Chapter 10

TRIGONOMETRIC IDENTITIES

FUNDAMENTAL LAW OF TRIGONOMETRY

Let α, β any two angles (real number), then

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

DEDUCTIONS FROM FUNDAMENTAL LAW

$$\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$$

$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$$

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

DOUBLE ANGLE IDENTITIES

(i) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

(ii) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$ (Lahore Board 2005)

(iii) $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

TRIPLE ANGLE IDENTITIES

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ (Gujranwala Board 2005)

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

HALF ANGLE IDENTITIES

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

SUM, DIFFERENCE & PRODUCT OF SINES & COSINES

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

and

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

where $P = \alpha + \beta$, $Q = \alpha - \beta$

EXERCISE 10.1

Q.1 Without using the table, find values of:

Solution:

(i) $\sin(-780^\circ)$ (ii) $\cot(-855^\circ)$ (iii) $\text{cosec}(2040^\circ)$

(iv) $\sec(-960^\circ)$ (v) $\tan(1110^\circ)$ (vi) $\sin(-300^\circ)$

$780° = 8 \times 90° + 60°$

$-\sin(780°) = -\sin(8 \times 90° + 60°)$

$= -\sin 60°$

$= -\dfrac{\sqrt{3}}{2}$

$90° \overline{)\,780°\,}(8$
$720°$
$60°$

(ii) $-\cot 855°$

Solution:

$855° = 9 \times 90° + 45°$

$-\cot(855°) = -\cot(9 \times 90° + 45°)$

$= -\cot 45°$

$= -(-1)$

$= 1$

$90° \overline{)\,855°\,}(9$
$810°$
$45°$

(iii) $\csc(2040°)$

Solution:

$2040° = 22 \times 90° + 60°$

Apply 'csc' both side.

$\csc(2040°) = \csc(22 \times 90° + 60°)$

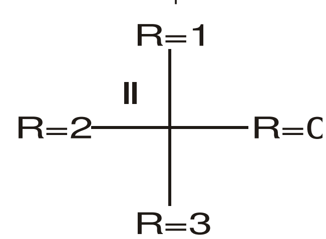
$= -\csc 60°$

$= -\dfrac{2}{\sqrt{3}}$

$90° \overline{)\,2040°\,}(22$
$+1980°$
$60°$

For Quadrant

$4 \overline{)\,9\,}(2$
8
1



Note:

(i) When R = 0, then quad. I or IV

(ii) When R = 1, then quad. II or I

(iii) When R = 2, then quad. III or II

(iv) When R = 3, then quad. IV or III

(iv)

Solution:

$\sec(-960^\circ) = \sec(960^\circ)$

$960^\circ = 10 \times 90^\circ + 60^\circ$

Apply 'sec' both sides.

$\text{secf}(780^\circ) = \sec(10 \times 90^\circ + 60^\circ)$

$= -\sec 60^\circ$

$= -2$

$90^\circ \,)\, 960^\circ \,($ 8

$\pm 900^\circ$

60°

$\therefore \quad \dfrac{960^\circ}{90^\circ} = 10.666$

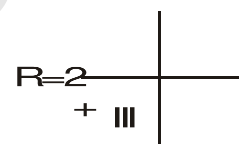
We take only 10.

For Quadrant

$4 \,|\, 10 \,|\, 2$

8

$2 \rightarrow R$



(v) tan 1110°

Solution:

$1110^\circ = 12 \times 90^\circ + 30$

Apply 'tan' both sides.

$\tan(1110^\circ) = \tan(12 \times 90^\circ + 30^\circ)$

$= \tan 30^\circ = \dfrac{1}{\sqrt{3}}$

$90^\circ \,)\, 1110^\circ \,($ 12

$\pm 1080^\circ$

30°

$\therefore \quad \dfrac{1110^\circ}{90^\circ} = 120$

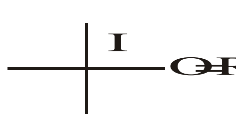
We take only 12.

