Lecture Advanced Investments, September 8th, 2025

Portfolio theory (cont.)
CAPM, theory and testing

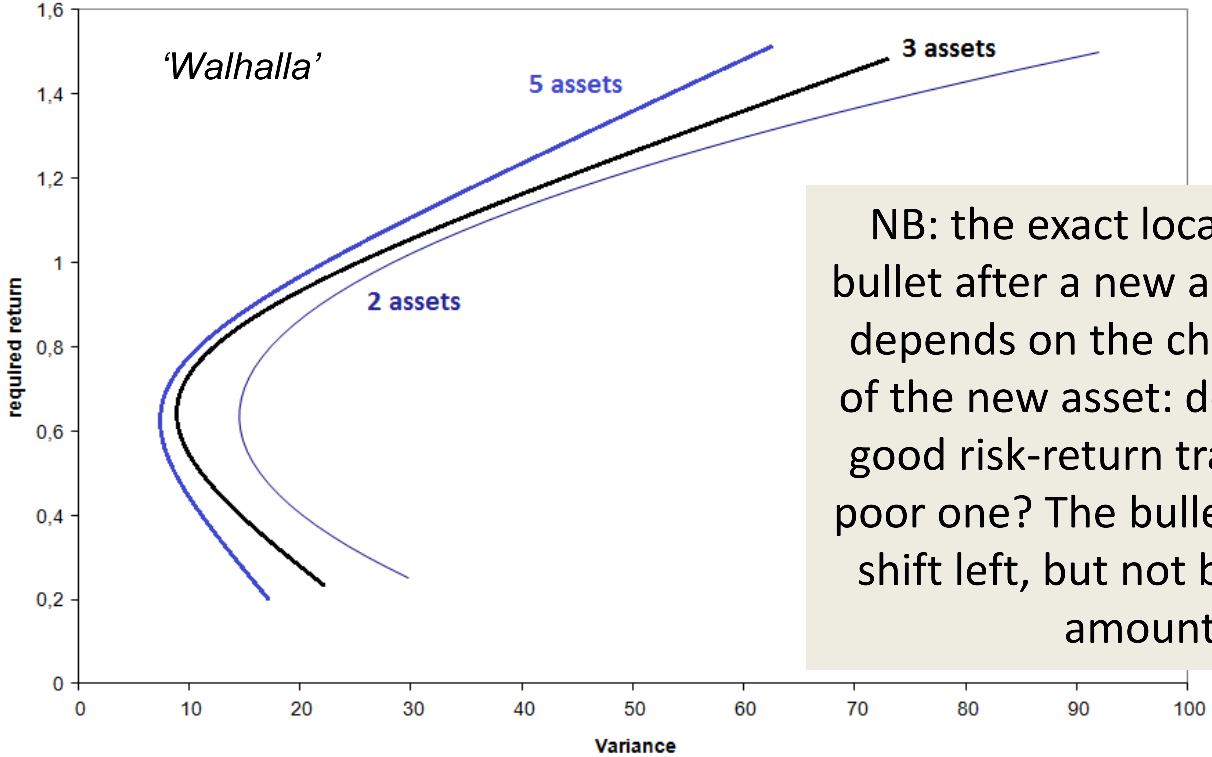
Dr. P.J.P.M. (Philippe) Versijp

Program

- Portfolio theory
  - 4 slides from last week
  - ETFs: additional insight based on last week
  - Restrictions (skipping that during the lecture, see clip 1)
  - The risk free object (clip 2, will do a quick recap and Q&A on portfolio theory)
- The CAPM
  - Refresher
  - CAPM and SDF
  - Fama & MacBeth 1973: testable implications
  - Why do we want to test the CAPM
- Formula of the week

Portfolio theory : the bullet (1b)

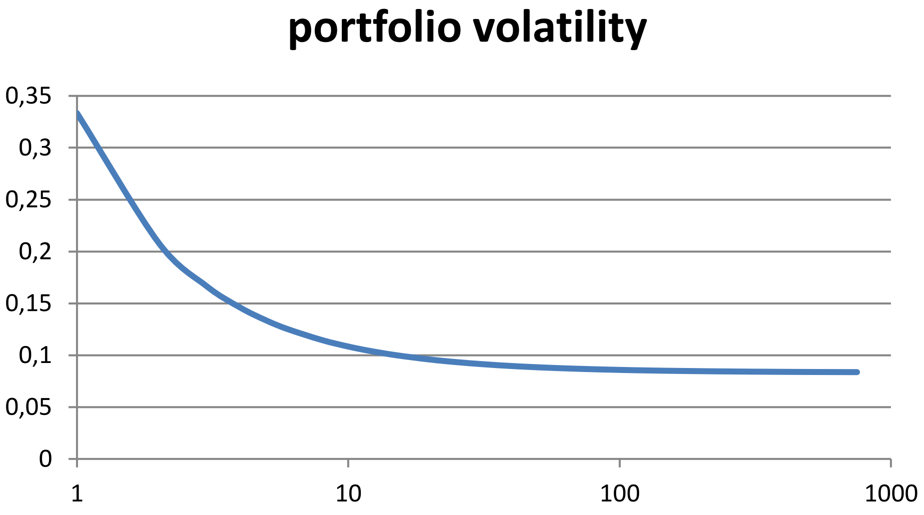
- And this of course also applies to more than 2 assets….



