EXERCISE 1.1

Q.1 Given that:

(a) $f(x) = x^2 - x$ **(b)** $f(x) = \sqrt{x+4}$

Find: (i) $f(-2)$ **(ii)** $f(0)$

(iii) $f(x-1)$ **(iv)** $f(x^2+4)$

Solution:

(a) $f(x) = x^2 - x$

(i) $f(-2) = (-2)^2 - (-2)$

$= 4 + 2 = 6$ Ans.

(ii) $f(0) = (0)^2 - 0$

$= 0$ Ans.

(iii) $f(x-1) = (x-1)^2 - (x-1)$

$= x^2 - 2x + 1 - x + 1$

$= x^2 - 3x + 2$ Ans.

(iv) $f(x^2+4) = (x^2+4)^2 - (x^2+4)$

$= x^4 + 8x^2 + 16 - x^2 - 4$

$= x^4 + 7x^2 + 12$ Ans.

(b) $f(x) = \sqrt{x+4}$

(i) $f(-2) = \sqrt{-2+4} = \sqrt{2}$ Ans.

(ii) $f(0) = \sqrt{0+4} = \sqrt{4} = 2$ Ans.

(iii) $f(x-1) = \sqrt{x-1+4} =$

(iv) $f(x^2+4) = \sqrt{x^2+4+4} = \sqrt{x^2+8}$ Ans.

Q.2 Find $\frac{f(a+h) - f(a)}{h}$ and simplify where,

(i) $f(x) = 6x - 9$ **(ii)** $f(x) = \sin x$

(iii) $f(x) = x^3 + 2x^2 - 1$ **(iv)** $f(x) = \cos x$

Solution:

(i) $f(x) = 6x - 9$

$f(a+h) = 6(a+h) - 9$

$f(a+h) = 6a + 6h - 9$

$f(a) = 6a - 9$

$$\frac{f(a+h)-f(a)}{h} = \frac{6a+6h-9-(6a-9)}{h}$$
$$= \frac{6a+6h-9-6a+9}{h}$$
$$= \frac{6h}{h}$$
$$= 6 \quad \text{Ans.}$$

(ii) $\mathbf{f(x) = \sin x}$ ***(Lahore Board 2008)***

$$f(a+h) = \sin(a+h)$$
$$f(a) = \sin a$$
$$\frac{f(a+h)-f(a)}{h} = \frac{\sin(a+h)-\sin a}{h}$$
$$= \frac{2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{h} \quad \because \sin p - \sin q = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$$
$$= \frac{2}{h}\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)$$
$$= \frac{2}{h}\cos\left(\frac{2a}{2}+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$
$$= \frac{2}{h}\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right) \quad \text{Ans.}$$

(iii) $\mathbf{f(x) = x^3 + 2x^2 - 1}$

$$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1$$
$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2(a^2 + 2ah + h^2) - 1 \quad \because [(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2]$$
$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1$$
$$f(a) = a^3 + 2a^2 - 1$$
$$\frac{f(a+h)-f(a)}{h} = \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - (a^3 + 2a^2 - 1)}{h}$$
$$= \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - a^3 - 2a^2 + 1}{h}$$
$$= \frac{h^3 + 3a^2h + 3ah^2 + 4ah + 2h^2}{h}$$
$$= \frac{h(h^2 + 3a^2 + 3ah + 4a + 2h)}{h}$$
$$= h^2 + (3a+2)h + 3a^2 + 4a \quad \text{Ans.}$$

(iv)

$$\mathbf{f(x) = \cos x}$$

$$f(a+h) = \cos(a+h)$$

$$f(a) = \cos a$$

$$\frac{f(a+h)-f(a)}{h} = \frac{\cos(a+h)-\cos a}{h}$$

$$= \frac{-2\sin\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{h}$$

$$\because \left[\cos p - \cos q = -2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)\right]$$

$$= \frac{-2}{h}\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$= \frac{-2}{h}\sin\left(\frac{2a}{2}+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$= \frac{-2}{h}\sin\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right) \quad \text{Ans.}$$

Q.3 Express the following: ***(Lahore Board 2009-2010)***

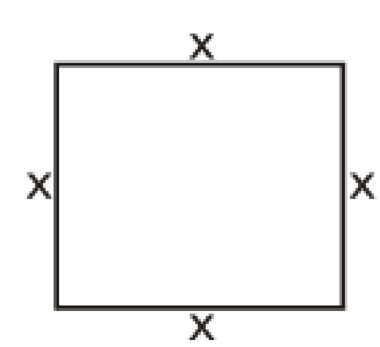
(a) The perimeter P of square as a function of its area A.

(b) The area A of a circle as a function of its circumference C.

(c) The volume V of a cube as a function of the area A of its base.

Solution:

(a) Let,



Length of square $= x$

Width of square $= x$

Perimeter of a square $= P = x + x + x + x$

$$P = 4x \quad \ldots\ldots\ldots (1)$$

Area of a square $= A = x \times x$

$$A = x^2$$

$$x = \sqrt{A}$$

Put $x = \sqrt{A}$ in equation (1)

$$\boxed{P = 4\sqrt{A}}$$

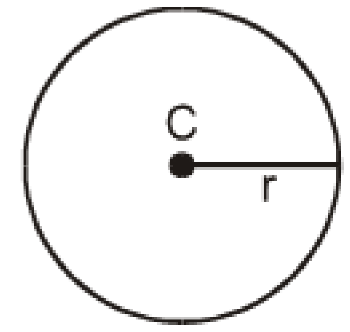
Shows perimeter P of a square as a function of its area A.

(b) Let 'r' be the radius of the circle.

Area of a circle $= A = \pi r^2$ .... (1)

Circumference of a circle $= C = 2\pi r$

$$r = \frac{C}{2\pi}$$



Put $r = \dfrac{C}{2\pi}$ in equation (1)

$$A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\pi C^2}{4\pi^2}$$

$$A = \frac{C^2}{4\pi}$$

Shows area A of a circle as a function of its circumference C.

