EXERCISE 10.2

Q.1 Prove that

(i) $\sin(180+\theta) = -\sin\theta$

(ii) $\cos(180^o+\theta) = -\cos\theta$

(iii) $\tan(270^o-\theta) = \cot\theta$

(iv) $\cos(\theta-180^o) = -\cos\theta$

(v) $\cos(270^o+\theta) = \sin\theta$ **(Lahore Board 2009)**

(vi) $\sin(\theta+270^o) = -\cos\theta$

(vii) $\tan(180^o+\theta) = \tan\theta$

(viii) $\cos(360^o-\theta) = \cos\theta$

Solution:

(i) $\sin(180+\theta) = -\sin\theta$

$$\text{L.H.S} = \sin(180^o+\theta) = \sin 180^o\cos\theta + \cos 180^o\sin\theta$$
$$= 0 \times \cos\theta + (-1)\sin\theta$$
$$= -\sin\theta = \text{R.H.S.}$$

Hence proved.

(ii) $\cos(180^o+\theta) = -\cos\theta$

$$\text{L.H.S.} = \cos(180^o+\theta)$$
$$= \cos 180^o\cos\theta - \sin 180^o\sin\theta$$
$$= -1 \times \cos\theta - (0)\times\sin\theta$$
$$= -\cos\theta = \text{R.H.S.}$$

Hence proved.

(iii) $\tan(270^o-\theta) = \cot\theta$

$$\text{L.H.S.} = \tan(270^o-\theta)$$
$$= \frac{\sin(270^o-\theta)}{\cos(270^o-\theta)}$$
$$= \frac{\sin 270^o\cos\theta - \sin\theta\cos 270^o}{\cos 270^o\cos\theta + \sin 270^o\sin\theta}$$
$$= \frac{-\cos\theta}{-\sin\theta} = \cot\theta = \text{R.H.S.}$$

Hence proved.

(iv) $\mathbf{\cos(\theta - 180^o) = -\cos\theta}$

L.H.S. $= \cos(\theta - 180^o)$

$= \cos\theta\cos 180^o + \sin\theta\sin 180^o$

$= -\cos\theta = \text{R.H.S.}$

Hence proved.

(v) $\mathbf{\cos(270^o + \theta) = \sin\theta}$ **(Lahore Board 2009)**

L.H.S. $= \cos(270^o + \theta)$

$= \cos 270^o \cos\theta - \sin 270^o \sin\theta$

$= 0 \times (\cos\theta) - (-1)\sin\theta$

$= \sin\theta = \text{R.H.S.}$

Hence proved.

(vi) $\mathbf{\sin(\theta + 270^o) = -\cos\theta}$

L.H.S. $= \sin(\theta + 270^o)$

$= \sin\theta\cos 270^o + \cos\theta\sin 270^o$

$= \sin\theta \times 0 + \cos\theta(-1)$

$= -\cos\theta = \text{R.H.S.}$

Hence proved.

(vii) $\mathbf{\tan(180^o + \theta) = \tan\theta}$

L.H.S. $= \tan(180^o + \theta)$

$$= \frac{\tan 180^o + \tan\theta}{1 - \tan 180^o \tan\theta}$$

$$= \frac{0 + \tan\theta}{1 - 0} = \frac{\tan\theta}{1} = \tan\theta = \text{R.H.S.}$$

Hence proved.

(viii) $\mathbf{\cos(360^o - \theta) = \cos\theta}$

L.H.S. $= \cos(360^o - \theta)$

$= \cos 360^o \cos\theta + \sin 360^o \sin\theta$

$= 1 \times \cos\theta + 0 \times \sin\theta$

$= \cos\theta = \text{R.H.S.}$

Hence proved.

Q.2 Find the values of the following:

(i) sin 15° (ii) cos 15° (iii) tan 15° (iv) sec 15°

(v) sin 105° (vi) cos 105° (vii) tan 105° (viii) sec 105°

Solution:

(i) sin 15°

$= \sin(60^\circ - 45^\circ)$

$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ Ans.

(ii) cos 15°

$= \cos(60^\circ - 45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ Ans.

(iii) tan 15°

$= \tan(60^\circ - 45^\circ)$

$= \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$

$= \frac{\sqrt{3}-1}{1+\sqrt{3}}$ Ans.

