

# Universality of the chiral soliton lattice in the low energy limit of QCD

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In this paper, we show that the chiral soliton lattice (ChSL) is, in a precise sense, a universal feature of the low-energy limit of QCD minimally coupled to Maxwell theory. Here, we disclose that not only can the ChSL be obtained from the gauged Skyrme model in 3+1 dimensions, including the back-reaction of the Maxwell  $U(1)$  gauge field, we also show that the ChSL remains unchanged if the sub-leading corrections to the Skyrme model in the 't Hooft large  $N_c$  expansion are included. Taking into account the highly non-linear character of such corrections, this is quite a surprising result. By considering a suitable ansatz adapted to describe topological solitons at finite baryon density in a constant magnetic field, the generalized Skyrme model coupled to the Maxwell theory is reduced to the effective Lagrangian of the ChSL phase, describing a lattice of domain walls made of hadrons. One of the key points in this construction is the fact that even when the usual topological charge density vanishes, the presence of the Callan-Witten term in the topological charge density allows for a non-vanishing baryon number. In the present approach, the magnetic field can be external, as is usually assumed for the ChSL, or it can be self-consistently generated by the hadronic layers themselves. Finally, we show how our formulation allows us to study the coupling of the chiral soliton lattice with quark matter.

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## I. INTRODUCTION

The ChSL is a hadronic phase consisting of a lattice of aligned parallel domain walls -two-dimensional topological solitons- separated at equal space intervals. This phase appears in many contexts, from condensed matter physics to high-energy physics [1]-[4]. In recent years, the ChSL has attracted a lot of attention due to its role in quantum chromodynamics (QCD) under extreme conditions. In fact, it turns out that the ChSL is the ground state of QCD at finite baryon chemical potential in the presence of a critical homogeneous magnetic field [5]-[8]. Beyond a sufficiently strong magnetic field, the ChSL becomes the energetically preferred hadronic phase, stabilized by the presence of a topological charge that emerges from the chiral anomaly. Relevant properties of the ChSL have been studied, such as the identification of the sector of the phase diagram where this phase can exist [9]-[11], its extension to spinning baryonic matter phases [12]-[14], the formation of topological solitons [15]-[18] and its relation to holography [19]-[23] (see also [24]-[26]).

The chiral soliton lattice can be described by the effective sine-Gordon theory (which admits topological multi-soliton solutions) in an external magnetic field and supplemented by the Wess-Zumino-Witten term. Although this effective action describes the chiral soliton lattice phase well, its derivation from quantum field theories is not yet completely clear. Obtaining the phase from first principles is the novelty of this article. In particular, the first question we will answer (affirmatively) is the following: what happens if the magnetic field is not an external field? Is it possible to construct the chiral soliton lattice even when the magnetic field is a dynamical field minimally coupled to the hadronic field?

In the context of effective field theories, the Skyrme model describes the low-energy limit of QCD at leading order in the large  $N_c$  expansion [27], [28]. Their topological soliton solutions are fermionic degrees of freedom recognized as baryons [29], [30], which arise due to the non-linear interaction between mesons (see Refs. [31] and [32]). This theory can be minimally coupled to Maxwell's electrodynamics to describe low-energy charged hadrons. In Ref. [33] (see also [34]), it was shown that when this coupling is performed, the usual expression for the topological charge density must be modified in order to ensure its conservation and to maintain invariance under gauge transformations.

In this manuscript, we will derive the ChSL from the gauged Skyrme model. At first glance, this construction does not seem to be possible for the following reason: the effective action of the ChSL involves dependence on only one spatial coordinate for the soliton profile (as expected for domain walls), while the existence of a non-zero topological charge usually requires an Ansatz involving all three spatial coordinates, as can be seen from Eq. (5) below. The key point in our construction is the fact that it is possible to construct solutions with non-vanishing topological charge even when the pion field only depends on one spatial coordinate, as long as the Skyrme model is minimally coupled to the Maxwell theory. In fact, we show that the Callan-Witten contribution present in the topological charge density is non-null for domain-wall configurations. Furthermore, the fact that the topological charge is by construction an integer number naturally leads to a quantization condition for the magnetic field.

