Chapter 9

FUNDAMENTALS OF TRIGONOMETRY

TRIGONOMETRY

Trigonometry means measurement of triangle.

ANGLE

Two rays with a common starting point form an angle. Angle is measured positive if rotation is anticlockwise and negative if rotation is clockwise.

MEASUREMENT OF ANGLES

1. Sexagesimal System

In this system an angle is measured in degrees, minutes and seconds.

2. Radian or Circular Measure

Radian **(Gujranwala Board 2007)**

Radian is the measure of the angle subtended at the centre of the circle by an arc whose length is equal to radius of the circle.

Degree

If the circumference of circle is divided into 360 arcs of equal lengths, then central angle subtended by each arc is of one degree measure and is denoted by 1^o.

NOTE

$$\pi \text{ radians} = 180^o$$

$$1 \text{ radian} = \frac{180^o}{\pi}$$

$$\Rightarrow \quad 1 \text{ radian} = \frac{180^o}{3.1416} = 57.296^o$$

$$1^o = \frac{\pi}{180} \text{ radians}$$

$$= \frac{3.1416}{180}$$

$1^o = 0.0175$ radian

Conterminal Angles (General Angles)

The angles with the same initial and terminal sides are called co–terminal angles i.e.,

$\theta, \theta \pm 2\pi, \theta \pm 4\pi$……. are co–terminal angles.

$\theta + 2k\pi$, $k \in z$ or $\theta + k(360^o)$ $k \in Z$ is called general angle where $0 \leq \theta \leq 2\pi$

Angle in standard Position

An angle is said to be in standard position if its vertex lie at the origin and its initial side along the positive x–axis.

Quadrantal Angle

If the terminal side of an angle falls on x–axis or y–axis it is called quadrantal angle.

i.e., $\theta^o, \pm 90^o, \pm 180^o, \pm 270^o, \pm 360^o$

Allied Angles

The angles associated with basic angles of measure "θ" to a right angle or its multiple are called Allied Angles.

So the angles of measure $90^o \pm \theta$, $180^o \pm \theta$, $270^o \pm \theta$, $360^o \pm \theta$ are known as Allied Angles.

Basic Angles OR Reference Angles:

Def. A +ve acute angle with x-axis is called basic angle or reference angle.

EXERCISE 9.1

Q.1 Express the following sexagesimal measures of angle in radians

(i) 0^o (ii) 45^o (iii) 60^o (iv) 75^o (v) 90^o

(vi) 105^o (vii) 120^o (viii) 135^o (ix) 150^o

(x) $10^o 15'$ (xi) $35^o 20'$ (xii) $75^o 6' 30'$

(xiii) $120^o 40''$ (xiv) $154^o 20''$ (xv) 0^o

(xvi) $3''$

Solution:

(i) $30° = 30^o \times \frac{\pi}{180}$ radian $= \frac{\pi}{6}$ radian

(ii) $45^o = 45^o \cdot \frac{\pi}{180}$ radian $= \frac{\pi}{4}$ radian

(iii) $60^o = 60 \times \frac{\pi}{180}$ radian $= \frac{\pi}{3}$ radian

(iv) $75^\circ = 75^\circ \times \frac{\pi}{180}$ radian $= 5\frac{\pi}{12}$ radian

(v) $90^\circ = 90^\circ \times \frac{\pi}{180}$ radian $= \frac{\pi}{2}$ radian

(vi) $105^\circ = 105^\circ \times \frac{\pi}{180}$ radian $= \frac{7\pi}{12}$ radian

(vii) $120^\circ = 120^\circ \times \frac{\pi}{180}$ radian $= \frac{2\pi}{3}$ radian

(viii) $135^\circ = 135^\circ \times \frac{\pi}{180}$ radian $= \frac{3\pi}{4}$ radian

(ix) $150^\circ = 150^\circ \times \frac{\pi}{180}$ radian $= \frac{5\pi}{6}$ radian

(x) $10^\circ\, 15' = 10^\circ + \frac{15^\circ}{60}$

$$= 10^\circ + \frac{1}{4}$$

$$= \frac{40^\circ + 1^\circ}{4} = \frac{41^\circ}{4}$$

$$= \frac{41}{4} \times \frac{\pi}{180} \text{ radian}$$

$$= \frac{41\,\pi}{720} \text{ radian}$$

(xi) $35^\circ\, 20' = 35^\circ + \frac{20^\circ}{60} = 35^\circ + \frac{1^\circ}{3}$

$$= \frac{105^\circ + 1^\circ}{3} = \frac{106^\circ}{3}$$

$$= \frac{106}{3} \times \frac{\pi}{180} \text{ radian} = \frac{53\pi}{270} \text{ radian}$$