(iv) sin 105°

$= \sin(60^\circ + 45^\circ)$

$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ Ans.

(v) cos 105°

$= \cos(60^\circ + 45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ Ans.

(vi) $\tan 105^o$

$$= \tan(60^o + 45^o)$$

$$= \frac{\tan 60° + \tan 45^o}{1 - \tan 60^o \tan 45^o}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \text{ Ans.}$$

Q.3 Prove that

(i) $\sin(45^o + \alpha) = \frac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha)$

(ii) $\cos(\alpha + 45^o) = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

Solution:

(i) $\sin(45^o + \alpha) = \frac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha)$

$$\text{L.H.S.} = \sin(45^o + \alpha) = \sin 45^o \cos\alpha + \cos 45^o \sin\alpha$$

$$= \frac{1}{\sqrt{2}}\cos\alpha + \frac{1}{\sqrt{2}}\sin\alpha$$

$$= \frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha) = \text{R.H.S.}$$

Hence proved.

(ii) $\cos(\alpha + 45^o) = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$

$$\text{L.H.S.} = \cos(\alpha + 45^o)$$

$$= \cos\alpha\cos 45^o - \sin\alpha \sin 45^o$$

$$= \frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha$$

$$= \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha) = \text{R.H.S.} \quad \text{Hence proved.}$$

Q.4 Prove that

(i) $\tan(45^o + A)\tan(45^o - A) = 1$

(ii) $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

(iii) $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

(iv) $\dfrac{\sin\theta - \cos\theta \cdot \tan\frac{\theta}{2}}{\cos\theta + \sin\theta \cdot \tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$

(v) $\dfrac{1 - \tan\theta\tan\phi}{1 + \tan\theta\tan\phi} = \dfrac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$

Solution:

(i) $\tan(45^o + A)\tan(45^o - A) = 1$

$$\text{L.H.S.} = \tan(45^o + A)\tan(45^o - A)$$

$$= \frac{\tan 45^o + \tan A}{1 - \tan 45^o \tan A} \cdot \frac{\tan 45^o - \tan A}{1 + \tan 45^o \tan A}$$

$$= \frac{1 + \tan A}{1 - \tan A} \cdot \frac{1 - \tan A}{1 + \tan A}$$

$$= 1 = \text{R.H.S.} \quad \text{Hence proved.}$$

(ii) $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

$$\text{L.H.S.} = \tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta} + \frac{\tan\frac{3\pi}{4} + \tan\theta}{1 - \tan\frac{3\pi}{4}\tan\theta}$$

$$= \frac{1 - \tan\theta}{1 + \tan\theta} + \frac{-1 + \tan\theta}{1 + \tan\theta}$$

$$= \frac{1 - \tan\theta - 1 + \tan\theta}{1 + \tan\theta}$$

$$= \frac{0}{1 + \tan\theta} = 0 = \text{R.H.S.}$$

Hence proved.

(iii) $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

$$\text{L.H.S.} = \sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right)$$

$$= \left[\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right] + \left[\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right]$$

$$= \left(\sin\theta\frac{\sqrt{3}}{2} + \cos\theta\frac{1}{2}\right) + \left(\cos\theta\frac{1}{2} - \sin\theta\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

$$= \cos\theta = \text{R.H.S.}$$

Hence proved.

(iv) $$\frac{\sin\theta - \cos\theta\,.\,\tan\frac{\theta}{2}}{\cos\theta + \sin\theta\,.\,\tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\text{R.H.S.} = \tan\frac{\theta}{2}$$

$$= \tan\left(\theta - \frac{\theta}{2}\right)$$

$$= \frac{\tan\theta - \tan\frac{\theta}{2}}{1 + \tan\theta\tan\frac{\theta}{2}} = \frac{\frac{\sin\theta}{\cos\theta} - \tan\frac{\theta}{2}}{1 + \frac{\sin\theta}{\cos\theta}\tan\frac{\theta}{2}}$$

$$= \frac{\sin\theta - \cos\theta\tan\frac{\theta}{2}}{\cos\theta + \sin\tan\frac{\theta}{2}} = \text{L.H.S.}$$

Hence proved.