In order to perform such a derivation, here we will use a modification of the Ansatz presented in Refs. [35]-[43], which have allowed the construction of exact solutions that describe baryonic crystalline structures at finite volume in the gauged Skyrme and Yang-Mills theories. In particular, using the exponential representation of  $SU(2)$  (being, in this context, the isospin global symmetry group), we will show the explicit form of the Skyrme field  $U(x)$  in terms of a single one-dimensional soliton profile. Even more, by knowing the exact form of this complex scalar field, it becomes possible to couple the ChSL to quark matter in a straightforward way, as we will discuss in the last part of this article.

On the other hand, a natural question arises. It is well known that, in the 't Hooft expansion (see [44]-[49] and references therein), sub-leading corrections to the Skyrme model appear. Such corrections are extremely complicated and (at first glance) could destroy or, at the very least, substantially modify the neat analytic form of the chiral soliton lattice. Quite surprisingly, we will show that regardless of how many sub-leading terms are included in the action, the chiral soliton lattice remains unchanged. This shows in a very clear way the universal nature of ChSL. In order to improve the clarity of the presentation, in the main text we will only quote the relevant results about the universality of the ChSL, while the technical discussion will be included in the Appendix.

This paper is organized as follows. In Sec. II we give a brief review of the gauged  $SU(2)$ -Skyrme model. In Sec. III we show how to construct exact topological soliton solutions at finite volume. Then, in Sec. IV we derive the effective Lagrangian of the ChSL from the gauged Skyrme model using the Ansatz that allows the exact domain-wall configurations previously constructed. We explore the main characteristics of this inhomogeneous condensate and derive the critical value of the magnetic field. In Sec. V we show how these hadronic configurations can be coupled with quark matter. Sec. VI is devoted to the conclusions.

## II. PRELIMINARIES

The gauged generalized Skyrme model is described by the action

$$\begin{aligned} I(U, A_\mu) &= \frac{f_\pi^2}{4} \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \text{Tr} \left( R_\mu R^\mu + \frac{\lambda}{8} G_{\mu\nu} G^{\mu\nu} \right) - m_\pi^2 \text{Tr}(2\mathbb{I} - U - U^{-1}) \right] - \frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + I_{\text{corr}} , \\ R_\mu &= U^{-1} D_\mu U , \quad G_{\mu\nu} = [R_\mu, R_\nu] , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \end{aligned} \quad (1)$$

where  $U(x) \in SU(2)$  is the Skyrme field, and the covariant derivative associated to the  $U(1)$  gauge field  $A_\mu$  is defined as  $D_\mu U = \nabla_\mu U + A_\mu [t_3, U]$ , where  $t_a = i\sigma_a$  are the  $SU(2)$  generators written in terms of the Pauli matrices  $\sigma_a$ . Here, the coupling  $f_\pi$  is the pion decay constant,  $m_\pi$  is the pion mass, and  $\lambda$  is a positive number fixed experimentally. Moreover,  $I_{\text{corr}}$  represents the sub-leading corrections to the Skyrme model that come from the large  $N_c$  expansion of QCD. In this section, we only consider the usual gauged Skyrme model, however, our construction also works when such terms are included; see the Appendix for the details.

