EXERCISE 8.3

BINOMIAL SERIES **(Lahore Board 2009, 11)**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

Here index n is negative integer or a fraction.

Q.1 Expand the following upto 4 terms, taking value of x such that the expansion in each case is valid.

(i) $(1-x)^{1/2}$ **(ii)** $(1+2x)^{-1}$

(iii) $(1+x)^{-1/3}$ **(iv)** $(4-3x)^{1/2}$

(v) $(8-2x)^{-1}$ **(Lahore Board 2008)** **(vi)** $(2-3x)^{-2}$ **(Lahore Board 2010)**

(vii) $\dfrac{(1-x)^{-1}}{(1+x)^2}$ **(viii)** $\dfrac{\sqrt{1+2x}}{1-x}$

(ix) $\dfrac{(4+2x)^{1/2}}{(2-x)}$ **(x)** $(1+x-2x^2)^{1/2}$

(xi) $(1-2x+3x^2)^{-1/3}$

Solution:

(i) $(1-x)^{1/2}$

By binomial series

$$= \left(1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \ldots\ldots\right)$$

$$= 1 - \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\times\frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\times\frac{1}{6}(-x^3) + \ldots\ldots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots\ldots$$

Valid if $|x| < 1$

(ii) $(1+2x)^{-1}$

$$1 + (-1)(2x) + \frac{(-1)(-1-1)}{2!}(2x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(2x)^3 + \ldots$$

$$= 1 - 2x + 4x^2 - 8x^3 + \ldots\ldots$$

Valid if $|2x| < 1$

$2|x| < 1$

$\Rightarrow$ $|x| < \frac{1}{2}$

(iii) $\mathbf{(1 + x)^{-1/3}}$

$$1 + \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}x^3 + \ldots\ldots.$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \ldots\ldots.$$

Valid if $|x| < 1$

(iv) $\mathbf{(4 - 3x)^{1/2}}$

$$(4)^{1/2}\left(1 - \frac{3x}{4}\right)^{1/2}$$

$$= 2\left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{3x}{4}\right)^3 + \ldots\ldots\right]$$

$$= 2\left[1 - \frac{3x}{8} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} \times \frac{9x^2}{16} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(\frac{-27x^3}{64}\right) + \ldots\ldots\right]$$

$$= 2 - \frac{3x}{4} - \frac{9}{64}x^2 - \frac{27}{512}x^3 + \ldots\ldots.$$

Expansion is valid if

$$\left|\frac{3}{4}x\right| < 1$$

$$\Rightarrow \quad \frac{3}{4}|x| < 1$$

$$\Rightarrow \quad |x| < \frac{4}{3}$$

(v) $\mathbf{(8 - 2x)^{-1}}$ **(Lahore Board 2008)**

$$= 8^{-1}\left(1 - \frac{2x}{8}\right)^{-1}$$

$$= \frac{1}{8}\left[1 - \frac{x}{4}\right]^{-1}$$

$$= \frac{1}{8}\left[1 + \frac{1}{4}x + \frac{-1 \times -2}{2 \times 1}\ \frac{1}{16}x^2 + \frac{(-1)\times(-2)\times(-3)}{3 \times 2 \times 1} \times \frac{-1}{64} \times^3 + \ldots\right]$$

$$= \frac{1}{8}\left[1 + \frac{1}{4}x + \frac{1}{16}x^2 + \frac{1}{64}x^3 + \ldots\ldots.\right]$$

$$= \frac{1}{8} + \frac{1}{32}x + \frac{1}{128}x^2 + \frac{1}{512}x^3 + \ldots\ldots.$$

The expansion valid only if

$$\left|\frac{x}{4}\right| < 1$$

$$\Rightarrow \quad \frac{1}{4}|x| < 1$$

$$\Rightarrow \quad |x| < 4$$

(vi) $\mathbf{(2-3x)^{-2}}$ **(Lahore Board 2010)**

$$2^{-2}\left(1-\frac{3x}{2}\right)^{-2}$$

$$= \frac{1}{4}\left[1+(-2)\left(\frac{-3}{2}x\right)+\frac{(-2)(-2-1)}{2!}\left(\frac{-3x}{2}\right)^2+\frac{(-2)(-2-1)(-2-2)}{3!}\left(\frac{-3}{2}x\right)^3+\ldots\ldots\ldots\right]$$

$$= \frac{1}{4}\left[1+3x+\frac{-2x-3}{2}\times\frac{9x^2}{4}+\frac{(-2)(-3)(-4)}{6}\times\frac{-27x^3}{8}+\ldots\ldots\right]$$

$$= \frac{1}{4}\left[1+3x+\frac{27x^2}{4}+\frac{27x^3}{2}+\ldots\ldots\right]$$

$$= \frac{1}{4}+\frac{3}{4}x+\frac{27x^2}{16}+\frac{27x^3}{8}+\ldots\ldots$$

The above expansion is valid only if

$$\left|\frac{3x}{2}\right| < 1$$

$$\Rightarrow \quad \frac{3}{2}|x| < 1$$

$$\Rightarrow \quad |x| < \frac{2}{3}$$

(vii) $\mathbf{\dfrac{(1-x)^{-1}}{(1+x)^2}}$

$$= (1-x)^{-1}(1+x)^{-2}$$

$$= \left[1+x+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+\ldots\ldots\right]$$

$$\left[1-2x+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}(+x)^3+\ldots\ldots\right]$$

$$= \left[1+x+\frac{(-1)(-2)}{2}(x)^2+\frac{(-1)(-2)(-3)}{6}(-x^3)+\ldots\ldots\right]$$

$$\left[1-2x+\frac{(-2)(-3)}{2}x^2+\frac{(-2)(-3)(-4)}{6}x^3+\ldots\ldots\right]$$

