

Spatially flat cosmological quantum spacetimes

Christian Gaß¹ and Harold C. Steinacker¹

¹Faculty of Physics, University of Vienna
Boltzmannstrasse 5, A-1090 Vienna, Austria
email: christian.gass@univie.ac.at, harold.steinacker@univie.ac.at

October 27, 2025

Abstract

We recently described a cosmological quantum spacetime of vanishing spatial curvature, which can be considered as background for the IKKT matrix model, assuming that the resulting gauge theory couples weakly. Building on this example, we construct a large class of spatially flat cosmological quantum spacetimes. We also elaborate on various details of their algebraic and semi-classical structure as well as the higher spin modes present in these models. In particular, we introduce the notion of approximate diffeomorphisms on the cosmological quantum spacetimes that stem from gauge transformations of the underlying matrix model, and investigate how different gauges are related in the semi-classical regime by approximate diffeomorphisms. Finally, we briefly outline how the described quantum spacetimes could be incorporated into the full IKKT model.

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1 Introduction

It has been known for some time that the unitary irreducible representations of $\mathfrak{so}(4, 2)$ in the minimal discrete series [11], labeled by $n \in \mathbb{N}_0$ and sometimes called *doubleton representations*, can be used to model the *fuzzy hyperboloid* H_n^4 . This is possible because the doubleton representations descend to irreducible representations of $\mathfrak{so}(4, 1)$. In a next step, the fuzzy hyperboloid can be used to model covariant quantum spacetimes via certain projections [27, 28, 34]. Among those quantum spacetimes, the ones that are currently best understood are FLRW spacetimes with negative spatial curvature ($k = -1$) and a Big Bounce geometry [4, 23, 24, 27–29, 34]. Such spacetimes have a spatial $SO(3, 1)$ -symmetry.

We recently extended the set of examples by describing a spatially flat ($k = 0$) FLRW spacetime with a Big Bang [12]. In the current paper, we generalize this example to describe a much larger class of spatially flat FLRW quantum spacetimes, which have a spatial $E(3)$ -symmetry.

All these cosmological quantum spacetimes should be viewed in the context of the IKKT matrix model [19] formulated in $D = 10$ dimensions. In particular, the cosmological $k = 0$ quantum spacetime discussed in [12] defines an action

$$S_0[\Phi] = \text{Tr} (([T^\mu, \Phi^\dagger][T_\mu, \Phi] - m^2 \Phi^\dagger \Phi)), \quad (1.1)$$

where Φ is a free, non-commutative scalar field on the quantum spacetime, and where the matrices T^μ form a static background that transforms under an irreducible representation of $E(3)$. $S_0[\Phi]$ can be seen as a toy model for the bosonic part of the action of the IKKT model (1.5) below – with a static instead of dynamical background T^μ in the regime where the matrices describing the extra dimensions, represented by several scalars, couple weakly.

Our paper should also be viewed as part of an ongoing program to construct examples of quantum spacetimes and gravity from matrix theory. We are aware of constructions of other

cosmological spacetimes [22–24, 27–29] in the framework of the IKKT model, as well as fuzzy de Sitter space [8, 13, 14]. For other related work on the origin of quantum spacetime from matrix theory see e.g. [2, 6, 7, 9, 10, 16, 26]. Let us also mention recent work on the polarized IKKT model [15, 20, 21, 25], a deformation of IKKT that may extend the scope of consistent backgrounds.

We consider quantum spaces defined as quantized symplectic spaces \mathcal{M} via hermitian operators or matrices acting on a separable Hilbert space \mathcal{H} . The space of "nice" operators in $\text{End}(\mathcal{H})$ is interpreted in terms of quantized functions on the underlying symplectic manifold \mathcal{M} , related by a quantization map

$$\begin{aligned}\mathcal{Q} : \quad & \mathcal{C}(\mathcal{M}) \rightarrow \text{End}(\mathcal{H}), \\ & \phi(\cdot) \mapsto \Phi.\end{aligned}\tag{1.2}$$

In a certain semi-classical regime, the map \mathcal{Q} should be invertible and satisfy

$$\begin{aligned}\mathcal{Q}(\phi\psi) &\sim \mathcal{Q}(\phi)\mathcal{Q}(\psi), \\ i\mathcal{Q}(\{\phi, \psi\}) &\sim [\mathcal{Q}(\phi), \mathcal{Q}(\psi)], \\ \text{and} \quad \text{Tr}(\mathcal{Q}(\phi)) &\sim \int_{\mathcal{M}} \frac{\Omega}{(2\pi)^d} \phi,\end{aligned}\tag{1.3}$$

where \sim indicates approximate equality in the semi-classical regime and where Ω is the symplectic volume form on \mathcal{M} (which has dimension $2d$). Such quantization maps are not unique, but preferred maps can often be found by requiring their compatibility with some symmetry group, notably for quantized coadjoint orbits such as the fuzzy hyperboloid described in the following. Another way to describe quantization maps is via (quasi-)coherent states $|\phi\rangle$ as $\Phi = \int \Omega \phi(x) |\phi\rangle\langle\phi|$; see e.g. [30, 32, 34] for an introduction to this framework.

In the case of covariant quantum spacetimes originating from the fuzzy hyperboloid, it turns out that the underlying symplectic space \mathcal{M} is the six-dimensional projective space $\mathbb{C}P^{2,1}$ [27, 28], which can be written as a local bundle

$$\mathcal{M} \cong \mathcal{M}^{1,3} \tilde{\times} S^2,\tag{1.4}$$

where $\mathcal{M}^{1,3}$ is a cosmological spacetime (in particular, the metric has Minkowskian signature) and where $\tilde{\times}$ indicates that the isometry group of the spacetime acts non-trivially on the local fiber. The presence of the internal two-sphere S^2 implies the existence of higher-spin degrees of freedom in the model.

Due to the non-trivial bundle structure of the symplectic space, one encounters a problem. Gauge transformations of the matrix model correspond to symplectomorphisms on \mathcal{M} [34] but not necessarily to diffeomorphisms on $\mathcal{M}^{1,3}$. However, it may happen that gauge transformations are related to *approximate diffeomorphisms* on $\mathcal{M}^{1,3}$ in regimes where the contribution of the higher spin variables is negligible. The study of such approximate diffeomorphisms on cosmological $k = 0$ quantum spacetimes is one of the main points of this paper. Our definitions and results may be extended to other covariant quantum spacetimes.

The metric of Minkowskian signature on $\mathcal{M}^{1,3}$ is encoded in the *matrix d'Alembertian* \square_T associated to the action (1.1). Its semi-classical version $\square \sim \square_T$ is in turn related to a geometric d'Alembertian \square_G via a *dilaton* ρ^2 , $\square = \rho^2 \square_G$. This geometric d'Alembertian then yields the Minkowskian metric $G_{\mu\nu}$ on $\mathcal{M}^{1,3}$, which is in general higher-spin valued.

We will see that on a timelike curve, the dependence of the metric tensor and other physical quantities on higher-spin variables can be removed by going to local normal coordinates. The residual higher spin components should be small in the semi-classical regime, more precisely in a local patch at late times near this timelike curve.

The description of this semi-classical structure of cosmological $k = 0$ quantum spacetimes forms the main body of our paper. However, we also outline how these spacetimes may be incorporated into the IKKT model, whose full action is given by

$$S_{\text{IKKT}}[T, \Psi] = \text{Tr} ([T^a, T^b][T_a, T_b] + \bar{\Psi}\Gamma_a[T^a, \Psi]). \quad (1.5)$$

Here a and b run from 0 to 9, T^a are hermitian matrices and Ψ is a Majorana-Weyl spinor of $SO(9, 1)$ with Grassmann-valued matrices as entries.

While the true IKKT model (1.5) has no quadratic "mass" term, it is sometimes necessary to add an $E(3)$ -invariant mass term to the action (1.5) by hand to stabilize spacetime. For example, the background discussed in [12] satisfies the IKKT equations with a non-vanishing mass term.

To embed the background T^μ , $\mu = 0, \dots, 3$, describing cosmological quantum spacetimes into the full IKKT model, and to recover gravity [33], one needs to take into account the coupling of T^μ to the matrices describing the extra dimensions as well as their self-coupling. This has been studied to some extent in the case of cosmological $k = -1$ quantum spacetimes [5, 22, 34]. In the present paper, we will only outline how this may be extended to the case of $k = 0$ quantum spacetimes. We expect that the classical stabilization of extra dimensions via R charge discussed in [22] will carry over to the $k = 0$ case, while quantum effects coming from the one-loop effective action need to be taken into account so that the matrices defining spacetime lead to a physically reasonable cosmic evolution.

1.1 Outline of the paper and summary of the main results

Complementing several recent constructions in the case of $k = -1$ cosmological quantum spacetimes [4, 5, 22–24, 27–29, 31, 34], the main aim of this paper is to elaborate on and generalize the example given by the action $S_0[\Phi]$ from (1.1), which was discussed in [12].

In Section 2, we briefly recall how covariant quantum spacetimes arise from the fuzzy hyperboloid H_n^4 . We work out the algebraic and semi-classical structure as well as the higher-spin theory on the cosmological quantum spacetime $\mathcal{M}^{1,3}$ that arises from the local bundle structure $\mathcal{M} = \mathcal{M}^{1,3} \tilde{\times} S^2$. We also discuss how gauge transformations can implement approximate diffeomorphisms on $\mathcal{M}^{1,3}$ in the semi-classical regime.

The background T^μ in (1.1), which we work out in Section 3 and which was partially investigated in the previous paper [12], can be used as a *reference background* for cosmological $k = 0$ quantum spacetimes. We discuss its algebraic structure, which is richer in the present case than it is for cosmological $k = -1$ quantum spacetimes: there are two distinct but important actions of $E(3)$, while the respective actions of $SO(3, 1)$ in the $k = -1$ case coincide. We introduce appropriate derivations on $\mathcal{M}^{1,3}$ that act also on higher-spin valued functions. In complete analogy to the $k = -1$ case, this derivation is used to relate vector fields on $\mathcal{M}^{1,3}$ to tangential vector fields on H_n^4 .

Using the reference background T^μ as a starting point, we describe in Section 4 dynamical backgrounds that respect the $E(3)$ symmetry by adding time dependent prefactors

$$T^0 \rightarrow \tilde{T}^0 := \alpha(Y^0)T^0, \quad T^i \rightarrow \tilde{T}^i := \beta(Y^0)T^i, \quad (1.6)$$

where $y^0 \sim Y^0$ parametrizes time on $\mathcal{M}^{1,3}$, and α, β are well-behaved functions. However, this description is redundant. We show that different choices (α, β) are related via gauge transformation generated by $\Lambda(Y^0)T^0$, corresponding to approximate diffeomorphisms along the time direction under mild assumptions. Distinguished gauges are the *timelike gauge* $\beta \equiv 1$, where formulas become particularly simple, and the *covariant gauge* $\alpha = \beta$, in which the generators do not only transform under a representation of $E(3)$ but also under a representation of $SO(3, 1)$.

For the dynamical backgrounds, we determine various quantities that are important to describe the physics in the non-commutative and the semi-classical regime: (i) the matrix d'Alembertian that governs the IKKT equations of motion, (ii) various geometric quantities that describe the resulting semi-classical spacetime, (iii) local normal coordinates, which allow to eliminate higher-spin contributions along a timelike curve.

In particular, the cosmological $k = 0$ spacetimes described with the background \tilde{T}^μ define a FLRW geometry with cosmic scale factor that depends on α and β . We illustrate the relation between gauge transformation and diffeomorphisms by verifying that certain distinguished, gauge-equivalent backgrounds yield equivalent FLRW line elements in the timelike and covariant gauge. One of them is the background where the dilaton is constant, which is shown to yield a *shrinking* FLRW spacetime for $k = 0$, rather than an expanding spacetime as in the $k = -1$ case. However, we can obtain a variety of expanding $k = 0$ FLRW spacetimes for different α and/or β . This computation also demonstrates the use and the consistency of local normal coordinates in the higher-spin framework [34].

Finally, we outline in Section 5 how the background \tilde{T}^μ can be incorporated in the full IKKT model. The details of such an incorporation are beyond the scope of the present paper and are left for future research. However, we concisely discuss the general recipe, which is based on a stabilization of dynamical extra-dimensions and corrections to the IKKT equations at one-loop and which should work similarly to the $k = -1$ case discussed in [2, 5, 22].

In Appendix A, we list a number of pertinent properties of cosmological $k = -1$ quantum spacetimes that are needed in our construction of cosmological $k = 0$ spacetimes. Appendix B contains a derivation of the concrete form of the $SO(3, 1)$ generators in various cases.

2 Fuzzy hyperboloid and cosmological quantum spacetimes

Let us describe how to obtain cosmological quantum spacetimes from matrix models. Within the scope of this paper, such a construction cannot be fully self-contained. We refer to the book [34] for a comprehensive introduction.

2.1 Fuzzy hyperboloid

Let M^{ab} , $a, b = 0, \dots, 5$ denote the generators of $SO(4, 2)$, which are skew-symmetric and satisfy the commutation relations

$$[M^{ab}, M^{cd}] = i(M^{ac}\delta^{bd} - M^{ad}\delta^{bc} - M^{bc}\delta^{ad} + M^{bd}\delta^{ac}). \quad (2.1)$$

Unless specified otherwise, Latin indices a, b, \dots will run from 0 to 5, Latin indices i, j, \dots will run from 1 to 3 and Greek indices will run from 0 to 3. We use the conventions $(\eta^{ab}) = \text{diag}(-1, 1, 1, 1, 1, -1)$ and $(\eta^{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$.

In this paper, we will only consider doubleton representations \mathcal{H}_n of $\mathfrak{so}(4, 2)$, which are labeled by $n \in \mathbb{N}_0$, and assume that the M^{ab} are given in one of these representations. There is a standard oscillator construction of the doubleton representations, which is described in detail in [34]. In the doubleton representation labeled by n , we have the important identity [34]

$$\eta_{ab}M^{ac}M^{bd} + (c \leftrightarrow d) = \frac{n^2 - 4}{2}\eta^{cd} =: 2\frac{R^2}{r^2}\eta^{cd}, \quad n = 0, 1, 2, \dots, \quad (2.2)$$

where we introduced two dependent parameters r^2 , $r > 0$, and $R^2 = r^2 \frac{n^2 - 4}{4}$, which will introduce a scale on the fuzzy hyperboloid and the cosmological quantum spacetimes. For further details on doubleton representations, we refer to the existing literature [11, 34].