Alternative Method:

$$\text{L.H.S} = \frac{\sin\theta - \cos\theta\tan\theta/2}{\cos\theta + \sin\theta\tan\theta/2}$$

$$= \frac{\sin\theta - \cos\theta\frac{\sin\theta/2}{\cos\theta/2}}{\cos\theta + \sin\theta\frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\sin\theta\cos\theta/2 - \cos\theta\sin\theta/2}{\cos\theta\cos\theta/2 + \sin\theta\sin\theta/2}$$

$$= \frac{\sin(\theta - \theta/2)}{\cos(\theta - \theta/2)} = \frac{\sin\theta/2}{\cos\theta/2}$$

$$= \tan\theta/2$$

$$= \text{R.H.S}$$

(v) $$\frac{1 - \tan\theta\tan\phi}{1 + \tan\theta\tan\phi} = \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$$

$$\text{L.H.S.} = \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$$

$$= \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$$

Alternative Method:

$$\text{L.H.S} = \frac{1 - \tan\theta\tan\theta}{1 + \tan\theta\tan\theta}$$

Dividing up & down by $\cos\theta\cos\phi$

$$= \frac{\frac{\cos\theta\cos\phi}{\cos\phi\cos\theta} - \frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}}{\frac{\cos\theta\cos\phi}{\cos\theta\cos\phi} + \frac{\sin\theta\sin\phi}{\cos\theta\cos\phi}}$$

$$= \frac{1 - \tan\theta\tan\phi}{1 + \tan\theta\tan\phi}$$

$=$ L.H.S. Hence proved.

$$= \frac{1 - \frac{\sin\theta}{\cos\theta}\frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin\theta}{\cos\theta}\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\cos\theta\cos\theta - \sin\theta\sin\theta}{\cos\theta\cos\theta}}{\frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos\theta\cos\theta}}$$

$$= \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$$

$=$ R.H.S

Q.5 Show that

$$\mathbf{\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha}$$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \cos(\alpha+\beta)\cos(\alpha-\beta) \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= (\cos\alpha\ \cos\beta)^2 - (\sin\alpha\sin\beta)^2 \\
&= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta \\
&= \cos^2\alpha(1-\sin^2\beta) - (1-\cos^2\alpha)\sin^2\beta \\
&= \cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\beta + \cos^2\alpha\sin^2\beta \\
&= \cos^2\alpha - \sin^2\beta = \text{R.H.S.}
\end{aligned}$$

Again,

$$\begin{aligned}
\text{L.H.S.} &= \cos(\alpha+\beta)\cos(\alpha-\beta) \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\text{in}\beta) \\
&= (\cos\alpha\cos\beta)^2 - (\sin\alpha\sin\beta)^2 \\
&= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta \\
&= (1-\sin^2\alpha)\cos^2\beta - \sin^2\alpha(1-\cos^2\beta) \\
&= \cos^2\beta - \sin^2\alpha\cos^2\beta - \sin^2\alpha + \sin^2\alpha\cos^2\beta \\
&= \cos^2\beta - \sin^2\alpha = \text{R.H.S.}
\end{aligned}$$

Q.6 Show that $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

Solution:

$$\text{L.H.S.} = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$= \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \tan\alpha = \text{R.H.S}$$

Hence proved.

Q.7 Show that

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

(ii) $\cot(\alpha-\beta) = \dfrac{\cos\alpha\cos\beta+1}{\cos\beta-\cos\alpha}$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:

(i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

$$\text{L.H.S.} = \cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$$

$$= \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Dividing up & down by $\sin\alpha\sin\beta$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

$$= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$= \text{R.H.S.}$ Hence proved

Alternative Method:

$$\text{R.H.S} = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha}\dfrac{\cos\beta}{\sin\beta}-1}{\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\cos\beta}{\sin\beta}}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \cot(\alpha+\beta) = \text{L.H.S}$$

(ii) $\cot(\alpha - \beta) = \dfrac{\cot\alpha \cot\beta + 1}{\cot\beta - \cot\alpha}$

$$\text{L.H.S.} = \cot(\alpha-\beta) = \frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)}$$

$$= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

Dividing up & down by $\sin\alpha\sin\beta$

$$= \frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} + \dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} - \dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$$

= R.H.S. Hence proved.