The variation of the action in Eq. (1) with respect to the fields  $U$  and  $A_\mu$  leads to the following field equations

$$D_\mu \left( R^\mu + \frac{\lambda}{4} [R_\nu, G^{\mu\nu}] \right) - \frac{m_\pi^2}{2} (U - U^{-1}) = 0 , \quad \nabla_\mu F^{\mu\nu} = -\frac{f_\pi^2}{4} \text{Tr} \left[ [t_3, U] \left( R_\mu + \frac{\lambda}{4} [R^\nu, G_{\mu\nu}] \right) U^{-1} \right] . \quad (2)$$

The energy-momentum tensor of the gauged Skyrme model is

$$\begin{aligned} T_{\mu\nu} &= -\frac{f_\pi^2}{2} \text{Tr} \left[ R_\mu R_\nu - \frac{1}{2} g_{\mu\nu} R^\alpha R_\alpha + \frac{\lambda}{4} \left( g^{\alpha\beta} G_{\mu\alpha} G_{\nu\beta} - \frac{1}{4} g_{\mu\nu} G_{\sigma\rho} G^{\sigma\rho} \right) - \frac{m_\pi^2}{2} g_{\mu\nu} (2\mathbb{I} - U - U^{-1}) \right] + \\ &\quad + g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} . \end{aligned} \quad (3)$$

The topological current  $\rho^\mu$  is written as

$$\rho^\mu = \epsilon^{\mu\nu\alpha\beta} \text{Tr} \left[ R_\nu R_\alpha R_\beta - 3\nabla_\nu [A_\alpha t_3 (U^{-1} \nabla_\beta U + (\nabla_\beta U) U^{-1})] \right] , \quad (4)$$

where the first term is the usual topological density, while the second is the so-called Callan-Witten term, which must be added in order to preserve the current conservation and gauge invariance. The integral of the temporal component of the topological current on a space-like hypersurface represents the topological (baryonic) charge of the configuration

$$n_B = \frac{1}{24\pi^2} \int_{\Sigma} \rho^0 , \quad \rho^0 = \epsilon^{ijk} \text{Tr} \left[ R_i R_j R_k - 3\nabla_i [A_j t_3 (U^{-1} \nabla_k U + (\nabla_k U) U^{-1})] \right] . \quad (5)$$

It is also possible to add a baryon chemical potential,  $\mu_B$ , via the Wess-Zumino-Witten term, given by

$$I_{\text{WZW}} = \frac{1}{24\pi^2} \int d^4x \left( \mu_B \rho^0 - \frac{1}{2} A_\mu \rho^\mu \right) . \quad (6)$$

A general element of  $SU(2)$  in the exponential representation is written as

$$U^{\pm 1}(x^\mu) = \cos(\alpha) \mathbf{1}_2 \pm \sin(\alpha) n^i t_i \quad \text{with} \quad \begin{cases} n^1 = \sin \Theta \cos \Phi , \\ n^2 = \sin \Theta \sin \Phi , \\ n^3 = \cos \Theta , \end{cases} \quad (7)$$

where  $\alpha$ ,  $\Theta$  and  $\Phi$  are the three degrees of freedom of the  $SU(2)$  field. A relevant fact comes from the Callan-Witten term in this representation. One can check that, replacing Eq. (7) into Eq. (4), both contributions of the topological charge density are written in terms of the degrees of freedom  $\alpha$ ,  $\Theta$  and  $\Phi$ , as follows

$$\rho^0 = 12 \sin^2 \alpha \sin \Theta d\alpha d\Theta d\Phi + 12 d \left[ \left( \alpha - \frac{1}{2} \sin(2\alpha) \right) \sin \Theta A d\Theta - \alpha \cos \Theta F \right] , \quad (8)$$

where  $A = A_i dx^i$  and  $F = \frac{1}{2} F_{ij} dx^i dx^j$ . It is evident that fixing any one of the degrees of freedom to a constant implies that the contribution of the usual topological charge vanishes. However, the Callan-Witten term is not necessarily zero under this assumption. For example, fixing  $\Theta = \pi$  the topological charge density get a non-vanishing contribution from the Callan-Witten term, reading

$$\rho^0 = -12F d\alpha . \quad (9)$$

Therefore, one can construct topological solitons considering only two non-trivial degrees of freedom, as long as there is a non-zero electromagnetic field strength.

