NOTE:

(i) The rule or formula of a binomial for expansion raised to any positive integral power n.

(ii) It is finite series.

(iii) Number of terms in the expansion of $(a + x)^n$ is $n + 1$.

(iv) $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots\ldots, \binom{n}{n}$ are called binomial coefficients.

(v) $\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \ldots\ldots \binom{n}{n}$ are called even binomial coefficients.

(vi) $\binom{n}{1}, \binom{n}{3}, \binom{n}{5}, \ldots\ldots \binom{n}{n-1}$ are called odd binomial coefficients.

SUM OF BIN BINOMIAL COEFFICIENTS

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots\ldots + \binom{n}{n} = 2^n$$

SUM OF EVEN BINOMIAL COEFFICIENTS

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\ldots + \binom{n}{n} = 2^{n-1}$$

SUM OF ODD BINOMIAL COEFFICIENTS

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\ldots + \binom{n}{n-1} = 2^{n-1}$$

REMARK

Sum of even binomial coefficients = Sum of odd binomial coefficients.

EXERCISE 8.2

Q.1 Using binomial theorem, expand the following:

(i) $(a + 2b)^5$ **(ii)** $\left(\frac{x}{2} - \frac{2}{x^2}\right)^6$

(iii) $\left(3a - \frac{x}{3a}\right)^4$ **(iv)** $\left(2a - \frac{x}{a}\right)^7$

(v) $\left(\frac{x}{2y} - \frac{2y}{x}\right)^8$ **(vi)** $\left[\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right]^6$

Solution:

(i) $(a + 2b)^5$

$$= \binom{5}{0} a^5 (2b)^0 + \binom{5}{1} a^4 (2b)^1 + \binom{5}{2} a^3 (2b)^2 + \binom{5}{3} a^2 (2b)^3 + \binom{5}{4} a^1 (2b)^4 + \binom{5}{5} a^0 (2b)^5$$

$$= a^5 + 5a^4 (2b) + 10a^3 (4b^2) + 10a^2 (8b^3) + 5a (16b^4) + 32b^5$$

$$= a^5 + 10a^4 b + 40 a^3 b^2 + 80 a^2 b^3 + 80 a b^4 + 32 b^5$$

(ii) $\left(\frac{x}{2}-\frac{2}{x^2}\right)^6$

$$= \binom{6}{0}\left(\frac{x}{2}\right)^6 - \binom{6}{1}\left(\frac{x}{2}\right)^{6-1}\left(\frac{2}{x^2}\right)^1 + \binom{6}{2}\left(\frac{x}{2}\right)^{6-2}\left(\frac{2}{x^2}\right)^2 - \binom{6}{3}\left(\frac{x}{2}\right)^{6-3}\left(\frac{2}{x^2}\right)^3$$

$$+ \binom{6}{4}\left(\frac{x}{2}\right)^{6-4}\left(\frac{2}{x^2}\right)^4 - \binom{6}{5}\left(\frac{x}{2}\right)^{6-5}\left(\frac{2}{x^2}\right)^5 + \binom{6}{6}\left(\frac{x}{2}\right)^{6-6}\left(\frac{2}{x^2}\right)^6$$

$$= \frac{x^6}{64} - 6\frac{x^5}{32}\times\frac{2}{x^2} + 15\times\frac{x^4}{16}\times\frac{4}{x^4} - 20\times\frac{x^3}{8}\times\frac{8}{x^6} + 15\times\frac{x^2}{4}\times\frac{16}{x^8} - 6\times\frac{x}{2}\times\frac{32}{x^{10}} + \frac{64}{x^{12}}$$

$$= \frac{x^6}{64} - \frac{3}{8}x^3 + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$$

(iii) $\left(3a - \frac{x}{3a}\right)^4$

$$\binom{4}{0}(3a)^4\left(\frac{x}{3a}\right)^0 - \binom{4}{1}(3a)^{4-1}\left(\frac{x}{3a}\right)^1 + \binom{4}{2}(3a)^{4-2}\left(\frac{x}{3a}\right)^2 - \binom{4}{3}(3a)^{4-3}\left(\frac{x}{3a}\right)^3$$

$$+ \binom{4}{4}(3a)^{4-4}\left(\frac{x}{3a}\right)^4$$

$$= 81a^4 - 4\times 27a^3\times\frac{x}{3a} + 6\times 9a^2\times\frac{x^2}{9a^2}\times\frac{x^2}{9a^2} - 4\times 3a\times\frac{x^3}{27a^3} + \frac{x^4}{81a^4}$$

$$= 81a^4 - \frac{108}{3}a^2x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}$$

$$= 81a^4 - 36a^2x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}$$

(iv) $\left(2a - \frac{x}{a}\right)^7$

$$= \binom{7}{0}(2a)^7 - \binom{7}{1}(2a)^6\left(\frac{x}{a}\right)^1 + \binom{7}{2}(2a)^5\left(\frac{x}{a}\right)^2 - \binom{7}{3}(2a)^4\left(\frac{x}{a}\right)^3$$

$$+ \binom{7}{4}(2a)^3\left(\frac{x}{a}\right)^4 - \binom{7}{5}(2a)^2\left(\frac{x}{a}\right)^5 + \binom{7}{6}(2a)^1\left(\frac{x}{a}\right)^6 - \binom{7}{7}(2a)^0\left(\frac{x}{a}\right)^7$$

$$= 128a^7 - 7(64a^6)\left(\frac{x}{a}\right) + 21\times 32a^5\frac{x^2}{a^2} - 35(16a^4)\left(\frac{x^3}{a^3}\right)$$

$$+ 35(8a^3)\left(\frac{x^4}{a^4}\right) - 21(4a^2)\left(\frac{x^5}{a^5}\right) + 7(2a)\left(\frac{x^6}{a^6}\right) - \frac{x^7}{a^7}$$

$$= 128a^7 - 448\,x\,a^5 + 672\,a^3x^2 - 560\,a\,x^3 + \frac{280\,x^4}{a} - \frac{84\,x^5}{a^3} + \frac{14x^6}{a^5} - \frac{x^7}{a^7}$$

