EXERCISE 5.4

Resolve the following partial fractions.

Q.1 $\dfrac{\mathbf{x^3 + 2x + 2}}{\mathbf{(x^2 + x + 1)^2}}$

Solution:

Let

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{(Ax + B)}{(x^2 + x + 1)} + \frac{(Cx + D)}{(x^2 + x + 1)^2} \quad \ldots\ldots\ldots (1)$$

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{(Ax + B)(x^2 + x + 1) + (Cx + D)}{(x^2 + x + 1)^2}$$

$$x^3 + 2x + 2 = (Ax + B)(x^2 + x + 1) + (Cx + D)$$

$$x^3 + 2x + 2 = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + Cx + D \quad \ldots\ldots\ldots (2)$$

Equating coefficients of x^3, x^2, x and constant term in equation (2), we get

x^3 ; $\boxed{A = 1}$

x^2 ; $A + B = 0$

$1 + B = 0 \Rightarrow \boxed{B = -1}$

x ; $A + B + C = 2$

$1 - 1 + C = 2 \Rightarrow \boxed{C = 2}$

Cons ; $B + D = 2$

$-1 + D = 2 \Rightarrow \boxed{D = 3}$

Put values of A, B, C, D in equation (1) we get

$$\frac{x^3 + 2x + 2}{(x^2 + x + 1)^2} = \frac{x - 1}{(x^2 + x + 1)} + \frac{2x + 3}{(x^2 + x + 1)^2}$$

are required partial fraction.

Q.2 $\dfrac{\mathbf{x^2}}{\mathbf{(x^2 + 1)^2 (x - 1)}}$

Solution:

Let

$$\frac{x^2}{(x - 1)(x^2 + 1)^2} = \frac{A}{(x - 1)} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} \quad (1)$$

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)}{(x-1)(x^2+1)^2}$$

$$x^2 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \quad \ldots\ldots\ldots (2)$$

$$x^2 = A(x^4+1+2x^2) + (Bx+C)(x^3+x-x^2-1) + (Dx+E)(x-1)$$

$$x^2 = Ax^4 + A + 2Ax^2 + Bx^4 + Bx^2 - Bx^3 - Bx + Cx^3 + Cx - Cx^2 - C$$
$$+ Dx^2 - Dx + Ex - E \quad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$(1)^2 = A\left((1)^2+1\right)^2 + 0 + 0$$

$$1 = A(1+1)^2$$

$$1 = A(2)^2 \quad \Rightarrow \quad = \boxed{A = \frac{1}{4}}$$

Equating coefficients of x^4, x^3, x^2 and constant term in equation (3), we get

x^4 ; $A + B = 0$

$$B = -A$$

$$\boxed{B = -\frac{1}{4}}$$

x^3 ; $-B + C = 0$

$$C = B$$

$$\boxed{C = -\frac{1}{4}}$$

x^2 ; $2A + B - C + D = 1$

$$D = 1 - 2A - B + C$$

$$= 1 - 2\left(\frac{1}{4}\right) + \frac{1}{4} - \frac{1}{4}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{D = \frac{1}{2}}$$

x ; $-B + C - D + E = 0$

$$\frac{1}{4} - \frac{1}{4} - \frac{1}{2} + E = 0$$

$$\boxed{E = \frac{1}{2}}$$

Put values of A, B, C, D and E in equation (1) we get

$$\frac{x^2}{(x^2+1)^2 (x-1)} = + \frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2}$$

$$= \frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2}$$

are required partial fraction.

