Chapter 4

QUADRATIC EQUATIONS

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A Quadratic Equation in x is an equation that can be written in the form of $ax^2 + bx + c = 0$; where a, b, c are real numbers and $a \neq 0$, for example

$x^2 - 7x + 10 = 0$; $a = 1$, $b = -7$, $c = 10$

$x^2 = 4$; $a = 1$, $b = 0$, $c = -4$

Another name for a quadratic equation in x is 2nd degree polynomial.

There are three basic techniques for solving a quadratic equation.

(i) By Factorization

(ii) By Completing Squares

(iii) By Applying the Quadratic Formula

EXERCISE 4.1

Solve the following equation by factorization:

Q.1 $3x^2 + 4x + 1 = 0$

Solution:

$3x^2 + 4x + 1 = 0$

$\Rightarrow \quad 3x^2 + 3x + x + 1 = 0$

$\Rightarrow \quad 3x(x + 1) + 1(x + 1) = 0$

$\Rightarrow \quad (3x + 1)(x + 1) = 0$

Either $(3x + 1) = 0$ or $x + 1 = 0$

$3x = -1$ $\qquad$ $x = -1$

$x = -\frac{1}{3}$ $\qquad$ $x = -1$

Hence the solution set $= \left\{-\frac{1}{3}, -1\right\}$

Q.2 $x^2 + 7x + 12 = 0$ **(Lahore Board 2007)**

Solution:

$x^2 + 7x + 12 = 0$

$\Rightarrow \quad x^2 + 4x + 3x + 12 = 0$

$\Rightarrow \quad x(x+4) + 3(x+4) = 0$

$\Rightarrow \quad x(x+4) + (x+3) = 0$

Either $x + 4 = 0$ or $x + 3 = 0$

$x = -4$ $x = -3$

Hence the solution set $= \{-4, -3\}$

Q.3 $9x^2 - 12x - 5 = 0$

Solution:

$9x^2 - 12x - 5 = 0$

$\Rightarrow \quad 9x^2 - 15x + 3x - 5 = 0$

$\Rightarrow \quad 3x(3x-5) + 1(3x-5) = 0$

$\Rightarrow \quad (3x-5)(3x+1) = 0$

$\Rightarrow$ Either $3x - 5 = 0$ or $3x + 1 = 0$

$\Rightarrow \quad 3x = 5$ $3x = -1$

$\Rightarrow \quad x = \frac{5}{3}$ $x = -\frac{1}{3}$

Hence the solution set $= \left\{-\frac{1}{3}, \frac{5}{3}\right\}$

Q.4 $x^2 - x = 2$

Solution:

$x^2 - x = 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

$\Rightarrow$ Either $x - 2 = 0$ or $x + 1 = 0$

$\Rightarrow \quad x = 2,$ $x = -1$

Hence the solution set $= \{-1, 2\}$

Q.5 $x(x+7) = (2x-1)(x+4)$ **(Lahore Board 2011)**

Solution:

$x(x+7) = (2x-1)(x+4)$

$\Rightarrow \quad x^2 + 7x = 2x^2 + 8x - x - 4$

$\Rightarrow \quad 2x^2 + 8x - x - 4 - x^2 - 7x = 0$

$\Rightarrow \quad x^2 - 4 = 0$

$\Rightarrow \quad (x^2) - (2)^2 = 0$

$\Rightarrow \quad (x+2)(x-2) = 0$

$\Rightarrow$ Either $x + 2 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = -2,$ $x = 2$

Hence the solution set $= \{-2, 2\}$

Q.6 $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{5}{2}; \quad x \neq -1, 0$