### III. GAUGED SOLITONS AT FINITE VOLUME

Let us consider a finite volume system described by the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 , \quad (10)$$

where the spatial coordinates  $\{x, y, z\}$  have the ranges

$$-l/2 \leq x \leq l/2 , \quad 0 \leq y \leq \pi , \quad 0 \leq z \leq 2\pi . \quad (11)$$

According to the discussion in the previous section, a simple choice for the matter field that considerably reduces the field equations and, at the same time, leads to a non-vanishing topological charge, is

$$\alpha = \alpha(t, x) , \quad \Theta = \pi , \quad (12)$$

together with the Maxwell potential that leads to a constant magnetic field in the  $x$  direction, that is

$$A_\mu = (0, 0, Bz, 0) . \quad (13)$$

For this Ansatz, the Maxwell equations are automatically satisfied, while the Skyrme equations are reduced to the sine-Gordon one

$$\square\alpha - m_\pi^2 \sin(\alpha) = 0 , \quad (14)$$

where  $\square = -\partial_t^2 + \partial_x^2$ . The non-trivial topological charge density in Eq. (9) comes from the Callan-Witten term,  $\rho^0 = 12B\partial_x\alpha/\pi$ , resulting in the topological charge

$$n_B = \frac{1}{24\pi^2} \int 12B d\alpha dy dz = B(\alpha(x_f) - \alpha(x_i)) . \quad (15)$$

Assuming the boundary conditions  $\alpha(-l/2) = 0$  and  $\alpha(l/2) = n\pi$ , with  $n$  an integer (condition that guarantees the periodicity of the  $U$  matrix:  $U(-l/2) = \pm U(l/2)$ ), the topological charge turns out to be

$$n_B = n\pi B . \quad (16)$$

Note that, as the topological charge is the baryon number, namely an integer,  $\pi B$  must also be an integer.

In the static case,  $\alpha = \alpha(x)$ , the sine-Gordon equation is reduced to

$$\frac{d^2\alpha}{dx^2} - m_\pi^2 \sin(\alpha) = 0 . \quad (17)$$

This equation can be rewritten as the following quadrature,

$$\left( \frac{d\alpha}{dx} \right)^2 + 2m_\pi^2 \cos \alpha = \ell , \quad (18)$$

where  $\ell$  is an integration constant with dimensions of energy squared. A general solution for this equation is given by the Jacobi amplitude  $\text{Am}(x, m)$ , as

$$\alpha(x) = 2\text{Am}\left(\frac{1}{2}\sqrt{\ell - 2m_\pi^2} \left(x + \frac{l}{2}\right), \frac{4m_\pi^2}{2m_\pi^2 - \ell}\right) . \quad (19)$$

The resulting baryonic density  $\rho^0$  reads

$$\pi\rho^0 = 12B\sqrt{\ell - 2m_\pi^2} \text{dn}\left(\frac{1}{2}\sqrt{\ell - 2m_\pi^2} \left(x + \frac{l}{2}\right), \frac{4m_\pi^2}{2m_\pi^2 - \ell}\right) , \quad (20)$$

where  $\text{dn}(x, m)$  is the Jacobi elliptic function. It is periodic in  $x$  with a period given by the complete elliptic integral of the first kind  $K(m)$  as  $4K[4m_\pi^2/(2m_\pi^2 - \ell)]/\sqrt{\ell - 2m_\pi^2}$ . This period is real while  $\ell > 2m_\pi^2$ , being divergent at  $\ell = 2m_\pi^2$ , and approaches  $2\pi$  as  $\ell \rightarrow \infty$ , which means increasingly shorter  $K(m)$  periods for the baryonic density.