The generators M^{ab} can be used to introduce a set of non-commuting coordinate matrices

$$X^a := rM^{a5}, \quad a = 0, \dots, 4. \quad (2.3)$$

Due to the identity (2.2), the X^a satisfy the $SO(4, 1)$ -invariant constraint

$$X^\mu X_\mu + X^{4^2} = -R^2 \mathbf{1}, \quad (2.4)$$

which means that they can be interpreted as non-commutative coordinates describing the four-dimensional fuzzy hyperboloid H_n^4 of radius R . Strictly speaking, R^2 is negative for $n = 0, 1$ and vanishes for $n = 2$. In the minimal case $n = 0$, one can nevertheless describe meaningful spacetimes in some late-time regime [23], but to have a global semi-classical structure one should assume $n \gg 0$ [34]. Except for a few instances where we also consider the minimal case $n = 0$, we will always assume $n \gg 0$.

2.1.1 Semi-classical structure

It turns out that the semi-classical structure describes six-dimensional projective space $\mathbb{C}P^{2,1} := \{z \in \mathbb{C}^4 \mid \bar{z}z = 1\}$, which can be viewed as coadjoint orbit of $SO(4, 2)$. In other words, the fuzzy hyperboloid H_n^4 can be viewed as quantization of (the symplectic space) $\mathbb{C}P^{2,1}$, which has a (local) bundle structure

$$\mathcal{M} := \mathbb{C}P^{2,1} \cong H^4 \tilde{\times} S^2. \quad (2.5)$$

Here, H^4 is the four-hyperboloid, S^2 the two-sphere and $\tilde{\times}$ indicates a local bundle structure and that the isometry group $SO(4, 1)$ of H^4 acts non-trivially on the local fiber in a way respecting the bundle map. The internal two-sphere corresponds to the existence of higher-spin degrees of freedom in the model [27, 28, 34].

We write $x^a \sim X^a$ and $m^{ab} \sim M^{ab}$ for the semi-classical versions of X^a and M^{ab} for $a, b = 0, \dots, 4$. Moreover, we have $\{\cdot, \cdot\} \sim -i[\cdot, \cdot]$.

The algebra $\mathcal{C} := \mathcal{C}(\mathcal{M})$ of functions on \mathcal{M} has a decomposition [34]

$$\mathcal{C} = \bigoplus_{s=0}^{\infty} \mathcal{C}^s, \quad (2.6)$$

where \mathcal{C}^s can be interpreted as describing the spin- s degrees of freedom. The subspace \mathcal{C}^0 of *classical (or scalar) functions* consists of functions that only depend on x^μ . The higher-spin degrees of freedom are described by two independent variables parametrizing the internal S^2 .

A useful derivation on \mathcal{C} is the *tangential derivative*

$$\eth^a := \frac{1}{R^2} m^{ab} \{x_b, \cdot\} = -\frac{1}{R^2} x_b \{m^{ab}, \cdot\}, \quad a, b = 0, \dots, 4, \quad (2.7)$$

which satisfies $x_a \eth^a = 0$. We will see later how \eth^a serves to define appropriate derivations on the cosmological spacetimes. The trace on $\text{End}(\mathcal{H}_n)$ becomes the integral over \mathcal{M} with respect to the symplectic density as in (1.3). In order to have finite kinetic energy and inner products, functions $\phi \in \mathcal{C}$ are subject to certain integrability conditions, which we tacitly assume in this work, cf. [34] for details.

Note that (2.2) implies

$$\eth^a x^b = \eta^{ab} + \frac{x^a x^b}{R^2}. \quad (2.8)$$

Moreover, for any $\phi \in \mathcal{C}$, the vector field $V^a := \{x^a, \phi\}$ is conserved with respect to \eth^a ,

$$\eth^a V_a = 0. \quad (2.9)$$

The conservation law (2.9) can be verified using the Jacobi identity for the Poisson bracket, the skew-symmetry of m^{ab} and the identity (2.2).

2.2 Cosmological quantum spacetimes

To associate a quantum spacetime to the fuzzy hyperboloid H_n^4 , one needs an appropriate metric structure of Minkowskian signature. This structure is encoded in a *matrix d'Alembertian*

$$\square_T := [T^\mu, [T_\mu, \cdot]], \quad (2.10)$$

where T^μ is a suitable set of matrices. In the choices yielding cosmological spacetimes, the matrix algebra can be identified in the semi-classical regime with the algebra of functions on a bundle over spacetime as in (1.4): $\mathcal{M} \cong \mathcal{M}^{1,3} \tilde{\times} S^2$ in analogy to (2.5). Here $\mathcal{M}^{1,3}$ is the cosmological spacetime, whose coordinate functions are described by the semi-classical $x^\mu \sim X^\mu$, and the internal two-sphere S^2 is parametrized by $t^\mu \sim T^\mu$ with (locally) two independent entries. The variables t^μ are interpreted as *higher spin variables*.

Specifically, the metric is constructed in the following way. For $\phi \in \mathcal{C}(\mathcal{M})$, let $[\phi]_0 \in \mathcal{C}(\mathcal{M}^{1,3})$ denote its fiber projection. By (1.3), we have

$$\text{Tr}(Q(\phi)) \sim \int_{\mathcal{M}^{1,3}} \Omega^{1,3} [\phi]_0 =: \int_{\mathcal{M}^{1,3}} \rho_M [\phi]_0 d^4x, \quad (2.11)$$

where the volume form $\Omega^{1,3} := \rho_M d^4x$ on $\mathcal{M}^{1,3}$ is the pull-back of the unique $SO(4, 1)$ -invariant volume form on the 4-hyperboloid $H^4 \subset \mathbb{R}^{4,1}$ spanned by the x^a generators. It can be written explicitly as [12, 27]

$$\Omega^{1,3} = \frac{dx^0 \dots dx^3}{R^3 x^4} = \frac{dy^0 dy^1 \dots dy^3}{R^3 y^0}, \quad (2.12)$$

where $y^0 := x^0 + x^4$ and $y^i = x^i$. The $SO(3, 1)$ -invariant quantity x^4 and the coordinates x^μ are well-adjusted to cosmological $k = -1$ spacetimes, while the $E(3)$ -invariant quantity y^0 and the coordinates y^μ are well-adjusted to $k = 0$ spacetimes.

Now consider a free, non-commutative scalar field $\Phi = \Phi(X)$ described by the action $S_0[\Phi]$ from (1.1), which is equivalent to

$$S_0[\Phi] = -\text{Tr}(\Phi^\dagger (\square_T + m^2) \Phi). \quad (2.13)$$

In the semi-classical regime, the action reduces to

$$S_0[\Phi] \sim - \int_{\mathcal{M}} \Omega \phi^* (\square + m^2) \phi = - \int_{\mathcal{M}^{1,3}} \Omega^{1,3} [\phi^* (\square + m^2) \phi]_0, \quad (2.14)$$

where the semi-classical d'Alembertian, acting on a suitable space of functions on \mathcal{M} , is given by

$$\square := -\{t^\mu, \{t_\mu, \cdot\}\}. \quad (2.15)$$

It turns out that the right-hand side of (2.14) can be given a geometric meaning when restricted to scalar expressions by introducing a *dilaton* ρ^2 so that

$$\square = \rho^2 \square_G, \quad (2.16)$$

where \square_G is a geometric d'Alembertian corresponding to a metric $G_{\mu\nu}$ of Minkowskian signature,

$$\square_G = -\frac{1}{\sqrt{-G}} \partial_\mu \sqrt{-G} G^{\mu\nu} \partial_\nu. \quad (2.17)$$

It is most convenient to describe $G^{\mu\nu}$ via a frame

$$E^\alpha := \{t^\alpha, \cdot\}, \quad E^{\alpha\mu} := \{t^\alpha, y^\mu\}, \quad (2.18)$$

where y^μ are the coordinates of $\mathcal{M}^{1,3}$ as above. Then, defining $\gamma^{\mu\nu} := \eta_{\alpha\beta} E^{\alpha\mu} E^{\beta\nu}$, one finds [34]

$$\rho^2 = \rho_M \sqrt{|\gamma^{\mu\nu}|} \quad \text{and} \quad G_{\mu\nu} = \rho^2 \gamma_{\mu\nu}. \quad (2.19)$$

For more general backgrounds, the so-defined frame and metric will typically be higher-spin valued. Locally, one can remove their dependence on the higher-spin variables by changing to suitable local normal coordinates. In the present case of (general) cosmological $k = 0$ quantum spacetimes, the metric and frame in local normal coordinates will not depend on the higher-spin variables along a timelike curve, and we argue that their dependence on the higher-spin variables is small in the semi-classical regime.

2.3 Physical scales and semi-classical regime

The minimal requirement that we should impose on our model is that the semi-classical approximation is valid in a local patch at late times. In the case of cosmological $k = 0$ quantum spacetimes, that means that for $y^0 \gg 0$, there must exist a sufficiently large region where the higher-spin degrees of freedom and other non-observed features are suppressed if we express physical quantities in terms of local normal coordinates.

Additionally, there are various physical scales that are relevant to determine the range of validity of the semi-classical physics in our matrix model description of cosmological quantum spacetimes. These are

1. the effective scale of non-commutativity,

$$L_{\text{NC}}^{\overline{G}} = |\overline{G}_{\mu\kappa} \overline{G}_{\nu\lambda} \theta^{\mu\nu} \theta^{\kappa\lambda}|^{\frac{1}{4}}, \quad (2.20)$$

set by $\theta^{\mu\nu} = \{y^\mu, y^\nu\}$ and the classical part of the effective metric $\overline{G}_{\mu\nu}$ in local normal coordinates;

2. the cosmic curvature scale $L_{\text{cosm}} := H^{-1}$ set by the Hubble rate of the FLRW spacetime associated to $\bar{G}_{\mu\nu}$ – see (4.47) below;
3. the order of magnitude of the higher spin components of the frame and metric – see (4.36) below.

For the semi-classical approximation to be valid, the cosmic curvature scale should be much larger than the effective scale of non-commutativity,

$$L_{\text{cosm}} \gg L_{\text{NC}}, \quad (2.21)$$

and the higher-spin components should be negligible in a reasonably large patch. We make these requirements precise in Subsection 4.7, specifically in (4.51), for concrete choices of the background.

2.4 Gauge transformations and symplectomorphisms

On the non-commutative level, a gauge transformation on the fuzzy hyperboloid H_n^4 is implemented by a hermitian operator $\Lambda \in \text{End}(\mathcal{H}_n)$. Accordingly, a real-valued function $\Lambda \in \mathcal{C}(\mathcal{M})$ defines an infinitesimal symplectomorphism, i.e., an infinitesimal diffeomorphism preserving the symplectic form, on \mathcal{M} [34],

$$\delta_\Lambda \phi := \{\Lambda, \phi\}, \quad \phi \in \mathcal{C}(\mathcal{M}). \quad (2.22)$$

This infinitesimal diffeomorphism is nothing but the Lie derivative of ϕ with respect to the Hamiltonian vector field $\xi := \{\Lambda, \cdot\}$,

$$\delta_\Lambda \phi = \mathcal{L}_\xi \phi. \quad (2.23)$$

Accordingly, a Hamiltonian vector field $V = \{v, \cdot\}$ on \mathcal{M} transforms under a gauge transformation $\delta_\Lambda v$ of the generator as

$$\begin{aligned} (\delta_\Lambda V)[\phi] &= \{\{\Lambda, v\}, \phi\} = \{\Lambda, \{v, \phi\}\} - \{v, \{\Lambda, \phi\}\} \\ &= \xi[V[\phi]] - V[\xi[\phi]] = [\xi, V][\phi] \\ &= \mathcal{L}_\xi V[\phi] \end{aligned} \quad (2.24)$$

for any $\phi \in \mathcal{C}(\mathcal{M})$. Here, $[\xi, V] = \mathcal{L}_\xi V$ is the Lie bracket of the vector fields ξ and V , which is precisely the Lie derivative of V along ξ . Thus, V transforms covariantly (i.e., as the Lie derivative) under the symplectomorphism generated by ξ .

In the case of cosmological quantum spacetimes arising from the fuzzy hyperboloid, we encounter a complication: the spacetime $\mathcal{M}^{1,3}$ is not the same as the symplectic space \mathcal{M} , as described in Subsection 2.2. Instead, we have $\mathcal{M} \sim \mathcal{M}^{1,3} \tilde{\times} S^2$, where the internal sphere accounts for higher spin degrees of freedom in the model. Hence higher-spin valued quantities might not transform covariantly from the spacetime point of view.

The most important example is the non-covariant transformation behavior of higher-spin valued frames:

Consider the frame $E^{\alpha\mu} := E^\alpha[y^\mu] = \{t^\alpha, y^\mu\}$ given by (2.18), and $\xi^\mu := \{\Lambda, y^\mu\} = \delta_\Lambda y^\mu$. Assume that both of them admit decompositions

$$\xi^\mu = [\xi^\mu]_0 + [\xi^\mu]_{\mathfrak{hs}}, \quad E^{\alpha\mu} = [E^{\alpha\mu}]_0 + [E^{\alpha\mu}]_{\mathfrak{hs}} \quad (2.25)$$

into a classical and a higher-spin component. We will show below that the deviation between the *classical Lie derivative of the classical frame* $\mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0$ from the projection onto the spin-0 part of the gauge variation $(\delta_\Lambda E^\alpha)[y^\mu]$ is given by

$$\mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0 - [(\delta_\Lambda E^\alpha)[y^\mu]]_0 = \left[\{t^\alpha, [\xi^\mu]_{\mathfrak{hs}}\} - \delta_\Lambda [E^{\alpha\mu}]_{\mathfrak{hs}} \right]_0.$$

(2.26)

That is to say that higher-spin components both of ξ and E contribute to an obstruction to tensorial transformation behavior (where the rhs would vanish). We will assume that the classical part $[\xi^\mu]_0$ of ξ^μ is non-vanishing.