NB: the exact location of the bullet after a new asset is added depends on the characteristics of the new asset: does it offer a good risk-return trade-off, or a poor one? The bullet will always shift left, but not by the same amount.

Portfolio theory : the bullet (3)

- And: 'a little diversification goes a long way': the first few assets shift the bullet leftward much more than later ones. (or, within mutual funds, move the mean-variance characteristics closer to the bullet)



Portfolio volatility as a function of the number of stocks in portfolio.

NB: exact shape depends on stock characteristics.

What can we learn from this?

Portfolio theory : the bullet (4)

- Main influences on the bullets:
  - Mean returns (see lecture 4)
  - Correlations (see lecture 6)

    NB: these will be discussed in a more general context
  - Maintained assumptions, especially A5-A7. Transaction costs, information costs and indivisibility make the problem non-smooth (picture), unless more and very specific assumptions are made. Optimization techniques may fail, some participants may have arbitrage opportunities (in which case you don’t care much about bullets)

Portfolio theory : the bullet (5)

- From bullet to optimal portfolio:
  - This normative part requires you to know the preference structure, either in the form of indifference curves (combinations of mean and variance with the same utility) or the utility function itself.
  - You need to take a riskfree object (if available) and restrictions (for example no shortselling) into account. This will be discussed ~~next~~ this week.

Portfolio theory : the risk free object (1)

- The problem becomes somewhat different if we add an object that has *no variance* (so it's risk free, under almost any preference structure).
- The possible portfolio's will generally have *a higher utility,* especially for risk averse agents.

Portfolio theory : the risk free object (2)

- The formula derived last week breaks down if a row of the covariance matrix consists of zeros; the inverse does not exist.
- Actually, the problem becomes *simpler,* instead of adding a risk free object we can drop the restriction the portfolio weights sum to one ($\omega^{*\prime}e=1$); the remainder is the proportion invested in the risk free asset. (and keep the matrix as it was)
- The formula for the portfolio return becomes:

NB: just as last time, 'e' is a vector consisting solely of 1's, in order to make the dimensions match

$$\overline{r_p} = r_f + \omega'(\bar{r} - r_f e)$$

- Variance stays the same ($\omega' V \omega$), so the objective function is now:

$$\min_{\omega} \frac{1}{2}\omega' V \omega + \lambda\left\{\overline{R_p} - [r_f + \omega'(\bar{r} - r_f e)]\right\}$$

Portfolio theory : the risk free object (3)

- We can use the same techniques as without the risk free object (set first derivatives to zero, substitute to get rid of the Lagrange multiplier, and rearrange terms to get an expression for the optimal portfolio weights.

$$\min_{\omega} \frac{1}{2}\omega' \mathrm{V} \omega + \lambda \left\{ \overline{\mathrm{R}_{\mathrm{p}}} - [\mathrm{r}_{\mathrm{f}} + \omega'(\bar{\mathrm{r}} - \mathrm{r}_{\mathrm{f}}\mathrm{e})] \right\}$$

$$\omega^{*} = \frac{\overline{R_p} - r_f}{\bar{r}' V^{-1} \bar{r} - 2\bar{r}' V^{-1} e(r_f) + e' V^{-1} e(r_f)^2} V^{-1}(\bar{r} - r_f e)$$

$$\sigma_{\mathrm{p}}^{2} = \omega^{*\prime} \mathrm{V} \omega^{*} = \frac{(\overline{\mathrm{R}_{\mathrm{p}}} - \mathrm{r}_{\mathrm{f}})^{2}}{\bar{\mathrm{r}}' \mathrm{V}^{-1} \bar{\mathrm{r}} - 2\bar{\mathrm{r}}' \mathrm{V}^{-1} \mathrm{e}(\mathrm{r}_{\mathrm{f}}) + \mathrm{e}' \mathrm{V}^{-1} \mathrm{e}(\mathrm{r}_{\mathrm{f}})^{2}}$$