(v) $\left(\dfrac{x}{2y}-\dfrac{2y}{x}\right)^8$

$$= \binom{8}{0}\left(\frac{x}{2y}\right)^8 - \binom{8}{1}\left(\frac{x}{2y}\right)^7\left(+\frac{2y}{x}\right) + \binom{8}{2}\left(\frac{x}{2y}\right)^6\left(\frac{2y}{x}\right)^2 - \binom{8}{3}\left(\frac{x}{2y}\right)^5\left(\frac{2y}{x}\right)^3$$

$$+ \binom{8}{4}\left(\frac{x}{2y}\right)^4\left(\frac{2y}{x}\right)^4 - \binom{8}{5}\left(\frac{x}{2y}\right)^3\left(\frac{2y}{x}\right)^5 + \binom{8}{6}\left(\frac{x}{2y}\right)^2\left(\frac{2y}{x}\right)^6$$

$$- \binom{8}{7}\left(\frac{x}{2y}\right)^1\left(\frac{2y}{x}\right)^7 + \binom{8}{8}\left(\frac{x}{2y}\right)^0\left(\frac{2y}{x}\right)^8$$

$$= \frac{x^8}{256\,y^8} - 8\,\frac{x^7}{128\,y^7}\times\frac{2y}{x} + 28\left(\frac{x^6}{64y^6}\right)\left(\frac{4y^2}{x^2}\right) - 56\left(\frac{x^5}{32y^5}\right)\left(\frac{8y^3}{x^3}\right)$$

$$+ 70\left(\frac{x^4}{16y^4}\right)\left(\frac{16y^4}{x^4}\right) - 56\left(\frac{x^3}{8y^3}\right)\left(\frac{32y^5}{x^5}\right) + 28\,\frac{x^2}{4y^2}\times\frac{64y^6}{x^6} - 8\times\frac{x}{2y}\times\frac{128y^7}{x^7} + \frac{256\,y^8}{x^8}$$

$$= \frac{x^8}{256y^8} - \frac{x^6}{8y^6} + \frac{7x^4}{16y^4} - 14\,\frac{x^2}{y^2} + 70 - \frac{224y^2}{x^2} + \frac{448y^4}{x^4} - \frac{512y^6}{x^6} + \frac{256y^8}{x^8}$$

(vi) $\left[\sqrt{\dfrac{a}{x}}-\sqrt{\dfrac{x}{a}}\right]^6$

$$= \binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^6 - \binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^{6-1}\left(\sqrt{\frac{x}{a}}\right)^1$$

$$+ \binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^{6-2}\left(\sqrt{\frac{x}{a}}\right)^2 - \binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^{6-3}\left(\sqrt{\frac{x}{a}}\right)^3$$

$$+ \binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^{6-4}\left(\sqrt{\frac{x}{a}}\right)^4 - \binom{6}{5}\left(\sqrt{\frac{a}{x}}\right)^{6-5}\left(-\sqrt{\frac{x}{a}}\right)^5 + \binom{6}{6}\left(\sqrt{\frac{a}{x}}\right)^0\left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^{-1} + 15\left(\sqrt{\frac{a}{x}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^{-2}$$

$$- 20\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^{-3} + 15\left(\sqrt{\frac{a}{x}}\right)^2\left(\sqrt{\frac{a}{x}}\right)^{-4} - 6\left(\sqrt{\frac{a}{x}}\right)\left(\sqrt{\frac{a}{x}}\right)^{-5} + \left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^4 + 15\left(\sqrt{\frac{a}{x}}\right)^2 - 20\left(\sqrt{\frac{a}{x}}\right)^0$$

$$+ 15\left(\sqrt{\frac{a}{x}}\right)^{-2} - 6\left(\sqrt{\frac{a}{x}}\right)^{-4} + \left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\frac{a}{x}\right)^{6/2} - 6\left(\frac{a}{x}\right)^{4/2} + 15\left(\frac{a}{x}\right) - 20 + 15\left(\frac{a}{x}\right)^{-2/2} - 6\left(\frac{a}{x}\right)^{-4/2} + \left(\frac{x}{a}\right)^{6/2}$$

$$= \frac{a^3}{x^3} - 6\,\frac{a^2}{x^2} + 15\,\frac{a}{x} - 20 + 15\left(\frac{a}{x}\right)^{-1} - 6\left(\frac{a}{x}\right)^{-2} + \left(\frac{x}{a}\right)^3$$

$$= \frac{a^3}{x^3} - 6\,\frac{a^2}{x^2} + 15\,\frac{a}{x} - 20 + 15\,\frac{x}{a} - 6\,\frac{x^2}{a^2} + \frac{x^3}{a^3}$$

Q.2 Calculate the following by means of binomial theorem.

