Lecture Advanced Investments, September 3rd, 2025

Basics of risk and return, portfolio theory

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Advanced Investments: *objectives of the course*

- Understanding modern asset pricing and portfolio models:
  - Risk-return trade-off
  - Extensions of, and deviations from, the standard models
  - Measurement issues
  - In theory and practice
  - Realising where the challenges for Finance lie, and explain them

  NB: we'll focus on general patterns, not stock picking.

- Secondary: being able to work with the above using empirical data
  - Excel / EVIEWS / STATA / Python / R exercises

  (i.e. refeshing and expanding skills for your thesis; use any software package you prefer, though I'd advise against SPSS)

Advanced Investments: contents

- Program for the course
  - Portfolio theory
  - CAPM: assumptions, empirics, testing
  - Anomalies
  - Return; influence and measurement
  - Portfolio management & evaluation
  - Risk: what *is* risk
  - Foreign investments, derivatives
  - Role of information and behavioral aspects

Contact

Dr. Philippe Versijp

- versijp@ese.eur.nl
- Office hours **by appointment only**, always send an email (no fixed office)
- Freelance, so Canvas works much better (especially the discussion feature)

Canvas / SIN-online

- Canvas: canvas.eur.nl
  - Announcements (all)
  - Sheets
  - Material computer exercises
  - Forum
  - Course blog

- SIN-online: ese.sin-online.nl
  - Changes in location, times etc.

- Course code: FEM 11074

Literature

- *Compulsory*:
  - Cochrane (“Asset Pricing: revised edition”)
  - Papers (links on Canvas)
  - Anything mentioned during class
- *Necessary but substitutable*: see the schedule, 2 basic books for which you can use the textbook you undoubtedly own from your bachelor, and Pennacchi, "Theory of Asset Pricing" - would be good if you can trace that one. Read the FAQ for further info/considerations.
- *Recommended:*
  - Non compulsory parts of books
  - Extra articles (links once again on Canvas)