$= [1 + x + x^2 + x^3 + \ldots\ldots] [1 - 2x + 3x^2 - 4x^3 + \ldots\ldots]$

$= 1 - 2x + 3x^2 - 4x^3 + x - 2x^2 + 3x^3 + x^2 - 2x^3 + x^3 + \ldots\ldots$

$= 1 - x + 2x^2 - 2x^3 + \ldots\ldots$

The above expansion are valid if

$|x| < 1$

(viii) $\dfrac{\sqrt{1 + 2x}}{1 - x}$

$(1 + 2x)^{1/2} (1 - x)^{-1}$

$= (1 - x)^{-1} (1 + 2x)^{1/2}$

$$= \left[1 + x + \frac{(-1)(-1-1)}{2!}(-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 + \ldots\ldots \right]$$

$$\left[1 + \frac{1}{2}2x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(2x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2x)^3 + \ldots\ldots \right]$$

$$= \left[1 + x + \frac{-1 \times -2}{2}x^2 + \frac{-1 \times -2 \times -3}{6}(-x^3) + \ldots\ldots \right]$$

$$\left[1 + x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}4x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}8x^3 + \ldots\ldots \right]$$

$$= [1 + x + x^2 + x^3 + \ldots\ldots]\left[1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots\ldots \right]$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + x + x^2 - \frac{1}{2}x^3 + x^2 + x^3 + x^3 + \ldots\ldots$$

$$= 1 + 2x + \frac{3}{2}x^2 + 2x^3 + \ldots\ldots$$

The above expansion valid if

$|x| < \frac{1}{2}$ and $|x| < 1$

(ix) $\dfrac{(4 + 2x)^{1/2}}{(2 - x)}$

$(4 + 2x)^{1/2} (2 - x)^{-1}$

$$= 4^{1/2}\left(1 + \frac{2}{4}x\right)^{1/2} 2^{-1}\left(1 - \frac{x}{2}\right)^{-1}$$

$$= 2\left(1 + \frac{1}{2}\,\frac{x}{2} + \frac{1}{2}\,\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{x^2}{4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\ldots\right)$$

$$\frac{1}{2}\left[1 + \frac{x}{2} + \frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{-x}{2}\right)^3 + \ldots\ldots\right]$$

$$= \left[1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + \ldots\ldots\right]\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \ldots\ldots\right]$$

$$= 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{1}{32}x^2 - \frac{1}{64}x^3 + \frac{1}{128}x^3$$

$$= 1 + \frac{3}{4}x + \frac{11}{32}x^2 + \frac{23}{128}x^3 + \ldots\ldots$$

The expansion of $\left(1 + \frac{x}{2}\right)^{1/2}$ and $\left(1 - \frac{x}{2}\right)^{-1}$ are valid if

$$\left|\frac{x}{2}\right| < 1$$

$\Rightarrow \quad |x| < 2$

(x) $\quad \mathbf{(1 + x - 2x^2)^{1/2}}$

$$= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x - 2x^2)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x - 2x^2)^3 + \ldots\ldots$$

$$= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2}\times -\frac{1}{2}}{2}(x^2 + 4x^4 - 4x^3) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(x^3 - 8x^6 - 6x^4 + 12x^5) + \ldots$$

$$= 1 + \frac{1}{2}x - x^2 - \frac{1}{8}x^2 - \frac{1}{2}x^4 + \frac{1}{2}x^3 + \frac{1}{16}x^3 - \frac{3}{8}x^4 + \frac{3}{4}x^5 - \frac{1}{2}x^6 + \ldots\ldots$$

$$= 1 + \frac{1}{2}x - \frac{9}{8}x^2 + \frac{9}{16}x^3 + \ldots\ldots$$

The above expansion is valid only if $|x - 2x^2| < 1$ that is either

$x - 2x^2 < 1$ or $-(x - 2x^2) < 1$

$\Rightarrow \quad -2x^2 + x - 1 < 0 \quad \ldots\ldots\ldots (1) \qquad 2x^2 - x - 1 < 0 \quad \ldots\ldots\ldots (2)$

$2x^2 - 2x + x - 1 < 0$

$(x - 1)(2x + 1) < 0$

$-\frac{1}{2} < x < 1$

(xi) $\mathbf{(1 - 2x + 3x^2)^{-1/3}}$

$[1 + (3x^2 - 2x)]^{-1/3}$

$$= \left[1 + \left(\frac{-1}{3}\right)(3x^2 - 2x) + \frac{\left(-\frac{1}{3}\right)\left(\frac{-1}{3} - 1\right)}{2!}(3x^2 - 2x)^2 + \frac{\frac{-1}{3}\left(\frac{-1}{3} - 1\right)\left(\frac{-1}{3} - 2\right)}{3!}(3x^2 - 2x)^3 + \ldots\right]$$

$$= 1 - \frac{1}{3}(3x^2 - 2x) - \frac{1}{3} \times \frac{-4}{3} \times \frac{1}{2}(9x^4 + 4x^2 - 12x^3) + \frac{-1}{3} \times \frac{-4}{3} \times \frac{-7}{2} \times \frac{1}{6}$$
$$(27x^6 - 8x^3 - 54x^5 + 36\,x^4) + \ldots\ldots$$

$$= 1 - x^2 + \frac{2}{3}x + \frac{2}{9}(9x^4 + 4x^2 - 12x^3) - \frac{7}{27}(27x^6 - 8x^3 - 54x^5 + 36x^4) + \ldots\ldots$$

$$= 1 - x^2 + \frac{2}{3}x + 2x^4 + \frac{8}{9}x^2 - \frac{24}{9}x^3 - 7x^6 + \frac{56}{27}x^3 + 14x^5 + \ldots\ldots$$

$$= 1 + \frac{2}{3}x - \frac{1}{9}x^2 - \frac{16}{27}x^3 + \ldots\ldots$$

The above expansion is valid only if

$|3x^2 - 2x| < 1$

$3x^2 - 2x < 1 \qquad -(3x^2 - 2x) < 1$

$3x^2 - 2x - 1 < 1$

$3x^2 - 3x + x - 1 < 1$

$(3x + 1)(x - 1) < 1$

$\frac{-1}{3} < x < 1$

Q.2 Using Binomial theorem find the value of the following to three places of decimals.

