Expected Values Estimated via Mean-Field Approximation are 1/N-Accurate

Extended Abstract*

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ABSTRACT

In this paper, we study the accuracy of mean-field approximation. We show that, under general conditions, the expectation of any performance functional converges at rate $O(1/N)$ to its mean-field approximation. Our result applies for finite and infinite-dimensional mean-field models. We provide numerical experiments that demonstrate that this rate of convergence is tight.

CCS CONCEPTS

• **General and reference → Performance**; • **Mathematics of computing → Stochastic processes**; *Queueing theory*; Ordinary differential equations;

KEYWORDS

Mean-field approximation, queueing theory, accuracy of approximation, supermarket model, power-of-two-choice

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1 INTRODUCTION

Mean-field approximation is a powerful tool for studying systems composed of a large number of interacting objects. The idea of mean-field approximation is to replace a complex stochastic system by a simpler deterministic dynamical system. This approximation is widely used to study the performance of computer-based systems. This approximation is known to be asymptotically exact for many systems, in which the fraction of objects in a given state i, $X_i^{(N)}$, converges at rate $O(1/\sqrt{N})$ to a deterministic quantity x_i, as the number of objects N goes to infinity [2, 5].

In this paper, we show that, when one wants to estimate expected values, the rate of convergence is much faster. More precisely, we show that for a smooth function h, the expectation $\mathbb{E}[h(X^{(N)})]$ converges at rate $1/N$ to its mean-field approximation $h(x)$:

$$\left|\mathbb{E}\left[h(X^{(N)})\right] - h(x)\right| = O\left(\frac{1}{N}\right). \tag{1}$$

We show that this essentially holds for the transient regime as soon as the drift of the system is twice-differentiable. It holds for the stationary regime if in addition the differential equation has a unique stable point that is exponentially stable. We also exhibit an example that shows that when the drift of the system is only Lipschitz-continuous, the convergence can be slower.

As an example, we study in detail the convergence rate of the classical power-of-two-choice model (*a.k.a.* supermarket model) of [3, 4]. These papers show that, as the number of servers N go to infinity, the average queue length goes to $m^\infty(\rho) = \Theta(\log 1/(1-\rho))$.

Our results show that an average value estimated via mean-field approximation is $1/N$-accurate. In a queuing network such as the two-choice model, the average queue length can be expressed as $\mathbb{E}[h(X^{(N)})]$. Equation (1) shows that the average queue length converges at rate $O(1/N)$ to its mean-field approximation. We provide numerical evidence that for this model, for any finite value of N, the average queue length is $m^N(\rho) \approx m^\infty(\rho) + \frac{\rho^2}{2N(1-\rho)}$. We illustrate this in Table 1 where we report the average queue length for $\rho = 0.9$. We observe that $m^N(0.9) \approx m^\infty(0.9) + 4/N$.

Number of servers (N)	10	100	1000	$+\infty$
Average queue length (m^N)	2.81	2.39	2.36	2.35
Error ($m^N - m^\infty$)	0.45	0.039	0.004	0

Table 1: Average queue length for the two-choice model.

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*Full version of the paper : https://github.com/ngast/meanFieldAccuracy [1].

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