Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \frac{d}{dx}(\tanh x)$$

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \sec h^2 x$$

$$\frac{dy}{dx} = \frac{\frac{1}{\cosh^2 x}}{\frac{\sinh x}{\cosh . x}}$$

$$\frac{dy}{dx} = \frac{1}{\sinh x \cosh x}$$

$$\frac{dy}{dx} = \frac{2}{2 \sinh x \cosh x} = \frac{2}{\sinh 2x}$$

$$\boxed{\frac{dy}{dx} = 2 \operatorname{cosech} 2x}$$ Ans.

(vi) $\mathbf{y = \sinh^{-1}\left(\frac{x}{2}\right)}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\frac{x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4+x^2}{4}}} \cdot \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{4+x^2}}{2}} \cdot \frac{1}{2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{4+x^2}}}$$ Ans.

EXERCISE 2.7

Q.1: Find y_2 if

(i) $\mathbf{y = 2x^5 - 3x^4 + 4x^3 + x - 2}$ **(ii)** $\mathbf{y = (2x+5)^{\frac{3}{2}}}$

(iii) $\mathbf{y = \sqrt{x} + \frac{1}{\sqrt{x}}}$

Solution:

(i) $\mathbf{y = 2x^5 - 3x^4 + 4x^3 + x - 2}$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

$$\boxed{y_2 = 40x^3 - 36x^2 + 24x} \quad \text{Ans.}$$

(ii) $\mathbf{y = (2x+5)^{\frac{3}{2}}}$

$$y_1 = \frac{3}{2}(2x+5)^{\frac{3}{2}-1} \cdot \frac{d}{dx}(2x+5)$$

$$y_1 = \frac{3}{2}(2x+5)^{\frac{1}{2}-1} \cdot 2$$

$$y_1 = 3(2x+5)^{\frac{1}{2}}$$

$$y_2 = \frac{3}{2}(2x+5)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(2x+5)$$

$$y_2 = \frac{3}{2}(2x+5)^{\frac{-1}{2}} \cdot 2$$

$$\boxed{y_2 = \frac{3}{\sqrt{2x+5}}} \quad \text{Ans.}$$

(iii) $\mathbf{y = \sqrt{x} + \frac{1}{\sqrt{x}}}$

$$y = x^{\frac{1}{2}} + x^{\frac{-1}{2}}$$

Diff. w.r.t. 'x'

$$y_1 = \frac{1}{2}x^{\frac{-1}{2}} + \left(\frac{-1}{2}\right)x^{\frac{-3}{2}}$$

$$y_1 = \frac{1}{2}x^{\frac{-1}{2}} - \frac{-1}{2}x^{\frac{-3}{2}}$$

$$y_2 = \frac{1}{2}\left(\frac{-1}{2}\right)x^{\frac{-3}{2}} - \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{\frac{-5}{2}}$$

$$y_2 = \frac{-1}{4x^{\frac{3}{2}}} + \frac{3}{4\,x^{\frac{5}{2}}}$$

$$\boxed{y_2 = \frac{-x+3}{4\,x^{\frac{5}{2}}}}$$ Ans.

Q.2: **Find y_2 if**

(i) $y = x^2 . e^{-x}$ ***(L.B 2011) (L.B 2008)***

(ii) $y = \ell n\left(\frac{2x+3}{3x+2}\right)$ ***(L.B 2009)***

Solution:

(i) $y = x^2 . e^{-x}$

$$y_1 = x^2 \frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(x^2)$$

$$y_1 = x^2 . e^{-x}\ (-1) + e^{-x}\,2x$$

$$y_1 = -x^2e^{-x} + 2xe^{-x}$$

$$y_1 = e^{-x}(2x - x^2)$$

$$y_2 = e^{-x}\frac{d}{dx}(2x - x^2) + (2x - x^2) . \frac{d}{dx}(e^{-x})$$

$$y_2 = e^{-x}(2 - 2x) + (2x - x^2) . e^{-x} . (-1)$$

$$y_2 = e^{-x}(2 - 2x - 2x + x^2)$$

$$\boxed{y_2 = e^{-x}(x^2 - 4x + 2)}$$ Ans.

