We can represent complex numbers by points of the coordinate plane. In this representation every complex number will be represented by one and only one point of the coordinate plane.

In this representation the x–axis is called the real axis and the y–axis is called the imaginary axis. The figure representing one or more complex numbers on the complex plane is called an Argand diagram.

EXERCISE 1.3

Q.1 Graph the following numbers on the complex plane:

(i) $2 + 3i = (2, 3)$

(ii) $2 - 3i = (2, -3)$

(iii) $-2 - 3i = (-2, -3)$

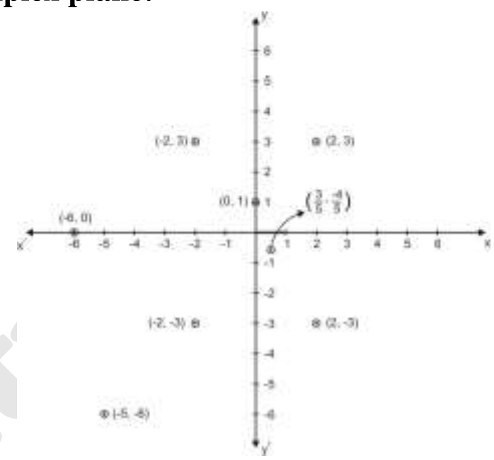
(iv) $-2 + 3i = (-2, 3)$

(v) $-6 = (-6, 0)$

(vi) $i = (0, 1)$

(vii) $\frac{3}{5} - \frac{4}{5}i = \left(\frac{3}{5}, -\frac{4}{5}\right) = (0.6, -0.8)$

(viii) $-5 - 6i = (-5, -6)$



Q.2 Find the multiplicative inverse of the following numbers:

(i) $-3i$

Solution:

$-3i = (0 - 3)$

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(0, -3) = \left(\frac{0}{(0)^2 + (-3)^2}, \frac{3}{(0)^2 + (-3)^2}\right)$

$$= \left(0, \frac{3}{9}\right) = \left(0, \frac{1}{3}\right) = 0 + \frac{1}{3}i = \frac{1}{3}i$$

(ii) $1 - 2i$ **(Lahore Board 2006)**

Solution:

$1 - 2i = (1, -2)$

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(1, -2) = \left(\frac{1}{(1)^2 + (-2)^2}, \frac{-(-2)}{(1)^2 + (-2)^2}\right)$

$$= \left(\frac{1}{1+4}, \frac{2}{1+4}\right)$$

$$= \left(\frac{1}{5}, \frac{2}{5}\right) = \frac{1}{5} + \frac{2}{5}i$$

(iii) $-3 - 5i$ **(Gujranwala Board 2004)**

Solution:

$-3 - 5i = (-3, -5)$

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(-3, -5) = \left(\frac{-3}{(-3)^2 + (-5)^2}, \frac{-(-5)}{(-3)^2 + (-5)^2}\right)$

$$= \left(\frac{-3}{9+25}, \frac{5}{9+25}\right)$$

$$= \left(\frac{-3}{34}, \frac{5}{34}\right) = \frac{-3}{34} + \frac{5}{34}i$$

(iv) (1, 2)

Solution:

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(1, 2) = \left(\frac{1}{(1)^2 + (2)^2}, \frac{-2}{(1)^2 + (+2)^2}\right)$

$$= \left(\frac{1}{1+4}, \frac{-2}{1+4}\right)$$

$$= \left(\frac{1}{5}, \frac{-2}{5}\right)$$

Q.3 Simplify

(i) i^{101} **(Gujranwala Board 2007)**

Solution:

$i^{101} = i^{100} . i = (i^2)^{50} . i = (-1)^{50} . i = i$ $\qquad \because i^2 = -1$

(ii) $(-ai)^4$, $a \in R$

Solution:

$(-ai)^4 = (-1)^4 . a^4 i^4 = a^4 (i^2)^2 = a^4 (-1)^2 = a^4$ $\qquad \because i^2 = -1$