(i) $(0.97)^3$ **(Lahore Board 2010)**

(ii) $(2.\ 02)^4$ **(Lahore Board 2011)**

(iii) $(9.98)^4$

(iv) $(2.9)^5$

Solution:

(i) $(0.97)^3 = (1-0.03)^3$

$$= \binom{3}{0}(1)^3 - \binom{3}{1}(1)^2(.03)^1 + \binom{3}{2}(1)^1(.03)^2 - \binom{3}{3}(1)^0(.03)^3$$

$= 1 - 0.09 + 0.0027 - 0.000027$

$= 0.9127$

(ii) $(2.02)^4 = (2+0.02)^4$

$$= \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(.02)^1 + \binom{4}{2}(2)^2(.02)^2 + \binom{4}{3}(2)^1(.02)^3 + \binom{4}{4}(2)^0(.02)^4$$

$= 16 + 4\ (8)\ (.02) + 6\ (4)\ (0.0004) + 4\ (2)\ (0.000008) + 0.00000016$

$= 16.64 + 0.0096 + 0.000064$

$= 16.64$

(iii) $(9.98)^4 = (10-0.02)^4$

$$= \binom{4}{0}(10)^4(.02)^0 - \binom{4}{1}(10)^3(.02)^1 + \binom{4}{2}(10)^2(.02)^2 - \binom{4}{3}(10)^1(.02)^3 + \binom{4}{4}(10)^0(.02)^4$$

$= 10000 - 80 + 600\ (0.0004) - 40\ (0.000008) + 0.00000016$

$= 9920.24$

(iv) $(2.9)^5 = (3-0.1)^5$

$$= \binom{5}{0}(3)^5 - \binom{5}{1}(3)^4(.01) + \binom{5}{2}(3)^3(.01)^2 - \binom{5}{3}(3)^2(.01)^3 + \binom{5}{4}(3)^1(.01)^4 - \binom{5}{5}(3)^0(.01)^5$$

$= 243 - 4.05 + 10\ (27)\ (0.0001) - 10\ (9)\ (0.000001) + 15\ (0.00000001) - 0.0000000001$

$= 24.3 + 5 \times 81 - 0.01 + 10 \times 8 \times 0.0001$

$= 205.2$

Q.3 Expand and simplify the following:

(i) $(a+\sqrt{2}\,x)^4 + (a-\sqrt{2}\,x)^4$

(ii) $(2+\sqrt{3})^5 + (2-\sqrt{3})^5$

(iii) $(2+i)^5 - (2-i)^5$

(iv) $(x+\sqrt{x^2-1})^3 + (x-\sqrt{x^2-1})^3$

Solution:

(i) $(a+\sqrt{2}\,x)^4 + (a-\sqrt{2}\,x)^4$

$$= \binom{4}{0} a^4 + \binom{4}{1}(a)^3(\sqrt{2}\,x)^1 + \binom{4}{2}(a)^2(\sqrt{2}\,x)^2 + \binom{4}{3} a(\sqrt{2}\,x)^3 + \binom{4}{4} a^0(\sqrt{2}\,x)^4$$

$$(a+\sqrt{2}\,x)^4 = a^4 + 4a^3\sqrt{2}\,x + 6a^2(\sqrt{2}\,x)^2 + 4a(\sqrt{2}\,x)^3 + (\sqrt{2}\,x)^4 \quad \ldots\ldots\ldots \text{(i)}$$

$$(a-\sqrt{2}\,x)^4 = a^4 - 4a^3\sqrt{2}\,x + 6a^2(\sqrt{2}\,x)^2 - 4a(\sqrt{2}\,x)^3 + (\sqrt{2}\,x)^4 \quad \ldots\ldots\ldots \text{(ii)}$$

By adding (i) and (ii)

$$(a+\sqrt{2}\,x)^4 + (a-\sqrt{2}\,x)^4 = 2a^4 + 12a^2(\sqrt{2}\,x)^2 + 2(\sqrt{2}\,x)^4$$

$$= 2a^4 + 12a^2(2x^2) + 2(4x^4)$$

$$= 2a^4 + 24a^2x^2 + 8x^4$$

(ii) $(2+\sqrt{3})^5 + (2-\sqrt{3})^5$

$$(2+\sqrt{3})^5 = \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(\sqrt{3})^1 + \binom{5}{2}(2)^3(\sqrt{3})^2 + \binom{5}{3}(2)^2(\sqrt{3})^3$$

$$+ \binom{5}{4}(2)(\sqrt{3})^4 + \binom{5}{5}(2)^0(\sqrt{3})^5$$

$$= 32 + 5\times 16\sqrt{3} + 10\times 8(\sqrt{3})^2 + 10\times 4(\sqrt{3})^3 + 5(2)(\sqrt{3})^4 + (\sqrt{3})^5$$

$$= 32 + 80\sqrt{3} + 80(\sqrt{3})^2 + 40(\sqrt{3})^3 + 10(\sqrt{3})^4 + (\sqrt{3})^5$$

$$(2-\sqrt{3})^5 = 32 - 80\sqrt{3} + 80(\sqrt{3})^2 - 40(\sqrt{3})^3 + 10(\sqrt{3})^4 - (\sqrt{3})^5$$

Adding

$$(2+\sqrt{3})^5 + (2-\sqrt{3})^5 = 64 + 480 + 180 = 724$$

(iii) $(2+i)^5 - (2-i)^5$

$$(2+i)^5 = \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(i) + \binom{5}{2}(2)^3(i)^2 + \binom{5}{3}(2)^2(i)^3$$

$$+ \binom{5}{4}(2)^1(i)^4 + \binom{5}{5}(2)^0(i)^5$$

$$= 32 + 5 \times 16\,(i) + 10 \times 8 \times (i)^2 + 10 \times 4\,(i)^3 + 5 \times 2\,(i)^4 + i^5$$