Finally, since  $|x| < l/2$ , we see that the integration constant can be related to both the boundary conditions and the baryonic charge, by imposing that  $\rho^0$  must be  $n$ -periodic inside the interval, that is

$$4n K\left(\frac{4m_\pi^2}{2m_\pi^2 - \ell}\right) = l\sqrt{\ell - 2m_\pi^2} . \quad (21)$$

This equation has a finite number of solutions for  $\ell$  as long as  $l > 2\pi$ . In Fig. 1 we have represented some of these solutions, which intersect the blue curve  $K\left(\frac{4m_\pi^2}{2m_\pi^2 - \ell}\right)/\sqrt{\ell - 2m_\pi^2}$ .

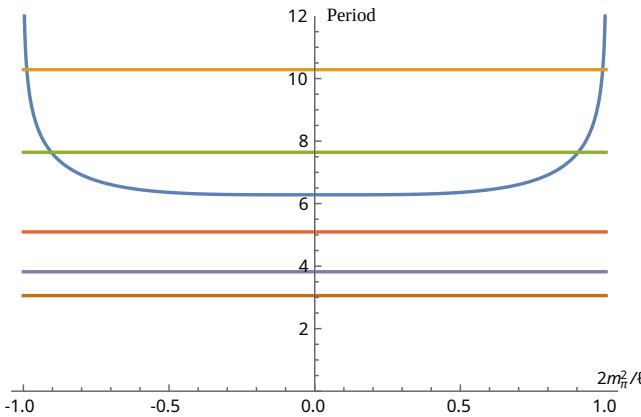


FIG. 1: Solutions of Eq. (21). The horizontal lines correspond to  $l/n$  with  $l = 2\pi + 9$  and  $n = 1, 2, 3, 4, 5$  from top to bottom.

#### IV. EMERGENT CHIRAL SOLITON LATTICE

The field configuration presented in the previous section, through the description of topological solitons in a finite volume in the context of the generalized Skyrme theory, is nothing less than (a generalization of) the ChSL. To see this more clearly, we can calculate the Hamiltonian density from our Ansatz, obtaining the following

$$\mathcal{H} = f_\pi^2 \left( \frac{1}{2} (\alpha')^2 + m_\pi^2 (1 - \cos(\alpha)) \right) - \frac{\mu_B B}{4\pi^2} \alpha' + \frac{B^2}{2}. \quad (22)$$

From the above expression, we can see that, in the absence of the last term (which is proportional to  $\vec{B}^2$ ) the Hamiltonian corresponds to the free energy density of the ChSL; this is the sine-Gordon theory with the coupling between the baryon chemical potential, the magnetic field, and the gradient of the pion field. This configuration describes a periodic array of topological solitons carrying baryon charge, magnetic moment, and breaking the parity symmetry [6], [7].

Using the elementary properties of the Jacobi elliptic functions, one can write the energy in terms of the elliptic modulus (see [6] and references therein). Then, by minimizing the energy, it is possible to derive the critical value for the magnetic field above which the ChSL is formed;  $B > B_{\text{ChSL}}$ . This well-known result is given by

$$B_{\text{ChSL}} = \frac{16\pi m_\pi f_\pi^2}{\mu_B}. \quad (23)$$

Now, in the general case when  $\vec{B}^2$  is different from zero (see Eq. (22)), the solution can be interpreted as a periodic array of hadronic layers that generates its own magnetic field, this due to the minimal coupling of the Skyrme model with the Maxwell potential. This configuration is, of course, much more difficult to analyze, however, its chiral limit can be studied. In fact, we see that when  $m_\pi \rightarrow 0$ , the configuration minimizing the free energy satisfies

$$(\alpha') = \frac{\mu_B B}{4\pi^2 f_\pi^2}. \quad (24)$$

Notice that the right-hand side of the above equation provides with an estimate of the integration constant  $\ell$  in Eq. (18). In this case, the free energy density of the system becomes

$$\lim_{m_\pi \rightarrow 0} \mathcal{H} = -\frac{1}{2} \left( \frac{\mu_B B}{4\pi^2 f_\pi^2} \right)^2 + \frac{B^2}{2}. \quad (25)$$

The chiral limit of the system that includes the term proportional to  $B^2$  in Eq. (22) was explored in Ref. [15], showing the similarity of this system with the behavior of ordinary type-II superconductors.