Solution:

$$\frac{x}{x+1}+\frac{x+1}{x}=\frac{5}{2}$$

$$\Rightarrow \quad \frac{x^2+(x+1)^2}{x(x+1)}=\frac{5}{2}$$

$$\Rightarrow \quad \frac{x^2+x^2+2x+1}{x^2+x}=\frac{5}{2}$$

$$\Rightarrow \quad \frac{2x^2+2x+1}{x^2+x}=\frac{5}{2}$$

$$\Rightarrow \quad 5(x^2+x) = 2(2x^2+2x+1)$$

$$\Rightarrow \quad 5x^2+5x = 4x^2+4x+2$$

$$\Rightarrow \quad 5x^2+5x-4x^2-4x-2 = 0$$

$$\Rightarrow \quad x^2+x-2 = 0$$

$$\Rightarrow \quad x^2+2x-x-2 = 0$$

$$\Rightarrow \quad x(x+2)-1(x+2) = 0$$

$$\Rightarrow \quad (x+2)(x-1) = 0$$

$$\Rightarrow \quad \text{Either} \quad x+2 = 0 \quad \text{or} \quad x-1 = 0$$

$$\Rightarrow \quad x = -2 \quad \text{or} \quad x = 1$$

Hence the solution set $= \{-2, 1\}$

Q.7 $\dfrac{1}{x+1}+\dfrac{2}{x+2}=\dfrac{7}{x+5}; \quad x \neq -1, -2, -5$

Solution:

$$\frac{1}{x+1}+\frac{2}{x+2}=\frac{7}{x+5}$$

$$\Rightarrow \quad \frac{x+2+2(x+1)}{(x+1)(x+2)}=\frac{7}{x+5}$$

$$\Rightarrow \quad \frac{x+2+2x+2}{x^2+2x+x+2}=\frac{7}{x+5}$$

$$\Rightarrow \quad \frac{3x+4}{x^2+3x+2}=\frac{7}{x+5}$$

$$\Rightarrow \quad 7(x^2+3x+2) = (x+5)(3x+4)$$

$\Rightarrow \quad 7x^2 + 21x + 14 = 3x^2 + 4x + 15x + 20$

$\Rightarrow \quad 7x^2 + 21x + 14 - 3x^2 - 4x - 15x - 20 = 0$

$\Rightarrow \quad 4x^2 + 2x - 6 = 0$

$\Rightarrow \quad 2(2x^2 + x - 3) = 0$

$\Rightarrow \quad 2x^2 + x - 3 = 0$

$\Rightarrow \quad 2x^2 + 3x - 2x - 3 = 0$

$\Rightarrow \quad x(2x + 3) - 1(2x + 3) = 0$

$\Rightarrow \quad (2x + 3)(x - 1) = 0$

$\Rightarrow$ Either $2x + 3 = 0$ or $x - 1 = 0$

$\Rightarrow \quad 2x = -3 \qquad x = 1$

$\Rightarrow \quad x = -\frac{3}{2}$

Hence the solution set $= \left\{-\frac{3}{2}, 1\right\}$

Q.8 $\frac{a}{ax-1} + \frac{b}{bx-1} = a + b; \quad x \neq \frac{1}{a}, \frac{1}{b}$ **(Lahore Board 2008)**

Solution:

$$\frac{a}{ax-1} + \frac{b}{bx-1} = a + b$$

$$\Rightarrow \quad \frac{a}{ax-1} - b + \frac{b}{bx-1} - a = 0$$

$$\Rightarrow \quad \frac{a - b(ax-1)}{ax-1} + \frac{b - a(bx-1)}{bx-1} = 0$$

$$\Rightarrow \quad \frac{a - abx + b}{ax-1} + \frac{b - abx + a}{bx-1} = 0$$

$$\Rightarrow \quad (a + b - abx)\left[\frac{1}{ax-1} + \frac{1}{bx-1}\right] = 0$$

$$\Rightarrow \quad (a + b - abx)\left[\frac{bx - 1 + ax - 1}{(ax-1)(bx-1)}\right] = 0$$

$$\Rightarrow \quad (a + b - abx)\left[\frac{ax + bx - 2}{(ax-1)(bx-1)}\right] = 0$$