Q.3 $\mathbf{\dfrac{2x-5}{(x^2+2)^2 (x-2)}}$

Solution:

Let

$$\frac{2x-5}{(x-2)(x^2+2)^2} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \qquad \ldots\ldots\ldots (1)$$

$$\frac{2x-5}{(x-2)(x^2+2)^2} = \frac{A(x^2+2)^2 + (Bx+C)(x-2)(x^2+1) + (Dx+E)(x-2)}{(x-2)(x^2+2)^2}$$

$$2x-5 = A(x^2+2)^2 + (Bx+C)(x-2)(x^2+2) + (Dx+E)(x-2) \qquad \ldots\ldots\ldots (2)$$

$$2x-5 = A(x^4+4+4x^2) + (Bx+C)(x^3-2x^2+2x-4) + (Dx+E)(x-2)$$

$$2x-5 = Ax^4 + 4A + 4Ax^2 + Bx^4 - 2Bx^3 + 2Bx^2 - 4Bx + Cx^3 - 2Cx^2 + 2Cx - 4C$$
$$+ Dx^2 - 2Dx + Ex - 2E \qquad \ldots\ldots\ldots (3)$$

Put $x = 2$ in equation (2), we get

$$2(2) - 5 = A\left((2)^2+2\right)^2 + 0 + 0$$

$$4 - 5 = A(4+2)^2$$

$$-1 = A(6)^2 \quad \Rightarrow \quad = \boxed{A = -\frac{1}{36}}$$

Equating coefficients of x^4, x^3, x^2 and x in equation (3), we get

$x^4 \quad ; \quad A + B = 0$

$$B = -A = -\left(-\frac{1}{36}\right)$$

$$\boxed{B = \frac{1}{36}}$$

$$x^3 \quad ; \quad -2B + C = 0$$

$$-2\left(-\frac{1}{36}\right) + C = 0$$

$$\boxed{C = \frac{1}{18}}$$

$$x^2 \quad ; \quad 4A + 2B - 2C + D = 0$$

$$4\left(-\frac{1}{36}\right) + 2\left(-\frac{1}{36}\right) - 2\left(-\frac{1}{18}\right) + D = 0$$

$$-\frac{1}{9} + \frac{1}{18} - \frac{1}{9} + D = 0$$

$$D = \frac{1}{9} + \frac{1}{9} - \frac{1}{18}$$

$$= \frac{2+2-1}{18} = \frac{3}{18}$$

$$\boxed{D = \frac{1}{6}}$$

$$x \quad ; \quad -4B + 2C - 2D + E = 0$$

$$E = 2 + 4B - 2C + 2D$$

$$= 2 + 4\left(\frac{1}{36}\right) - \left(\frac{1}{18}\right) + 2\left(\frac{1}{6}\right)$$

$$= 2 + \frac{1}{9} - \frac{1}{9} + \frac{1}{3}$$

$$= 2 + \frac{1}{3} = \frac{6+1}{3}$$

$$\boxed{E = \frac{7}{3}}$$

Put values of A, B, C, D, E in equation (1) we get

$$\frac{2x-5}{(x^2+2)^2\,(x-2)} = \frac{-\frac{1}{36}}{(x-2)} + \frac{\frac{1}{36}x + \frac{1}{18}}{x^2+2} + \frac{\frac{1}{6}x + \frac{7}{3}}{(x^2+2)^2}$$

$$= \frac{-1}{36\,(x-2)}\,\frac{x+2}{36\,(x^2+2)} + \frac{x+14}{6\,(x^2+2)^2}$$

Q.4 $\dfrac{8x^2}{(x^2+1)^2(1-x^2)}$

Solution:

As

$$\frac{8x^2}{(1-x^2)(x^2+1)^2} = \frac{8x^2}{(1-x)(1+x)(x^2+1)^2}$$

Let

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \quad \text{.........} (1)$$

