EXERCISE 4.6

Q.1 If α, β are the roots of $3x^2 - 2x + 4 = 0$. Find the values of

(i) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solution:

As α, β are roots of $3x^2 - 2x + 4 = 0$

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} = \frac{-(-2)}{3} = \frac{2}{3} \qquad \ldots\ldots\ldots (1)$$

and $$\alpha\beta = \frac{c}{a} = \frac{4}{3} \qquad \ldots\ldots\ldots (2)$$

Now $$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2 \beta^2}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{(\alpha\beta)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

Put values from (1) and (2)

$$= \frac{\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)}{\left(\frac{4}{3}\right)^2}$$

$$= \frac{\frac{4}{9} - \frac{8}{3}}{\frac{16}{9}} = \frac{\frac{4-24}{9}}{\frac{16}{9}} = \frac{-20}{16} = \frac{-5}{4}$$

(ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\,\alpha\beta}{\alpha\,\beta} = \frac{(\alpha+\beta)^2 - 2\,\alpha\,\beta}{\alpha\,\beta}$$

Put values from (1) and (2)

$$= \frac{\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)}{\frac{4}{3}}$$

$$= \frac{\frac{4}{9} - \frac{8}{3}}{\frac{4}{3}} = \frac{\frac{4-24}{9}}{\frac{4}{3}}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{-20}{9} \times \frac{3}{4} = \frac{-5}{3}$$

(iii) $\boldsymbol{\alpha^4 + \beta^4}$

$$\begin{aligned}
\alpha^4 + \beta^4 &= (\alpha^2)^2 + (\beta^2)^2 + 2\alpha^2\beta^2 - 2\alpha^2\beta^2 \\
&= (\alpha^2 + \beta^2)^2 - 2\,(\alpha\beta)^2 \\
&= (\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta)^2 - 2\,(\alpha\beta)^2 \\
&= [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2\,(\alpha\beta)^2
\end{aligned}$$

Putting values from (1) and (2)

$$= \left[\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)\right]^2 - 2\left(\frac{4}{3}\right)^2$$

$$= \left[\frac{4}{9} - \frac{8}{3}\right]^2 - 2\left(\frac{16}{9}\right)$$

$$= \left[\frac{4-24}{9}\right]^2 - \left(\frac{32}{9}\right)$$

$$= \left[\frac{-20}{9}\right]^2 - \left(\frac{32}{9}\right)$$

$$= \frac{400}{81} - \frac{32}{9} = \frac{400-288}{81} = \frac{112}{81}$$

(iv) $\boldsymbol{\alpha^3 + \beta^3}$

$$\begin{aligned}
\alpha^3 + \beta^3 &= (\alpha+\beta)\,(\alpha^2 - \alpha\beta + \beta^2) \\
&= (\alpha+\beta)\,(\alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta - \alpha\beta)
\end{aligned}$$

$$= (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]$$

Putting values from (1) and (2)

$$= \left(\frac{2}{3}\right)\left[\left(\frac{2}{3}\right)^2 - 3\left(\frac{4}{3}\right)\right]$$

$$= \frac{2}{3}\left[\frac{4}{9} - 4\right] = \frac{2}{3}\left[\frac{4-36}{9}\right]$$

$$= \frac{2}{3}\left[\frac{-32}{9}\right] = \frac{-64}{27}$$

(v) $\boldsymbol{\frac{1}{\alpha^3} + \frac{1}{\beta^3}}$

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3} = \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3}$$

$$= \frac{(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha+\beta)(\alpha^2 + 2\alpha\beta + \beta^2 - 3\alpha\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]}{(\alpha\beta)^3}$$

Put values from (1) and (2)

$$= \frac{\left(\frac{2}{3}\right)\left[\left(\frac{2}{3}\right)^2 - 3\left(\frac{4}{3}\right)\right]}{\left(\frac{4}{3}\right)^3} = \frac{\frac{2}{3}\left[\frac{4}{9} - 4\right]}{\frac{64}{27}}$$

$$= \frac{\left(\frac{2}{3}\right)\left(\frac{-32}{9}\right)}{\frac{64}{27}} = -\frac{2}{3}\left(\frac{32}{9}\ \frac{27}{64}\right) = -1$$