Solution:

(i) $\sqrt{99}$

$$= (99)^{1/2} = (100 - 1)^{1/2} = (100)^{1/2}\left(1 - \frac{1}{100}\right)^{1/2}$$

$$= 10\left[1 + \left(\frac{1}{2}\right)\left(\frac{-1}{100}\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{-1}{100}\right)^2 + \ldots\ldots\right]$$

$$= 10\left[1 - \frac{1}{2 \times 100} + \frac{1}{2} \times \frac{-1}{2} \times \frac{1}{2} \times \frac{1}{100 \times 100} + \ldots\ldots\right]$$

$= 10 - \frac{1}{20} - \frac{1}{8000} + \ldots\ldots.$

$= 10 - 0.05 - 0.000125 + \ldots\ldots. = 9.950$

(ii) $\mathbf{(0.98)^{1/2}}$

$= (1 - .02)^{1/2}$

$= 1 + \frac{1}{2}(-.02) + \frac{1}{2}\left(\frac{1}{2} - 1\right)(-.02)^2 + \ldots\ldots.$

$= 1 - .01 + \frac{1}{2}\left(-\frac{1}{2}\right)(.0004) + \ldots\ldots.$

$= 1 - .01 - .00005 + \ldots\ldots. = .990$

(iii) $\mathbf{(1.03)^{1/3}}$

$= (1 + .03)^{1/3}$

$= \left(1 + \frac{1}{3}(.03) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}(.03)^2 + \ldots\ldots.\right)$

$= 1 + .01 + \frac{1}{3} \times \frac{-2}{3} \times \frac{1}{2} \times .0009 + \ldots\ldots.$

$= 1 + .01 - .0001 + \ldots\ldots. = 1.010$

(iv) $\mathbf{\sqrt[3]{65}}$

$= (65)^{1/3} = (64 + 1)^{1/3}$

$= (64)^{1/3}\left(1 + \frac{1}{64}\right)^{1/3}$

$= 4\left[1 + \frac{1}{3}\left(\frac{1}{64}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{1}{64}\right)^2 + \ldots\ldots.\right]$

$= 4\left[1 + \frac{1}{192} + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2} \times \frac{1}{4096} + \ldots\ldots.\right]$

$= 4 + 0.021 - 0.0001 + \ldots\ldots. = 4.021$

(v) $\mathbf{\sqrt[4]{17}}$

$= (17)^{1/4} = (16 + 1)^{1/4} = 16^{1/4}\left(1 + \frac{1}{16}\right)^{1/4}$

$$= 2^{4 \times (1/4)}\left[1 + \frac{1}{4}\left(\frac{1}{16}\right) + \ldots\ldots\right]$$

$$= 2 + \frac{1}{2 \times 16} + \ldots\ldots = 2 + \frac{1}{32} + \ldots\ldots$$

$$= 2 + 0.031 + \ldots\ldots = 2.031$$

(vi) $\sqrt[5]{31}$

$$= (31)^{1/5} = (32 - 1)^{1/5}$$

$$= 32^{1/5}\left(1 - \frac{1}{32}\right)^{1/5}$$

$$= 2^{5 \times (1/5)}\left[1 + \left(\frac{1}{5}\right)\left(\frac{-1}{32}\right) + \ldots\ldots\right]$$

$$= 2 - \frac{1}{5 \times 16} + \ldots\ldots = 2 - 0.013 = 1.987$$

(vii) $\dfrac{1}{\sqrt[3]{998}}$

$$= \left(\frac{1}{(998)^{1/3}}\right) = (998)^{-1/3}$$

$$= (1000 - 2)^{-1/3} = (1000)^{-1/3}\left[1 - \frac{2}{1000}\right]^{-1/3}$$

$$= 10^{3 \times (-1/3)}\left[1 - \frac{1}{500}\right]^{-1/3}$$

$$= 10^{-1}\left[1 + \left(\frac{1}{3}\right)\left(\frac{1}{500}\right) + \ldots\ldots\right]$$

$$= \frac{1}{10}\left[1 + \frac{1}{1500} + \ldots\ldots\right] = \frac{1}{10} + \frac{1}{15000} + \ldots\ldots$$

$$= 0.1 + 0.000067 + \ldots\ldots = 0.1000$$

(viii) $\dfrac{1}{\sqrt[5]{252}}$

$$= \frac{1}{(252)^{1/5}} = (252)^{-1/5} = (243 + 9)^{-1/5}$$

$$= 243^{-1/5}\left[1+\frac{9}{243}\right]^{-1/5} = 3^{5\times -1/5}\left[1+\left(\frac{-1}{5}\right)\left(\frac{9}{243}\right)+\ldots\ldots\right]$$

$$= 3^{-1}\left[1-\frac{1}{5}\times\frac{1}{27}+\ldots\ldots\right] = \frac{1}{3}\left[1-\frac{1}{135}+\ldots\ldots\right]$$

$$= \frac{1}{3}[1-0.007+\ldots\ldots] = \frac{1}{3}[0.993] = 0.331$$

(ix) $\mathbf{\dfrac{\sqrt{7}}{\sqrt{8}}}$

$$= \left(\frac{7}{8}\right)^{1/2} = \left(1-\frac{1}{8}\right)^{1/2}$$

$$= 1+\left(\frac{1}{2}\right)\left(\frac{-1}{8}\right)+\ldots\ldots$$

$$= 1-\frac{1}{16}+\ldots\ldots = 1-0.063+\ldots\ldots = 0.938$$

(x) $\mathbf{(.998)^{-1/3}}$

$$= (1-0.002)^{-1/3}$$

$$= 1+\left(\frac{-1}{3}\right)(-0.002)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2\times 1}(-0.002)^2+\ldots\ldots$$

$$= 1+0.001+\frac{2}{9}(0)+\ldots\ldots$$

$$= 1+0.001+0+\ldots\ldots = 1.001$$

(xi) $\mathbf{\dfrac{1}{\sqrt[6]{486}}}$

$$= \frac{1}{(486)^{1/6}} = (486)^{-1/6} = (729-243)^{-1/6}$$

$$= (729)^{-1/6}\times\left[1-\frac{243}{729}\right]^{-1/6} = (3^6)^{-1/6}\left[1-\frac{1}{3}\right]^{-1/6}$$

$$= \frac{1}{3}\left[1-\frac{1}{3}\right]^{-1/6} \Rightarrow = \frac{1}{3}\left[1+\left(\frac{-1}{6}\right)\left(\frac{-1}{3}\right)+\frac{\left(\frac{-1}{6}\right)\left(\frac{-1}{6}-1\right)}{2!}\left(\frac{-1}{3}\right)^2+\ldots\right]$$

$$= \frac{1}{3}[1+0.0555+0.0108+\ldots\ldots]$$

$$= \frac{1}{3}[1.06895] = 0.356$$

(xii) $\mathbf{(1280)^{1/4}}$

$$= (1296 - 16)^{1/4} = 1296^{1/4}\left(1 - \frac{16}{1296}\right)^{1/4}$$

$$= 6^{4 \times (1/4)}\left[1 + \left(\frac{1}{4}\right)\left(-\frac{16}{296}\right) + \ldots\ldots\right]$$

$$= 6\left[1 - \frac{1}{324} + \ldots\ldots\right] = 6\,[1 - 0.003 + \ldots\ldots] = 6\,[0.997] = 5.981$$