Alternative Method:

$$\text{R.H.S} = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$$

$$= \frac{\dfrac{\cos\alpha}{\sin\alpha}\dfrac{\cos\beta}{\sin\beta} + 1}{\dfrac{\cos\beta}{\sin\beta} - \dfrac{\cos\alpha}{\sin\alpha}}$$

$$= \frac{\dfrac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$= \frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)}$$

$$= \cot(\alpha-\beta)$$

= L.H.S

(iii) $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

$$\text{R.H.S.} = \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha in\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

Dividing up & down by $\cos\alpha\cos\beta$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$$

= L.H.S. Hence proved

Alternative Method:

$$\text{L.H.S} = \frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \text{R.H.S}$$

Q.8 **If $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{40}{41}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.**

Show that $\sin(\alpha - \beta) = \frac{133}{205}$

Solution:

$\sin\alpha = \frac{4}{5}$ $\quad 0 < \alpha < \frac{\pi}{2}, 0 < \beta < \pi/2 \Rightarrow \alpha$ and β are in I quadrant.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$\cos\alpha = \frac{3}{5}$ $\quad (\because \ \alpha$ is in I Quadrant)

$$\cos\beta = \frac{40}{41}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{1600}{1681}} = \sqrt{\frac{1681-1600}{1681}} = \sqrt{\frac{81}{1681}} = \pm\frac{9}{41}$$

$$\sin\beta = \pm\frac{9}{41}$$

$\sin\beta = \frac{9}{41}$ $\quad (\because \ \beta$ is in I Quadrant)

we know that

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{4}{5} \times \frac{40}{41} - \frac{3}{5} \times \frac{9}{41}$$

$$= \frac{160}{205} - \frac{27}{205} = \frac{160-27}{205}$$

$\sin(\alpha-\beta) = \frac{133}{205}$ $\quad$ Ans.

Q.9 **If $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{12}{13}$ where $\frac{\pi}{2} < \alpha < \pi$ & $\frac{\pi}{2} < \beta < \pi$. Find**

(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$ (iii) $\tan(\alpha+\beta)$

(iv) $\sin(\alpha-\beta)$ (v) $\cos(\alpha-\beta)$ (vi) $\tan(\alpha-\beta)$

Solution:

(i) $\sin(\alpha+\beta)$

$\sin\alpha = \frac{4}{5}$ ($\frac{\pi}{2} < \alpha < \pi \Rightarrow \alpha$ is in II Quadrant)

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$\cos\alpha = \frac{-3}{5}$ ($\because$ α is in II Quadrant)

$\sin\beta = \frac{12}{13}$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

$\cos\beta = \frac{-5}{13}$ (as β lie in II Quadrant).

we know $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{4}{5} \times \frac{-5}{13} + \frac{-3}{5} \times \frac{12}{13} = \frac{-4}{13} - \frac{36}{65} = \frac{-20-36}{65} = -\frac{56}{65} \quad \text{Ans.}$$

(ii) $\cos(\alpha+\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= -\frac{3}{5} \times \frac{-5}{13} - \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} - \frac{48}{65} = \frac{15-48}{65} = \frac{-33}{65} \quad \text{Ans.}$$

(iii) $\tan(\alpha+\beta)$

$$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\frac{-56}{65}}{\frac{-33}{65}} = \frac{56}{33} \quad \text{Ans.}$$

(iv) $\sin(\alpha-\beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{4}{5} \times \frac{-5}{13} - \frac{-3}{5} \times \frac{12}{13} = \frac{-20}{65} + \frac{36}{65} = \frac{-20+36}{65} = \frac{16}{65} \quad \text{Ans.}$$

(v) $\cos(\alpha - \beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{-3}{5} \times \frac{-5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} + \frac{48}{65}$

$\cos(\alpha - \beta) = \frac{63}{65}$ Ans.

(vi) $\tan(\alpha - \beta)$

$$= \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\frac{16}{65}}{\frac{63}{65}}$$

$\tan(\alpha - \beta) = \frac{16}{63}$ Ans.

Since $\sin(\alpha - \beta)$, $\cos(\alpha - \beta)$, $\tan(\alpha - \beta)$ are all +ve, so terminal side of $(\alpha - \beta)$ is in I Quadrant.