(vi) $\boldsymbol{\alpha^2 - \beta^2}$

$$\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$$

$$= (\alpha+\beta)\sqrt{(\alpha-\beta)^2}$$

$$= (\alpha+\beta)\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta}$$

$$= (\alpha+\beta)\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta - 2\alpha\beta}$$

$$= (\alpha+\beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

Putting values from (1) and (2)

$$= \frac{2}{3}\sqrt{\left(\frac{2}{3}\right)^2 - 4\,\frac{4}{3}} = \frac{2}{3}\sqrt{\frac{4}{9} - \frac{16}{3}}$$

$$= \frac{2}{3}\sqrt{\frac{-44}{9}} = \frac{2}{3}\,\frac{2\sqrt{-11}}{3} = \frac{4\sqrt{11}}{9}\,i$$

Q.2 If 'α' β are the roots of $x^2 - px - p - c = 0$.
Prove that $(1+\alpha)(1+\beta) = 1 - c$ (Lahore Board 2008, 2010, 2011)

Solution:

As α, β are the roots of $x^2 - px - p - c = 0$

$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-p)}{1} = p \quad \ldots\ldots\ldots (1)$

$\alpha\beta = \frac{c}{a} = \frac{-p-c}{1} = -p-c \quad \ldots\ldots\ldots (2)$

To prove $(1+\alpha)(1+\beta) = 1 - c$

Take L.H.S. $= (1+\alpha)(1+\beta) = 1 + \beta + \alpha + \alpha\beta$

$= 1 + (\alpha+\beta) + \alpha\beta$

Put values from (1) and (2)

$= 1 + p - p - c$

$= 1 - c = \text{R.H.S.}$

$\Rightarrow$ Hence proved.

Q.3 Find the condition that one root of $x^2 + px + q = 0$ is

Solution:

(i) Double the other

Let $\alpha, 2\alpha$ be roots of $x^2 + px + q = 0$ then

Sum of the roots $= \alpha + 2\alpha = \frac{-b}{a} = \frac{-p}{1} = -p$

Product of roots $= \alpha \,.\, 2\alpha = \frac{c}{a} = \frac{q}{1} = q$

$\Rightarrow \quad \alpha + 2\alpha = -p \quad \Rightarrow \quad 3\alpha = -p \quad \ldots\ldots\ldots (1)$

$\alpha \,.\, 2\alpha = q \quad \Rightarrow \quad 2\alpha^2 = q \quad \ldots\ldots\ldots (2)$

Putting $\alpha = \frac{-p}{3}$ from (1) and (2)

$\Rightarrow \quad 2\left(\frac{-p}{3}\right)^2 = q \quad \Rightarrow \quad 2 \cdot \frac{p^2}{9} = q$

$\Rightarrow \quad 2p^2 = 9q$ is the required condition.

(ii) Square of the other

Let α, α^2 be roots of $x^2 + px + q = 0$ then

$\alpha + \alpha^2 = \frac{-b}{a} = \frac{-p}{1} = -p$

$\Rightarrow \quad \alpha + \alpha^2 = -p \quad \text{.........} (1)$

and $\quad \alpha \cdot \alpha^2 = \frac{c}{a} = \frac{q}{1}$

$\alpha^3 = q \quad \text{.........} (2)$

Cubing equation (1)

$(\alpha + \alpha^2)^3 = (-p)^3$

$\Rightarrow \quad \alpha^3 + (\alpha^2)^3 + 3(\alpha)(\alpha^2)(\alpha + \alpha^2) = -p^3$

$\Rightarrow \quad \alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -p^3$

Put $\alpha^3 = q$ and $\alpha + \alpha^2 = -p$ from equation (1) and (2)

$\Rightarrow \quad q + q^2 + 3q(-p) = -p^3$

$\Rightarrow \quad q^2 + q - 3pq + p^3 = 0$

$\Rightarrow \quad p^3 + q^2 + q = 3pq$

is required condition.