(ii) $y = \ell n\left(\frac{2x+3}{3x+2}\right)$

$$y = \ell n\,(2x+3) - \ell n\,(3x+2)$$

$$y_1 = \frac{1}{2x+3} . 2 - \frac{1}{3x+2} . 3$$

$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

$$y_2 = 2\,(-1\,(2x+3)^{-2})\,(2) - 3\,(-1\,(3x+2)^{-2} . 3)$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$y_2 = \frac{-4\,(3x+2)^2 + 9(2x+3)^2}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-4\,(9x^2 + 4 + 12x)^2 + 9(4x^2 + 9 + 12x)}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-36x^2 - 16 - 48x + 36x^2 + 81 + 108x}{(2x+3)^2(3x+2)^2}$$

$$\boxed{y_2 = \frac{60x + 65}{(2x+3)^2(3x+2)^2}} \qquad \text{Ans.}$$

Q.3: Find y_2 if

(i) $x^2 + y^2 = a^2$ **(ii)** $x^3 - y^3 = a^3$

(iii) $x = a\cos\theta \;,\; y = a\sin\theta$ ***(G.B 2006)***

(iv) $x = at^2, \; y = bt^4$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution:

(i) $x^2 + y^2 = a^2$

$$2x + 2yy_1 = 0$$
$$2yy_1 = -2x$$
$$y_1 = \frac{-2x}{2y}$$
$$y_1 = \frac{-x}{y}$$
$$y_2 = -\frac{y\,.\,1 - x.y_1}{y^2}$$
$$y_2 = -\frac{y - x\left(\frac{-x}{y}\right)}{y^2}$$
$$y_2 = -\frac{y + \frac{x^2}{y}}{y^2}$$
$$= -\frac{\frac{y^2 + x^2}{y}}{y^2}$$
$$y_2 = -\frac{y^2 + x^2}{y^3}$$
$$\boxed{y_2 = \frac{-a^2}{y^3}} \qquad \because x^2 + y = a^2 \qquad \text{Ans.}$$

(ii) $x^3 - y^3 = a^3$

$$3x^2 - 3y^2y_1 = 0$$
$$-3y^2y_1 = -3x^2$$

$$y_1 = \frac{-3x^2}{-3y^2}$$

$$y_1 = \frac{x^2}{y^2}$$

$$y_2 = \frac{y^2\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(y^2)}{(y^2)^2}$$

$$y_2 = \frac{y^2 . 2x - x^2 . 2yy_1}{y^4}$$

$$y_2 = \frac{2xy^2 - 2x^2y\left(\frac{x^2}{y^2}\right)}{y^4}$$

$$y_2 = \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

$$y_2 = \frac{\frac{2xy^3 - 2x^4}{y}}{y^4}$$

$$y_2 = \frac{-2x(x^3 - y^3)}{y^5}$$

$$\boxed{y_2 = \frac{-2x\,a^3}{y^5}} \quad \text{Ans.} \qquad \because\ x^3 - y^3 = a^3$$

(iii) $\mathbf{x = a\cos\theta}$, $\mathbf{y = a\sin\theta}$

$x = a\cos\theta$

Diff. w.r.t. 'θ'

$$\frac{dx}{d\theta} = -a\sin\theta$$

$y = a\sin\theta$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = a\cos\theta$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} . \frac{d\theta}{dx}$$

$$y_1 = a\cos\theta \ . \ \frac{-1}{a\sin\theta}$$

$$y_1 = -\cot\theta$$

$$y_2 = -(-\text{cosec}^2\theta) \ . \ \frac{-1}{a\sin\theta}$$

$$\boxed{y_2 = \frac{-1}{a\sin^3\theta}} \quad \text{Ans.}$$

(iv) $\mathbf{x = at^2 \quad , \quad y = bt^4}$

$x = at^2$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = 2\,at$$

$y = bt^4$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = 4\,bt^3$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$y_1 = 4\,bt^3 \;.\; \frac{1}{2at}$$

$$y_1 = \frac{2b}{a}t^2$$

$$y_2 = \frac{x}{a}(2t)\frac{dt}{dx} \Rightarrow y_2 = \frac{4\,bt}{a} \cdot \frac{1}{2at}$$