We aim to work on backgrounds (or at least in regimes) where the right-hand side of (2.26) is small compared to the individual terms on the left-hand side, in order to preserve the classical notions of differential geometry. In particular, this is expected to hold in local normal coordinates, where the \mathfrak{hs} components of the frame are small.

To derive (2.26), note first that

$$[\delta_\Lambda [E^{\alpha\mu}]_0]_0 = [\xi^\rho \partial_\rho [E^{\alpha\mu}]_0]_0 = [\xi^\rho]_0 \partial_\rho [E^{\alpha\mu}]_0 = \mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0 + [E^{\alpha\rho}]_0 \partial_\rho [\xi^\mu]_0. \quad (2.27)$$

Adding a zero-term in the form of the Jacobi identity, we obtain

$$\begin{aligned} \mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0 &= [\delta_\Lambda [E^{\alpha\mu}]_0]_0 - [E^{\alpha\rho}]_0 \partial_\rho [\xi^\mu]_0 \\ &\quad - [\delta_\Lambda E^{\alpha\mu}]_0 + [(\delta_\Lambda E^\alpha)[y^\mu]]_0 + [\{t^\alpha, \xi^\mu\}]_0 \\ &= [(\delta_\Lambda E^\alpha)[y^\mu]]_0 - [\delta_\Lambda [E^{\alpha\mu}]_{\mathfrak{hs}}]_0 + [\{t^\alpha, \xi^\mu\} - E^{\alpha\rho} \partial_\rho [\xi^\mu]_0]_0 \\ &= [(\delta_\Lambda E^\alpha)[y^\mu] - \delta_\Lambda [E^{\alpha\mu}]_{\mathfrak{hs}} + \{t^\alpha, [\xi^\mu]_{\mathfrak{hs}}\}]_0. \end{aligned} \quad (2.28)$$

If, on $\mathcal{M}^{1,3}$, the deviation from (2.24) is small in a reasonable sense, then we say that the gauge transformation determined by Λ implements an *approximate diffeomorphism on $\mathcal{M}^{1,3}$* . More precisely, we compare (2.26) to the classical Lie derivative of the classical frame $\mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0$. That is, we use the quantity

$$\Delta_{\delta_\Lambda} E^{\alpha\mu} := \left| \frac{\mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0 - [(\delta_\Lambda E^\alpha)[y^\mu]]_0}{\mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0} \right|, \quad \mathcal{L}_{[\xi]_0}[E^{\alpha\mu}]_0 \neq 0 \quad (2.29)$$

to estimate the deviation from tensoriality in the semi-classical regime. We will elaborate this in an important example in Section 4.6.

3 Static reference background for a $k = 0$ quantum spacetime

Before describing more general cosmological $k = 0$ spacetimes, let us first recall and elaborate on the reference background considered in [12]. Some relevant identities from the $k = -1$ case are collected in Appendix A.

The generators

$$T^\mu := \frac{1}{R}(M^{\mu 0} + M^{\mu 4}), \quad Y^\mu := r(M^{\mu 5} + \delta^{\mu 0}M^{45}), \quad (3.1)$$

where M^{ab} , $a, b = 0, \dots, 5$, are the generators of $SO(4, 2)$ described in Section 2, were shown to yield the metric $G_{\mu\nu}$ of a spatially flat FLRW spacetime with a Big Bang [12, 34],

$$G_{\mu\nu} = \frac{y^0}{R}\eta_{\mu\nu}. \quad (3.2)$$

Here, $y^0 > 0$ is the conformal time. Using the cosmological time $t := \frac{2}{3\sqrt{R}}y^{0\frac{3}{2}}$, the metric $G_{\mu\nu}$ corresponds to the line element

$$ds_G^2 = -dt^2 + a(t)^2 dy^2 \quad \text{with} \quad a(t) = \left(\frac{3t}{2R}\right)^{\frac{1}{3}}. \quad (3.3)$$

The T^μ generators will define a matrix background¹ in the IKKT model, while the Y^μ generators are considered as (quantized) local coordinate functions on spacetime. The matrices T^μ and Y^μ satisfy the commutation relations²

$$[T^i, T^j] = 0, \quad [T^0, T^i] = -\frac{i}{R}T^i, \quad [T^\mu, Y^\nu] = i\frac{Y^0}{R}\eta^{\mu\nu} \quad (3.4)$$

as well as

$$[Y^0, Y^i] = ir^2RT^i, \quad [Y^i, Y^j] = -ir^2M^{ij}. \quad (3.5)$$

The identity (2.2) implies the following constraints on the T^μ and Y^μ generators:

$$T^i T_i - \frac{1}{r^2 R^2} Y^{02} = 0, \quad T^\mu Y_\mu + Y^\mu T_\mu = 0. \quad (3.6)$$

3.1 $E(3)$ -subalgebra, invariant quantities and Casimir operators

Clearly, the set $\{M^{ij}, T^i \mid i, j = 1, 2, 3\}$ can be interpreted as generators of an $E(3)$ -subalgebra because

$$[M^{ij}, T^k] = i(\delta^{ik}T^j - \delta^{jk}T^i). \quad (3.7)$$

¹These are special cases of the "pure backgrounds" considered in [34].

²These commutation relations are very similar to those of $k = -1$ quantum spacetime upon zooming in to a locally flat patch [17].

They act via commutators on the space of matrices $\text{End}(\mathcal{H})$. A priori, it is clear that all quantities invariant under $SO(3) \subset E(3)$ must be functions of

$$Y^0, T^0, Y^k Y_k, T^k T_k, Y^k T_k \quad \text{and} \quad T^k Y_k. \quad (3.8)$$

However, there are several redundancies due to the constraints (3.4) and the identity

$$\frac{4i}{R} Y^0 = [T^\mu, Y_\mu] = 2T^\mu Y_\mu. \quad (3.9)$$

Eliminating the redundancies, we are left with three independent linear or quadratic $SO(3)$ -invariant quantities, in terms of which all $SO(3)$ -invariant quantities can be expressed:

$$Y^0, T^0 \quad \text{and} \quad Y^k Y_k. \quad (3.10)$$

Among these three, only Y^0 is invariant under $E(3)$. To see this, note that by (3.4), Y^0 commutes with the generators T^i of the translations, while T^0 and $Y^k Y_k$ don't. Therefore, any $E(3)$ -invariant quantity must be a function of Y^0 only.

Note that both T^i and T^μ transform as vectors under $E(3)$. That is, both of them carry a representation of $E(3)$. In the four-dimensional representation on the space with basis T^μ , the translations are implemented by a function $c_k T^k$, so that

$$\left. \begin{aligned} \delta T^0 &= i[c_k T^k, T^0] = -\frac{1}{R} c_k T^k \\ \delta T^i &= i[c_k T^k, T^i] = 0 \end{aligned} \right\} \Rightarrow \delta T^\mu = \Lambda_\nu^\mu T^\nu. \quad (3.11)$$

In terms of the generators of $E(3)$, the two quadratic Casimir operators of $E(3)$ are given by

$$C_1 = T^i T_i \quad \text{and} \quad C_2 = \varepsilon_{ijk} T^i M^{jk}. \quad (3.12)$$

Their values on the $SO(4, 2)$ -doubleton representation labeled by n are

$$C_1 = \frac{Y^0{}^2}{r^2 R^2} \quad \text{and} \quad C_2 = -\frac{n}{rR} Y^0. \quad (3.13)$$

The first Casimir simply follows from the normalization of the T^i , while the second follows from the identity (A.3).³

Remark 3.1. We want to point out an interesting coincidence. Observe that a Yang-Mills type action of the form

$$S[T] := \text{Tr} ([T^\mu, T^\nu] [T_\mu, T_\nu]) \quad (3.14)$$

³Note that to derive C_2 from (A.3), one needs to keep track of signs carefully because the Minkowski metrics relating X^0 to $X_0 = -X^0$ and Y^0 to $Y_0 = -Y^0$ do not coincide.

is manifestly invariant under the global $E(3)$ symmetry

$$S[T^0, T^i + c^i \mathbf{1}] = S[T^0, T^i] \quad \text{and} \quad S[T^0, R_j^i T^j] = S[T^0, T^i], \quad (3.15)$$

where c^i are constants and where R_j^i is a rotation. This transformation is distinct from the above gauge invariance under $E(3)$, in contrast to the $SO(3, 1)$ symmetry of the $k = -1$ case. The transformation

$$T^i \rightarrow T^i + \delta T^i := T^i + c^i \mathbf{1} = T^i + i[\Lambda_c, T^i] \quad (3.16)$$

for $k = 0$ is also equivalent to a gauge transformation, generated by $\Lambda_c = \frac{R}{Y^0} c_i Y^i$. Therefore this symmetry does not lead to physical zero modes.

3.2 Explicit form of the $SO(3, 1)$ generators

It was shown in [12] that in the minimal representation $n = 0$, the generators of rotations M^{ij} are given by⁴

$$M^{ij} = \frac{R}{Y^0} (T^i Y^j - T^j Y^i), \quad n = 0. \quad (3.17)$$

In fact, this form can only be correct in the case $n = 0$ because M^{ij} from (3.17) satisfies $C_2 = 0$, where C_2 is the second Casimir operator from (3.12) or (3.13), respectively.

Determining M^{ij} explicitly for generic n is beyond the scope of this paper. It is clear, that M^{ij} must be of the form

$$M^{ij} = c(T^i Y^j - T^j Y^i) + d\varepsilon^{ijk} Y_k + f\varepsilon^{ijk} T_k, \quad (3.18)$$

where c, d and f are functions of the $SO(3)$ -invariant quantities Y^0, T^0 and $Y^k Y_k$. However, their semi-classical limits $m^{ij} \sim M^{ij}$ for $n \gg 0$ and also in the minimal case $n = 0$ can be determined explicitly. In Appendix B, we derive

$$\begin{aligned} m^{i0} &= \frac{R}{y^0} \times \begin{cases} \frac{y^{02} + y^2 + R^2}{2y^0} t^i - t^0 y^i, & n = 0; \\ \frac{y^{02} + y^2 - R^2}{2y^0} t^i - t^0 y^i + \frac{R}{y^0} \varepsilon^{ijk} y_j t_k, & n \gg 0; \end{cases} \\ m^{ij} &= \frac{R}{y^0} \times \begin{cases} t^i y^j - t^j y^i, & n = 0; \\ t^i y^j - t^j y^i + R \varepsilon^{ijk} t_k, & n \gg 0. \end{cases} \end{aligned} \quad (3.19)$$

A simple explicit realization of the minimal $n = 0$ representation in terms of three canonical generators is given in [18].

⁴Note that the generators T^μ are no longer hermitian for $n = 0$ because $R = ir$ in this case. The $SO(3, 1)$ generators, however, remain hermitian. T^μ can be made hermitian for $n = 0$ via the redefinition $T^\mu = \frac{1}{r}(M^{\mu 0} + M^{\mu 4})$, and adjusting the formulas for the $SO(3, 1)$ generators accordingly.

3.3 Fiber projection

We would like to average higher-spin valued quantities over the internal S^2 . As introduced in Subsection 2.2, we denote this average by $[\cdot]_0$. The starting point is to average linear quadratic expressions in the t^μ variables.

Using the semi-classical version of the identity (3.6), we parametrize $\mathbf{t} = \frac{y^0}{rR}\mathbf{u}$, with $\mathbf{u}^2 = 1$, i.e., $\mathbf{u} \in S^2 \hookrightarrow \mathbb{R}^3$, and $t^0 = \frac{1}{rR}(y\mathbf{u})$. Therefore, $[t^\mu]_0 = 0$ and, using the identity

$$[u^i u^j]_0 = \frac{1}{4\pi} \int_{S^2} d\mathbf{u} u^i u^j = \frac{1}{3} \delta^{ij}, \quad (3.20)$$

we find

$$[t^\mu t^\nu]_0 = \frac{1}{3r^2 R^2} \kappa^{\mu\nu} \quad (3.21)$$

with $\kappa^{00} = y^2$, $\kappa^{i0} = \kappa^{0i} = y^0 y^i$ and $\kappa^{ij} = y^{02} \delta^{ij}$.

Note that $\kappa^{\mu\nu}$ is transversal in the sense $\kappa^{\mu\nu} y_\nu = 0$. As a consequence of (3.21) and (3.19), the various components of the average $[t^\alpha m^{\mu\nu}]_0$ are given by

$$\begin{aligned} [t^0 m^{i0}]_0 &= \frac{1}{3r^2 R y^0} \times \begin{cases} -\frac{y^\mu y_\mu - R^2}{2} y^i; \\ -\frac{y^\mu y_\mu + R^2}{2} y^i; \end{cases} \\ [t^l m^{i0}]_0 &= \frac{1}{3r^2 R} \times \begin{cases} \frac{y^{02} + y^2 + R^2}{2} \delta^{li} - y^l y^i; \\ \frac{y^{02} + y^2 - R^2}{2} \delta^{li} - y^l y^i - R \varepsilon^{ilj} y_j; \end{cases} \\ [t^0 m^{ij}]_0 &= \frac{1}{3r^2 R} \times \begin{cases} 0; \\ R \varepsilon^{ijk} y_k; \end{cases} \\ [t^l m^{ij}]_0 &= \frac{y^0}{3r^2 R} \times \begin{cases} \delta^{li} y^j - \delta^{lj} y^i; \\ \delta^{li} y^j - \delta^{lj} y^i + R \varepsilon^{ijl}, \end{cases} \end{aligned} \quad (3.22)$$

where in all four equations, the upper case corresponds to the minimal representation $n = 0$ and the lower case is valid for $n \gg 0$.

Fiber projections of polynomials of higher order in t^μ can be computed using Wick type formulas as in [31, 34].

