Chapter 7

VECTORS

Vectors:

A vector quantity is that possesses both magnitude and direction i.e. displacement, velocity, weight, force etc.

Scalar:

A scalar quantity is that possesses only magnitude. It can be specified by a number i.e. mass, time, density, length, volume etc.

Magnitude/Length/Norm/Modulus of a Vector:

The positive real number, which is measure of the length of the vector, is called modulus, length, magnitude or norm of a vector.

Formula $\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$

Zero Vector:

If terminal point B of a vector $\overrightarrow{AB}$ concides with its initial point A, then $|\overrightarrow{AB}| = 0$ called zero vector or Null vector.

Position vector:

The vector, whose initial point O is origin & whose terminal point is P, is called position vector of OP.

EXERCISE 7.1

Q.1 Write the vector $\overrightarrow{PQ}$ in the form xi + yj.

(i) P (2, 3), Q (6, -2)

Solution:

P(2, 3) , Q (6, -2)

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ – position vector of P

$= (6-2)\,\underline{i} + (-2-3)\,\underline{j} = 4\underline{i} - 5\underline{j}$

(ii) P (0, 5), Q (-1, -6)

Solution:

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (-1-0)\underline{i} + (-6-5)\underline{j} = -\underline{i} - 11\underline{j}$

Q.2: Find the magnitude of the vector $\underline{u}$.

Formula $\boxed{\text{Magnitude or length or Norm of } \underline{v} = x\underline{i} + y\underline{j} + z\underline{k} \text{ is } |V| = \sqrt{x^2+y^2+z^2}}$

(i) $\underline{u} = 2\underline{i} - 7\underline{j}$

Solution:

$\underline{u} = 2\underline{i} - 7\underline{j}$

$|\underline{u}| = \sqrt{(2)^2 + (-7)^2} = \sqrt{4+49} = \sqrt{53}$

(ii) $\underline{u} = \underline{i} + \underline{j}$

Solution:

$\underline{u} = \underline{i} + \underline{j}$

$|\underline{u}| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

(ii) $\underline{u} = [3, -4]$ **(Lahore Board 2005)**

Solution:

$\underline{u} = 3\underline{i} - 4\underline{j}$

$|\underline{u}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Q.3 If $\underline{u} = 2\underline{i} - 7\underline{j}$, $\underline{v} = \underline{i} - 6\underline{j}$ & $\underline{w} = -\underline{i} + \underline{j}$, find the following vectors.

(i) $\underline{u} + \underline{v} - \underline{w}$

Solution:

$\underline{u} + \underline{v} - \underline{w} = (2\underline{i} - 7\underline{j}) + (\underline{i} - 6\underline{j}) - (-\underline{i} + \underline{j})$

$= 2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} + \underline{i} - \underline{j} = 4\underline{i} - 14\underline{j}$ Ans.

(ii) $2\underline{u} - 3\underline{v} + 4\underline{w}$

Solution:

$2\underline{u} - 3\underline{v} + 4\underline{w}$

$= 2(2\underline{i} - 7\underline{j}) - 3(\underline{i} - 6\underline{j}) + 4(-\underline{i} + \underline{j})$

$= 4\underline{i} - 14\underline{j} - 3\underline{i} + 18\underline{j} - 4\underline{i} + 4\underline{j} = -3\underline{i} + 8\underline{j}$

(iii) $\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} + \frac{1}{2}\underline{w}$

Solution:

$= \frac{1}{2}[\underline{u} + \underline{v} + \underline{w}]$

$= \frac{1}{2}[2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} - \underline{i} + \underline{j}]$

$= \frac{1}{2}[2\underline{i} - 12\underline{j}]$

$$= \quad \frac{2}{2}[\,\underline{i} - 6\underline{j}\,] \;=\; \underline{i} - 6\underline{j}$$

Q.4 Find the sum of the vectors $\overrightarrow{AB}$ & $\overrightarrow{CD}$, given the four points A(1, –1), B (2, 0), C(-1, 3) & D (-2, 2)

Solution:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (2-1)\,\underline{i} + (0+1)\,\underline{j} \quad = \underline{i} + \underline{j}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$
$$= (-2+1)\,\underline{i} + (2-3)\,\underline{j} \quad = -\underline{i} - \underline{j}$$

$$\text{Sum} = \overrightarrow{AB} + \overrightarrow{CD} = \underline{i} + \underline{j} - \underline{i} - \underline{j} \quad = 0\underline{i} + 0\underline{j} \quad = \text{Null vector}$$

Q.5 Find the vector from the point A to the origin, where $\overrightarrow{AB} = 4\underline{i} - 2\underline{j}$ and B is the point (–2, 5).

