$= \sin 36^o$

$$\boxed{\sin 144^o = \sqrt{\frac{10 - 2\sqrt{5}}{16}}} \quad \text{Ans.}$$

$\cos 144^o = \cos(180^o - 36^o) = -\cos 36^o$

$$\boxed{\cos 144^o = -\left(\frac{1+\sqrt{5}}{4}\right)} \quad \text{Ans.}$$

Next $\mathbf{\cos 36^o . \cos 72^o . \cos 108^o . \cos 144^o = \frac{1}{16}}$

L.H.S. $= \cos 36^o \cos 72^o \cos(180^o - 72^o) \cos 144$

$= \cos 36^o \cos 72^o (-\cos 72^o) \cos 144^o$

$$= \left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\left(-\left(\frac{\sqrt{5}-1}{4}\right)\right)\left(\frac{1+\sqrt{5}}{4}\right)$$

$$= \left(\frac{1+\sqrt{5}}{4}\right)^2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left[\frac{(1+\sqrt{5})(\sqrt{5}-1)}{16}\right]^2$$

$$= \left[\frac{(\sqrt{5})^2 - (1)^2}{16}\right]^2 = \left(\frac{5-1}{16}\right)^2$$

$$= \left(\frac{4}{16}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{R.H.S.}$$

Hence proved.

EXERCISE 10.4

Q.1 Express the following product as sums and differences

(i) $2 \sin 3\theta \cos\theta$ **(Lahore Board 2006)**

(ii) $2 \cos 5\theta \sin 3\theta$

(iii) $\sin 5\theta \cos 2\theta$ **(Gujranwala Board 2004)**

(iv) $2 \sin 7\theta \sin 2\theta$

(v) $\cos(x + y) \sin(x - y)$ **(vi)** $\cos(2x + 30) \cos(2x - 30)$

(vii) $\sin 12^o \sin 46^o$ **(viii)** $\sin(x + 45^o) \sin(x - 45^o)$

Solutions:

(i) **$2 \sin 3\theta \cos\theta$**

$= \sin(3\theta + \theta) + \sin(3\theta - \theta)$ $\quad (\because 2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta))$

$= \sin 4\theta + \sin 2\theta$ Ans.

(ii) **$2 \cos 5\theta \sin 3\theta$**

$= \sin(5\theta + 3\theta) - \sin(5\theta - 3\theta)$ $\quad (\because 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta))$

$= \sin 8\theta - \sin 2\theta$ Ans.

(iii) **$\sin 5\theta \cos 2\theta$** **(Gujranwala Board 2004)**

multiple & divide by 2

$= \frac{1}{2}[2\sin 5\theta \cos 2\theta]$

$= \frac{1}{2}[\sin(5\theta + 2\theta) + \sin(5\theta - 2\theta)]$

$= \frac{1}{2}[\sin 7\theta + \sin 3\theta]$ Ans.

(iv) **$2 \sin 7\theta \sin 2\theta$**

$= -[-2\sin 7\theta \sin 2\theta]$

$= -\cos[(7\theta + 2\theta) - \cos(7\theta - 2\theta)]$ $\quad (\because -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

$= -[\cos 9\theta - \cos 5\theta]$ Ans.

(v) **$\cos(x + y)\sin(x - y)$**

multiply & dividing by 2

$= \frac{1}{2}[2\cos(x+y)\sin(x-y)]$ $\quad (\because 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta))$

$= \frac{1}{2}\left[\sin(x + y + x - y) - \sin(x + y - x + y\right]$

$= \frac{1}{2}[\sin(2x) - \sin 2y]$ Ans.

(vi) **$\cos(2x + 30^o)\cos(2x - 30^o)$**

multiply & divide by 2

$= \frac{1}{2}[2\cos(2x + 30^o)\cos(2x - 30^o)]$ $\quad (\because 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta))$

$= \frac{1}{2}\left[\cos(2x + 30^o + 2x - 30^o) + \cos(2x + 30^o - 2x + 30^o)\right]$

$= \frac{1}{2}[\cos 4x + \cos 60^o]$ Ans.

