As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Therefore, given triples can be direction angles of a vector.

The Scalar Product of Two vectors

Definition:

Let two non zero vectors $\underline{u}$ & $\underline{v}$ in the plane or in space, have same initial point. The dot product of $\underline{u}$ and $\underline{v}$, written as $\underline{u} \cdot \underline{v}$, is defined by

$$\underline{u} \cdot \underline{v} = |\underline{u}|\ |\underline{v}| \cos\theta$$ where θ is angle between $\underline{u}$ & $\underline{v}$ and $0 \leq \theta \leq \pi$.

Orthogonal / Perpendicular vectors:

The two vectors $\underline{u}$ & $\underline{v}$ are orthogonal / perpendicular if and only if $\underline{u} \cdot \underline{v} = o$

Remember:

(i) Dot product, inner product, scalar product are same.

(ii) $\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$

(iii) $\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0$

(iv) Scalar product is commutative i.e., $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

EXERCISE 7.3

Q.1 Find the Cosine of the angle θ between $\underline{u}$ and $\underline{v}$.

(i) $\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$ $\quad \underline{v} = 2\underline{i} - \underline{j} + \underline{k}$

Formula

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}|\ |\underline{v}|}$$

Solution:

$$\underline{u} = 3\underline{i} + \underline{j} - \underline{k}, \quad \underline{v} = 2\underline{i} - \underline{j} + \underline{k}$$

$$\underline{u} \cdot \underline{v} = (3\underline{i} + \underline{j} - \underline{k}) \cdot (2\underline{i} - \underline{j} + \underline{k})$$

$$\underline{u} \cdot \underline{v} = 6 - 1 - 1 = 4$$

$$|\underline{u}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9 + 1 + 1} = \sqrt{11}$$

$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}|\ |\underline{v}|} = \frac{4}{\sqrt{11}\sqrt{6}} = \frac{4}{\sqrt{66}}$$

$$\cos\theta = \frac{4}{\sqrt{66}}$$ Ans.

(ii) $\underline{\mathbf{u}} = \underline{i} - 3\underline{\mathbf{j}} + 4\underline{\mathbf{k}}$, $\underline{\mathbf{v}} = 4\underline{i} - \underline{\mathbf{j}} + 3\underline{\mathbf{k}}$

Solution:

$$\underline{u}\cdot\underline{v} = (\underline{i} - 3\underline{j} + 4\underline{k}) \cdot (4\underline{i} - \underline{j} + 3\underline{k})$$
$$= 4 + 3 + 12$$
$$\underline{u}\cdot\underline{v} = 19$$
$$|\underline{u}| = \sqrt{(1)^2 + (-3)^2 + (4)^2} = \sqrt{1 + 9 + 16} = \sqrt{26}$$
$$|\underline{v}| = \sqrt{(4)^2 + (-1)^2 + (3)^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$$
$$\cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|\,|\underline{v}|} = \frac{19}{\sqrt{26}\sqrt{26}}$$
$$\cos\theta = \frac{19}{26} \qquad \text{Ans.}$$

(iii) $\underline{\mathbf{u}} = [-3, 5]$, $\underline{\mathbf{v}} = [6, -2]$

Solution:

$$\underline{u} = -3\underline{i} + 5\underline{j}, \quad \underline{v} = 6\underline{i} - 2\underline{j}$$
$$\underline{u}\cdot\underline{v} = (-3\underline{i} + 5\underline{j}) \cdot (6\underline{i} - 2\underline{j})$$
$$= -18 - 10$$
$$\underline{u}\cdot\underline{v} = -28$$
$$|\underline{u}| = \sqrt{(-3)^2 + (5)^2} = \sqrt{9 + 25} = \sqrt{34}$$
$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2} = \sqrt{36 + 4} = \sqrt{40}$$
$$\cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|\,|\underline{v}|}$$
$$\cos\theta = \frac{-28}{\sqrt{34}\sqrt{40}} = \frac{-28}{\sqrt{34}\sqrt{10}} = \frac{-28}{\sqrt{2\times 17}\sqrt{2\times 5}} = \frac{-14}{2\sqrt{85}}$$
$$\cos\theta = \frac{-7}{\sqrt{85}} \qquad \text{Ans.}$$

(iv) $\underline{\mathbf{u}} = [2, -3, 1]$, $\underline{\mathbf{v}} = [2, 4, 1]$

Solution:

$$\underline{u} = 2\underline{i} - 3\underline{j} + \underline{k}, \quad \underline{v} = 2\underline{i} + 4\underline{j} + \underline{k}$$
$$\underline{u}\cdot\underline{v} = (2\underline{i} - 3\underline{j} + \underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k})$$
$$= 4 - 12 + 1$$
$$\underline{u}\cdot\underline{v} = -7$$
$$|\underline{u}| = \sqrt{(2)^2 + (-3)^2\,(1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$
$$|\underline{v}| = \sqrt{(2)^2 + (4)^2 + (1)^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$\cos\theta = \frac{\underline{u}\,.\underline{v}}{|\underline{u}|\,|\underline{v}|}$$

$$\cos\theta = \frac{-7}{\sqrt{14}\sqrt{21}} = \frac{-7}{\sqrt{2\times7\times3\times7}} = \frac{-7}{7\sqrt{6}} = \frac{-1}{\sqrt{6}}$$

$$\cos\theta = \frac{-1}{\sqrt{6}} \qquad \text{Ans.}$$

Q.2 Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when

(i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$

Solution:

Formula Projection of $\underline{a}$ along $\underline{b} = \dfrac{\underline{a}\,.\underline{b}}{|\underline{b}|}$

Project of $\underline{b}$ along $\underline{a} = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|}$

$$\underline{a} = \underline{i} + 0\underline{j} - \underline{k} \quad , \quad \underline{b} = 0\underline{i} + \underline{j} + \underline{k}$$

$$\underline{a}\,.\,\underline{b} = (\underline{i} + 0\underline{j} - \underline{k})\;.\;(0\underline{i} + \underline{j} + \underline{k})$$

$$= 0 + 0 - 1$$

$$\underline{a}\,.\,\underline{b} = -1$$

$$|\underline{a}| = \sqrt{(1)^2 + 0 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$|\underline{b}| = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

Projection of $\underline{a}$ along $\underline{b} = \dfrac{\underline{a}.\underline{b}}{|\underline{b}|} = \dfrac{-1}{\sqrt{2}}$ Ans.

And Projection of $\underline{b}$ along $\underline{a} = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|} = \dfrac{-1}{\sqrt{2}}$ Ans.

(ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$ (Gujranwala Board 2004, 2007)

Solution:

$$\underline{a}\,.\,\underline{b} = (3\underline{i} + \underline{j} - \underline{k})\;.\;(-2\underline{i} - \underline{j} + \underline{k})$$

$$\underline{a}\,.\,\underline{b} = -6 - 1 - 1 = -8$$

$$|\underline{a}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$$

$$|\underline{b}| = \sqrt{(-2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

Projection of $\underline{a}$ along $\underline{b} = \dfrac{\underline{a}.\underline{b}}{|\underline{b}|} = \dfrac{-8}{\sqrt{6}}$ Ans.

Projection of $\underline{b}$ along $\underline{a} = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|} = \dfrac{-8}{\sqrt{11}}$ Ans.

Q.3 Find a real number α so that the vectors $\underline{u}$ & $\underline{v}$ are perpendicular.

(i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$ $\quad \underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$ (Lahore Board 2010,11)

Solution:

Since $\underline{u}$ & $\underline{v}$ are perpendicular so

$$\underline{u} \cdot \underline{v} = 0$$

$$(2\alpha\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$$

$$2\alpha + \alpha - 4 = 0$$

$$3\alpha - 4 = 0$$

$$\alpha = \frac{4}{3} \qquad \text{Ans.}$$

(i) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$ $\quad \underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$ (Lahore Board 2006)

Solution:

$$\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k} \quad , \quad \underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$$

Since $\underline{u}$ & $\underline{v}$ are perpendicular so

$$\underline{u} \cdot \underline{v} = 0$$

$$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$$

$$\alpha + 2\alpha^2 - 3 = 0$$

$$2\alpha^2 + \alpha - 3 = 0$$

$$2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$$

$$\alpha\,(2\alpha + 3) - 1\,(2\alpha + 3) = 0$$

$$(\alpha - 1)(2\alpha + 3) = 0$$

$$\alpha - 1 = 0 \quad , \quad 2\alpha + 3 = 0$$

$$\alpha = 1 \quad , \quad \alpha = \frac{-3}{2} \qquad \text{Ans.}$$

Q.4 Find the number Z so that the triangle with vertices A (1, –1, 0), B (–2,2,1) and C(0, 2, Z) is a right triangle with right angle at C.

Solution:

Given A (1, –1, 0) , B (–2, 2, 1) , C (0, 2, Z)

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= (0 - 1)\,\underline{i} + (2 + 1)\,\underline{j} + (Z - 0)\,\underline{k}$$

$$\overrightarrow{AC} = -\underline{i} + 3\underline{j} + Z\underline{k}$$

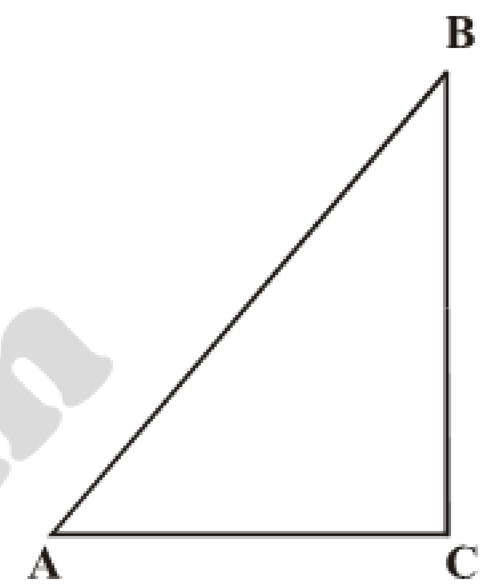
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= (0 + 2)\underline{i} + (2 - 2)\underline{j} + (Z - 1)\underline{k}$$

