Lecture Advanced Investments, September 26th, 2025

Risk and utility functions

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Program

Role of utility functions in Asset Pricing models

- Pre-SDF: Risk aversion coefficients
  - CARA, DARA, CRRA

See Clip 1

- Marginal utility and risk
  - Orders of stochastic dominance
  - CAPM vs. SD
  - Why and when MV?
- Stochastic Dominance
  - Higher orders
  - Usefulness
- Lower partial moments

Extra: Some thoughts on ESG risk

ESG risk (1)

- In many publications, and perhaps even more so in practice, ESG [environmental, social and governance] risks are mentioned.
- But do risks like that fit into our models/understanding of risk?

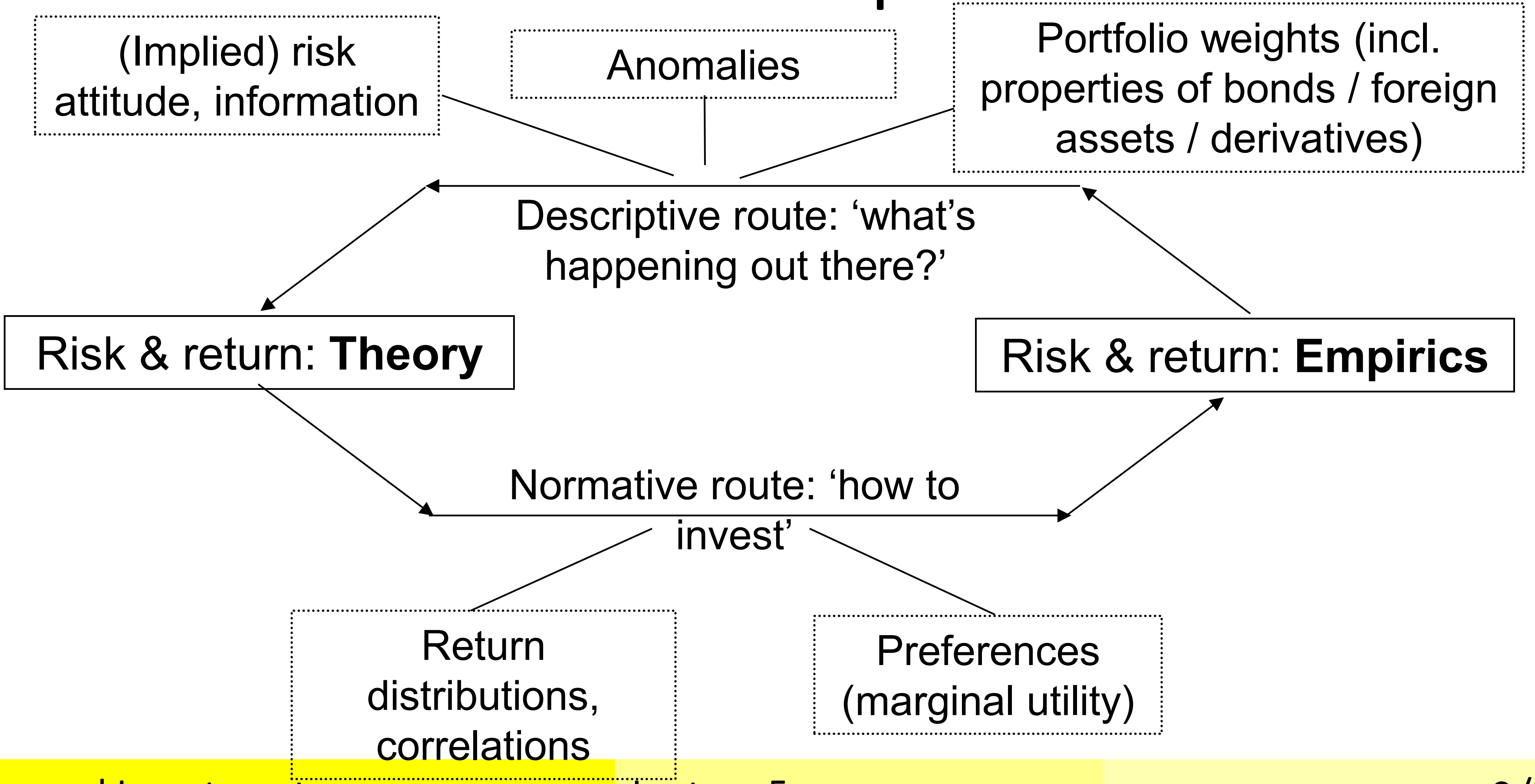
- Yes, if and in so far they – someday – affect returns of the corresponding stocks. We can even make a factor out of it, long high ESG scores, short low ESG scores. (or more complex versions).
- Problem with ESG is that typically, we're talking about risks that rarely materialize, but when they do (especially governance) they can be severe. Yet if everyone bids up green stocks, the realized return is very much a short term affair.