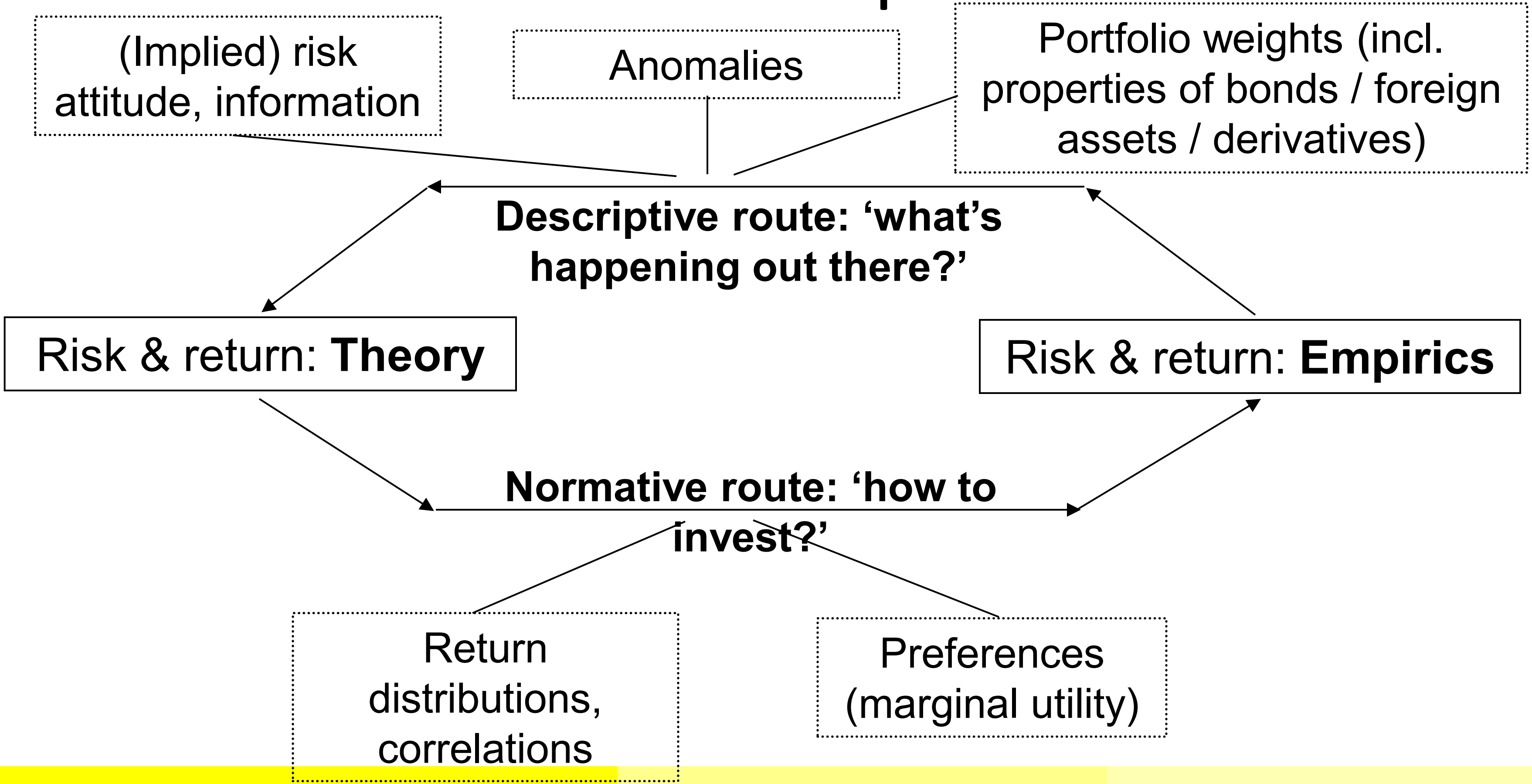
Exam

- Written exam at the end.
  - Counts for 100% of your grade.
- Open questions, closed book.
  - May include (simple) calculations; a sheet with formulas will be provided.
  - Graphic and/or programmable calculators are not allowed.
  - The examination board has forbidden the use of dictionaries.
  - All questions revolve around ‘why is this happening?’ I’m far less interested in mere facts than in the reasons why a situation occurs. The former tends to get a small fraction of the points available, the latter the majority of the points. Take the phrase ‘explain your answer’ seriously!

Program for today

- Introduction (done)
- Roadmap
- Trade-off risk and return
  - Refresher
  - Stochastic Discount Factor *(our workhorse for this course!)*
- Portfolio theory
  - Refresher
  - Maths
  - The bullet (also in excel)
- Formula of the week

Roadmap



(Implied) risk attitude, information
Anomalies
Portfolio weights (incl. properties of bonds / foreign assets / derivatives)
Descriptive route: 'what's happening out there?'
Risk & return: Theory
Risk & return: Empirics
Normative route: 'how to invest?'
Return distributions, correlations
Preferences (marginal utility)

Refresher: risk & return (1)

- Investments: a trade-off between *risk* and *return*.
  - Observation 1: People have money to invest.
  - Observation 2: There are several assets to invest in, and we have a choice between different assets.
  - Observation 3: We assume that people have preferences.

  In other words, there are different possibilities, and they are not all equal, some are better than others (and maybe one is best).

See also: Levy-Post chapter 8 (and 9+10) ; Markowitz (1952)

Refresher: risk & return (2)

- We tend to model that decision as a trade-off between return (expected gain) and risk (probability and magnitude of loss / shortfall). Assume:
  - A1. Agents prefer more over less (nonsatiation)
  - A2. Agents dislike risk (risk-aversion). The definition of risk is an important matter.