For Quadrant

$4 \,|\, 12 \,|\, 3$

12

$0 \rightarrow R$



(vi) $\sin(-300°) = -\sin 300°$

Solution:

$$300° = 3 \times 90° + 30°$$

$$\begin{array}{r|l} 90° & 300° \; (3 \\ & \pm 270° \\ \hline & 30° \end{array}$$

Apply '−sin' both sides.

$$-\sin(300°) = -\sin(3 \times 90° + 30°)$$

$$= -\cos 30°$$

$\therefore$ Angle is in II quadrant cos is the in IV quadrant. So

$$= -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} \qquad \text{Ans.}$$

Q.2 Express each of the following as a trigonometric function of an angle of positive degree measure of less than 45°.

(i) $\sin 196^o$ (ii) $\cos 147^o$ (iii) $\sin 319^o$

(iv) $\cos 254^o$ (v) $\tan 294^o$ (vi) $\cos 728^o$

(vii) $\sin(-625^o)$ (viii) $\cos(-435^o)$ (ix) $\sin(150^o)$

Solution:

(i) $\sin 196^o$

$= \sin(180^o + 16^o)$

$= \sin 180^o \cos 16^o + \cos 180^o \sin 16^o$ $\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

$= 0 \times \cos 16^o + (-1)\sin 16^o$

$= -\sin 16^o$

Alternative Method:

$$\sin 196° = \sin(180° + 16°)$$

$$= \sin[2(90°) + 16°]$$

$$= -\sin 16°$$

(ii) $\mathbf{\cos 147^o}$

$= \cos(180^o - 33^o)$ $\boxed{\cos(\alpha - \beta) = \cos\beta\cos\beta + \sin\alpha\sin\beta}$

$= \cos 180^o \cos 33^o + \sin 180^o \sin 33^o$

$= -1 \times \cos 33^o + 0 \times \sin 33^o$

$= -\cos 33^o$

Alternative Method:

$$\begin{aligned}\cos 147° &= \cos(180° - 33°)\\ &= \cos[2(90°) - 33°]\\ &= -\cos 33°\end{aligned}$$

(iii) $\mathbf{\sin 319^o}$

$= \sin(360^o - 41^o)$

$= \sin 360^o \cos 41^o - \sin 41^o \cos 360^o$

$= 0 \times \cos 41^o - \sin 41^o \times 1$

$= -\sin 41^o$

Alternative Method:

$$\begin{aligned}\sin 319° &= \sin(360° - 41°)\\ &= \sin[4(90°) - 41°]\\ &= -\sin 41°\end{aligned}$$

(iv) $\mathbf{\cos 254^o}$

$\cos(270^o - 16^o)$

$= \cos 270^o \cos 16^o + \sin 270^o \sin 16^o$

$= 0 \times \cos 16^o + (-1) \times \sin 16^o$

$= -\sin 16^o$

Alternative Method:

$$\begin{aligned}\cos 254° &= \cos(270° - 16°)\\ &= \cos[3(90°) - 16°]\\ &= -\sin 16°\end{aligned}$$

(v) $\mathbf{\tan 294^o}$

$$\begin{aligned}\tan 294° &= \tan(270° + 24°)\\ &= \tan[3(90°) + 24°]\\ &= -\cot 24°\end{aligned}$$

(vi) $\mathbf{\cos 728^o}$

$$\begin{aligned}\cos 728° &= \cos(720° + 8°)\\ &= \cos[8(90°) + 8°]\\ &= \cos 8°\end{aligned}$$

(vii) $\sin(-625^o)$

$$\begin{aligned}\sin(-625°) &= -\sin 625° \\ &= -\sin(630° - 5°) \\ &= -\sin[7(90°) - 5°] \\ &= -(-\cos 5°) \\ &= \cos 5°\end{aligned}$$

(viii) $\cos(-435^o)$

$$\begin{aligned}\cos(-435) &= \cos 435° \\ &= \cos(450° - 15°) \\ &= \cos[5(90°) - 15°] \\ &= \sin 15°\end{aligned}$$

(ix) $\sin 150^o$

$= \sin(180^o - 30^o)$

$= \sin 180^o \cos 30^o - \sin 30^o \cos 180^o$

$= 0 \times \cos 30^o - \sin 30^o (-1)$

$= \sin 30^o$

Alternative Method:

$$\begin{aligned}\sin 150° &= \sin(180° - 30°) \\ &= \sin[2(90°) - 30°] \\ &= \sin 30°\end{aligned}$$

