Integrate

$$\int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)}\,dx = \int \frac{2x + 1 + 2}{x^2 + x + 1}\,dx + \frac{1}{2}\int \frac{2x - 8 + 2 - 2}{x^2 + 2x + 3}\,dx$$

$$= \int \frac{2x + 1}{x^2 + x + 1}\,dx + 2\int \frac{dx}{x^2 + x + 1} + \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x + 3}\,dx - \frac{10}{2}\int \frac{dx}{x^2 + 2x + 3}$$

$$= \ln|x^2+x+1| + 2\int \frac{dx}{x^2 + x + \frac{1}{4} - \frac{1}{4} + 1} + \frac{1}{2}\ln|x^2+2x+3| - 5\int \frac{dx}{x^2 + 2x + 1 - 1 + 3}$$

$$= \ln|x^2 + x + 1| + 2\int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}} + \ln|x^2 + 2x + 3)^{1/2}| - 5\int \frac{dx}{(x + 1)^2 + 2}$$

$$= \ln|(x^2 + x + 1)\sqrt{x^2 + 2x + 3}| + 2\int \frac{dx}{(x + \frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - 5\int \frac{dx}{(x + 1)^2 + (\sqrt{2})^2}$$

$$= \ln|(x^2+x+1)\sqrt{x^2 + 2x + 3}| + 2 \cdot \frac{1}{\sqrt{3/2}}\tan^{-1}\left(\frac{x + \frac{1}{2}}{\sqrt{3/2}}\right) - 5 \cdot \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) + c$$

$$= \ln|(x^2 + x + 1)\sqrt{x^2 + 2x + 3}| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) - \frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x + 1}{\sqrt{2}}\right) + c$$

EXERCISE 3.6

Q.1 $\displaystyle\int_1^2 (x^2 + 1)\,dx$

Solution:

$$\int_1^2 (x^2 + 1)\,dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 dx$$

$$= \left[\frac{x^3}{2}\right]_1^2 + [x]_1^2$$

$$\because \int_a^b f(x)\,dx = [\varphi(x)]_a^b = \varphi(b) - \varphi(a)$$

$$= \frac{1}{3}(2^3 - 1^3) + (2 - 1)$$

$$= \frac{1}{3}(8 - 1) + 1 = \frac{7}{3} + 1$$

$$= \frac{7+3}{3} = \frac{10}{3} \quad \text{Ans.}$$

Q.2 $\displaystyle \int_{-1}^{1} (x^{1/3} + 1)\, dx$

Solution:

$$\int_{-1}^{1} (x^{1/3} + 1)\, dx$$

$$= \int_{-1}^{1} x^{1/3}\, dx + \int_{-1}^{1} dx$$

$$= \left[\frac{x^{2/3}}{\frac{2}{3}}\right]_{-1}^{1} + [x]_{-1}^{1}$$

$$= \frac{3}{2}\left[(1)^{2/3} - (-1)^{2/3}\right] + [1 - (-1)]$$

$$= \frac{3}{2}(1 - 1) + 2$$

$$= \frac{3}{2}(0) + 2 = 2 \quad \text{Ans.}$$

Q.3 $\displaystyle \int_{-2}^{0} \frac{1}{(2x - 1)^2}\, dx$

Solution:

$$\int_{-2}^{0} \frac{1}{(2x - 1)^2}\, dx$$

$$= \frac{1}{2}\int_{-2}^{0} (2x - 1)^{-2} \,.\, 2\, dx$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{-2+1}}{-2+1}\right]_{-2}^{0}$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{-1}}{-1}\right]_{-2}^{0}$$

$$= \frac{-1}{2}\left[\frac{1}{2x-1}\right]_{-2}^{0}$$

$$= \frac{-1}{2}\left[\frac{1}{2(0)-1} - \frac{1}{2(-2)-1}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{0-1} - \frac{1}{-4-1}\right]$$