Now $\sin(\alpha + \beta)$ is $-$ ve, $\cos(\alpha + \beta)$ is $-$ ve and $\tan(\alpha + \beta)$ is +ve. Thus terminal side of $\alpha + \beta$ is in III Quadrant.

Q.10 Find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ given that

(i) $\tan\alpha = \frac{3}{4}$, $\cos\beta = \frac{5}{13}$ and neither the terminal side of angle of measure α nor that of β in I quadrant. (Lahore Board 2005)

(ii) $\tan\alpha = -\frac{15}{18}$, $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor β is in the IV quadrant.

Solution:

(i) $\tan\alpha = \frac{3}{4}$

$\tan\alpha = \frac{3}{4} \Rightarrow \cot\alpha = \frac{4}{3}$

i.e. is + ve, so terminal side in the III Quadrant

we know that

$$\sec\alpha = \sqrt{1 + \tan^2\alpha} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$$

$\sec\alpha = \frac{-5}{4} \Rightarrow \cos\alpha = \frac{-5}{4}$

$\cos ec\ \alpha = \sqrt{1+\cot^2\alpha} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{9+16}{9}} = \sqrt{\frac{25}{9}} = \pm\frac{5}{3}$

$\text{cosec}\ \alpha = \frac{-5}{3} \Rightarrow \sin\alpha = \frac{-3}{5}$, $\sin\beta = \sqrt{1-\frac{25}{169}} = \pm\frac{12}{13}$ $\sin\beta = -\frac{12}{13}$

Now

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{-3}{5} \times \frac{5}{13} + \frac{-4}{5} \times \frac{-12}{13}$

$= \frac{-3}{13} + \frac{48}{65} = \frac{-15+48}{65}$

$\boxed{\sin(\alpha+\beta) = \frac{33}{65}}$ Ans.

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{-4}{5} \times \frac{5}{13} - \frac{-3}{5} \times \frac{-12}{3}$

$= \frac{-4}{13} - \frac{36}{65} = \frac{-20-36}{65}$

$\boxed{\cos(\alpha+\beta) = \frac{-56}{65}}$ Ans.

(ii) $\tan\alpha = -\frac{15}{18}$, $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor β is in the IV quadrant.

$\tan\alpha = \frac{-15}{8} \Rightarrow \cot\alpha = \frac{-8}{15}$

since tan is – ve. Thus terminal side in the II Quadrant

$\sec\alpha = \sqrt{1+\tan^2\alpha} = \sqrt{1+\frac{225}{64}} = \sqrt{\frac{64+225}{64}} = \sqrt{\frac{289}{64}} = \pm\frac{17}{8}$

$\sec\alpha = \frac{-17}{8} \Rightarrow \cos\alpha = \frac{-8}{17}$

$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}} = \pm\frac{15}{17}$

$\sin\alpha = \frac{15}{17}$

$\sin\beta = \frac{-7}{25} \quad \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{49}{625}} = \sqrt{\frac{625-49}{625}} = \pm\frac{24}{25}$

$\cos\beta = \frac{-24}{25}$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{15}{17} \times \frac{-24}{25} + \frac{-8}{17} \times \frac{-7}{25}$$

$$= \frac{-360}{425} + \frac{56}{425} = \frac{-360+56}{425} = \frac{-304}{425} \qquad \text{Ans.}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{-8}{17} \times \frac{-24}{25} - \frac{15}{17} \times \frac{-7}{25}$$

$$= \frac{192}{425} + \frac{105}{425} = \frac{192+105}{425}$$

$$\boxed{\cos(\alpha+\beta) = \frac{297}{425}} \qquad \text{Ans.}$$

Q.11 Prove that $\frac{\cos 8^o - \sin 8^o}{\cos 8^o + \sin 8^o} = \tan 37^o$

Solution:

$$\text{R.H.S.} = \tan 37^o$$

$$= \tan(45^o - 8^o)$$

$$= \frac{\tan 45^o - \tan 8^o}{1 + \tan 45^o \tan 8^o} = \frac{1-\tan 8^o}{1+\tan 8^o}$$

$$= \frac{1-\frac{\sin 8^o}{\cos 8^o}}{1+\frac{\sin 8^o}{\cos 8^o}} = \frac{\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}{\frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}} = \frac{\cos 8^o - \sin 8^o}{\cos 8^o + \sin 8^o} = \text{L.H.S.}$$

Hence proved.