ESG risk (2)

- A typical result is then that *realized returns* are high (the additional inflow is not a one-off, but continues over many years, each time giving the return an upwards nudge) but *expected* returns get lower and lower (less risk means less reward; being green often costs money, unless… it’s marketing or efficiency)
- See for example Pastor, Stambaugh & Taylor (2021, 2022, both JFE).

- Taking this one step further: it is quite likely – though good research seems missing – that investors in green/brown assets disagree, in a self-selecting way, not just about expected returns, but also the likelihood and magnitude of the downside that creates the risk…

Roadmap



(Implied) risk attitude, information
Anomalies
Portfolio weights (incl. properties of bonds / foreign assets / derivatives)
Descriptive route: 'what's happening out there?'
Risk & return: **Theory**
Risk & return: **Empirics**
Normative route: 'how to invest'
Return distributions, correlations
Preferences (marginal utility)

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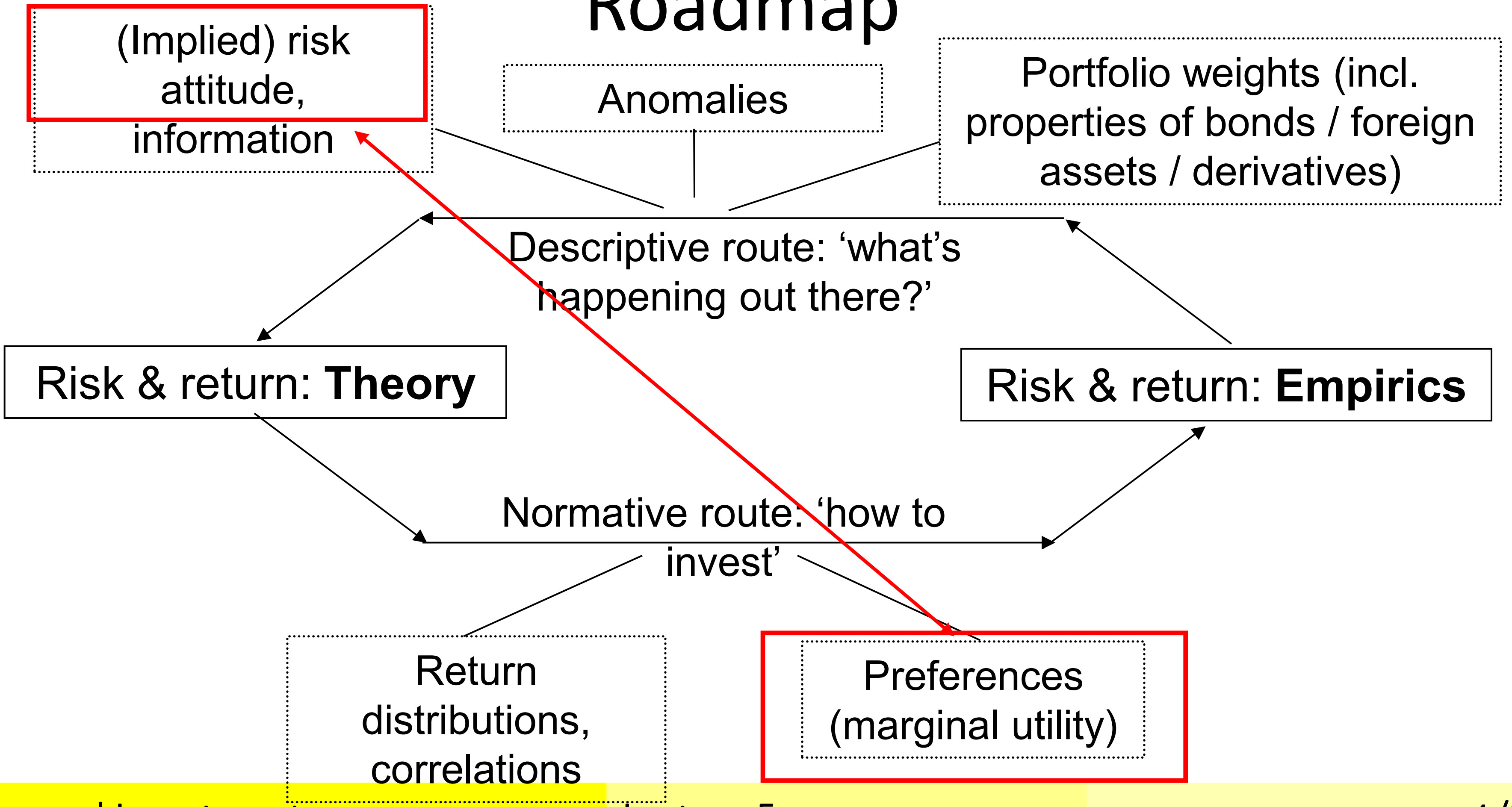
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Traditional measures of risk (1)