(iii) i^{-3}

Solution:

$$i^{-3} = \frac{1}{i^3} = \frac{1}{i^2 \, . \, i} = \frac{1}{-i} \cdot \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i$$

(iv) $\mathbf{i^{-10}}$

Solution:

$$i^{-10} = \frac{1}{i^{10}} = \frac{1}{(i^2)^5} = \frac{1}{(-1)^5} = \frac{1}{-1} = -1$$

Q.4 Prove that $\bar{z} = z$ iff z is real. **(Lahore Board 2009)**

Solution:

Let $z = a + ib$ then $\bar{z} = a - ib$

Suppose $z = \bar{z}$

$\Rightarrow$ $a + ib = a - ib$

$\Rightarrow$ $a + ib - a + ib = 0$

$\Rightarrow$ $2ib = 0 \Rightarrow b = 0 \qquad \because 2i \neq 0$

$\Rightarrow$ $z = a + i0$

$\Rightarrow$ $z = a$

$\Rightarrow$ z is real.

Now conversely;

Suppose that z is real

Such that $z = a$

then $\bar{z} = a$

$\Rightarrow$ $z = \bar{z}$

Hence proved.

Q.5 Simplify by expressing in form a + bi

(i) $\mathbf{5 + 2\sqrt{-4}}$

Solution:

$$5 + 2\sqrt{-4} = 5 + 2\sqrt{-1} \, . \sqrt{4} = 5 + 2i\,(2) = 5 + 4i \qquad \because \sqrt{-1} = 1$$

(ii) $\mathbf{(2 + \sqrt{-3})(3 + \sqrt{-3})}$

Solution:

$$\begin{aligned}(2 + \sqrt{-3})(3 + \sqrt{-3}) &= 6 + 2\sqrt{-3} + 3\sqrt{-3} + (\sqrt{-3})^2 \\ &= 6 + 2\sqrt{-1} \, . \sqrt{3} + 3\sqrt{-1} \, . \sqrt{3} + (-3) \\ &= 6 + 2i\sqrt{3} + 3i\sqrt{3} - 3 \\ &= 3 + 5\sqrt{3}\, i\end{aligned}$$

(iii) $\mathbf{\dfrac{2}{\sqrt{5} + \sqrt{-8}}}$

$$\frac{2}{\sqrt{5}+\sqrt{-8}} = \frac{2}{\sqrt{5}+\sqrt{-8}}\cdot\frac{\sqrt{5}-\sqrt{-8}}{\sqrt{5}-\sqrt{-8}} \qquad \text{Rationalizing}$$

$$= \frac{2\,(\sqrt{5}-\sqrt{-8})}{(\sqrt{5})^2-(\sqrt{-8})^2} = \frac{2\sqrt{5}-2\sqrt{-8}}{5-(-8)}$$

$$= \frac{2\sqrt{5}-2\sqrt{-1}\,.\sqrt{8}}{5+8} = \frac{2\sqrt{5}-2i\sqrt{8}}{13}$$

$$= \frac{2\sqrt{5}}{13}-\frac{4i\sqrt{2}}{13}$$

(iv) $\mathbf{\dfrac{3}{\sqrt{6}-\sqrt{-12}}}$

Solution:

$$\frac{3}{\sqrt{6}-\sqrt{-12}} = \frac{3}{\sqrt{6}-\sqrt{12}} \times \frac{\sqrt{6}+\sqrt{-12}}{\sqrt{6}+\sqrt{-12}} \qquad \text{Rationalizing}$$

$$= \frac{3\,(\sqrt{6}+\sqrt{12}\,)}{(\sqrt{6}\,)^2-(\sqrt{-12})^2}$$

$$= \frac{3\sqrt{6}+3\sqrt{-12}}{6-(-12)} = \frac{3\sqrt{6}+3\sqrt{-1}\,.\sqrt{12}}{6+12}$$

$$= \frac{3\sqrt{6}+3i\sqrt{12}}{18} = \frac{3\sqrt{6}}{18}+\frac{3i\,.\,2\sqrt{3}}{18}$$

$$= \frac{\sqrt{6}}{6}+\frac{\sqrt{3}}{3}\,i = \frac{1}{\sqrt{6}}+\frac{1}{\sqrt{3}}\,i$$