$$= \frac{-1}{2}\left(-1 + \frac{1}{5}\right)$$

$$= \frac{-1}{2}\left(\frac{-5+1}{5}\right)$$

$$= \frac{-1(-4)}{10}$$

$$= \frac{2}{5} \qquad \text{Ans.}$$

Q.4 $\displaystyle \int_{-6}^{2} \sqrt{3-x}\ dx$

Solution:

$$\int_{-6}^{2} \sqrt{3-x}\ dx$$

$$= -\int_{-6}^{2} (3-x)^{1/2}\ -dx$$

$$= -\left[\frac{(3-x)^{2/3}}{\frac{3}{2}}\right]_{-6}^{2}$$

$$= \frac{-2}{3}\left[(3-2)^{2/3} - (3+6)^{2/3}\right]$$

$$= \frac{-2}{3}\left[(1)^{2/3} - (9)^{2/3}\right]$$

$$= \frac{-2}{3}[1-(3^2)^{2/3}]$$

$$= \frac{-2}{3}(1-27)$$

$$= \frac{-2}{3}(-26)$$

$$= \frac{52}{3} \qquad \text{Ans.}$$

Q.5 $\displaystyle\int_1^{\sqrt{5}} \sqrt{(2t-1)^3}\, dt$

Solution:

$$A = \int_1^{\sqrt{5}} \sqrt{(2t-1)^3}\, dt$$

$$= \frac{1}{2}\int_1^{\sqrt{5}} (2t-1)^3 \,.\, 2dt$$

$$= \frac{1}{2}\left[\frac{(2t-1)^{5/2}}{\frac{5}{2}}\right]_1^{\sqrt{5}}$$

$$= \frac{1}{5}[(2\sqrt{5}-1)^{5/2}-(2-1)^{5/2}]$$

$$= \frac{1}{5}[(2\sqrt{5}-1)^{5/2}-1] \qquad \text{Ans.}$$

Q.6 $\displaystyle\int_2^{\sqrt{5}} x\sqrt{x^2-1}\, dx$

Solution:

$$\int_2^{\sqrt{5}} x\sqrt{x^2-1}\, dx$$

$$= \frac{1}{2}\int_2^{\sqrt{5}} (x^2-1)^{1/2} \,.\, 2xdx$$

$$= \frac{1}{2}\left[\frac{(x^2-1)^{3/2}}{3/2}\right]_2^{\sqrt{5}}$$

$$= \frac{1}{3}\left[(5-1)^{3/2}-(4-1)^{3/2}\right]$$

$$= \frac{1}{3}[4^{3/2} - 3^{3/2}] \qquad = \frac{1}{3}[(2^2)^{3/2} - 3\sqrt{3}\,]$$

$$= \frac{1}{3}(8 - 3\sqrt{3}\,) \qquad \text{Ans.}$$

Q.7 $\displaystyle\int_1^2 \frac{x}{x^2+2}\,dx$ **(Lhr. Board 2011, Guj. Board 2008)**

Solution:

$$\int_1^2 \frac{x}{x^2+2}\,dx$$

$$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+2}\,dx$$

$$= \frac{1}{2}\left[\ln(x^2+2)\right]_1^2$$

$$= \frac{1}{2}\ \ln(4+2) - \ln(1+2)]$$

$$= \frac{1}{2}(\ln 6 - \ln 3)$$

$$= \frac{1}{2}\ln\frac{6}{3}$$

$$= \frac{1}{2}\ln 2 \qquad \text{Ans.}$$

Q.8 $\displaystyle\int_2^3 \left(x - \frac{1}{x}\right)^2 dx$

Solution:

$$\int_2^3 \left(x - \frac{1}{x}\right)^2 dx$$

$$= \int_2^3 \left(x^2 - 2 + \frac{1}{x^2}\right) dx$$

$$= \int_2^3 x^2 dx - 2\int_2^3 dx + \int_2^3 x^{-2}\,dx$$

$$= \left[\frac{x^3}{3}\right]_2^3 - 2\,[x]_2^3 + \left[\frac{x-1}{-1}\right]_2^3$$

$$= \frac{1}{3}(3^3 - 2^3) - 2(3-2) - \left[\frac{1}{x}\right]_2^3$$

$$= \frac{1}{3}(27-8) - 2\,(1) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{19}{3} - 2 - \frac{1}{3} + \frac{1}{2}$$

$$= \frac{38 - 12 - 2 + 3}{6}$$

$$= \frac{27}{6} = \frac{9}{2} \quad \text{Ans.}$$

Q.9 $\displaystyle\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1}\; dx$ **(Guj. Board 2007)**