	Asset 1	Asset 2	Asset 3	Asset 4
State 1	-5	-3	-7	0
State 2	+10	+19	+25	2

Refresher: risk & return (3)

- This gives rise to (the) 2 central problems of investments:
  - How *should* investors, given their preferences, invest their money?
  - What can we say about how the market and its participants actually operate (and invest)?
- The first point is normative, the second positive/descriptive. Yet both revolve around the risk-return relationship, and both interact: information about how markets work influences investment decisions, which influences the market in its turn.

  Of course, there's also the feedback between *financial theory* and practice.

Refresher: risk & return (4)

- Both positive and normative approaches require the use of assumptions. If practice deviates from the theory, there's something 'wrong' with the theory, and the first place to look is at the set of assumptions. (a second place would be the data, a third the methodology. More about that later, assume those are OK for now).
- One cannot test all assumptions at the same time; there are too many of them, there are related, and the alternative (no assumptions) does not give any predictions at all.
- Hence, it makes sense to have a look not only at the assumptions themselves, but *why* we make them.

Risk & return: ‘the formula’ (1)

- Asset returns (price changes) also determine the other part of the question, namely risk. For instance, we can separate:
  - The chance at a return (high or low)
  - The utility of a return – the same return can have different utility in different situations: if other assets in your portfolio have big negative returns, you like a positive return more. If your consumption is low (e.g. due to a recession), a wipe-out of your investments may hurt more.

- Conclusion: the amount matters (mean and variance), and the relation with other factors (covariance) matters.

Risk & return: 'the formula' (2)

Conclusion: the amount matters, and the relation with other factors matters. In a formula:

$$p_t = E(m_{t+1} x_{t+1})$$

P stand for price, m is called the stochastic discount factor, x is simply the return(distribution).

The *Stochastic Discount Factor* (SDF) m_{t+1} captures the relation with other factors and the reward required to bear the risk inherent in x. Today's price (p_t) is determined by "tomorrow's" possible returns (x_{t+1}) and their riskiness (m_{t+1})

See also: Cochrane, Chapter 6, p. 106-108

Risk & return: ‘the formula’ (3)

The SDF can be derived form the utility function (hence the relation between risk and (marginal) utility). Suppose utility is given by

$$U(c_t, c_{t+1}) = u(c_t) + E_t[\varphi u(c_{t+1})]$$

and an investor has a current endowment (‘income’) e_t and a future one e_{t+1}. Suppose he buys an amount γ of an asset at price p_t to get the return x_{t+1}:

$$c_t = e_t - \gamma p_t$$

$$c_{t+1} = e_{t+1} + \gamma x_{t+1}$$

Now the investor maximizes his total utility (first derivative w.r.t. γ set to zero)

Risk & return: ‘the formula’ (4)

Solving this optimalisation (substitute c into the utility function, apply the rules for taking derivatives) leads to:

$$-p_t u`(c_t) + E_t[\varphi x_{t+1} u`(c_{t+1})] = 0$$

$$p_t = E_t\left[\varphi \frac{u`(c_{t+1})}{u`(c_t)} x_{t+1}\right]$$

Sounds familiar!

$$\varphi \frac{u`(c_{t+1})}{u`(c_t)} = m_{t+1}$$

(In this relatively simple model)

Risk & return: ‘the formula’ (5)

- The SDF, which translates returns to prices by judging its riskiness, depends on marginal utility. We’ll get back to this several times.

  The big question is: what is the shape of marginal utility and/or m?