(c) Let x be the each side of cube. ***(Gujranwala Board 2008)***

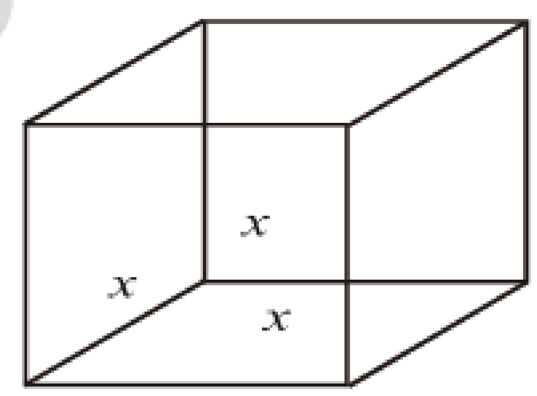
Volume of cube $= V = x \times x \times x$

$V = x^3$ .... (1)

Area of base $= A = x \times x$

$A = x^2$

$x = \sqrt{A}$



Put, $x = \sqrt{A}$ in equation (1)

$V = (\sqrt{A})^3$

$V = A^{3/2}$

Shows volume V of a cube as a function of the area A of its base.

Q.4 Find the domain and the range of the function g defined below and sketch of graph of g.

(i) $g(x) = 2x - 5$ **(ii)** $g(x) = \sqrt{x^2 - 4}$

(iii) $g(x) = \sqrt{x+1}$ ***(Lhr.Board-2011)*** **(iv)** $g(x) = |x - 3|$

(v) $g(x) = \begin{cases} 6x + 7, & x \le -2 \\ x - 3, & x > -2 \end{cases}$ **(vi)** $g(x) = \begin{cases} x - 1, & x < 3 \\ 2x + 1, & 3 \le x \end{cases}$

(vii) $g(x) = \dfrac{x^2 + 3x + 2}{x + 1}, x \ne -1$ **(viii)** $g(x) = \dfrac{x^2 - 16}{x - 4}, x \ne 4$

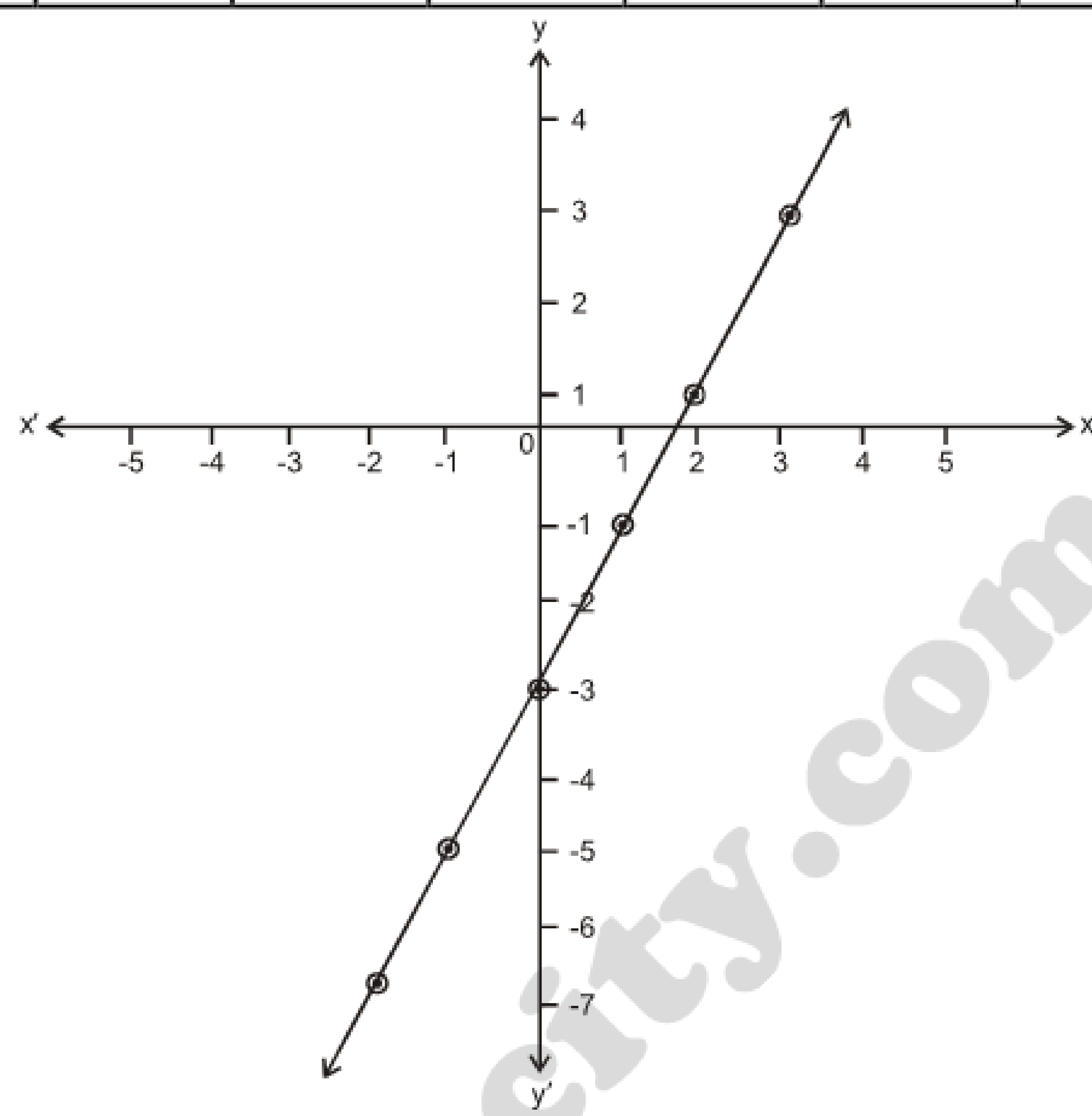
Solution:

(i) $g(x) = 2x - 5$

Domain of g(x) = Set of all real numbers

Range of g(x) = Set of all real numbers

x	– 3	– 2	– 1	0	1	2	3
g(x) = 2x – 5	–7	–9	–7	–5	–3	–1	1

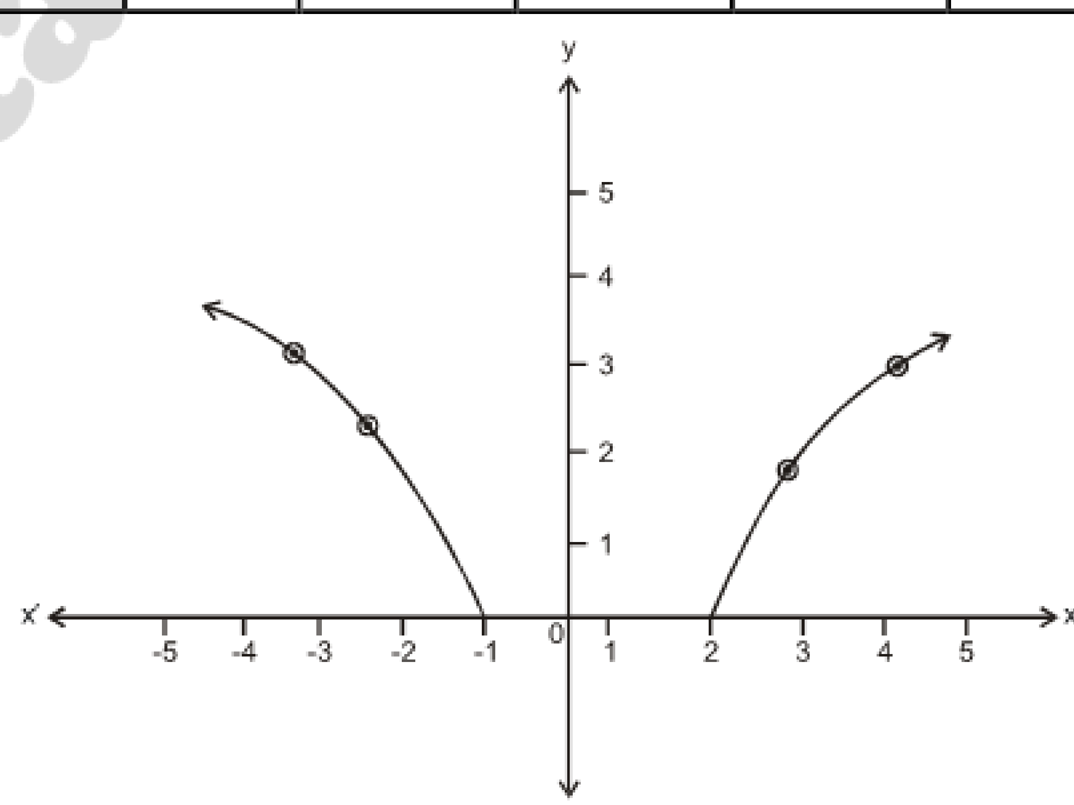


(ii) **g(x)** = $\sqrt{\mathbf{x^2 - 4}}$ ***(Lahore Board 2008)***

Domain of g(x) = R – (–2, 2)

Range of g(x) = [0, + ∞)

x	– 4	– 3	– 2	2	3	4
$g(x) = \sqrt{x^2 - 4}$	$2\sqrt{3}$	$\sqrt{5}$	0	0	$\sqrt{5}$	$2\sqrt{3}$

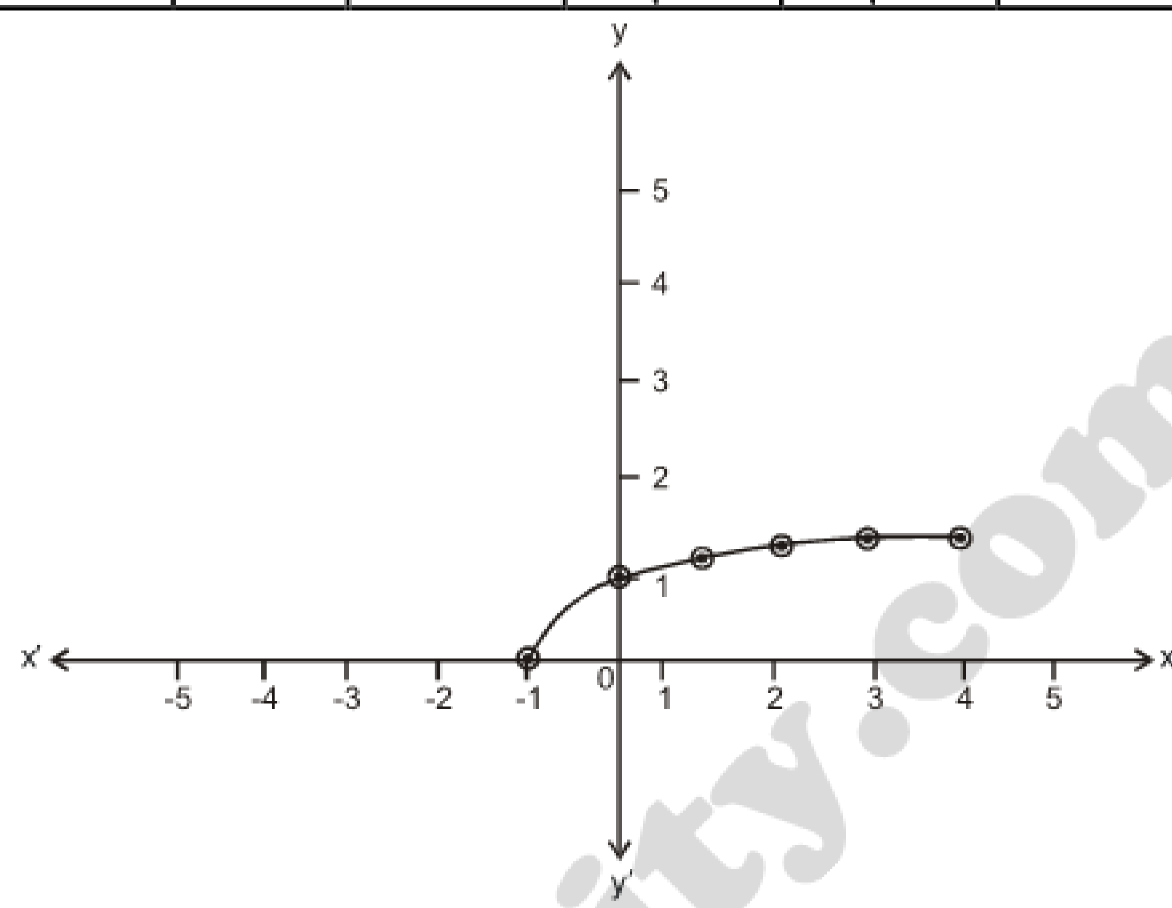


(iii) $\mathbf{g(x) = \sqrt{x+1}}$

Domain of g(x) = $[-1, +\infty)$

Range of g(x) = $[0, +\infty)$

x	-1	0	1	2	3	4
$g(x) = \sqrt{x+1}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$