- Traditionally, asset pricing / investment models would start with an assumption on the utility function, choosing from the following categories:
  - Constant relative risk aversion (CRRA)
  - Constant absolute risk aversion (CARA)
  - Decreasing absolute risk aversion (DARA)

- In models without an explicit SDF these describe the risk attitude, and often provide just enough information to get a (portfolio) decision out of a model.
- NB: every model has an SDF equivalent. However, up to 15 years or so it was not customary to make that explicit. Investments have been working with the CAPM for 30 years before the SDF approach became popular, and some areas of Finance still sidestep it).

Traditional measures of risk (2)

- Constant Relative Risk Aversion (CRRA) utility functions are assuming that risk aversion stays the same as a percentage of wealth, i.e. an asset with either a -10% or a +15% return is equally risky regardless of the underlying amount (-10% of 3 euros or of 3 million)
- Within limited ranges this is plausible.
- Formula:

$$\mathrm{RRA} = -\frac{x U''(x)}{U'(x)} = \text{constant}$$

- The x is needed for the relative element, the second derivative enters since the shape of risk aversion is determined at that level. (decreasing marginal utility)
- Example functions: Power utility: $U(x)= (1/\alpha)X^{\alpha}$; log utility: $U(x) = \ln(x)$.

Traditional measures of risk (3)

- Constant Absolute Risk Aversion (CARA) utility functions on the other hand are assuming that risk <u>aversion stays the same if the amount at stake</u> or your underlying wealth <u>stays the same</u>. In other words: a millionaire will find it equally ‘bad’ to risk losing 100 euros as a broke student.
- May work better for big risks.
- Formula:

$$\mathrm{ARA} = -\frac{U''(x)}{U'(x)} = \mathrm{constant}$$

- Example functions: (negative) exponential utility: $U(x) = -e^{-bx}$, with $b > 0$.
- NB: quadratic utility has *increasing* ARA. (negative number that approaches zero)

Traditional measures of risk (4)

- Decreasing Absolute Risk Aversion (DARA) utility functions are supported by the economic argument that a rich person will care less amount losing a fixed amount than a poor person (they may still have CRRA).
- Formula:

$$\mathrm{ARA} = -\frac{U''(x)}{U'(x)} = \text{declining in x}$$

- Example functions: U(x)=ln(x).
- This setup is reasonably popular.

Marginal utility and risk (1)

- But let's tackle preferences in a more structured way.
- The SDF tells us when we like returns, so it must account for risk as well: we like risky returns less than safe returns, even though we'd still have to define what risk is – *something that the SDF also does.*
- A crucial point in this is the relation between risk and *marginal* utility – at time t+1 (current consumption is ***certain***).

$$p_t = E_t\left[\varphi \frac{u`(c_{t+1})}{u`(c_t)} x_{t+1}\right] \qquad \varphi \frac{u`(c_{t+1})}{u`(c_t)} = m_{t+1}$$

- Intuitively, this makes sense as well. In the optimum <u>an extra unit of return</u> is (almost) worth <u>the extra unit of risk</u> – or vice versa. We compare incremental changes, so it's logical that marginal utility is the driving factor.

Marginal utility and risk (2)

- In the CAPM, the SDF is shaped by the return on the market portfolio ($m_{t+1}=a+bR_M$), and again we see the importance of the assumption that one invests in everything: otherwise the relation between R_M and either consumption or wealth would be weak.
- But is it realistic to judge risk solely in relation to the market portfolio? Isn’t that an assumption that’s too strict?
- In fact, many if not all investors have some sort of poorly diversified holdings, which means that their utility is influenced by other factors than the market – the ‘when do we like returns’ is not wholly dependent on the R_M.
- Examples include real estate (own home), human capital, ownership of a business, etc. etc. etc. In one word: messy.

Marginal utility and risk (3)

- And how about the assumption that risk = variance?

  There is no guarantee that variance is the appropriate risk measure – in fact, as we'll see later on, it must be defective.
- Still, even in a more general setting we can say something about the SDF, based on the fact that marginal utility is the derivate of the utility function, which must make economic sense.