- In many cases, m is a *linear function* of a factor:

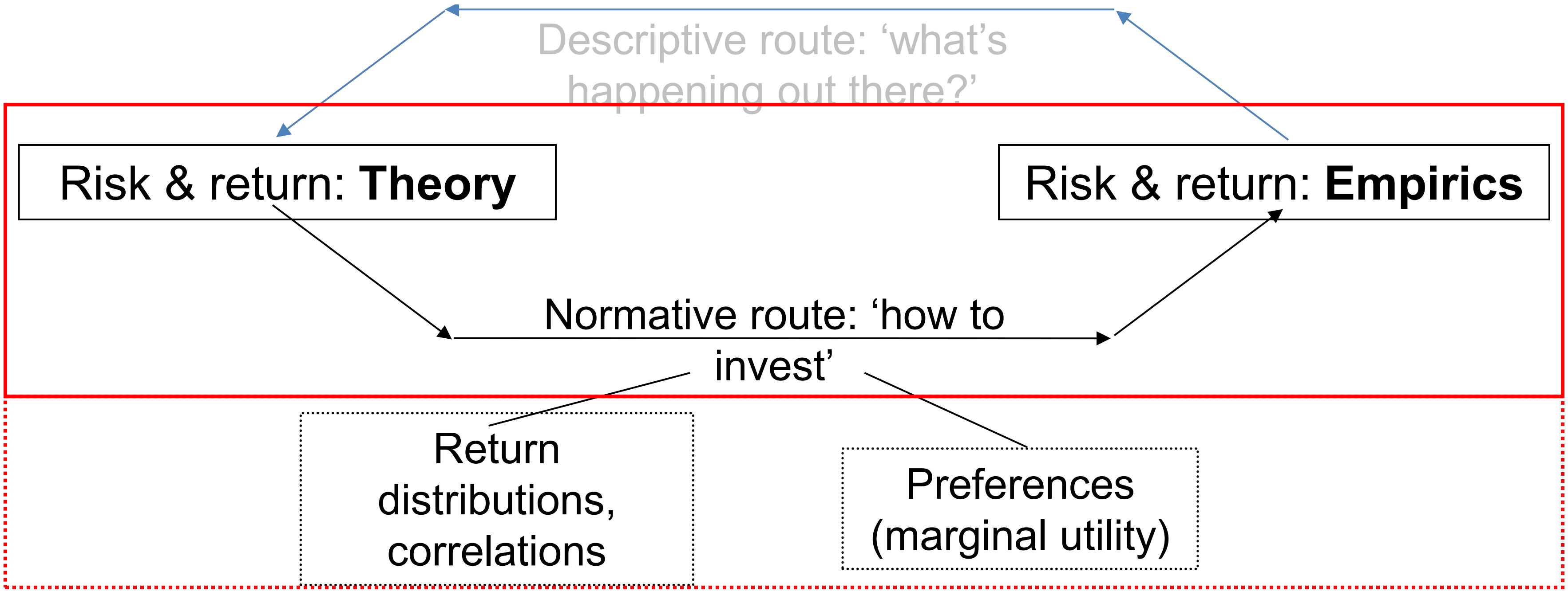
$$m_{t+1} = a + b`f$$

- That factor captures the ‘when’: returns in situation A may be more pleasant than the same returns in situation B. (think for example: ‘boom’ and ‘recession’. ***Or remember it through this joke.....***). But before this can be shown, we need to review portfolio theory and the CAPM.

Roadmap : Today

Portfolio Theory: ***How to invest*** given certain assumptions

(on return distributions and/or marginal utility)



Refresher : portfolio theory (1)

- Portfolio theory is normative: which portfolio of assets is best for a given set of preferences. It focuses on a single agent, and therefore needs fewer assumptions than a model of the entire market.
- We already had:
  - A1. Agents prefer more over less (nonsatiation)
  - A2. Agents dislike risk (risk-aversion).

    These are necessary to determine the 2 parts of the trade-off; what’s good and what’s bad.

  Additionally, we assume about the investors:
  - A3. They maximize utility, and do so for 1 period.
  - A4. Utility is a function of *expected return* and *variance*, and nothing else.

Refresher : portfolio theory (2)

- A3. They maximize utility, and do so for 1 period.

  Maximizing utility is a rephrasing of an element of *rationality*: agents are capable of finding the very best solution for their problem, and are willing to do so. Also see A6.