Solution:

$$\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1}\; dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2 + x + 1)^{1/2} \,.\, 2\left(x + \frac{1}{2}\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2 + x + 1)^{1/2} \,.\, (2x + 1)\, dx$$

$$= \frac{1}{2}\left[\frac{(x^3 + x + 1)^{3/2}}{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{1}{3}\left[(1 + 1 + 1)^{3/2} - (1 - 1 + 1)^{3/2}\right]$$

$$= \frac{1}{3}[3^{3/2} - 1^{3/2}]$$

$$= \frac{1}{3}(3\sqrt{3} - 1)$$

$$= \frac{3\sqrt{3}}{3} - \frac{1}{3}$$

$$= \sqrt{3} - \frac{1}{3} \quad \text{Ans.}$$

Q.10 $\displaystyle\int_0^3 \frac{dx}{x^2+9}$

Solution:

$$\int_0^3 \frac{dx}{x^2+9}$$

$$= \int_0^3 \frac{dx}{(x)^2+(3)^2}$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$= \frac{1}{3}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{12} \qquad \text{Ans.}$$

Q.11 $\displaystyle\int_{\pi/6}^{\pi/3} \cos t\, dt$

Solution:

$$\int_{\frac{\pi}{0}}^{\frac{\pi}{3}} \cos t\, dt = [\sin t]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \text{Sin}\frac{\pi}{3} - \sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} \qquad \text{Ans.}$$

Q.12 $\displaystyle\int_2^1 (x + \frac{1}{x})^{½} (1 - \frac{1}{x^2})\, dx$ **(Lhr. Board 2011)**

Solution:

$$\int_2^1 (x + \frac{1}{x})^{½} (1 - \frac{1}{x^2})\, dx$$

$$= \left[\frac{(x+\frac{1}{x})^{3/2}}{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{3}\left[(2+\frac{1}{2})^{3/2} - (1+\frac{1}{1})^{3/2}\right]$$

$$= \frac{2}{3}\left[\left(\frac{4+1}{2}\right)^{3/2} - 2^{3/2}\right]$$

$$= \frac{2}{3}\left[\left(\frac{5}{2}\right)^{3/2} - 2\sqrt{2}\right]$$

$$= \frac{2}{3}\left[\frac{5\sqrt{5}-8^{3/2}}{2\sqrt{2}} - 2\sqrt{2}\right] = \frac{2}{3}\left[\frac{5\sqrt{5}-8}{2\sqrt{2}}\right]$$

$$= \frac{1}{3\sqrt{2}}(5\sqrt{5} - 8) \qquad \text{Ans.}$$

Q.13 $\displaystyle\int_1^2 \ln x \, dx$

Solution:

$$\int_1^2 \ln x \, dx$$

$$= \int_1^2 1 \, . \ln x \, dx$$

$$= [\ln x \, . \, x]_1^2 - \int_1^2 x \, . \, \frac{1}{x} \, dx$$

$$= [2 \ln 2 - 1 \ln 1] - \int_1^2 dx$$

$$= (2 \ln 2 - 0) - [x]_1^2$$

$$= 2 \ln 2 - (2 - 1)$$

$$= 2 \ln 2 - 1 \qquad \text{Ans.}$$

Q.14 $\displaystyle\int_0^2 (e^{x/2} - e^{-x/2}) \, dx$

Solution:

$$\int_0^2 (e^{x/2} - e^{-x/2}) \, dx$$

$$= \int_0^2 e^{x/2}\,dx - \int_0^2 e^{-x/2}\,dx$$

$$= \left[\frac{e^{x/2}}{\frac{1}{2}}\right]_0^2 - \left[\frac{e^{-x/2}}{\frac{-1}{2}}\right]_0^2$$

$$= 2(e^{2/2} - e^{0/2}) + 2\,(e^{-2/2} - e^{-0/2})$$

$$= 2\,(e - 1 + e^{-1} - 1)$$

$$= 2\,(e + \frac{1}{e} - 2)$$

$$= 2\left(\frac{e^2 + 1 - 2e}{e}\right)$$

$$= \frac{2}{e}\,(e - 1)^2 \qquad \text{Ans.}$$

Q.15 $\displaystyle\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1}\,d\theta$

Solution:

$$\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1}\,d\theta$$

$$= \int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{2\cos^2\theta} \qquad \left(\begin{array}{l} \because \cos 2\theta = 2\cos^2\theta - 1 \\ \quad 1 + \cos 2\theta = 2\cos^2\theta \end{array}\right)$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{\cos\theta} + \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sec\theta\,d\theta + \int_0^{\pi/4} \sec\theta\tan\theta\,d\theta]$$

$$= \frac{1}{2}\,[[\ln|\sec\theta + \tan\theta|\,]_0^{\pi/4} + [\sec\theta]_0^{\pi/4}\,]$$

$$= \frac{1}{2}\,[\,\ln|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\,| - \ln|\sec 0 + \tan 0| + \sec\frac{\pi}{4} - \sec 0]$$

$$= \frac{1}{2}\,[\ln(\sqrt{2} + 1) - \ln(1 + 0) + \sqrt{2} - 1]$$

$$= \frac{1}{2}\,[\ln(\sqrt{2} + 1) + \sqrt{2} - 1] \qquad \text{Ans.}$$

Q.16 $$\int_0^{\frac{\pi}{6}} \cos^3\theta \; d\theta$$

Solution:

$$\int_0^{\frac{\pi}{6}} \cos^3\theta \; d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos\theta \, . \, \cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos\theta \, (1 - \sin^2\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos\theta d\theta - \int_0^{\frac{\pi}{6}} \sin^2\theta \cos\theta d\theta$$

$$= [\sin\theta]_0^{\pi/6} - \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{6} - \sin 0 - \frac{1}{3}\left(\sin^3\frac{\pi}{6} - \sin^3 0\right)$$

$$= \frac{1}{2} - 0 - \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right]$$

$$= \frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right) = \frac{1}{2} - \frac{1}{24}$$

$$= \frac{12 - 1}{24} = \frac{11}{24} \qquad \text{Ans.}$$

Q.17 $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta \cot^2\theta \; d\theta$$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta \cot^2\theta \; d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos\theta\ (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} (\cos^2\theta \operatorname{cosec}^2\theta - \cos^2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos^2\theta}{\sin^2\theta} - \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cot^2\theta - \frac{1+\cos 2\theta}{2})\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\operatorname{cosec}^2\theta - 1 - \frac{1+\cos 2\theta}{2})\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{2\operatorname{cosec}^2\theta - 2 - 1 - \cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\operatorname{cosec}^2\theta - 3 - \cos 2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta d\theta - \frac{3}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta d\theta$$

$$= [-\cot\theta]_{\pi/6}^{\pi/4} - \frac{3}{2}\ [\theta]_{\pi/6}^{\pi/4} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/4}$$

$$= -[\cot\frac{\pi}{4} - \cot\frac{\pi}{6}] - \frac{3}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) - \frac{1}{4}[\sin 2\left(\frac{\pi}{4}\right) - \sin 2\ (\frac{\pi}{6})]$$

$$= -1 + \sqrt{3} - \frac{3}{2}\left(\frac{3\pi - 2\pi}{12}\right) - \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= -1 + \sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8}$$

$$= \frac{-8 + 8\sqrt{3} - \pi - 2 + \sqrt{3}}{8}$$

$$= \frac{-10 - \pi + 9\sqrt{3}}{8} \qquad \text{Ans.}$$

Q.18 $\displaystyle\int_0^{\frac{\pi}{4}} \cos^4 t\, dt$

Solution:

$$\int_0^{\frac{\pi}{4}} \cos^4 t\, dt$$

$$= \int_0^{\frac{\pi}{4}} (\cos^2 t)^2\, dt$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1 + \cos 2t}{2}\right)^2 dt$$