$$\boxed{y_2 = \frac{2b}{a^2}} \quad \text{Ans.}$$

(v) $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$

$$2x^2 + 2yy_1 + 2g + 2fy_1 = 0$$

$$2(y + f)y_1 = -2x - 2g$$

$$y_1 = \frac{-2(x+g)}{2(y+f)}$$

$$y_1 = -\frac{x+g}{y+f}$$

$$y_2 = -\frac{(y+f)\frac{d}{dx}(x+g) - (x+g)\frac{d}{dx}(y+f)}{(y+f)^2}$$

$$y_2 = -\frac{(y+f) - (x+g)\,.\,y_1}{(y+f)^2}$$

$$y_2 = -\frac{(y+f) - (x+g)\,.\left(\frac{-(x+g)}{y+f}\right)}{(y+f)^2}$$

$$y_2 = -\frac{\dfrac{(y+f)^2+(x+g)^2}{y+f}}{(y+f)^2}$$

$$y_2 = -\frac{y^2+f^2+2fy+x^2+g^2+2gx}{(y+f)^3}$$

$$y_2 = -\frac{-c+f^2+g^2}{(y+f)^3} \qquad \begin{bmatrix} \because & x^2+y^2+2gx+2fy+c=0 \\ & x^2+y^2+2gx+2fy=-c \end{bmatrix}$$

$$\boxed{y_2 = \frac{c-f^2-g^2}{(y+f)^3}} \qquad \text{Ans.}$$

Q.4: Find y_4 if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ell n\,(x^2-9)$

Solution:

(i) $\mathbf{y = \sin 3x}$

$y_1 = \cos 3x.3$

$y_1 = 3\cos 3x$

$y_2 = 3(-\sin 3x)\,.\,3$

$y_2 = -9\sin 3x$

$y_3 = -9\cos 3x\,.\,3$

$y_3 = -27\cos 3x$

$y_4 = -27(-\sin 3x)(3)$

$\boxed{y_4 = 81\sin 3x}$ Ans.

(ii) $\mathbf{y = \cos^3 x}$ **.............. (1)** ***(L.B 2008)***

$\cos 3x = \cos(2x + x)$

$\cos 3x = \cos 2x\cos x - \sin 2x\sin x$

$\cos 3x = (2\cos^2 x - 1)\cos x - 2\sin x\cos x\,.\,\sin x$

$\cos 3x = 2\cos^3 x - \cos x - 2\cos x\sin^2 x$

$\cos 3x = 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)$

$\cos 3x = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$

$4\cos^3 x = \cos 3x + 3\cos x$

$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

$\therefore$ From equation (1)

$$y = \frac{1}{4}(\cos 3x + 3\cos x)$$

$$y_1 = \frac{1}{4}[-\sin 3x \,.\, 3 + 3(-\sin x)]$$

$$y_1 = \frac{1}{4}[-3\sin 3x - 3\sin x]$$

$$y_2 = \frac{1}{4}[-3\cos 3x \,.\, 3 - 3\cos x]$$

$$y_2 = \frac{1}{4}[-9\cos 3x - 3\cos x]$$

$$y_3 = \frac{1}{4}[-9(-\sin 3x) \,.\, 3 - 3(-\sin x)]$$

$$y_3 = \frac{1}{4}[27\sin 3x + 3\sin x]$$

$$y_4 = \frac{1}{4}[27\cos 3x \,.\, 3 + 3\cos x]$$

$$y_4 = \frac{1}{4}[81(4\cos^3 x - 3\cos x) + 3\cos x]$$

$$y_4 = \frac{1}{4}[324\cos^3 x - 243\, x + 3\cos x]$$

$$y_4 = \frac{1}{4}[324\cos^3 x - 240\cos x]$$

$$y_4 = \frac{4}{4}[81\cos^3 x - 60\cos x]$$

$$\boxed{y_4 = -60\cos x + 81\cos^3 x} \qquad \text{Ans.}$$

(iii) $\mathbf{y = \ell n\,(x^2 - 9)}$

$$y = \ell n\,(x+3)(x-3)$$

$$y = \ell n\,(x+3) + \ell n\,(x-3)$$

$$y_1 = \frac{1}{(x+3)} + \frac{1}{(x-3)}$$

$$y_1 = (x+3)^{-1} + (x-3)^{-1}$$

$$y_2 = -(x+3)^{-2} - (x-3)^{-2}$$

$$y_3 = 2(x+3)^{-3} + 2(x-3)^{-3}$$

$$y_4 = -6(x+3)^{-4} - 6(x-3)^{-4}$$

$$y_4 = \frac{-6}{(x+3)^4} - \frac{6}{(x-3)^4}$$

$$\boxed{y_4 = -6\left[\frac{1}{(x+3)^4} + \frac{1}{(x-3)^4}\right]} \quad \text{Ans.}$$

