Balls in the 3^{rd} layer $= 6+5+4+3+2+1 = 21$

Balls in the 4^{th} layer $= 5+4+3+2+1 = 15$

Balls in the 5^{th} layer $= 4+3+2+1 = 10$

Balls in the 6^{th} layer $= 3+2+1 = 6$

Balls in the 7^{th} layer $= 2+1 = 3$

Balls in the 8^{th} layer $= 1 = 1$

Total Balls $= 36+28+21+15+10+6+3+1 = 120$

GEOMETRIC PROGRESSION OR SEQUENCE (G.P):

A sequence $\{a_n\}$ is a geometric sequence or geometric progression if $\dfrac{a_n}{a_{n-1}}$ is the same non–zero number for all $n \in N$ and $n > 1$. The quotient $\dfrac{a_n}{a_{n-1}}$ is usually denoted by r and is called common ratio of the G.P.

General Term of G.P.

General term or nth term of G.P. is given by

$a_n = a_1 r^{n-1}$

EXERCISE 6.6

Q.1 Find the 5^{th} term of G.P. 3, 6, 12, ...

(Lahore Board 2006, Gujranwala Board 2007)

Solution:

Given sequence

3, 6, 12, ...

$a_1 = 3, \quad r = \dfrac{6}{3} = 2, \quad n = 5$

As $a_n = a_1 r^{n-1}$

$a_5 = (3)(2)^{5-1} = 3(2)^4 = 3(16) = 48$

Q.2 Find the 11^{th} term of the sequence $1+i, 2, \dfrac{4}{1+i}$ (Lahore Board 2011)

Solution:

Given sequence

$1+i,\ 2,\ \dfrac{4}{1+i}, \ldots\ldots.$

$a_1 = 1+i, \quad r = \dfrac{1}{1+i}, \quad n = 11$

As $\quad a_n = a_1 r^{n-1}$

$$a_{11} = (1+i)\left(\frac{2}{1+i}\right)^{10}$$

$$= (1+i)\frac{2^{10}}{(1+i)^{10}} = \frac{2^{10}(1+i)}{\left[(1+i)^2\right]^5}$$

$$= \frac{2^{10}(1+i)}{(1+i^2+2i)^5} = \frac{2^{10}(1+i)}{(1-1+2i)^5}$$

$$= \frac{2^{10}(1+i)}{(2i)^5} = \frac{2^{10}(1+i)}{2^5 \cdot i^5}$$

$$= 2^5\,\frac{1+i}{i} = 2^5\,\frac{1+i}{i}\times\frac{i}{i}$$

$$= 2^5 \cdot \frac{(1+i)\,i}{i^2} = -32\,(i+i^2)$$

$$= -32\,(i-1) = 32\,(1-i)$$

Q.3 Find the 12^{th} term of $1+i, 2i, -2+2i, \ldots\ldots$

Solution:

Given sequence

$1+i,\ 2i,\ -2+2i, \ldots\ldots$

$a_1 = 1+i, \quad r = \dfrac{2i}{1+i}, \quad n = 12, \quad a_{12} = ?$

As $\quad a_n = a_1 r^{n-1}$

$$a_{12} = (1+i)\left(\frac{2i}{1+i}\right)^{11}$$

$$= \frac{(1+i)(2i)^{11}}{(1+i)^{11}}$$

$$= \frac{2^{11} \cdot i^{11}}{(1+i)^{10}}$$

$$= \frac{2^{11} \cdot (i^2)^5 \cdot i}{\left[(1+i)^2\right]^5} = \frac{2^{11} \cdot (-1)^5 \cdot i}{(1+i^2+2i)^5}$$

$$= \frac{-2^{11} \cdot i}{(1-1+2i)^5} = \frac{-i \cdot 2^{11}}{2^5 \cdot i^5}$$

$$= -\frac{2^6}{i^4} = -\frac{2^6}{(-1)^2} = -2^6 = -64$$

Q.4 Find the 11th term of the sequence $1 + i, 2, 2(1 - i), \ldots\ldots.$

(Lahore Board 2005)

Solution:

Given sequence

$1 + i, 2, 2(1 - i), \ldots\ldots.$

$a_1 = 1 + i, \quad r = \dfrac{2}{1 + i}, \quad n = 11$

As $a_n = a_1 r^{n-1}$

$$a_{11} = (1 + i)\left(\frac{2}{1 + i}\right)^{10}$$

$$= \frac{(1 + i) \cdot 2^{10}}{(1 + i)^{10}} = \frac{2^{10}(1 + i)}{\left[(1 + i)^2\right]^5}$$

$$= \frac{2^{10}(1 + i)}{(1 + i^2 + 2i)^5} = \frac{2^{10}(1 + i)}{(1 - 1 + 2i)^5}$$

$$= \frac{2^{10}(1 + i)}{(2i)^5} = \frac{2^{10}(1 + i)}{2^5 \cdot i^5} = \frac{2^5(1 + i)}{(i^2)^2 \cdot i}$$

$$= \frac{3^2(1 + i)}{(-1)^2 \cdot i} = \frac{32(1 + i)}{i} \cdot \frac{i}{i}$$

$$= \frac{32(i + i^2)}{i^2} = -32(i - 1) = 32(1 - i)$$

Q.5 If an automobile depreciates in values 5% every year, at the end of 4 years. What is the value of the automobile purchased for Rs. 12,000?

Solution:

$5\% = \dfrac{5}{100} = 0.05$

The value in first year = Rs. 12000

At the end of 1st year = 12000 – 5% of 12000 = 12000 – (0.05) (12000) = 11400

At the end of 2nd year = 11400 – 5% 11400 = 11400 – 570 = 10830

At the end of 3rd year = 10830 – 5% 10830 = 10288.5

At the end of 4th year = 10288.5 – 5% 10288.5 = 9774

So at the end of 4 year its value is = Rs. 9774

Q.6 Which term of the sequence is $x^2 - y^2, x + y, \frac{x+y}{x-y}, ...$ is $\frac{x+y}{(x-y)^9}$?

Solution:

Given sequence is

$x^2 - y^2, x + y, \frac{x+y}{x-y}, ...$

$a_1 = x^2 - y^2, \quad r = \frac{x+y}{x^2-y^2} = \frac{1}{x-y}, \quad a_n = \frac{x+y}{(x-y)^9}, \quad n = ?$

As $a_n = a_1 r^{n-1}$

$\Rightarrow \quad \frac{x+y}{(x-y)^9} = (x^2 - y^2) . \left(\frac{1}{x-y}\right)^{n-1}$

$\Rightarrow \quad \frac{x+y}{(x-y)^9} = \frac{(x+y)(x-y)}{(x-y)^{n-1}}$

$\Rightarrow \quad \frac{x+y}{(x-y)^9} = \frac{x+y}{(x-y)^{n-2}}$

$\Rightarrow \quad \frac{1}{(x-y)^9} = \frac{1}{(x-y)^{n-2}}$

$\Rightarrow \quad 9 = n - 2$

$\Rightarrow \quad \boxed{n = 11}$

Q.7 If a, b, c, d are in G.P. Prove that (Gujranwala Board 2003)

(i) $a - b, \ b - c, \ c - d$, are in G.P.

(ii) $a^2 - b^2, \ b^2 - c^2, \ c^2 - d^2$, are in G.P.

(iii) $a^2 + b^2, \ b^2 + c^2, \ c^2 + d^2$, are in G.P.

Solution:

Given that

a, b, c, d are in G.P.

$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r \text{ (say)}$

$\Rightarrow \quad b = ar \quad .... (1)$

$\Rightarrow \quad c = br = ar . r = ar^2 \quad .... (2)$

$\Rightarrow \quad d = cr = ar^2 . r = ar^3 \quad .... (3)$

(i) To prove $a - b, \ b - c, \ c - d$ are in G.P. we will prove that

$\frac{b-c}{a-b} = \frac{c-d}{b-c}$

Take L.H.S. $= \dfrac{b-c}{a-b}$

$= \dfrac{ar - ar^2}{a - ar}$ from (1) and (2)

$= \dfrac{ar\,(1-r)}{a\,(1-r)} = r$

Now R.H.S. $= \dfrac{c-d}{b-c}$

$= \dfrac{ar^2 - 2r^3}{ar - ar^2}$ from (1), (2), (3)

$= \dfrac{ar^2\,(1-r)}{ar\,(1-r)} = r$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $a-b,\ b-c,\ c-d$ are in G.P.