$$= 32 + 80i + 80i^2 + 40i^3 + 10i^4 + i^5$$

$$(2 - i)^5 = 32 - 80i + 80i^2 - 40i^3 + 10i^4 - i^5$$

$$- \quad + \quad - \quad + \quad - \quad +$$

Subtracting

$$(2 + i)^5 - (2 - i)^5 = 160i + 80i^3 + 2i^5$$

$$= 160i - 80i + 2i = 82i$$

(iv) $\left(x + \sqrt{x^2 - 1} \right)^3 + \left(x - \sqrt{x^2 - 1} \right)^3$

First we take $\left(x + \sqrt{x^2 - 1})\right)^3$

$$= x^3 + \binom{3}{1}(x)^{3-1}\left(\sqrt{x^2 - 1}\right) + \binom{3}{2}(x)^1\left(\sqrt{x^2 - 1}\right)^2 + \binom{3}{3}(x)^0\left(\sqrt{x^2 - 1}\right)^3$$

$$\left(x + \sqrt{x^2 - 1} \right)^3 = x^3 + 3x^2\sqrt{x^2 - 1} + 3x\,(x^2 - 1) + \left(\sqrt{x^2 - 1}\right)^3$$

$$\left(x - \sqrt{x^2 - 1} \right)^3 = x^3 - 3x^2\sqrt{x^2 - 1} + 3x\,(x^2 - 1) - \left(\sqrt{x^2 - 1}\right)^3$$

Adding

$$\left(x + \sqrt{x^2 - 1} \right)^3 + \left(x - \sqrt{x^2 - 1} \right)^3 = 2x^3 + 6x\,(x^2 - 1)$$

$$= 2x^3 + 6x^3 - 6x$$

$$= 8x^3 - 6x$$

Q.4 Expand the following in ascending power of x

(i) $(2 + x - x^2)^4$ **(ii)** $(1 - x + x^2)^4$

(iii) $(1 - x - x^2)^4$

Solution:

(i) $(2 + x - x^2)^4$

Let $2 + x = y$

$$(y - x^2)^4 = \binom{4}{0}(y)^4(x^2)^0 - \binom{4}{1}(y^3)(x^2)^1 + \binom{4}{2}(y^2)(x^2)^2 - \binom{4}{3}(y)(x^2)^3 + \binom{4}{4}(y)^0(x^2)^4$$

$$= y^4 - 4y^3x^2 + 6y^2x^4 - 4y\,x^6 + x^8$$

Putting value $y = 2 + x$ again

$$= (2 + x)^4 - 4\,(2 + x)^3x^2 + 6\,(2 + x)^2x^4 - 4\,(2 + x)\,x^6 + x^8$$

$$= \left[\binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(x) + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4\right]$$

$$- 4\,[8 + x^3 + 6x^2 + 12x]\,x^2 + 6\,(4 + x^2 + 4x)\,x^4 - (8 + 4x)\,x^6 + x^8$$

$$= 16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$$

(ii) $\mathbf{(1 - x + x^2)^4}$

Let $1 - x = y$

$$(y + x^2)^4 = \binom{4}{0} y^4 + \binom{4}{1} y^3 x^2 + \binom{4}{2} y^2 (x^2)^2 + \binom{4}{3} y (x^2)^3 + \binom{4}{4} y^0 (x^2)^4$$

$$= y^4 + 4y^3 x^2 + 6y^2 x^4 + 4y x^6 + x^8$$

Putting value of y

$$= (1 - x)^4 + 4 (1 - x)^3 x^2 + 6 (1 - x)^2 x^4 + 4 (1 - x) x^6 + x^8$$

$$= \left[\binom{4}{0} (1)^4 (x)^0 - \binom{4}{1} (1)^3 (x)^1 + \binom{4}{2} (1)^2 (x)^2 - \binom{4}{3} (1)^1 (x)^3 + \binom{4}{4} (1)^0 (x)^4 \right]$$

$$+ 4 [1 - x^3 - 3x + 3x^2] x^2 + 6 (1 + x^2 - 2x) x^4 + 4 (x^6 - x^7) + x^8$$

$$= 1 - 4x + 6x^2 - 4x^3 + x^4 + 4x^2 - 12x^3 + 12x^4 - 4x^5 + 6x^4 - 12x^5 + 10x^6 - 4x^7 + x^8$$

$$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 4x^5 + 10x^6 - 4x^7 + x^8$$

(iii) $\mathbf{(1 - x - x^2)^4}$

Let $1 - x = y$

$$(y - x^2)^4 = \binom{4}{0} y^4 - \binom{4}{1} (y^3) x^2 + \binom{4}{2} (y)^2 (x^2)^2 - \binom{4}{3} y (x^2)^3 + \binom{4}{4} y^0 (x^2)^4$$

$$= y^4 - 4y^3 x^2 + 6y^2 x^4 - 4 x^6 y + x^8$$

Putting value of y

$$= (1 - x)^4 - 4 (1 - x)^3 x^2 + 6 (1 - x)^2 x^4 - 4x^6 (1 - x) + x^8$$

$$= \left[\binom{4}{0} (1)^4 - \binom{4}{1} (1)^3 (x) + \binom{4}{2} (1)^2 (x)^2 - \binom{4}{3} (1) (x)^3 + \binom{4}{4} (1)^0 (x)^4 \right]$$

$$- 4 [1 - x^3 - 3x + 3x^2] x^2 + 6 (1 + x^2 - 2x) x^4 - 4 (x^6 - x^7) + x^8$$

$$= 1 - 4x + 6x^2 - 4x^3 + x^4 - 4x^2 (1 - 3x + 3x^2 - x^3) + 6x^4 - 12x^5 + 6x^6 - 4x^6 + 4x^7 + x^8$$

$$= 1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$$

Q.5 Expand the following in descending power of x

(i) $\mathbf{(x^2 + x - 1)^3}$ **(ii)** $\mathbf{\left(x - 1 - \frac{1}{x}\right)^3}$

Solution:

(i) $\mathbf{(x^2 + x - 1)^3}$

Let $x - 1 = y \Rightarrow (x^2 + y)^3$

$$= \binom{3}{0} (x^2)^3 + \binom{3}{1} (x^2)^2 (y) + \binom{3}{2} (x^2) (y)^2 + \binom{3}{3} (x^2)^0 (y)^3$$

$$(x^2 + y)^3 = x^6 + 3x^4 y + 3x^2 y^2 + y^3$$

Putting value of y

$$\begin{aligned}(x^2+x-1)^3 &= x^6+3x^4(x-1)+3x^2(x-1)^2+(x-1)^3\\ &= x^6+3x^5-3x^4+3x^2(x^2+1-2x)+x^3-1-3x^2+3x\\ &= x^6+3x^5-3x^4+3x^4+3x^2-6x^3+x^3-1-3x^2+3x\\ &= x^6+3x^5-5x^3+3x-1\end{aligned}$$

(i) $\left(\mathbf{x-1-\frac{1}{x}}\right)^3$

Let $x-1 = y \Rightarrow \left(y-\frac{1}{x}\right)^3$

$$= \binom{3}{0}(y^3)-\binom{3}{1}(y^2)\left(\frac{1}{x}\right)+\binom{3}{2}(y)\left(\frac{1}{x}\right)^2-\binom{3}{3}(y)^0\left(\frac{1}{x}\right)^3$$

$$= y^3-\frac{3y^2}{x}+\frac{3y}{x^2}-\frac{1}{x^3}$$

Putting value of y

$$\begin{aligned}\left(x-1-\frac{1}{x}\right)^3 &= (x-1)^3-\frac{3(x-1)^2}{x}+\frac{3(x-1)}{x^2}-\frac{1}{x^3}\\ &= x^3-1-3x^2+3x-\frac{3}{x}(x^2+1-2x)+\frac{1}{x^2}(3x-3)-\frac{1}{x^3}\\ &= x-3x^2+5-\frac{3}{x^2}-\frac{1}{x^3}\end{aligned}$$

GENERAL TERM OF EXPANSION $(a+x)^n$

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r$$

Q.6 Find the term involving:

(i) x^4 **in the expansion of** $(3-2x)^7$ **(Lahore Board 2008)**

(ii) x^{-2} **in the expansion of** $\left(x-\frac{2}{x^2}\right)^{13}$

(iii) a^4 **in the expansion of** $\left(\frac{2}{x}-a\right)^9$ **(Lahore Board 2004)**

(iv) y^3 **in the expansion of** $(x-\sqrt{y})^{11}$

Solution:

(i) x^4 **in the expansion of** $(3-2x)^7$

We know that general term formula is

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r$$

Since $n = 7$, $a = 3$, $x = (-2x)$

$$T_{r+1} = \binom{7}{r} a^{7-r} (-2x)^r$$

$$T_{r+1} = \binom{7}{r} 3^{7-r} (-2)^r x^r \quad \ldots\ldots\ldots (1)$$

We have to find term involving x^4, so comparing the powers of x, we have $r = 4$

Putting $r = 4$ in (1)

$$T_{4+1} = \binom{7}{4} 3^{7-4} (-2)^4 x^4$$

$$= 35 \times 27 \times 16x^4$$

$$T_{4+1} = 15120\, x^4$$

(ii) $\mathbf{x^{-2}}$ **in the expansion of** $\left(x - \frac{2}{x^2}\right)^{13}$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$a = x,\ n = 13,\ x = \left(-\frac{2}{x^2}\right)$$

$$T_{r+1} = \binom{13}{r} (x)^{13-r} \left(-\frac{2}{x^2}\right)^r$$

$$= \binom{13}{r} x^{13-r-2r} (-2)^r$$

$$= \binom{13}{r} x^{13-3r} (-2)^r \quad \ldots\ldots\ldots (1)$$

We have to find term involving x^{-2} so comparing the power of x in (1)

$$13 - 3r = -2$$

$$13 + 2 = 3r$$

$$15 = 3r$$

$$\boxed{r = 5}$$

Put in (1)

$$T_{5+1} = \binom{13}{5} x^{13-3(5)} (-2)^5$$

$$T_6 = 1287 \times x^{-2} \times -32 = -41184\, x^{-2}$$

(iii) **a^4 in the expansion of $\left(\frac{2}{x} - a\right)^9$**

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$n = 9, \ a = \frac{2}{x}, \ x = (-a)$$

$$T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r$$

$$= \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-1)^r a^r \quad \ldots\ldots\ldots (1)$$

We have to find term involving a^4, so comparing the powers of a, we get

$$T_{4+1} = \binom{9}{4}\left(\frac{2}{x}\right)^{9-4}(-1)^4 (a)^4$$

$$= (126) \times \frac{2^5}{x^5} \times 1 \times a^4 = 126 \times \frac{32}{x^5} a^4$$

$$T_5 = 4032 \frac{a^4}{x^5}$$

(iv) **y^3 in the expansion of $\left(x - \sqrt{y}\right)^{11}$**

$$a = x, \ x = (-\sqrt{y}), \ n = 11$$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{11}{r}(x)^{11-r}(-\sqrt{y})^r$$

$$= \binom{11}{r} x^{11-r}(-1)^r y^{r/2} \quad \ldots\ldots\ldots (1)$$

We have to find term involving y^3, so comparing the powers of y we get

$$\frac{r}{2} = 3 \quad \Rightarrow \quad r = 6 \ \text{ Put in (1)}$$

$$T_{6+1} = \binom{11}{6} x^{11-6}(-1)^6 y^{6/2}$$

$$T_7 = 462\, x^5 \times 1 \times y^3$$

$$T_7 = 462\, x^5 y^3$$

Q.7 Find the coefficient of

(i) x^5 **in the expansion of** $\left(x^2 - \frac{3}{2x}\right)^{10}$

(ii) x^n **in the expansion of** $\left(x^2 - \frac{1}{x}\right)^{2n}$

Solution:

(i) x^5 **in the expansion of** $\left(x^2 - \frac{3}{2x}\right)^{10}$ **(Lahore Board 2003-04)**

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$n = 10, \; a = x^2, \; x = \left(-\frac{3}{2x}\right)$$

$$T_{r+1} = \binom{10}{r} (x^2)^{10-r} \left(-\frac{3}{2x}\right)^r$$

$$= \binom{10}{r} (x)^{20-2r} \left(-\frac{3}{2}\right)^r \frac{1}{x^r}$$

$$= \binom{10}{r} (x)^{20-2r-r} \left(-\frac{3}{2}\right)^r$$

$$= \binom{10}{r} (x)^{20-3r} \left(-\frac{3}{2}\right)^r \quad \ldots\ldots\ldots (1)$$

we have to find the coefficient of x^5, so comparing the powers of x, we get

$$20 - 3r = 5$$

$$15 = 3r \Rightarrow \boxed{r = 5}$$

Put in (1)

$$T_{5+1} = \binom{10}{5} x^{20-15} \left(-\frac{3}{2}\right)^5$$

$$T_6 = 252 \times x^5 \times \frac{-243}{32} = \frac{-15309}{8} x^5$$

Coefficient of x^5 is $\frac{-15309}{8}$

(ii) $\quad$ **x^n in the expansion of $\left(x^2 - \frac{1}{x}\right)^{2n}$**

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{2n}{r} (x^2)^{2n-r} \left(-\frac{1}{x}\right)^r$$

$$= \binom{2n}{r} (x)^{4n-2r} \frac{(-1)^r}{x^r}$$

$$= \binom{2n}{r} x^{4n-3r} (-1)^r \quad \ldots\ldots\ldots (1)$$

we have to find the coefficient of x^n, so comparing the powers of x, we get

$$4n - 3r = n$$

$$4n - n = 3r$$

$$3n = 3r$$

$$\boxed{n = r}$$

Put in (1)

$$T_{n+1} = \binom{2n}{n} x^{4n-3n} (-1)^n = \frac{(2n)!}{n!\,(2n-n)!} x^n (-1)^n$$

$$= \frac{(2n)!}{n!\,n!} x^n (-1)^n$$

Coefficient of x^n is $\frac{(-1)^n (2n)!}{(n!)^2}$

Q.8 $\quad$ **Find 6^{th} term in the expansion of $\left(x^2 - \frac{3}{2x}\right)^{10}$**

Solution:

$$a = x^2, \quad x = \frac{-3}{2x}, \quad n = 10, \quad r = 5$$

We know by general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{10}{5} (x^2)^{10-5} \left(\frac{-3}{2x}\right)^5$$

$$T_6 = 252 \times x^{10} \times \frac{-243}{32\,x^5}$$

$$T_6 = \frac{-15309}{8} x^5$$

Q.9 Find the term independent of x in the following expansions.

(i) $\left(x - \frac{2}{x}\right)^{10}$ **(Gujranwala Board 2003, Lahore Board 2008)**

(ii) $\left(\sqrt{x} + \frac{1}{2x^2}\right)^{10}$

(iii) $(1 + x^2)^3\left(1 + \frac{1}{x^2}\right)^4$

Solution:

(i) $\left(x - \frac{2}{x}\right)^{10}$

$a = x, \quad x = \frac{-2}{x} \quad n = 10, \quad r = ?$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$= \binom{10}{r} x^{10-r} \left(\frac{-2}{x}\right)^r$$

$$= \binom{10}{r} x^{10-r} \frac{(-2)^r}{x^r}$$

$$= \binom{10}{r} x^{10-r-r} (-2)^r$$

$$= \binom{10}{r} x^{10-2r} (-2)^r \quad \ldots\ldots\ldots (1)$$

We have to find the term independent of x i.e., x^0 so comparing the powers of x, we have

$10 - 2r = 0$

$10 = 2r \quad \Rightarrow \quad \boxed{r = 5}$ Put in (1)

$$T_{5+1} = \binom{10}{5} x^{10-2(5)} (-2)^5$$

$$T_6 = 252 \times x^{10-10} \times (-32) = -8064\, x^0 = -8064$$

Equating Index of x to 0 to get expression independent of x

(ii) $\left(\sqrt{x}+\dfrac{1}{2x^2}\right)^{10}$

$a = \sqrt{x}, \quad x = \dfrac{1}{2x^2}, \quad n = 10, \quad r = ?$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$= \binom{10}{r} (\sqrt{x})^{10-r} \left(\frac{1}{2x^2}\right)^r$$

$$= \binom{10}{r} x^{\frac{10-r}{2}} \frac{1}{2^r x^{2r}}$$

$$= \binom{10}{r} x^{\frac{10-r}{2}-2r} \left(\frac{1}{2^r}\right) \qquad \ldots\ldots\ldots (1)$$