(iii) Additive inverse of the other: **(Gujranwala Board 2005)**

Let $\alpha, -\alpha$ be roots of $x^2 + px + q = 0$ then

$\alpha + (-\alpha) = \frac{-b}{a} = \frac{-p}{1}$

$\Rightarrow \quad \alpha - \alpha = -p$

$\Rightarrow \quad -p = 0 \quad \Rightarrow \quad p = 0$

and $\quad \alpha \cdot (-\alpha) = \frac{c}{a} = \frac{q}{1}$

$\Rightarrow \quad -\alpha^2 = q \quad \Rightarrow \quad \alpha^2 = -q$

$\Rightarrow \quad p = 0$ is the required condition.

(iv) Multiplicative inverse of the other:

Let $\alpha, \frac{1}{\alpha}$ be the roots of $x^2 + px + q = 0$ then

$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \frac{-b}{a} = \frac{-p}{1}$

$\Rightarrow \quad \alpha + \frac{1}{\alpha} = -p \qquad \ldots\ldots\ldots (1)$

and $\quad \alpha \cdot \frac{1}{\alpha} = \frac{c}{a} = \frac{q}{1}$

$\Rightarrow \quad 1 = q \quad \Rightarrow \quad q = 1$ is required condition.

Q.4 If the roots of the equation $x^2 - px + q = 0$ differ by unity prove that $p^2 = 4q + 1$.

Solution:

Let 'α', $(\alpha - 1)$ be the roots of $x^2 - px + q = 0$, then

$\alpha + (\alpha - 1) = \frac{-(-p)}{1}$

$\Rightarrow \quad 2\alpha - 1 = p \quad \Rightarrow \quad 2\alpha = p + 1 \quad \Rightarrow \quad \alpha = \frac{p+1}{2} \qquad \ldots\ldots\ldots (1)$

and $\quad \alpha(\alpha - 1) = \frac{q}{1}$

$\Rightarrow \quad \alpha^2 - \alpha = q \qquad \ldots\ldots\ldots (2)$

Put the value of 'α' from equation (1) in equation (2)

$\left(\frac{p+1}{2}\right)^2 - \left(\frac{p+1}{2}\right) = q \quad \Rightarrow \quad \frac{p^2 + 1 + 2p}{4} - \frac{p+1}{2} = q$

$\Rightarrow \quad \frac{p^2 + 1 + 2p - 2p - 2}{4} = q$

$\Rightarrow \quad \frac{p^2 - 1}{4} = q \quad \Rightarrow \quad p^2 - 1 = 4q \quad \Rightarrow \quad p^2 = 4q + 1$

Hence proved.

Q.5 Find the condition that $\frac{a}{x-a}+\frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in signs.

Solution:

$$\frac{a}{x-a}+\frac{b}{x-b} = 5$$

$$\frac{a(x-b)+b(x-a)}{(x-a)(x-b)} = 5$$

$$\Rightarrow \quad \frac{ax-ab+bx-ab}{x^2-bx-ax+ab} = 5$$

$$\Rightarrow \quad ax+bx-2ab = 5(x^2-bx-ax+ab)$$

$$\Rightarrow \quad ax+bx-2ab = 5x^2-5bx-5ax+5ab$$

$$\Rightarrow \quad 5x^2-5bx-5ax+5ab-ax-bx+2ab = 0$$

$$\Rightarrow \quad 5x^2-6bx-6ax+7ab = 0$$

$$\Rightarrow \quad 5x^2-6x(a+b)+7ab = 0$$

Let $\alpha, -\alpha$ be the roots of this equation.

$$\Rightarrow \quad \alpha+(-\alpha) = -\frac{-6(a+b)}{5}$$

$$\Rightarrow \quad \alpha-\alpha = \frac{6(a+b)}{5}$$

$$\Rightarrow \quad 0 = \frac{6(a+b)}{5} \quad \Rightarrow \quad a+b = 0$$

is the required condition.