(vii) $\sin 12^o \sin 46^o$

multiply & divide by -2

$= \frac{-1}{2}[-2\sin 12^o \sin 46^o)$ $\left(\because\ -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)\right)$

$= \frac{-1}{2}[\cos(12^o + 46^o) - \cos(12^o - 46^o)]$

$= \frac{-1}{2}[\cos 58^o - \cos(-34^o)]$

$= \frac{-1}{2}[\cos 58^o - \cos 34^o]$ Ans. $(\because\ \cos(-\theta) = \cos\theta)$

(viii) $\sin(x + 45^o)\sin(x - 45^o)$

multiply & dividing by -2

$= \frac{-1}{2}[-2\sin(x + 45^o)\sin(x - 45^o)]$ $\left(\because\ -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)\right)$

$= \frac{-1}{2}[\cos(x + 45^o + x - 45^o) - \cos(x + 45^o - x + 45^o)]$

$= -\frac{1}{2}[\cos 2x - \cos 90^o]$

Q.2 Express the following sums and differences as product

(i) $\sin 5\theta + \sin 3\theta$ (Lahore Board 2006,2007)

(ii) $\sin 8\theta - \sin 4\theta$

(iii) $\cos 6\theta + \cos 3\theta$

(iv) $\cos 7\theta - \cos\theta$ (Lahore Board 2009)

(v) $\cos 12 + \cos 48$ (Lahore Board 2010)

(vi) $\sin(x + 30) + \sin(x - 30)$

Solution:

(i) $\sin 5\theta + \sin 3\theta$

$= 2\sin\frac{5\theta + 3\theta}{2}\cos\frac{5\theta - 3\theta}{2}$ $\left(\therefore\ \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}\right)$

$= 2\sin 4\theta\cos\theta$ Ans.

(ii) $\sin 8\theta - \sin 4\theta$

$$= 2\cos\frac{8\theta + 4\theta}{2}\sin\frac{8\theta - 4\theta}{2} \qquad \because \left(\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}\right)$$

$= 2\cos 6\theta \sin 2\theta$ Ans.

(iii) $\cos 6\theta + \cos 3\theta$

$$= 2\cos\frac{6\theta + 3\theta}{2}\cos\frac{6\theta - 3\theta}{2} \qquad \because \left(\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}\right)$$

$= 2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$ Ans.

(iv) $\cos 7\theta - \cos\theta$ **(Lahore Board 2009)**

$$= -2\sin\frac{7\theta + \theta}{2} . \sin\frac{7\theta - \theta}{2} \qquad \because \left(\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}\right)$$

$= -2\sin 4\theta \sin 3\theta$ Ans.

(v) $\cos 12^o + \cos 48^o$ **(Lahore Board 2010)**

$$= 2\cos\frac{12^o + 48^o}{2}\cos\frac{12^o - 48^o}{2}$$

$$= 2\cos\frac{60^o}{2}\cos\frac{-36^o}{2} \qquad (\because \cos(-\theta) = \cos\theta)$$

$= 2\cos 30^o \cos 18^o$ Ans.

(vi) $\sin(x + 30^o) + \sin(x - 30^o)$

$$= 2\sin\frac{x + 30^o + x - 30^o}{2}\cos\frac{x + 30^o - x + 30^o}{2}$$

$$= 2\sin\frac{2x}{2}\cos\frac{60^o}{2}$$

$= 2\sin x \cos 30^o$ Ans.

Q.3 Prove the following identities

(i) $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

(ii) $\dfrac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

(iii) $\dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\left(\dfrac{\alpha - \beta}{2}\right)\cot\left(\dfrac{\alpha + \beta}{2}\right)$ **(Lahore Board 2007)**

Solution:

(i) $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

$$\text{L.H.S.} = \frac{\sin 3x - \sin x}{-(\cos 3x - \cos x)} \qquad \left(\because \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{-\left(-2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}\right)} \left(\because \cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\cos 2x \sin x}{2\sin 2x \sin x} = \frac{\cos 2x}{\sin 2x} = \cot 2x = \text{R.H.S.}$$

Hence proved.

(ii) $\dfrac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

$$\text{L.H.S.} = \frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} \qquad \left(\because \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\sin\frac{8x+2x}{2}\cos\frac{8x-2x}{2}}{2\cos\frac{8x+2x}{2}\cos\frac{8x-2x}{2}} \qquad \left(\because \cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\sin 5x\cos 3x}{2\cos 5x\cos 3x} = \frac{\sin 5x}{\cos 5x} = \tan 5x = \text{R.H.S.}$$

Hence proved.