- A4. Utility is a function of *expected return* and *variance*, and nothing else.

  This defines both <u>return</u> (as its expectation, that is, the average over all possible outcomes; we tend to take historical returns as a *proxy* for those possible outcomes) and <u>risk</u>: we assume variance (squared deviations from the mean) is a sufficient measure of risk. A lot more on this later.

Refresher : portfolio theory (3)

- A3 and A4 combine to form a central tenet of portfolio theory: agents minimize risk at a given level of return, or maximize return at a given level of risk. If A5 to A7 hold, the two approaches give the exact same set of portfolios (the *efficient set)*. The exact amount of risk aversion will determine which portfolio of that set is ultimately chosen.

- For the marketconditions, we assume:
  - A5. No distortion from costs, transaction fees, inflation or taxes.
  - A6. All information is available at no cost.
  - A7. All investments are infinitely divisible.

Refresher : portfolio theory (4)

- A5. No distortion from (transaction) costs, inflation or taxes.

  If trading has costs, the optimum shifts: investing a small amount in an asset becomes less attractive, so costs favor investing in fewer assets. Inflation and taxes tend to disrupt the choice between consumption now and investments (for consumption later) or between assets (e.g. tax breaks for 'green' investments)

- A6. All information is available at no cost.

  If information is costly, it again shifts the optimum (towards assets which look good with little or cheap information)

- A7. All investments are infinitely divisible.

  If you can only buy gold by the kilo, the minimum investment will be too high for some (or at least granulated), again shifting the optimum.

Portfolio theory : mathematics (1)

- Q: What does portfolio theory say?
  A: Diversify, diversify, diversify.

- If we're optimizing a portfolio given a certain set of assets, and mean-variance preferences, chances are that the optimum will consist of a lot of relatively small investments.

- Mathematically: $$\overline{r_p} = \omega' \bar{r}$$

  the return of a portfolio is the return of the assets in the portfolio times the portfolio weights (ω). NB: matrix notation!

$$\sigma_p^2 = \omega' V \omega$$

  the variance of the portfolio is determined by the weights and the covariance matrix of the assets.

Portfolio theory : mathematics (2)

$$\overline{r_p} = \omega' \overline{r} \qquad \sigma_p^2 = \omega' V \omega$$

If we assume there is no risk-free asset (actually a more realistic assumption than it seems) we should add the constraint that portfolio weights sum to 1:

NB: here, 'e' is a vector consisting solely of 1's.

$$\omega' e = 1$$

Optimization can be achieved by constructing the Lagrangian:

$$\min_{\omega} \frac{1}{2} \omega' V \omega + \lambda[\overline{R_p} - \omega' \overline{r}] + \gamma[1 - \omega' e]$$

We minimize the variance by adjusting the weights, and subject to the restrictions that the portfolio return equals R_p and the weights sum to 1.

We then construct the first order conditions (set derivatives to zero) w.r.t. ω and the lagrange multipliers λ and γ, and solve 2 equations in 2 unknowns.

Portfolio theory : mathematics (3)

Solution:

$$\omega^* = \frac{e'V^{-1}e\overline{R_p} - \overline{r}'V^{-1}e}{\overline{r}'V^{-1}\overline{r}e'V^{-1}e - \left(\overline{r}'V^{-1}e\right)^2} V^{-1}\overline{r} + \frac{\overline{r}'V^{-1}\overline{r} - \overline{r}'V^{-1}e\overline{R_p}}{\overline{r}'V^{-1}\overline{r}e'V^{-1}e - \left(\overline{r}'V^{-1}e\right)^2} V^{-1}e$$

Pennachi contains a text which simplifies this – *admittedly horrible* - expression by using the fact many combinations in the formula are in fact scalars (that is, just numbers, not vectors or matrices), also see the next slide.