Q.6 Show that $\forall\, z \in C$

(i) $z^2+\bar{z}^2$ is real number

Solution:

Let $z = a + bi$

Then $\bar{z} = a - bi$

Now

$$z^2+\bar{z}^2 = (a+bi)^2+(a-bi)^2$$

$$= (a)^2+(ib)^2+2\,abi+a^2+(ib)^2-2abi$$

$$= a^2+i^2b^2+a^2+i^2b^2 \qquad \because\ i^2 = -1$$

$$= a^2-b^2+a^2-b^2 = 2a^2-2b^2 = 2\,(a^2-b^2)$$

which is real number

Hence proved.

(ii) $(z-\overline{z})^2$ is real number **(Lahore Board 2009)**

Solution:

Let $z = a + bi$

then $\overline{z} = a - ib$

$$\Rightarrow (z-\overline{z})^2 = [(a+ib)-(a-ib)]^2 \qquad \because i^2 = -1$$

$$= [a+ib-a+ib]^2 = (2ib)^2 = 4i^2b^2 = -4b^2$$

which is a real number.

Hence proved.

Q.7 Simplify the followings

(i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$ **(Gujranwala Board 2003, 2005)**

Solution:

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \qquad \because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$$

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^3 = \frac{(-1+\sqrt{3}\,i)^3}{(2)^3}$$

$$= \frac{1}{8}\left[(-1)^3+(\sqrt{3}i)^3+3(-1)^2(\sqrt{3}i)+3(-1)(\sqrt{3}i)^2\right]$$

$$= \frac{1}{8}\left[-1+(\sqrt{3})^3 i^3+3(1)\sqrt{3}\,i-3(3i^2)\right]$$

$$= \frac{1}{8}\left[-1+\sqrt{3^3}\,i^2 . i+3\sqrt{3}\,i-3(3)(-1)\right]$$

$$= \frac{1}{8}\left[-1+\sqrt{27}(-1)i+3\sqrt{3}\,i+9\right]$$

$$= \frac{1}{8}\left[-1-3\sqrt{3}\,i+3\sqrt{3}\,i+9\right]$$

$$= \frac{1}{8}(8) = 1$$

(ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

Solution:

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{-1-\sqrt{3}}{2}\right)^3 = \left(\frac{(-1)(1+\sqrt{3}i)}{2}\right)^3 = \frac{(-1)^3(1+\sqrt{3}\,i)^3}{2^3}$$

$$= -\frac{1}{8}\left[(1)^3+(\sqrt{3}i)^3+3(1)^2(\sqrt{3}i)+3(1)(\sqrt{3}i)^2\right]$$

$$= -\frac{1}{8}\left[1 + (\sqrt{3})^3 i^3 + 3\sqrt{3}\, i + 3\,(3i^2)\right]$$

$$= -\frac{1}{8}\left[1 + \sqrt{3^3}\, i^2 . i + 3\sqrt{3}\, i + 9\,(-1)\right]$$

$$= -\frac{1}{8}\left[1 + \sqrt{27}\,(-1)\, i + 3\sqrt{3}\, i - 9\right]$$

$$= -\frac{1}{8}\left[1 - 3\sqrt{3}\, i + 3\sqrt{3}\, i - 9\right]$$

$$= -\frac{1}{8}[-8] = 1$$

(iii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

Solution:

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-1}$$

$$= \left(\frac{-1-\sqrt{3}i}{2}\right)^{-1}$$

$$= \frac{2}{-1-\sqrt{3}i} . \frac{-1+\sqrt{3}i}{-1+\sqrt{3}i} \qquad \text{Rationalizing}$$

$$= \frac{2\left(-1+\sqrt{3}\, i\right)}{(-1)^2-\left(\sqrt{3}\, i\right)^2} = \frac{-2+2\sqrt{3}\, i}{1-3i^2} = \frac{-2+2\sqrt{3}\, i}{1+3} \qquad \because i^2 = 1$$

$$= \frac{-2+2\sqrt{3}\, i}{4} = \frac{-2}{4}+\frac{2\sqrt{3}\, i}{4} = -\frac{1}{2}+\frac{\sqrt{3}\, i}{2}$$