Q.3 Find the coefficient of x^n in the expansion

(i) $\dfrac{1 + x^2}{(1 + x)^2}$ **(ii)** $\dfrac{(1 + x)^2}{(1 - x)^2}$

(iii) $\dfrac{(1 + x)^3}{(1 - x)^2}$ **(iv)** $\dfrac{(1 + x)^2}{(1 - x)^3}$

(v) $(1 - x + x^2 - x^3 + \ldots\ldots)$ **(Gujranwala Board 2005)**

Solution:

(i) $\dfrac{1 + x^2}{(1 + x)^2}$

$$= (1 + x^2)(1 + x)^{-2}$$

$$= (1 + x^2)\left[1 + (-2)(x) + \frac{(-2)(-2-1)x^2}{2!} + \frac{(-2)(-2-1)(-3-1)(x)^3}{3!} + \ldots\ldots\right]$$

$$= (1 + x^2)\left[1 + (-2)(x) + \frac{(-2)(-3)x^2}{2!} + \frac{(-2)(-3)(-4)}{3 \times 2 \times 1}x^3 + \ldots\ldots\right]$$

$$= (1 + x^2)\,[1 + (-1)\,2x + (-1)^2\,3x^2 + (-1)^3\,4x^3 + \ldots\ldots$$

$$+ \ldots\ldots + (-1)^{n-2}(n-1)^{n-2}x + (-1)^{n-1}\,n\,x^{n-1} + (-1)^n(n+1)\,x^n$$

$$= (-1)^{n-2}(n-1)\,x^n + (-1)^n(n+1)\,x^n$$

$$= \left[(-1)^n(-1)^2(n-1) + (-1)^n(n+1)\right]x^n = (-1)^n\,[n - 1 + n + 1]\,x^n$$

$$= (-1)^n\,.\,(2n)\,x^n$$

Coefficient of x^n is, $(-1)^n \times (2n)$

(ii) $\dfrac{(1 + x)^2}{(1 - x)^2}$

$$= (1 + x)^2(1 - x)^{-2}$$

$$= (1 + 2x + x^2)\left(1 + 2x + \frac{(-2)(-2-1)(-x)^2}{2!} + \ldots\ldots\right)$$

$$= (1 + 2x + x^2)\left(1 + 2x + \frac{(-2)(-3)}{2 \times 1}x^2 + \ldots\ldots\right)$$

$$= (1 + 2x + x^2)\left[1 + 2x + 3x^2 + \ldots.. + (n-1)x^{n-2} + n\,x^{n-1} + (n+1)x^n\right]$$

Now multiplying the terms to get terms involving x^n.

$$= (n+1)x^n + 2n\,x^{n-1+1} + (n-1)x^{n-2+2}$$

$$= (n+1)x^n + 2nx^n + (n-1)x^n$$

$$= (n + 1 + 2n + n - 1)x^n$$

$$= 4\,n\,x^n$$

Hence coefficient of x^n is $4n$

(iii) $\dfrac{(1+x)^3}{(1-x)^2}$

$$= (1+x)^3(1-x)^{-2}$$

$$= \left[1 + 3x + \frac{(3)(3-1)}{2!}(x)^2 + \frac{3(3-1)(3-2)}{3!}x^3\right]$$

$$\left[1 + 2x + \frac{(-2)(-2-1)(-x^2)}{2!} + \frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3 + \ldots\ldots\right]$$

$$= \left[1 + 3x + \frac{3 \times 2}{2 \times 1}x^2 + \frac{3 \times 2 \times 1}{3 \times 2 \times 1}x^3\right]$$

$$\left[1 + 2x + \frac{(-2)(-3)}{2 \times 1}x^2 + \frac{(-2)(-3)(-4)}{3 \times 2 \times 1}(-x)^3 + \ldots\ldots\right]$$

$$= \left[1 + 3x + 3x^2 + x^3\right]$$

$$\left[1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (n-2)x^{n-3} + (n-1)x^{n-2} + n\,x^{n-1} + (n+1)x^n\right]$$

$$= (n+1)x^n + 3n\,x^{n-1+1} + 3(n-1)x^{n-2+2} + (n-2)x^{n-3+3}$$

$$= (n+1)x^n + 3n\,x^n + 3(n-1)x^n + (n-2)x^n$$

$$= (n + 1 + 3n + 3n - 3 + n - 2)x^n$$

$$= (8n - 4)\,.\,x^n = 4(2n-1)x^n$$

Hence coefficient of x^n is $4(2n - 1)$.

(iv) $\dfrac{\mathbf{(1+x)^2}}{\mathbf{(1-x)^3}}$

$$= (1+x)^2(1-x)^{-3}$$

$$= \left[(1+2x+x^2)\left(1+3x+\frac{(-3)(-3-1)}{2!}(-x)^2+\ldots\ldots\right)\right]$$

$$= (1+2x+x^2)\left(1+3x+\frac{-3x-4}{2\times 1}x^2+\ldots\ldots\right)$$

$$= (1+2x+x^2)$$

$$\left(1+3x+\frac{3\times 4}{2}x^2+\frac{4\times 5}{2}x^3+\ldots\ldots+\frac{(n-1)(n)}{2}x^{n-2}+\frac{n(n+1)}{2}x^{n-1}+\frac{(n+1)(n+2)}{2}x^n\right)$$

$$\Rightarrow \quad = \frac{(n+1)(n+2)}{2}x^n+\frac{2(n)(n+1)}{2}x^{n-1+1}+\frac{(n-1)(n)}{2}x^{n-2+2}$$

$$= \left(\frac{n^2+3n+2+2n^2+2n+n^2-n}{2}\right)x^n$$

$$= \left(\frac{4n^2+4n+2}{2}\right)x^n = (2n^2+2n+1)x^n$$

Hence coefficient of x^n is $2n^2+2n+1$.