$\because \cos 2t = 2\cos^2 t - 1$

$2\cos^2 t = 1 + \cos 2t$

$\cos^2 t = 1 \cos 2t$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 + \cos^2 2t + 2\cos 2t)\, dt$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(1 + \frac{1 + \cos 4t}{2} + 2\cos 2t\right) dt$$

$\because \mathrm{Cos}^2 t = \dfrac{1 + \mathrm{Cos}\, 2t}{2}$

$\mathrm{Cos}^2 2t = \dfrac{1 + \mathrm{Cos}\, 4t}{2}$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\frac{2 + 1 + \cos 4t + 4\cos 2t}{2}\right) dt$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (3 + \cos 4t + 4\cos 2t)\, dt$$

$$= \frac{1}{8}\left[3\int_0^{\frac{\pi}{4}} dt + \int_0^{\frac{\pi}{4}} \cos 4t\, dt + 4\int_0^{\frac{\pi}{4}} \cos 2t\, dt\right]$$

$$= \frac{1}{8}\left[3\,[t]_0^{\pi/4} + \left[\frac{\sin 4t}{4}\right]_0^{\pi/4} + 4\left[\frac{\sin 2t}{2}\right]_0^{\pi/4}\right]$$

$$= \frac{1}{8}\left[3\left(\frac{\pi}{4}-0\right)+\frac{1}{4}\left(\sin 4 \cdot \frac{\pi}{4}-\sin 0\right)+2\left(\sin 2 \cdot \frac{\pi}{4}-\sin 0\right)\right]$$

$$= \frac{1}{8}\left[\frac{3\pi}{4}+\frac{1}{4}(0-0)+2(1-0)\right]$$

$$= \frac{1}{8}\left(\frac{3\pi}{4}+2\right)$$

$$= \frac{1}{8}\left(\frac{3\pi+8}{4}\right)$$

$$= \frac{3\pi+8}{32} \quad \text{Ans.}$$

Q.19 $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2\theta \sin\theta \, d\theta$

Solution:

$$\int_0^{\frac{\pi}{3}} \cos^2\theta \sin\theta \, d\theta$$

$$= -\int_0^{\frac{\pi}{3}} \cos^2\theta (-\sin\theta) \, d\theta$$

$$= -\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/3}$$

$$= \frac{-1}{3}\left[\cos^3\frac{\pi}{3}-\cos^3 0\right]$$

$$= \frac{-1}{3}\left[\left(\frac{1}{2}\right)^3-1\right]$$

$$= \frac{-1}{3}\left(\frac{1}{8}-1\right)$$

$$= \frac{-1}{3}\left(\frac{1-8}{8}\right)$$

$$= \frac{-1}{3}\left(\frac{-7}{8}\right)$$

$$= \frac{7}{24} \quad \text{Ans.}$$

Q.20 $\int_0^{\frac{\pi}{4}} (1 + \cos^2\theta) \tan^2 d\theta$

Solution:

$$\int_0^{\frac{\pi}{4}} (1 + \cos^2\theta) \tan^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta \cos^2\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \frac{\sin^2\theta}{\cos^2\theta} \, . \cos^2\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \sin^2\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1 + \frac{1 - \cos 2\theta}{2}) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{2\sec^2\theta - 2 + 1 - \cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (2 \sec^2\theta - 1 - \cos 2\theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2\theta d\theta$$