Q.6 If the roots of $px^2+qx+q = 0$ are α and β then prove that

$$\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}} = 0$$

Solution:

Since α, β be the roots of $px^2+qx+q = 0$, then

$$\alpha+\beta = \frac{-q}{p} \qquad \ldots\ldots\ldots (1)$$

and $$\alpha\beta = \frac{q}{p} \qquad \ldots\ldots\ldots (2)$$

To prove

$$\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}} = 0$$

Take

$$\text{L.H.S.} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\sqrt{\alpha}}{\sqrt{\beta}} + \frac{\sqrt{\beta}}{\sqrt{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\alpha + \beta}{\sqrt{\beta}\sqrt{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{q}{p}}$$

Put values from (1) and (2)

$$= \frac{-\frac{q}{p}}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}} \qquad ¤\ \frac{q}{p} = \sqrt{\frac{q}{p}}.\sqrt{\frac{q}{p}}$$

$$= -\frac{\sqrt{\frac{q}{p}}.\sqrt{\frac{q}{p}}}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}$$

$$= -\sqrt{\frac{q}{p}} + \sqrt{\frac{q}{p}} = 0 = \text{R.H.S.}$$

Hence proved.

Q.7 If α, β are the roots the equation $ax^2 + bx + c = 0$, form the equations whose roots are.

Solution:

(i) α^2, β^2 **(Lahore Board 2006)**

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a}$ ……… (1)

and $\quad \alpha\beta = \frac{c}{a}$ ……… (2)

Let the required equation is

$y^2 - sy + p = 0$ ……… (3)

whose roots are α^2, β^2

$\Rightarrow \quad s = \alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$ from equations (1) and (2)

$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

and $\quad p = \alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}$ from equation (2)

Put values of s and p in equation (3)

$\Rightarrow \quad y^2 - \left(\frac{b^2 - 2ac}{a^2}\right) y + \frac{c^2}{a^2} = 0$

$\Rightarrow \quad a^2y^2 - (b^2 - 2ac)\, y + c^2 = 0$

is required equation.

(ii) $\mathbf{\frac{1}{\alpha}, \frac{1}{\beta}}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\frac{b}{a}$ ……… (1)

and $\quad \alpha\beta = \frac{c}{a}$ ……… (2)

Let the required equation is

$y^2 - sy + p = 0$ ……… (3)

whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

$\Rightarrow \quad s = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$= \frac{-\frac{b}{a}}{\frac{c}{a}}$ from equations (1) and (2)

$= -\frac{b}{a}$

and $\quad p = \frac{1}{\alpha}\cdot\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$ from equation (2)

Put these values in equation (3)

$$y^2 - \left(-\frac{b}{c}\right)y + \frac{a}{c} = 0$$

$$\Rightarrow \quad y^2 + \frac{b}{c}y + \frac{a}{c} = 0$$

$$\Rightarrow \quad cy^2 + by + a = 0$$

is required condition.

(iii) $\quad \mathbf{\frac{1}{\alpha^2}, \frac{1}{\beta^2}}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha + \beta = -\frac{b}{a} \quad \ldots\ldots\ldots (1)$$

and $\quad \alpha\beta = \frac{c}{a} \quad \ldots\ldots\ldots (2)$

Let the required equation is

$$y^2 - sy + p = 0 \quad \ldots\ldots\ldots (3)$$

whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

$$\Rightarrow \quad s = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{(\alpha\beta)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

Put values from equation (1) and (2)

$$= \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)^2}$$

$$= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}} = \frac{\frac{b^2 - 2ac}{a^2}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ac}{c^2}$$

and $\quad p = \frac{1}{\alpha^2}\cdot\frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2} = \frac{1}{\left(\frac{c}{a}\right)^2} = \frac{a^2}{c^2}$ from equation (2)

Put these values of s and p in equation (3)

$$y^2 - \left(\frac{b^2-2ac}{c^2}\right)y + \frac{a^2}{c^2} = 0$$

$\Rightarrow \quad c^2y^2 - (b^2-2ac)\,y + a^2 = 0$ is required equation.

(iv) $\alpha^3,\ \beta^3$ **(Lahore Board 2005)**

Since α, β are the roots of the equation $ax^2+bx+c = 0$

$\Rightarrow \quad \alpha+\beta = -\frac{b}{a}$ .... (1)

and $\quad \alpha\beta = \frac{c}{a}$ .... (2)