3.4 Derivatives

In this subsection, we will construct suitable derivative operators acting on the algebra of \mathfrak{hs} -valued functions on spacetime, analogous to the construction for $k = -1$ in [34]. We will always assume $n \gg 0$. To obtain more transparent formulas, it is sometimes convenient to use the variables x^a suitable to describe the fuzzy hyperboloid and $k = -1$ spacetimes instead

of y^μ . However, in all formulas of the following considerations, x^a should be understood as $x^a(y^\mu)$ via the relations

$$x^0 = \frac{y^{0^2} + \mathbf{y}^2 + R^2}{2y^0}, \quad \mathbf{x} = \mathbf{y} \quad \text{and} \quad x^4 = \frac{y^{0^2} - \mathbf{y}^2 - R^2}{2y^0}. \quad (3.23)$$

In terms of the variables y^μ and x^4 , the tangential derivatives (2.7) on H^4 become

$$\begin{aligned} \bar{\partial}^0 &= -\frac{1}{R^2} \left(m^{j0} \{y_j, \cdot\} - Rt^0 \{x^4, \cdot\} \right); \\ \bar{\partial}^i &= -\frac{1}{R^2} \left(m^{i0} \{y^0, \cdot\} - m^{ij} \{y_j, \cdot\} - Rt^i \{x^4, \cdot\} \right); \\ \bar{\partial}^4 &= -\frac{1}{R^2} \left(-Rt^0 \{y^0, \cdot\} + (Rt^i - m^{i0}) \{y_i, \cdot\} + Rt^0 \{x^4, \cdot\} \right), \end{aligned} \quad (3.24)$$

where we may insert m^{ij} and m^{i0} for $n \gg 0$ from (3.19). The action of $\bar{\partial}^a$ on y^μ is given by

$$\begin{aligned} \bar{\partial}^0 y^0 &= -1 + \frac{x^0 y^0}{R^2}, & \bar{\partial}^i y^0 &= \frac{y^0 y^i}{R^2}, & \bar{\partial}^4 y^0 &= 1 + \frac{x^4 y^0}{R^2}, \\ \text{and } \bar{\partial}^a y^j &= \eta^{aj} + \frac{x^a y^j}{R^2}. \end{aligned} \quad (3.25)$$

Moreover, Eq. (3.24) and the Poisson brackets (A.4) yield

$$\begin{aligned} \bar{\partial}^0 t^0 &= \frac{x^0 t^0}{R^2}; \\ \bar{\partial}^0 t^j &= \frac{1}{R^2} \left(x^0 t^j + \frac{R}{y^0} (-Rt^j + \varepsilon^{jkl} y_k t_l) \right); \\ \bar{\partial}^i t^0 &= -\frac{1}{R^2} \left(-t^0 y^i + \frac{R}{y^0} (-Rt^i + \varepsilon^{ikl} y_k t_l) \right); \\ \bar{\partial}^i t^j &= -\frac{1}{R^2} \left(-y^i t^j + R \varepsilon^{ijk} t_k \right); \\ \bar{\partial}^4 t^0 &= \frac{x^4 t^0}{R^2}; \\ \bar{\partial}^4 t^j &= -\frac{1}{R^2} \left(-x^4 t^j + \frac{R}{y^0} (-Rt^j + \varepsilon^{jkl} y_k t_l) \right). \end{aligned} \quad (3.26)$$

A derivative ∂^μ on $\mathcal{M}^{1,3}$ that is appropriately applicable to \mathfrak{hs} -valued functions should satisfy

$$\partial^\mu y^\nu = \eta^{\mu\nu}. \quad (3.27)$$

It is easy to see that

$$\partial^\mu := \bar{\partial}^\mu - \frac{x^\mu}{y^0} (\bar{\partial}^0 + \bar{\partial}^4) = \bar{\partial}^\mu - \frac{y^\mu - \eta^{\mu 0} x^4}{y^0} (\bar{\partial}^0 + \bar{\partial}^4) \quad (3.28)$$

fulfills this condition. Note that $\partial^0 = \frac{1}{y^0}(x^4\eth^0 - x^0\eth^4)$. Applied to the higher-spin variables t^μ , we obtain

$$\partial^0 t^0 = 0, \quad \partial^0 t^j = \frac{1}{y^0} \left(-t^j + \frac{1}{R} \varepsilon^{jkl} y_k t_l \right) = -\partial^j t^0 \quad \text{and} \quad \partial^i t^j = -\frac{1}{R} \varepsilon^{ijk} t_k. \quad (3.29)$$

We thus have $\partial^\mu t^\nu + \partial^\nu t^\mu = 0$.

Remark 3.2. Note that (3.25) is only true for $n \gg 0$. Inserting instead m^{ij} and m^{i0} for $n = 0$ given in (3.19) yields $\eth^0 y^0 = \frac{x^0 y^0}{R^2}$ and $\eth^4 y^0 = \frac{x^4 y^0}{R^2}$. Hence there is no straightforward analog of (3.28) that satisfies (3.27) for $n = 0$. The problem arises because the minimal representation $n = 0$ admits a semi-classical structure only in the limit $y^0 \gg 0$, so ± 1 is subleading to $\frac{x^a y^0}{R^2}$ and we need to make a suitable approximation for the generators $m^{\mu\nu}$ to obtain reasonable tangential derivatives. This is beyond the scope of the present paper.

3.5 Reconstruction of divergence-free vector fields

In this subsection, we will again assume $n \gg 0$. In the case of a cosmological $k = -1$ spacetime, it has been shown that the pushforward V^μ of a tangential vector field V^a on H^4 to the spacetime $\mathcal{M}_{k=-1}^{1,3}$ satisfies

$$\eth_a V^a = \sinh(\eta) \partial_\mu (\rho_{M,k=-1} V^\mu), \quad (3.30)$$

where \eth_a is the tangential derivative on H^4 defined in (2.7); and that conversely any vector field V^μ on $\mathcal{M}_{k=-1}^{1,3}$ possesses a lift to H^4 so that (3.30) is satisfied [34]. Note that the vector fields may be \mathfrak{hs} valued. Therefore, there is a correspondence between tangential and divergence-free vector fields on H^4 to divergence-free vector fields on $\mathcal{M}_{k=-1}^{1,3}$. In the following, we establish the analogous correspondence for the $k = 0$ case.

To do so, suppose that V^μ is a (possibly \mathfrak{hs} -valued) vector field on $\mathcal{M}^{1,3}$. We can lift V^μ to a vector field \tilde{V}^a on H^4 by defining:

$$\tilde{V}^0 := V^0 + \frac{x_\mu V^\mu}{y^0}, \quad \tilde{V}^i := V^i, \quad i = 1, 2, 3, \quad \text{and} \quad \tilde{V}^4 := -\frac{x_\mu V^\mu}{y^0}. \quad (3.31)$$

The so-defined \tilde{V}^a is tangential on H^4 :

$$x_a \tilde{V}^a = x_\mu V^\mu + \frac{\eta^{00} x^0 - \eta^{44} x^4}{y^0} x_\mu V^\mu = 0. \quad (3.32)$$

Moreover, we have

$$\eth_a \tilde{V}^a = y^0 \partial_\mu \left(\frac{1}{y^0} V^\mu \right) = R^3 y^0 \partial_\mu (\rho_M V^\mu)$$

(3.33)

with $\partial_\mu = \eta_{\mu\nu}\partial^\nu$ as in (3.28). The relation (3.33) can be proved by a short computation, which uses (2.2):

$$\begin{aligned}
y^0\partial_\mu\left(\frac{1}{y^0}V^\mu\right) &= \partial_\mu V^\mu - \frac{V^0}{y^0} \\
&= \eth_\mu V^\mu - \frac{1}{y^0}\left(V^0 + x_\mu(\eth^0 + \eth^4)V^\mu\right) \\
&= \eth_\mu V^\mu - \frac{1}{y^0}\left(V^0 + (\eth^0 + \eth^4)x_\mu V^\mu - V_\mu(\eth^0 + \eth^4)x^\mu\right) \\
&= \eth_\mu V^\mu - \frac{1}{y^0}\left((\eth^0 + \eth^4)x_\mu V^\mu - \frac{y^0}{R^2}x_\mu V^\mu\right) \\
&= \eth_a \tilde{V}^a + x_\mu V^\mu \left((\eth^0 + \eth^4)\frac{1}{y^0} + \frac{1}{R^2} \right) \\
&= \eth_a \tilde{V}^a. \tag{3.34}
\end{aligned}$$

Next, consider a – possibly \mathfrak{hs} -valued – function $\phi = \phi(y, t) \in \mathcal{C}$, which defines a vector field $V^\mu := \{y^\mu, \phi\}$. The so-defined V^μ is conserved in the sense

$$\boxed{\partial_\mu(\rho_M V^\mu) = 0, \quad V^\mu = \{y^\mu, \phi\}.} \tag{3.35}$$

This follows as a corollary from (3.33). Instead of computing $\partial_\mu(\rho_M V^\mu)$, we may compute $\eth_a \tilde{V}^a$. Then, expressing y^0 in terms of x^0 and x^4 , a short computation yields

$$\eth_a \tilde{V}^a = -\frac{m^{0b} + m^{4b}}{R^2} \left\{ x_b, \frac{1}{2y^0} \{x^a x_a, \phi\} \right\} = 0. \tag{3.36}$$

For $\phi = y^\nu$ and $\theta^{\mu\nu} = \{y^\mu, y^\nu\}$, (3.35) implies in particular

$$\partial_\mu(\rho_M \theta^{\mu\nu}) = 0. \tag{3.37}$$

4 Dynamical background and general $k = 0$ quantum spacetime

We now describe more general cosmological $k = 0$ quantum spacetimes that arise from the fuzzy hyperboloid. To do so, we start by considering more general matrix backgrounds respecting the $E(3)$ symmetry.

4.1 Generalized $E(3)$ -covariant background

The generalized translation generators \tilde{T}^i of $E(3)$ should commute with each other and with the $E(3)$ -invariant quantity Y^0 ; and additionally transform as vector under the $SO(3)$ -subgroup generated by the M^{ij} . Any such vector \tilde{T}^i must be a linear combination of T^i , Y^i

and $\varepsilon^{ikl}Y_kT_l$ with coefficients that may depend on Y^0 . That is, we make the ansatz

$$\tilde{T}^i := \beta(Y^0)T^i + \gamma(Y^0)Y^i + \delta(Y^0)\varepsilon^{ikl}Y_kT_l. \quad (4.1)$$

Using the commutation relations (3.4) and (3.5), one readily verifies that

$$[\tilde{T}^i, Y^0] = [\tilde{T}^i, \tilde{T}^j] = 0 \quad (4.2)$$

together imply $\gamma = \delta = 0$.

We will see below that the function β is sufficient to describe a generic cosmological $k = 0$ spacetime – as should be, because the latter is entirely determined by the time-dependent scale factor. However, it is convenient to consider as well a deformed generator $\tilde{T}^0 := \alpha(Y^0)T^0$.

Summing up, we define a new, dynamical (i.e., Y^0 -dependent) background \tilde{T}^μ via

$$\boxed{\tilde{T}^0 = \alpha(Y^0)T^0; \quad \tilde{T}^i = \beta(Y^0)T^i.} \quad (4.3)$$

In fact, we will show in Subsection 4.6 that there are gauge transformations – defining approximate diffeomorphisms on $\mathcal{M}^{1,3}$ – which relate well-behaved gauge configurations (α, β) to other well-behaved gauge configurations $(\tilde{\alpha}, \tilde{\beta})$. Here, "well-behaved" essentially means that α and β should, at late times, satisfy some monotonicity conditions and not behave too differently. Thus, the semi-classical structures for two gauge configurations related by such an approximate diffeomorphism are equivalent.

The choices $\alpha \equiv 1$ or $\beta \equiv 1$ yield simple expressions for many interesting objects like the frame, the effective metric or the local normal coordinates. On the other hand, one might say that the choice $\alpha \equiv \beta$ is preferred because then, as in the reference case, the generators additionally carry a representation of $SO(3, 1)$.

We call $\beta \equiv 1$ the *timelike gauge*, $\alpha \equiv 1$ the *spacelike gauge* and $\alpha \equiv \beta$ the *covariant gauge*.

4.2 Gauge symmetries

An important problem is to understand how diffeomorphisms of spacetime arise from gauge transformations. The present setting suggests to focus on diffeos which respect $E(3)$. These arise from gauge transformations generated by $\Lambda(Y^0)T^0$, which preserve the structure of the background due to the commutation relations (3.4):

$$\begin{aligned} e^{-i\Lambda(Y^0)T^0} \alpha(Y^0)T^0 e^{i\Lambda(Y^0)T^0} &= \tilde{\alpha}(Y^0)T^0; \\ e^{-i\Lambda(Y^0)T^0} \beta(Y^0)T^i e^{i\Lambda(Y^0)T^0} &= \tilde{\beta}(Y^0)T^i. \end{aligned} \quad (4.4)$$

Since $[\Lambda T^0, [\Lambda T^0, \tilde{T}^\mu]] \neq 0$ for a generic function Λ , it is not easy to determine the full action of the gauge transformation on the generators \tilde{T}^μ . However, it is sufficient to determine the

infinitesimal action

$$\delta_{\Lambda T^0} \tilde{T}^\mu = i[\tilde{T}^\mu, \Lambda(Y^0)T^0]. \quad (4.5)$$

In detail, we have

$$\delta_{\Lambda T^0} \tilde{T}^0 = \frac{Y^0 W(\alpha, \Lambda)}{R} T^0 \quad \text{and} \quad \delta_{\Lambda T^0} \tilde{T}^i = -\frac{\Lambda(Y^0 \beta)'}{R} T^i, \quad (4.6)$$

where $W(f, g) := fg' - f'g$ is the Wronskian and where the prime indicates a derivative with respect to Y^0 .

We note the interesting fact that for non-trivial Λ , we have $\delta_{\Lambda T^0} \tilde{T}^i = 0$ if and only if $\beta = \frac{\text{const.}}{Y^0}$. The semi-classical structure corresponding to this particular choice of β is therefore not related to the structures for other β by an approximate diffeomorphism. In fact, we will see that $\beta = \frac{\text{const.}}{Y^0}$ corresponds to a degenerate frame and metric.

In all other cases, the infinitesimal gauge transformation is precisely the one relating backgrounds described by (α, β) to backgrounds described by $(\tilde{\alpha}, \tilde{\beta})$ for appropriate $\tilde{\alpha}, \tilde{\beta}$.