$$= [\tan\theta]_0^{\pi/4} - \frac{1}{2}[\theta] - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) - \frac{1}{2}\left(\frac{\pi}{4} - 0\right) - \frac{1}{4}\left[\sin 2\left(\frac{\pi}{4} - \sin 2\,(0)\right)\right]$$

$$= \quad 1 - 0 - \frac{\pi}{8} - \frac{1}{4}(1 - 0) \quad = \quad 1 - \frac{\pi}{8} - \frac{1}{4}$$

$$= \quad \frac{8 - \pi - 2}{8} = \frac{6 - \pi}{8} \qquad \text{Ans.}$$

Q.21 $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$ **(Guj. Board 2008)**

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$$

$$= \quad \int_0^{\frac{\pi}{4}} \frac{1/\cos^2\theta}{\tan\theta + 1}\, d\theta$$

$$= \quad \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\tan\theta + 1}\, d\theta$$

$$= \quad [ln\,|\tan\theta + 1|\,]_0^{\pi/4} \qquad = \quad ln\,|\tan\frac{\pi}{4} + 1\,| - ln\,|\tan 0 + 1|$$

$$= \quad ln\,|1 + 1| - ln\,|0 + 1| \qquad = \quad ln\,2 - 0 \; = \; ln\,2 \qquad \text{Ans.}$$

Q.22 $\displaystyle\int_{-1}^{5} |x - 3|\, dx$ **(Lhr. Board 2009 (S))**

Solution:

We know that

$$|x - 3| \quad = \begin{cases} +(x - 3) & \text{If } x > 3 \\ -(x - 3) & \text{If } x < 3 \\ 0 & \text{If } x = 3 \end{cases}$$

$$= \quad \int_{-1}^{5} |x - 3|\, dx + \int_{3}^{5} (x - 3)\, dx$$

$$= \quad -\int_{-1}^{3} |(x - 3)\, dx + \int_{3}^{5} (x - 3)\, dx$$

$$= \quad -\left[\frac{(x-3)^2}{2}\right]_{-1}^{3} + \left[\frac{(x-3)^2}{2}\right]_{3}^{5}$$

$$= -\left[0 - \frac{16}{2}\right] + \left[\frac{4}{2} - 0\right] = \frac{16}{2} + \frac{4}{2} = 8 + 2 = 10$$

Q.23 $\displaystyle\int_{\frac{1}{8}}^{1} \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{1}{3}}}\, dx$

Solution:

$$\int_{\frac{1}{8}}^{1} \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{1}{3}}}\, dx$$

$$= 3\int_{\frac{1}{8}}^{1} (x^{\frac{1}{3}} + 2)^2 \cdot \frac{1}{3} x^{\frac{-2}{3}}\, dx$$

$$= 3\left[\frac{(x+2)^3}{3}\right]_{1/8}^{1}$$

$$= (1+2)^3 - \left((1/8)^{\frac{1}{3}} + 2\right)^3$$

$$= 27 - \left(\frac{1}{(2^3)^{\frac{1}{3}}} + 2\right)^3$$

$$= 27 - \left(\frac{1}{2} + 2\right)^3 = 27 - \left(\frac{1+4}{2}\right)^3$$

$$= 27 - \frac{125}{8} = \frac{216 - 125}{8} = \frac{91}{8} \qquad \text{Ans.}$$

Q.24 $\displaystyle\int_{1}^{3} \frac{x^2 - 2}{x + 1}\, dx$

Solution:

$$\int_{1}^{3} \frac{x^2 - 2}{x + 1}\, dx$$

$$= \int_{1}^{3} \left(x - 1 - \frac{1}{x+1}\right) dx$$

$$\begin{array}{r|l} & x - 1 \\ x + 1 & x^2 - 2 \\ & \underline{-x^2 \quad \pm x} \\ & -x - 2 \end{array}$$

$$= \int_{1}^{3} x\,dx - \int_{1}^{3} dx - \int_{1}^{3} \frac{dx}{x+1}$$

$$= \left[\frac{x^2}{2}\right]_1^3 - [x]_1^3 - [ln|x+1|]_1^3 \qquad \mp x \mp 1 \quad -1$$