Q.5: If $x = \sin\theta$, $y = \sin m\theta$, Show that $(1 - x^2)\, y_2 - xy_1 + m^2y = 0$
(G.B 2006)

Solution:

$$x = \sin\theta \quad , \quad y = \sin m\theta$$

$$\theta = \sin^{-1}x$$

$$y = \sin m\theta$$

$$y = \sin m\,(\sin^{-1} x)$$

$$\sin^{-1}y = m \sin^{-1} x$$

$$\frac{1}{\sqrt{1-y^2}} \cdot y_1 = \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\; y_1 = m\sqrt{1-y^2}$$

Squaring on both sides

$$(1-x^2)y_1^2 = m^2(1-y^2)$$

$$(1-x^2).\, 2y_1 y_2 + y_1^2 \,.\, (-2x) = m^2(-2yy_1)$$

$$2y_1\,[(1-x^2).\, y_2 - xy_1] = -2m^2yy_1$$

$$(1-x^2).\, y_2 - xy_1 = \frac{-2m^2yy_1}{2y_1}$$

$$(1-x^2).\, y_2 - xy_1 = -m^2y$$

$$\boxed{(1-x^2)y_2 - xy_1 + m^2y = 0} \quad \text{Hence proved.}$$

Q.6: If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ *(L.B 2009)*

Solution:

$$y = e^x \sin x$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin \frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \cos x + \sin x\, e^x$$

$$= e^x(\cos x + \sin x)$$

Diff. again w.r.t. 'x'

$$\frac{d^2y}{d^2x} = e^x \frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x)\frac{d}{dx}(e^x)$$

$$\frac{d^2y}{d^2x} = e^x(-\sin x + \cos x) + (\cos x + \sin x)\, e^x$$

$$\frac{d^2y}{d^2x} = e^x(-\sin x + \cos x + \cos x + \sin x)$$

$$\frac{d^2y}{d^2x} = 2e^x \cos x$$

Taking

$$\frac{d^2y}{d^2x} - 2\frac{dy}{dx} + 2y = 2e^x\cos x - 2e^x(\cos x + \sin x) + 2e^x \sin x$$

$$= 2e^x\cos x - 2e^x\cos x - 2e^x\sin x + 2e^x \sin x$$

$$\boxed{\frac{d^2y}{d^2x} - 2\frac{dy}{dx} + 2y = 0}$$ Hence proved.

Q.7: If $y = e^{ax}\sin bx$, show that $\frac{d^2y}{d^2x} - 2a\frac{d^2y}{d^2x} + (a^2 + b^2)\,y = 0$

Solution:

$$y = e^{ax}\sin bx$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\sin bx) + \sin bx\frac{d}{dx}(e^{ax})$$

$$\frac{dy}{dx} = e^{ax}\cos bx\,.\,b + \sin bx\,.\,e^{ax}\,.\,a$$

$$\frac{dy}{dx} = e^{ax}(b\cos bx + a\sin bx)$$

Diff. again w.r.t. 'x'

$$\frac{d^2y}{d^2x} = e^{ax}(-b\sin bx\,.\,b + a\cos bx\,.\,b) + (b\cos bx + a\sin bx)\,e^{ax}\,.\,a$$

$$\frac{d^2y}{d^2x} = e^{ax}(-b^2\sin bx + ab\cos bx + ab\cos bx + a^2\sin bx)$$

$$\frac{d^2y}{d^2x} = e^{ax}(-b^2\sin bx + 2ab\cos bx + a^2\sin bx)$$

Taking

$$\frac{d^2y}{d^2x} - 2a\frac{dy}{dx} + (a^2 + b^2)y = e^{ax}(-b^2\sin bx + 2ab\cos bx + a^2\sin bx) - 2ae^{ax}$$

$(b\cos bx + a\sin bx) + (a^2 + b^2)\, e^{ax} \sin bx$

$$\frac{d^2y}{d^2x} - 2a\frac{dy}{dx} + (a^2+b^2)y = -e^{ax}\sin bx + 2e^{ax}ab\cos bx + e^{ax}a^2\sin bx - 2abe^{ax}\cos bx - 2a^2e^{ax}\sin bx + a^2e^{ax}\sin bx + b^2e^{ax}\sin bx$$

$$\boxed{\frac{d^2y}{d^2x} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0}$$ Hence proved.