Solution:

$$\overrightarrow{AB} = 4\underline{i} - 2\underline{j}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} - \overrightarrow{OB} = -\overrightarrow{OA}$$

$$\overrightarrow{AB} - \overrightarrow{OB} = \overrightarrow{AO} \qquad \because \overrightarrow{AO} = -\overrightarrow{OA}$$

$$\overrightarrow{AO} = (4\underline{i} - 2\underline{j}) - (-2\underline{i} + 5\underline{j})$$

$$\overrightarrow{AO} = 6\underline{i} - 7\underline{j}$$

Q.6 Find a unit vector in the direction of the vector given below

(i) $\underline{v} = 2\underline{i} - \underline{j}$ (Lahore Board 2009, 2010)

Solution:

$$v = 2\underline{i} - \underline{j}$$

$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2}$$

$$|\underline{v}| = \sqrt{4+1} = \sqrt{5}$$

Required unit vector is $\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - \underline{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}$

(ii) $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

Solution:

$$\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$$

$$|\underline{v}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

Required unit vector is $\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$ Ans.

(iii) $\underline{v} = \frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

Solution:

$$\underline{v} = \frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$$

$$|\underline{v}| = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

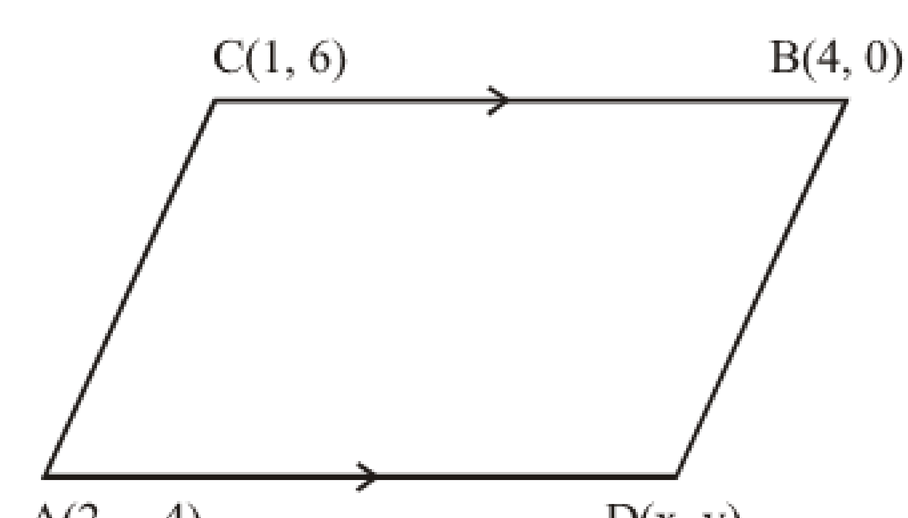
Required unit vector $\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{\frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = \frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$ Ans.

Q.7 If A, B and C are respectively the points (2, – 4), (4, 0) (1, 6). Use vectors to find coordinates of point D if

(i) ABCD is a parallelogram

Solution:

Let D (x, y) be the required vertex.
Since ABCD is a parallelogram



So $\overrightarrow{AB} = \overrightarrow{DC}$

$(4-2)\,\underline{i} + (0+4)\,\underline{j} = (1-x)\,\underline{i} + (6-y)\,\underline{j}$