4.3 Commutation relations and matrix d'Alembertian

The new generators (4.3) satisfy the commutation relations

$$\begin{aligned} [\tilde{T}^i, \tilde{T}^j] &= 0; & [\tilde{T}^0, \tilde{T}^i] &= -\frac{i\alpha}{R} (Y^0 \beta)' T^i; \\ [\tilde{T}^0, Y^0] &= -i\frac{Y^0 \alpha}{R}; & [\tilde{T}^0, Y^j] &= ir^2 R \alpha' T^j T^0; \\ [\tilde{T}^i, Y^0] &= 0; & [\tilde{T}^i, Y^j] &= i\frac{Y^0}{R} \beta \delta^{ij} + ir^2 R \beta' T^i T^j, \end{aligned} \quad (4.7)$$

and the double-commutators are given by (no summation)

$$\begin{aligned} [\tilde{T}^0, [\tilde{T}^0, \tilde{T}^j]] &= -\frac{\alpha}{R^2} (Y^0 \alpha (Y^0 \beta)')' T^j \\ [\tilde{T}^i, [\tilde{T}^i, \tilde{T}^j]] &= [\tilde{T}^i, [\tilde{T}^i, \tilde{T}^0]] = [\tilde{T}^0, [\tilde{T}^0, \tilde{T}^0]] = 0. \end{aligned} \quad (4.8)$$

Thus, applying the matrix d'Alembertian $\square_{\tilde{T}} := [\tilde{T}^\mu, [\tilde{T}_\mu, \cdot]]$ on the background yields

$\square_{\tilde{T}} \tilde{T}^0 = 0 \quad \text{and} \quad \square_{\tilde{T}} \tilde{T}^j = \frac{\alpha}{R^2 \beta} (Y^0 \alpha (Y^0 \beta)')' \tilde{T}^j.$

(4.9)

The specific generators U^μ obtained by setting $\alpha = \beta = \frac{rR}{Y^0}$ yield a degenerate frame and metric, and we will not consider them in the following as physical background. However, they

are very convenient to parametrize the internal S^2 because they are normalized, $U^i U_i = 1$ (see also Subsection 3.3). Therefore, if \mathfrak{hs} -valued quantities are expressed in terms of $u^\mu \sim U^\mu$, it is easy to estimate the contribution of the \mathfrak{hs} part. We will thus use u^μ instead of t^μ to expand the \mathfrak{hs} components of physical quantities.

We observe that the reference background T^μ corresponding to $\alpha \equiv \beta \equiv 1$ satisfies the equations $\square_T T^0 = 0$ and $\square_T T^i = \frac{1}{R^2} T^i$. It is therefore a classical solution of the IKKT model with mass terms for the spatial generators T^i .

4.4 Higher-spin valued frame, dilaton and effective metric

The frame arising from the general background $\tilde{T}^\mu \sim \tilde{t}^\mu$ reads

$$(E^{\alpha\mu}) := \{\tilde{t}^\alpha, y^\mu\} = \frac{y^0}{R} \begin{bmatrix} -\alpha & \alpha'(\mathbf{y}\mathbf{u})\mathbf{u}^t \\ \mathbf{0} & \beta\mathbf{1} + y^0\beta'\mathbf{u}\mathbf{u}^t \end{bmatrix} \quad (4.10)$$

with $u^\mu = \frac{rR}{y^0} t^\mu$ so that $\mathbf{u}^2 = 1$ and $u^0 = \frac{(\mathbf{u}\mathbf{y})}{y^0}$. Note that the inverse of a rank one perturbation of the identity is

$$(\mathbf{1} + a\mathbf{v}\mathbf{v}^t)^{-1} = \mathbf{1} - \frac{a\mathbf{v}\mathbf{v}^t}{1 + a\mathbf{v}^2}, \quad (4.11)$$

so that the inverse of the frame is formally given by

$$(E_{\alpha\mu}) = \frac{R}{y^0} \begin{bmatrix} -\frac{1}{\alpha} & \frac{\alpha'}{\alpha(y^0\beta)'}(\mathbf{y}\mathbf{u})\mathbf{u}^t \\ \mathbf{0} & \frac{1}{\beta}\left(\mathbf{1} - \frac{y^0\beta'}{(y^0\beta)'}\mathbf{u}\mathbf{u}^t\right) \end{bmatrix}. \quad (4.12)$$

We now see explicitly that the frame is non-degenerate if $(y^0\beta)' \neq 0$, hence if $\beta \neq \frac{\text{const.}}{y^0}$. In the following, we will exclude this choice of β , although we will sometimes comment on it. The determinant of the frame and the dilaton can easily be determined using (2.19),

$$\det(E^{\alpha\mu}) = -\frac{y^{04}\alpha\beta^2(y^0\beta)'}{R^4} \quad \text{and} \quad \rho^2 = \frac{y^{03}\alpha\beta^2(y^0\beta)'}{R^3}. \quad (4.13)$$

We obtain the (higher-spin valued) effective metric

$$\begin{aligned} G^{\mu\nu} = & \frac{R}{y^0\alpha\beta^2(y^0\beta)'} \left(\begin{bmatrix} -\alpha^2 & \mathbf{0}^t \\ \mathbf{0} & \beta^2\mathbf{1} \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} 0 & \alpha\alpha'(\mathbf{y}\mathbf{u})\mathbf{u}^t \\ \alpha\alpha'(\mathbf{y}\mathbf{u})\mathbf{u} & (y^0\beta'(y^0\beta' + 2\beta) - ((\mathbf{y}\mathbf{u})\alpha')^2)\mathbf{u}\mathbf{u}^t \end{bmatrix} \right). \end{aligned} \quad (4.14)$$

The frame $E^{\alpha\mu}$ has a spin-0 and a spin-2 part, which can easily be computed using (3.21). From the latter equation, we obtain the pure spin-0 part of the frame,

$$[(E^{\alpha\mu})]_0 = \frac{y^0}{R} \begin{bmatrix} -\alpha & \frac{\alpha'}{3} \mathbf{y}^t \\ \mathbf{0} & (\beta + \frac{y^0 \beta'}{3}) \mathbf{1} \end{bmatrix}, \quad (4.15)$$

while the pure spin-2 part is given by

$$[(E^{\alpha\mu})]_2 := (E^{\alpha\mu}) - [(E^{\alpha\mu})]_0 = \frac{y^0}{R} \begin{bmatrix} 0 & \alpha'((\mathbf{y}\mathbf{u})\mathbf{u}^t - \frac{\mathbf{y}^t}{3}) \\ \mathbf{0} & y^0 \beta'(\mathbf{u}\mathbf{u}^t - \frac{1}{3} \mathbf{1}) \end{bmatrix}. \quad (4.16)$$

The effective metric has a spin-0, a spin-2 and a spin-4 part, which we will not work out explicitly here.

4.5 Local normal coordinates

Due to the \mathfrak{hs} components, the physical meaning of a metric or frame is not clear *a priori*. As shown in [34], it is always possible to locally eliminate these \mathfrak{hs} components, in local normal coordinates. We illustrate this in the following for the present backgrounds.

We fix the point $\xi = (\tau, 0, 0, 0)$ as a reference point. We want to find local normal coordinates \tilde{y}^μ such that the frame has no higher-spin components at ξ and satisfies

$$\tilde{E}^{\mu\nu}|_\xi = \{\tilde{t}^\mu, \tilde{y}^\nu\}|_\xi = \frac{\tau}{R} \begin{bmatrix} -\alpha(\tau) & \mathbf{0}^t \\ \mathbf{0} & \beta(\tau) \mathbf{1} \end{bmatrix}. \quad (4.17)$$

We make the ansatz

$$\tilde{y}^0 = y^0, \quad \tilde{y}^i = y^i + b(y^0)(\mathbf{y}\mathbf{u})u^i = y^i + y^0 b(y^0)u^0 u^i, \quad (4.18)$$

which respects the local $SO(3)$. Note that we have $\mathbf{y}|_\xi = u^0|_\xi = 0$. This ansatz yields

$$\begin{aligned} \tilde{E}^{00} &= -\frac{y^0 \alpha}{R}, \\ \tilde{E}^{0j} &= \frac{1}{R} W(y^0(1+b), y^0 \alpha) \frac{(\mathbf{y}\mathbf{u})}{y^0} u^j, \\ \tilde{E}^{i0} &= 0, \\ \tilde{E}^{ij} &= \frac{y^0}{R} \left(\beta \delta^{ij} + (y^0 \beta' + b(y^0 \beta')) u^i u^j \right) \end{aligned} \quad (4.19)$$

where W is the Wronskian. Because $\mathbf{y}|_\xi = 0$, the requirement (4.17) is fulfilled for

$$b(\tau) = -\frac{y^0 \beta'}{(y^0 \beta')'} \Bigg|_{y^0=\tau}. \quad (4.20)$$

The local normal coordinates with b as in (4.20) naturally extend for all $\tau > 0$, i.e. on a time-like worldline. Then the frame $\tilde{E}^{\mu\nu}$ is globally given by

$$\begin{aligned} (\tilde{E}^{\mu\nu}) &= (\bar{E}^{\mu\nu}) + \frac{\mathcal{W}}{R} \begin{bmatrix} 0 & \frac{(\mathbf{y}\mathbf{u})}{y^0} \mathbf{u}^t \\ \mathbf{0} & 0_{3 \times 3} \end{bmatrix}, \\ \text{where } (\bar{E}^{\mu\nu}) &= \frac{y^0}{R} \begin{bmatrix} -\alpha & \mathbf{0}^t \\ \mathbf{0} & \beta \mathbf{1} \end{bmatrix} \end{aligned} \quad (4.21)$$

and where the Wronskian $\mathcal{W} = \mathcal{W}(y^0) := W\left(\frac{y^0\beta}{(y^0\beta)'}, y^0\alpha\right)$ determines the size of the higher-spin contributions. The determinant of the frame in local normal coordinates is then given by

$$\det(\tilde{E}^{\mu\nu}) = -\frac{y^0\alpha(y^0\beta)^3}{R^4}. \quad (4.22)$$

The symplectic density $\tilde{\rho}_M$ in the new coordinates reads

$$\rho_M d^4y = \tilde{\rho}_M d^4\tilde{y} = \frac{R}{y^0 \left| \det \frac{\partial \tilde{y}^\mu}{\partial y^\nu} \right|} d^4\tilde{y}, \quad (4.23)$$

and the Jacobian is readily computed to be

$$\det \frac{\partial \tilde{y}^\mu}{\partial y^\nu} = 1 + b = \frac{\beta}{(y^0\beta)'} \Rightarrow \tilde{\rho}_M = R \frac{(y^0\beta)'}{y^0\beta}. \quad (4.24)$$

The dilaton, which can be computed from the symplectic density and the determinant (4.22), is not affected by the change of coordinates: we have

$$\tilde{\rho}^2 = \rho^2 \quad (4.25)$$

with ρ^2 as in (4.13). Consequently, the effective metric is obtained as

$$\tilde{G}^{\mu\nu} = \frac{1}{R^2 \tilde{\rho}^2} \left(\begin{bmatrix} -(y^0\alpha)^2 & \mathbf{0}^t \\ \mathbf{0} & (y^0\beta)^2 \mathbf{1} \end{bmatrix} + \alpha \mathcal{W}(\mathbf{y}\mathbf{u}) \begin{bmatrix} 0 & \mathbf{u}^t \\ \mathbf{u} & -\mathcal{W} \frac{(\mathbf{y}\mathbf{u})}{\alpha y^0} \mathbf{u} \mathbf{u}^t \end{bmatrix} \right), \quad (4.26)$$

so that

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \bar{G}_{\mu\nu} + \frac{(y^0\beta)' \mathcal{W}(\mathbf{y}\mathbf{u})}{R y^0} \begin{bmatrix} \frac{\mathcal{W}(\mathbf{y}\mathbf{u})}{y^0 \alpha} & \mathbf{u}^t \\ \mathbf{u} & 0_{3 \times 3} \end{bmatrix}, \\ \text{where } \bar{G}_{\mu\nu} &= \frac{y^0 \alpha (y^0 \beta)'}{R} \begin{bmatrix} -\frac{\beta^2}{\alpha^2} & \mathbf{0}^t \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \\ \text{and } \det \bar{G}_{\mu\nu} &= -\frac{y^{04} \alpha^2 \beta^2 ((y^0 \beta)')^4}{R^4}. \end{aligned} \quad (4.27)$$

If the higher spin entries of $\tilde{G}_{\mu\nu}$ are small in a neighborhood of $L = \{(y^0, \mathbf{0}) \mid y^0 > 0\}$, we may use the metric $\bar{G}_{\mu\nu}$ in this neighborhood. The line element corresponding to $\bar{G}_{\mu\nu}$ is

$$ds_{\bar{G}}^2 = \frac{y^0 \beta^2 (y^0 \beta)'}{R \alpha} \left(-dy^{02} + \frac{\alpha^2}{\beta^2} d\mathbf{y}^2 \right). \quad (4.28)$$

To recognize this as a cosmological Friedman-Lemaître-Robertson-Walker (FLRW) metric, we define the cosmological time t via

$$t := \int_0^{y^0} \left(\frac{\beta(z)}{\alpha(z)} \frac{z\beta(z)(z\beta(z))'}{R} \right)^{\frac{1}{2}} dz, \quad dt = \left(\frac{\beta}{\alpha} \frac{y^0 \beta (y^0 \beta)'}{R} \right)^{\frac{1}{2}} dy^0. \quad (4.29)$$

Expressed in terms of t , the line element (4.28) becomes

$$ds_{\bar{G}}^2 = -dt^2 + a^2(t) d\mathbf{y}^2, \quad (4.30)$$

where

$$a(t) = \sqrt{\frac{y^0 \alpha (y^0 \beta)'}{R}} \quad (4.31)$$

is the cosmic scale factor with $y^0 = y^0(t)$, where the prime indicates a derivative with respect to y^0 as usual. If we indicate the derivative with respect to t by a dot, then the Hubble parameter H and the scalar curvature \mathcal{R} are given by

$$\begin{aligned} H &= \frac{\dot{a}}{a} = \frac{\dot{y}^0}{2} \frac{(y^0 \alpha (y^0 \beta)')'}{y^0 \alpha (y^0 \beta)'}, \\ \mathcal{R} &= \frac{\ddot{a}}{a} + H^2 = \frac{\dot{y}^0 (y^0 \alpha (y^0 \beta)')' + \dot{y}^{02} (y^0 \alpha (y^0 \beta)')''}{2y^0 \alpha (y^0 \beta)'}, \end{aligned} \quad (4.32)$$

where we always assume $y^0 = y^0(t)$. In Subsection 4.7, we will determine the asymptotic behavior of H and \mathcal{R} at late times assuming that α and β grow polynomially.