(v) $\mathbf{(1-x+x^2-x^3+\ldots\ldots\ldots)}$

We know that,

$$(1+x)^{-1} = 1-x+x^2-x^3+\ldots\ldots$$

Hence given expression becomes

$$\left[(1+x)^{-1}\right]^2 = (1+x)^{-2}$$

$$= 1+(-2)x+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{31}(-x)^3+\ldots\ldots$$

$$= 1+(-1)2x+\frac{(-2)(-3)}{2\times 1}x^2+\frac{(-2)(-3)(-4)}{3\times 2\times 1}(-x)^3+\ldots\ldots$$

$$= 1+(-1)2x+(-1)^2 3x^2+(-1)^3 4x^2+\ldots\ldots+(-1)^{n-2}(n-2)x^{n-2}$$

$$+(-1)^{n-1}n\,x^{n-1}+(-1)^n(n+1)x^n$$

Hence coefficient of x^n is only, $(-1)^n(n+1)$

Q.4 If x is so small that its square and higher power can be neglected, then show that

(i) $\dfrac{1-x}{\sqrt{1-x}} \approx 1 - \dfrac{3}{2}x$ **(Lahore Board 2009)**

(ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \dfrac{3}{2}x$

(iii) $\dfrac{(9+7x)^{1/2} - (16+3x)^{1/4}}{4+5x} \approx \dfrac{1}{4} - \dfrac{17}{284}x$

(iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \dfrac{25}{4}x$

(v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx \left(1 - \dfrac{5}{6}x\right)$ **(Gujranwala Board 2006)**

(vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2} - \dfrac{61}{48}x$

(vii) $\dfrac{\sqrt{4-x} + (8-x)^{1/3}}{(8-x)^{1/3}} \approx 2 - \dfrac{1}{12}x$

Solution:

(i) $\dfrac{1-x}{\sqrt{1-x}} \approx 1 - \dfrac{3}{2}x$

$$\begin{aligned}
\text{L.H.S.} &= (1-x)(1+x)^{-1/2} \\
&= (1-x)\left(1 - \frac{1}{2}x\right) \quad \text{(neglecting square and heigher power of } x) \\
&= 1 - \frac{1}{2}x - x \\
&= 1 - \frac{3}{2}x \\
&= \text{R.H.S.}
\end{aligned}$$

(ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \dfrac{3}{2}x$

$$\begin{aligned}
\text{L.H.S.} &= (1+2x)^{1/2}(1-x)^{-1/2} \\
&= \left(1 + \frac{1}{2}2x\right)\left(1 + \frac{1}{2}x\right)
\end{aligned}$$

$$= (1+x)\left(1+\frac{1}{2}x\right)$$

$$= 1+\frac{1}{2}x+x$$

$$= 1+\frac{3}{2}x = \text{R.H.S.}$$

(iii) $\dfrac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x} \approx \dfrac{1}{4}-\dfrac{17}{284}x$

$$\text{L.H.S.} = \left[(9+7x)^{1/2}-(16+3x)^{1/4}\right](4+5x)^{-1}$$

$$= \left[9^{1/2}\left(1+\frac{7}{9}x\right)^{1/2}-16^{1/4}\left(1+\frac{3x}{16}\right)^{1/4}\right].4^{-1}\left(1+\frac{5x}{4}\right)^{-1}$$

$$= \left[3\left(1+\frac{7}{8}x\right)-2\left(1+\frac{3}{64}x\right)\right]\frac{1}{4}\left(1-\frac{5x}{4}\right)$$

$$= \frac{1}{4}\left[3+\frac{7}{6}x-2-\frac{3}{32}x\right]\left(1-\frac{5}{4}x\right)$$

$$= \frac{1}{4}\left[\left(1+\frac{103}{96}x\right)\left(1-\frac{5}{4}x\right)\right]$$

$$= \frac{1}{4}\left(1-\frac{5}{4}x+\frac{103}{96}x\right) \qquad [\because \text{ neglecting heigher power of } x]$$

$$= \frac{1}{4}\left(1-\frac{17}{96}x\right)$$

$$= \frac{1}{4}-\frac{17}{384}x = \text{R.H.S.}$$

(iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2+\dfrac{25}{4}x$

$$\text{L.H.S.} = (4+x)^{1/2}(1-x)^{-3}$$

$$= 4^{1/2}\left(1+\frac{x}{4}\right)^{1/2}(1-x)^{-3}$$

$$= 2\left(1+\frac{1}{8}x\right)(1+3x)$$

$$= 2\left(1+3x+\frac{1}{8}x\right)$$

$$= 2\left(1+\frac{25}{8}x\right)$$

$$= 2+\frac{25}{4}x = \text{R.H.S.}$$

(v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx \left(1-\dfrac{5}{6}x\right)$

$$\text{L.H.S.} = (1+x)^{1/2}(4-3x)^{3/2}(8+5x)^{-1/3}$$

$$= (1+x)^{1/2}\,4^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}(8)^{-1/3}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= \left(1+\frac{1}{2}x\right)2^3\left(1-\frac{9}{8}x\right)2^{-1}\left(1-\frac{5}{24}x\right)$$

$$= 2^3\,2^{-1}\left(1+\frac{1}{2}x\right)\left(1-\frac{9}{8}x\right)\left(1-\frac{5}{24}x\right)$$

$$= 2^2\left(1+\frac{1}{2}x\right)\left(1-\frac{5}{24}x-\frac{9}{8}x\right)$$

$$= 4\left(1-\frac{5}{24}x-\frac{9}{8}x+\frac{1}{2}x\right)$$

$$= 4\left(1-\frac{5}{6}x\right)$$

$$= \text{R.H.S.}$$

(vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

$$\text{L.H.S.} = (1-x)^{1/2}(9-4x)^{1/2}(8+3x)^{-1/3}$$

$$= (1-x)^{1/2}\,9^{1/2}\left(1-\frac{4x}{9}\right)^{1/2}8^{-1/3}\left(1+\frac{3x}{8}\right)^{-1/3}$$

$$= \left(1-\frac{1}{2}x\right)3\left(1-\frac{4}{18}x\right)2^{-1}\left(1-\frac{3x}{24}\right)$$