$2\underline{i} + 4\underline{j} = (1-x)\,\underline{i} + (6-y)\,\underline{j}$

By comparing

$2 = 1 - x, \quad 4 = 6 - y$

$x = 1 - 2 \quad , \quad y = 6 - 4$

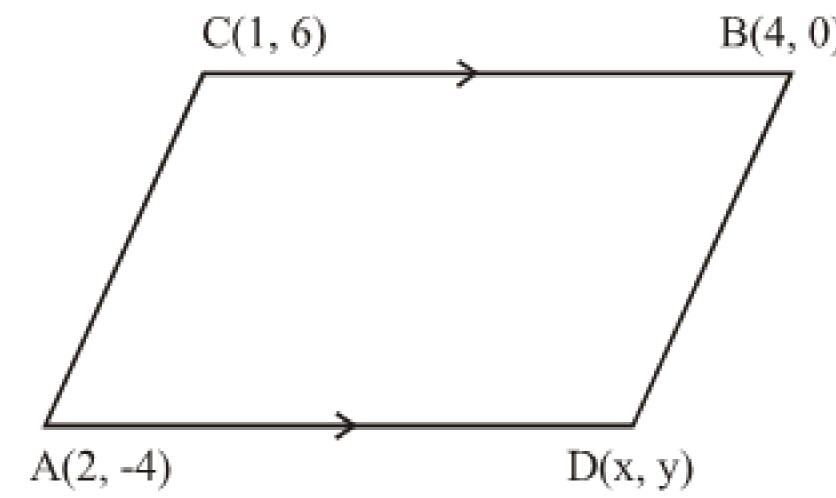
$x = -1 \quad , \quad y = 2$

Required coordinates of D are $(-1, 2)$

(ii) ADBC is a parallelogram.

Solution:

Since ADBC is a parallelogram



So $\overrightarrow{AD} = \overrightarrow{CB}$

$(x-2)\,\underline{i} + (y+4)\,\underline{j} = (4-1)\,\underline{i} + (0-6)\,\underline{j}$

$(x-2)\,\underline{i} + (y+4)\,\underline{i} = 3\underline{i} - 6\underline{j}$

By comparing

$x - 2 = 3 \quad , \quad y + 4 = -6$

$x = 5 \quad , \quad y = -10$

Required coordinates of D are $(5, -10)$

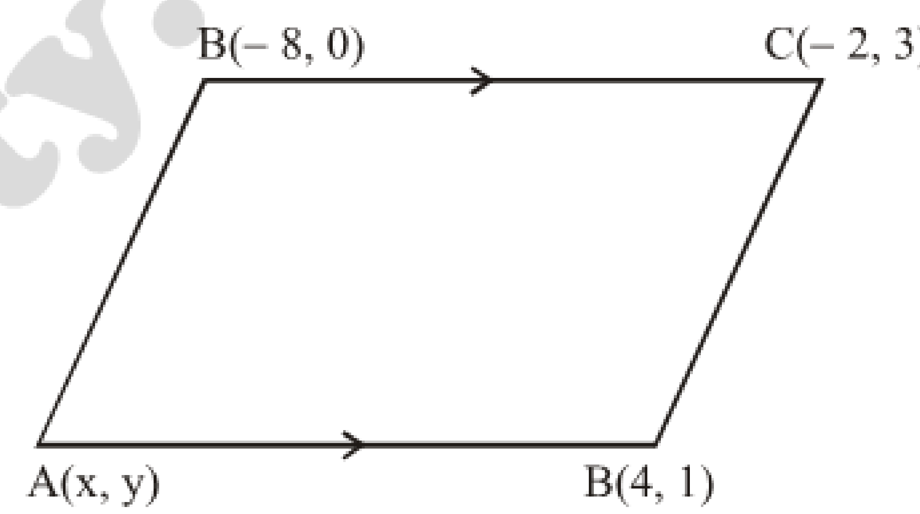
Q.8 If B, C and D are respectively (4, 1), (–2, 3) & (–8, 0). Use vector method to find the coordinates of the point

(i) A if ABCD is a parallelogram

Solution:

Let the coordinates of point A be (x, y)

Since ABCD is a parallelogram



Thus, $\overrightarrow{AB} = \overrightarrow{DC}$

$(4-x)\,\underline{i} + (1-y)\,\underline{j} = (-2+8)\,\underline{i} + (3-0)\,\underline{j}$

$(4-x)\,\underline{i} + (1-y)\,\underline{j} = 6\,\underline{i} + 3\,\underline{j}$