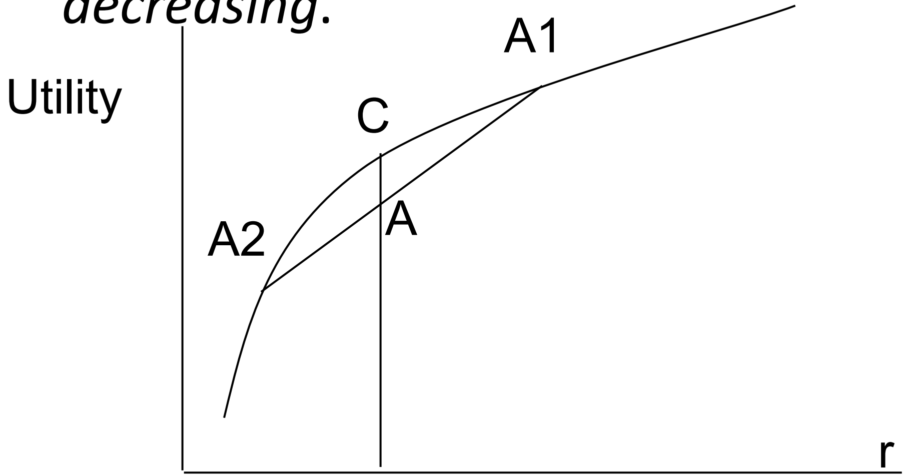
- This leads us to the orders of *Stochastic Dominance*; a framework that categorizes utility functions according to the assumptions they require.
- So rather than fitting the SDF 'from the bottom up' by finding factors, we allow the SDF to be merely a set of numbers, and see which restrictions should be imposed on that vector for it to make economic sense.

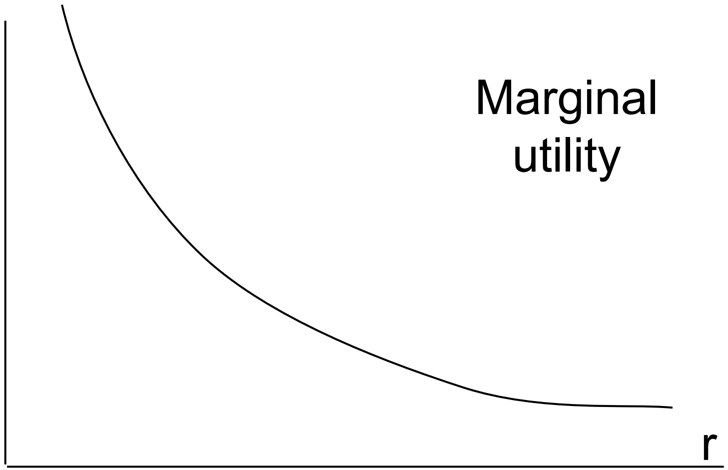
Orders of Stochastic Dominance (1)

- The first economic assumption we tend to make is that of *non-satiation:* one prefers more over less.
- Non-satiation requires a positive marginal utility at each point and hence a positive SDF; an additional unit of return should increase total utility.
- This corresponds to the assumptions of *first order stochastic dominance* (FSD); if a given portfolio is better for all investors who adhere to the non-satiation assumption, it is said to FSD dominate the alternative(s).
- However, finding a portfolio that is better for **all** investors in the FSD category (so everyone who prefers more over less) is very hard indeed. It rarely exists.

Orders of Stochastic Dominance (2)

- The second economic assumption we tend to make is that of *risk-aversion:* one prefers a certain alternative over a variable one given the same expectation.
- Risk aversion requires a marginal utility and hence the SDF to be *decreasing*.





Orders of Stochastic Dominance (3)

- Adding the assumption of risk aversion brings us to *second order stochastic dominance* (SSD); the definition for dominance (better for all risk averse investors) is merely an adjustment to that for FSD. To be precise: it **adds** the requirement of a decreasing SDF.

- NB: finding an asset or even a portfolio that is best for all investors isn't the point: the idea is to find a set of restrictions on the SDF *that make sense*. If we would find an SDF that way, we could use it in an asset pricing model, and <u>then</u> see how big the alpha's are, what the efficient set is, and so on.

- **Conclusions**: the SDF as a whole must be positive (FSD, nonsatiation), and decreasing (SSD, risk-aversion). If the SDF is flat, we have risk-neutral preferences, an increasing SDF means risk-seeking behavior!

Orders of Stochastic Dominance (4)

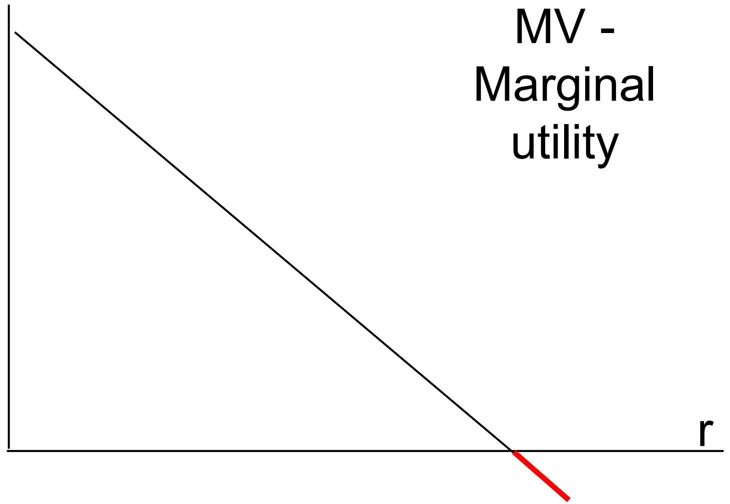
- Based on the criteria given by SSD (second order stochastic dominance) three questions now arise:
  - We have used the same 2 arguments for the MV framework in lecture 1 which led to the CAPM. Yet the SD approach is far more general. Why? And what does SD allow that MV doesn't?
  - Can we find more economic arguments to further restrict the SDF?
  - How/To what extent can we use these SDFs?