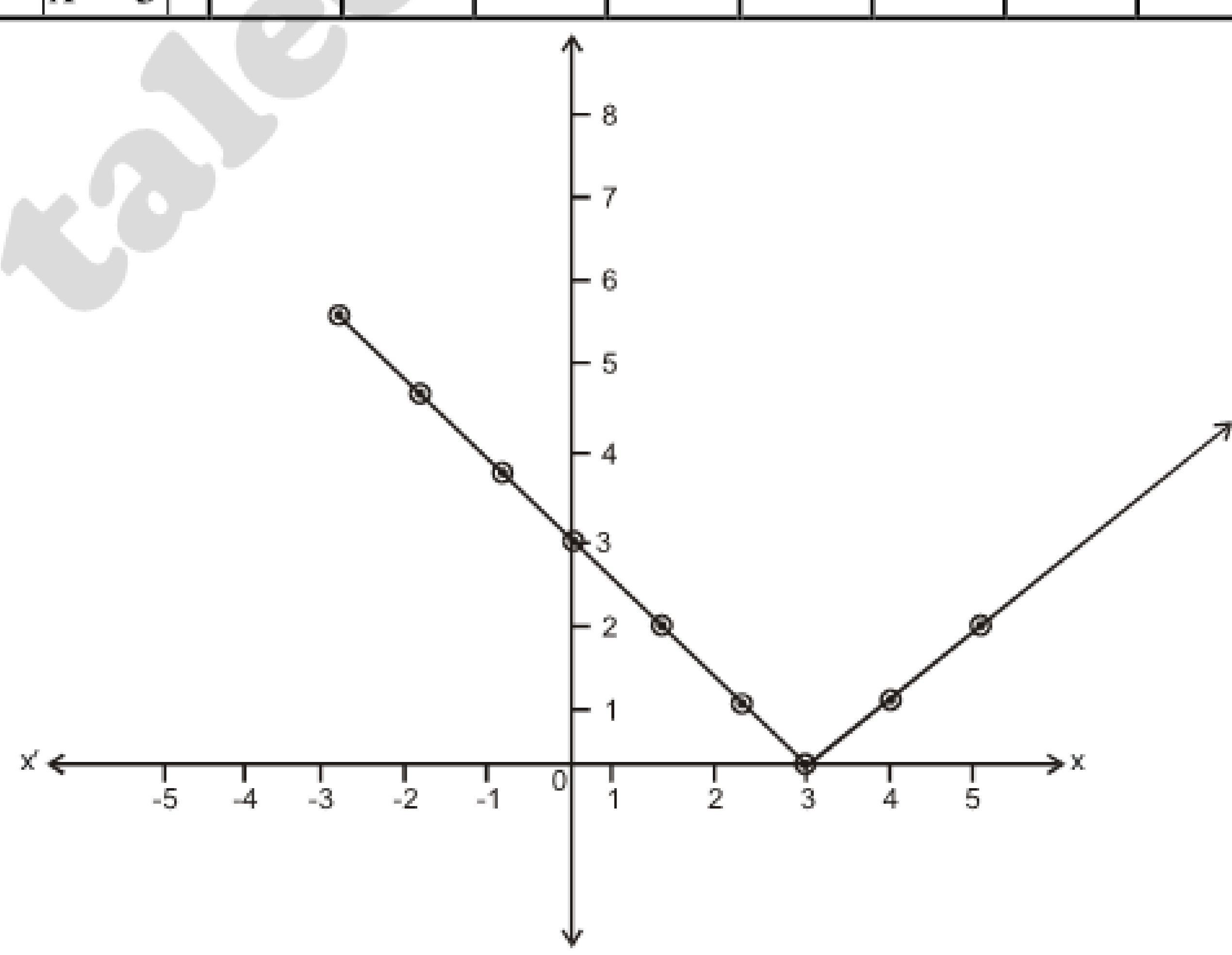


(iv) $\mathbf{g(x) = |x-3|}$

Domain of g(x) = $(-\infty, +\infty)$

Range of g(x) = $[0, +\infty)$

x	-3	-2	-1	0	1	2	3	4	5
$g(x) = \|x-3\|$	6	5	4	3	2	1	0	1	2



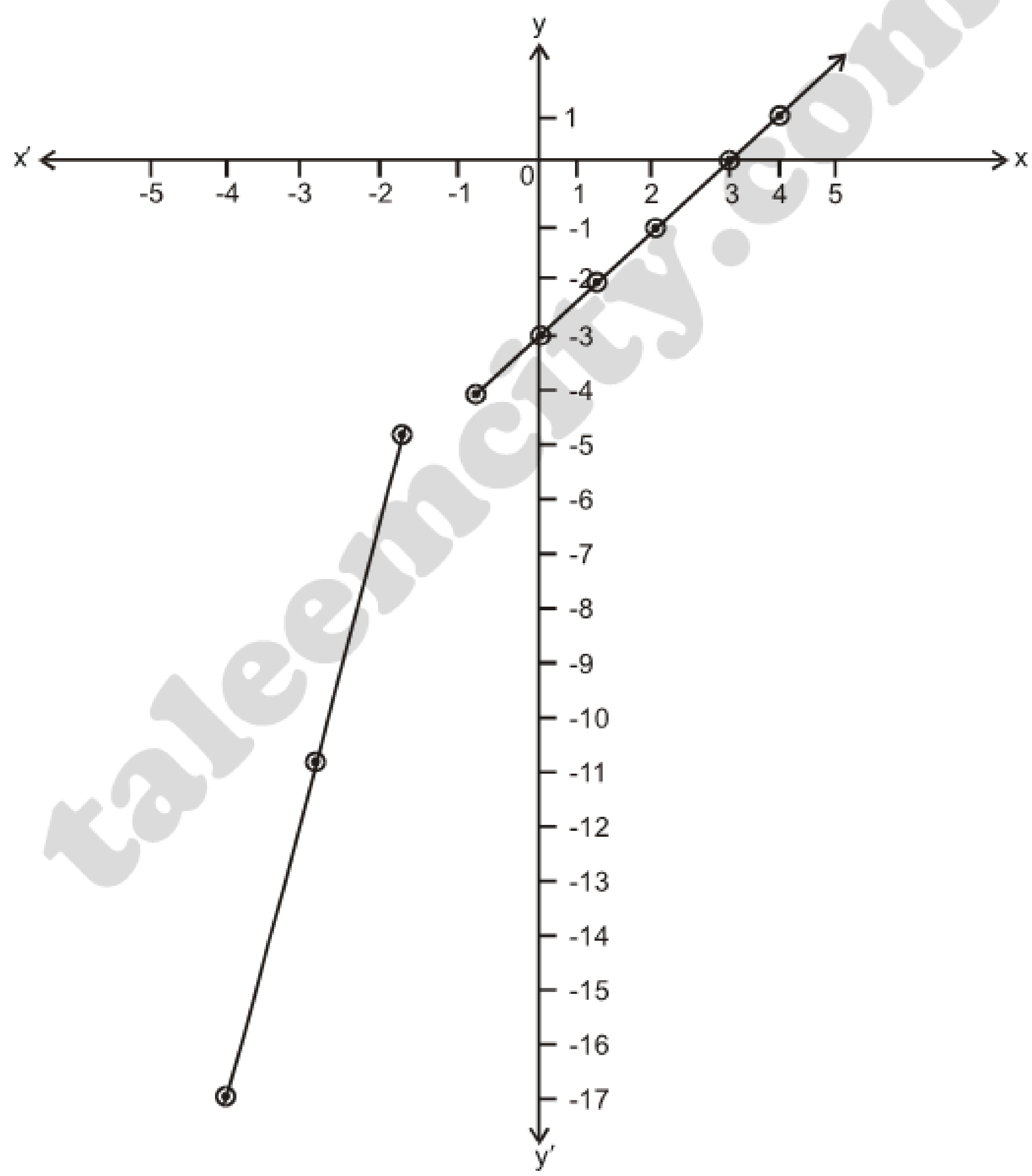
(v) $\mathbf{g(x) = \begin{cases} 6x + 7, & x \le -2 \\ x - 3, & x > -2 \end{cases}}$