Q.8: If $y = (\cos^{-1}x)^2$, prove that $(1-x^2)\,y_2 - xy_1 - 2 = 0$ ***(G.B 2007)***

Solution:

$$y = (\cos^{-1}x)^2$$

$$y_1 = 2(\cos^{-1}x)\,.\,\frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = -2(\cos^{-1}x)$$

Squaring on both sides

$$(1-x^2)\,y_1^2 = 4(\cos^{-1}x)^2 \qquad \because\ y = (\cos^{-1}x)^2$$

$$(1-x^2)\,y_1^2 = 4y$$

$$(1-x^2)\,.\,2y_1y_2 + y_1^2\,.\,(-2x) = 4y_1$$

$$2y_1[(1-x^2)\,y_2 - xy_1] = 4y_1$$

$$(1-x^2)\,y_2 - xy_1 = \frac{4y_1}{2y_1}$$

$$(1-x^2)\,y_2 - xy_1 = 2$$

$$\boxed{(1-x^2)\,y_2 - xy_1 - 2 = 0}$$ Hence proved.

Q.9: If $y = a\cos(\ell nx) + b\sin(\ell nx)$ Prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution:

$$y = a\cos(\ell nx) + b\sin(\ell nx)$$

$$\frac{dy}{dx} = -a\sin(\ell nx)\,.\,\frac{1}{x} + b\cos(\ell nx)\,.\,\frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\,[-a\sin(\ell nx) + b\cos(\ell nx)]$$

$$\frac{d^2y}{d^2x} = \frac{1}{x}\,\left[-a\cos(\ell nx)\,.\,\frac{1}{x} - b\sin(\ell nx)\,.\,\frac{1}{x}\right]$$

$$+ [(-a\sin(\ell nx) + b\cos(\ell nx)]\left(\frac{-1}{x^2}\right)$$

$$\frac{d^2y}{d^2x} = \frac{1}{x^2}\ [-a\cos(\ell nx) - b\sin(\ell nx) - \frac{1}{x^2}\ [-a\sin(\ell nx) + b\cos(\ell nx)]$$

$$\frac{d^2y}{d^2x} = \frac{1}{x^2}\ [-a\cos(\ell nx) - b\sin(\ell nx) + a\sin(\ell nx) - b\cos(\ell nx)]$$

Taking

$$x^2\frac{d^2y}{d^2x} + x\frac{dy}{dx} + y = x^2 . \frac{1}{x^2}\ [-a\cos(\ell nx) - b\sin(\ell nx) + a\sin(\ell nx)$$

$$-b\cos(\ell nx)] + x.\frac{1}{x}\ [-a\sin(\ell nx) + b\cos(\ell nx)] + a\cos(\ell nx) + b\sin(\ell nx)$$

$$= \quad -a\cos(\ell nx) - b\sin(\ell nx) + a\sin(\ell nx) - b\cos(\ell nx) - a\sin(\ell nx)$$

$$+ b\cos(\ell nx) + a\cos(\ell nx) + b\sin(\ell n\, x)$$

$$\boxed{x^2\frac{d^2y}{d^2x} + x\frac{dy}{dx} + y = 0}$$ Hence proved.

EXERCISE 2.8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ell n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$ ***(L.B 2005)***

(ii) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots\ldots\ldots\ldots$

(iii) $\sqrt{1 + x} = 1 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots\ldots\ldots\ldots$

(iv) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots\ldots\ldots$ ***(L.B 20011)***

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \ldots\ldots\ldots\ldots$

Solution:

(i) $\ell\, n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$

Let

$$f(x) = \ell n\,(1 + x)$$

$$f(0) = \ell n\,(1 + 0) = \ell n\, 1 = 0$$