Let the required equation is

$y^2 - sy + p = 0$ .... (3)

whose roots are α^3, β^3

$\Rightarrow \quad s = \alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) = (\alpha+\beta)(\alpha^2+\beta^2+2\alpha\beta-3\alpha\beta)$

$$= (\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right]$$

Put values from equations (1) and (2)

$$= -\frac{b}{a}\left[\left(-\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right)\right]$$

$$= -\frac{b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right] = -\frac{b}{a}\left[\frac{b^2-3ac}{a^2}\right] = \frac{-b\,(b^2-3ac)}{a^3}$$

and $\quad p = \alpha^3.\beta^3 = (\alpha\beta)^3 = \left(\frac{c}{a}\right)^3$ from equation (2)

$$= \frac{c^3}{a^3}$$

$\Rightarrow \quad$ equation (3) becomes

$$y^2 - \frac{-b\,(b^2-3ac)}{a^3}\,y + \frac{c^3}{a^3} = 0$$

$\Rightarrow \quad y^2 + \dfrac{b(b^2 - 3ac)}{a^3} y + \dfrac{c^3}{a^3} = 0$

$\Rightarrow \quad a^3 y^2 + b(b^2 - 3ac) y + c^3 = 0$

is required equation.

(v) $\quad \mathbf{\dfrac{1}{\alpha^3}, \dfrac{1}{\beta^3}}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\dfrac{b}{a} \qquad \ldots\ldots\ldots (1)$

and $\quad \alpha\beta = \dfrac{c}{a} \qquad \ldots\ldots\ldots (2)$

Let the required equation is

$y^2 - sy + p = 0 \qquad \ldots\ldots\ldots (3)$

whose roots are $\dfrac{1}{\alpha^3}, \dfrac{1}{\beta^3}$

$$\Rightarrow \quad s = \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}$$

$$= \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} = \frac{(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha+\beta)(\alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right]}{(\alpha\beta)^3}$$

Put values from equations (1) and (2)

$$= \frac{\frac{-b}{a}\left[\left(\frac{-b}{a}\right)^2 - 3\left(\frac{c}{a}\right)\right]}{\left(\frac{c}{a}\right)^3}$$

$$= \frac{\frac{-b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right]}{\frac{c^3}{a^3}}$$

$$= \frac{\frac{-b}{a}\left[\frac{b^2-3ac}{a^2}\right]}{\frac{c^3}{a^3}}$$

$$s = \frac{-b\,(b^2-3ac)}{a^3}\,\frac{a^3}{c^3} = \frac{-b\,(b^2-3ac)}{c^3}$$

and $\quad p = \frac{1}{\alpha^3}\cdot\frac{1}{\beta^3} = \frac{1}{(\alpha\beta)^3} = \frac{1}{\left(\frac{c}{a}\right)^3} = \frac{a^3}{c^3}$ from equation (2)

Put these values in equation (3)

$$y^2 - \left[\frac{-b\,(b^2-3ac)}{c^3}\right]y + \frac{a^3}{c^3} = 0$$

$$\Rightarrow \quad y^2 + \frac{b\,(b^2-3ac)}{c^3}\,y + \frac{a^3}{c^3} = 0$$

$$c^3y^2 + b\,(b^2-3ac)\,y + a^3 = 0$$

(vi) $\quad \boldsymbol{\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{-b}{a} \qquad \text{……… (1)}$$

and $$\alpha\beta = \frac{c}{a} \qquad \text{……… (2)}$$

Let the required equation is

$$y^2 - sy + p = 0 \qquad \text{……… (3)}$$

whose roots are $\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}$, then

$$\Rightarrow \quad s = \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} = (\alpha+\beta) + \frac{\beta+\alpha}{\alpha\beta}$$

Put values from equations (1) and (2)

$$s = \frac{-b}{a} + \frac{\frac{-b}{a}}{\frac{c}{a}} = -\frac{b}{a} - \frac{b}{c} = \frac{-bc-ab}{ac} = \frac{-b\,(a+c)}{ac}$$

and $$p = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$$

$$= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{\alpha\beta}$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

Put values from equations (1) and (2)

$$p = \frac{c}{a} + \frac{1}{\frac{c}{a}} + \frac{\left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\frac{c}{a}}$$

$$= \frac{c}{a} + \frac{a}{c} + \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c}{a}} = \frac{c}{a} + \frac{a}{c} + \frac{b^2 - 2ac}{a^2}\,\frac{a}{c}$$

$$= \frac{c}{a} + \frac{a}{c} + \frac{b^2 - 2ac}{ac} = \frac{c^2 + a^2 + b^2 - 2ac}{ac} = \frac{(a - c)^2 + b^2}{ac}$$

Put these values in equation (3)

$$y^2 - \left[\frac{-b\,(a + c)}{ac}\right]y + \frac{(a - c)^2 + b^2}{ac} = 0$$

$$y^2 + \frac{b\,(a + c)}{ac}\,y + \frac{(a - c)^2 + b^2}{ac} = 0$$

is the required equation.