$$= \frac{1}{2}(9-1) - (3-1) - [ln|3+1| - ln|1+1|]$$

$$= \frac{8}{2} - 2 - (ln\,4 - ln2)$$

$$= 4 - 2 - ln\left(\frac{4}{2}\right)$$

$$= 2 - ln\,2 \qquad \text{Ans.}$$

Q.25 $\displaystyle\int_2^3 \frac{3x^2 - 2x + 1}{(x - 1)(x^2 + 1)}\,dx$

Solution:

$$\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)}\,dx$$

$$= \int_2^3 \frac{3x^2 - 2x + 1}{x^3 - x^2 + x - 1}\,dx \quad = [ln|x^3 - x^2 + x - 1|]_2^3$$

$$= ln|27 - 9 + 3 - 1| - ln|8 - 4 + 2 - 1|$$

$$= ln\,20 - ln\,5 \quad = ln\frac{20}{5} \quad = ln\,4 \quad \text{Ans.}$$

Q.26 $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\,dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx - \int_0^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= [\sec x]_0^{\pi/4} - [\tan x]_0^{\pi/4}$$

$$= \left(\sec\frac{\pi}{4} - \sec 0\right) - \left(\tan\frac{\pi}{4} - \tan 0\right)$$

$$= \sqrt{2} - 1 - (1 - 0) = \sqrt{2} - 1 - 1 = \sqrt{2} - 2 \qquad \text{Ans.}$$

Q.27 $\displaystyle \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1-\sin x}{1-\sin^2 x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$$

$$= [\tan x]_0^{\pi/4} - [\sec x]_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) - \left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= (1 - 0) - (\sqrt{2} - 1) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2} \qquad \text{Ans.}$$

Q.28 $\displaystyle \int_0^{1} \frac{3x}{\sqrt{4-3x}}\,dx$ **(Lhr. Board 2008)**

Solution:

$$\int_0^{1} \frac{3x}{\sqrt{4-3x}}\,dx$$

$$= \quad -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}}\,dx$$

$$= \quad -\int_0^1 \left(\frac{4-3x}{\sqrt{4-3x}} - \frac{4}{\sqrt{4-3x}}\right) dx$$

$$= \quad -\int_0^1 \sqrt{4-3x}\,dx + 4\int_0^1 \frac{dx}{\sqrt{4-3x}}$$

$$= \quad \frac{1}{3}\int_0^1 (4-3x)^{1/2} - 3dx - \frac{4}{3}\int_0^1 (4-3x)^{-1/2} - 3dx$$

$$= \quad \frac{1}{3}\left[\frac{(4-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right] - \frac{4}{3}\left[\frac{(4-3x)^{1/2}}{\frac{1}{2}}\right]_0^1$$

$$= \quad \frac{2}{9}\left[(4-3)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(4-3)^{1/2} - (4-0)^{1/2}\right]$$

$$= \quad \frac{2}{9}\,[1-(2^2)^{\frac{3}{2}}] - \frac{8}{3}\,[1-(2^2)^{1/2}]$$

$$= \quad \frac{2}{9}(1-8) - \frac{8}{3}(1-2)$$

$$= \quad \frac{2}{9}(-7) - \frac{8}{3}(-1) = \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9} \qquad \text{Ans.}$$

Q.29 $\displaystyle \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x\,(2+\sin x)}\,dx$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x\,(2+\sin x)}\,dx$$

Put $\sin x = t$

$\cos x\,dx = dt \quad \Rightarrow \quad dx = \dfrac{dt}{\cos x}$

When $x = \dfrac{\pi}{6}, \quad t = \dfrac{1}{2}$

When $x = \dfrac{\pi}{2}$, $t = 1$

$$= \int_{\frac{1}{2}}^{1} \frac{\cos x}{t\,(2+t)} \times \frac{dt}{\cos x} \qquad = \int_{\frac{1}{2}}^{1} \frac{dt}{t(2+t)}$$