By comparing

$4 - x = 6, \quad 1 - y = 3$

$4 - 6 = x, \quad 1 - 3 = y$

$-2 = x, \quad -2 = y$

Therefore, required point A is $(-2, -2)$

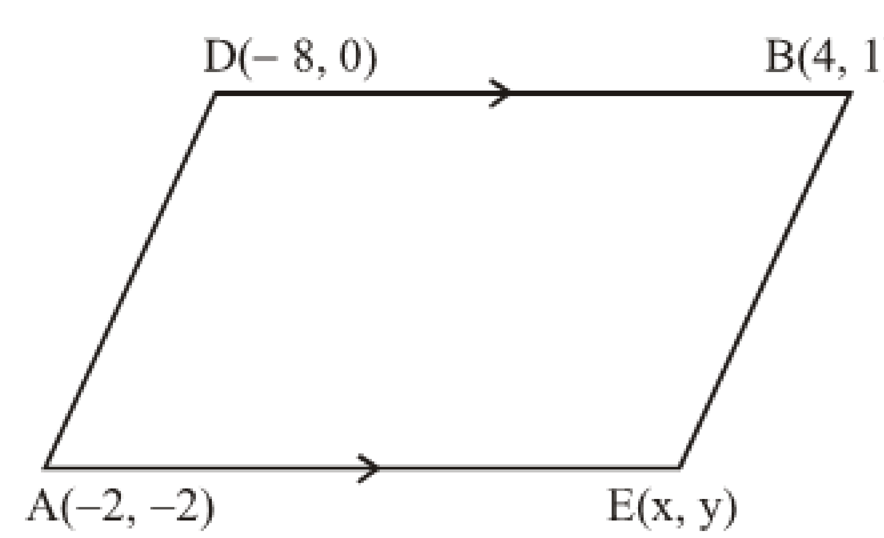
(ii) E, if AEBD is a parallelogram

Solution:

Let the coordinates of E be = (x, y)

B (4, 1), A (–2, –2), D (– 8, 0), E (x, y)

Since AEBD is a parallelogram



So $\overrightarrow{AE} = \overrightarrow{DB}$

$(x+2)\,\underline{i} + (y+2)\,\underline{j} = (4+8)\,\underline{i} + (1-0)\,\underline{j}$

$(x+2)\,\underline{i} + (y+2)\,\underline{j} = 12\,\underline{i} + \underline{j}$

By comparing

$x + 2 = 12, \quad y + 2 = 1$

$x = 12 - 2, \quad y = 1 - 2$

$x = 10, \quad y = -1$

Coordinates of E are (10, –1)

Q.9 If D is origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point, where A and B are (– 3,7) & (1, 0) respectively.

Solution:

Let the coordinates of point P be (x, y)

Therefore

O (0, 0), P (x, y), A (–3, 7), B (1, 0)

Since

$$\overrightarrow{OP} = \overrightarrow{AB}$$

$$(x-0)\,\underline{i} + (y-0)\,\underline{j} = (1+3)\,\underline{i} + (0-7)\,\underline{j}$$

$$x\,\underline{i} + y\,\underline{j} = 4\,\underline{i} - 7\underline{j}$$

$(x, y) = (4, -7)$ required point.

Q.10 Use vector to show that ABCD is a parallelogram when the points A,B,C & D are respectively (0, 0), (a, 0), (b, c) & (b – a, c).

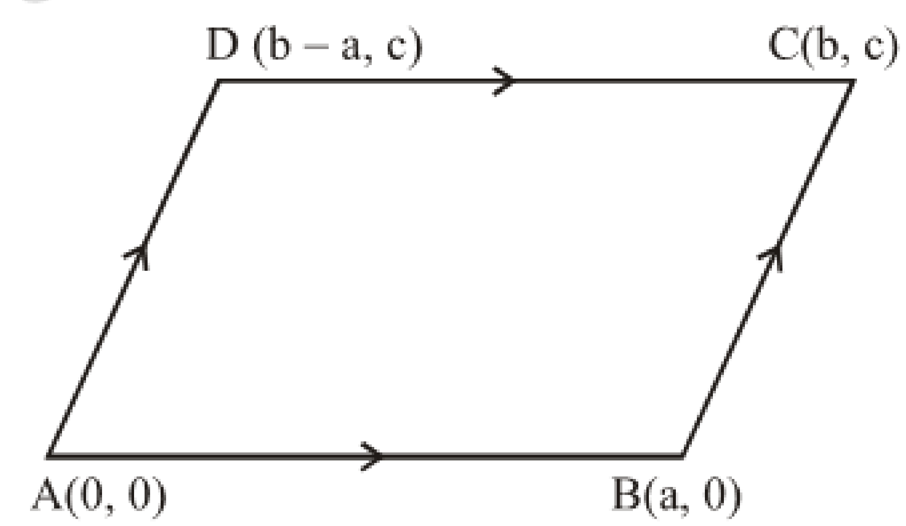
(Lahore Board 2009 (supply))