We have to find independent of x i.e., x^0 so comparing the powers of 'x', we get

$$\frac{10-r}{2} - 2r = 0$$

$$10 - r - 4r = 0$$

$$10 - 5r = 0$$

$$10 = 5r$$

$2 = r$ Put in (1)

$$T_{2+1} = \binom{10}{2} x^{\frac{10-2}{2}-2(2)} \left(\frac{1}{2^2}\right)$$

$$= 45\, x^{4-4} \left(\frac{1}{4}\right) = \frac{45}{4} x^0 = \frac{45}{4}$$

(iii) $(1+x^2)^3 \left(1+\dfrac{1}{x^2}\right)^4$

$$(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4 = (1+x^2)^3 \frac{(1+x^2)^4}{x^8}$$

$$= \frac{1}{x^8}(1+x^2)^7 \qquad \ldots\ldots\ldots (1)$$

Now $(1+x^2)^7$, we have $a = 1,\ x = x^2,\ n = 7,\ r = ?$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r = \binom{7}{r} (1)^{7-r} (x^2)^r$$

$$= \binom{7}{r} x^{2r} \qquad \text{equation (1) becomes}$$

$$= \frac{1}{x^8}\binom{7}{r} x^{2r}$$

$$= \binom{7}{r} x^{2r-8} \qquad \ldots\ldots\ldots (2)$$

We have to find term independent of x.

i.e. , x^0 so, comparing the powers of x.

$2r - 8 = 0$

$2r = 8 \Rightarrow r = 4$ put in (2).

$$\binom{7}{4} x^{8-8} = \frac{7!}{4! \times (7-4)!} x^0$$

$$= \frac{7!}{4! \times 3!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1} = 35$$

MIDDLE TERM

(1) If n is even then $\left(\frac{n}{2}+1\right)^{th}$ term will be only one middle term.

(2) If n is odd then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms will be the two middle terms.

Q.10 Determine the middle term in the following expansions.

(i) $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$ **(ii)** $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

(iii) $\left(2x - \frac{1}{2x}\right)^{2m+1}$

Solution:

(i) $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$

Since $n = 12$ is even so $\left(\frac{n}{2}+1\right)^{th}$ term i.e.,

$$\left(\frac{12}{2}+1\right)^{th} \text{ term } = 7^{th} \text{ term is the middle term}$$

Thus $r = 6$. Also $a = \frac{1}{x}$, $x = \left(\frac{-x^2}{2}\right)$, $n = 12$

We know that the general formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{6+1} = \binom{12}{6} \left(\frac{1}{x}\right)^{12-6} \left(\frac{-x^2}{2}\right)^6$$

$$= 924 \frac{1}{x^6} \frac{x^{12}}{64}$$

$$T_7 = \frac{231}{16} x^6$$

(ii) $\left(\mathbf{\frac{3}{2} x - \frac{1}{3x}}\right)^{\mathbf{11}}$

Since $n = 11$ is odd so $\left(\frac{11+1}{2}\right)^{th}$ term and $\left(\frac{11+3}{2}\right)^{th}$ term i.e., 6^{th} & 7^{th} terms will be the two middle terms.

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{11}{5} \left(\frac{3}{2} x\right)^{11-5} \left(-\frac{1}{3x}\right)^5$$

$$T_6 = 462 \times \left(\frac{3}{2} x\right)^6 \frac{(-1)^5}{(3x)^5}$$

$$= 462 \times \frac{(3x)^{6-5}}{64} \times -1$$

$$= \frac{-462 \times 3x}{64} = \frac{-693x}{32}$$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{6+1} = \binom{11}{6} \left(\frac{3}{2} x\right)^{11-6} \left(-\frac{1}{3x}\right)^6$$

$$= 462 \times \frac{(3x)^5}{(2)^5} \times \frac{1}{(3x)^6}$$

$$= 462 \times \frac{1}{32} \times \frac{1}{(3x)^{6-5}}$$

$$T_6 = \frac{462}{32 \times 3x} = \frac{77}{16x}$$

Hence two middle terms are $-\frac{693x}{32}$ and $\frac{77}{16x}$

(iii) $\left(\mathbf{2x - \frac{1}{2x}}\right)^{\mathbf{2m+1}}$

As $2m+1$ is odd, so there are two middle terms i.e., $\left(\frac{2m+1+1}{2}\right)$ and $\left(\frac{2m+1+3}{2}\right)$ are two middle terms.

$(m+1)^{th}$ and $(m+2)^{th}$ terms

For $(m+1)^{th}$ term

$$r = m, \quad n = 2m+1, \quad a = 2x, \quad x = \left(-\frac{1}{2x}\right)$$

$$T_{m+1} = \binom{2m+1}{m} (2x)^{2m+1-m} \left(-\frac{1}{2x}\right)^m$$

$$= \frac{(2m+1)!}{m!\,[2m+1-m]!} (2x)^{m+1-m} (-1)^m$$

$$= \frac{(2m+1)!}{m!\,(m+1)!} 2x\,(-1)^m$$

$$T_{m+1} = 2\,(-1)^m \frac{(2m+1)!}{m!\,(m+1)!} x$$

For $(m+2)^{th}$ term

$$r = m+2-1 = m+1$$

$$n = 2m+1, \quad a = 2x, \quad x = \left(-\frac{1}{2x}\right)$$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$= \binom{2m+1}{m+1} (2x)^{2m+1-m-1} \left(-\frac{1}{2x}\right)^{m+1}$$

$$= \frac{(2m+1)!}{(m+1)!\,[(2m+1-m-1)]!} (2x)^m \frac{(-1)^{m+1}}{(2x)^{m+1}} = \frac{(2m+1)!}{(m+1)!(m)!} \frac{(-1)^{m+1}}{(2x)^{m+1-m}}$$

$$T_{m+2} = \frac{(2m+1)!\,(-1)^{m+1}}{m!\,(m+1)!\,2x}$$

T_{m+1} and T_{m+2} are two middle terms.