$\overrightarrow{BC} \quad = \quad 2\underline{i} + 0\underline{j} + (Z-1)\underline{k}$

Since $\overrightarrow{AC}$ & $\overrightarrow{BC}$ are perpendicular
So,

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$$



$$(-\underline{i} + 3\underline{j} + Z\underline{k}) \cdot (2\underline{i} + 0\underline{j} + (Z-1)\underline{k}) = 0$$

$$-2 + 0 + Z(Z-1) = 0$$

$$-2 + Z^2 - Z = 0$$

$$Z^2 - Z - 2 = 0$$

$$Z^2 - 2Z + Z - 2 = 0$$

$$Z(Z-2) + 1(Z-2) = 0$$

$$(Z-2)(Z+1) = 0$$

$\Rightarrow \quad Z - 2 = 0 \qquad Z + 1 = 0$

$\qquad Z = 2, \qquad Z = -1$

Q.5 If V is a vector for which
$\underline{v} \cdot \underline{i} = 0 \quad \underline{v} \cdot \underline{j} = 0, \ \underline{v} \cdot \underline{k} = 0$, find $\underline{v}$ **(Lahore Board 2009)**

Solution:

Let $\underline{v} = x\underline{i} + y\underline{j} + z\underline{k} \quad \text{........ (1)}$

Now

$$\underline{v} \cdot \underline{i} = 0$$

$$(x\underline{i} + y\underline{j} + z\underline{k}) \cdot (\underline{i}) = 0$$

$$x = 0$$

Next

$$\underline{v} \cdot \underline{j} = 0$$

$$(x\underline{i} + y\underline{j} + z\underline{k}) \cdot (0\underline{i} + \underline{j} + 0\underline{k}) = 0$$

$$0 + y + 0 = 0 \qquad \Rightarrow \quad y = 0$$

$$\underline{v} \cdot \underline{k} = 0$$

$$(x\underline{i} + y\underline{j} + z\underline{k}) \cdot (0\underline{i} + \underline{j} + 0\underline{k}) = 0$$

$$0 + 0 + z = 0$$

$\Rightarrow \quad z = 0$

Substitute all values in (1)

$$\underline{v} = 0\underline{i} + 0\underline{j} + 0\underline{k}$$

$\Rightarrow \quad \underline{v} = 0 \quad$ (Null vector) $\qquad$ Ans.

Q.6(i) Show that the vectors $3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} + 5\underline{k}$ & $2\underline{i} + \underline{j} - 4\underline{k}$ form a right angle triangle.

Solution:

Let $\underline{u} = 3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{v} = \underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{w} = 2\underline{i} + \underline{j} - 4\underline{k}$

$$\underline{v} + \underline{w} = \underline{i} - 3\underline{j} + 5\underline{k} + 2\underline{i} + \underline{j} - 4\underline{k}$$
$$= 3\underline{i} - 2\underline{j} + \underline{k}$$
$$\underline{v} + \underline{w} = \underline{u}$$

Hence $\underline{u}$, $\underline{v}$, $\underline{w}$ from a triangle

$$\underline{u} \cdot \underline{w} = (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (2\underline{i} + \underline{j} - 4\underline{k})$$
$$= 6 - 2 - 4 = 0$$

$\Rightarrow$ $\underline{u}$ and $\underline{w}$ are perpendicular to each other.

Therefore, given triangle is right angled triangle.

(ii) Show that the set of points P(1, 3, 2), Q (4, 1, 4). R (6, 5, 5) from a right triangle.

Solution:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (4 - 1)\ \underline{i} + (1 - 3)\underline{j} + (4 - 2)\ \underline{k} = 3\underline{i} - 2\underline{j} + 2\underline{k}$$

$$\overrightarrow{QR} = (6 - 4)\ \underline{i} + (5 - 1)\underline{j} + (5 - 4)\ \underline{k} = 2\underline{i} + 4\underline{j} + \underline{k}$$

$$\overrightarrow{PR} = (6 - 1)\ \underline{i} + (5 - 3)\underline{j} + (5 - 2)\ \underline{k} = 5\underline{i} + 2\underline{j} + 3\underline{k}$$

$$\overrightarrow{QR} = \overrightarrow{PR}$$

Now $\overrightarrow{PQ} + \overrightarrow{QR} = 3\underline{i} - 2\underline{j} + 2\underline{k} + 2\underline{i} + 4\underline{j} + \underline{k}$

$$= 5\underline{i} + 2\underline{j} + 3\underline{k}$$

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

Thus,

P, Q, R from a triangle

$$\overrightarrow{PQ} \cdot \overrightarrow{QR} = (3\underline{i} - 2\underline{j} + 2\underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k})$$
$$= 6 - 8 + 2 = 0$$

Therefore $\overrightarrow{PQ}$ & $\overrightarrow{QR}$ are perpendicular to each other

Thus, given triangle is right-angled triangle.