(xii) $75^\circ\, 6'\, 30'' = 75^\circ + \frac{6^\circ}{60} + \frac{30^\circ}{3600}$

$$= 75^\circ + \frac{1^\circ}{10} + \frac{1^\circ}{120}$$

$$= \left(75 + \frac{1}{10} + \frac{1}{120}\right)^\circ$$

$$= \left(\frac{9000 + 12 + 1}{120}\right)^\circ$$

$$= \left(\frac{9013}{120}\right)^\circ$$

$$= \frac{9013}{120} \times \frac{\pi}{180} \text{ radian} = \frac{9013\,\pi}{21600} \text{ radian}$$

(xiii) $120'\,40'' = \dfrac{120^{o}}{60} + \dfrac{40^{o}}{3600}$

$$= \left(\frac{120}{60} + \frac{40}{3600}\right)^{o}$$

$$= \left(2 + \frac{1}{90}\right)^{0}$$

$$= \left(\frac{180 + 1}{90}\right)^{o}$$

$$= \frac{181^{o}}{90}$$

$$= \frac{181}{90} \times \frac{\pi}{180} \text{ radian} = \frac{181}{16200}\pi \text{ radian}$$

(xiv) $154^{o}\,20'' = 154^{o} + \dfrac{20^{o}}{3600}$

$$= 154^{o} + \frac{20^{o}}{3600}$$

$$= \left(154 + \frac{1}{180}\right)^{o}$$

$$= \left(\frac{27720 + 1}{180}\right)^{o}$$

$$= \frac{27721^{o}}{180}$$

$$= \frac{27721}{180} \times \frac{\pi}{180} \text{ radian}$$

$$= \frac{27721}{32400} \text{ radian}$$

(xv) $0^{o} = 0 \times \dfrac{\pi}{180}$ radian

$$= 0 \text{ radian}$$

(xvi) $3'' = \dfrac{3^{o}}{3600}$

$$= \frac{1^{o}}{1200}$$

$$= \frac{1}{1200} \times \frac{\pi}{180} \text{ radian}$$

$$= \frac{\pi}{216000} \text{ radian}$$

Q.2 Convert the following radian measures of angles into the measures of sexagesimal system.

(i) $\frac{\pi}{8}$ **(ii)** $\frac{\pi}{6}$ **(iii)** $\frac{\pi}{4}$ **(iv)** $\frac{\pi}{3}$ **(v)** $\frac{\pi}{2}$ **(vi)** $\frac{2\pi}{3}$ **(vii)** $\frac{3\pi}{4}$ **(viii)** $\frac{5\pi}{6}$

(ix) $\frac{7\pi}{12}$ **(x)** $\frac{9\pi}{5}$ **(xi)** $\frac{11\pi}{27}$ **(xii)** $\frac{13\pi}{16}$ **(xiii)** $\frac{17\pi}{24}$ **(xiv)** $\frac{25\pi}{36}$ **(xv)** $\frac{19\pi}{32}$

Solution:

(i) $\frac{\pi}{8} = \frac{\pi}{8} \times \frac{180}{\pi}$ degree

$= 22.5$ degrees

$= 22^{o}\, 30'$

(ii) $\frac{\pi}{6} = \frac{\pi}{6} \cdot \frac{180}{\pi}$ degrees

$= 30^{o}$

(iii) $\frac{\pi}{4} = \frac{\pi}{4} \times \frac{180}{\pi}$ degrees

$= 45^{o}$

(iv) $\frac{\pi}{3} = \frac{\pi}{3} \times \frac{180}{\pi}$ degrees

$= 60^{o}$

(v) $\frac{\pi}{2} = \frac{\pi}{2} \times \frac{180}{\pi} = 90^{o}$

(vi) $\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180}{\pi}$ degrees

$= 120^{o}$

(vii) $\frac{3\pi}{4} = \frac{3\pi}{4} \times \frac{180}{\pi}$ degrees

$= 135^{o}$

(viii) $\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180}{\pi}$ degrees

$= 150^{o}$

(ix) $\frac{7\pi}{12} = \frac{7\pi}{12} \times \frac{180}{\pi}$ degrees

$= 105^{o}$

(x) $\frac{9\pi}{5} = \frac{9\pi}{5} \times \frac{180}{\pi}$ degrees

$= 324^{o}$

(xi) $\frac{11\pi}{27} = \frac{11\pi}{27} \times \frac{180}{\pi}$ degrees

$= 73^\circ 20'$

(xii) $\frac{13\pi}{16} = \frac{13\pi}{16} \times \frac{180}{\pi}$ degrees

$= 146^\circ 15'$

(xiii) $\frac{17\pi}{24} = \frac{17\pi}{24} \times \frac{180}{\pi}$ degrees

$= 127^\circ 30'$

(xiv) $\frac{25\pi}{36} = \frac{25\pi}{36} \times \frac{180}{\pi}$ degrees

$= 125^\circ$

(xv) $\frac{19\pi}{32} = \frac{19\pi}{32} \times \frac{180}{\pi}$ degrees

$= 106^\circ 52' 30''$

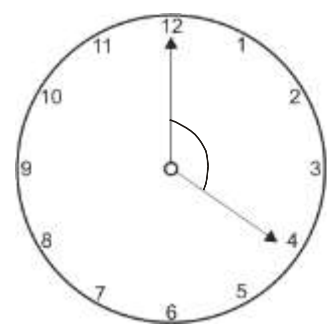
Q.3 What is the circular measure of angle between the hands of a watch at 4'O clock?

Solution:

Circular measure of angle for one complete revolution = 2π

Circular measure of angle between hands of a watch at 4'O clock $= 2\pi \times \frac{4}{12}$

$= \frac{2\pi}{3}$ radians



Q.4 Find θ when

(i) $l = 1.5$cm r = 2.5cm **(Lahore Board 2011)**

(ii) $l = 3.2$m r = 2m

Solution:

(i) $l = 1.5$cm r = 2.5cm

we know $l = r\,\theta$

$\Rightarrow \theta = \frac{l}{r}$

$\theta = \frac{1.5}{2.5} = 0.6$ radian.

(ii) $l = 3.2\text{m} \quad r = 2\text{m}$

we know $l = r\theta$

$$\Rightarrow \quad \theta = \frac{l}{r}$$

$$\theta = \frac{3.2}{2} = 1.6 \text{ radians.}$$

Q.5 Find l when

(i) $\theta = \pi$ **radian** $r = 6\text{cm}$

(ii) $\theta = 65^o\,20'$ $r = 18\text{mm}$

Solution:

(i) $\theta = \pi$ **radian** $r = 6\text{cm}$

We know

$$l = r\theta$$

$$= 6\pi = 6 \times 3.1416 = 18.85 \text{ cm}$$

(ii) $\theta = 65^o\,20'$ $r = 18\text{mm}$

$$\theta = = 65^o + \frac{20^o}{60} = \left(65 + \frac{20}{60}\right)^o = \frac{196^o}{3}$$

$$\theta = \frac{196}{3} \times \frac{\pi}{180} \text{ radian} = \frac{49}{3 \times 45}(3.1415)$$

$$l = r\theta$$

$$= 18 \times \frac{49}{3 \times 45}(3.1415) = 20.53 \text{ mm}$$

Q.6 Find r when

(i) $l = 5\text{cm}$ $\theta = \frac{1}{2}$ **radian** **(Gujranwala Board 2007, Lahore Board 2010)**

(ii) $l = 56\text{cm}$ $\theta = 45^o$ **(Lahore Board 2004 , 2007)**

Solution:

(i) $l = 5\text{cm}$ $\theta = \frac{1}{2}$ **radian**

we know that $l = r\theta$

$$r = \frac{l}{\theta} = \frac{5}{\frac{1}{2}} = 10\text{cm}$$

(ii) $l = 56\text{cm} \quad \theta = 45^o$

First we convert 45^o into radian measure

$$\theta = 45^o \Rightarrow 45 \times \frac{\pi}{180} \text{ radian}$$

we know that $l = r\theta$

$$\Rightarrow \quad r = \frac{l}{\theta} \Rightarrow \frac{56}{45 \times \frac{\pi}{180} \text{ radian}} = \frac{56}{0.785} = 71.33 \text{ cm}$$

Q.7 What is the length of the arc intercepted on a circle of radius 14cm by the arms of central angle of 45^o? (Gujranwala Board 2006)

Solution:

$l = ? \quad r = 14\text{cm} \quad \theta = 45^o$

$$\theta = 45 \times \frac{\pi}{180} \text{ radian}$$

$\theta = 0.785$

we know that $l = r\theta$

$l = 14 \times 0.785 = 10.99$ or 11 cm approximately.