Solution:

Let ABCD be a parallelogram

We have to prove that

$$\overrightarrow{AB} = \overrightarrow{DC} \quad \& \quad \overrightarrow{AD} = \overrightarrow{BC}$$



Now

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (a-0)\,\underline{i} + (0-0)\,\underline{j} = a\,\underline{i} + 0\underline{j} \quad \text{...... (i)}$$

$$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$$

$$= (b-b+a)\,\underline{i} + (c-c)\,\underline{j} = a\,\underline{i} + 0\underline{j} \quad \text{...... (ii)}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$

$$= (b-a-0)\,\underline{i} + (c-0)\,\underline{j}$$

$$\overrightarrow{AD} = (b-a)\,\underline{i} + c\underline{j} \quad \text{...... (iii)}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (b-a)\,\underline{i} + (c-0)\,\underline{j}$$

$$\overrightarrow{BC} = (b-a)\,\underline{i} + c\underline{j} \quad \text{...... (iv)}$$

from (i) (ii) (iii) & (iv)

$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$ Shows ABCD is a parallelogram.

Q.11 If $\overrightarrow{AB} = \overrightarrow{CD}$. Find coordinates of the point A when B, C, D are (1, 2), (-2, 5), D (4, 11) respectively.

Solution:

Let Coordinates of A be (x, y)

A (x, y), B (1, 2), C (– 2, 5), D (4, 11)

i.e.; $\overrightarrow{AB} = \overrightarrow{CD}$

$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OC}$

$(1 - x)\underline{i} + (2 - y)\underline{j} = (4 + 2)\underline{i} + (11 - 5)\underline{j}$

By comparing

$1 - x = 6, \qquad 2 - y = 6$

$1 - 6 = x, \qquad -y = 6 - 2$

$\Rightarrow \quad x = -5 \qquad y = -4$

Hence required point is (– 5, – 4)

Q.12 Find the position vector of the point of division of the line segments joining the following pair of points.

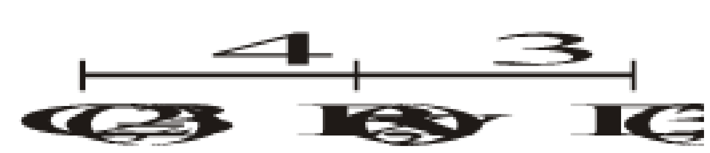
Formula $\underline{r} = \dfrac{q\underline{a} + P\underline{b}}{p + q}$

(i) Point C with position vector $2\underline{i} - 3\underline{j}$ and point D with position vector $3\underline{i} + 2\underline{j}$ in ratio 4 : 3. (Lahore Board 2009)

Solution:1

Let the position vector of the required point P be $\underline{r}$ which divides the points C and D in ratio 4:3 By ratio formula

$\underline{r} = \dfrac{P\underline{b} + q\underline{a}}{P + q}$



$= \dfrac{3(2\underline{i} - 3\underline{j}) + 4(3\underline{i} + 2\underline{j})}{4 + 3} = \dfrac{6\underline{i} - 9\underline{j} + 12\underline{i} + 8\underline{j}}{7} = \dfrac{18\underline{i} - \underline{j}}{7} = \dfrac{18}{7}\underline{i} - \dfrac{1}{7}\underline{j}$

(ii) Point E with position vector $5\underline{i}$ and point F with position vector $4\underline{i} + \underline{j}$ in ratio 2 : 5.

Solution:

Let the position vector of point P be $\underline{r}$ which divides the points E & F in ratio 2:5.