(vii) $\mathbf{(\alpha - \beta)^2,\ \ (\alpha + \beta)^2}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{-b}{a} \qquad \text{.........} (1)$$

and $\quad \alpha\beta = \dfrac{c}{a}$ .... (2)

Let the required equation is

$$y^2 - sy + p = 0 \qquad \text{......... (3)}$$

whose roots are $(\alpha-\beta)^2,\ (\alpha+\beta)^2$, then

$$\Rightarrow \quad s = (\alpha-\beta)^2 + (\alpha+\beta)^2$$

$$= (\alpha^2+\beta^2-2\alpha\beta) + (\alpha+\beta)^2$$

$$= (\alpha^2+\beta^2+2\alpha\beta-4\alpha\beta) + (\alpha+\beta)^2$$

$$= \left[(\alpha+\beta)^2 - 4\alpha\beta\right] + (\alpha+\beta)^2$$

Put values from equations (1) and (2)

$$s = \left[\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}\right] + \left(\frac{-b}{a}\right)^2$$

$$= \left[\frac{b^2}{a^2} - \frac{4c}{a}\right] + \frac{b^2}{a^2} = \frac{b^2-4ac}{a^2} + \frac{b^2}{a^2} = \frac{b^2-4ac+b^2}{a^2} = \frac{2b^2-4ac}{a^2}$$

and $\quad p = (\alpha-\beta)^2(\alpha+\beta)^2$

$$= (\alpha^2+\beta^2-2\alpha\beta)(\alpha+\beta)^2$$

$$= (\alpha^2+\beta^2+2\alpha\beta-4\alpha\beta)(\alpha+\beta)^2$$

$$= \left[(\alpha+\beta)^2 - 4\alpha\beta\right](\alpha+\beta)^2$$

Put values from equations (1) and (2)

$$= \left[\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}\right]\left(\frac{-b}{a}\right)^2 = \left[\frac{b^2}{a^2} - \frac{4c}{a}\right]\frac{b^2}{a^2}$$

$$p = \frac{b^2-4ac}{a^2}\,\frac{b^2}{a^2} = \frac{b^2(b^2-4ac)}{a^4}$$

Put these values in equation (3)

$$y^2 - \frac{2\,b^2-4ac}{a^2}\,y + \frac{b^2(b^2-4ac)}{a^4} = 0$$

$$a^4y^2 - 2a^2(b^2-2ac)\,y + b^2(b^2-4ac) = 0$$

is required equation.

(viii) $-\dfrac{1}{\alpha^3} - \dfrac{1}{\beta^3}$

Since α, β are the roots of the equation $ax^2 + bx + c = 0$

$\Rightarrow \quad \alpha + \beta = -\dfrac{b}{a}$ .... (1)

and $\quad \alpha\beta = \dfrac{c}{a}$ .... (2)

Let the required equation is

$y^2 - sy + p = 0$ .... (3)

whose roots are $-\dfrac{1}{\alpha^3}, -\dfrac{1}{\beta^3}$

$$\Rightarrow \quad s = -\frac{1}{\alpha^3} - \frac{1}{\beta^3} = \frac{-\beta^3 - \alpha^3}{\alpha^3\beta^3}$$

$$= -\frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} = -\frac{(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\alpha\beta)^3}$$

$$= -\frac{(\alpha + \beta)(\alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta)}{(\alpha\beta)^3}$$

$$= \frac{-(\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right]}{(\alpha\beta)^3}$$