(ii) To prove $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G.P. we will prove that

$$\frac{b^2-c^2}{a^2-b^2} = \frac{c^2-d^2}{b^2-c^2}$$

Take L.H.S. $= \dfrac{b^2-c^2}{a^2-b^2}$

$= \dfrac{(ar)^2-(ar^2)^2}{a^2-(ar)^2}$ from (1) and (2)

$= \dfrac{a^2r^2-a^2r^4}{a^2-a^2r^2} = \dfrac{a^2r^2\,(1-r^2)}{a^2\,(1-r^2)} = r^2$

Now R.H.S. $= \dfrac{c^2-d^2}{b^2-c^2}$

$= \dfrac{(ar^2)^2-(ar^3)^2}{(ar)^2-(ar^2)^2}$ from (1), (2), (3)

$= \dfrac{a^2r^4-a^2r^6}{a^2r^2-a^2r^4} = \dfrac{a^2r^4\,(1-r^2)}{a^2r^2\,(1-r^2)} = r^2$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G.P.

(iii) To prove $a^2+b^2,\ b^2+c^2,\ c^2+d^2$ are in G.P. we will prove that

$$\frac{b^2+c^2}{a^2+b^2} = \frac{c^2+d^2}{b^2+c^2}$$

Take L.H.S. $= \dfrac{b^2+c^2}{a^2+b^2}$

$= \dfrac{(ar)^2+(ar^2)^2}{a^2+(ar)^2}$ from (1) and (2)

$= \dfrac{a^2r^2+a^2r^4}{a^2+a^2r^2)} = \dfrac{a^2r^2(1+r^2)}{a^2(1+r^2)} = r^2$

Now R.H.S. $= \dfrac{c^2+d^2}{b^2+c^2}$

$= \dfrac{(ar^2)^2+(ar^3)^2}{(ar)^2+(ar^2)^2}$ from (1), (2) and (3)

$= \dfrac{a^2r^4+a^2r^6}{a^2r^2+a^2r^4} = \dfrac{a^2r^4(1+r^2)}{a^2r^2(1+r^2)} = r^2$

$\Rightarrow$ L.H.S. = R.H.S.

$\Rightarrow$ $a^2+b^2,\ b^2+c^2,\ c^2+d^2$ are in G.P.

Q.8 Show that the reciprocals of the terms of geometric sequence a_1, a_1r^2, a_1r^4 ...... form another geometric sequence.

Solution:

Given G.P. is $a_1, a_1r^2, a_1r^4, \ldots\ldots.$

We will show

$$\frac{1}{a_1}, \frac{1}{a_1r^2}, \frac{1}{a_1r^4} \ldots\ldots.. \text{ is G.P.}$$

For this take $r = \dfrac{\frac{1}{a_1r^2}}{\frac{1}{a_1}} = \dfrac{1}{a_1r^2}\times a_1 = \dfrac{1}{r^2}$

and $r = \dfrac{\frac{1}{a_1r^4}}{\frac{1}{a_1r^2}} = \dfrac{1}{a_1r^4}\times a_1r^2 = \dfrac{1}{r^2}$

$\Rightarrow$ $\dfrac{1}{a_1}, \dfrac{1}{a_1r^2}, \dfrac{1}{a_1r^4}, \ldots\ldots.$ is also G.P.

Q.9 Find the nth term of geometric sequence, if $\frac{a_5}{a_3} = \frac{4}{9}$ and $a_2 = \frac{4}{9}$

Solution:

Given that

$$\frac{a_5}{a_3} = \frac{4}{9}$$

$$\frac{a_1 r^4}{a_1 r^2} = \frac{4}{9} \Rightarrow r^2 = \frac{4}{9} \Rightarrow \boxed{r = \pm\frac{2}{3}}$$

Also given that

$$a_2 = \frac{4}{9} \Rightarrow a_1 r = \frac{4}{9} \Rightarrow a_1 = \frac{4}{9r}$$

when $r = \frac{2}{3} \Rightarrow a_1 = \frac{4}{9 \cdot \left(\frac{2}{3}\right)} = \frac{2}{3}$

when $r = -\frac{2}{3} \Rightarrow a_1 = \frac{4}{9\left(-\frac{2}{3}\right)} = -\frac{2}{3}$

so when $a_1 = \frac{2}{3}$, $r = \frac{2}{3}$

then $a_n = a_1 r^{n-1} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^n$

when $a_1 = -\frac{2}{3}$, $r = -\frac{2}{3}$

then $a_n = a_1 r^{n-1} = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1} = (-1)\left(\frac{2}{3}\right)(-1)^{n-1}\left(\frac{2}{3}\right)^{n-1} = (-1)^n\left(\frac{2}{3}\right)^n$

Q.10 Find three consecutive numbers in G.P whose sum is 26 and their product is 216.