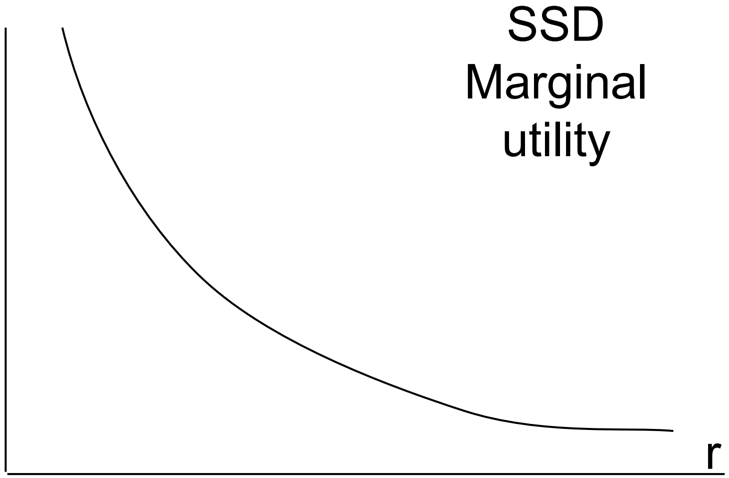
CAPM vs. SD (1)

- In the CAPM framework, risk aversion is still present, as long as certain conditions on parameters are met: SDF= $a+bR_M$, so for the SDF to be decreasing b<0 suffices.
- If the market portfolio is not the (only) factor that's relevant, we still only need to concern us with the question if *all* elements of the vector b are negative. (remember: $m = a + b'f$)
- The real problem lies in the fact that non-satiation can be violated. However one draws a straight line with a nonzero slope, with the right value for the x-variable (here R_M) we will get a negative value for the SDF at some point.
- This implies that at some returns, investors would prefer to have less return. This is rather painful.

CAPM vs. SD (2)

- MV can and will imply violations of nonsatiation:





- The other big difference is that the SDF under SD assumptions *need not be linear*.
- Also, the CAPM model, constructed from preferences, implies *quadratic utility* - *w*hich can only be valid over a limited range.

Intermezzo : top 10 silliest errors

CAPM vs. SD (3)

- The SDF shows one problem of quadratic utility, but there's more.
- Consider an asset with 2 possible states of the world:

	Asset 1	Asset 2
State 1	-5	-3
State 2	+10	+19
Mean	+2.5	+8
Variance	56.25	121

- The MV criterion cannot choose between the 2 (higher mean, but also higher variance), while asset 2 FSD dominates asset 1.

Then why / when MV?

- The mean-variance (MV) framework is clearly not a good description of reality:
  - It requires normally distributed returns (which is not the case), or quadratic utility (which violates non-satiation)
  - Regardless on how it is achieved, it will violate non-satiation again in certain cases, where it will invest in portfolios that are FSD dominated.
- Does the MV framework have anything to speak for it?

- Actually, yes. As an *approximation* it can still be valuable. As Tsiang (1972) defends, over a limited range Taylor's Theorem is still applicable, and for the vast majority of returns differences between the quadratic approximation and the real utility function will be small. FSD dominance is very rare in practice.
- MV primarily fails when explaining *extreme* returns.

Beyond SSD (1)

- But extreme returns can be very interesting, so back to SD.
- We impose very little structure on the SDF under SD rules. This is fortunate for a descriptive approach: we can see if a vast number of possible risk-return tradeoffs will capture the market. But perhaps we want to narrow it down a bit more.

- The preference for (positive) skewness can also be added, though the motivation for this is less straightforward at first sight.
- Skewness is related to asymmetries (in any place on the distribution). Suppose one buys fire insurance; one does so because one severely dislikes the asymmetric nature of the risk (burnt house is a big loss, the situation has no corresponding upside), even though insurance gives a negative return (insurer has to make a profit).

Beyond SSD (2)

- Likewise, one can look at lotteries. These are heavily skewed (big payoff, yet with a small probability). In fact, the expected return is negative (between -30 and -60% usually), yet still the lottery tickets find buyers.
- Of course, one could see this as ‘consumption’ (of the dream that just maybe you’ll be driving to work in a Aston Martin Valkyrie after the next draw), but one sees similar effects other situations too. There probably is a real skewness preference.

