$$\frac{d^2y}{d^2x} = \frac{1}{x^2}\left[-a\cos(\ell nx) - b\sin(\ell nx) - \frac{1}{x^2}\left[-a\sin(\ell nx) + b\cos(\ell nx)\right]\right.$$

$$\frac{d^2y}{d^2x} = \frac{1}{x^2}\left[-a\cos(\ell nx) - b\sin(\ell nx) + a\sin(\ell nx) - b\cos(\ell nx)\right]$$

Taking

$$x^2\frac{d^2y}{d^2x} + x\frac{dy}{dx} + y = x^2 . \frac{1}{x^2}\left[-a\cos(\ell nx) - b\sin(\ell nx) + a\sin(\ell nx)\right.$$

$$\left. - b\cos(\ell nx)\right] + x.\frac{1}{x}\left[-a\sin(\ell nx) + b\cos(\ell nx)\right] + a\cos(\ell nx) + b\sin(\ell nx)$$

$$= \quad -a\cos(\ell nx) - b\sin(\ell nx) + a\sin(\ell nx) - b\cos(\ell nx) - a\sin(\ell nx)$$

$$+ b\cos(\ell nx) + a\cos(\ell nx) + b\sin(\ell n\, x)$$

$$\boxed{x^2\frac{d^2y}{d^2x} + x\frac{dy}{dx} + y = 0}$$ Hence proved.

EXERCISE 2.8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$ ***(L.B 2005)***

(ii) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots\ldots\ldots$

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots\ldots\ldots\ldots$

(iv) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots\ldots\ldots$ ***(L.B 20011)***

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \ldots\ldots\ldots$

Solution:

(i) $\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$

Let

$$f(x) = \ell n(1+x)$$

$$f(0) = \ell n(1+0) = \ell n\, 1 = 0$$

$$f^{I}(x) = \frac{1}{1+x} \qquad f^{I}(0) = \frac{1}{1+0} = 1$$

$$f^{I}(x) = (1+x)^{-1}$$

$$f^{II}(x) = -(1+x)^{-2} \qquad f^{II}(0) = -(1+0)^{-2} = -1$$

$$f^{III}(x) = 2(1+x)^{-3} \qquad f^{III}(0) = 2(1+0)^{-3} = 2$$

$$f^{IV}(x) = -6(1+x)^{-4} \qquad f^{IV}(0) = -6(1+0)^{-4} = -6$$

The Maclaurin series expansion is

$$f(x) = f(0) + x\, f^{I}(0) + \frac{x^2}{2!} f^{II}(0) + \frac{x^3}{3!} f^{III}(0) + \frac{x^4}{4!} f^{IV}(0) + \ldots\ldots\ldots$$

$$= 0 + x\,(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \ldots\ldots\ldots\ldots$$

$$\boxed{\ell n\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots}$$ Hence proved.

(ii) $\mathbf{\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots\ldots\ldots\ldots}$

Let $f(x) = \cos x$

$f(0) = \cos 0 = 1$

$$f^{I}(x) = -\sin x \qquad f^{I}(0) = -\sin 0 = 0$$

$$f^{II}(x) = -\cos x \qquad f^{II}(0) = -\cos 0 = -1$$

$$f^{III}(x) = \sin x \qquad f^{III}(0) = \sin 0 = 0$$

$$f^{IV}(x) = \cos x \qquad f^{IV}(0) = \cos 0 = 1$$

$$f^{V}(x) = -\sin x \qquad f^{V}(0) = -\sin 0 = 0$$

$$f^{VI}(x) = -\cos x \qquad f^{VI}(0) = -\cos 0 = -1$$

The Maclaurin series expansion is

$$f(x) = f(0) + x\, f^{I}(0) + \frac{x^2}{\lfloor 2} f^{II}(0) + \frac{x^3}{\lfloor 3} + f^{III}(0) + \frac{x^4}{\lfloor 4} f^{IV}(0) + \frac{x^5}{\lfloor 5} f^{V}(0) + \frac{x^6}{\lfloor 6} f^{VI}(0) + ..$$

$$= 1 + x(0) + \frac{x^2}{\lfloor 2}(-1) + \frac{x^3}{\lfloor 3}(0) + \frac{x^4}{\lfloor 4}(1) + \frac{x^5}{\lfloor 5}(0) + \frac{x^6}{\lfloor 6}(-1) + \ldots\ldots.$$

$$\boxed{\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots\ldots\ldots\ldots}$$ Hence proved.

(iii) $\mathbf{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^6}{16} + \ldots\ldots\ldots\ldots}$

Let $f(x) = \sqrt{1+x}$

$$f(0) = \sqrt{1+0}$$
$$= \sqrt{1} = 1$$

$$f'(x) = \frac{1}{2}(x+1)^{\frac{-1}{2}} \qquad f'(0) = \frac{1}{2}(1+0)^{\frac{-1}{2}} = \frac{1}{2}$$

$$f^{II}(x) = \frac{-1}{4}(1+x)^{\frac{-3}{2}} \qquad f^{II}(0) = \frac{-1}{4}(1+0)^{\frac{-3}{2}} = \frac{-1}{4}$$

$$f^{III}(x) = \frac{3}{8}(1+x)^{\frac{-5}{2}} \qquad f^{III}(0) = \frac{3}{8}(1+0)^{\frac{-5}{2}} = \frac{3}{8}$$

The Maclaurin series expansion is

$$f(x) = f(0) + xf^{I}(0) + \frac{x^2}{2!}f^{II}(0) + \frac{x^3}{3!}f^{III}(0) + \ldots\ldots\ldots\ldots$$
$$= 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2}\left(\frac{-1}{4}\right) + \frac{x^3}{6}\left(\frac{3}{8}\right) + \ldots\ldots\ldots\ldots$$

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots\ldots\ldots\ldots}$$ Hence proved.