(iii) $\dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\dfrac{\alpha-\beta}{2}\cot\dfrac{\alpha+\beta}{2}$ **(Lahore Board 2007)**

$$\text{L.H.S.} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} \qquad \left(\because \sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)\right)$$

$$= \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} \qquad \left(\because \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right)$$

$$= \cot\frac{\alpha+\beta}{2} . \tan\frac{\alpha-\beta}{2} = \text{R.H.S.}$$

Hence the proof.

Q.4 Prove that

(i) $\cos 20^o + \cos 100^o + \cos 140^o = 0$ **(Lahore Board 2008)**

(ii) $\sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2}\cos 2\theta$ **(Lahore Board 2005)**

(iii) $\frac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

Solution:

(i) $\cos 20^o + \cos 100^o + \cos 140^o = 0$

$$\begin{aligned}
\text{L.H.S.} &= \cos 20^o + \cos 100^o + \cos 140^o \\
&= [\cos 140^o + \cos 20^o] + \cos 100^o \left(\because \cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right) \\
&= 2\cos\frac{140^o + 20^o}{2}\cos\frac{140^o - 20^o}{2} + \cos 100^o \\
&= 2\cos 80^o \cos 60^o + \cos 100^o \\
&= 2\cos 80^o \frac{1}{2} + \cos 100^o \\
&= \cos 80^o + \cos 100^o \\
&= 2\cos\frac{80^o + 100^o}{2}\cos\frac{80^o - 100^o}{2} \\
&= 2\cos 90^o \cos(-10^o) \\
&= 2 \times 0 \times \cos 10^o \\
&= 0 = \text{R.H.S.}
\end{aligned}$$

Hence proved.

(ii) $\sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2}\cos 2\theta$

$$\text{L.H.S.} = \sin\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) \qquad \left(\begin{array}{l}\because \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \quad\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{array}\right)$$

$$\begin{aligned}
&= \left(\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\right)\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right) \\
&= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) \\
&= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) \\
&= \frac{1}{2}(\cos^2\theta - \sin^2\theta) \\
&= \frac{1}{2}\cos 2\theta = \text{R.H.S.}
\end{aligned}$$

Hence proved.

(iii) $$\frac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$$

$$\text{L.H.S.} = \frac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} \quad \left(\begin{array}{l} \because \ \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \\ \cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \end{array}\right)$$

$$= \frac{[\sin 7\theta + \sin\theta] + [\sin 5\theta + \sin 3\theta]}{[\cos 7\theta + \cos\theta] + [\cos 5\theta + \cos 3\theta]}$$

$$= \frac{2\sin\frac{(7\theta + \theta)}{2}\cos\frac{7\theta - \theta}{2} + 2\sin\frac{5\theta + 3\theta}{2}\cos\frac{5\theta - 3\theta}{2}}{2\cos\frac{7\theta + \theta}{2}\cos\frac{7\theta - \theta}{2} + 2\cos\frac{5\theta + 3\theta}{2}\cos\frac{5\theta - 3\theta}{2}}$$

$$= \frac{2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta}$$

$$= \frac{2\sin 4\theta(\cos 3\theta + \cos\theta)}{2\cos 4\theta(\cos 3\theta + \cos\theta)}$$

$$= \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta = \text{R.H.S.}$$

Hence proved.

Q.5 Prove that

(i) $\cos 20^o \cos 40^o \cos 60^o \cos 80^o = \frac{1}{16}$

(Gujranawala Board 2004, Lahore Board 2008)

(ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}$

(iii) $\sin 10^o \sin 30^o \sin 50^o \sin 70^o = \frac{1}{16}$ **(Lahore Board 2011)**

Solution:

(i) $\cos 20^o \cos 40^o \cos 60^o \cos 80^o = \frac{1}{16}$

$$\text{L.H.S.} = \cos 20^o \cos 40^o \cos 60^o \cos 80^o$$

$$= \frac{1}{2}\cos 40^o \cos 20^o \cos 80^o$$

$$= \frac{1}{2}[\cos 40^o \cos 20^o]\cos 80^o \qquad (\because \ 2\cos\alpha\cos\beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$= \frac{1}{2}\left[\frac{2}{2}\cos 40^\circ \cos 20^\circ\right]\cos 80^\circ$$

$$= \frac{1}{2} \times \frac{1}{2}[\cos(40^o + 20^o) + \cos(40^o - 20^o)]\cos 80^o$$