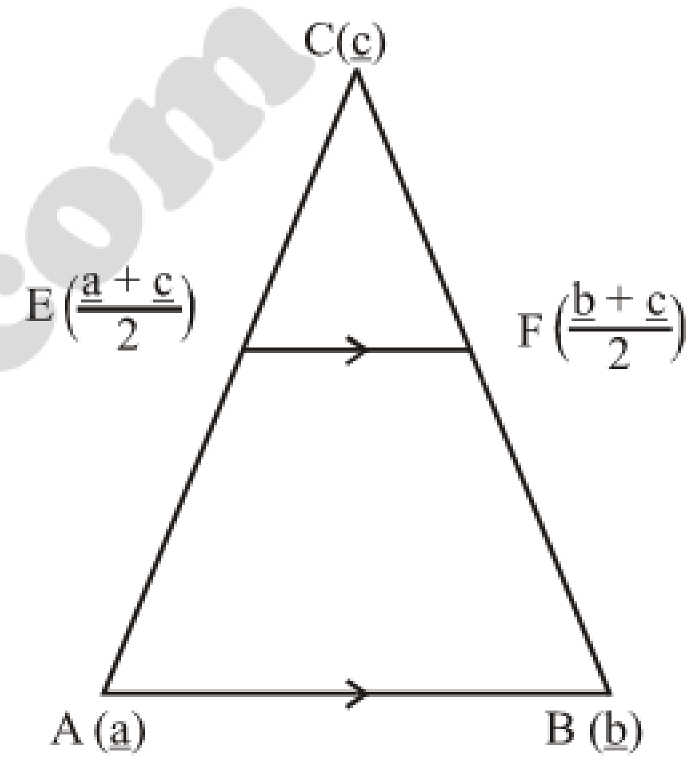
By ratio formula

$$\underline{r} = \frac{P\underline{b} + q\,\underline{a}}{P + q}$$

$$\underline{r} = \frac{5(5\underline{i}) + 2(4\underline{i} + \underline{j})}{2+5} = \frac{25\underline{i} + 8\underline{i} + 2\underline{j}}{7} = \frac{33\underline{i} + 2\underline{j}}{7} = \frac{33}{7}\underline{i} + \frac{2}{7}\underline{j} \quad \text{Ans.}$$

Q.14 Prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and half as long. (Lahore Board 2011)

Solution:



Let ABC be any triangle and Let E & F be the mid points of the two sides AC & BC respectively. Let $\underline{a}$, $\underline{b}$, and $\underline{c}$ be position vector of A, B and C. Therefore position vectors of E & F are $\left(\frac{\underline{a}+\underline{c}}{2}\right)$ and $\left(\frac{\underline{b}+\underline{c}}{2}\right)$ respectively.

We have to show that (i) $\overrightarrow{AB}$ is parallel to $\overrightarrow{EF}$

(ii) $\frac{1}{2}\overrightarrow{AB} = \overrightarrow{EF}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \underline{b} - \underline{a} \quad \text{............ (i)}$$

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE}$$

$$= \frac{\underline{b}+\underline{c}}{2} - \frac{\underline{a}+\underline{c}}{2} = \frac{\underline{b}+\underline{c}-\underline{a}-\underline{c}}{2} = \frac{\underline{b}-\underline{a}}{2}$$

$$\overrightarrow{EF} = \frac{1}{2}(\underline{b}-\underline{a}) = \frac{1}{2}\overrightarrow{AB} \quad \text{using (i)}$$

$$\overrightarrow{EF} = \lambda\,\overrightarrow{AB} \quad \text{where } \lambda = \frac{1}{2}.$$