Domain of g(x) = $(-\infty, +\infty)$

Range of g(x) = $(-\infty, +\infty)$

$x \le -2$	-2	-3	-4	-5
$g(x) = 6x + 7$	-5	-11	-17	-23

$x > -2$	-1	0	1	2	3	3
$g(x) = x - 3$	-4	-3	-2	-1	0	1



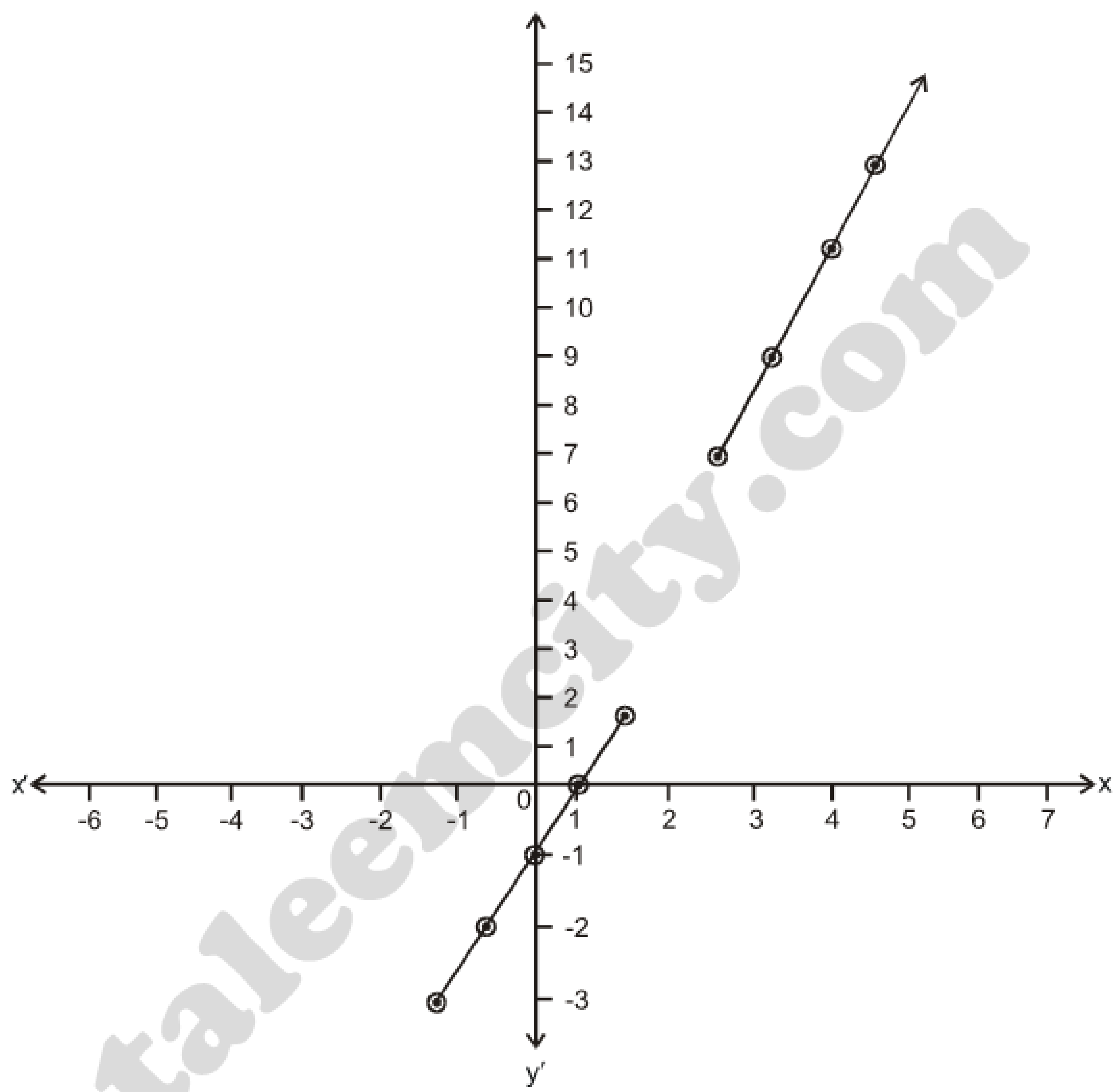
(vi) $\mathbf{g(x) = \begin{cases} x - 1, & x < 3 \\ 2x + 1, & 3 \le x \end{cases}}$