$\Rightarrow$ Either $a + b - abx = 0$ or $\frac{ax + bx - 2}{(ax-1)(bx-1)} = 0$

$\Rightarrow$ $abx = a + b$ or $ax + bx - 2 = 0$

$\Rightarrow$ $x = \frac{a+b}{ab}$ or $x(a+b) - 2 = 0$

$x(a+b) = 2$

$x = \frac{2}{a+b}$

Hence the solution set $= \left\{ \frac{2}{a+b}, \frac{a+b}{ab} \right\}$

Solve the following equations by completing the square:

Q.9 $x^2 - 2x - 899 = 0$ **(Lahore Board 2007)**

Solution:

$x^2 - 2x - 899 = 0$

$\Rightarrow$ $x^2 - 2x = 899$

add $\left(\frac{2}{2}\right)^2 = (1)^2$ on both sides

$\Rightarrow$ $x^2 - 2x + (1)^2 = 899 + (1)^2$

$\Rightarrow$ $(x-1)^2 = 900$

$\Rightarrow$ $(x-1)^2 = (30)^2$

$\Rightarrow$ $(x-1) = \pm 30$

$\Rightarrow$ Either $x - 1 = 30$ or $x - 1 = -30$

$\Rightarrow$ $x = 31$ or $x = -29$

Hence the solution set $= \{-29, 31\}$

Q.10 $x^2 + 4x - 1085 = 0$

Solution:

$x^2 + 4x - 1085 = 0$

$\Rightarrow$ $x^2 + 4x = 1085$

add $\left(\frac{4}{2}\right)^2 = (2)^2$ to both sides

$\Rightarrow$ $x^2 + 4x + (2)^2 = 1085 + (2)^2$

$\Rightarrow$ $(x+2)^2 = 1089$

$\Rightarrow$ $(x+2)^2 = (33)^2$

$\Rightarrow \quad x + 2 = \pm 33$

$\Rightarrow$ Either $x + 2 = 33$ or $x + 2 = -33$

$x = 31$ or $x = -35$

Hence the solution set $= \{-35, 31\}$

Q.11 $\mathbf{x^2 + 6x - 567 = 0}$

Solution:

$x^2 + 6x - 567 = 0$

$\Rightarrow \quad x^2 + 6x = 567$

add $\left(\frac{6}{2}\right)^2 = (3)^2$ to both sides

$\Rightarrow \quad x^2 + 6x + (3)^2 = 567 + (3)^2$

$\Rightarrow \quad (x + 3)^2 = 567$

$\Rightarrow \quad (x + 3)^2 = (24)^2$

$\Rightarrow \quad x + 3 = \pm 24$

$\Rightarrow$ Either $x + 3 = 24$ or $x + 3 = -24$

$\Rightarrow \quad x = 21$ or $x = -27$

Hence the solution set $= \{-27, 21\}$

Q.12 $\mathbf{x^2 - 3x - 648 = 0}$

Solution:

$x^2 - 3x - 648 = 0$

$\Rightarrow \quad x^2 - 3x = 648$

add $\left(\frac{3}{2}\right)^2$ to both sides

$\Rightarrow \quad x^2 - 3x + \left(\frac{3}{2}\right)^2 = 648 + \left(\frac{3}{2}\right)^2$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = 648 + \frac{9}{4}$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = \frac{2601}{4}$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = \left(\frac{51}{2}\right)^2$

$\Rightarrow \quad x - \frac{3}{2} = \pm\frac{51}{2}$

$\Rightarrow \quad$ Either $x - \frac{3}{2} = \frac{51}{2}$ or $x - \frac{3}{2} = -\frac{51}{2}$

$\Rightarrow \quad x = \frac{51}{2} + \frac{3}{2}$ or $x = \frac{-51}{2} + \frac{3}{2}$

$\Rightarrow \quad x = \frac{54}{2} = 27$ or $x = \frac{-48}{2} = -24$

Hence the solution set $= \{-24, 27\}$

Q.13 $\mathbf{x^2 - x - 1806 = 0}$

Solution:

$x^2 - x - 1806 = 0$

$\Rightarrow \quad x^2 - x = 1806$

add $\left(\frac{1}{2}\right)^2$ to both sides

$\Rightarrow \quad x^2 - x + \left(\frac{1}{2}\right)^2 = 1806 + \left(\frac{1}{2}\right)^2$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = 1806 + \frac{1}{4}$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = \frac{7225}{4} = \left(\frac{85}{2}\right)^2$

$\Rightarrow \quad x - \frac{1}{2} = \pm\frac{85}{2}$

$\Rightarrow \quad$ Either $x - \frac{1}{2} = \frac{85}{2}$ or $x - \frac{1}{2} = -\frac{85}{2}$

$\Rightarrow \quad x = \frac{85}{2} + \frac{1}{2}$ or $x = \frac{-85}{2} + \frac{1}{2}$

$\Rightarrow \quad x = \frac{86}{2} = 43$ or $x = -\frac{84}{2} = -42$

Hence the solution set $= \{-42, 43\}$

Q.14 $\mathbf{2x^2 + 12x - 110 = 0}$

Solution:

$2x^2 + 12x - 110 = 0$

$\Rightarrow \quad 2(x^2 + 6x - 55) = 0$

$\Rightarrow \quad x^2 + 6x - 55 = 0$

$\Rightarrow \quad x^2 + 6x = 55$

add $\left(\frac{6}{2}\right)^2 = (3)^2$ to both sides

$\Rightarrow \quad x^2 + 6x + (3)^2 = 55 + (3)^2$

$\Rightarrow \quad (x + 3)^2 = 55 + 9$

$\Rightarrow \quad (x + 3)^2 = 64$

$\Rightarrow \quad (x + 3)^2 = (8)^2$

$\Rightarrow \quad (x + 3) = \pm 8$

$\Rightarrow$ Either $x + 3 = 8$ or $x + 3 = -8$

$x = 5$ or $x = -11$

Hence the solution set $= \{-11, 5\}$

Find roots of the following equations by using quadratic formula.

Q.15 $5x^2 - 13x + 6 = 0$

Solution:

$5x^2 - 13x + 6 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 5, \quad b = -13, \quad c = 6$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(5)(6)}}{2(5)}$$

$$x = \frac{13 \pm \sqrt{169 - 120}}{10} = \frac{13 \pm \sqrt{49}}{10} = \frac{13 \pm 7}{10}$$

$\Rightarrow$ Either $x = \frac{13 + 7}{10}$ or $x = \frac{13 - 7}{10}$

$\Rightarrow \quad x = \frac{20}{10}$ or $x = \frac{6}{10}$

$\Rightarrow \quad x = 2$ or $x = \frac{3}{5}$

Hence the solution set $= \left\{\frac{3}{5}, 2\right\}$

Q.16 $4x^2 + 7x - 1 = 0$

Solution:

$4x^2 + 7x - 1 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 4, \quad b = 7, \quad c = -1$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(4)(-1)}}{2(4)} = \frac{-7 \pm \sqrt{49 + 16}}{8} = \frac{-7 \pm \sqrt{65}}{8}$$

$\Rightarrow$ Either $x = \frac{-7 + \sqrt{65}}{8}$ or $x = \frac{-7 - \sqrt{65}}{8}$

Hence the solution set $= \left\{ \frac{-7 + \sqrt{65}}{8}, \frac{-7 - \sqrt{65}}{8} \right\}$

Q.17 $15x^2 + 2ax - a^2 = 0$

Solution:

$15x^2 + 2ax - a^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 15 \quad b = 2a \quad c = -a^2$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2a \pm \sqrt{(2a)^2 - 4(15)(-a^2)}}{2(15)}$$

$$= \frac{-2a \pm \sqrt{4a^2 + 60a^2}}{30} = \frac{-2a \pm \sqrt{64a^2}}{30} = \frac{-2a \pm 8a}{30}$$