Also note that with the optimal portfolio for a given required return, we can also find it's variance:

$$\sigma_p^2 = \omega^{*\prime} V \omega^*$$

Portfolio theory : mathematics (4)

Simplification:

$$\omega^* = \frac{e'V^{-1}e\overline{R_p} - \bar{r}'V^{-1}e}{\bar{r}'V^{-1}\bar{r}e'V^{-1}e - \left(\bar{r}'V^{-1}e\right)^2} V^{-1}\bar{r} + \frac{\bar{r}'V^{-1}\bar{r} - \bar{r}'V^{-1}e\overline{R_p}}{\bar{r}'V^{-1}\bar{r}e'V^{-1}e - \left(\bar{r}'V^{-1}e\right)^2} V^{-1}e$$

$$= \frac{\delta\overline{R_p} - \alpha}{\varsigma\delta - \alpha^2} V^{-1}\bar{r} + \frac{\varsigma - \alpha\overline{R_p}}{\varsigma\delta - \alpha^2} V^{-1}e$$

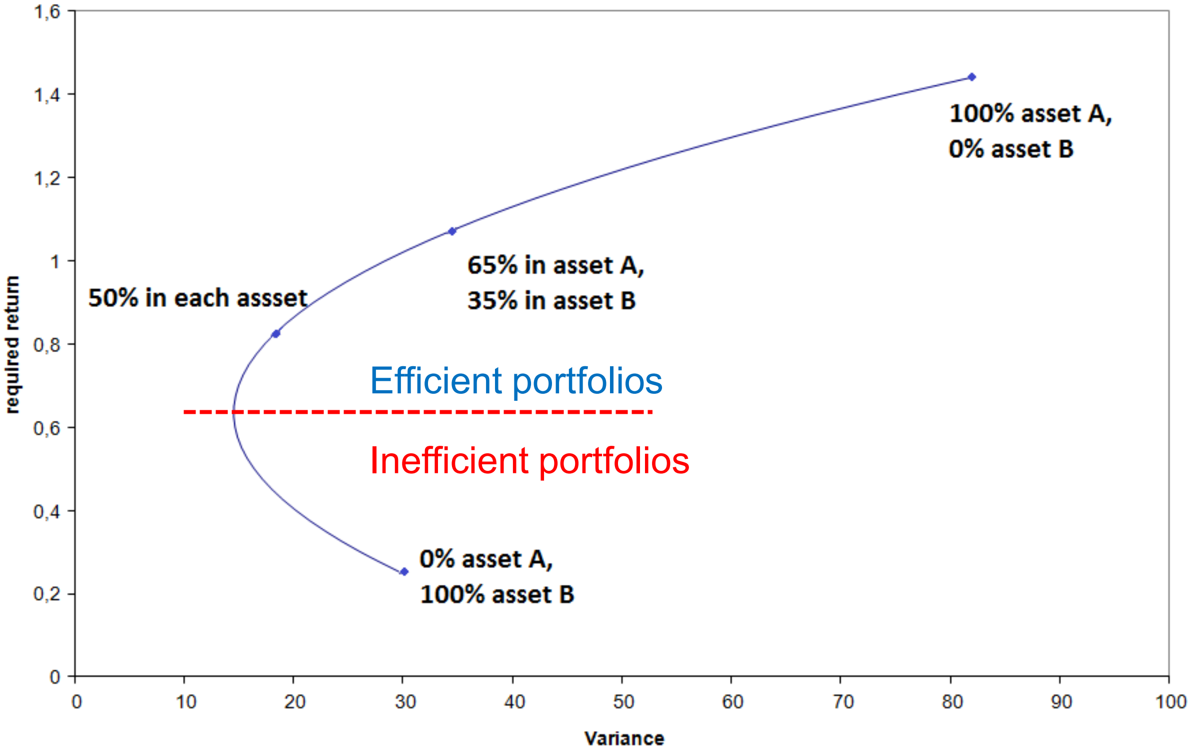
(Where every greek letter is just a number)

Both formula’s give the same message: the optimal portfolio based on 3 factors:

1) What are the average returns of the assets (the vector r)
2) How are they related (based on matrix V, that is correlations!)
3) How much return do I want / how much risk am I willing to bear (R_p)

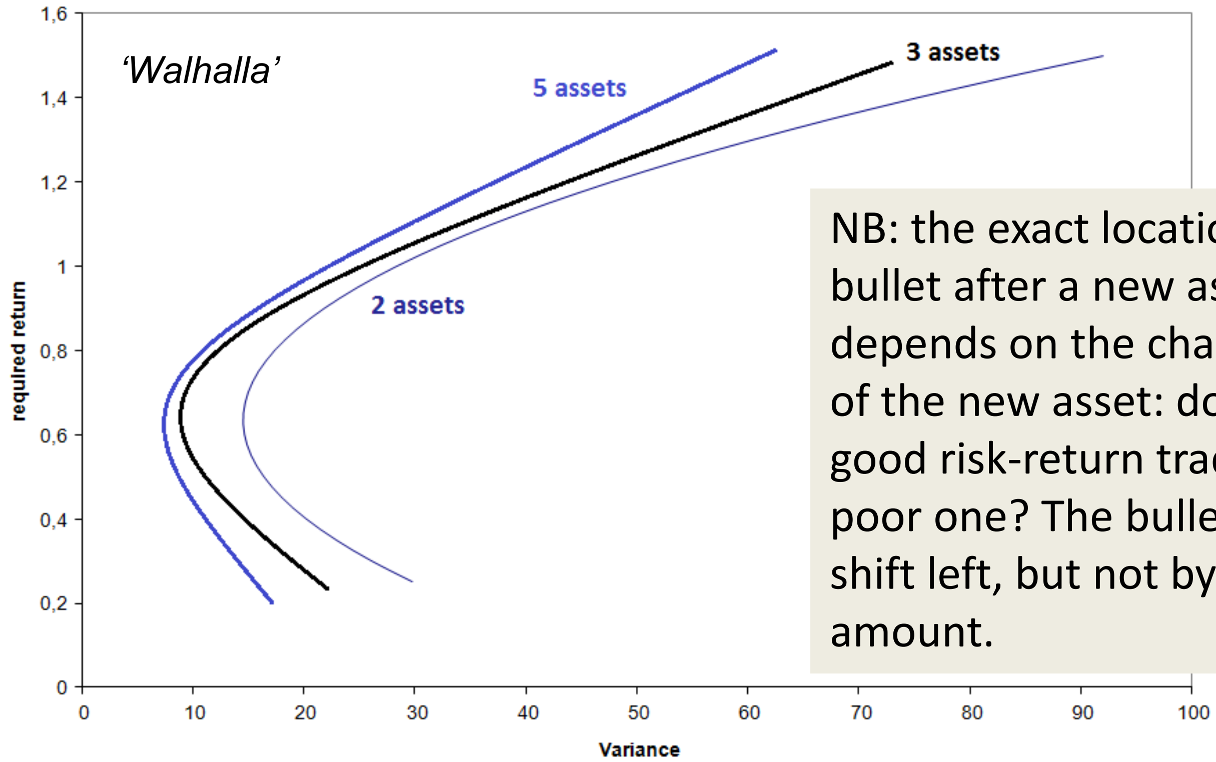
Portfolio theory : the bullet (1)

- Which bring us to the classic bullet of mean-variance analysis.



Portfolio theory : the bullet (1b)

- And this of course also applies to more than 2 assets....



NB: the exact location of the bullet after a new asset is added depends on the characteristics of the new asset: does it offer a good risk-return trade-off, or a poor one? The bullet will always shift left, but not by the same amount.