$$= 3^1\,2^{-1}\left(1-\frac{1}{2}x\right)\left(1-\frac{2}{9}x\right)\left(1-\frac{1}{8}x\right)$$

$$= \frac{3}{2}\left(1-\frac{1}{2}x\right)\left(1-\frac{1}{8}x-\frac{2}{9}x\right)$$

$$= \frac{3}{2}\left(1-\frac{1}{8}x-\frac{2}{9}x-\frac{1}{2}x\right)$$

$$= \frac{3}{2}\left(1-\frac{61}{72}x\right)$$

$$= \frac{3}{2} - \frac{3}{2} \times \frac{61}{72} x$$

$$= \frac{3}{2} - \frac{61}{48}$$

$$= \text{R.H.S.}$$

(vii) $\dfrac{\sqrt{4-x} + (8-x)^{1/3}}{(8-x)^{1/3}} \approx 2 - \dfrac{1}{12} x$

$$\text{L.H.S.} = \left[(4-x)^{1/2} + (8-x)^{1/3}\right](8-x)^{-1/3}$$

$$= \left[4^{1/2}\left(1-\frac{x}{4}\right)^{1/2} + 8^{1/3}\left(1-\frac{x}{8}\right)^{1/3}\right](8)^{-1/3}\left(1-\frac{x}{8}\right)^{-1/3}$$

$$= \left[2\left(1-\frac{1}{8}x\right) + 2\left(1-\frac{x}{24}\right)\right]2^{-1}\left(1+\frac{1}{24}x\right)$$

$$= \left[2-\frac{1}{4}x + 2 - \frac{x}{12}\right]\frac{1}{2}\left(1+\frac{1}{24}x\right)$$

$$= \frac{1}{2}\left(4-\frac{1}{3}x\right)\left(1+\frac{1}{24}x\right)$$

$$= \frac{1}{2}\left(4+\frac{1}{6}x - \frac{1}{3}x\right)$$

$$= \frac{1}{2}\left(4-\frac{1}{6}x\right)$$

$$= 2 - \frac{1}{12}x$$

$$= \text{R.H.S.}$$

Q.5 If x is so small that its cube and heigher power can be neglected, show that

(i) $\sqrt{1-x-2x^2} \approx 1 - \dfrac{1}{2}x - \dfrac{9}{8}x^2$

(ii) $\sqrt{\dfrac{1+x}{1-x}} \approx 1 + x + \dfrac{1}{2}x$

Solution:

(i) $\sqrt{1-x-2x^2} \approx 1 - \dfrac{1}{2}x - \dfrac{9}{8}x^2$

$$\text{L.H.S.} = (1-x-2x^2)^{1/2}$$

$$= \left[1-(x+2x^2)\right]^{1/2}$$

$$= 1 - \frac{1}{2}(x + 2x^2) + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left[-(x + 2x^2)^2\right]$$

$$= 1 - \frac{1}{2}(x + 2x^2) + \frac{1}{2}\left(-\frac{1}{2}\right) \times \frac{1}{2}(x^2 + 4x^4 + 4x^3)$$

$$= 1 - \frac{1}{2}(x + 2x^2) - \frac{1}{8}(x^2 + 4x^4 + 4x^3)$$

$$= 1 - \frac{1}{2}x - x^2 - \frac{1}{8}x^2 \qquad \text{(neglecting cube and heigher power of } x)$$

$$= 1 - \frac{1}{2}x - \frac{9}{8}x^2$$

$$= \text{R.H.S.}$$

(ii) $\sqrt{\dfrac{\mathbf{1+x}}{\mathbf{1-x}}} \approx \mathbf{1 + x + \dfrac{1}{2}x}$

$$\text{L.H.S.} = (1 + x)^{1/2}(1 - x)^{-1/2}$$

$$= \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}x^2\right]\left[1 + \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2} - 1\right)}{2!}(-x)^2\right]$$

$$= \left[1 + \frac{1}{2}x - \frac{1}{8}x^2\right]\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{3}{16}x^3 - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{3}{64}x^4$$

$$= 1 + \frac{1}{2}x + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{1}{8}x^2 + \frac{1}{4}x^2$$

$$= 1 + x + \frac{1}{2}x^2$$

$$= \text{R.H.S.}$$

Q.6 If x is very nearly equal to 1, then prove that

$Px^p - qx^q \approx (p - q)\,x^{p+q}$

(Gujranwala Board 2005, 2003), (Lahore Board 2003, 2009, 2011)

Solution:

Since $x \approx 1$

Let $x = 1 + h$ where h is so small that its square and heigher powers can be neglected.

$$\text{L.H.S.} = P\,x^p - q\,x^q$$

$$\approx P(1 + h)^p - q(1 + h)^q$$

$$\approx P(1 + ph) - q(1 + qh)$$
$$\approx P + p^2 h - q - q^2 h$$
$$\approx (p - q) + (p^2 - q^2) h$$
$$\approx (p - q) + (p - q)(p + q) h$$
$$\approx (p - q)[1 + (p + q) h] \quad \text{......... (1)}$$
$$\text{R.H.S.} = (P - q) x^{p+q}$$
$$\approx (P - q)(1 + h)^{p+q}$$
$$\approx (P - q)\left[1 + (p + q) h\right] \quad \text{......... (2)}$$

From (1) and (2) we have

L.H.S. = R.H.S.

Hence proved.