Now, it is important to note that the above results can also be derived by considering the partition function of the system  $Z = \sum_n \exp\{-\beta(E_{\text{Cl}} - \mu_B n_B)\}$ . Here, the sum runs over the integer number  $n$  in the boundary conditions,  $E_{\text{Cl}}$  is the classic energy density in Eq. (22) that does not include the contribution of the WZW term. In order to have a convergent sum, the baryon chemical potential must satisfy  $\mu_B < E_{\text{Cl}}/n_B$ . Finally, the dispersion relation of pions in this setup can also be derived in a straightforward way by following Ref. [50].

## V. ABOUT THE COUPLING WITH QUARK MATTER

In the previous sections, we have shown that the ChSL emerges naturally from the low-energy limit of QCD by considering topological solitons in flat space-time described by a matter field given by Eqs. (7) and (12). This opens the possibility of coupling the ChSL directly to quark matter, via the usual Yukawa coupling. This issue has been treated many times in the literature (see [51] [52], [53], [54], [55], [56], [57] and references therein, or [58] for a more modern treatment); however, due to the difficulty of the problem, few exact solutions of the Dirac equation coupled to Skyrme are known (see, for example, [59] and [60], [61] and [62]).

In the following, we consider an interaction term in the Lagrangian of the form,

$$\mathcal{L}_{\text{int}} = \bar{\psi} (i\gamma^\mu D_\mu + gU^{\gamma_5}) \psi , \quad (26)$$

where  $\psi$  is a spin-isospin spinor, whose mass is negligible compared to the energy scale of the coupling with the  $U$  field (and for the same reason, we ignore backreaction effects), and  $D_\mu = (\partial_\mu - ieU^{-1}[t_3, U]A_\mu)$  is the usual Maxwell-covariant derivative acting on the space of spinors, where  $U^{-1}[t_3, U]$  is a charge matrix, which acts only on the isospin components. The matter field is coupled in the minimal parity-invariant manner, as

$$U^{\gamma_5} = \left( \frac{1 + \gamma_5}{2} \right) U + \left( \frac{1 - \gamma_5}{2} \right) U^{-1} = \cos(\alpha) - i\gamma_5\tau_3 \sin(\alpha) , \quad (27)$$

where the ChSL Ansatz dramatically simplifies the expression for the coupling. We write the Pauli matrices as  $\tau_a$  here, emphasizing they act on the isospin components of the spinor.

The Dirac Hamiltonian reads,

$$\hat{H} = \vec{\alpha} \cdot \vec{\pi} + g\hat{\beta}U^{\gamma_5} , \quad (28)$$

where  $\vec{\pi} = (\vec{p} - eU^{-1}[t_3, U]\vec{A})$  is the canonical momentum operator,  $\vec{A} = (0, Bz, 0)$ . We see that, actually, all operators involved in the Hamiltonian are diagonal in the isospin space; hence, we can consider the Dirac spinor as an eigenvalue of  $\tau_3$ , and separate obtaining two decoupled equations,

$$\hat{H}^{(i)}\Psi^{(i)} = \epsilon\Psi^{(i)} , \quad (29)$$

where  $i$  is the isospin index.

Now, we can consider the chiral rotations of the spinor,

$$\psi \rightarrow \hat{S}\psi = \exp\left(\frac{i}{2}\gamma_5\tau_3\alpha(x)\right)\psi , \quad (30)$$

then, the Lagrangian transforms into,

$$\mathcal{L}' = \bar{\psi} (i\gamma^\mu D_\mu + i\gamma_1\partial_x\alpha(x)\gamma_5 + g) \psi , \quad (31)$$

where now, the ChSL contributes both as a mass term through the coupling constant and as a pseudo-vector potential. This modified system leads to a Hamiltonian where the coupling is manifested only via a mass term.