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2}$$

$$= \frac{A(1+x)(x^2+1)^2 + B(1-x)(x^2+1)^2 + (Cx+D)(x^2+1)(1-x)(1+x) + (Ex+F)(1-x)(1+x)}{(1-x)(1+x)(x^2+1)^2}$$

$$8x^2 = A(1+x)(x^2+1)^2 + B(1-x)(x^2+1)^2 + (Cx+D)(x^2+1)(1-x)(1+x)$$
$$+ (Ex+F)(1-x)(1+x) \quad \text{.........} (2)$$

$$8x^2 = A(x^5+x^4+2x^3+2x^2+x) + B(-x^5+x^4-2x^3+2x^2-x) + C(-x^5+x)$$
$$+ D(-x^4+1) + (Ex+F)(1-x^2)$$

$$8x^2 = Ax^5 + Ax^4 + 2Ax^3 + 2Ax^2 + Ax - Bx^5 + Bx^4 - 2Bx^3 + 2Bx^2 - Bx$$
$$- Cx^5 + Cx - Dx^4 + D + Ex - Ex^3 + F - Fx^2 \quad \text{.........} (3)$$

Put $x = 1$ in equation (2), we get

$$8(1)^2 = A(1+1)\left((1)^2+1\right)^2$$

$$8 = A(2)(1+1)^2$$

$$8 = A(2)(4) \quad \Rightarrow \quad \boxed{A = 1}$$

Put $x = -1$ in equation (2), we get

$$8(-1)^2 = 0 + B(+1+1)\left((-1)^2+1\right)^2$$

$$8 = B(+2)(1+1)^2$$

$$8 = B(8)$$

$$\boxed{B = 1}$$

Equating coefficients of x^5, x^4, x^3 in equation (3) we get

x^5 ; $A - B - C = 0$

$C = A - B = 1 - 1 = 0$

$\boxed{C = 0}$

$x^4 \quad ; \quad A + B - D = 0$

$D = A + B = 1 + 1$

$\boxed{D = 2}$

$x^3 \quad ; \quad 2A - 2B - E = 0$

$E = 2A - 2B$

$= 2(1) - 2(1)$

$\boxed{E = 0}$

Put values of A, B, C, D and E in equation (1) we get

$$\frac{8x^2}{(1-x)(1+x)(x^2+1)^2} = \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{x^2+1} - \frac{4}{(x^2+1)^2}$$

are required partial fractions.

Q.5 $\mathbf{\dfrac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2}}$

Solution:

Let

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)} + \frac{Dx+E}{(x^2+x+1)^2} \quad \ldots\ldots\ldots (1)$$

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1)}{(x-1)(x^2+x+1)^2}$$

$$4x^4 + 3x^3 + 6x^2 + 5x = A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1) \quad \ldots\ldots\ldots (2)$$

$$4x^4 + 3x^3 + 6x^2 + 5x = Ax^4 + 2Ax^3 + 3Ax^2 + 2Ax + A + Bx^4 - Bx + Cx^3 - C + Dx^2 - Dx + Ex - E \quad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$4(1)^4 + 3(1)^3 + 6(1)^2 + 5(1) = A\left((1)^2 + 1 + 1\right)^2 + 0 + 0$$

$$4 + 3 + 6 + 5 = A(1 + 1 + 1)^2$$

$18 = 9A \Rightarrow \boxed{A = 2}$

$x^4 \quad ; \quad A + B = 4$

$B = 4 - A = 4 - 2$

$\boxed{B = 2}$

$x^3 \quad ; \quad 2A + C = 3$

$C = 3 - 2A = 3 - 2(2) = 3 - 4$

$\boxed{C = -1}$

$x^2 \quad ; \quad 3A + D = 6$

$D = 6 - 3A = 6 - 3(2) = 6 - 6 = 0$

$\boxed{D = 0}$

$x \quad ; \quad 2A - B - D + E = 5$

$E = 5 - 2A + B + D$

$= 5 - 2(2) + 2 + 0$

$\boxed{E = 3}$

Put values of A, B, D, D and E in equation (1)

$$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{2}{(x-1)} + \frac{2x-1}{(x^2+x+1)} + \frac{3}{(x^2+x+1)^2}$$

are required partial fractions.