- The corresponding SD rule is *third order stochastic dominance* (TSD), which says the SDF is positive, decreasing and decreasing at an decreasing rate.

Beyond SSD (3)

- In other words: U’ >0 (positive SDF), U’’ <0 (decreasing) and U’’’>0 (at a decreasing rate).
- Looks like a pattern, perhaps U’’’’<0?
- Yes, one can even go to 4th order SD using arguments based on kurtosis aversion, but beyond that the economic argument becomes difficult to make. Also, the difference between the orders become smaller as the new restrictions eliminate fewer portfolios.
- One can also impose other restrictions on marginal utility, for instance based on risk aversion coefficients (see later slides)

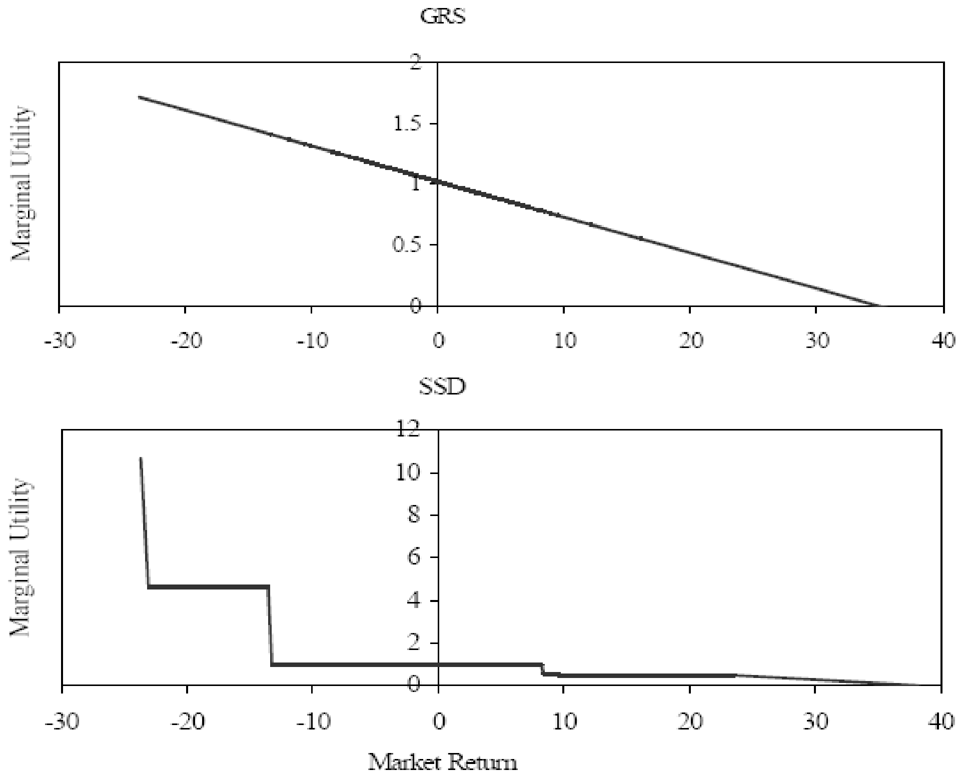
Usefulness of SD SDFs (1)

- It is possible to build descriptive models based on SD principles. The idea is that we simply let the PC calculate which SDF best fits the available data, so instead of imposing a parametric risk-return trade-off, we let the 'data speak'. Results are mixed, some anomalies are easier to explain than other (e.g., momentum remains a hard one).

- Methodologically, the idea is that we use the relation $E(m_{t+1}r)=1$; when using excess returns it becomes $E(m_{t+1}r)=0$, so the left-hand side can be directly interpreted as a pricing error. Then one can minimize (again) the sum of weighted pricing errors - my PhD thesis in a nutshell.

Usefulness of SD SDFs (2)

- Figure from Post & Versiip, 2007 (JFQA)



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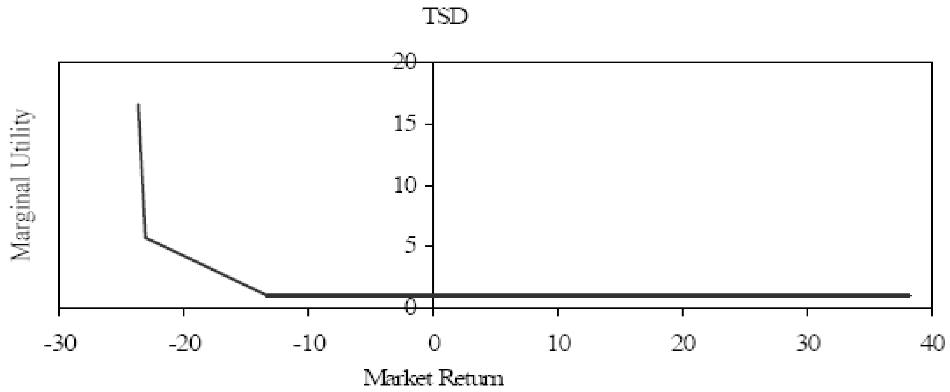