Q.7 Show that mid point of hypotenuse a right triangle is equidistant from its vertices.

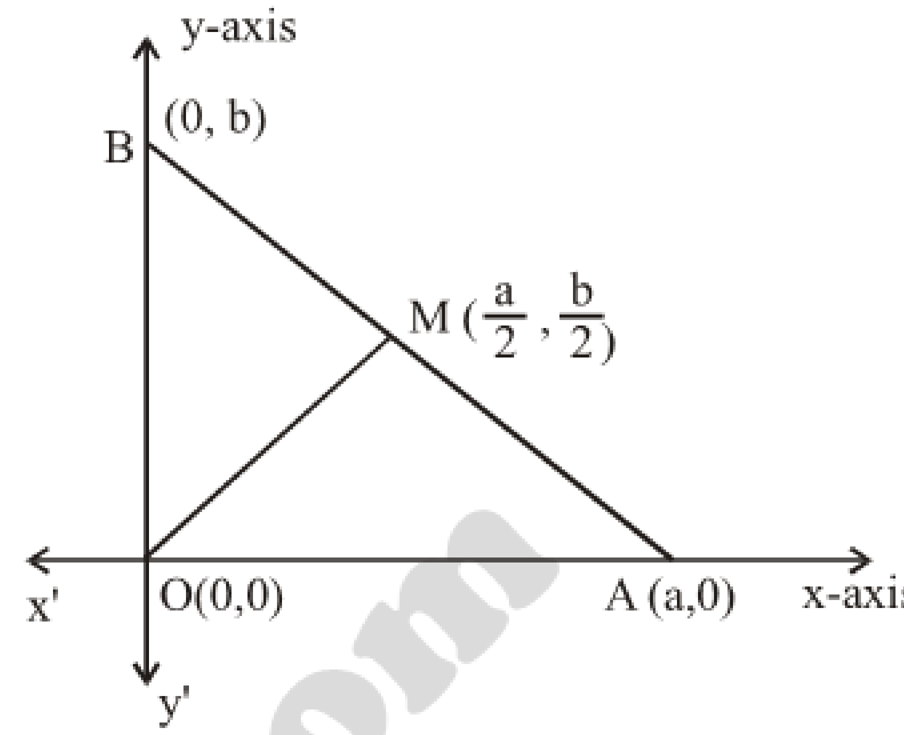
Solution:

Let AOB be any triangle with vertex O is at origin.

Therefore, coordinates of O,A, and B will be O (0, 0), A (a, o) B (o, b).

Coordinates of mid point M are =

$$\left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$$



We have to prove that mid point of hypotenous is equidistant from its vertical i.e.

$$|\overrightarrow{OM}| = |\overrightarrow{AM}| = |\overrightarrow{BM}|$$

$$\overrightarrow{OM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j} = \frac{a}{2}\underline{i} + \frac{b}{2}\underline{j}$$

$$|\overrightarrow{OM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2} \quad \text{——— (i)}$$

$$\overrightarrow{AM} = \left(\frac{a}{2} - a\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j} = \frac{-a}{2}\underline{i} + \frac{b}{2}\underline{j}$$

$$|\overrightarrow{AM}| = \sqrt{\left(\frac{-a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \sqrt{\frac{a^2 + b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2} \quad \text{——— (ii)}$$

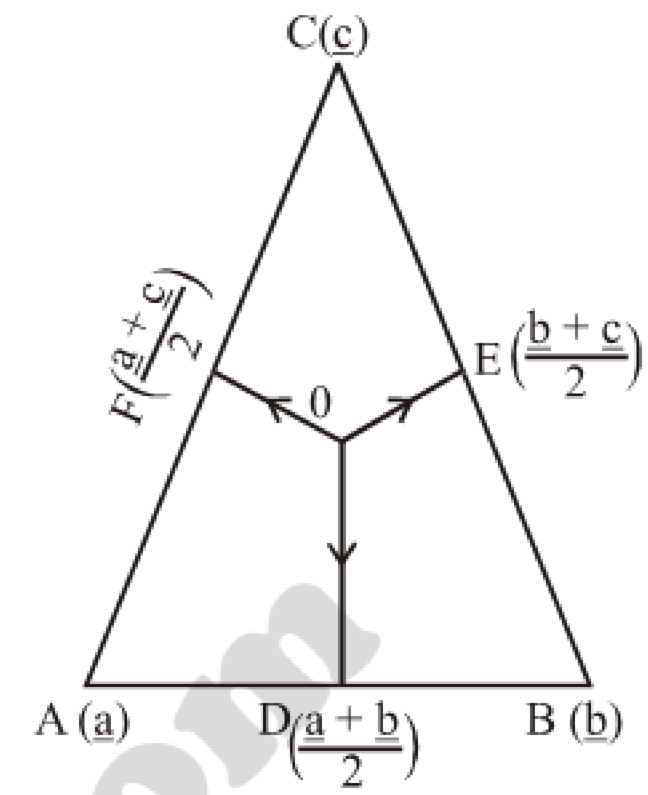
$$\overrightarrow{BM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - b\right)\underline{j} = \frac{a}{2}\underline{i} - \frac{b}{2}\underline{j}$$

$$|\overrightarrow{BM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{-b}{2}\right)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2 + b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2} \quad \text{——— (iii)}$$

From (i) (ii) & (iii) M is equidistant from its vertices.