Q.11 Find $(2n+1)^{th}$ term from the end in the expansion of $\left(x - \dfrac{1}{2x}\right)^{3n}$

Solution:

To find $(2n+1)^{th}$ terms, we have $r = 2n$

And for the term from the end, we have

$$a = -\frac{1}{2x} \quad \text{and} \quad x = x$$

By general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{2n+1} = \binom{3n}{2n} \left(-\frac{1}{2x}\right)^{3n-2n} (x)^{2n}$$

$$= \frac{3n!}{2n!\,(3n-2n)!} \left(\frac{-1}{2x}\right)^n x^{2n}$$

$$= \frac{3n!}{(2n)!\,n!} \frac{(-1)^n}{2^n x^n} x^{2n}$$

$$= \frac{3n!\,(-1)^n}{2n!\,n!\,2^n} x^{2n-n}$$

$$= \frac{(3n)!\,(-1)^n x^n}{2n!\,n!\,2n}$$

Q.12 Show that middle term of $(1+x)^{2n}$ is $\dfrac{1.3.5 \ldots\ldots (2n-1)}{n!} 2^n . x^n$

Solution:

As $2n$ is even so $\left(\dfrac{2n}{2}+1\right)^{th}$ term is the middle term i.e., $(n+1)^{th}$ term $r = n$

General term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{n+1} = \binom{2n}{n} (1)^{2n-n} x^n$$

$$= \frac{(2n)!}{n!\,[2n-n]!} (1)^n x^n = \frac{(2n)!}{n!\,n!} x^n$$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4) \ldots\ldots 5 \times 4 \times 3 \times 2 \times 1}{n!\,n!} x^n$$

$$= \frac{[(2n)(2n-2)(2n-4) \ldots . 4 \times 2][(2n-1)(2n-3)(2n-5) \ldots . 5 \times 3 \times 1] x^n}{n!\, n!}$$

$$= \frac{[2^n (n)(n-1)(n-2) \ldots (n-2) \ldots \times 2 \times 1][(2n-1)(2n-3) \ldots 5 \times 3 \times 1] x^n}{n!\, n!}$$

$$= \frac{2^n n! [(2n-1)(2n-3) \ldots . 5 \times 3 \times 1] x^n}{n!\, n!}$$

$$T_{n+1} = \frac{2^n [1 \times 3 \times 5 \times \ldots . (2n-3)(2n-1)] x^n}{n!}$$

Q.13 Show that: $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots .. + \binom{n}{n-1} = 2^{n-1}$

Solution:

We know that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \ldots \ldots . + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n \quad (1)$$

Put $x = 1$ in equation (1)

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \ldots \ldots . + \binom{n}{n-1} + \binom{n}{n}$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \ldots \ldots .. + \binom{n}{n-1} + \binom{n}{n} \quad (2)$$

Next put $x = -1$ in equation (1)

$$(1-1)^n = \binom{n}{0} - \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \ldots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$$

if n is even then

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n \binom{n}{n}$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots \ldots \ldots + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots \ldots + \binom{n}{n-1} \ldots (3)$$

We can write (2) as.

$$2^n = \left\{\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{n}\right\} + \left\{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}\right\} \quad (4)$$

Using (3) in (4)

$$2^n = \left\{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}\right\} + \left\{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}\right\}$$

$$2^n = 2\left\{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}\right\}$$

$$\frac{2^n}{2} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{n-1}$$

$$\Rightarrow \quad \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots . + \binom{n}{n-1} = 2^{n-1}$$

Q.14 Show that $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+.....+\frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

Solution:

$$\text{L. H. S.} = \binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+.....+\frac{1}{n+1}\binom{n}{n}$$

$$= \frac{n!}{0!\,(n-0)!}+\frac{1}{2}\,\frac{n!}{1!\,(n-1)!}+\frac{1}{3}\,\frac{n!}{2!\,(n-2)!}+\frac{1}{4}\,\frac{n!}{3!\,(n-3)!}+\ldots+\frac{1}{n+1}\,\frac{n!}{n!\,(n-n)!}$$

Taking common $n!$

$$= n!\left[\frac{1}{n!}+\frac{1}{2!\,(n-1)!}+\frac{1}{3!\,(n-2)!}+\frac{1}{4!\,(n-3)!}+\ldots.+\frac{1}{(n+1)\,n!}\right]$$

Now multiplying and dividing by $n+1$

$$= \frac{(n+1)\,n!}{(n+1)}\left[\frac{1}{n!}+\frac{1}{2!\,(n-1)!}+\frac{1}{3!\,(n-2)!}+\frac{1}{4!\,(n-3)!}+\ldots+\frac{1}{(n+1)\,n!}\right]$$

$$= \frac{(n+1)!}{n+1}\left[\frac{1}{n!}+\frac{1}{2!\,(n-1)!}+\frac{1}{3!\,(n-2)!}+\frac{1}{4!\,(n-3)!}+\ldots+\frac{1}{(n+1)\,n!}\right]$$

$$= \frac{1}{n+1}\left[\frac{(n+1)!}{n!}+\frac{(n+1)!}{2!\,(n-1)!}+\frac{(n+1)!}{3!\,(n-2)!}+\frac{(n+1)!}{4!\,(n-3)!}+\ldots+\frac{(n+1)!}{(n+1)\,n!}\right]$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots.+\binom{n+1}{n+1}\right]$$

Adding and subtracting $\binom{n+1}{0}$

$$= \frac{1}{n+1}\left[\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\ldots.+\binom{n+1}{n+1}-\binom{n+1}{0}\right] \quad (1)$$

We know that

$$\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\ldots.+\binom{n+1}{n+1} = 2^{n+1}$$

and $\binom{n+1}{0} = 1$

So (1) becomes

$$\frac{1}{n+1}\,[2^{n+1}-1] = \frac{2^{n+1}-1}{n+1} = \text{R.H.S.}$$

Hence proved.