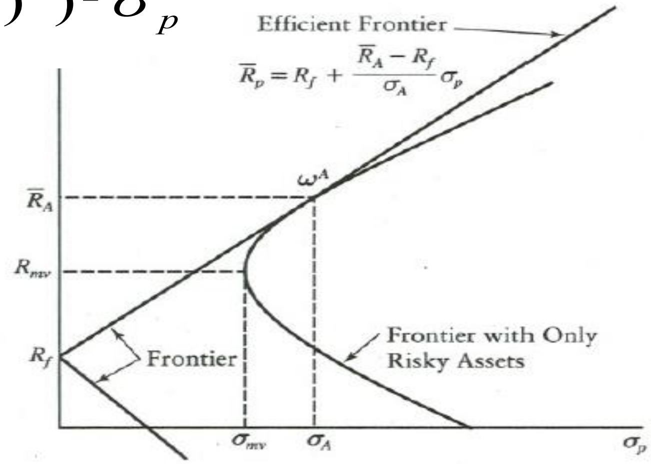
Portfolio theory : the risk free object (4)

- Next, take the square root of the portfolio variance, and we get a *linear* expression in r , σ space. So if we plot the return against the standard deviation (instead of variance), we get a straight line. (it's still a bullet in r , σ^2 space)

$$\overline{r_p} = r_f \pm (\bar{r}'V^{-1}\bar{r} - 2\bar{r}'V^{-1}e(r_f) + e'V^{-1}e(r_f)^2)^{\frac{1}{2}}\sigma_p$$

(NB: this is the Capital Market Line in the CAPM)

Also see: Pennacchi, Chapter 2, p. 44-48



Portfolio theory : the risk free object (5)

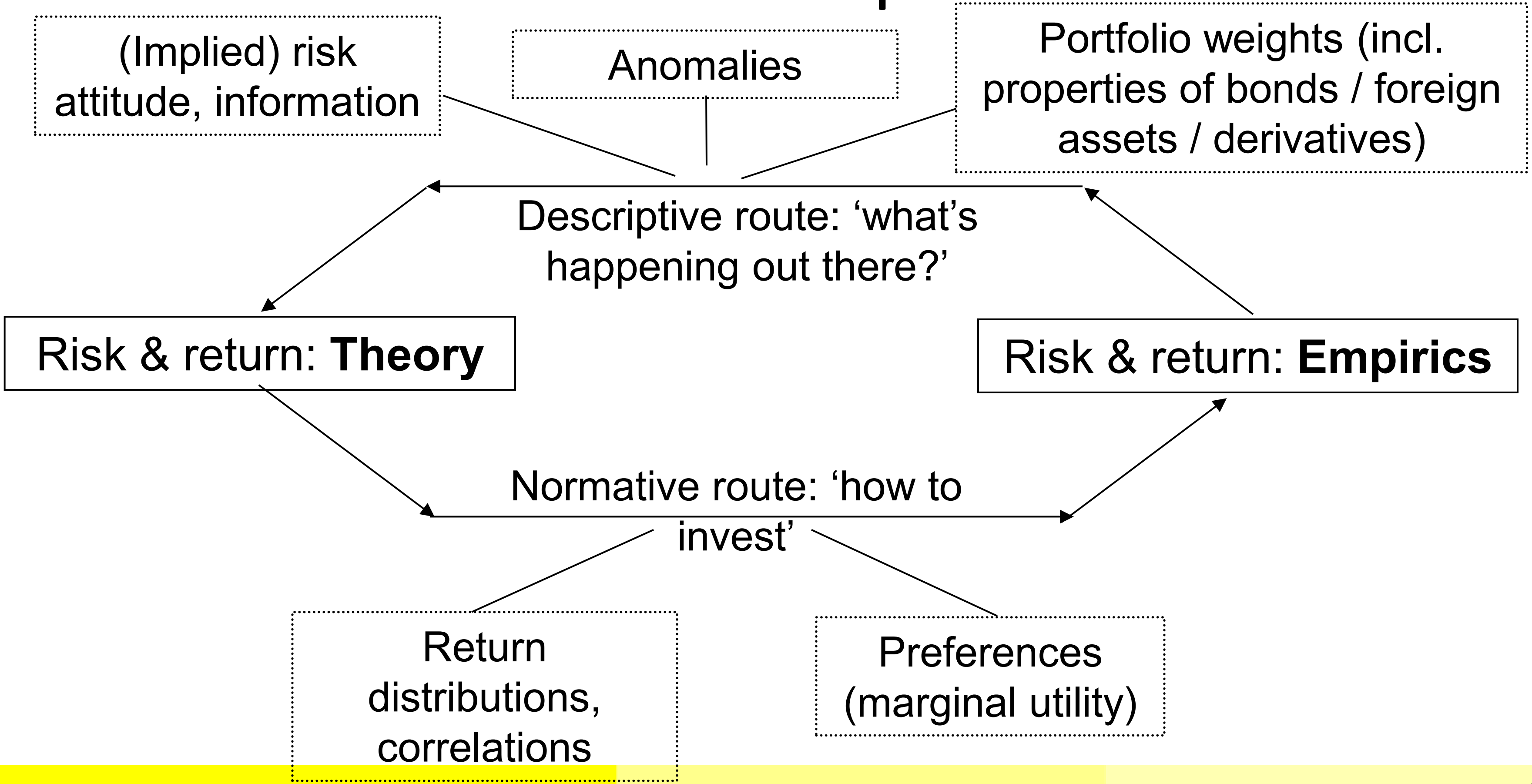
- The next crucial observation is that portfolio separation still holds, but now between 2 special assets:
  - The risk free object
  - The tangency portfolio
- Each investor can (and will) maximize his or her personal utility function by combining one portfolio of risky assets, and the risk free object.
- *If* all agents agree on the parameters (expected return, covariance matrix), and A5-A7 hold, they will all have the same tangency portfolio.