Q.6 $\mathbf{\dfrac{2x^4 - 3x^3 - 4x}{(x^2+2)^2 (x+1)^2}}$

Solution:

Let

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2}$$

$$= \frac{A(x+1)(x+2)^2 + B(x^2+2)^2 + (Cx+D)(x+1)^2(x+2) + (Ex+F)(x+1)^2}{(x+1)^2(x^2+2)^2}$$

$$2x^4 - 3x^3 - 4x = A(x+1)(x+2)^2 + B(x^2+2)^2 + (Cx+D)(x+1)^2(x+2) + (Ex+F)(x+1)^2 \quad \ldots\ldots (2)$$

$$2x^4 - 3x^3 - 4x = A(x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4) + B(x^4 + 4 + 4x^2)$$
$$+ C(x^5 + 2x^4 + 3x^3 + 4x^2 + 2x)\ D(x^4+2x^3+3x^2+4x+2) + E(x^3+2x^2+x)$$
$$+ F(x^2 + 2x + 1)$$

$$2x^4 - 3x^3 - 4x = Ax^5 + Ax^4 + 4Ax^3 + 4Ax^2 + 4Ax + 4A + Bx^4 + 4B + 4Bx^2$$
$$+ Cx^5 + 2Cx^4 + 3Cx^3 + 4Cx^2 + 2Cx + Dx^4 + 2Dx^3 + 3Dx^2 + 4Dx$$
$$+ 2D + Ex^3 + 2Ex^2 + Ex + Fx^2 + 2Fx + F \qquad (3)$$

Put $x = -1$ in equation (2), we get

$$2(-1)^4 - 3(-1)^3 - 4(-1) = 0 + B\left((-1)2 + 2\right)^2 + 0 + 0$$

$$2 + 3 + 4 = B(1 + 2)^2$$

$$9 = B(3)^2$$

$$9 = 9B \quad \Rightarrow \quad \boxed{B = 1}$$

Equating coefficients of x^5, x^4, x^3, x^2, x and constant in equation (3), we get

x^5 ; $A + C = 0$ (i)

x^4 ; $A + B + 2C + D = 2$ (ii)

x^3 ; $4A + 3C + 2D + E = -3$ (iii)

x^2 ; $4A + 4B + 4C + 3D + 2E + F = 0$ (iv)

x ; $4A + 2C + 4D + E + 2F = -4$ (v)

cons ; $4A + 4B + 2D + F = 0$ (vi)

from (i)

$$C = -A$$

from (ii)

$$D = 2 - A - B - 2C = 2 - A - 1 + 2A = 1 + A \qquad (vii)$$

from (iii)

$$E = -3 - 4A - 3C - 2D = -3 - 4A + 3A - 2 - 2A = -3A - 5 \qquad (viii)$$

Subtracting (vi) from (iv), we get

$$4C + D + 2E = 0$$

$$\Rightarrow \quad -4A + 1 + A + 2(-3A - 5) = 0$$

$$-9A = 9$$

$$\Rightarrow \quad \boxed{A = -1}$$

Put this value of A in (i), (vii), (viii) we get

(i) $\Rightarrow$ $C = -(-1) = 1$ $\Rightarrow$ $\boxed{C = -1}$

(vii) $\Rightarrow$ $D = 1 + (-1) = 0$ $\Rightarrow$ $\boxed{D = 0}$

(viii) $\Rightarrow$ $E = -3(-1) - 5 = 3 - 5 = -2$

$\Rightarrow$ $\boxed{E = -2}$

from equation (vi)

$4(-1) + 4(1) + 2(0) + F = 0$

$-4 + 4 + 0 + F = 0$

$\boxed{F = 0}$

Put values of A, B, C, D, E and F in equation (1), we get

$$\frac{2x^4 - 3x^3 - 4x}{(x+1)^2 (x^2+2)^2} = \frac{-1}{x+1} + \frac{1}{(x+1)^2} + \frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}$$

are required partial fractions.