Q.12 If α, β, γ are the angles of a triangle ABC, show that

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}.\cot\frac{\gamma}{2}$$ **(Lahore Board 2004)**

Solution:

Since α, β, γ are the angles of a triangle ABC

$$\alpha + \beta + \gamma = 180^o$$

$$\frac{\alpha + \beta + \gamma}{2} = 90^o$$

$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^o$$

$$\frac{\alpha}{2} + \frac{\beta}{2} = 90^o - \frac{\gamma}{2} \quad \text{......... (1)}$$

taking tan on both sides

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(90^o - \frac{\gamma}{2}\right)$$

$$\frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2}$$

$$\frac{\frac{1}{\cot\frac{\alpha}{2}} + \frac{1}{\cot\frac{\beta}{2}}}{1 - \cot\frac{\alpha}{2}.\cot\frac{\beta}{2}} = \cot\frac{\gamma}{2}$$

$$\frac{\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1} = \cot\frac{\gamma}{2}$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} = \cot\frac{\gamma}{2}\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1\right)$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} = \cot\frac{\alpha}{2}\ \cot\frac{\beta}{2}\ \cot\frac{\gamma}{2} - \cot\frac{\gamma}{2}$$

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\ \cot\frac{\beta}{2}\ \cot\frac{\gamma}{2}$$

Hence proved.

Q.13 If $\alpha + \beta + \gamma = 180^o$, show that
$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$ (Gujranwala Board 2005)

Solution:

$\alpha + \beta + \gamma = 180^o$

$\alpha + \beta = 180^o - \gamma$

taking tan on both sides

$\tan(\alpha + \beta) = \tan(180^o - \gamma)$

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma$$

$$\frac{\frac{1}{\cot\alpha} + \frac{1}{\cot\beta}}{1 - \frac{1}{\cot\alpha}\frac{1}{\cot\beta}} = -\frac{1}{\cot\gamma}$$

$$\frac{\cot\alpha + \cot\beta}{\cot\alpha\cot\beta - 1} = \frac{-1}{\cot\gamma}$$

$\cot\gamma(\cot\alpha + \cot\beta) = -(\cot\alpha\cot\beta - 1)$

$\cot\alpha\cot\gamma + \cot\beta\cot\gamma = -\cot\alpha\cot\beta + 1$

$\cot\alpha\cot\gamma + \cot\beta\cot\gamma + \cot\alpha\cot\beta = 1$

Hence proved.

Q.14 Express the following in form $\gamma\sin(\theta + \phi)$ where terminal sides of the angles of measure θ and ϕ are in first quadrant

(i) $12\sin\theta + 5\cos\theta$ **(ii) $3\sin\theta - 4\cos\theta$**

(iii) $\sin\theta - \cos\theta$

(iv) $5\sin\theta - 4\cos\theta$ **(Gujranwala Board 2007)**

(v) $\sin\theta + \cos\theta$ **(vi) $3\sin\theta - 5\cos\theta$**

Solution:

(i) $12\sin\theta + 5\cos\theta$

Since $r\sin(\theta + \phi) = r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= r\sin\theta\cos\phi + r\cos\theta\sin\phi$

Let $r\cos\phi = 12$, $r\sin\phi = 5$

Squaring & adding

$r^2\cos^2\phi + r^2\sin^2\phi = 144 + 25$

$r^2(\cos^2\phi + \sin^2\phi) = 169$

$r^2 = 169 \quad \Rightarrow \quad r = 13$

$\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{5}{12}$

$$12\sin\theta + 5\cos\theta = \frac{13}{13}(12\sin\theta + 5\cos\theta)$$

$$= 13\left(\frac{12}{13}\sin\theta + \frac{5}{13}\cos\theta\right)$$

$$= r(\cos\phi\sin\theta + \sin\phi\cos\theta)$$

$$= r\sin(\theta + \phi)$$

$$= \boxed{\text{where } \phi = \tan^{-1}\frac{5}{12}} \qquad \text{Ans.}$$

(ii) $3\sin\theta - 4\cos\theta$

$r\sin(\theta - \phi) = r\sin\theta\cos\phi - r\cos\theta\sin\phi$

Let $r\cos\phi = 3$ .... (i)

$r\sin\phi = +4$ .... (ii)