Q.8 Find the radius of the circle, in which the arms of a central angle of measure 1 radian cut off an arc of length 35cm.

Solution:

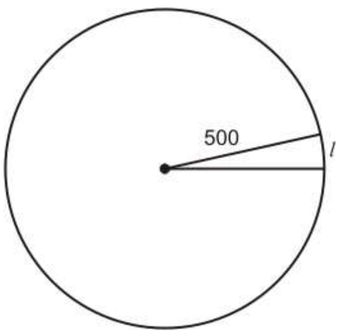
$r = ? \quad \theta = 1 \text{ radian} \quad l = 35\text{cm}$

we know that $l = r\theta$

$$\Rightarrow \quad r = \frac{l}{\theta} = \frac{35}{1} = 35\text{cm}$$

Q.9 A railway train is running on a circular track of radius 500 meters at the rate of 30 km/h. Through what angle will it turn in 10 sec.

Solution:



$\theta = ?, \quad r = 500 \text{ m}, \quad \text{Speed} = 30\text{km/h}$

$$\text{Speed} = \frac{30 \times 1000}{3600} = \frac{25}{3} \text{ m/sec}$$

Distance covered in 10 sec

Distance = speed × time

$$l = \frac{25}{3} \times 10 \text{ m}$$

$$= \frac{250}{3}\text{ m}$$

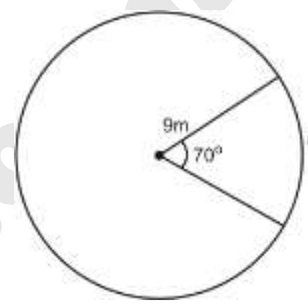
We know that $l = r\,\theta$

$$\Rightarrow \quad \theta = \frac{l}{r}$$

$$= \frac{250}{3} \times \frac{1}{500} = \frac{1}{6}\text{ radian}$$

Q.10 A horse is tethered to a peg by a rope of 9m in length and it can move in a circle with the peg as centre. If the horse moves along the circumference of the circle, keeping the rope tight, how far will it have gone when the rope has turned through an angle of 70°?

Solution:



$r = 9\text{m}$, $\theta = 70^o = 70 \times \frac{\pi}{180}$ radians, $l = ?$

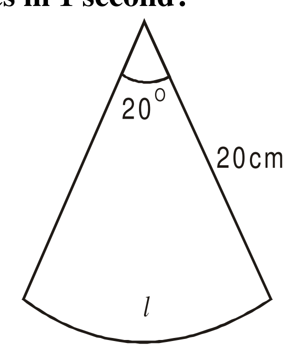
we know that

$l = r\,\theta$

$= 9 \times 70 \times \frac{3.1416}{180} = 10.99\text{m}$

Q.11 The pendulum of a clock is 20cm long and it swings through an angle of 20^o each second. How far does the tip of the pendulum moves in 1 second?

Solution:



$r = 20\text{ cm}$, $\theta = 20^o = 20 \times \frac{\pi}{180}$ radian, $l = ?$

we know that

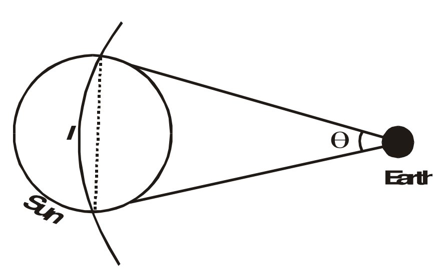
$l = r\,\theta$

$l = 20 \times 20 \times \frac{3.1416}{180}$

$l = 6.89\text{ cm}$

Q.12 Assuming average distance of the earth from the sun to be 148×10^6 km and angle subtended by the sun at the eye of the person on the earth measure 9.3×10^{-3} radian. Find the diameter of the sun.

Solution:



$l = ?$, $r = 148 \times 10^6\text{ km}$, $\theta = 9.3 \times 10^{-3}$ radian

we know that

$l = r\,\theta$

$l = 148 \times 10^6 \times 9.3 \times 10^{-3}$

$l = 1376.4 \times 10^3$ km.

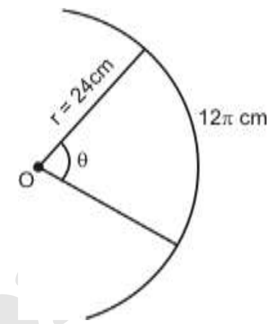
$l = 1.3764 \times 10^3 \times 10^3$ km

$l = 1.3764 \times 10^6$km

Q.13 A circular wire of radius 6cm is cut straightened and then bent so as to lie along the circumference of hoop of radius 24cm. Find the measure of the angle which it subtends at the centre of hoop.