Domain of g(x) = $(-\infty, +\infty)$

Range of g(x) = $(-\infty, 2) \cup [7, +\infty)$

x < 3	– 2	– 1	0	1	2
g(x) = x – 1	– 3	– 2	– 1	0	1

x ≥ 3	3	4	5	6
g(x) = 2x + 1	7	9	11	13



(vii) $$\mathbf{g(x) = \frac{x^2+3x+2}{x+1}\quad , \quad x \neq -1}$$

$$g(x) = \frac{x^2+2x+x+2}{x+1}$$

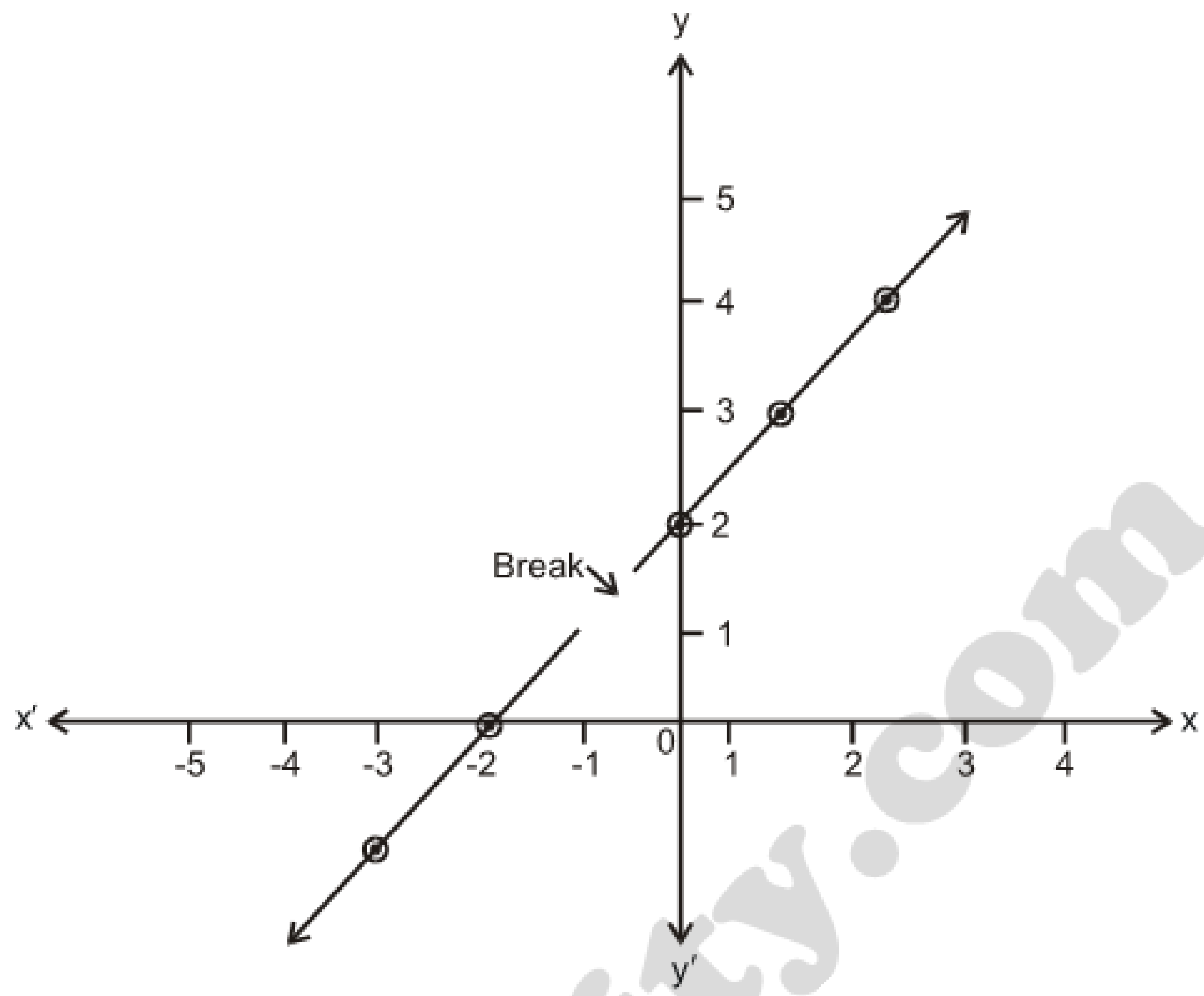
$$= \frac{x(x+2)+1(x+2)}{x+1}$$

$$= \frac{(x+2)(x+1)}{x+1} = x+2$$

Domain of g(x) = R – {–1}

Range of g(x) = R – {1}

x	-3	-2	0	1	2
$g(x) = x + 2$	-1	0	2	3	4



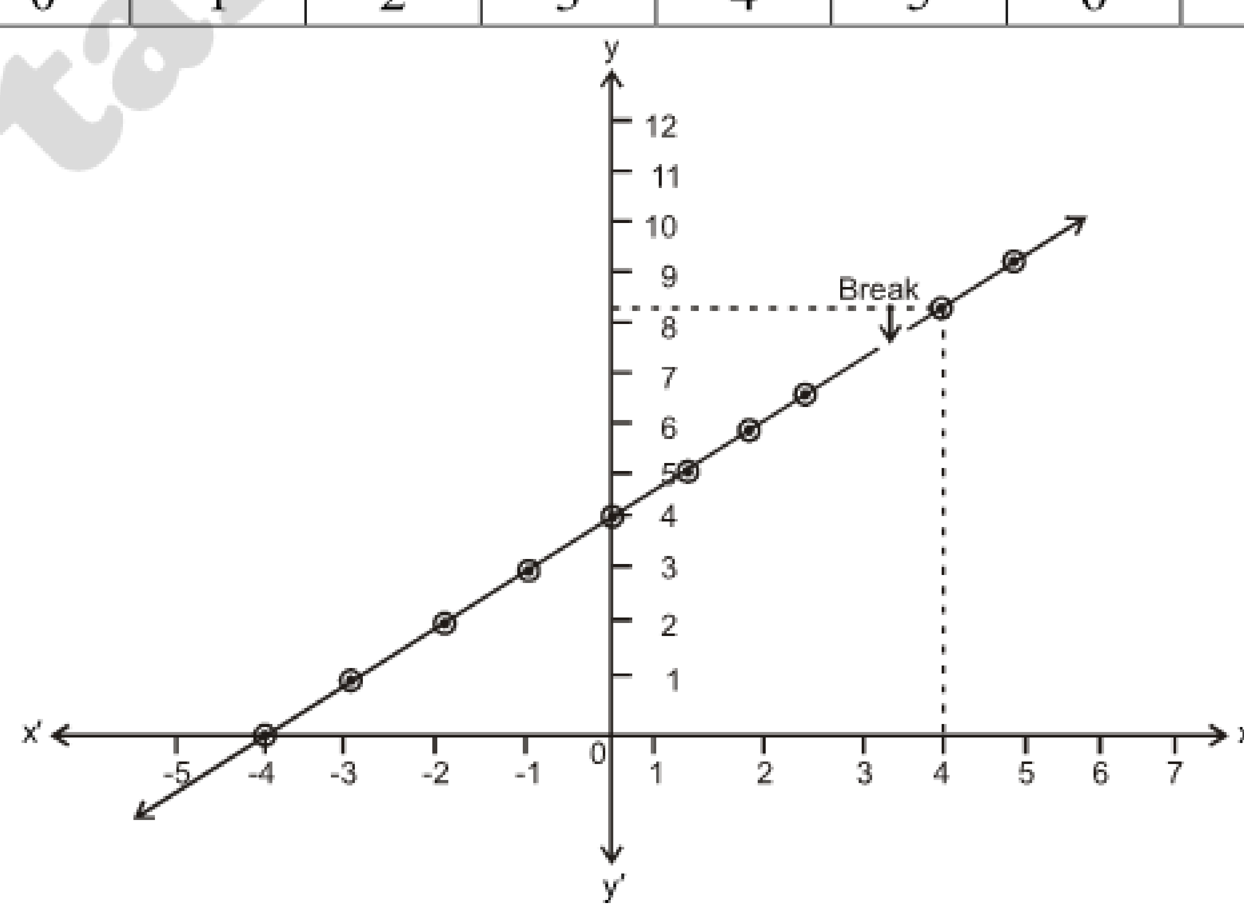
(viii) $$g(x) = \frac{x^2 - 16}{x - 4}, \quad x \neq 4$$

$$= \frac{(x + 4)(x - 4)}{x - 4} = x + 4$$

Domain of $g(x) = R - \{4\}$

Range of $g(x) = R - \{8\}$

x	-4	-3	-2	-1	0	1	2	3	5	6
$g(x) = x + 4$	0	1	2	3	4	5	6	7	9	10



Q.5 Given $f(x) = x^3 - ax^2 + bx + 1$.
If $f(2) = -3$ and $f(-1) = 0$. Find the values of a and b.

Solution:

$$f(x) = x^3 - ax^2 + bx + 1 \quad \ldots\ldots\ldots (1)$$

Put $x = 2$ in equation (1)

$$f(2) = (2)^3 - a(2)^2 + b(2) + 1$$
$$= 8 - 4a + 2b + 1$$
$$f(2) = 9 - 4a + 2b$$

Put $x = -1$ in equation (1)

$$f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$$
$$= -1 - a - b + 1$$
$$f(-1) = -a - b$$

Since $f(2) = -3$

$$9 - 4a + 2b = -3$$
$$-4a + 2b = -3 - 9$$
$$-2(2a - b) = -12$$