$\Rightarrow$ Either $x = \frac{-2a + 8a}{30}$ or $\frac{-2a - 8a}{30}$

$\Rightarrow \quad x = \frac{6}{30}$ or $\frac{-10a}{30}$

$\Rightarrow \quad x = \frac{a}{5}$ or $-\frac{a}{3}$

Hence the solution set $= \left\{ -\frac{a}{3}, \frac{a}{5} \right\}$

Q.18 $16x^2 + 8x + 1 = 0$

Solution:

$16x^2 + 8x + 1 = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = 16, \quad b = 8, \quad c = 1$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-8 \pm \sqrt{(8)^2 - 4(16)(1)}}{2(16)} = \frac{-8 \pm \sqrt{64 - 64}}{32} = \frac{-8 \pm 0}{32} = \frac{-8}{32}$$

$$\Rightarrow \quad x = -\frac{1}{4}$$

Hence the solution set $= \left\{-\frac{1}{4}\right\}$

Q.19 $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

Solution:

$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

$$\Rightarrow \quad x^2 - bx - ax + ab + x^2 - cx - bx + bc + x^2 - ax - cx + ac = 0$$

$$\Rightarrow \quad 3x^2 - 2ax - 2bx - 2cx + ab + bc + ac = 0$$

$$\Rightarrow \quad 3x^2 - 2(a + b + c)x + ab + bc + ac = 0$$

Comparing with $ax^2 + bx + c = 0$, we have $a = 3$, $b = -2(a + b + c)$, $c = ab + bc + ac$

So using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2(a + b + c) \pm \sqrt{4(a + b + c)^2 - 4(3)(ab + bc + ac)}}{2(3)}$$

$$= \frac{2(a + b + c) \pm \sqrt{4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ac) - 12(ab + bc + ac)}}{6}$$

$$= \frac{2(a + b + c) \pm 2\sqrt{4a^2 + 4b^2 + 4c^2 + 8ab + 8bc + 8ac - 12ab - 12bc - 12ac}}{6}$$

$$= \frac{2(a + b + c) \pm \sqrt{4a^2 + 4b^2 + 4c^2 - 4ab - 4bc - 4ac}}{6}$$

$$= \frac{2(a+b+c) \pm 2\sqrt{a^2+b^2+c^2-ab-bc-ac}}{6}$$

$$= \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ac}}{3}$$

Hence the solution set $= \left\{ \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ac}}{3} \right\}$

Q.20 $(a+b)x^2 + (a+2b+c)x + b + c = 0$

Solution:

$(a+b)x^2 + (a+2b+c)x + b + c = 0$

Comparing with $ax^2 + bx + c = 0$, we have $a = a+b$, $b = a+2b+c$, $c = b+c$

So using

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(a+2b+c) \pm \sqrt{(a+2b+c)^2 - 4(a+b)(b+c)}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm \sqrt{a^2+4b^2+c^2+4ab+4bc+2ac-4ab-4ac-4b^2-4bc}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm \sqrt{a^2+c^2-2ac}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm \sqrt{(a-c)^2}}{2(a+b)}$$

$$= \frac{-(a+2b+c) \pm (a-c)}{2(a+b)}$$

$\Rightarrow$ Either $x = \frac{-(a+2b+c)+(a-c)}{2(a+b)}$ or $x = \frac{-(a+2b+c)-(a-c)}{2(a+b)}$

$= \frac{-a-2b-c+a-c}{2(a+b)}$ or $= \frac{-a-2b-c-a+c}{2(a+b)}$

$= \frac{-2b-2c}{2(a+b)}$ or $= \frac{-2a-2b}{2(a+b)}$

$= \frac{-2(b+c)}{2(a+b)}$ or $= \frac{-2(a+b)}{2(a+b)}$

$= -\frac{b+c}{a+b}$ or $= -1$

Hence the solution set $= \left\{ -1, -\frac{b+c}{a+b} \right\}$