Q.8 Prove that perpendicular bisectors of the sides of a triangle are concurrent.

Solution:



Let $\overrightarrow{OD}$ & $\overrightarrow{OE}$ be the perpendicular bisectors of the sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Let F be mid point of AC. Join F with O.

Let O is taken as origin.

Since $\overrightarrow{OD}$ is perpendicular to $\overrightarrow{AB}$

$$\overrightarrow{OD} \cdot \overrightarrow{AB} = 0$$

$$\left(\frac{\underline{a} + \underline{b}}{2}\right) \cdot (\underline{b} - \underline{a}) = 0$$

$$(\underline{b} + \underline{a}) \cdot (\underline{b} - \underline{a}) = 0 \times 2$$

$$\Rightarrow \quad b^2 - a^2 = 0 \qquad \text{............. (i)}$$

Again $\overrightarrow{OE}$ is perpendicular to $\overrightarrow{BC}$

$$\overrightarrow{OE} \cdot \overrightarrow{BC} = 0$$

$$\left(\frac{\underline{b} + \underline{c}}{2}\right) \cdot (\underline{c} - \underline{b}) = 0 \times 2$$

$$(\underline{c} + \underline{b}) \cdot (\underline{c} - \underline{b}) = 0 \times 2$$

$$\Rightarrow \quad c^2 - b^2 = 0 \qquad \text{............. (ii)}$$

Adding (i) & (ii) we have

$$b^2 - a^2 + c^2 - b^2 = 0$$

$$c^2 - a^2 = 0$$

$$(\underline{c} + \underline{a}) \cdot (\underline{c} - \underline{a}) = 0$$

$$\left(\frac{\underline{c} + \underline{a}}{2}\right) \cdot (\underline{c} - \underline{a}) = 0$$

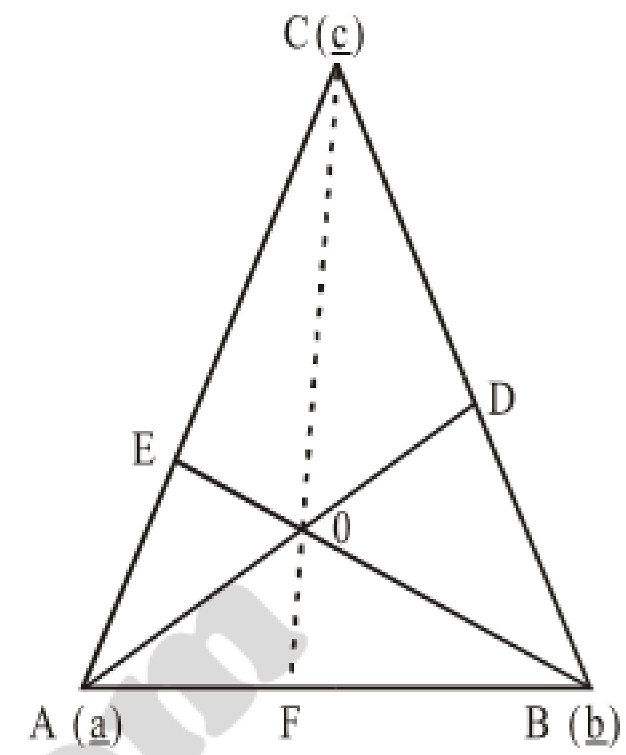
$$\overrightarrow{OF} \cdot \overrightarrow{AC} = 0$$

Which shows that $\overrightarrow{OF}$ is perpendicular to $\overrightarrow{AC}$. Hence perpendicular bisectors of the sides of a triangle are concurrent.

Q.9 Prove that the attitudes of a triangle are concurrent. (Lahore Board 2009)

Solution:

Let AD, BE be the attitudes drawn from vertices A, B, respectively. Join C to O & produce it meet AB at F.