$$2a - b = \frac{-12}{-2}$$
$$2a - b = 6 \quad \ldots\ldots\ldots (2)$$

And

$$f(-1) = 0$$
$$-a - b = 0$$
$$-a = b$$
$$a = -b \quad \ldots\ldots\ldots (3)$$

Put $a = -b$ in equation (2)

$$2(-b) - b = 6$$
$$-2b - b = 6$$
$$-3b = 6$$
$$b = \frac{6}{-3} = -2$$

Put $b = -2$ in equation (2)

$$a = -(-2)$$
$$a = 2$$

$\therefore$ $\boxed{a = 2\ ,\ b = -2}$ Ans.

Q.6 A stone falls from a height of 60m on the ground, the height h after x second is approximately given by $h(x) = 40 - 10x^2$.

(i) What is the height of the stone when.

(a) $x = 1$ sec (b) $x = 1.5$ sec (c) $x = 1.7$ sec

(ii) When does the stone strike the ground?

Solution:

(i) $h(x) = 40 - 10x^2$ ……… (1)

(a) Put $x = 1$ sec in equation (1)

$$h(1) = 40 - 10(1)^2$$
$$= 40 - 10$$
$$= 30\text{m} \quad \text{Ans.}$$

(b) Put $x = 1.5$ sec in equation (1)

$$h(1.5) = 40 - 10\,(1.5)^2$$
$$= 40 - 10\,(2.25)$$
$$= 40 - 22.5$$
$$= 17.5\text{m} \quad \text{Ans.}$$

(c) Put $x = 1.7$ sec in equation (1)

$$h(1.7) = 40 - 10\,(1.7)^2$$
$$= 40 - 10\,(2.89)$$
$$= 40 - 28.9$$
$$= 11.1\text{ m} \quad \text{Ans.}$$

(ii) When then the stone strike the ground.

then $h(x) = 0$

$$0 = 40 - 10x^2$$
$$10x^2 = 40$$
$$x^2 = \frac{40}{10}$$
$$x^2 = 4$$
$$x = 2 \text{ sec Ans.}$$

Q.7: Show that the Parametric equations.

