The Overlapped Hyperrectangle Problem

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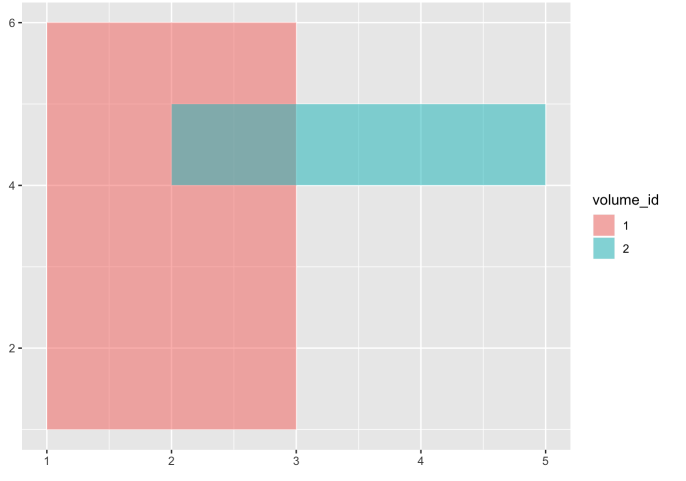
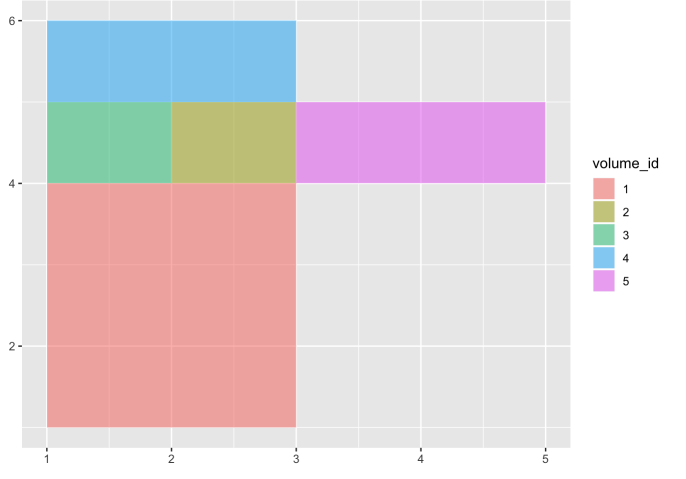
The Problem

First, a concise definition: Given a set of hyperrectangles, find any set of hyperrectangles which occupy the same space with no overlap & contain at least all the original boundaries. A hyperrectangle is the space defined by the cartesian product of ranges. With less jargon, axis-aligned rectangles/prisms/volumes in arbitrary dimensions.

Take an example - 2 dimension rectangles with some overlap:



Here's a solution:



This is a valid solution because:

- the orginal space is exactly covered
- all the covering volumes are rectangles
- there is no overlap between rectangles
- none of the original boundaries are crossed in the solution.

Motivation

Before describing a solution to this (possibly vacuous seeming) problem, I'd like to motivate it with an example.

The RuleFit algorithm (my implementation here) is a predictive modeling method. At a very high level, its purpose is to produce a set of conjunctive ranges on the predictor set (imagine them as a set of real valued vectors) which explain variation in the response vector. Take for example a predictor set of {age, weight, height} with a response of jumping height (i.e. we're predicting how high someone can jump). One might imagine that lower age, lower weight, and larger height produce larger jumping height, and vice versa; RuleFit may pick up on such a pattern & produce, for example, the following set of "rules":

- {height:[130-200] & weight:[50-60] & age:[10-25]}
- {height:[175-225] & weight:[55-75] & age:[20-35]}
- {height:[100-250] & weight:[70-100] & age:[45-55]}
- etc.

with the idea that a person qualifying for one or more of these rules provides useful information for predicting jumping height. In reality, each rule is assigned a real value ("effect") which are summed up to produce the jumping height prediction.

Notice that many of these ranges are overlapped, which means a person may qualify for many rules. That makes interpretation & inference somewhat challenging, since we cannot longer can glance at a single rule as a single "nugget" of information. The practitioner is forced to contextualize rules & compute the high-dimensional intersections, which a human mind (or at least mine) isn't good at. Wouldn't it be great if these rules were disjoint, so that a person can only qualify for one rule? Solving the overlapped hyperrectangle problem does just that.