Since $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{AO} \perp \overrightarrow{BC}$ also

$$\overrightarrow{AO} \,.\, \overrightarrow{BC} = 0$$
$$-a \,.\, (\underline{c} - \underline{b}) = 0$$
$$-\underline{a} \,.\, \underline{c} + \underline{a} \,.\, \underline{b} = 0$$
$$\underline{a} \,.\, \underline{b} = \underline{a} \,.\, \underline{c} \quad \text{............. (i)}$$

Since $\overrightarrow{BE} \perp \overrightarrow{AC}$

$\Rightarrow \quad \overrightarrow{BO} \perp \overrightarrow{AC} \quad \Rightarrow \quad \overrightarrow{BO} \,.\, \overrightarrow{AC} = 0$

$\Rightarrow \quad -\underline{b} \,.\, (\underline{c} - \underline{a}) = 0$

$\Rightarrow \quad -\underline{b} \,.\, \underline{c} + \underline{b} \,.\, \underline{a} = 0$

$$\underline{a} \,.\, \underline{b} = \underline{b} \,.\, \underline{c} \quad \text{........... (ii)}$$

from (i) & (ii) we have

$$\underline{a} \,.\, \underline{c} = \underline{b} \,.\, \underline{c}$$
$$\underline{a} \,.\, \underline{c} - \underline{b} \,.\, \underline{c} = 0$$
$$-\underline{c} \,.\, (\underline{b} - \underline{a}) = 0$$
$$\overrightarrow{OF} \,.\, \overrightarrow{AB} = 0$$

Thus $\overrightarrow{OF} \perp \overrightarrow{AB}$

$\Rightarrow \quad \overrightarrow{CF} \perp \overrightarrow{AB}$

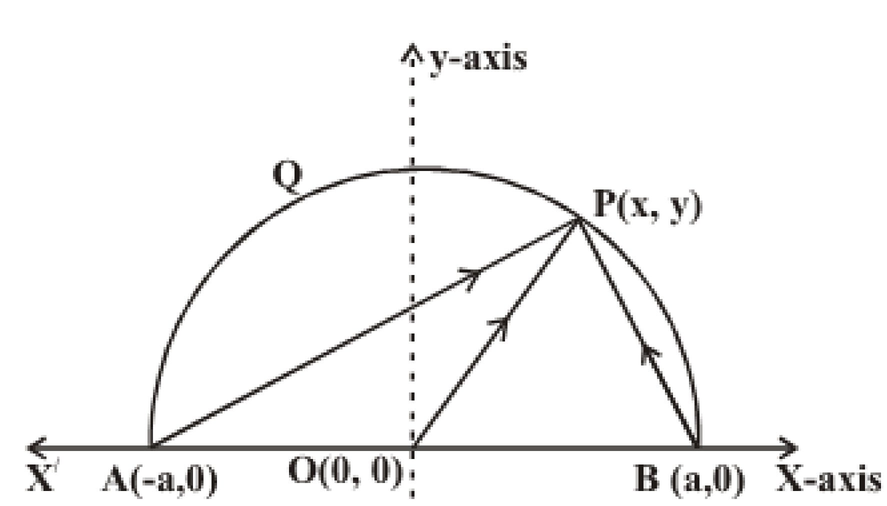
Shows altitudes of a triangle are concurrent.

Q.10 Proved that the angle is a semi circle is a right angle.

(Gujranwala Board 2006, Lahore Board, 2007)

Solution:

Let AQB be a semi circle of radius a with center at origin. Take x-axis along AB. Let P(x,y) be any point on semicircle. Join A and B with P join O and P.



Now

$$\overrightarrow{OA} \quad = \quad -a\, \underline{i}$$

$\overrightarrow{OB} = a\,\underline{i}$

$\overrightarrow{OP} = a$ (radius given)

$|\overrightarrow{OP}|^2 = a^2$ .... (i)

$\overrightarrow{OP} = x\,\underline{i} + y\underline{j}$

$|\overrightarrow{OP}| = \sqrt{x^2+y^2} \Rightarrow |\overrightarrow{OP}| = x^2 + y^2$

$= x^2 + y^2 = a^2$ .... (ii) using (i)

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$\overrightarrow{AP} = (x+a)\underline{i} + (y-0)\underline{j} = (x+a)\underline{i} + y\underline{j}$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$

$= (x-a)\underline{i} + (y-0)\underline{j} = (x-a)\underline{i} + y\underline{j}$

$\overrightarrow{AP}.\overrightarrow{BP} = ((x+a)\underline{i} + y\underline{j}).((x-a)\underline{i} + y\underline{j})$

$= x^2 - a^2 + y^2$

$= x^2 + y^2 - a^2$

$= a^2 - a^2 = 0$ (Using ii)

Hence $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$.