$$= \frac{1}{4}[\cos 60^o + \cos 20^o]\cos 80^o$$

$$= \frac{1}{4}\left[\frac{1}{2} + \cos 20^o\right]\cos 80^o$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^o + \cos 80^o \cos 20^o\right] \quad (\because\ 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta))$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^\circ + \frac{2}{2}\cos 80^\circ \cos 20^\circ\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^o + \frac{1}{2}\{\cos(80^o + 20^o) + \cos(80^o - 20^o)\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80^o + \frac{1}{2}\cos 100^o + \frac{1}{2}\cos 60^o\right]$$

$$= \frac{1}{8}[\cos 80^o + \cos 100^o + \cos 60^o]$$

$$= \frac{1}{8}[\cos 80^\circ + \cos(180^\circ - 80^\circ) + \cos 60^\circ] \quad (\because\ \cos(\pi - \theta) = -\cos\theta)$$

$$= \frac{1}{8}[\cos 80^\circ - \cos 80^\circ + \cos 60^\circ]$$

$$= \frac{1}{8}[\cos 60^\circ]$$

$$= \frac{1}{8}\left[\frac{1}{2}\right] = \frac{1}{16} = \text{R.H.S.}$$

Hence proved.

(ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}$ **(Lahore Board 2011)**

$$\text{L.H.S.} = \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}$$

$$= \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\frac{\sqrt{3}}{2}\sin\frac{4\pi}{9}$$

$$= \frac{\sqrt{3}}{2}\sin 20^o \sin 40^o \sin 80^o$$

$$= \frac{\sqrt{3}}{2}[\sin 40^o \sin 20^o]\sin 80^\circ \quad (\because\ -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta))$$

$$= \frac{\sqrt{3}}{2}\left[\frac{-2}{-2}\sin40° \sin20°\right]\sin80°$$

$$= \frac{-\sqrt{3}}{2}\left[\frac{1}{2}\{\cos(40^o + 20^o) - \cos(40^o - 20^o)\}\sin 80^o\right]$$

$$= \frac{-\sqrt{3}}{4}[\{\cos 60^o - \cos 20^o\}\sin 80^o]$$

$$= \frac{-\sqrt{3}}{4}\left[\left\{\frac{1}{2} - \cos20°\right\}\sin80°\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80° - \sin80° \cos20°\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80° - \frac{2}{2}\sin80° \cos20°\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80° - \frac{1}{2}\sin(80° + 20°) + \sin(80° - 20°)\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80° - \frac{1}{2}\sin100° - \frac{1}{2}\sin60°\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80° - \frac{1}{2}\sin(180° - 80°) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right]$$

$$= \frac{-\sqrt{3}}{4}\left[\frac{1}{2}\sin80° - \frac{1}{2}\sin 80° - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{-\sqrt{3}}{4}\left(-\frac{\sqrt{3}}{4}\right)$$

$$= \frac{3}{16} = \text{R.H.S}$$

Hence proved.

(iii) $\mathbf{\sin 10^o \sin 30^o \sin 50^o \sin 70^o = \frac{1}{16}}$

L.H.S. $= \sin 10^o \sin 30^o \sin 50^o \sin 70^o$

$$= \frac{1}{2}\sin 10^o \sin 50^o \sin 70^o$$

$$= \frac{-1}{2 \times 2}[(-2\sin 10^o \sin 50^o) . \sin 70^o]$$

$$= \frac{-1}{4}[\{\cos(10^o + 50^o) - \cos(10^o - 50^o)\}\sin 70^o]$$

$$= \frac{-1}{4}[(\cos 60^o - \cos 40^o)\sin 70^o]$$

$$= \frac{-1}{8}[\,2\cos 60^o \sin 70^o - 2\cos 40^o \sin 70^o\,] \quad \because 2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$= \frac{-1}{8}\left[\,2\,\frac{1}{2}\sin 70^o - \{\,\sin(40^o+70^o) - \sin(40^o-70^o)\,\}\,\right]$$

$$= \frac{-1}{8}[\,\sin 70^o - \{\,\sin 110^o - \sin(-30^o)\,\}\,]$$

$$= \frac{-1}{8}[\,\sin 70^o - \{\,\sin 110^o + \sin 30^o\}\,]$$

$$= \frac{-1}{8}[\,\sin(180^o - 110^o) - \sin 110^o - \sin 30^o\,]$$

$$= \frac{-1}{8}[\,\sin 110^o - \sin 110^o - \sin 30^o\,]$$

$$= \frac{-1}{8}\left[-\frac{1}{2}\right]$$

$$= \frac{1}{16} = \text{R.H.S.}$$

Hence proved.