Portfolio theory : the risk free object (6)

- The next logical step would take us to an equilibrium model. *If* <u>everyone</u> needs the same portfolio for their optimum, and *if* all assets are to be held (returns get adjusted if they are so low that no-one wants that asset), that tangency portfolio must contain all assets – it must in fact be the <u>market portfolio.</u>

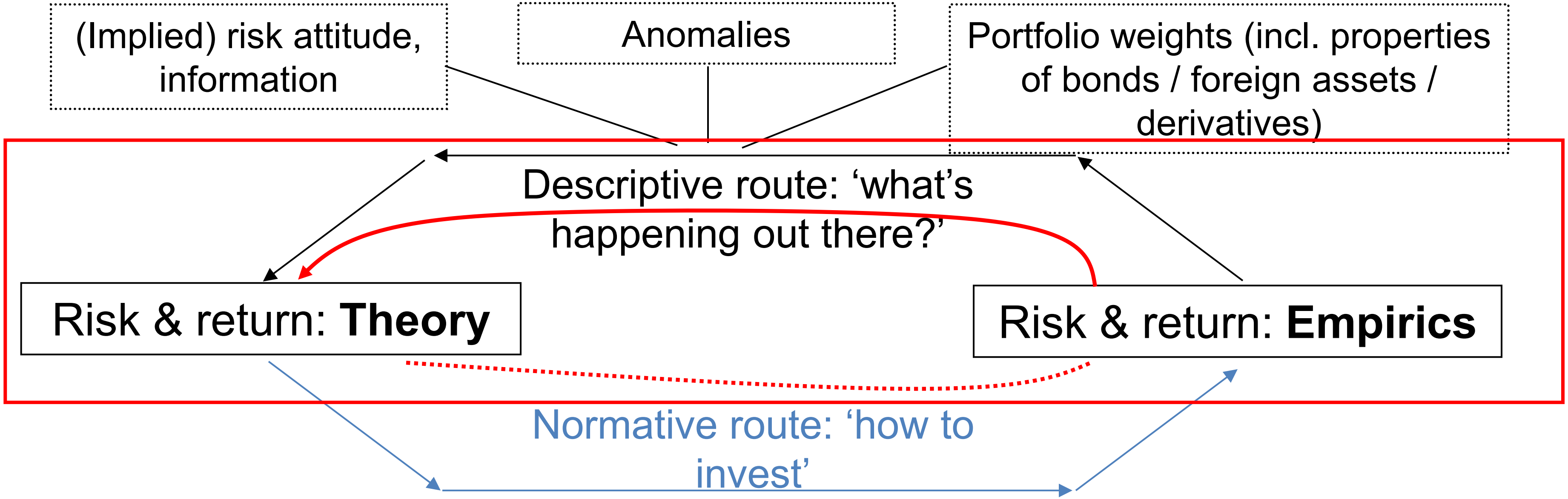
- And that brings us very close to the CAPM…

Roadmap


(Implied) risk attitude, information
Anomalies
Portfolio weights (incl. properties of bonds / foreign assets / derivatives)
Descriptive route: 'what's happening out there?'
Risk & return: **Theory**
Risk & return: **Empirics**
Normative route: 'how to invest'
Return distributions, correlations
Preferences (marginal utility)

Roadmap : today

The CAPM is strictly speaking descriptive, but it is based on such a wealth of assumptions (mostly from normative portfolio theory!) that it's descriptive qualities are questionable. Yet it forms the basis for much of what follows.