Due to the dependence on the  $x$  coordinate, boosts and chiral rotations are clearly related to each other. Doing a chiral rotation is then equivalent to choosing some chiral basis for the fermions [63], and then we could work with the Dirac equation rendered from the above Lagrangian instead. In general, we have a system composed both of a Dirac particle in a constant, uniform magnetic field (which has already been visited many times in the literature; see, for example, [64] and references therein) and which also features a discrete symmetry along the  $x$  direction. Exploiting these two features could allow us to determine analytically the spectrum of the Dirac equation and the corresponding band structure. We hope to return to this subject in a future publication.

## VI. FINAL REMARKS

In this paper, we reveal the surprising universal character of the ChSL in the low-energy limit of QCD. First, we have discussed how these configurations can be obtained in the gauged-Skyrme model minimally coupled with the Maxwell gauge field. The key step has been an Ansatz for the hadronic matter with the properties that the only non-vanishing contribution to the baryonic density arises from the Callan-Witten term. Such an Ansatz describes

topological solitons at a finite baryon density. Moreover, the fact that the topological charge is an integer (the baryon number in this context) leads directly to a quantization condition for the magnetic field. A quite remarkable result of the present analysis is that the chiral soliton lattice remains unchanged if we include the sub-leading corrections to the Skyrme model in the 't Hooft large  $N_c$  expansion. Taking into account the highly non-linear character of such sub-leading corrections, this is a very intriguing result, which is a strong manifestation of the universality of the ChSL. As a future perspective, the present approach allows to study the coupling of the ChSL with quark matter using the Dirac equation coupled to the chiral field. We hope to come back to this issue in a future publication [65].

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### Appendix A: Generalized Skyrme terms

In this Appendix we show that the construction performed above also works when higher-order corrections coming from the large  $N_c$  expansion are supplemented to the Skyrme action, showing that the ChSL has a sort of universal character within the low-energy limit of QCD.

The so-called generalized Skyrme model includes the higher-order corrections that comes from the large  $N_c$  limit of QCD, which are supplemented in the original Skyrme action. The first two sub-leading terms are given by

$$\begin{aligned}\mathcal{L}_6 &= \frac{c_6}{96} \text{Tr}[G_\mu^\nu G_\nu^\rho G_\rho^\mu] , \\ \mathcal{L}_8 &= -\frac{c_8}{256} \left( \text{Tr}[G_\mu^\nu G_\nu^\rho G_\rho^\sigma G_\sigma^\mu] - \text{Tr}[\{G_\mu^\nu, G_\rho^\sigma\} G_\nu^\rho G_\sigma^\mu] \right) .\end{aligned}$$

Taking into account the above new contributions, the Skyrme equations are given by

$$\begin{aligned}\frac{f_\pi^2}{2} \left( D_\mu \left( R^\mu + \frac{\lambda}{4} [R_\nu, G^{\mu\nu}] \right) - \frac{m_\pi^2}{2} (U - U^{-1}) \right) + 3c_6 [R_\mu, D_\nu [G^{\rho\nu}, G_\rho^\mu]] \\ + 4c_8 \left[ R_\mu, D_\nu \left( G^{\nu\rho} G_{\rho\sigma} G^{\sigma\mu} + G^{\mu\rho} G_{\rho\sigma} G^{\nu\sigma} + \{G_{\rho\sigma}, \{G^{\mu\rho}, G^{\nu\sigma}\}\} \right) \right] = 0 .\end{aligned}\quad (\text{A1})$$