Let us estimate the magnitude of the higher spin-components of the frame $\tilde{E}^{\mu\nu}$ and metric $\tilde{G}^{\mu\nu}$ in local normal coordinates away from the line L . The only entries with higher-spin components are \tilde{E}^{0j} , and we have

$$|\tilde{E}^{0j}| \leq \left| \frac{\mathcal{W}}{\sqrt{R y^0 \alpha (y^0 \beta)'}} \right| \|\mathbf{y}\|_{\bar{G}}, \quad (4.33)$$

where

$$\|\mathbf{y}\|_{\bar{G}} := (\bar{G}_{ij} y^i y^j)^{\frac{1}{2}} = \left| \frac{y^0 \alpha (y^0 \beta)'}{R} \delta_{ij} y^i y^j \right|^{\frac{1}{2}} \quad (4.34)$$

is the length of \mathbf{y} measured in terms of the metric \overline{G} . Similar estimates on the higher spin components of the metric yield

$$\begin{aligned} |\tilde{G}_{00} - \overline{G}_{00}| &\leq \frac{\mathcal{W}^2}{y^{04}\alpha^2} \|\mathbf{y}\|_{\overline{G}}^2; \\ |\tilde{G}_{0j} - \overline{G}_{0j}| &\leq \left| \frac{(y^0\beta)' \mathcal{W}^2}{Ry^{03}\alpha} \right|^{\frac{1}{2}} \|\mathbf{y}\|_{\overline{G}}. \end{aligned} \quad (4.35)$$

The higher-spin components of the metric and frame are negligible in the semi-classical regime if the right-hand sides of (4.33) and (4.35) become small compared to the reference frame $\overline{E}^{\mu\nu}$ and metric $\overline{G}_{\mu\nu}$. That is,

$$\begin{aligned} \Delta \tilde{E}^{0j} &:= \frac{|\tilde{E}^{0j}|}{\left| \det \overline{E}^{\mu\nu} \right|^{\frac{1}{4}}} \leq \left| \frac{R\mathcal{W}^2}{y^{05}(\alpha\beta)^{\frac{3}{2}}(y^0\beta)'} \right|^{\frac{1}{2}} \|\mathbf{y}\|_{\overline{G}}; \\ \Delta \tilde{G}_{00} &:= \frac{|\tilde{G}_{00} - \overline{G}_{00}|}{\left| \det \overline{G}_{\mu\nu} \right|^{\frac{1}{4}}} \leq \left| \frac{R\mathcal{W}^2}{y^{05}\alpha^{\frac{5}{2}}\beta^{\frac{1}{2}}(y^0\beta)'} \right|^{\frac{1}{2}} \|\mathbf{y}\|_{\overline{G}}^2; \\ \Delta \tilde{G}_{0j} &:= \frac{|\tilde{G}_{0j} - \overline{G}_{0j}|}{\left| \det \overline{G}_{\mu\nu} \right|^{\frac{1}{4}}} \leq \left| \frac{R\mathcal{W}^2}{y^{05}\alpha^2\beta(y^0\beta)'} \right|^{\frac{1}{2}} \|\mathbf{y}\|_{\overline{G}} \end{aligned} \quad (4.36)$$

must become small in a local patch at late times. Therefore, the bounds (4.36) can be used to estimate the size of the patch in which the semi-classical approximation is valid.

4.6 Approximate diffeomorphisms relating different backgrounds

In this section, we show that the gauge transformation (4.4) corresponds to an approximate diffeomorphism on $\mathcal{M}^{1,3}$, under mild assumptions.

For $\Psi \in \mathcal{C}(\mathcal{M})$, we introduce the notation $\delta\Psi := \{\Lambda(y^0)t^0, \Psi\}$. If Φ is a scalar function of the variables y^μ , then its variation $\delta\Phi = \xi^\mu \partial_\mu \Phi$ is described by the higher-spin valued vector

$$\xi^\mu = \frac{y^0}{R} (-\Lambda, \Lambda'(\mathbf{y}\mathbf{u})\mathbf{u}), \quad (4.37)$$

which contains a classical (spin-0) and a spin-2 part, $\xi^\mu = [\xi^\mu]_0 + [\xi^\mu]_2$,

$$\begin{aligned} [\xi^\mu]_0 &= \frac{y^0}{R} (-\Lambda, \frac{\Lambda'}{3}\mathbf{y}), \\ [\xi^\mu]_2 &= \frac{y^0\Lambda'}{R} (0, (\mathbf{y}\mathbf{u})\mathbf{u} - \frac{\mathbf{y}}{3}). \end{aligned} \quad (4.38)$$

The classical part describes a translation along the time direction y^0 near $y = 0$. Note that a time translation cannot implement the gauge transformation globally because the vector field is volume-preserving in the sense

$$\partial_\mu(\rho_M[\xi^\mu]_0) = 0. \quad (4.39)$$

Now we want to check the Lie derivatives of the frame (2.26). Using the explicit form of the spin-2 part of the frame given by (4.16), we find

$$\begin{aligned} \{\tilde{t}^\alpha, [\xi^0]_2\} - \delta[E^{\alpha 0}]_2 &= 0; \\ \{\tilde{t}^0, [\xi^j]_2\} - \delta[E^{0j}]_2 &= -\frac{y^0 y_k}{R^2} (y^0 W(\alpha, \Lambda))' [u^k u^j]_2 \in \mathcal{C}^2; \\ \{\tilde{t}^i, [\xi^j]_2\} - \delta[E^{ij}]_2 &= \frac{2y^{03}\Lambda'\beta'}{3R^2} u^i u^j + \frac{y^{02}}{R^2} (W(\beta, \Lambda) + 2y^0\Lambda\beta'') [u^i u^j]_2. \end{aligned} \quad (4.40)$$

Thus, only the purely spatial part contributes to the right-hand side of (2.26), which parametrizes the deviation from a proper transformation behavior of the frame under diffeomorphisms:

$$\begin{aligned} \mathcal{L}_{[\xi]_0}[E^{0\mu}]_0 - [(\delta E^0)[y^\mu]]_0 &= \mathcal{L}_{[\xi]_0}[E^{i0}]_0 - [(\delta E^i)[y^0]]_0 = 0; \\ \mathcal{L}_{[\xi]_0}[E^{ij}]_0 - [(\delta E^i)[y^j]]_0 &= \frac{2y^{03}}{9R^2} \beta' \Lambda' \delta^{ij}. \end{aligned} \quad (4.41)$$

The relative error (2.29) becomes

$$\left| \frac{\mathcal{L}_{[\xi]_0}[E^{ij}]_0 - [(\delta_\Lambda E^i)[y^j]]_0}{\mathcal{L}_{[\xi]_0}[E^{ij}]_0} \right| = \frac{\frac{2}{9} \left| \frac{y^0 \beta'}{(y^0 \beta')'} \right| \left| \frac{\Lambda'}{\Lambda} \right|}{\left| 1 + \frac{y^0 \Lambda'}{3\Lambda} + \frac{y^0 (y^0 \beta)''}{3(y^0 \beta)'} - \frac{2}{9} \frac{y^0 \beta'}{(y^0 \beta)'} \frac{\Lambda'}{\Lambda} \right|}. \quad (4.42)$$

Suppose that $\beta \sim y^{0\mu}$ as $y^0 \gg 0$ for some $\mu \in \mathbb{R} \setminus \{-1\}$. Then

$$\left| \frac{\mathcal{L}_{[\xi]_0}[E^{ij}]_0 - [(\delta_\Lambda E^i)[y^j]]_0}{\mathcal{L}_{[\xi]_0}[E^{ij}]_0} \right| = \frac{2}{9} \frac{\left| \frac{\mu}{\mu+1} \right| \left| \frac{\Lambda'}{\Lambda} \right|}{\left| 1 + \frac{y^0 \Lambda'}{3\Lambda} + \frac{\mu}{3} - \frac{2}{9} \frac{\mu}{\mu+1} \frac{\Lambda'}{\Lambda} \right|}. \quad (4.43)$$

Therefore, any $\Lambda \sim y^{0a}$ as $y^0 \gg 0$ will make the error decay as $y^0 \rightarrow \infty$ ⁵, and the frame transforms as expected.

Suppose we are given a fixed gauge configuration (α, β) . In order to determine which other gauge configurations $(\tilde{\alpha}, \tilde{\beta})$ lie in the same gauge orbit as (α, β) , one needs to compute the finite gauge transformations implemented by $\Lambda(y^0)t^0$ such that (4.42) becomes small at late times. This is beyond the scope of our paper, but our evaluation of physical consistency conditions in Subsection 4.7 below suggests that the ratios $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ must not grow too fast. That is to say, α and β should not behave too differently. We will describe more concrete bounds on their behavior in the following two subsections.

⁵With the exception $\Lambda = cy^{0-3-\mu}$.

4.7 Physical scales and consistency conditions

Let us now explicitly work out constraints on the background \tilde{T}^μ so that the conditions for reasonable semi-classical physics described in Subsection 2.3 are fulfilled. We restrict our considerations to backgrounds where α and β (and their derivatives) behave monotonously like powers in the late time regime,

$$\alpha \sim y^{0\lambda}, \quad \beta \sim y^{0\mu} \quad \text{for} \quad y^0 \gg 0. \quad (4.44)$$

On the non-commutative level, the leading contribution to $|[Y^\mu, Y^\nu][Y_\mu, Y_\nu]|$ is the mixed spatial-temporal entry [34]. This should remain true on the semi-classical level, which is easy to verify if the ratio $\frac{\alpha}{\beta}$ does not grow too fast. In this case, we generically find

$$L_{\text{NC}}^{\overline{G}} \sim 2|\overline{G}_{00}\overline{G}_{ij}\{y^0, y^i\}\{y^0, y^j\}|^{\frac{1}{4}} = \sqrt{\frac{4r}{R}\beta(y^0\beta)'y^0} \sim y^{0^{1+\mu}}. \quad (4.45)$$

The cosmological time t defined in (4.29) and the derivatives of y^0 with respect to t satisfy

$$\begin{aligned} t &\sim \frac{2}{3\mu - \lambda + 3}y^{0\frac{3\mu - \lambda + 3}{2}}, & y^0 &\sim \left(\frac{3\mu - \lambda + 3}{2}t\right)^{\frac{2}{3\mu - \lambda + 3}}, \\ \dot{y}^0 &\sim y^{0\frac{-3\mu + \lambda - 1}{2}}, & \ddot{y}^0 &\sim \frac{-3\mu + \lambda - 1}{2}y^{0-3\mu + \lambda - 2}. \end{aligned} \quad (4.46)$$

Consequently, the scale factor, Hubble rate and Ricci curvature behave like

$$\begin{aligned} a &\sim y^{0\frac{1+\lambda+\mu}{2}} \sim t^{\frac{1+\lambda+\mu}{3\mu-\lambda+3}}; \\ H &\sim \frac{1+\lambda+\mu}{2}y^{0\frac{-3\mu+\lambda-3}{2}} \sim \frac{1+\lambda+\mu}{3\mu-\lambda+3} \frac{1}{t}; \\ \mathcal{R} &\sim -\frac{(1+\lambda+\mu)(\mu-3\lambda+1)}{4}y^{0-3\mu+\lambda-3} \\ &\sim -\frac{(1+\lambda+\mu)(\mu-3\lambda+1)}{(3\mu-\lambda+3)^2} \frac{1}{t^2}. \end{aligned} \quad (4.47)$$

In a physically reasonable situation, the minimum requirements should be that the Hubble rate and Ricci curvature decay with y^0 , and that the cosmic scale L_{cosm} is large compared to the non-commutativity scale $L_{\text{NC}}^{\overline{G}}$,

$$\frac{L_{\text{cosm}}}{L_{\text{NC}}^{\overline{G}}} = \frac{1}{H L_{\text{NC}}^{\overline{G}}} \sim y^{0\frac{1+\mu-\lambda}{2}} \gg 1 \quad \text{for} \quad y^0 \gg 0. \quad (4.48)$$

The decay of the Hubble rate implies $\lambda < 3(1 + \mu)$, which is equivalent to t increasing with y^0 . The requirement that the cosmic scale becomes large compared to the scale of

non-commutativity implies $\lambda < 1 + \mu$. Both constraints are upper bounds on the growth of the ratio $\frac{\alpha}{\beta}$, as expected.

The second bound $\lambda < 1 + \mu$ also guarantees that the scale of non-commutativity is dominated by the mixed spatial-temporal entries, as should be:

$$\left| \frac{G_{ij}G_{kl}\theta^{ik}\theta^{jl}}{G_{ij}G_{00}\theta^{i0}\theta^{j0}} \right| \sim y^{0-2(1+\mu-\lambda)}. \quad (4.49)$$

Finally, the estimates (4.36) for the higher-spin components of the frame and metric read

$$\begin{aligned} \Delta \tilde{E}^{0j} &:= \frac{|\tilde{E}^{0j}|}{|\det \bar{E}^{\mu\nu}|^{\frac{1}{4}}} \leq \left| \frac{\sqrt{R}\lambda}{(1+\mu)^{\frac{3}{2}}} \right| y^{0-\frac{6+5\mu-\lambda}{4}} \|\mathbf{y}\|_{\bar{G}}; \\ \Delta \tilde{G}_{00} &:= \frac{|\tilde{G}_{00} - \bar{G}_{00}|}{|\det \bar{G}_{\mu\nu}|^{\frac{1}{4}}} \leq \left| \frac{R\lambda^2}{(1+\mu)^3} \right| y^{0-\frac{6+3\mu+\lambda}{2}} \|\mathbf{y}\|_{\bar{G}}^2; \\ \Delta \tilde{G}_{0j} &:= \frac{|\tilde{G}_{0j} - \bar{G}_{0j}|}{|\det \bar{G}_{\mu\nu}|^{\frac{1}{4}}} \leq \left| \frac{\sqrt{R}\lambda}{(1+\mu)^{\frac{3}{2}}} \right| y^{0-\frac{3+2\mu}{2}} \|\mathbf{y}\|_{\bar{G}}. \end{aligned} \quad (4.50)$$

These bounds can be used to estimate the size of the patch at late times in which the semi-classical approximation is valid: if all powers of y^0 on the right-hand side are negative, this patch will be large at late times. However, if at least one estimate grows with y^0 , then the semi-classical approximation is only valid in a small patch.