Refresher : CAPM derivation (1)

- The CAPM builds on portfolio theory, but goes (much) further: it is an equilibrium model; describing the entire asset market. It is of a more descriptive nature, as virtually no-one thinks it works well enough to warrant using it in a normative way.

- The CAPM requires more assumptions:
  - A8. All investors have the same expectation regarding returns and covariances.
  - A9. All investors can lend and borrow at the riskfree rate
  - A10. Asset markets are characterized by perfect competition: no-one is big enough to have an influence on asset prices (and hence returns), everyone is a price taker.

Refresher : CAPM derivation (2)

- A8. All investors have the same expectation regarding returns and covariances.
  - This assumption is needed (as discussed in the first hour) to arrive at the same MV-bullet for everyone. Remember portfolio separation (also known as two-fund separation): we can construct the efficient frontier using just the risk free object and the tangency portfolio. The CAPM only obtains if that tangency portfolio is the same for everyone.
- A9. All investors can lend and borrow at the riskfree rate.
  - If this assumption fails, other optima than combinations of the tangency portfolio and r_f will be chosen (picture). Even difference between lending and borrowing rates will lead to other optima.

Refresher : CAPM derivation (3)

- A10. Asset markets are characterised by perfect competition: no-one is big enough to have an influence on asset prices (and hence returns), everyone is a price taker.
  - If this assumption fails, life becomes a lot more complicated. It may be possible (and optimal) to try to corner the market; we need to take aspects of game-theory into account if there are multiple agents with pricing power, and so on.
  - Of all assumptions made in asset pricing, this is probably the one researchers like to drop least. We like it when prices (and hence returns) are *given*. If not, you get a lot of 'chicken-or-egg' effects cascading throughout the market, as everyone's decisions will influence prices and hence everyone else's decisions.

Refresher : CAPM derivation (4)

- We do NOT assume all investors have the same preferences or even similar preferences; we only need them to follow the MV-criterion (more on how to reach this with any type of preference later).

- Remember: If everyone needs the same portfolio for their optimum, and *if* all assets are to be held (returns get adjusted if they are so low that no-one wants that asset, which happens in perfect competition), that tangency portfolio must contain all assets – <u>it must in fact be the market portfolio.</u>

Refresher : CAPM derivation (5)

- Characteristics of tangency portfolio (Pennacchi 3.1.1):
  - The tangency portfolio can be obtained form these results; it's the only one in the effcient set which has zero investment in the risk free asset, so

$$e'\omega^* = 1 \Rightarrow r_p = \bar{r}'\omega^*$$

  - Pluggin this in the formula (try this at home) gives:

$$\omega^M = [\bar{r}'V^{-1}e - e'V^{-1}e(r_f)]^{-1} V^{-1}(\bar{r} - r_f e) = sV^{-1}(\bar{r} - r_f e)$$

$$r_M = \omega^M{}'\bar{r}; \qquad \sigma_M^2 = s(r_M - r_f)$$

  - And rearranging this leads to

$$(\bar{r} - r_f e) = \frac{V\omega^M}{\sigma_M^2}(r_M - r_f) = \frac{Cov(r_m, r)}{\sigma_M^2}(r_M - r_f) = \beta(r_M - r_f)$$

  - Which sound look vaguely familiar.....

Refresher : CAPM derivation (6)

$$(\bar{r} - r_f e) = \beta(r_M - r_f)$$

- The relation of the return of *any* asset with the tangency portfolio (and in equilibrium, the market portfolio) makes the importance of diversification even more clear: diversifiable risk is not priced (*you can get rid of it, so no-one is willing to pay for it*).

See also: Levy-Post 10.1.5; Pennacchi 3.1.2.

- In re-arranged form the formula above is called the '**Security Market Line**', as it *describes* the return on each asset, efficient and inefficient alike. Taking expectations, looking at a single stock (subscript i), and moving the risk-free rate to the risht hand side:

$$E(r_i) = r_f + \overbrace{(E(r_M) - r_f)}^{\text{Market risk premium}}\beta_i$$

Refresher : CAPM derivation (7)

$$E(r_i) = r_f + \beta_i (E(r_M) - r_f)$$

- A few notes regarding this formula:
  - Beta = covariance/variance. But this is actually (also) a regression coefficient! Take $E(r_i)$ is the dependent variable (y) and $(E(r_m)-r_f)$ as independent variable (x), add an error term, and you're done!
  - In turn, this implies

    $$\beta_i = [(E(r_M) - r_f)' (E(r_M) - r_f)]^{-1} (E(r_M) - r_f)' E(r_i)$$

    which we'll need in a few slides.
  - This formula applies to all assets, if they belong to the efficient set or not. Remember that the CAPM is an equilibrium model, so any asset that has a return that's too low for the risk it bears will be sold (short) up to the point the price has dropped so much that the return will have risen (pay less for the same payoff = higher return). Likewise if a return is too high.

