also be justified in this way.

Modifying or replacing an existing structure can also aim to unify or simplify arguments from separate contexts. Given the rapid development of mathematical knowledge in many different directions, unification is a quite serious task. We will discuss two principles of unification separately here. The transfer of arguments to other areas, illustrated by homological algebra as a technology of structure comparison, and the transfer of structural elements, illustrated by the concept of the group object for the formalization of symmetries. Chapter 8 is dedicated to these illustrations. In Chap. 9, we then discuss specialized structures as well as possible generalizations and unifications.

Since very different contents are only treated very briefly in this part, each section contains its own annotated bibliography.