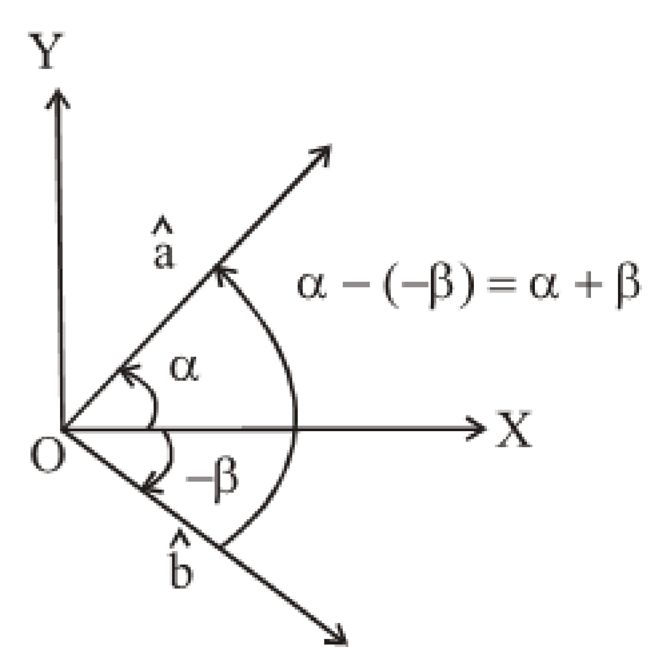
$\therefore \angle APB = 90^o$

Q.11 Prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (Lahore Board 2007,2011)

Solution:

Let $\hat{a}$ and $\hat{b}$ be two unit vectors making angles α and β with x-axis

Therefore, we can write



$\hat{a} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\hat{b} = \cos\beta\,\underline{i} - \sin\beta\,\underline{j}$

$\hat{a}.\hat{b} = (\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}).(\cos\beta\,\underline{i} - \sin\beta\,\underline{j})$

$|\hat{a}|\,|\hat{b}|\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

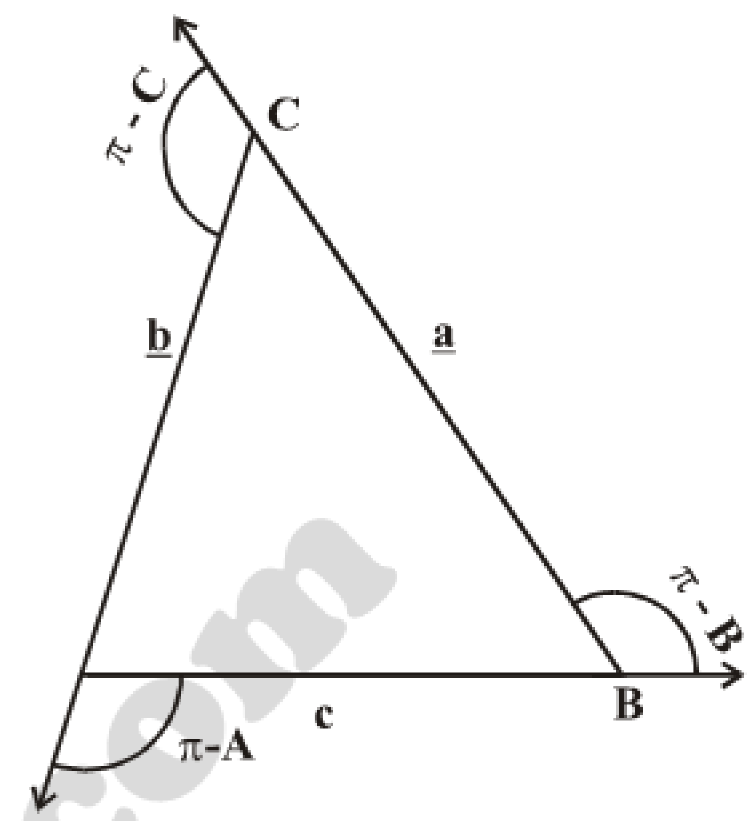
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\quad (\because |\hat{b}| = 1, |\hat{a}| = 1)$

Hence proved

Q.12 Prove that in any triangle ABC

(i) b = ccosA + a cos C

Solution:



$b = \cos A + a\cos C$

For any triangle $\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c}$

$\underline{b} = -(\underline{a} + \underline{c})$ .... (i)

Taking dot product with $\underline{b}$, we have

$\underline{b}\,.\,\underline{b} = -\underline{b}\,.\,(\underline{a} + \underline{c})$

$= -\underline{b}\,.\,\underline{a} - \underline{b}\,.\,\underline{c}$

$b^2 = -|\underline{b}|\,|\underline{a}|\cos(\pi - C) - |\underline{b}|\,|\underline{c}|\cos(\pi - A)$

$= -ba\,(-\cos(+c) - bc\,(-\cos(+A))$

$b^2 = ba\cos C + bc\cos A$

$b = a\cos C + c\cos A$

(Dividing throughout by b) Hence proved

(ii) c = a cos B + b cos A

Solution:

For triangle ABC, we have

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{c} = -\underline{a} - \underline{b}$

$\underline{c} = -(\underline{a} + \underline{b})$ .... (i)

Taking dot product with $\underline{c}$

$\underline{c}\,.\,\underline{c} = -\underline{c}\,.\,(\underline{a} + \underline{b})$

$c^2 = -\underline{c}\,.\,\underline{a} - \underline{c}\,.\,\underline{b}$

$= -|\underline{c}|\,|\underline{a}|\cos(\pi - B) - |\underline{c}|\,|\underline{b}|\cos(\pi - A)$

$c^2 = -ac\,(-\cos B) - cb(-\cos A)$ $\quad (\because \cos(\pi - \theta) = -\cos\theta)$

$c^2 = ac\cos B + bc\cos A$

$c = a\cos B + b\cos A$ (dividing by c) Hence proved.