One small doubt....

- The market portfolio is very central to the CAPM. It should contain all assets there are (including hard-to-access ones) in proportion to market capitalization.
- Now every asset is certainly not feasible (0.001% of a priceless old master painting?) but ...

We've already seen that diversification benefits decrease once you have a lot of assets. If a similar set of correlations is available, there wouldn't be too much of a problem.

- But how about market capitalization as weight? Especially with crypto's and commodities, that can shift rapidly these days. After all, (market cap = price x quantity) and prices can be very volatile. If the price doubles/halves, an investor who already owns R_m will not need to adjust anything, but an investor who just starts or wasn't mimicking R_m perfectly suddenly needs to trade – big time.

These considerations aren't a problem in a one period model, but once you add day-to-day adjustments, it can get quite messy.

CAPM as a factor model (1)

$$p_t = E(m_{t+1} x_{t+1})$$

- Remember that the Stochastic Discount Factor was a general tool for translating *probabilities*, *returns* and *the situation in which these returns occur* to prices. It's also applicable to the CAPM.
- We first rewrite the SDF formula in terms of returns (x_{t+1} is strictly speaking a payoff, so we need to divide everything by p_t), and assume a linear structure for m_{t+1}:

$$1 = E(m_{t+1} r_i) \qquad\qquad m = a + b' f$$

- This is actually equivalent to

$$E(r_i) = \gamma + \lambda` \beta_i$$

It now remains to determine a, b and f such that $\gamma = r_f$ and $\lambda = E(r_m) - r_f$.

See also: Cochrane p. 106-108

CAPM as a factor model (2)

- A quick excursion to show why this works:
  - Ingredients:

$$1 = E(m_{t+1} r_i) \qquad m = a + b' f$$

$$E(m_{t+1} r_i) = E(m_{t+1}) E(r_i) + \mathrm{cov}(m_{t+1}, r_i)$$

  - Some stirring:

$$E(r_i) E(m_{t+1}) = 1 - \mathrm{cov}(m_{t+1}, r_i)$$

$$E(r_i) = \frac{1}{E(m_{t+1})} - \frac{\mathrm{cov}(m_{t+1}, r_i)}{E(m_{t+1})}$$

  - Now use eq. 1.6 from Cochrane (p.11): since the SDF also has to price the risk free object, which has no covariances (r_f is a constant, or it isn't risk free)

$$E(r_f) = r_f = \frac{1}{E(m_{t+1})}$$

  - So $\gamma = r_f$ checks out – and not only for the CAPM, but for *any* linear factor model.

CAPM as a factor model (3)

- But that's just the appetizer:

$$E(r_i) = \frac{1}{E(m_{t+1})} - \frac{\text{cov}(m_{t+1}, r_i)}{E(m_{t+1})} = \frac{1}{a} - \frac{\text{cov}(m_{t+1}, r_i)}{a} = \frac{1}{a} - \frac{\boxed{E(r_i f`)}b}{a}$$

- Now let's take a 'wild guess' and use (R_m-r_f) as f. NB: the constant r_f doesn't matter here, the 'action' is because of R_m

$$\beta_i = [(E(r_M) - r_f)' (E(r_M) - r_f)]^{-1} \boxed{(E(r_M) - r_f)' E(r_i)}$$

- So let's stir this sauce some more:

$$E(r_i) = \frac{1}{a} - \frac{E(r_i f`)b}{a} = \frac{1}{a} - \beta` \frac{E(ff`)b}{a}; \qquad f = E(r_M)$$

Which gives us an expression for λ, namely $\frac{E(ff`)b}{a}$

CAPM as a factor model (4)

- The previous slide shows that the CAPM is consistent with a linear factor model.
- What remains to be done is to explain the choice of the market return as the factor.
  - In the CAPM, one invests in a combination of the risk free asset and the tangency portfolio.
  - Since all assets are held (prices adjust if they aren't), the tangency portfolio is the market portfolio, containing *every asset there is*.
  - Hence, if the market return changes, consumption changes. Marginal utility of consumption was the basis of the SDF:

$$\phi \frac{u`(c_{t+1})}{u`(c_t)} = m_t$$

NB: φ denotes time-preference

CAPM as a factor model (5)

- What remains to be done is to explain the choice of the market return as a factor.
  - Actually, $m_{t+1} = a + bR_M$ can be obtained in various ways; all of them ways to justify that people care about the mean return and variance only. Lecture 5 will get back to this in more detail.
  - Also note that (as explained in Penacchi 4.2) there is a perfect negative correlation between marginal utility and the return on the market portfolio.
    - Perfect, since there is nothing you don't invest in.
    - Negative, since the extra utility from each further unit of return (=consumption) decreases. This is a consequence of risk aversion: marginal utility is decreasing. This means that in $m_{t+1} = a + bR_M$, we should have $b<0$.

