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Answer 1

(a) (i) $\mathcal{T}_1$ is a topology. Since the entire space and the empty set are both open ($A, \emptyset \in \mathcal{T}_1$), and there are no other elements other than those, we can say that $\mathcal{T}_1$ is a topology.

(ii) $\mathcal{T}_2$ is not a topology, because "the union of any number of open sets is open" property does not hold for $\mathcal{T}_2$. For example, $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_2$. Therefore, we can say $\mathcal{T}_2$ is not a topology.

(iii) $\mathcal{T}_3$ is a topology. $\emptyset$ and A are in the $\mathcal{T}_3$. The union of the elements of any subset of $\mathcal{T}_3$ is in $\mathcal{T}_3$. Also the intersection of the elements of any subset of $\mathcal{T}_3$ is in $\mathcal{T}_3$. Therefore, we can say that $\mathcal{T}_3$ is a topology.

(iv) $\mathcal{T}_4$ is not a topology. $\emptyset$ and A are in the $\mathcal{T}_3$, but we cannot say the union of any collection of sets in $\mathcal{T}_4$ is also in $\mathcal{T}_4$. For example, $\{a, c\} \cup \{b\} \notin \mathcal{T}_4$, so $\mathcal{T}_4$ is not a topology.

(b) (i)

(ii)

(iii)

Answer 2

(a) Yes, f is injective. Let's pick random different points from, Ax(0,1] such that (a1,b1) and (a2,b2). Assume f(a1,b1) = f(a2,b2), so this means a1+b1 = a2+b2. This can be written as a1-a2 = b1-b2. Since the least possible difference for a1-a2 is 1, this equation does not hold, because $|b2 - b1| < 0$ for all values of b1 and b2 in (0,1).

(b) No, f is not surjective. Note a+b∈[0,∞), and f(a,b) $\neq 0$ ∀(a,b) ∈ Ax(0,1). If it was, then f(a,b)=0 → a+b=0 → a = -b. So a and b both should be zero, or b should be equal to -a, but (0,0) ∉Ax(0,1) and there are no a such that -a∈(0,1). Therefore f is not surjective.

(c) By Cantor-Schroder-Bernstein theorem, if A and B are sets with $|A| \leq |B|\ and\ |B| \leq |A|$, then $|A| = |B|$. Namely if we can show there are 1-to-1 functions f from A to B and B to A, cardinalities of A and B are same, because there is one-to-one correspondence. Since we have defined f function that is injective and given g function which is also injective, we have bijection between $\mathrm{Ax}(0, 1)$ and $[0, \infty)$. Therefore, we can say that $\mathrm{Ax}(0, 1)$ and $[0, \infty)$ have the same cardinality.

Answer 3

(a) Since the domain of function is finite, and co-domain is countable; we can list all the function elements like,{(0,1),(1,1),(0,2),(1,2),(0,3),(1,3),...}, we can say that this set of function is countable.

(b) In this function set, our domain is finite and co-domain is countable, so we can list all the function set as {(0,1),(1,1),(2,1),...,(n,1),(0,2),(1,2),...,(n,2),(0,3)...}, this function set is countable.

(c) We count the elements of $\mathbb{Z}^+$ like $\mathbb{Z}^+ = \{1, -1, 2, -2, 3, -3, ...\}$
Since, we can define a bijective function such that f: $\mathbb{Z}^+ \rightarrow \mathbb{Z}^+$: f(x) = x $\forall x \in \mathbb{Z}^+$
we can say that the set C of all functions f: $\mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ is countable.

(d) Let's assume there are countable infinite functions from N→{0,1}. Functions that are defined can be written as $f_1 : 00010101...$, $f_2 : 110101010...$, $f_3 : 01010111101...$, ...
If we pick n^{th} element of n^{th} function for different n values, and construct a function, which is not in this countable infinite list, we prove by contradiction, the set D is uncountably infinite.

(e)

Answer 4

(a) We start by showing that log(n!) is $O(n^n)$. Then we should show that $|log(n!)| \leq C|log(n^n)|$ and in the end, n! $\leq$ Cn^n for some C and all n $\geq$ k; k, C $\in \mathbb{Z}^+$. If we choose k=C=1, we will have

log(n!) = log(n·(n-1)...2·1)
= log(1) + log(2) + ...+ log(n) $\leq$ log(n) + log(n) + ... + log(n) = n·log(n) = $log(n^n)$
Therefore, $n! < n^n$ and n! is $O(n^n)$.

Now we will show that log(n!) is $\Omega(n^n)$. If we can show $C|log(n^n)| \leq |log(n!)|$ for some C and all n$\leq$; k, C$\in \mathbb{Z}^+$. We choose C=k=1.

log(n!) = log(1) + log(2) + ... + log(n/2) + ... + log(n) $\geq$ log(n/2) + ... + log(n) = n/2 ·log(n)
Then log(n!) $\geq log(n^{n/2})$. When take exponent 10 of both sides, we can write this as,
n! $\geq (n/2)^{n/2}$ so, n! is $\Omega(n^n)$

Since n! is $O(n^n)$ and n! is $\Omega(n^{n/2})$ We can say that n! is $\Theta(n^n)$.

(b) If we can show that $(n+a)^b$ is $O(n^b)$ and $(n+a)^b$ is $\Omega(n^b)$, then we can say $(n+a)^b$ is $\Theta(n^b)$.

for n $\geq |a|$ $(n+a)^b \leq (2n)^b = 2^b n^b = Cn^b$
Therefore, for C = 2^b, we can say $(n+a)^b$ is $O(n^b)$

for n $\geq |a|$ $(n+a)^b \geq (n/2)^b = 2^{-b} n^b = Cn^b$
Therefore, for C = 2^{-b}, we can say $(n+a)^b$ is $\Omega(n^b)$

From these results, we concluded $(n+a)^b$ is $\Theta(n^b)$.

Answer 5

(a) Let x = y·q+r, and r = x(mod y). We know that $a^b - 1 = (a-1)\cdot(a^{b-1} + a^{b-2} + ... + 1)$, $\forall k\, k \geq 1$. Namely, $(a-1)|(a^b - 1)$.

If we choose 2^y as "a" value, we get $(2^y - 1)|(2^{y\cdot q} - 1)$.
Therefore,
$(2^x - 1)\; mod(2^y - 1) = 2^r - 1 = 2^{xmod(y)} - 1$

(b) For the Bézout's theorem, if a and b are positive integers, then there exist integers s and t such that gcd(a,b) = sa + tb

Let $gcd(2^m - 1, 2^n - 1) = k$ (1), then,
2^m-1 ≡ 0 (mod k) and 2^n-1 ≡ 0 (mod k)
2^m ≡ 1 (mod k) and 2^n ≡ 1 (mod k)

and thus,
$2^{ms+nt} \equiv 1$(mod k) $\quad \forall s, t \in Z$

As we mentioned above, ms+nt = gcd(m,n)
$2^{ms+nt} \equiv 1$(mod k) $= 2^{gcd(m,n)} - 1 \equiv 0$(mod k)
Therefore,
$gcd(2^m - 1, 2^n - 1) = 2^{gcd(m,n)} - 1$