Solution:

Let the required numbers are $\frac{a}{r}$, a, ar

then $\frac{a}{r} + a + ar = 26$ ……… (1)

and $\frac{a}{r} . a . ar = 216$

$a^3 = 216 \Rightarrow \boxed{a = 6}$

Put $a = 6$ in equation (1), we get

$\frac{6}{r} + 6 + 6r = 26$

$\Rightarrow \quad 6r^2 + 6r + 6 = 26r$

$\Rightarrow \quad 6r^2 + 6r - 26r + 6 = 0$

$\Rightarrow \quad 6r^2 - 20r + 6 = 0$

$\Rightarrow \quad 3r^2 - 10r + 3 = 0$

$\Rightarrow \quad 3r^2 - 9r - r + 3 = 0$

$\Rightarrow \quad 3r(r - 3) - 1(r - 3) = 0$

$\Rightarrow \quad (3r - 1)(r - 3) = 0$

$\Rightarrow \quad 3r - 1 = 0 \quad \text{or} \quad 6r - 2 = 0$

$\Rightarrow \quad r = \frac{1}{3} \quad \text{or} \quad r = 3$

when $a = 6, \quad r = \frac{1}{3}$

$\frac{a}{r} = \frac{6}{\frac{1}{3}} = 18$

$a = 6$

$ar = 6 . \frac{1}{3} = 2$

when $a = 6, \quad r = 3$ then

$\frac{a}{r} = \frac{6}{3} = 2$

$a = 6$

$ar = 6(3) = 18$

$\Rightarrow$ required numbers are 2, 6, 18, or 18, 6, 2

Q.11 If the sum of four consecutive terms of G.P is 80 and A.M of second and fourth of them is 30. Find the term.

Solution:

Given that

$$a + ar + ar^2 + ar^3 = 80$$

or $$a + ar^2 + ar + ar^3 = 80 \qquad \text{......... (1)}$$

also

$$\frac{ar + ar^3}{2} = 30$$

$\Rightarrow$ $$ar + ar^3 = 60 \qquad \text{......... (2)}$$

Put (2) in (1), we get

$$a + ar^2 + 60 = 80$$

$$a + ar^2 = 20 \qquad \text{......... (3)}$$

equation (2) can be written as

$$r(a + ar^2) = 60 \qquad \text{......... (4)}$$

Put (3) in (4), we have

$$r(20) = 60$$

$$\boxed{r = 3}$$

Put $r = 3$ in equation (3), we get

$$a + a(3)^2 = 20$$

$$a + 9a = 20$$

$$10a = 20 \Rightarrow \boxed{a = 2}$$

$\Rightarrow$ $$ar = 2(3) = 6$$

$$ar^2 = 2(3)^2 = 18$$

$$ar^3 = 2(3)^3 = 54$$

$\Rightarrow$ required terms are 2, 6, 18, 54.

Q.12 If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. Show that common ratio is $\pm\sqrt{\frac{a}{c}}$

(Lahore Board 2009)

Solution:

Given that

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P

$\Rightarrow \quad r = \frac{\frac{1}{b}}{\frac{1}{a}} = \frac{1}{b} . a = \frac{a}{b} \qquad \ldots\ldots\ldots (1)$

and $\quad r = \frac{\frac{1}{c}}{\frac{1}{b}} = \frac{1}{c} . b = \frac{b}{c} \qquad \ldots\ldots\ldots (2)$

multiply equations (1) and (2), we get

$r^2 = \frac{a}{b} . \frac{b}{c} = \frac{a}{c}$

$\Rightarrow \quad r^2 = \pm\sqrt{\frac{a}{c}}$

Q.13 If the numbers 1, 4, and 3 are subtracted from three consecutive terms of an A.P, the resulting numbers are in G.P. Find the numbers, if their sum is 21.