Fama & MacBeth, 1973 (1)

- A seminal paper regarding empirical tests of the CAPM, or as they call it, the 'two parameter model'. (Indicating investors care only about mean and dispersion around that mean, technically you can relax the assumptions a little bit)

- They focus on three testable implications from the CAPM:
  - The relationship $E(r_i) = r_f + \beta_i(E(r_M) - r_f)$ is linear,
  - No other factor than beta should explain E(r),
  - $E(r_m)$-r_f > 0, otherwise there would be no reward for risk and every risk averse investor (a maintained assumption) would invest in r_f only.

Fama & MacBeth, 1973 (2)

- Testing for these assumptions can be done with simple regression analysis:
  - Add non-linear terms (quadratic, following Taylor's theorem) to see if non-linearity occurs
  - Add non-beta terms to see if they explain anything
    - Problem: which one(s)?
  - Check the estimated coefficients; no only do you want $E(r_m)-r_f > 0$, but if slightly possible:
    - Intercept = r_f
    - Slope = market risk premium.

NB: We actually have (independent) data on those, so while the regression treats them as free parameters, they aren't really so. Unbelievable numbers still bode ill for the CAPM.

Fama & MacBeth, 1973 (3)

- F&McB test the following regression:

$$\tilde{R}_{it} = \tilde{\gamma}_{0t} + \tilde{\gamma}_{1t}\beta_i + \tilde{\gamma}_{2t}\beta_i^2 + \tilde{\gamma}_{3t}s_i + \tilde{\eta}_{it}. \qquad (7)$$

- Which is (allowing for small differences in notation) just our CAPM relation to which a quadratic term and an extra factor (mimicking idiosyncratic risk) is added.
  - Problem: we need data on returns (no problem), but also on beta.
  - Beta is calculated, more precisely, it's a result of another regression:

$$\tilde{R}_{it} = a_i + \beta^i \tilde{R}_{mt} + \tilde{\epsilon}_{it}. \qquad (8)$$

  - Now here's one of those nasty measurement issues.

Fama & MacBeth, 1973 (4)

- Now here's one of those nasty measurement issues.

$$\tilde{R}_{it} = a_i + \beta^i \tilde{R}_{mt} + \tilde{\epsilon}_{it}. \qquad (8)$$

  - If beta's are estimated, they will have some uncertainty themselves. Since they are used as independent variables later on (eq. 7), they are supposed to be error-free. They're not, and the methodological problem of errors-in-variables causes bias. This was not fully appreciated at this time, F&McB seemed more concerned about the fact that if the CAPM fails, this relation may in fact fail to include relevant variables, which also leads to bias.
  - If we are to estimate this, we assume that a stock has a constant beta over time. Not going to happen (leverage effects, take-overs, divestments; any policy change that changes the relation with the market portfolio [think 'general state of the economy'] is going to change beta. F&McB mitigate that by using *portfolios*.

    (a very sound idea)

In fact, you can also correct for this with econometrics: See Jay Shanken (1992) https://doi.org/10.1093/rfs/5.1.1

Fama & MacBeth, 1973 (5)

- Now here's one of those nasty measurement issues.

$$\tilde{R}_{it} = a_i + \beta^i \tilde{R}_{mt} + \tilde{\epsilon}_{it}. \qquad (8)$$

  – But worst of all, there's *Roll's critique* (1977). Since the CAPM only works if everyone holds the same portfolio which invests in absolutely everything, we need to estimate beta using that market portfolio. Which is utterly impossible, the data isn't available, and we know that this assumption is too stringent for reality.

  ❖ In fact, we have to use a proxy for M. So any test of the CAPM (or related theories) is in fact a test for *two* hypotheses:

    ➢ The CAPM is right

    ➢ Our choice of the market portfolio is right (or close enough).