- Note that this serves as an illustration only; the SDF above explains the market relative to a certain dataset. But it is remarkable that in many cases, we see a strong aversion to big losses. (-12% and more)

Usefulness of SD SDFs (4)

- Portfolio optimization is however not straightforward with this approach. Instead of a single utility function (or a class differing only in 2 parameters, as in the MV model) we have a whole collection of possible utility functions. Each dataset will potentially give a (completely) different SDF. Each combination has to be checked against all admissible preferences.
- Two solutions: assume (again) similar type of preferences, or use algorithms that try to find better portfolios in each step. Some progress has been made in this respect recently.
- Yet, a 'tangency portfolio' still seems out of reach. We can only draw normative conclusions if we have a parametric characterization of the SDF.

Lower Partial Moments (1)

- Another, more flexible way to parameterise the utility function is to use *Lower Partial Moments.*
- Papers: Bawa (1975), Bawa-Lindenberg (1977).

- The idea is that return below a certain threshold constitutes risk. This has the economic advantage that high positive returns are not considered to make the same contribution to the riskiness of an asset as very negative returns.

- One can combine various thresholds if need be, and one can also vary the ‘intensity’ of the riskiness by using another order of LPM.

Lower Partial Moments (2)

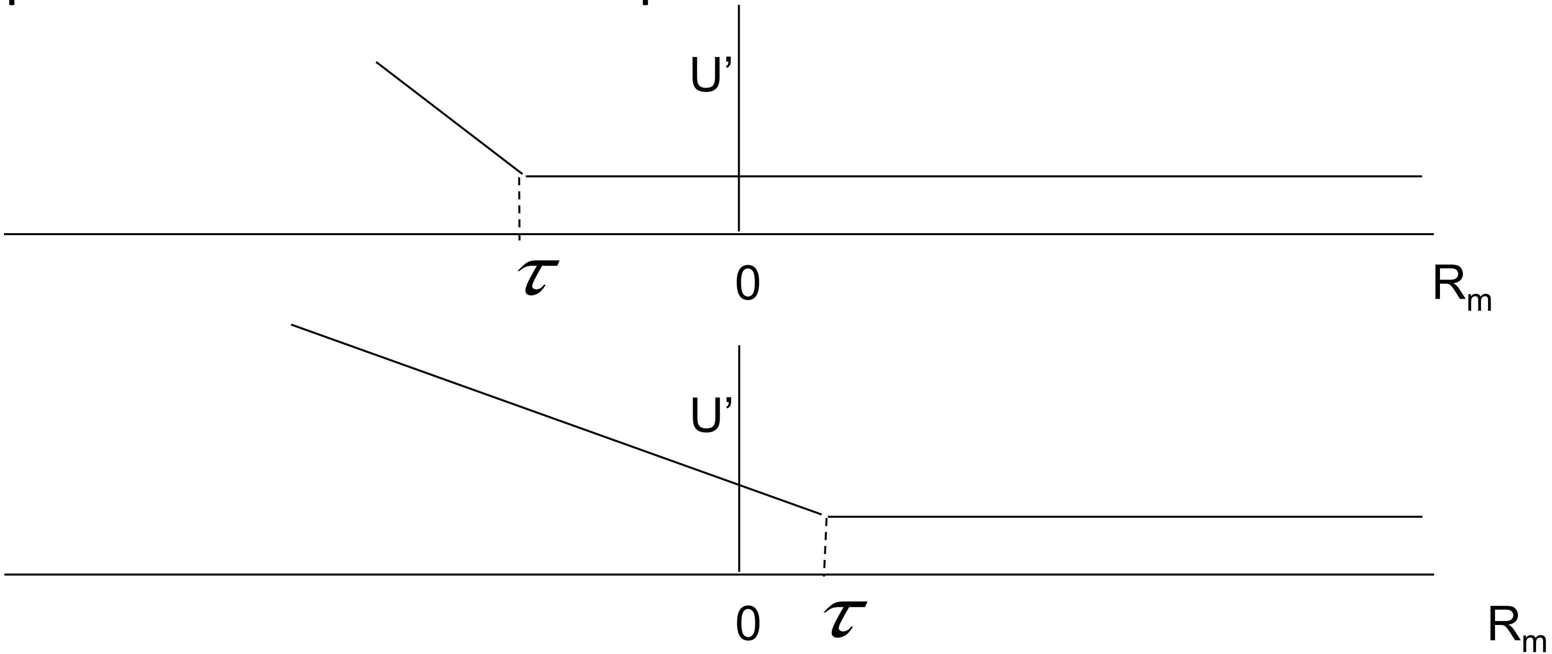
- The most appealing characteristic of LPMs is however that under certain conditions it can be expanded to an equilibrium model like the CAPM, see Bawa-Lindenberg (1977).