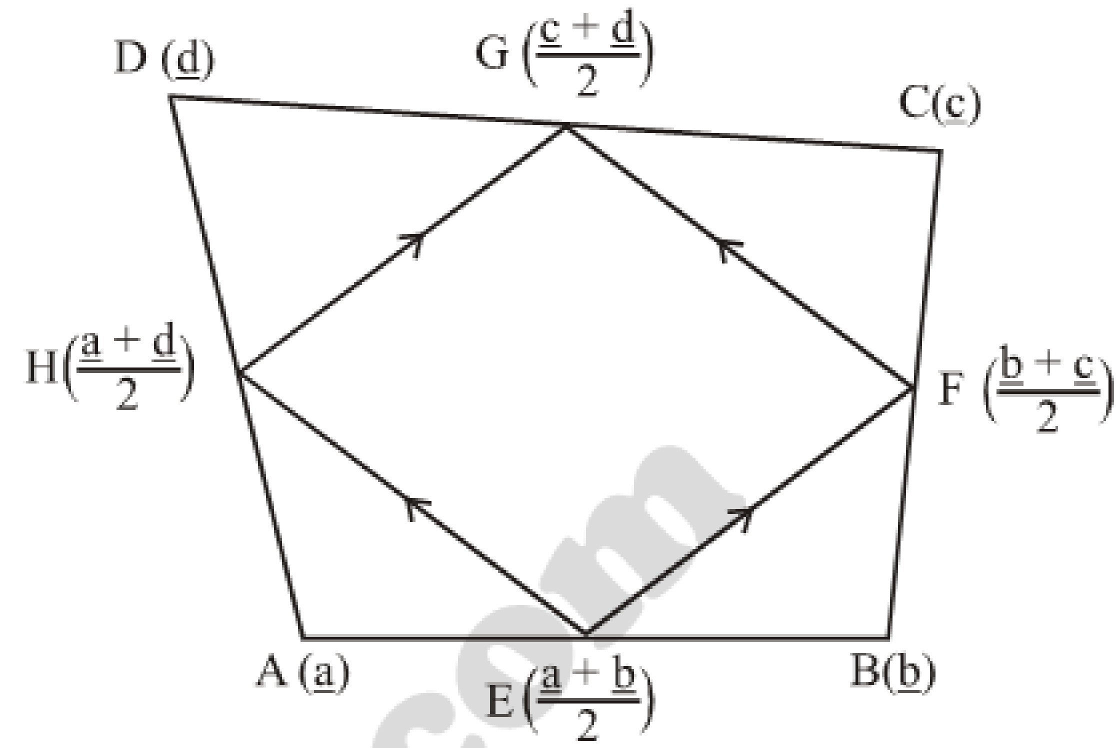
Hence $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are parallel & half as long. Hence proved.

Q.15 Prove that the line segments joining the mid points of the sides of a quadrilateral taken in order form a parallelogram.

(Gujranwala Board 2007, Lahore Board 2009)

Solution:

Let ABCD be any quadrilateral. Let E, F, G, H be mid points of the sides, $\underline{a}$, $\underline{b}$, $\underline{c}$ & $\underline{d}$ are the position vectors of A, B, C and D respectively. The position vectors of E, F, G, & H are $\frac{\underline{a}+\underline{b}}{2}$, $\frac{\underline{b}+\underline{c}}{2}$, $\frac{\underline{c}+\underline{d}}{2}$ & $\frac{\underline{a}+\underline{d}}{2}$ respectively.



We have to prove that EFGH is a parallelogram.

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE}$$
$$= \frac{\underline{b}+\underline{c}}{2} - \frac{\underline{a}+\underline{b}}{2}$$
$$\overrightarrow{EF} = \frac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2} = \frac{\underline{c}-\underline{a}}{2} \quad \text{.........(i)}$$
$$\overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH}$$
$$= \frac{\underline{c}+\underline{d}}{2} - \frac{\underline{a}+\underline{d}}{2}$$
$$= \frac{\underline{c}+\underline{d}-\underline{a}-\underline{d}}{2} = \frac{\underline{c}-\underline{d}}{2} \quad \text{......... (ii)}$$
$$\overrightarrow{FG} = \overrightarrow{OG} - \overrightarrow{OF}$$
$$= \frac{\underline{c}+\underline{d}}{2} - \frac{\underline{b}+\underline{c}}{2}$$
$$\overrightarrow{FG} = \frac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2} = \frac{\underline{d}-\underline{b}}{2} \quad \text{.........(iii)}$$
$$\overrightarrow{EH} = \overrightarrow{OH} - \overrightarrow{OE}$$
$$= \frac{\underline{a}+\underline{d}}{2} - \frac{\underline{a}+\underline{b}}{2} = \frac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2} = \frac{\underline{d}-\underline{b}}{2} \quad \text{........ (iv)}$$

from (i), (ii), (iii) & (iv)

$$\overrightarrow{EF} = \overrightarrow{HG} \text{ and } \overrightarrow{EH} = \overrightarrow{FG}$$

Shows EFGH is a parallogram.