Solution:

$\theta = ?$ Length of circular wire of radius 6 cm



$l = 2\pi r = 2\pi(6)$

$l = 12\pi = 12(3.1416) = 37.7$ cm $\quad r = 24$ cm

we know that $l = r\theta$

$\Rightarrow \quad \theta = \dfrac{l}{r}$

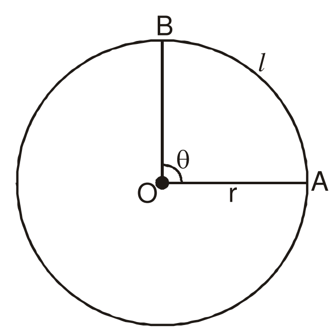
$\theta = \dfrac{37.7}{24}$

$\theta = 1.57$ radian

Q.14 Show that the area of a sector of a circular region of radius r is $\frac{1}{2}r^2\theta$, where θ is the circular measure of the central angle of the sector.

Solution:

Let us consider a circle with centre at O. Let r be radius of circle. Let l be length of arc AB.



Let θ be central angle in radian. We know that

Area of sector of circular region : Area of circle = Angle of sector : Angle of circle

$$\Rightarrow \quad \frac{\text{Area of sector of circular region}}{\text{area of circle}} = \frac{\text{Angle of sector}}{\text{Angle of circle}}$$

$$\frac{\text{Area of sector of circular region}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{Area of sector of circular region} = \frac{\theta}{2\pi}\cdot\pi r^2$$

$$= \frac{1}{2}r^2\theta$$

Hence proved.

Q.15 Two cities A and B lie on the equator such that their longitudes are 45^o E and 25^o W respectively. Find the distance between the two cities, taking radius of the earth as 6400 kms.

Solution:

θ = Total angle between two cities

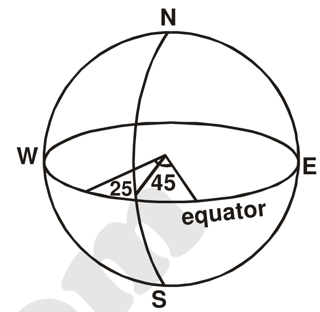
$\theta = 45^o + 25^\circ = 70^\circ$

$\theta = 70 \times \frac{\pi}{180}$ radian

$r = 6400$ km , $l = ?$

we know that $l = r\theta$

$= 6400 \times 70 \times \frac{3.1416}{180} = 7819.09$ km



Q.16 The moon subtends an angle of 0.5^o at the eye of an observer on earth. The distance of the moon from the earth is 3.844×10^5 km approx what is the length of diameter of the moon?

Solution:

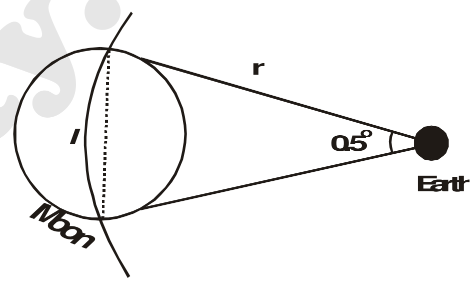
$l = ?$ $\theta = 0.5^o = 0.5\frac{\pi}{180}$ radian

$r = 3.844 \times 10^5$ km

we know that $l = r\theta$

$l = 3.844 \times 10^5 \times 0.5 \times \frac{\pi}{180}$

$l = 3354.52$ km



Q.17 The angle subtended by the earth at an eye of spaceman, landed on the moon, is $1^o 54'$. Find the radius of the earth is 6400km. Find the approximate distance between the moon and the earth.

Solution:

$r = ?$

θ = Angle subtended the spaceman on the moon

$\theta = 1^o 54'$

$\theta = 1^o + \frac{54^o}{60} = \frac{114^o}{60} = \frac{114}{60} \times \frac{\pi}{180}$ radian

l = diameter of earth

$l = r + r = 6400 + 6400 = 12800$ km

we know that $l = r\theta$

$\Rightarrow$ $r = \frac{l}{\theta} = \frac{12800}{\frac{114}{60} \times \frac{3.1416}{180}}$

$r = \frac{12800}{0.03316} = 386007.23$ km