Portfolio theory : the bullet (2)

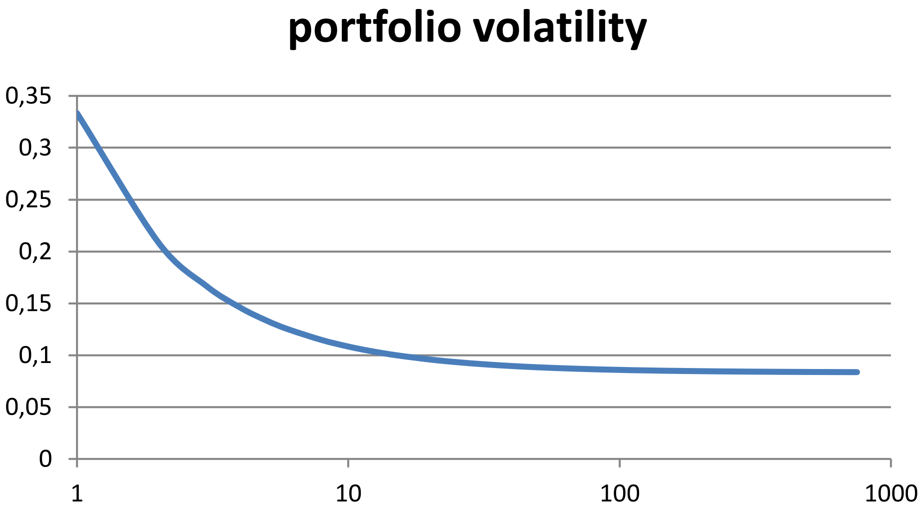
- **Excel exercise**: find the bullet for a dataset consisting of 10 portfolios (sorted on size, assume that this is the investment possibility set for this exercise) for 600 months (Jan 1950 – Dec 1999)
- Two ways of doing this:
  - Work out the formula. (done that already, see file)
  - ‘brute-force’: use the solver to find the portfolio weights that minimize variance for a given return. *This is your exercise for this week.*

    NB: the results should be the same as obtained through the formula for at least the first 3 decimal places.

Intermezzo : top 10 silliest errors

Portfolio theory : the bullet (3)

- And: ‘a little diversification goes a long way’: the first few assets shift the bullet leftward much more than later ones. (or, within mutual funds, move the mean-variance characteristics closer to the bullet)



Portfolio volatility as a function of the number of stocks in portfolio.

NB: exact shape depends on stock characteristics.

What can we learn from this?

Portfolio theory : the bullet (4)

- Main influences on the bullets:
  - Mean returns (see lecture 4)
  - Correlations (see lecture 6)

    NB: these will be discussed in a more general context
  - Maintained assumptions, especially A5-A7. Transaction costs, information costs and indivisibility make the problem non-smooth (picture), unless more and very specific assumptions are made. Optimization techniques may fail, some participants may have arbitrage opportunities (in which case you don’t care much about bullets)

Portfolio theory : the bullet (5)

- From bullet to optimal portfolio:
  - This normative part requires you to know the preference structure, either in the form of indifference curves (combinations of mean and variance with the same utility) or the utility function itself.
  - You need to take a riskfree object (if available) and restrictions (for example no shortselling) into account. This will be discussed next week.

➢ **Picture: bullet and indifference curves, with values for U.**

Formula of the week

Each week, I'll summarize the most important parts of the lecture using a 'formula of the week'. This week it is the formula for **normative portfolio theory**: $\omega^{*} = f(\overline{R_p}, V^{-1}, \overline{r})$

If: A1. Nonsatiation / A2. Risk-aversion / A3. Investors maximize utility, and do so for 1 period / A4. Utility is a function of *expected return* and *variance*, and nothing else / A5. No distortion from costs, transaction fees, inflation or taxes / A6. All information is available at no cost / A7. All investments are infinitely divisible, all hold, we have a neat conclusion:

The optimal portfolio will be determined by the **average return of the assets** you can invest in (r), **the relation between the assets** (V^{-1}; determines how much you gain through diversification), and how **much return you want** / risk you're willing to bear (R_p)