(Note: I've made some over-simplifications of RuleFit and encourage you to check out the above link if you're interested.)

Solution

A conceptually simple solution is to:

- Extend all boundaries to infinity (treating the boundary hyperplanes as fixed and allowing all other dimensions to be free)
- Re-create hyperrectangles according to the new boundaries
- Prune hyperrectangles outside the original covered space

To prove it's a solution:

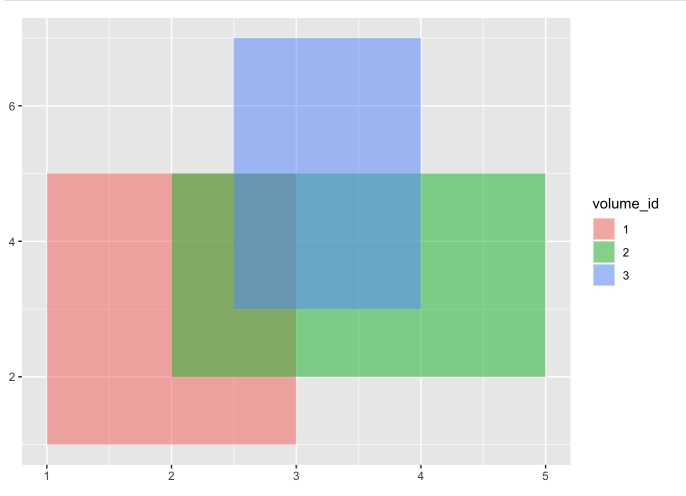
- It creates hyperrectangles, as all original boundaries are orthogonal by problem definition.
- Clearly all the original boundaries are preserved because they are simply scaled in size.
- It is capable of capturing a superset of the original space because it is partitioning the entire n-dimensional space.
  - The non-covered hyperrectangles are able to be cleanly removed, as in without eating into the covered space, since, as argued above, all the original boundaries are preserved.

Without ado, we step into some code.

Representation

Hyperrectangles will be represented in dimension-range format. In this representation, an n-dimensional hyperrectangle consists of n dimension-ranges. For this exercise, hyperrectangles are given an id and all placed into one data frame. Here we define and visualize a 2 dimensional example:

```
example_2d <- data.frame(
    dimension = rep(c('x','y'), 3),
  volume_id = rep(1:3, each=2),
  min = c(1,1,
          2,2,
          2.5,3),
  max = c(3,5,
          5,5,
          4,7)
)
```

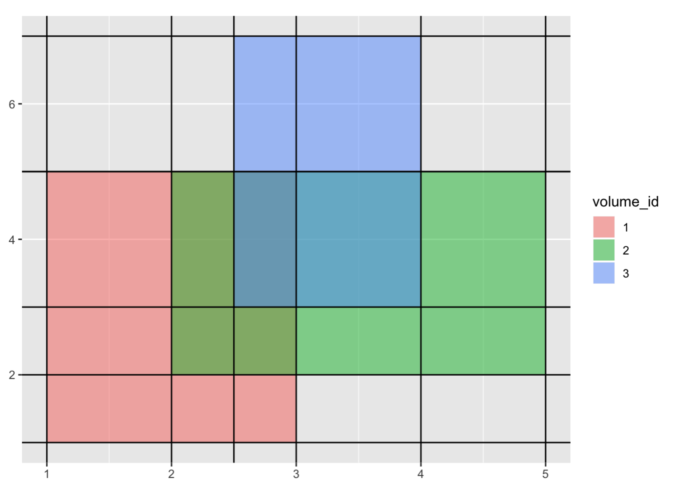


Extending Boundaries

This is as simple as picking "dimension=value" as a fixed point (while letting all other dimensions be free) for all range bounds.

```
build_fully_partitioned_space <- function(volumes) {
  volumes %>%
    mutate(bound = min) %>%
    select(dimension, bound) %>%
    rbind(
      volumes %>%
        mutate(bound = max) %>%
        select(dimension, bound),
      stringsAsFactors = FALSE
    )
}
```

Visualizing our example:

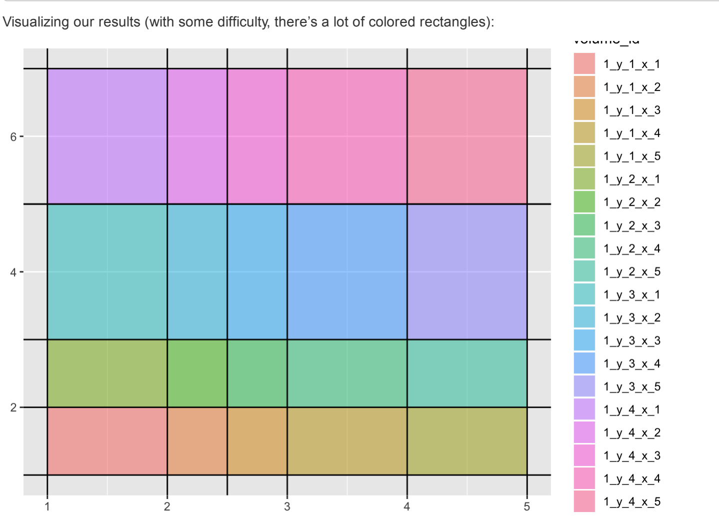


Recreating Hyperrectangles

The next step is to take the orthogonal hyperplanes from the above step and form then into hyperrectangles abutted against one another.

```
generate_volumes_from_partitioned_space <- function(partitioned_space, id_starter = 1) {
  if (nrow(partitioned_space) == 0) {
    return(data.frame())
  }

  # pick an arbtirary first dimension
  dimension_of_interest <- partitioned_space$dimension[1]
  dimension_bounds <- partitioned_space %>%
    filter(dimension == dimension_of_interest) %>%
    # this is a small optimization - equal bounds are redundant
    distinct() %>%
    arrange(bound)

  # there should always be 2 or more, since each bound corresponds to hyperrectangle edge
  stopifnot(nrow(dimension_bounds) > 1)

  # subspace meaning everything outside the dimension of interest
  partitioned_subspace <- partitioned_space %>% filter(dimension != dimension_of_interest)
  # recursively build ranges from the subspace before tacking on ranges for the dimension of interest in this s
tack frame
  subspace_volumes <- generate_volumes_from_partitioned_space(partitioned_subspace, id_starter = id_starter)

  # "expanded" by the dimension of interest, that is
  expanded_volumes <- data.frame()
  for (bound_ix in 1:(nrow(dimension_bounds) - 1)) {
    # note that we are iterating on the sorted bounds
    lower_bound <- dimension_bounds$bound[bound_ix]
    upper_bound <- dimension_bounds$bound[bound_ix + 1]

    if (nrow(subspace_volumes) == 0) {
      # case this is the first dimension - there's nothing to add onto
      new_volume_id <- paste0(id_starter, '_', dimension_of_interest, '_', bound_ix)
      new_dimension_bounds <- list(new_volume_id = new_volume_id,
                                   min = lower_bound,
                                   max = upper_bound,
                                   dimension = dimension_of_interest)
    }
    else {
      # case this is after the first dimension - create a new volume for each subspace volume with the new boun
ds added (cartesian product)
      new_dimension_bounds <- lapply(unique(subspace_volumes$new_volume_id), function(new_volume_id) {
        list(new_volume_id = paste0(new_volume_id, '_', dimension_of_interest, '_', bound_ix), # TODO this form
 of creating an ID could get costly in higher dimensions
            min = lower_bound,
            max = upper_bound,
            dimension = dimension_of_interest)
      }) %>% bind_rows() %>%
        rbind(subspace_volumes %>%
                mutate(new_volume_id = paste0(new_volume_id, '_', dimension_of_interest, '_', bound_ix)),
              stringsAsFactors= FALSE)
    }

    expanded_volumes <- rbind(expanded_volumes, new_dimension_bounds,
                              stringsAsFactors = FALSE)
  }

  return(expanded_volumes)
}
```

Visualizing our results (with some difficulty, there's a lot of colored rectangles):