- If returns are normally distributed, LPM and MV optimalisation will give the same result; the M-LPM model *nests* the MV model if the latter is obtained from assumptions on the return distribution.

- But if returns are not normally distributed, LPMs can still be valid and offer a more promising description of risk.

Lower Partial Moments (3)

- Typical SDFs with an LPM specification:



Lower Partial Moments (4)

- A Lower partial moment is defined as:

$$LPM_{n,\tau} = \frac{1}{T}\sum_{t=1}^{t=T}\left(\mathrm{Min}\left[(x-\tau),0\right]\right)^{n}$$

- This means we only look at returns below a threshold tau, and take them to the power n. However, if the return is above the threshold this return does not add any risk.

- If the threshold is taken to be zero, and the order (n) is 2, the risk measure that follows is called *semi-variance*, as it is the same as variance but only for negative returns.
- NB: this makes most (economic) sense if one uses excess returns, which is actually the same as using r_f as your threshold.

Lower Partial Moments (5)

- The M-LPM problem is as follows:

$$\min_{\omega}\ \mathrm{LPM}_{n,\tau} = \frac{1}{T}\sum_{t=1}^{t=T}\left(\mathrm{Min}\left[(\omega' X - e'\tau),0\right]\right)^{n}$$

$$\text{s.t.}\quad \omega'\bar{r} = r_p$$

- This leaves us with an efficient frontier not unlike that of Modern Portfolio Theory (though it will look different in mean, standard deviation space if that isn't the correct risk measure).
- There will be a tangency portfolio if a risk-free object is available, and portfolio separation still applies.
- The actual optimization is more involved, due to the discontinuity of the objective function at the threshold.

Lower Partial Moments (6)

- Assignments for this week:

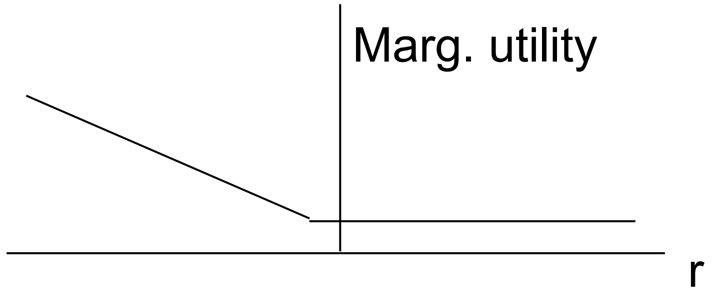
1. Calculate the LPMs for your solutions to the portfolio optimisation problem.
2. Redesign the excel file for portfolio optimisation so it uses the Mean – Lower partial moment criterion.

NB: actually solving the problem should be possible, but the solver software may not co-operate due to the discontinuity. It *may* help to use zero portfolio weights as a starting value.

Lower Partial Moments (7)

- Last but not least: *one can create a beta based on LPM that acts just as the beta in the CAPM*. Instead of covariances, one uses co-LPMs.

- To repeat: in SDF terms, the equilibrium model based on an LPM of order two and excess returns looks as follows: $m_t = a + b*Min(R_M - r_f, 0)$.



- This means that returns above the threshold are treated as if the investor would be *risk-neutral* there.

Lower Partial Moments (8)

- However, while this assumption as such is doubtful, there are 2 mitigating factors:
  - As an approximation, it may be satisfactory
  - One can combine LPMs with several different thresholds for a more realistic description.

- In fact, *any* positive and decreasing SDF (i.e., rules of at least SSD apply) can be constructed from combining LPMs with differing thresholds.
- LPMs can therefore be seen as ‘building blocks’.
- However, the equilibrium model which nests the CAPM cannot be (easily) obtained that way.

Formula of the week

- The formula of the week is that for the Lower partial moment:

$$\mathrm{LPM}_{n,\tau} = \frac{1}{T}\sum_{t=1}^{t=T}\left(\mathrm{Min}\left[(x-\tau),0\right]\right)^{n}$$

- LPMs are very suitable to look at alternative forms to describe risk:
  - MV and LPM are compatible under specific circumstances – normality of returns.
  - However, LPMs will not violate nonsatiation by constructing SDFs with negative values. Its SDF is decreasing (=risk-averse) or constant (=risk neutral) above the threshold.
  - LPM focusses on *downside risk*.
  - LPMs can be combined to form more realistic descriptions.