CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 1 Solutions

YILMAZ, Mert Kaan
e2381093@ceng.metu.edu.tr

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1. (a) **linear:** We know that, a difference equation is said to be linear if it has the form: (from the reference book Luenberger - Introduction to dynamic systems)
$a_n(k)y(k+n) + a_{n-1}(k)y(k+n-1) + ... + a_0(k)y(k) = g(k)$
As we can see, the system satisfy this property. Therefore, it's linear.
time invariant: To call a system is time invariant, time shift on the input should produce the corresponding shifted output. Therefore, if the system has only constant coefficients, that system is time invariant.
Since this property has been satisfied by this system, we can say that it is time invariant system.
forced: We can see that there is a forcing term g(k) in the system. Therefore, it's forced.

   (b) **non-linear:** Similar to part a of this question, a differential equation is said to be linear if it has the form:
$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + a_0(t)x = g(t)$
As we can see, the system does not satisfy this property. Therefore, it's non-linear.
time varying: To call a system is time invariant, time shift on the input should produce the corresponding shifted output. Therefore, if the system has only constant coefficients, that system is time invariant.
Since this property has not been satisfied by this system, we can say that it is time varying system.
unforced: Because there is no forcing term g(k), we can say that it's unforced.

   (c) **non-linear:** Similar to part a of this question, a differential equation is said to be linear if it has the form:
$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + a_0(t)x = g(t)$
As we can see, the system does not satisfy this property. Therefore, it's non-linear.
time varying: To call a system is time invariant, time shift on the input should produce the corresponding shifted output. Therefore, if the system has only constant coefficients, that system is time invariant.
Since this property has not been satisfied by this system, we can say that it is time varying system.
forced: Because there is a forcing term g(k), we can say that it's forced.

2. (a) The first condition we need to check is whether A matrix is diagonal or not.
To satisfy this, we need to find two different eigen values and show that $A = SJS^{-1}$

$$det(xI - A) = \begin{vmatrix} x-2 & 2 \\ 5 & x+1 \end{vmatrix} = 0 \rightarrow (x-4)\cdot(x+3) = 0$$

$\lambda_1 = 4$ and $\lambda_2 = -3$

$$When \lambda_1 = 4, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$When \lambda_2 = -3, v_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

At this point, we need to construct S and J matrices with the values that we found.

$$S = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}, S^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{-1}{7} & \frac{1}{7} \end{bmatrix}, J = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$$

Now we can check if $A = SJS^{-1}$ is satisfied or not.

$$\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{-1}{7} & \frac{1}{7} \end{bmatrix}$$

So this condition is satisfied.
It's time to put it into relevant places on the equation:
$A = SJS^{-1} \rightarrow$ x'(t)=Ax(t)+b $= SJS^{-1}$x(t)+b $\rightarrow$ Multiply from left with S^{-1}
$S^{-1}x'(t) = (S^{-1}x(t))' \rightarrow u = S^{-1}x(t)$
$u' = S^{-1}x(t)$
Since we know that $u' = Ju + c$, $u(0) = u_0$, and x $= S^{-1}x(t), u'(t) = S^{-1}x'(t)$

$$c = \begin{bmatrix} \frac{9}{7} \\ \frac{1}{7} \end{bmatrix}, u_0 = \begin{bmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{bmatrix}$$

From these values we can get:
$u_1'(t) = 4u_1(t) + \frac{9}{7}, u_1(0) = \frac{-1}{7}$
$u_2'(t) = -3u_2(t) + \frac{1}{7}, u_2(0) = \frac{3}{7}$
It's known that the exact solution of $x' = ax + b$ is $x(t) = eat(x_0 + \frac{b}{a}) - \frac{b}{a}$
Therefore we can calculate u_1 and u_2 as:
$u_1(t) = \frac{5e^{4t}-9}{28}$ and $u_2(t) = \frac{8e^{-3t}+1}{21}$
And for the last step, we can calculate the exact solution of the equation as:

$$x(t) = S \cdot u = \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{15e^{4t}-64e^{-3t}-35}{84} \\ \frac{15e^{4t}+160e^{-3t}-7}{84} \end{bmatrix}$$

(b) It's enough to check highest order term, which in this case it's e^{4t}.
As t$\rightarrow \infty$ x(t) is exploding in the direction of eigenvector with the eigenvalue $\lambda = 4$

3. Let's say that, f(x)=x'(t)=-7x(t)+5=0
   In order to find the fixed point, we need to solve the equation f(x)=0
   For the equation above, x(t) $= \frac{5}{7}$ is a fixed point.
   There are different cases we need to check while deciding a fixed point is stable or not. Firstly, we can conclude from the equation above, f'(x)=-7, so for all x f(x) is always decreasing.
   To call that a fixed point is a stable fixed point, as t$\rightarrow \infty$ also x(t)$\rightarrow \tilde{x}$.
   For x below $\tilde{x}$, we have x'=f(x)¿0 and opposite for x above $\tilde{x}$, f(x)¡0.
   This satisfies the condition I have mentioned above (x(t)$\rightarrow \tilde{x}$). Therefore, we can say that $\tilde{x} = \frac{5}{7}$ is a stable point.

4. $x''' + t^3x'' + (t+1)x' - x = 2t + 1$
   We need to write this equation in the form of: x' = Ax + b
   For this question, different x(t) variables should be represented as u(t) variables such that:
   $u_1(t) = x(t)$,
   $u_2(t) = x'(t)$,
   $u_3(t) = x''(t)$
   At this point we need to rearrange the equation a little bit.
   $x''' = 2t + 1 - t^3x'' - (t+1)x' + x$
   From these, we can drive:
   $u_1' = u_2$
   $u_2' = u_3$
   $u_3' = 2t + 1 - t^3u_3 - (t+1)u_2 + u_1$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(t+1) & -t^3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2t+1 \end{bmatrix}$$

5. (a)
   (b)
   (c)