  ❖ Some comfort is given by the diminishing benefits of diversification if you add extra assets, but we're likely to forget entire asset classes, especially if our approximation of M consists of only stocks. <u>Still, we have to make do with what we can get.</u>

Intermezzo: time-series and cross-section

(if time allows)

Month	SmBM1	SmBM2	SmBM3	BigBM1	BigBM2	BigBM3
196307	-1.08	-0.60	-1.16	-0.03	0.53	-1.85
196308	4.45	4.35	5.70	5.42	4.62	7.65
196309	-3.07	-0.53	-1.91	-1.06	-1.72	-1.79
200507	6.64	6.77	6.33	4.60	2.65	4.03
average	0.944	1.232	1.553	0.903	1.011	1.173

Dataset: different stocks during multiple periods

Intermezzo: time-series and cross-section

(if time allows)

Time-series

Month	SmBM1	SmBM2	SmBM3	BigBM1	BigBM2	BigBM3
196307	-1.08	-0.60	-1.16	-0.03	0.53	-1.85
196308	4.45	4.35	5.70	5.42	4.62	7.65
196309	-3.07	-0.53	-1.91	-1.06	-1.72	-1.79
200507	6.64	6.77	6.33	4.60	2.65	4.03
average	0.944	1.232	1.553	0.903	1.011	1.173

$$r_{\text{portfolio 1}} = \alpha + \beta_1 r_{\text{market}} + \beta_2 \text{something else?}$$

Estimate the coefficients; asses the different types of risk inherent in investing in 'portfolio 1'. Fama & MacBeth: $\beta_2 = 0$

Intermezzo: time-series and cross-section

(if time allows)

Month	SmBM1	SmBM2	SmBM3	BigBM1	BigBM2	BigBM3
196307	-1.08	-0.60	-1.16	-0.03	0.53	-1.85
196308	4.45	4.35	5.70	5.42	4.62	7.65
196309	-3.07	-0.53	-1.91	-1.06	-1.72	-1.79
200507	6.64	6.77	6.33	4.60	2.65	4.03
average	0.944	1.232	1.553	0.903	1.011	1.173

Cross-section

Average return = $\varrho_0 + \varrho_1\, \beta_{\text{market}} + \varrho_2$ other factors

Estimating *riskpremia*; a feature of ‘the market as a whole’

Most ‘anomalies’ are located in the cross-section

Fama & MacBeth, 1973 (6)

$$\tilde{R}_{it} = \tilde{\gamma}_{0t} + \tilde{\gamma}_{1t}\beta_i + \tilde{\gamma}_{2t}\beta_i^2 + \tilde{\gamma}_{3t}s_i + \tilde{\eta}_{it}. \qquad (7)$$

- Results: F&McB find that in their data, the testable implications from the CAPM are not violated.
  - *T-tests for $\gamma_2 = 0$, $\gamma_3 = 0$, not rejected, so no non-linear factor, no other factor*
  - *(should do an F-test for joint significance, but probably same results)*
  - *Mostly believable values for the parameters.*

- Assignment 2b: replicate the study from F&McB for a longer period (in excel, or, preferably, EVIEWS) for 2 datasets.
  - Results and detailed instructions will be made available through Canvas.

Why do we want to test? (1)

- CAPM in itself is descriptive: how good is this description (mostly, rather poor)
- Violations of the CAPM can point us in interesting directions:
  - Return that has no associated risk (or the wrong amount) is a golden opportunity for investors
  - Risk-return relations different from the CAPM point to gaps in our theoretical understanding.
- Suppose Rf is 1%, the return on the market portfolio is 6.5%, and the beta of a portfolio is 1.4. We make 8.2% on our investment. Great deal, or not?

$$(r - r_f e) = \alpha + \beta(r_M - r_f)$$

$$8.2 - 1 = \alpha + 1.4(6.5 - 1)$$

$$7.2 = \alpha + 7.7$$

Why do we want to test? (2)

- CAPM in itself is descriptive: how good is this description (mostly, rather poor)
- Violations of the CAPM can point us in interesting directions:
  - Return that has no associated risk (or the wrong amount) is a golden opportunity for investors
  - Risk-return relations different from the CAPM point to gaps in our theoretical understanding.
- Note the contradictory focus: academics tend to try to explain and remove ‘anomalies’, practioners try to exploit them and earn money from it – ‘the hunt for alpha’. CAPM as a regression:

$$(\bar{r} - r_f e) = \alpha + \beta(r_M - r_f) + \varepsilon$$

Formula of the week

This week, the 'formula of the week' is the **Security Market Line** in the CAPM: $E(r_i) = r_f + \beta_i (E(r_M) - r_f)$

If: Normative portfolio theory holds / A8. (same expectations for returns and covariances) / A9. Lend and borrow at the riskfree rate / A10. Asset markets have perfect competition: everyone is a price taker.

Then: Everyone will invest in a combination of the risk free rate and the tangency portfolio. Since each asset must be held (otherwise prices adjust to restore the equilibrium) the tangency portfolio will consist of every asset in the market. This leads to the SDF in the CAPM :

$$p_t = E(m_{t+1} x_{t+1}); \qquad m_{t+1} = a + bR_M$$

Risk unrelated to movements in the market portfolio can be diversified and isn't priced. (which refers to the 'when' do we value returns part).