Q.7 If $p - q$ is small, when compared with p or q show that

$$\frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q} = \left[\frac{p+q}{2q}\right]^{1/n}$$

Solution:

$$\text{L.H.S.} = \frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q}$$

Let $p - q = h$

$p = q + h$, where 'h' is a small that it square and higher powers can be neglected.

$$= \frac{(2n+1)(q+h) + (2n-1)q}{(2n-1)(q+h) + (2n+1)q}$$
$$= \frac{2nq + 2nh + q + h + 2nq - q}{2nq + 2nh - q - h + 2nq + q}$$
$$= \frac{4nq + 2nh + h}{4nq + 2nh - h} = \frac{4nq + (2n+1)h}{4nq + (2n-1)h}$$
$$= \frac{4nq\left[1 + \left(\frac{2n+1}{4nq}\right)h\right]}{4nq\left[1 + \left(\frac{2n-1}{4nq}\right)h\right]}$$
$$= \left[1 + \left(\frac{2n+1}{4nq}\right)h\right]\left[1 + \left(\frac{2n-1}{4nq}\right)h\right]^{-1}$$
$$= \left[1 + \left(\frac{2n+1}{4nq}\right)h\right]\left[1 - \left(\frac{2n-1}{4nq}\right)h\right]$$

$$= 1 + \left(\frac{2n+1}{4nq}\right)h - \left(\frac{2n-1}{4nq}\right)h$$

$$= 1 + \frac{2nh + h - 2nh + h}{4nq}$$

$$= 1 + \frac{2h}{4nq} = 1 + \frac{h}{2nq} \qquad \ldots\ldots\ldots (1)$$

$$\text{R.H.S.} = \left[\frac{p+q}{2q}\right]^{1/n}$$

$$= \left[\frac{q+h+q}{2q}\right]^{1/n}$$

$$= \left[\frac{2q+h}{2q}\right]^{1/n} = \left[\frac{2q}{2q} + \frac{h}{2q}\right]^{1/n}$$

$$= \left[1 + \frac{h}{2q}\right]^{1/n} = 1 + \frac{h}{2nq} \qquad \ldots\ldots\ldots (2)$$

By (1) and (2)

L.H.S. = R.H.S.

Q.8 Show that $\left[\frac{n}{2(n+N)}\right]^{1/2} = \frac{8n}{9n-N} - \frac{n+N}{4n}$ where n and N are nearly equal.

Solution:

Since, $N \approx n$

$\Rightarrow$ $N = n + h$, where 'h' is so small that its square and higher powers can be neglected.

$$\text{L.H.S.} = \left[\frac{n}{2(n+N)}\right]^{1/2}$$

$$= \left[\frac{n}{2(n+n+h)}\right]^{1/2} = \left[\frac{n}{2(2n+h)}\right]^{1/2} = \left[\frac{n}{4n+2h}\right]^{1/2}$$

$$= \left[\frac{n}{4n\left(1+\frac{2h}{4n}\right)}\right]^{1/2} = \left[\frac{1}{4^{1/2}\left(1+\frac{2h}{4n}\right)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{4}}\left[1+\frac{2h}{4n}\right]^{-1/2} = \frac{1}{2}\left[1-\frac{2h}{8n}\right]$$

$$= \frac{1}{2}\left[1-\frac{h}{4n}\right] \qquad \ldots\ldots\ldots (1)$$

$$\text{R.H.S.} = \frac{8n}{9n - n} - \frac{n + N}{4n}$$

$$= \frac{8n}{9n - (n + h)} - \frac{(n + n + h)}{4n}$$

$$= \frac{8n}{9n - n - h} - \frac{2n + h}{4n}$$

$$= \frac{8n}{8n - h} - \frac{2n + h}{4n}$$

$$= \frac{8n}{8n\left(1 - \frac{h}{8n}\right)} - \frac{2n + h}{4n}$$

$$= \left(1 - \frac{h}{8n}\right)^{-1} - \frac{2n + h}{4n}$$

$$\approx 1 + \frac{h}{8n} - \left(\frac{2n + h}{4n}\right)$$

$$\approx \frac{8n + h - 4n - 2h}{8n}$$

$$\approx \frac{4n - h}{8n} \approx \frac{4n}{8n} - \frac{h}{8n} \approx \frac{1}{2} - \frac{h}{8n}$$

$$\approx \frac{1}{2}\left[1 - \frac{h}{4n}\right] \qquad \ldots\ldots\ldots (2)$$

From (1) and (2), we have

L.H.S. = R.H.S.

Q.9 Identify the following series as binomial expansion and find the sum in ease.

(i) $1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1 \cdot 3}{2! \cdot 4}\left(\frac{1}{4}\right)^2 - \frac{1 \cdot 3 \cdot 5}{3! \cdot 8}\left(\frac{1}{4}\right)^3 + \ldots\ldots$

(ii) $1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{2}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{2}\right)^3 + \ldots\ldots$

(iii) $1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \ldots\ldots$

(iv) $1 - \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{3}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{3}\right)^3 + \ldots\ldots$

Solution:

(i) $$\mathbf{1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1\,.\,3}{2!\,.\,4}\left(\frac{1}{4}\right)^2 - \frac{1\,.\,3\,.\,5}{3!\,.\,8}\left(\frac{1}{4}\right)^3 + \ldots\ldots}$$

Let $(1 + x)^n = 1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1.3}{2!\,4}\left(\frac{1}{4}\right)^2 - \ldots\ldots$

Also, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$

Now, comparing term by term of the above two equations, we have

$$nx = \frac{-1}{8} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{128} \qquad \ldots\ldots\ldots (2)$$

$$\therefore \quad \frac{1.3}{2!\,4}\left(\frac{1}{4}\right)^2 = \frac{1.3}{2\,.\,1\,.\,4}\,\frac{1}{16} = \frac{3}{128}$$

By (1), we have

$$x = \frac{-1}{8n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\cdot\frac{1}{64\,n^2} = \frac{3}{128}$$

$$\frac{n-1}{128n} = \frac{3}{128}$$

Multiplying both sides by 128

$$\frac{n-1}{n} = 3 \quad \Rightarrow \quad n - 1 = 3n$$

$$\Rightarrow \quad -1 = 2n$$

$$\Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting value of n in (3)

$$x = -\frac{1}{48\left(\frac{-1}{2}\right)}$$

$$\Rightarrow \quad \boxed{x = \frac{1}{4}}$$

Now, putting the values of x and n in,

$$(1+x)^n = \left(1+\frac{1}{4}\right)^{-1/2} = \left(\frac{5}{4}\right)^{-1/2}$$

$$\text{Required sum} = \left(\frac{4}{5}\right)^{1/2} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