Finally, we require the dilaton to be either constant or increasing with y^0 , corresponding to weak Yang-Mills coupling $g_{\text{YM}} \sim \rho^{-1}$ [35]. Using the formula (4.13), this means that $3(1+\mu) + \lambda \geq 0$. This condition might not be strictly necessary, but we will find later that it only excludes some *shrinking* FLRW spacetimes, which are anyway not interesting from a physical point of view.

Let us summarize all physical requirements:

t grows, H and \mathcal{R} decay with y^0 :	$\lambda < 3(1+\mu)$,
$L_{\text{cosm}} \gg L_{\text{NC}}$:	$\lambda < 1 + \mu$,
semi-class. approx. valid in large patch:	$\begin{cases} \lambda < 6 + 5\mu, \\ \lambda > -6 - 3\mu, \\ \mu > -\frac{3}{2}, \end{cases}$
dilaton constant or growing with y^0 :	$\lambda \geq -3(1+\mu)$.

These conditions can easily be satisfied simultaneously. For example, the undeformed background $\lambda = \mu = 0$ and small deformations thereof satisfy (4.51).

4.8 Distinguished gauges

Let us now consider special choices of one of the two parameters. We investigate the *timelike gauge* $\beta \equiv 1$ and the *covariant gauge* $\alpha \equiv \beta$. We will verify explicitly that these different gauges lead to the same classical geometry for some specific backgrounds, by computing the effective metric in local normal coordinates. This validates in particular the physical significance of local normal coordinates.

4.8.1 Timelike gauge

As outlined in Subsection 4.5, things become particularly simple in the gauge $\beta \equiv 1$ because then, the local normal coordinates coincide with the original coordinates y^μ . In this gauge, we have

$$\square_{\tilde{T}} \tilde{T}^0 = 0 \quad \text{and} \quad \square_{\tilde{T}} \tilde{T}^j = \frac{\alpha}{R^2} (Y^0 \alpha)' \tilde{T}^j. \quad (4.52)$$

On the semi-classical level, the frame and its determinant are given by

$$(E^{\alpha\mu}) = \frac{y^0}{R} \begin{bmatrix} -\alpha & \alpha'(\mathbf{y}\mathbf{u})\mathbf{u}^t \\ \mathbf{0} & 1 \end{bmatrix}, \quad \det(E^{\alpha\mu}) = -\frac{y^{04}\alpha}{R^4}, \quad (4.53)$$

and the effective metric reads

$$G_{\mu\nu} = \overline{G}_{\mu\nu} + \frac{y^0\alpha'}{R} (\mathbf{y}\mathbf{u}) \begin{bmatrix} \frac{\alpha'(\mathbf{y}\mathbf{u})}{\alpha} & \mathbf{u}^t \\ \mathbf{u} & 0_{3\times 3} \end{bmatrix}, \quad \overline{G}_{\mu\nu} = \frac{y^0\alpha}{R} \begin{bmatrix} -\frac{1}{\alpha^2} & 0^t \\ \mathbf{0} & 1 \end{bmatrix}. \quad (4.54)$$

If we assume a powerlike behavior $\alpha = \text{const.} \times y^{0\lambda}$, the physical consistency conditions (4.51) in the timelike gauge ($\mu = 0$) become

$$-3 \leq \lambda < 1. \quad (4.55)$$

Defining $t := \text{const.} \times y^{0\frac{3-\lambda}{2}}$, the line element associated to $\overline{G}_{\mu\nu}$ becomes

$$ds_{\overline{G}}^2 = \text{const.} \times \left(-dt^2 + t^{2\frac{1+\lambda}{3-\lambda}} d\mathbf{y}^2 \right), \quad (4.56)$$

which is the line element of a spatially flat FLRW spacetime with scale factor $a_\lambda(t) = t^{\frac{1+\lambda}{3-\lambda}}$. Different values of λ yield differently behaving scale factors:

- $\lambda \in (-1, 1)$: expanding FLRW;
- $\lambda = -1$: static case;
- $\lambda \in [-3, -1)$: shrinking FLRW.

The scale factor approaches linear growth as $\lambda \nearrow 1$, and the dilaton (4.13) satisfies

$$\tilde{\rho}^2 \sim y^{03+\lambda} \sim t^{\frac{6+2\lambda}{3-\lambda}}. \quad (4.57)$$

Let us mention two distinguished backgrounds.

(i) The classical solution. By *classical solution*, we mean the solution that makes \tilde{T}^μ a solution of the equation $\square_{\tilde{T}} \tilde{T}^\mu = 0$. In the current gauge, this means that $\alpha = \frac{\text{const.}}{y^0}$. If we define $t := \text{const.} \times y^{0^2}$, the line element (4.56) becomes the line element of the Minkowski metric,

$$ds_{\overline{G}}^2 = \text{const.} \times \left(-dt^2 + d\mathbf{y}^2 \right). \quad (4.58)$$

However, the dilaton is not constant, but grows like $\tilde{\rho}^2 \sim t$.

(ii) The constant dilaton background. The dilaton is constant for $\alpha = \frac{\text{const.}}{y^{0^3}}$. Defining $t := \text{const.} \times y^{0^3}$, the line element (4.56) becomes

$$ds_{\overline{G}}^2 = \text{const.} \times \left(-dt^2 + t^{-\frac{2}{3}} d\mathbf{y}^2 \right). \quad (4.59)$$

4.8.2 Covariant gauge

Let us now discuss the covariant gauge $\alpha \equiv \beta$. In this gauge, the frame is given by

$$(\tilde{E}^{\mu\nu}) = (\overline{E}^{\mu\nu}) + \frac{(y^0\beta)^2(y^0\beta)''}{R((y^0\beta)')^2} \begin{bmatrix} 0 & \frac{(\mathbf{y}\mathbf{u})}{y^0} \mathbf{u}^t \\ \mathbf{0} & 0_{3 \times 3} \end{bmatrix}, \quad \overline{E}^{\mu\nu} = \frac{y^0\beta}{R} \eta^{\mu\nu}, \quad (4.60)$$

and the effective metric reads

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \overline{G}_{\mu\nu} + \frac{y^0\beta^2(y^0\beta)''(\mathbf{y}\mathbf{u})}{R(y^0\beta)'} \begin{bmatrix} \frac{\beta(y^0\beta)''(\mathbf{y}\mathbf{u})}{y^0((y^0\beta)')^2} & \mathbf{u}^t \\ \mathbf{u} & 0_{3 \times 3} \end{bmatrix}, \\ \text{where } \overline{G}_{\mu\nu} &= \frac{((y^0\beta)^2)'}{2R} \eta_{\mu\nu}. \end{aligned} \quad (4.61)$$

If we assume a powerlike behavior $\beta = \text{const.} \times y^{0^\mu}$, the physical consistency conditions (4.51) in the timelike gauge ($\lambda = \mu$) reduce to the constraint

$$\mu \geq -\frac{3}{4}. \quad (4.62)$$

Then we define $t := \text{const.} \times y^{0^{\mu+\frac{3}{2}}}$, so that the line element becomes

$$ds_{\overline{G}}^2 = \text{const.} \times \left(-dt^2 + t^{\frac{4\mu+2}{2\mu+3}} d\mathbf{y}^2 \right). \quad (4.63)$$

The scale factor of this FLRW line element is $a_\mu(t) = t^{\frac{2\mu+1}{2\mu+3}}$, and in complete analogy to the timelike gauge, different values of μ yield differently behaving scale factors:

$$\begin{aligned} \mu \in (-\frac{1}{2}, \infty) &: \text{ expanding FLRW;} \\ \mu = -\frac{1}{2} &: \text{ static case;} \\ \mu \in [-\frac{3}{4}, -\frac{1}{2}] &: \text{ shrinking FLRW.} \end{aligned}$$

The scale factor approaches linear growth as $\mu \rightarrow \infty$, and the dilaton (4.13) satisfies

$$\tilde{\rho}^2 \sim y^{0+4\mu} \sim t^{\frac{6+8\mu}{3+2\mu}}. \quad (4.64)$$

Let us again discuss some distinguished solutions.

(i) The classical solution. For the dynamical background to satisfy the equations $\square_{\tilde{T}} \tilde{T}^\mu = 0$, β must be of the form

$$\beta(y^0) = \frac{\sqrt{cy^0 + d}}{y^0} \quad (4.65)$$

for some constants c, d . We assume $c > 0$ to exclude the degenerate solution and $d \geq 0$. For large times, we have $\beta \sim y^{0-\frac{1}{2}}$.

As in the timelike gauge, the metric \bar{G} is simply a multiple of the Minkowski metric – now even without a change of time variable,

$$ds_{\bar{G}}^2 = \text{const.} \times \left(-dy^{02} + dy^2 \right). \quad (4.66)$$

As in the timelike gauge, the dilaton grows like $\tilde{\rho}^2 = t$.

(ii) The constant dilaton background. The dilaton is constant for β of the form

$$\beta(y^0) = \frac{(cy^0 + d)^{\frac{1}{4}}}{y^0} \quad (4.67)$$

If we define $t := \text{const.} \times (cy^0 + d)^{\frac{3}{4}}$, then, again as in the timelike gauge, the line element corresponding to such β is given by

$$ds_{\bar{G}}^2 = \text{const.} \times \left(-dt^2 + t^{-\frac{2}{3}} dy^2 \right), \quad (4.68)$$

which describes the same shrinking, spatially flat FLRW spacetime as the constant dilaton solution in the timelike gauge.

(iii) Background with exactly vanishing hs components. Consider $\beta = \frac{d}{y^0} + c$ for some $c > 0$. The corresponding background is a sum of the reference background T^μ and the background U^μ (which on its own yields a degenerate frame and metric). For this background, the higher-spin components of the frame $\tilde{E}^{\mu\nu}$ and metric $\tilde{G}^{\mu\nu}$ in local normal coordinates vanish exactly because $(y^0\beta)'' = 0$. Moreover, the frame and metric are non-degenerate because $(y^0\beta)' = c \neq 0$.

Thus, the transition to local normal coordinates removes the higher-spin components globally for such β , and not only along the line L . We have

$$\tilde{G}_{\mu\nu} = \bar{G}_{\mu\nu}, \quad (4.69)$$

and the FLRW line element is the same as for the reference background, where $a(t) = t^{\frac{1}{3}}$ with $t = \text{const.} \times (cy^0 + d)^{\frac{3}{2}}$.

Remark 4.1. Note that by (4.26), a background where the \mathfrak{hs} components of the metric vanish exactly can be found for any gauge configuration $\alpha = f(\beta)$ such that

$$\mathcal{W} = W\left(\frac{y^0\beta}{(y^0\beta)'}, y^0f(\beta)\right) = 0. \quad (4.70)$$

We presented this background in more detail for the covariant gauge because then (4.70) becomes particularly simple: $(y^0\beta)'' = 0$, $(y^0\beta)' \neq 0$.

5 FLRW quantum spacetimes and the IKKT model

Let us finally describe how the cosmological quantum spacetimes described above could be incorporated into the IKKT model⁶. To this end, consider the bosonic part of the IKKT action (1.5),

$$S[T] := \text{Tr}([T^a, T^b][T_a, T_b]), \quad a, b = 0, \dots, 9. \quad (5.1)$$

We split the T^a into the matrices parametrizing the background and the extra dimensions, respectively,

$$T^\mu = \tilde{T}^\mu, \quad T^I = \Phi^I, \quad \mu = 0, \dots, 3, \quad I = 4, \dots, 9. \quad (5.2)$$

Then

$$S[T] = -\text{Tr}\left(\tilde{T}^\nu(\square_{\tilde{T}} + \square_\Phi)\tilde{T}_\nu + \Phi^J(\square_{\tilde{T}} + \square_\Phi)\Phi_J\right), \quad (5.3)$$

and the equations of motion for the background and extra dimensions become

$$(\square_{\tilde{T}} + \square_\Phi)T^a = 0, \quad a = 0, \dots, 9. \quad (5.4)$$

Taking into account (4.9), the background \tilde{T}^μ and the non-trivial extra-dimensions Φ^I satisfy the *classical* IKKT equations of motion if and only if

$$\begin{aligned} \square_\Phi \tilde{T}^0 &= 0; \\ \square_\Phi \tilde{T}^j &= -\frac{\alpha}{R^2\beta}(Y^0\alpha(Y^0\beta)')'\tilde{T}^j; \\ (\square_{\tilde{T}} + \square_\Phi)\Phi^I &= 0. \end{aligned} \quad (5.5)$$

Since $\square_\Phi = [\Phi^I, [\Phi_I, \cdot]]$ is a positive operator, non-trivial solutions for the Φ^I are possible only if $\square_{\tilde{T}}\Phi^I \neq 0$. Following [22], the latter is naturally achieved by introducing some

⁶It is sometimes questioned whether the IKKT mode admits any non-trivial backgrounds or saddle points with reduced dimension. These doubts are resolved by the example of flat D3 BPS branes realized by the Moyal-Weyl quantum plane $\mathbb{R}_\theta^{3,1}$, leading to noncommutative $\mathcal{N} = 4$ SYM [1].

time-dependence to the internal generators Φ_I , corresponding to a non-trivial R charge. This is achieved by an ansatz of the form

$$\begin{aligned} T_I^+ &:= \chi(Y^0) e^{i\omega(Y^0)} (\mathcal{K}_{2I} + i\mathcal{K}_{2I+1}), \\ T_I^- &:= \chi(Y^0) e^{-i\omega(Y^0)} (\mathcal{K}_{2I} - i\mathcal{K}_{2I+1}), \quad I = 2, 3, 4, \end{aligned} \quad (5.6)$$

where the \mathcal{K}_J are fixed matrices. This respects the full $E(3)$ symmetry of the cosmology, and it allows to stabilize a large hierarchy between the cosmic IR scale and a UV scale set by the Kaluza-Klein modes arising on the compact space \mathcal{K} generated by the Φ^j , as demonstrated for $k = -1$ in [22]. However, this will not affect the equations of motion for the background \tilde{T}^μ .