(iii) $b^2 = c^2 + a^2 - 2ac\cos B$

Solution:

For triangle ABC, by vector addition

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c}$

$\underline{b} = -(\underline{a} + \underline{c})$ .... (i)

Taking dot product with $\underline{b}$

$\underline{b}\,.\,\underline{b} = -(\underline{a} + \underline{c})\,.\,\underline{b}$

$b^2 = -(\underline{a} + \underline{c})\,.\,-(\underline{a} + \underline{c})$

$= \underline{a}\,.\,\underline{a} + \underline{a}\,.\,\underline{c} + \underline{c}\,.\,\underline{a} + \underline{c}\,.\underline{c}$

$b^2 = a^2 + 2\underline{a}\,.\,\underline{c} + c^2$ $\quad (\therefore \underline{a}\,.\,\underline{c} = \underline{c}\,.\,\underline{a})$

$b^2 = a^2 + 2\,|\underline{a}|\,|\underline{c}|\cos(\pi - B) + c^2$

$b^2 = a^2 + 2ac\,(-\cos B) + c^2$

$b^2 = a^2 + c^2 - 2ac\cos B$

Hence proved

(iv) $\mathbf{c^2 = a^2 + b^2 - 2ab\cos C}$

Solution:

For triangle ABC

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{c} = -\underline{a} - \underline{b}$

$\underline{c} = -(\underline{a} + \underline{b})$ .... (i)

Taking dot product by $\underline{c}$

$\underline{c}\cdot\underline{c} = -(\underline{a} + \underline{b})\cdot\underline{c}$

$c^2 = -(\underline{a} + \underline{b})\cdot -(\underline{a} + \underline{b})$

$= \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{a} + \underline{b}\cdot\underline{b}$

$c^2 = a^2 + 2\underline{a}\cdot\underline{b} + b^2$ $\quad (\because \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a})$

$c^2 = a^2 + 2\,|\underline{a}|\,|\underline{b}|\cos(\pi - C) + b^2$

$c^2 = a^2 + 2ab\,(-\cos C) + b^2$

$c^2 = a^2 + b^2 - 2ab\cos C$

Hence proved

The Cross Product or Vector Product of Two Vectors

Let $\underline{u}$ & $\underline{v}$ be two vectors. The cross or vector product of $\underline{u}$ and $\underline{v}$ is defined as

$\underline{u} \times \underline{v} = |\underline{u}|\,|\underline{v}|\sin\theta\,\hat{n}$

When $\hat{n}$ is unit vector perpendicular to the plane of $\underline{u}$ and $\underline{v}$.

$\underline{u} \times \underline{v} = |\underline{u}|\,|\underline{v}|\,\text{Sin}\,\theta\,\hat{n}$

Where

$$\hat{n} = \frac{\underline{u} \times \underline{v}}{|\underline{u} \times \underline{v}|}$$

$$\text{Sin}\theta = \frac{\underline{u} \times \underline{v}}{|\underline{u}|\,|\underline{v}|\,\hat{n}}$$

$$\sin\theta = \frac{|\underline{u} \times \underline{v}|}{|\underline{u}|\ |\underline{v}|}$$

Important Points;

(i) $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$

(ii) $\underline{i} \times \underline{j} = \underline{k}\ ,\ \ \underline{j} \times \underline{k} = \underline{i}\ ,\ \underline{k} \times \underline{i} = \underline{j}$

(iii) $\underline{i} \times \underline{j} \neq \underline{j} \times \underline{i}$ i.e., Cross product is not commutative

(vi) Area of parallelogram = $|\underline{u} \times \underline{v}|$

(v) Area of triangle = $\frac{1}{2}\ |\underline{u} \times \underline{v}|$

<u>Parallel vectors:</u>

If $\underline{u}$ & $\underline{v}$ area parallel vectors then $\underline{u} \times \underline{v} = 0$

EXERCISE 7.4

Q.1 Compute the cross product $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$, check your answer by showing that each $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$.

(i) $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}\ ,\quad \underline{b} = \underline{i} - \underline{j} + \underline{k}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \underline{i}\ (1-1) - \underline{j}\ (2+1) + \underline{k}\,(-2-1)$$

$$\underline{a} \times \underline{b} = 0\underline{i} - 3\underline{j} - 3\underline{k}$$

We will show that $\underline{a}$ is perpendicular to $\underline{a} \times \underline{b}$, for this we have $\underline{a} \cdot (\underline{a} \times \underline{b})$

$$= (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

$\therefore$ $\underline{a}$ and $\underline{a} \times \underline{b}$ are perpendicular.

Next, we will show that $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$. For this we have $\underline{b} \cdot (\underline{a} \times \underline{b})$

$$(\underline{i} - \underline{j} + \underline{k}) \quad . \quad (0\underline{i} - 3\underline{j} - 3\underline{k})$$