Taking

$$\frac{t}{t\,(2+t)} = \frac{A}{t} + \frac{B}{2+t} \qquad \text{———} \qquad (1)$$

Multiplying t (2 + t) on both sides in eq. (1)

$$1 = A\,(2+t) + Bt \qquad \text{———} \qquad (2)$$

To find A

Put $t = 0$ in equation (2)

$1 = A\,(2 + 0)$

$2A = 1$

$A = \dfrac{1}{2}$

To find B

Put $2 + t = 0$

$t = -2$ in equation (2)

$1 = B\,(-2)$

$B = \dfrac{-1}{2}$

$\therefore$ From equation (1)

$$\frac{1}{t(2+t)} = \frac{\frac{1}{2}}{t} + \frac{\frac{-1}{2}}{2+t}$$

Integrate from $\dfrac{1}{2}$ to 1

$$\int_{\frac{1}{2}}^{1} \frac{dt}{t\,(2+t)} = \frac{1}{2}\int_{\frac{1}{2}}^{1} \frac{dt}{t} - \frac{1}{2}\int_{\frac{1}{2}}^{1} \frac{dt}{2+t}$$

$$= \frac{1}{2}\,[\ln|t|]_{\frac{1}{2}}^{1} - \frac{1}{2}\,[\ln|2+t|]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{2}\left(\ln 1 - \ln\frac{1}{2}\right) - \frac{1}{2}\left(\ln 3 - \ln\left(2 + \frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left[-\ln\frac{1}{2} - \ln 3 + \ln\left(\frac{4+1}{2}\right)\right]$$

$$= \frac{1}{2}\left[-\ln\frac{1}{2} - \ln 3 + \ln\frac{5}{2}\right]$$

$$= \frac{1}{2}\ln\left(\frac{5/2}{\frac{1}{2}\times 3}\right)$$

$$= \frac{1}{2}\ln\left(\frac{5}{3}\right) \qquad \text{Ans}$$

Q.30 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)}\,dx$$

Put

$\cos x = t$

$-\sin x\,dx = dt$

$dx = \dfrac{dt}{-\sin x}$

When $x = 0$, $t = 1$

When $t = \frac{\pi}{2}$, $t = 0$

$$= \int_1^0 \frac{\sin x}{(1+t)(2+t)} \times \frac{dt}{-\sin x} = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

Taking

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t} \qquad \text{———} \qquad (1)$$

Multiplying $(1+t)(2+t)$ on both sides in eq. (1)

$$1 = A(2+t) + B(1+t) \qquad \text{———} \qquad (2)$$

To find A

Put

$1 + t = 0$

$t = -1$ in equation (2)

$1 = A(2-1)$

$$\boxed{A = 1}$$

To find B

Put $2 + t = 0$

$t = -2$ in equation (2)

$$1 = B(1 - 2)$$
$$-B = 1$$
$$\boxed{B = -1}$$

$\therefore$ From equation (1)

$$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$$

Integrate from 0 to 1

$$\int_0^1 \frac{dt}{(1+t)(2+t)} = \int_0^1 \frac{dt}{1+t} - \int_0^1 \frac{dt}{2+t}$$
$$= [\ln|1+t|]_0^1 - [\ln|2+t|]_0^1$$
$$= (\ln 2 - \ln 1) - (\ln 3 - \ln 2)$$
$$= \ln 2 - \ln 3 + \ln 2$$
$$= \ln \frac{2 \times 2}{3} = \ln \frac{4}{3} \quad \text{Ans}$$

EXERCISE 3.7

Q.1 Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$ (Lhr. Board 2005, 2008)

Solution:

$y = x^2 + 1$ from $x = 1$ to $x = 2$

$$\text{Required area} = \int_a^b y\,dx$$
$$= \int_1^2 (x^2 + 1)dx$$
$$= \int_1^2 x^2\,dx + \int_1^2 dx$$
$$= \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2$$