Q.3 Prove the following:

(i) $\sin(180^o + \alpha)\sin(90^o - \alpha) = -\sin\alpha\cos\alpha$

(ii) $\sin 780^o \sin 480^o + \cos 120^o \sin 30^o = \frac{1}{2}$

(iii) $\cos 306^o + \cos 234^o + \cos 162^o + \cos 18^o = 0$

(iv) $\cos 330^o \sin 600^o + \cos 120^o \sin 150^o = -1$

Solution:

(i) $\sin(180^o + \alpha)\sin(90^o - \alpha) = -\sin\alpha\cos\alpha$

$$\begin{aligned}\text{L.H.S.} &= \sin(180^o + \alpha)\sin(90^o - \alpha) \\ &= [\sin 180^o \cos\alpha + \cos 180^o \sin\alpha][\sin 90^o \cos\alpha - \cos 90^o \sin\alpha] \\ &= [0 \times \cos\alpha + (-1)\sin\alpha][1 \times \cos\alpha - 0 \times \sin\alpha] \\ &= (-\sin\alpha)(\cos\alpha)\end{aligned}$$

$= -\sin\alpha\cos\alpha$

$=$ R.H.S. Hence proved.

Alternative Method:

L.H.S $= \sin(180° + \alpha)\sin(90° - \alpha)$

$= \sin[2(90°) + \alpha]\sin(90° - \alpha)$

$= (-\sin\alpha)(\cos\alpha)$

$= -\sin\alpha\cos\alpha$

(ii) $\sin 780^o \sin 480^o + \cos 120^o \sin 30^o = \dfrac{1}{2}$

L.H.S. $= \sin 780^o \sin 480^o + \cos 120^o \sin 30^o$ .... (1)

$\text{Sin } 780^o = \sin(720° + 60°)$

$= \sin[8(90°) + 60°]$

$= \sin 60°$

$\sin 480^o = \sin(450° + 30°)$

$= \sin[5(90°) + 30°]$

$= \cos 30°$

$\cos 120° = \cos(180° - 60°)$

$= \cos[2(90°) - 60°]$

$= -\cos 60°$

L.H.S. $= \sin 780° \sin 480° + \cos 120° \sin 30°$

$= \sin 60° \cos 30° + (-\cos 60°)\sin 30°$

$= \left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) - \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)$

$= \dfrac{3}{4} - \dfrac{1}{4}$

$= \dfrac{3-1}{4} = \dfrac{2}{4}$

$= \dfrac{1}{2} =$ R.H.S

(iii) $\cos 306^o + \cos 234^o + \cos 162^o + \cos 18^o = 0$

L.H.S. $= \cos 306^o + \cos 234^o + \cos 162^o + \cos 18^o$

$= \cos(360^o - 54^o) + \cos(180^o + 54^o) + \cos(180^o - 18^o) + \cos 18^o$

$= \cos 54^o - \cos 54^o - \cos 18^o + \cos 18^o$ $\left(\begin{array}{rl} \therefore \cos(2\pi - \theta) &= \cos\theta \\ \cos(\pi + \theta) &= -\cos\theta \\ \cos(\pi - \theta) &= -\cos\theta \end{array}\right)$

$= 0$

$=$ R.H.S.

(iv) $\mathbf{\cos 330^o \sin 600^o + \cos 120^o \sin 150^o = -1}$

$$\text{L.H.S.} = \cos 330^o \sin 600^o + \cos 120^o \sin 150^o$$

$$= \cos(360^o - 30^o)\sin(540^o + 60^o) + \cos(180^o - 60^o)\sin(180^o - 30^o)$$

$$\cos[4(90°) - 30°]\ \sin[6(90°) + 60°] + \cos[2(90°) - 60°]\ \sin[2(90°) - 30°]$$

$$= \cos 30^o(-\sin 60^o) + (\cos 60^o)\sin 30^o$$

$$= \frac{-\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$= -\frac{3}{4} - \frac{1}{4} = \frac{-3-1}{4} = \frac{-4}{4} = -1 = \text{R.H.S.}$$

Q.4 Prove that

(i) $$\frac{\sin^2(\pi + \theta)\tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right)\cos^2(\pi - \theta)\,\text{cosec}\,(2\pi - \theta)} = \cos\theta$$