Taking square & adding

$r^2\cos^2\phi + r^2\sin^2\phi = (3)^2 + (+4)^2$

$r^2(\cos\phi + \sin\phi)^2 = 9 + 16$

$r^2 = 25 \quad \Rightarrow \quad r = 5$

dividing (ii) by (i)

$\dfrac{r\sin\phi}{r\cos\phi} = \dfrac{+4}{3}$

$\tan\phi = \dfrac{4}{3}$

$$3\sin\theta - 4\cos\theta = \frac{5}{5}(3\sin\theta - 4\cos\theta)$$

$$= 5\left(\frac{3}{5}\sin\theta - \frac{4}{5}\cos\theta\right)$$

$$= 5(\cos\phi\sin\theta - \sin\phi\cos\theta)$$

$$= r\sin(\theta - \theta)$$

(iii) $\sin\theta - \cos\theta$

$r\sin(\theta - \phi) = r\sin\theta\cos\phi - r\cos\theta\sin\phi$

Let $r\cos\phi = 1$ ……… (i)

$r\sin\phi = 1$ ……… (ii)

Taking square of (i) & (ii) & adding

$r^2(\cos^2\phi + \sin^2\phi) = 1 + 1$

$r^2 = 2$

$r = \sqrt{2}$

Dividing (ii) by (i) we have

$\frac{r\sin\phi}{r\cos\phi} = 1$

$\tan\phi = 1$

$\boxed{\sqrt{2}\,\sin(\theta - \phi)\ ,\ \tan\phi = 1}$ Ans.

(iv) $5\sin\theta - 4\cos\theta$

$r\sin(\theta - \phi) = r\sin\theta\cos\phi - r\sin\phi\cos\theta$

Let $r\cos\phi = 5$ ……… (i)

$r\sin\phi = 4$ ……… (ii)

Taking square & adding

$r^2(\cos^2\phi + \sin^2\phi) = (5)^2 + (4)^2$

$r^2 = 41$

$r = \sqrt{41}$

Dividing (ii) by (i)

$\frac{r\sin\phi}{r\cos\phi} = \frac{4}{5}$

$\tan\phi = \frac{4}{5}$

$\boxed{\sqrt{41}\,\sin(\theta - \phi)\ ,\ \tan\phi = \frac{4}{5}}$ Ans.

(v) $\sin\theta + \cos\theta$ **(Gujranwala Board 2007)**

$r\sin(\theta + \phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$

Let $r\cos\phi = 1$ ……… (i)

$r\sin\phi = 1$ ……… (ii)

Taking square of (i) & (ii) & adding

$r^2(\cos^2\phi + \sin^2\phi) = 1 + 1$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

Dividing (ii) by (i)

$\frac{r\sin\phi}{r\cos\phi} = 1$

$\tan\phi = 1$

$\boxed{r\sin(\theta+\phi) = \sqrt{2}\sin(\theta+\phi), \ \tan\phi = 1}$ Ans.

(vi) $3\sin\theta - 5\cos\theta$

$r\sin(\theta-\phi) = r\sin\theta\cos\phi - r\cos\theta\sin\phi$

Let $r\cos\phi = 3$ ……… (i)

$r\sin\phi = 5$ ……… (ii)

Taking square & adding of (i) and (ii)

$r^2(\cos^2\phi + \sin^2\phi) = (5)^2 + (3)^2$

$r^2 = 25 + 9$

$r^2 = 34 \Rightarrow r = \sqrt{34}$

$\tan\phi = \frac{5}{3}$ (dividing (ii) by (i))

$\boxed{r\sin(\theta-\phi) = \sqrt{34}\sin(\theta-\phi), \ \tan\phi = \frac{5}{3}}$ Ans.

EXERCISE 10.3

Q.1 Find the values of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ when

(i) $\sin\alpha = \frac{12}{13}$ **(ii) $\tan\alpha = \frac{12}{13}$**

(iii) $\cos\alpha = \frac{3}{5}$ where $0 < \alpha < \frac{\pi}{2}$

Solution

(i) $\sin\alpha = \frac{12}{13}$ $0 < \alpha < \frac{\pi}{2}$

$\sin\alpha = \frac{12}{13}$

$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}}$

$\cos\alpha = \frac{5}{13}$ (since α in I Quadrant)