CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 2 Solutions

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1. (a) We can condense all the given information from the question text into a transition matrix P, namely,

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

(b) We need to compute different possibilities here.
We have different state transitions for the given case:
i) unskilled laborer $\rightarrow$ unskilled laborer $\rightarrow$ professional man
ii) unskilled laborer $\rightarrow$ skilled laborer $\rightarrow$ professional man
iii) unskilled laborer $\rightarrow$ professional man $\rightarrow$ professional man

Let's model professionals to be the first state, skilled laborers to be the second state, and unskilled laborers to be the third state.
P(i) = P(3→3→1) = P(3→3)·P(3→1) = 0.5·0.1=0.05
P(ii) = P(3→2→1) = P(3→2)·P(2→1) = 0.4·0.2=0.02
P(iii) = P(3→1→1) = P(3→1)·P(1→1) = 0.1·0.7=0.07

At the end, P(3→j→1) = P(i) + P(ii) + P(iii) = 0.05 + 0.02 + 0.07 = 0.14

(c) Just like in the previous question we need to compute different possibilities here.
We have different state transitions for the given case:
i) professional man $\rightarrow$ unskilled laborer $\rightarrow$ professional man
ii) professional man $\rightarrow$ skilled laborer $\rightarrow$ professional man
iii) professional man $\rightarrow$ professional man $\rightarrow$ professional man

Let's model professionals to be the first state, skilled laborers to be the second state, and unskilled laborers to be the third state.
P(i) = P(1→3→1) = P(1→3)·P(3→1) = 0.1·0.1=0.01
P(ii) = P(1→2→1) = P(1→2)·P(2→1) = 0.2·0.2 = 0.04
P(iii) = P(1→1→1) = P(1→1)·P(1→1) = 0.7·0.7=0.49

At the end, P(1→j→1) = P(i) + P(ii) + P(iii) = 0.01 + 0.04 + 0.49 = 0.54

(d) Assume we have started at the state one, namely with a professional man. We will use general formula for calculations $(p(m) = p(0)P^m)$.
Therefore we can write, $(p(100) = p(0)P^{100}) = [1, 0, 0]P^{100})$.

I have used octave online to calculate big matrix multiplications here with these code lines:
a = [1, 0, 0]
b = [0.7 0.2 0.1; 0.2 0.6 0.2; 0.1 0.4 0.5]
a * b$\hat{1}$00
and the result is: [0.3529, 0.4118, 0.2353] which means if we start with a professional man 100th generation grandson is going to be professional man with probability 0.3529, skilled laborer with probability 0.4118, and lastly unskilled laborer with probability 0.2353.

2. (a) Because it has three states, we will get three terms in $M_c = [B, AB, A^2B]$
We need to calculate them first:

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$A^2B = A \cdot (AB) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The rank of the matrix is 3, so we can say that it's controllable.

(b)

3. (a) The definition tells us that a discrete-time system x(k + 1) = Ax(k), y(k) = C x(k) is said to be observable if there is a finite index N such that knowing the output sequence y(0), y(1), . . . , y(N-1) is sufficient to determine the initial state x(0).
Therefore, in this question, we will check if y(0) and y(1) can determine x(0).

We have given:

$$x(k+1) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x(k)$$

$x_1(1) = x_3(0)$
$x_2(1) = -2x_1(0) - x_3(0)$
$x_3(1) = x_2(0)$

y(0) $= -2x_2(0) - 4x_3(0)$
y(1) $= -2x_2(1) - 4x_3(1)$

(b) We are given,

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} ; C = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix};$$

And with these we want to calculate observability matrix:

$$CA = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix};$$

$$CA^2 = (CA) \cdot A = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 8 \end{bmatrix};$$

$$M_0 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix};$$

Since the matrix M_0 is rank 3/full rank, we can say that the system is observable.

4. $\dot{x} = 3x^2 - 3x^3$
We are given that the derivative of x(t) ($\frac{dx}{dt}$) is $3x^2 - 3x^3$. And we also know from the definition, a fixed point of a dynamical system is a state vector $\tilde{x}$ with the property that if the system is ever in the state $\tilde{x}$, it will remain in that state for all the time.
Therefore, if we are looking for fixed points for this equation, we can say:
$\dot{x} = (\frac{dx}{dt}) = 3x^2 - 3x^3 = 3 \cdot x^2 \cdot (1-x) = 0$
$\tilde{x}_1 = 0$, $\tilde{x}_2 = 1$

The formula for linearization is:
f(x) $\approx$ f($\tilde{x}$) + $\frac{d}{dx} f(\tilde{x})(x - \tilde{x})$
Since $\tilde{x}$ is a fixed point, $f(\tilde{x}) = 0$.

i) For $\tilde{x}_1 = 0$
f(x) = 0(x-0) = 0, coefficient of x = 0, so it has test fails.

ii) For $\tilde{x}_2 = 1$
f(x) = -3(x-1) = -3x + 3, coefficient of x ¡ 0, so it's stable.