(iv) $(a + bi)^2$

Solution:

$$(a + bi)^2 = (a)^2 + 2\,(a)\,(bi) + (bi)^2 = a^2 + b^2 i^2 + 2abi$$

(v) $(a + bi)^2$

Solution:

$$(a + bi)^2 = \frac{1}{(a+bi)^2} = \left(\frac{1}{a+bi} . \frac{a-bi}{a-bi}\right)^2 \qquad \text{Rationalizing}$$

$$= \left(\frac{a-bi}{a^2-b^2i^2}\right)^2 = \left(\frac{a-bi}{a^2+b^2}\right)^2 \qquad \because i^2 = -1$$

$$= \frac{(a-bi)^2}{(a^2+b^2)^2}$$

$$= \frac{a^2+b^2i^2-2abi}{(a^2+b^2)^2}$$

$$= \frac{a^2-b^2-2abi}{(a^2+b^2)^2} \qquad \because\ i^2 = -1$$

$$= \frac{a^2-b^2}{(a^2+b^2)^2} - \frac{2abi}{(a^2+b^2)^2}$$

(vi) $\quad \mathbf{(a + bi)^3}$

Solution:

$$(a+bi)^3 = a^3 + b^3i^3 + 3a^2(bi) + 3a(bi)^2$$
$$= a^3 + b^3 . i^2 . i + 3a^2bi + 3ab^2i^2$$
$$= a^3 - b^3i + 3a^2bi - 3ab^2 \qquad \because\ i^2 = -1$$
$$= a^3 - 3ab^2 + i(3a^2b - b^3)$$

(vii) $\quad \mathbf{(a - bi)^3}$

Solution:

$$(a-bi)^3 = (a)^3 - (bi)^3 - 3(a)^2(bi) + 3(a)(bi)^2$$
$$= a^3 - b^3 . i^3 - 3a^2bi + 3ab^2i^2 \qquad \because\ (a-b)^3 = a^3-b^3-3a^2b+3ab^2$$
$$= a^3 - b^3 i^2 . i - 3a^2bi + 3ab^2(-1)$$
$$= a^3 + b^3i - 3a^2bi - 3ab^2$$
$$= a^3 - 3ab^2 + i(b^3 - 3a^2b)$$

(viii) $\quad \mathbf{\left(3 - \sqrt{-4}\right)^{-3}}$

Solution:

$$\left(3-\sqrt{-4}\right)^{-3} = \left(3 - \sqrt{4} . \sqrt{-1}\right)^{-3}$$
$$= (3 - i\,2)^{-3} = (3-2i)^{-3}$$
$$= \frac{1}{(3-2i)^3} = \left(\frac{1}{3-2i} . \frac{3+2i}{3+2i}\right)^3 \qquad \text{Rationalizing}$$
$$= \left(\frac{3+2i}{(3)^2-(2i)^2}\right)^3$$
$$= \frac{(3+2i)^3}{(9-4i^2)^3} = \frac{(3+2i)^3}{(9+4)^3}$$
$$= \frac{1}{(13)^3}(3+2i)^3$$

$$= \frac{1}{2197}[(3)^3 + (2i)^3 + 3\,(3)^2\,(2i) + 3\,(3)\,(2i)^2]$$

$$= \frac{1}{2197}[27 + 8i^3 + 54i + 9\,.\,4\,i^2]$$

$$= \frac{1}{2197}[27 + 8i^2\,.\,i + 54i + 36\,(-1)] = \frac{1}{2197}\,[27 - 8i + 54i - 36]$$

$$= \frac{1}{2197}[-9 + 46i]$$

$$= \frac{-9}{2197} + \frac{46}{2197}\,i$$