Solution:

Let $a - d$, a, $a + d$ are in A.P.

then $\quad a - d + a + a + d = 21$

$\Rightarrow \quad 3a = 21 \quad \Rightarrow \quad \boxed{a = 7}$

also given that

$a - d - 1, \quad a - 4, \quad a + d - 3$ are in G.P

$\Rightarrow \quad \frac{a-4}{a-d-1} = \frac{a+d-3}{a-4}$

Put $a = 7$

$\Rightarrow \quad \frac{7-4}{7-d-1} = \frac{7+d-3}{7-4}$

$\Rightarrow \quad \frac{3}{6-d} = \frac{4+d}{3}$

$\Rightarrow \quad (6 - d)(4 + d) = 9$

$\Rightarrow \quad 24 + 6d - 4d - d^2 = 9$

$\Rightarrow \quad 24 + 2d - d^2 = 9$

$\Rightarrow \quad d^2 - 2d - 24 + 9 = 0$

$\Rightarrow \quad d^2 - 2d - 15 = 0$

$\Rightarrow \quad d^2 - 5d + 3d - 15 = 0$

$\Rightarrow \quad d(d - 5) + 3(d - 5) = 0$

$\Rightarrow \quad (d + 3)(d - 5) = 0$

$\Rightarrow \quad d = -3 \quad \text{or} \quad d = 5$

when $a = 7, \ d = -3$

$a - d = 7 - (-3) = 10$

$a = 7$

$a + d = 7 + (-3) = 4$

when $a = 7, \ d = 5$

$a - d = 7 - 5 = 2$

$a = 7$

$a + d = 7 + 5 = 12$

so required numbers are 10, 7, 4 or 2, 7, 12

Q.14 If three consecutive numbers in A.P are increased by 1, 4, 15 respectively, the resulting numbers are in G.P. Find original numbers of if their sum is 6.

Solution:

Let $a - d, \ a, \ a + d$ be the consecutive terms in A.P.

then $a - d + a + a + d = 6$

$3a = 6 \Rightarrow \boxed{a = 2}$

also given that

$a - d + 1, \quad a + 4, \quad a + d + 15$ are in G.P

$$\Rightarrow \quad \frac{a + 4}{a - d + 1} = \frac{a + d + 15}{a + 4}$$

Put $a = 2$

$$\Rightarrow \quad \frac{2 + 4}{2 - d + 1} = \frac{2 + d + 15}{2 + 4}$$

$$\Rightarrow \quad \frac{6}{3 - d} = \frac{d + 17}{6}$$

$\Rightarrow \quad 36 = (3 - d)(17 + d)$

$\Rightarrow \quad 36 = 3d + 51 - d^2 - 17d$

$\Rightarrow \quad 36 = -d^2 - 14d + 51$

$\Rightarrow \quad d^2 + 14d - 51 + 36 = 0$

$\Rightarrow \quad d^2 + 14d - 15 = 0$

$\Rightarrow \quad d^2 + 15d - d - 15 = 0$

$\Rightarrow \quad d(d + 15) - 1(d + 15) = 0$

$\Rightarrow \quad (d - 1)(d + 15) = 0$

$\Rightarrow \quad d = 1 \quad \text{or} \quad d = -15$

when $a = 2, \quad d = 1$

$a - d = 2 - 1 = 1$

$a = 2$

$a + d = 2 + 1 = 3$

when $a = 2, \quad d = -15$

$a - d = 2 - (-15) = 2 + 15 = 17$

$a = 2$

$a + d = 2 + (-15) = 2 - 15 = -13$

so the required numbers are $1, 2, 3$ or $17, 2, -13$

GEOMETRIC MEANS

A number G is said to be a geometric means (G.M) between two numbers a and b if a, G, b are in G.P. therefore

$$\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \pm\sqrt{ab}$$

EXERCISE 6.7

Q.1 Find G.M. between

(i) -2 and 8 **(Lahore Board 2007)**

(ii) $-2i$ and $8i$ **(Gujranwala Board 2007, Lahore Board 2008)**

Solution:

(i) -2 and 8

Let $a = -2$ and $b = 8$

as $\quad G.M. = \pm\sqrt{ab}$

$$= \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16} = \pm\sqrt{-1}\sqrt{16} = \pm 4i$$

(ii) $-2i$ and $8i$

Let $a = -2i$, $b = 8i$