The Maxwell equations will also be affected by including these new terms due to the  $U(1)$  connection are present in each of the corrections. Indeed, the electromagnetic current in the Maxwell equations in Eq. (2) now takes the form

$$\begin{aligned}J_\mu &= -\frac{f_\pi^2}{2} \text{Tr} \left[ \hat{O} \left( R_\mu + \frac{\lambda}{4} [R^\nu, G_{\mu\nu}] \right) \right] + \frac{c_6}{32} \text{Tr} \left[ \hat{O} \left( [R^\alpha, [G_{\mu\nu}, G_\alpha^\nu]] \right) \right] \\ &\quad - \frac{c_8}{64} \text{Tr} \left[ \hat{O} \left( [R^\alpha, G_\alpha^\nu G_\nu^\rho G_{\rho\mu} + G_{\rho\mu} G_\nu^\rho G_\alpha^\nu + \{G^{\nu\rho}, \{G_{\mu\nu}, G_{\rho\alpha}\}\}] \right) \right] ,\end{aligned}\quad (\text{A2})$$

where we have defined  $\hat{O} = U^{-1}[t_3, U]$ . Finally, we can compute the energy-momentum tensor of the gauged generalized Skyrme model, obtaining the following

$$T_{\mu\nu}^{\text{gen}} = T_{\mu\nu} + T_{\mu\nu}^{(6)} + T_{\mu\nu}^{(8)} , \quad (\text{A3})$$

where  $T_{\mu\nu}$  has been defined in Eq. (3), and

$$\begin{aligned}T_{\mu\nu}^{(6)} &= -\frac{c_6}{16} \text{Tr} \left( g^{\alpha\gamma} g^{\beta\rho} G_{\mu\alpha} G_{\nu\beta} G_{\gamma\rho} - \frac{1}{6} g_{\mu\nu} G_\alpha^\beta G_\beta^\rho G_\rho^\alpha \right) , \\ T_{\mu\nu}^{(8)} &= \frac{c_8}{32} \text{Tr} \left( g^{\alpha\rho} g^{\beta\gamma} g^{\delta\lambda} G_{\alpha\mu} G_{\nu\beta} G_{\gamma\delta} G_{\lambda\rho} + \frac{1}{2} \{G_{\mu\alpha}, G_{\lambda\rho}\} \{G_{\beta\nu}, G_{\gamma\delta}\} g^{\alpha\gamma} g^{\beta\rho} g^{\delta\lambda} \right. \\ &\quad \left. - \frac{1}{8} g_{\mu\nu} (G_\alpha^\beta G_\beta^\rho G_\rho^\sigma G_\sigma^\alpha - \{G_\alpha^\beta, G_\rho^\sigma\} G_\beta^\rho G_\sigma^\alpha) \right) .\end{aligned}$$

At a first glance, it looks a hopeless task to find analytic solutions to the above extremely non-linear field equations. In fact, this is not the case. It is important to note two very relevant facts. The first is that all new contributions to the electromagnetic current are proportional to  $\hat{O}$ . However, according to the Ansatz in Eqs. (7) and (12), the matrix  $U$  depends only on the identity matrix and the generator  $t_3$ , namely,

$$U(\alpha) = \cos(\alpha)\mathbf{1}_2 - \sin(\alpha)t_3 . \quad (\text{A4})$$

Thus, the tensor  $\hat{O}$  is identically zero, and therefore, all extra contributions to the current eventually cancel out. Something similar happens with the field equations in Eq. (A1) and the energy density in Eq. (A3). Since  $U$  is given by Eq. (A4), it follows that  $R_\mu$  depends only on one of the generators of the group. Indeed, in our Ansatz this tensor becomes

$$R_\mu = -\partial_\mu \alpha t_3 . \quad (\text{A5})$$

From the above, it is clear that the tensor  $G_\mu = [R_\mu, R_\nu]$  must be  $G_{\mu\nu} = 0$ . Furthermore, since all extra contributions in both the field equations and the energy-momentum tensor depend on this tensor (and not on  $R_\mu$  separately), all extra contributions eventually cancel out. From all the above, we can conclude that the solution presented here is not only a solution to the Skyrme model, but also to the generalized Skyrme model (similar arguments would also hold including further terms in the 't Hooft expansion). That is why the ChSL is a universal feature of the low energy limit of QCD.

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