(i) $x = at^2$, $y = 2at$ represent the equation of Parabola $y^2 = 4ax$

(ii) $x = a\cos\theta$, $y = b\sin\theta$ represent the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iii) $x = a\sec\theta$, $y = b\tan\theta$ represent the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution:

(i) $x = at^2$ .... (i) , $y = 2at$ .... (ii)

From equation (ii)

$$t = \frac{y}{2a}$$

Putting it in (i)

$$x = a\left(\frac{y}{2a}\right)^2 = a\left(\frac{y^2}{4a^2}\right)$$

$y^2 = 4ax$ Hence proved.

(ii) $x = a\cos\theta$

$$\frac{x}{a} = \cos\theta$$

Squaring on both sides

$$\frac{x^2}{a^2} = \cos^2\theta \quad \text{.... (i)}$$

$y = b\sin\theta$

$$\frac{y}{b} = \sin\theta$$

Squaring on both sides

$$\frac{y^2}{b^2} = \sin^2\theta \quad \text{.... (ii)}$$

Adding equation (i) & equation (ii)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

$= 1$ Hence proved.

(iii) $x = a\sec\theta$

$$\frac{x}{a} = \sec\theta$$

Squaring on both sides

$$\frac{x^2}{a^2} = \sec^2\theta \quad \text{.... (i)}$$

$y = b\tan\theta$

$$\frac{y}{b} = \tan\theta$$

Squaring on both sides

$$\frac{y^2}{b^2} = \tan^2\theta \quad \text{.... (ii)}$$

Subtracting equation (ii) from equation (i)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta$$

$= 1 + \tan^2\theta - \tan^2\theta$

$= 1$ Hence proved.

Q.8 Prove the identities:

(i) $\sinh 2x = 2 \sinh x \cos hx$ **(Lahore Board 2006)**

(ii) $\text{sech}^2 x = 1 - \tanh^2 x$

(iii) $\text{cosech}^2 x = \coth^2 x - 1$

Solution:

(i) $\sinh 2x = 2 \sinh x \cos hx$

R.H.S $= 2\sinh x \cosh x$

$$= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$$= \sin h2x$$

$= \text{L.H.S.}$ Hence proved.

(ii) $\textbf{sec } \mathbf{h^2x = 1 - tan\ h^2x}$

$$\text{R.H.S} = 1 - \tan h^2x$$

$$= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

$$= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x \cdot e^{-x} - (e^{2x} + e^{-2x} - 2e^x \cdot e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$= \left(\frac{2}{e^x + e^{-x}}\right)^2$$

$$= (\sec hx)^2$$

$$= \text{sech}^2x$$

$= \text{L.H.S}$ Hence proved

(iii) $\textbf{cosec } \mathbf{h^2x = cot\ h^2x - 1}$

$$\text{R.H.S} = \cot h^2x - 1$$

$$= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1$$

$$= \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} - 1$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x - e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x \cdot e^{-x})}{(e^x - e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x - e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$= \left(\frac{2}{e^x - e^{-x}}\right)^2$$

$$= (\text{cosech}x)^2 = \text{cosech}^2x$$

$= \text{L.H.S}$ Hence proved

Q.9 Determine whether the given function f is even or odd:

(i) $f(x) = x^3 + x$ **(ii)** $f(x) = (x + 2)^2$

(iii) $f(x) = x\sqrt{x^2 + 5}$ **(iv)** $f(x) = \dfrac{x - 1}{x + 1}, \; x \neq -1$

(v) $f(x) = x^{2/3} + 6$ **(vi)** $f(x) = \dfrac{x^3 - x}{x^2 + 1}$

Solution:

(i) $f(x) = x^3 + x$

$$f(-x) = (-x)^3 + (-x)$$
$$= -x^3 - x$$
$$= -(x^3 + x)$$
$$= -f(x)$$

$\therefore$ f(x) is an odd function.

(ii) $f(x) = (x + 2)^2$

$$f(-x) = (-x + 2)^2$$
$$\neq \pm f(x)$$

$\therefore$ f(x) is neither even nor odd function.

(iii) $f(x) = x\sqrt{x^2 + 5}$

$$f(-x) = -x\sqrt{(-x)^2 + 5}$$
$$= -x\sqrt{x^2 + 5}$$
$$= -f(x)$$

$\therefore$ f(x) is an odd function.

(iv) $f(x) = \dfrac{x - 1}{x + 1}, \; x \neq -1$

$$f(-x) = \frac{-x - 1}{-x + 1} = \frac{-(x + 1)}{-(x - 1)}$$

$$= \frac{x+1}{x-1} \neq \pm f(x)$$

$\therefore$ f(x) is neither even nor odd function.

(v) $\mathbf{f(x) = x^{2/3} + 6}$

$$\begin{aligned} f(-x) &= (-x)^{2/3} + 6 \\ &= [(-x)^2]^{1/3} + 6 \\ &= (x^2)^{1/3} + 6 \\ &= x^{2/3} + 6 \\ &= f(x) \end{aligned}$$

$\therefore$ f(x) is an even function.

(vi) $\mathbf{f(x) = \dfrac{x^3 - x}{x^2 + 1}}$

$$\begin{aligned} f(-x) &= \frac{(-x)^3 - (-x)}{(-x)^2 + 1} \\ &= \frac{-x^3 + x}{x^2 + 1} \\ &= \frac{-(x^3 - x)}{x^2 + 1} \\ &= -f(x) \end{aligned}$$

$\therefore$ f(x) is an odd function.

Composition of Functions:

Let f be a function from set X to set Y and g be a function from set Y to set Z. The composition of f and g is a function, denoted by gof, from X to Z and is defined by.

$$(gof)(x) = g(f(x)) = gf(x) \text{ for all } x \in X$$

Inverse of a Function:

Let f be one-one function from X onto Y. The inverse function of f, denoted by f^{-1}, is a function from Y onto X and is defined by.

$$x = f^{-1}(y), \ \forall\ y \in Y \text{ if and only if } y = f(x), \forall\ x \in X$$

EXERCISE 1.2

Q.1 The real valued functions f and g are defined below. Find

(a) fog (x) (b) gof (x) (c) fof (x) (d) gog (x)

(i) $\mathbf{f(x) = 2x + 1 \ ; \ g(x) = \dfrac{3}{x-1}, \ x \neq 1}$