(ii) $$\frac{\cos(90^o + \theta)\sec(-\theta)\tan(180^o - \theta)}{\sec(360^o - \theta)\sin(180^o + \theta)\cot(90^o - \theta)} = -1$$

Solution:

(i) $$\frac{\sin^2(\pi + \theta)\tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right)\cos^2(\pi - \theta)\,\text{cosec}\,(2\pi - \theta)} = \cos\theta$$

$$\text{L.H.S.} = \frac{\sin^2(\pi + \theta)\tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right)\cos^2(\pi - \theta)\,\text{cosec}\,(2\pi - \theta)}$$

$$= \frac{[\sin(\pi + \theta)]^2\tan\left(\frac{3\pi}{2} + \theta\right)}{\left[\cot\left(\frac{3\pi}{2} - \theta\right)\right]^2[\cos(\pi - \theta)]^2\,\text{cosec}\,(2\pi - \theta)}$$

$$= \frac{(-\sin\theta)^2(-\cot\theta)}{(\tan\theta)^2(-\cos\theta)^2(-\text{cosec}\,\theta)}$$

$$= \frac{\sin^2\theta\left(\frac{-\cos\theta}{\sin\theta}\right)}{-\tan^2\theta\cos^2\theta\,\text{cosec}\,\theta}$$

$$= \frac{-\sin\theta\cos\theta}{-\frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\cdot\frac{1}{\sin\theta}}$$

$$= \frac{\sin\theta\cos\theta}{\sin\theta}$$

$$= \cos\theta = \text{R.H.S.}$$

Hence proved.

(ii) $$\frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)} = -1$$

$$\text{L.H.S.} = \frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}$$

$$= \frac{-\sin\theta\sec\theta(-\tan\theta)}{\sec\theta(-\sin\theta)\tan\theta} \qquad \left(\begin{array}{c} \therefore \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta \\ \cos(-\theta) = \cos\theta \\ \tan(\pi-\theta) = -\tan\theta \end{array}\right)$$

$$= \frac{+1}{-1} = -1 = \text{R.H.S.}$$

Q.5 If α, β, γ are the angles of a triangle ABC, then prove that

(i) $\sin(\alpha+\beta) = \sin\gamma$ **(ii)** $\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$

(iii) $\cos(\alpha+\beta) = -\cos\gamma$ **(iv)** $\tan(\alpha+\beta) + \tan\gamma = 0$

Solution:

(i) $\sin(\alpha+\beta) = \sin\gamma$

For a triangle ABC we know that

$\alpha+\beta+\gamma = 180^\circ$

$\alpha+\beta = 180^\circ-\gamma$

$\sin(\alpha+\beta) = \sin(180^\circ-\gamma)$

$= \sin\gamma \qquad (\therefore \sin(\pi-\theta) = \sin\theta)$

Hence proved.

(ii) $\mathbf{\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}}$

Since α, β, γ are angles of triangle ABC

so $\alpha+\beta+\gamma = 180^o$

$\alpha+\beta = 180^o - \gamma$

$\Rightarrow \quad \frac{\alpha+\beta}{2} = \frac{180^o-\gamma}{2} = 90^o - \frac{\gamma}{2}$

$\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(90^o - \frac{\gamma}{2}\right)$

$\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$ Hence proved.

(iii) $\mathbf{\cos(\alpha+\beta) = -\cos\gamma}$

since α, β, γ are angles of triangle

so $\alpha+\beta+\gamma = 180^o$

$\alpha+\beta = 180^o - \gamma$

$\cos(\alpha+\beta) = \cos(180^o - \gamma)$

$\cos(\alpha+\beta) = -\cos\gamma$ Hence proved.

(iv) $\mathbf{\tan(\alpha+\beta) + \tan\gamma = 0}$

since α, β, γ are angles of triangle so

$\alpha+\beta+\gamma = 180^o$

$\alpha+\beta = 180^o - \gamma$

taking tan on both sides

$\tan(\alpha+\beta) = \tan(180^o - \gamma) = -\tan\gamma$

$\tan(\alpha+\beta) + \tan\gamma = 0$

Hence proved.