To find physically reasonable solutions for \tilde{T}^j , it thus appears that quantum corrections are required. Indeed, quantum corrections arising e.g. from the one-loop effective action will certainly modify the equations of motion, and a partial analysis of such effects was carried out for the $k = -1$ case in [5, 22]. Using covariance arguments, one may reason along the lines of [22] that the one-loop quantum corrections modify the classical equations of (5.5), so that the right-hand side is not vanishing but dependent on the dilaton and the scale $m_{\mathcal{K}}^2$ of the extra dimension, yielding in particular

$$-\frac{\alpha}{R^2 \beta} (Y^0 \alpha(Y^0 \beta)')' \sim F(\tilde{\rho}^2) m_{\mathcal{K}}^2 + \text{higher-loop corrections}, \quad (5.7)$$

where $F(\tilde{\rho})^2$ is a function of the dilaton. However, a detailed study of this issue is beyond the scope of this paper.

Let us also mention that the one-loop effective action on covariant spacetimes arising from the IKKT model was computed in [33] and worked out in some detail for cosmological $k = -1$ quantum spacetimes in [5] under the assumption that the full ten-dimensional background has a product structure $\mathcal{M}^{1,3} \times \mathcal{K}$. For related works, see [3, 4, 24, 34].

6 Discussion

We have described a large class of spatially flat cosmological quantum spacetimes arising from the doubleton representations of $\mathfrak{so}(4, 2)$. The obtained spacetimes can be expanding with a Big Bang geometry or shrinking with a Big Crunch geometry. We have elaborated on the algebraic and semi-classical structure of these spacetimes and the associated matrix configurations, including the appearing higher-spin degrees of freedom, the effective metric and geometry, the definition of appropriate derivation and the form of local normal coordinates which make the higher-spin components of the effective metric vanish along a timelike curve.

A particular focus was on the relation between gauge transformations of the matrix model and approximate diffeomorphisms on the spacetime. We elaborate two distinguished gauges, the timelike gauge and the covariant gauge. Among other results, we have shown that particular backgrounds – the ones solving the classical equations of motion and the ones yielding a

constant dilaton – yield equivalent geometries in these two gauges. Determining the finite gauge transformations relating arbitrary gauges is however beyond the scope of this work.

An interesting result is the existence of a background $\beta = c + \frac{d}{y^0}$ in the covariant gauge, where the higher-spin contributions to the effective metric can be eliminated globally by going to local normal coordinates. This background, which corresponds to an expanding FLRW spacetime with scale factor $a(t) \sim t^{\frac{1}{3}}$, is a sum of the reference background investigated in [12] and the background $\beta_0 \sim \frac{1}{y^0}$, which on its own yields a degenerate frame and (inverse) metric.

An important deviation from the case of cosmological $k = -1$ spacetimes is the observation that the background corresponding to a constant dilaton yields a shrinking spacetime, rather than an expanding spacetime. One might argue that the case of a constant dilaton is, in some sense, the "most physical". However, such a conclusion is premature without a more complete understanding of the resulting physics. Indeed, we also obtain a large class of expanding spacetimes with a Big Bang provided that the dilaton has some mild growing behavior during the cosmic evolution. This should ensure a weak coupling (at late times) for the low-energy gauge theory governing the internal fluctuation modes corresponding to \mathcal{K} .

Finally in Section 5, we have outlined how the cosmological $k = 0$ quantum spacetimes under consideration can be embedded into the full ten-dimensional IKKT model. The details of such an embedding are beyond the scope of the present paper. Although we expect that this works similar to the better-known case of $k = -1$ spacetimes [3–5, 22], there are some important differences. In particular, the fact that the $k = 0$ case is "more commutative" means that time-dependent extra-dimensions will not affect the classical equations of motion of spacetime, so that quantum effects are required to stabilize spacetime.

A Identities from $k = -1$ cosmological quantum spacetimes

It was shown previously that a reference metric of a cosmological spacetime with $k = -1$ and a Big Bounce is induced by the static choice

$$T_{k=-1}^\mu := \frac{1}{R} M^{\mu 4}, \quad (\text{A.1})$$

which can be generalized and made dynamical by replacing $T_{k=-1}^\mu$ by $\tilde{T}_{k=-1}^\mu := f(X^4) T_{k=-1}^\mu$. Here, f is a (well-behaved) function of X^4 , which can be expressed in terms of the cosmological time variable only [22, 27, 28, 34]. The X^μ and $T_{k=-1}^\mu$ satisfy the commutation relations

$$\begin{aligned} [X^\mu, X^\nu] &= -ir^2 M^{\mu\nu}; \\ [T_{k=-1}^\mu, T_{k=-1}^\nu] &= \frac{i}{R^2} M^{\mu\nu}; \\ [T_{k=-1}^\mu, X^\nu] &= \frac{iX^4}{R} \eta^{\mu\nu}. \end{aligned} \quad (\text{A.2})$$

The generators M^{ab} and the matrices X^a satisfy a number of $SO(4, 1)$ -invariant constraints in the doubleton representations. Among these, we will make use of the identity (2.3) and [34]

$$\varepsilon_{abcde} M^{ab} M^{cd} = \frac{4n}{r} X_e, \quad a, b, c, d, e = 0, \dots, 4. \quad (\text{A.3})$$

For large $n \rightarrow \infty$, we have in the semi-classical limit $R \sim \frac{nr}{2}$, $x^\mu \sim X^\mu$, $t_{k=-1}^\mu \sim T_{k=-1}^\mu$, $m^{ab} \sim M^{ab}$ and $\{\cdot, \cdot\} \sim -i[\cdot, \cdot]$. Moreover, the Poisson-brackets with $x^4 \sim X^4$ relate x^μ and $t_{k=-1}^\mu$,

$$\{x^4, x^\mu\} = r^2 R t_{k=-1}^\mu \quad \text{and} \quad \{x^4, t_{k=-1}^\mu\} = \frac{x^\mu}{R}. \quad (\text{A.4})$$

A standard parametrization of the x^a is

$$x^a = R \begin{pmatrix} \cosh(\eta) \cosh(\chi) \\ \cosh(\eta) \sinh(\chi) \Omega \\ \sinh(\eta) \end{pmatrix}, \quad (\text{A.5})$$

where $\Omega \in S^2 \hookrightarrow \mathbb{R}^3$, and where $\eta \in \mathbb{R}$ parametrizes time and $\chi \in \mathbb{R}$ is a spatial variable.

B Semi-classical generators of rotations and boosts

We determine the semi-classical form of the generators of rotations, $M^{ij} \sim m^{ij}$ and of the boosts, $M^{i0} \sim m^{i0}$ as functions of the vectors \mathbf{y} and \mathbf{t} and the $SO(3)$ -invariant variables y^0, t^0 and $y := \sqrt{\delta_{ij} y^i y^j}$, assuming that the parameter n parametrizing the representation is either large, $n \gg 0$, or vanishes $n = 0$. Starting point are the concrete forms $m_{k=-1}^{\mu\nu}$ of the semi-classical generators of $SO(3, 1)$ in the $k = -1$ case, which is in the large- n -limit given by [34, Eq. (5.4.13)]

$$m^{\mu\nu} = -\frac{R}{R^2 + x^4} \left(x^4 (x^\mu t_{k=-1}^\nu - x^\nu t_{k=-1}^\mu) + R \varepsilon^{\mu\nu\alpha\beta} x_\alpha t_{k=-1,\beta} \right), \quad n \gg 0, \quad (\text{B.1})$$

where $x^\mu = y^\mu - \delta^{\mu 0} x^4$, $t_{k=-1}^\mu = t^\mu - \frac{1}{R} m^{\mu 0}$ is the standard background for a cosmological $k = -1$ quantum spacetime, related to y^μ via (3.23).

In the case $n = 0$, we have the simpler form [23]

$$m^{\mu\nu} = \frac{R}{x^4} (t_{k=-1}^\mu x^\nu - t_{k=-1}^\nu x^\mu). \quad (\text{B.2})$$

The anti-symmetric tensors in three dimensions are spanned by the Hodge duals of the vectors \mathbf{y}, \mathbf{t} and $\mathbf{t} \times \mathbf{y}$. That is, they are spanned by

$$\begin{aligned} A^{ij} &:= t^i y^j - t^j y^i, & B^{ij} &:= \varepsilon^{ijk} y_k, & \text{and} & C^{ij} := \varepsilon^{ijk} t_k, \\ \text{or } A &= \mathbf{t} \wedge \mathbf{y} = *(\mathbf{t} \times \mathbf{y}), & B &= *\mathbf{y}, & \text{and} & C = *\mathbf{t}. \end{aligned} \quad (\text{B.3})$$

For convenience, note that the tensors

$$D^{ij} := A_{kl}(\varepsilon^{ikl}y^j - \varepsilon^{jkl}y^i) \quad \text{and} \quad F^{ij} := A_{kl}(\varepsilon^{ikl}t^j - \varepsilon^{jkl}t^i) \quad (\text{B.4})$$

satisfy

$$\begin{aligned} \frac{1}{2}D^{ij} &= y^0 t^0 B^{ij} - y^2 C^{ij}; \\ \frac{1}{2}F^{ij} &= \frac{y^{02}}{r^2 R^2} B^{ij} - y^0 t^0 C^{ij}. \end{aligned} \quad (\text{B.5})$$

Remark B.1. In the case of a $k = -1$ spacetime, where we look for antisymmetric rank two tensors in four spacetime dimensions, invariant under $SO(3, 1)$, we only have two independent candidates $A_4 = t \wedge y$ and $B_4 = * (t \wedge y)$, while in the case of $k = 0$, three spatial dimensions and $SO(3)$, we have the three candidates (B.3).

Minimal case $n = 0$. Starting from the form (B.2), the formula (3.19) was proven in [12] for the minimal case $n = 0$.

Large n case. Substituting y^μ and t^μ for x^μ and $t_{k=-1}^\mu$ into (B.1) yields⁷

$$\begin{aligned} m^{ij} &= \frac{Rx^4}{R^2 + x^{42}} A^{ij} - \frac{R^2 t^0}{R^2 + x^{42}} B^{ij} + \frac{R^2 x^0}{R^2 + x^{42}} C^{ij} \\ &\quad + \frac{x^4}{R^2 + x^{42}} (y^i m^{j0} - y^j m^{i0}) - \frac{Rx^0}{R^2 + x^{42}} \varepsilon^{ij}_k m^{k0}. \end{aligned} \quad (\text{B.6})$$

A similar substitution yields

$$\begin{aligned} m^{i0} &= \frac{Rx^4}{R^2 + x^{42}} (x^0 t^i - \frac{1}{R} x^0 m^{i0} - t^0 y^i) + \frac{R^2}{R^2 + x^{42}} \varepsilon^{ijk} y_j t_k - \frac{R}{R^2 + x^{42}} \varepsilon^{ij}_k y_j m^{k0} \\ &= \frac{1}{R^2 + x^4 y^0} \left(Rx^4 (x^0 t^i - t^0 y^i) + R^2 \varepsilon^{ijk} y_j t_k - R \delta^{il} \varepsilon_{ljk} y^j m^{k0} \right). \end{aligned} \quad (\text{B.7})$$

Now, the vector $V^i := m^{i0}$ must be a linear combination

$$m^{i0} = a y^i + b t^i + c \varepsilon^{ijk} y_j t_k, \quad (\text{B.8})$$

where a, b, c are functions of y^0, t^0, y . Inserting this form into the first form of m^{i0} and equating the two yields

$$\begin{aligned} &-R t^0 (x^4 + c y^0) y^i + R (c y^2 + x^4 x^0) t^i + R (R - b) \varepsilon^{ijk} y_j t_k \\ &= (R^2 + x^4 y^0) (a y^i + b t^i + c \varepsilon^{ijk} y_j t_k). \end{aligned} \quad (\text{B.9})$$

⁷Note that one needs to be careful raising and lowering indices: The indices of y^μ and x^μ are lowered with a different Minkowski metric. In the following, all indices are raised and lowered with the Minkowski metric corresponding to x^μ . This does not matter for the spatial components, but we make sure that x^0, t^0 and y^0 do only appear with upper indices in this derivation.

The corresponding system of equations is uniquely solved by

$$a = -\frac{Rt^0}{y^0}, \quad b = R \frac{y^{0^2} + y^2 - R^2}{2y^{0^2}}, \quad \text{and} \quad c = \frac{R^2}{y^{0^2}}, \quad (\text{B.10})$$

which implies (3.19).

Using (3.19), one then easily verifies $\{m^{ij}, y^0\} = \{m^{ij}, t^0\} = 0$ and $\{m^{ij}, t^k\} = \delta^{ik}t^j - \delta^{jk}t^i$. Finally, we have

$$\begin{aligned} \{m^{ij}, y^k\} &= \delta^{ik}y^j - \delta^{jk}y^i + \frac{r^2R^3}{y^{0^2}} \left(\varepsilon^{ikl}t^j - \varepsilon^{jkl}t^i - \varepsilon^{ijl}t^k \right) t_l + R\varepsilon^{ijk} \\ &= \delta^{ik}y^j - \delta^{jk}y^i, \end{aligned} \quad (\text{B.11})$$

which implies $\{m^{ij}, y\} = 0$.

Acknowledgments. This work is supported by the Austrian Science Fund (FWF) grant P36479. HS would like to thank Pei-Ming Ho and Hikaru Kawai for a related collaboration. We also thank Alessandro Manta for valuable discussions.

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