Pruning

You'll notice that the above is not actually our original space- actually it's the minimal bounding hyperrectangle. Here we prune away any new hyperrectangles not in the original covering space:

```
prune_uncovering_volumes <- function(new_volumes, original_volumes) {
    # we left join because not all new volumes belong to all old volumes
    # the range join prescribes that the original volumes contains the new volume
    original_to_new_volumes <- fuzzy_left_join(original_volumes, new_volumes,
                                               by = c('min' = 'min',
                                                      'max' = 'max',
                                                      'dimension' = 'dimension'),
                                               match_fun = c(`<=`, `>=`, `==`)) %>%
      # renaming some things in a reasonable way
      mutate(dimension = dimension.x) %>%
      select(-dimension.x, -dimension.y)

  covering_volumes <- data.frame()
  for (new_volume_id_to_check in unique(new_volumes$new_volume_id)) {
    volume <- new_volumes %>%
      filter(new_volume_id == new_volume_id_to_check)

    in_covering_space <- FALSE
    for (original_volume_id_to_check in unique(original_volumes$volume_id)) {
      original_volume_to_check <- original_to_new_volumes %>%
        filter(volume_id == original_volume_id_to_check)
      # here we make sure all dimensions are contained
      volume_dimensions_contained <- original_to_new_volumes %>%
        filter(volume_id == original_volume_id_to_check &
               new_volume_id == new_volume_id_to_check) %>%
        pull(dimension) %>%
        setequal(original_volume_to_check$dimension)

      if (volume_dimensions_contained) {
        in_covering_space <- TRUE
        break
      }
    }

    if (in_covering_space) {
      covering_volumes <- rbind(covering_volumes,
                                volume,
                                stringsAsFactors = FALSE)
    }
  }

  covering_volumes
}
```

And visualizing:

```
final_unoverlapped_hyperrectangles <- prune_uncovering_volumes(new_volumes, example_2d)

plot_rectangles(final_unoverlapped_hyperrectangles %>% mutate(volume_id = new_volume_id))
```

Putting It All Together

Here's out complete algorithm:

```
unoverlap_hyperrectangles <- function(volumes) {
  partitioned_space <- build_fully_partitioned_space(volumes)
  new_volumes <- generate_volumes_from_partitioned_space(partitioned_space)
  prune_uncovering_volumes(new_volumes, volumes)
}
```

A couple nice features to note:

- Although our example was 2 dimensional, everything extends nicely to many dimensions
- In the case of unbounded hyperrectangles (i.e. one or more bounds are infinite), this algorithm still works

And less nice features:

- The solution is frequently suboptimal. In the above example, many rectangles are superfluous, suggesting that there could be a "fusing" step after pruning.

Runtime

Take `V` as the number of hyperrectangles and `D` as their dimensionality. Looking at each step of the algorithm:

- Partitioning the space: `O(VD)` since every range is run through
- Generating volumes: Each level of the recursion, considering a d dimension space, does `dV^(2^(d-1) + 1)` work. The `V^(2^d)` term represents the recursively expanded boundaries, where `O(V)` bounds the number of bounds produced along any dimension, and the `2^d` term represents the cartesian product of the current dimension against all prior subspaces. Solving this recurrence across `1` to `D` recursions is beyond my current time limitations, so I'll give a non-tight bound of `O((DV^(2^(D-1) + 1))^D)`
- Pruning: At this point, `O(DV^(2^(d-1) + 1))` hyperrectangles exist. All original hyperrectangles, `V` are crossed with the new hyperrectangles across all dimensions, `D`, giving `O(D^2 * V^(2^(d-1) + 1))` performance.

As these steps are additive, the total runtime is dominated by the runtime of generating all volumes.

A Cheeky Extension

Ok, so this approach is pretty straight forward. Why did I spend the time writing about it? Besides to the fun application to RuleFit, it solves what is, at face value, a seemingly difficult problem: Given a set of potentially overlapped hyperrectangles, compute the hyper-volume they occupy.

If approached with the problem, you might stumble around with recursive inclusion/exclusion applications (as I did when I failed a simpler version of this problem interviewing at google). But, with overlaps between rectangles removed, the solution is exceedingly simple! Just compute the hyper-volume of all hyperrectangles and sum up.

```
compute_hypervolume <- function(volumes) {
  unoverlap_hyperrectangles(volumes) %>%
    group_by(new_volume_id) %>%
    summarize(
      hypervolume = Reduce(`*`, max - min)
    ) %>%
    summarize(
      total_hypervolume = sum(hypervolume)
    ) %>%
    pull(total_hypervolume)
}

compute_hypervolume(example_2d)
```

```
## [1] 17
```