Put values from equations (1) and (2)

$$= \frac{-\left(-\frac{b}{a}\right)\left[\left(-\frac{b}{a}\right)^2 - 3\frac{c}{a}\right]}{\left(\frac{c}{a}\right)^3}$$

$$= \frac{\frac{b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right]}{\frac{c^3}{a^3}} = \frac{b}{a}\left[\frac{b^2 - 3ac}{a^2}\right]\frac{a^3}{c^3}$$

$$s = \frac{b(b^2 - 3ac)}{c^3}$$

and $\quad p = \left(-\dfrac{1}{\alpha^3}\right)\left(-\dfrac{1}{\beta^3}\right) = \dfrac{1}{\alpha^3\beta^3} = \dfrac{1}{(\alpha\beta)^3} = \dfrac{1}{\left(\frac{c}{a}\right)^3} = \dfrac{a^3}{c^3}$ from equation (2)

Put these values in equation (3)

$$y^2 - \frac{b(b^2-3ac)}{c^3})\, y + \frac{a^3}{c^3} = 0$$

$\Rightarrow$ $$c^3 y^2 - b(b^2 - 3ac)\, y + a^3 = 0$$

is the required equation.

Q.8 If α, β are the roots of $5x^2 - x - 2 = 0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$. (Lahore Board 2007)

Solution:

Since α, β are the roots of the equation $5x^2 - x - 2 = 0$

$\Rightarrow$ $$\alpha + \beta = \frac{-(-1)}{5} = \frac{1}{5} \quad \dots\dots\dots (1)$$

and $$\alpha\beta = \frac{-2}{5} \quad \dots\dots\dots (2)$$

Let the required equation is

$$y^2 - sy + p = 0 \quad \dots\dots\dots (3)$$

whose roots are $\frac{3}{\alpha}, \frac{3}{\beta}$

$\Rightarrow$ $$s = \frac{3}{\alpha} + \frac{3}{\beta} = \frac{3\beta + 3\alpha}{\alpha\beta} = \frac{3(\alpha+\beta)}{\alpha\beta}$$

Put values from equations (1) and (2)

$$s = \frac{3\left(\frac{1}{5}\right)}{\frac{-2}{5}} = -3 \cdot \frac{1}{5} \cdot \frac{5}{2} = \frac{-3}{2}$$

$$p = \frac{3}{\alpha}\,\frac{3}{\beta} = \frac{9}{\alpha\beta} = \frac{9}{\frac{-2}{5}} = -9 \cdot \frac{5}{2} = -\frac{45}{2}$$ from equation (2)

Put these values in equation (3)

$\Rightarrow$ $$y^2 - \left(\frac{-3}{2}\right) y - \frac{45}{2} = 0$$

$\Rightarrow$ $$y^2 + \frac{3}{2} y - \frac{45}{2} = 0$$

$\Rightarrow$ $2y^2 + 3y - 45 = 0$ is required equation.

Q.9 If α, β are the roots of $x^2 - 3x + 5 = 0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$. **(Gujranwala Board 2003)**

Solution:

Since α, β are the roots of the equation $x^2 - 3x + 5 = 0$

$\Rightarrow \quad \alpha + \beta = \frac{-(-3)}{1} = 3 \quad \text{.........} (1)$

and $\quad \alpha\beta = \frac{5}{1} = 5 \quad \text{.........} (2)$

Let the required equation be

$y^2 - sy + p = 0 \quad \text{.........} (3)$

whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$

$$\Rightarrow \quad s = \frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta}$$

$$= \frac{(1-\alpha)(1+\beta) + (1+\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$$

$$= \frac{1+\beta-\alpha-\alpha\beta+1-\beta+\alpha-\alpha\beta}{1+\beta+\alpha+\alpha\beta} = \frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

Put values from equations (1) and (2)

$$s = \frac{2-2(5)}{1+3+5} = \frac{2-10}{9} = \frac{-8}{9}$$

and $$p = \frac{1-\alpha}{1+\alpha}\,\frac{1-\beta}{1+\beta}$$

$$= \frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)} = \frac{1-\beta-\alpha+\alpha\beta}{1+\alpha+\beta+\alpha\beta} = -\frac{1-(\alpha+\beta)+\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

Put these values in equations (1) and (2)

$$= \frac{1-3+5}{1+3+5} = \frac{3}{9}$$

$\Rightarrow$ equation (3) becomes

$$y^2 - \left(-\frac{8}{9}\right)y + \frac{3}{9} = 0 \quad \Rightarrow \quad 9y^2 + 8y + 3 = 0$$