(iv) $\mathbf{e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots\ldots\ldots}$

Let
$$f(x) = e^x$$
$$f(0) = e^o = 1$$
$$f^{I}(x) = e^x \qquad f^{I}(0) = e^0 = 1$$
$$f^{II}(x) = e^x \qquad f^{II}(0) = e^0 = 1$$
$$f^{III}(x) = e^x \qquad f^{III}(0) = e^o = 1$$

The Maclaurin series expansion is

$$f(x) = f(0) + x f^{I}(0) + \frac{x^2}{\lfloor 2} f^{II}(0) + \frac{x^3}{\lfloor 3} f^{III}(0) + \ldots\ldots\ldots\ldots$$
$$= 1 + x(1) + \frac{x^2}{\lfloor 2}(1) + \frac{x^3}{\lfloor 3}(1) + \ldots\ldots\ldots\ldots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots\ldots\ldots}$$ Hence proved.

(v) $\mathbf{e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \ldots\ldots\ldots}$

Let
$$f(x) = e^{2x}$$
$$f(0) = e^{2(0)} = e^0 = 1$$

$f^{I}(x) = 2e^{2x}$	$f^{I}(0) = 2e^{2(0)} = 2e^{0} = 2$
$f^{II}(x) = 4e^{2x}$	$f^{II}(0) = 4e^{2(0)} = 4e^{0} = 4$
$f^{III}(x) = 8e^{2x}$	$f^{III}(0) = 8e^{2(0)} = 8e^{0} = 8$

The Maclaurin series expansion is

$$f(x) = f(0) + x\, f^{I}(0) + \frac{x^2}{\lfloor 2}\, f^{II}(0) + \frac{x^3}{\lfloor 3}\, f^{III}(0) + \ldots\ldots\ldots\ldots\ldots$$

$$e^{2x} = 1 + x(2) + \frac{x^2}{\lfloor 2}(4) + \frac{x^3}{\lfloor 3}(8) + \ldots\ldots\ldots\ldots\ldots$$

$$\boxed{e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \ldots\ldots\ldots}$$ Hence proved.

Q.2: Show that

$$\mathbf{\cos(x+h) = \cos x - h \sin x - \frac{h^2}{\lfloor 2} \cos x + \frac{h^3}{\lfloor 3} \sin x + \ldots\ldots}$$ **and evaluate $\cos 61^{o}$.**

Solution:

Let $f(x+h) = \cos(x+h)$

then

$f(x) = \cos x$

$f^{I}(x) = -\sin x$

$f^{II}(x) = -\cos x$

$f^{III}(x) = \sin x$

The Taylor series expansion is

$$f(x+h) = f(x) + hf^{I}(x) + \frac{h^2}{\lfloor 2} f^{II}(x) + \frac{h^3}{\lfloor 3} f^{III}(x) + \ldots\ldots\ldots\ldots$$

$$\cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{\lfloor 2}(-\cos x) + \frac{h^3}{\lfloor 3} \sin x + \ldots\ldots\ldots$$

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{\lfloor 2} \cos x + \frac{h^3}{\lfloor 3} \sin x + \ldots\ldots\ldots$$

Put $x = 60^{o}$, $h = 1^{o} = \frac{\pi}{180}$ rad $= 0.01745$ rad

$$\cos(60^{o} + 1^{o}) = \cos 60^{o} - (0.01745) \sin 60^{o} - \frac{(0.01745)^2}{2} \cos 60^{o} + \frac{(0.01745)^3}{6} \sin 60^{o} + \ldots\ldots\ldots$$

$$\cos 61^\circ \approx 0.5 - 0.0151 - 0.000076 + 0.00000076 + \ldots\ldots\ldots\ldots$$

$$\boxed{\cos 61^\circ \approx 0.4848} \qquad \text{Ans}$$

Q.3: Show that

$$\mathbf{2^{x+h} = 2^x \left\{1 + (\ell n2)\, h + \frac{(\ell n2)^2}{\lfloor 2} h^2 + \frac{(\ell n2)^3}{\lfloor 3} h^3 + \ldots\ldots \right\}}$$

Solution:

Let

$$f(x+h) = 2^{x+h}$$

then

$$f(x) = 2^x$$

$$f^{I}(x) = (\ell n2)\, 2^x$$

$$f^{II}(x) = (\ell n2)^2\, 2^x$$

$$f^{III}(x) = (\ell n2)^3\, 2^x$$

The Taylor series expansion is

$$f(x+h) = f(x) + hf^{I}(x) + \frac{h^2}{\lfloor 2} f^{II}(x) + \frac{h^3}{\lfloor 3} f^{III}(x) + \ldots\ldots\ldots\ldots$$

$$2^{x+h} = 2x + h\,(\ell n2)\, 2^x + \frac{h^2}{\lfloor 2} (\ell n\, 2)^2 . 2^x + \frac{h^3}{\lfloor 3} (\ell n\, 2)^3 . 2^x + \ldots\ldots\ldots\ldots$$

$$2^{x+h} = 2x \left\{ 1 + (\ell n2)h + \frac{(\ell n2)^2}{\lfloor 2} h^2 + \frac{(\ell n2)^3}{\lfloor 3} h^3 + \ldots\ldots \right\}$$

Hence proved.