(ii) $$\mathbf{1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1\,.\,3}{2\,.\,4}\left(\frac{1}{2}\right)^2-\frac{1\,.\,3\,.\,5}{2\,.\,4.\,6}\left(\frac{1}{2}\right)^3+.....}$$

$$\text{Let } (1+x)^n = 1-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\frac{1\,.\,3}{2\,.\,4}\left(\frac{1}{2}\right)^2-\ldots\ldots.$$

$$\text{Also, } (1+x)^n = 1+nx+\frac{n\,(n-1)}{2!}x^2+\ldots\ldots.$$

Now comparing term by term of the above two equations, we have

$$nx = -\frac{1}{4} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n\,(n-1)}{2!}x^2 = \frac{3}{32} \qquad \ldots\ldots\ldots (2)$$

$$\therefore \qquad \frac{1\,.\,3}{2\,.\,4}\,\frac{1}{4} = \frac{3}{32}$$

By (1), we have

$$x = -\frac{1}{4n} \qquad \ldots\ldots\ldots (3)$$

[Putting the value of x in (2)]

$$\frac{n\,(n-1)}{2!}\,\frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\Rightarrow \qquad \frac{n-1}{n} = 3 \qquad (\because \text{ since multiply both sides by } 32)$$

$$\Rightarrow \qquad n-1 = 3n$$

$$\Rightarrow \qquad -1 = 3n-n$$

$$-1 = 2n$$

$$\Rightarrow \qquad \boxed{n = \frac{-1}{2}}$$

Putting the value of n in (3)

$$x = -\frac{1}{2+\left(-\frac{1}{2}\right)}$$

$$\boxed{x = \frac{1}{2}}$$

Now, putting the values of x and n in

$$(1+x)^n = \left(1+\frac{1}{2}\right)^{-1/2} = \left(\frac{3}{2}\right)^{-1/2} = \left(\frac{2}{3}\right)^{1/2} = \sqrt{\frac{2}{3}}, \text{ required sum}$$

(iii) $\mathbf{1+\frac{3}{4}+\frac{3\,.\,5}{4\,.\,8}+\frac{3\,.\,5\,.\,7}{4\,.\,8\,.\,12}+\ldots\ldots}$

Let $(1+x)^n = 1+\frac{3}{4}+\frac{3\,.\,5}{4\,.\,8}+\ldots\ldots$

Also, $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots$

Now comparing term by term above two equations, we have

$$nx = \frac{3}{4} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{15}{32} \qquad \ldots\ldots\ldots (2)$$

By (1), we have

$$x = \frac{3}{4n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\,\frac{9}{16n^2} = \frac{15}{32}$$

$$\frac{(n-1)\,9}{32n} = \frac{15}{32}$$

$$\frac{(n-1)\,9}{n} = 15 \qquad (\because \text{ multiplying both sides by } 32)$$

$$9n-9 = 15n$$

$$-9 = 15n-9n$$

$$-9 = 6n \quad \Rightarrow \quad n = \frac{-9}{6} \quad \Rightarrow \quad \boxed{n = \frac{-3}{2}}$$

Putting the value of n in (3).

$$x = \frac{3}{4\left(\frac{-3}{2}\right)}$$

$$\Rightarrow \quad \boxed{x = \frac{-1}{2}}$$

Now putting the values of x and n in,

$$(1+x)^n = \left(1-\frac{1}{2}\right)^{-3/2} = \left(\frac{1}{2}\right)^{-3/2} = (2)^{3/2} = 2\,.\sqrt{2}$$

(iv) $\mathbf{1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1\,.\,3}{2\,.\,4}\left(\frac{1}{3}\right)^2 - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(\frac{1}{3}\right)^3 + \ldots\ldots}$

Let $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots.$

Now comparing term by term of the above two equations, we have

$$nx = \frac{-1}{6} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{8}\,\frac{1}{9}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1}{24} \qquad \ldots\ldots\ldots (2)$$

From (1) $\quad x = \frac{-1}{6n}$ Put in

$$\frac{n^2-n}{2}\left(\frac{1}{36\,n^2}\right) = \frac{1}{24}$$

$$\frac{n(n-1)}{2} \times \frac{1}{36\,n^2} = \frac{1}{24}$$

$$\frac{n-1}{72n} = \frac{1}{24}$$

$$24n - 24 = 72n$$

$$-24 = 48n$$

$$\boxed{n = \frac{-1}{2}} \qquad \text{Put in (1)}$$

$$x = \frac{-1}{6} \times \frac{-2}{1}$$

$$x = \frac{1}{3}$$

Hence $\left(1 + \frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}$

Q.10 Use binomial theorem to show that, $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots\ldots = \sqrt{2}$

Solution:

L.H.S.

Let $(1 + x)^n = 1 + \frac{1}{4} + \frac{1\,.\,3}{4\,.\,8} + \ldots\ldots$

Also, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$

Now comparing term by term the above two equations, we have

$$nx = \frac{1}{4} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{32} \qquad \ldots\ldots\ldots (2)$$

By (1), we have

$$x = \frac{1}{4n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\,\frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\frac{n-1}{n} = 3 \qquad (\because \text{ multiplying both sides by 32})$$

$$n - 1 = 3n$$

$$-1 = 3n - n$$

$$-1 = 2n \qquad \Rightarrow \qquad \boxed{n = \frac{-1}{2}}$$

Putting the values of n and x in (3)

$$(1+x)^n = \left(1-\frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = (2)^{1/2} = \sqrt{2} \text{ R.H.S.}$$

Hence proved.

Q.11 If $y = \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^2 + \ldots\ldots$ then prove that $y^2 + 2y - 2 = 0$

Solution:

By adding '1' on both sides,

$$1 + y = 1 + \frac{1}{3} + \frac{1\,.\,3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1\,.\,3\,.\,5}{3!}\left(\frac{1}{3}\right)^2 + \ldots\ldots$$

Let $(1+x)^n = 1 + \frac{1}{3} + \frac{1\,.\,3}{2!}\left(\frac{1}{3}\right)^2 + \ldots\ldots$

Also, $(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2!}x^2 + \ldots\ldots$

Now comparing term by term above two equations, we have

$$nx = \frac{1}{3} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n\,(n-1)\,x^2}{2!} = \frac{1\,.\,3}{2!}\left(\frac{1}{3}\right)^2 = \frac{3}{2}\left(\frac{1}{9}\right) = \frac{3}{18} \qquad \ldots\ldots\ldots (2)$$

By (1) we have,

$$x = \frac{1}{3n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n\,(n-1)}{2}\left(\frac{1}{3n}\right)^2 = \frac{3}{18}$$

$$\frac{n\,(n-1)}{2} \times \frac{1}{9n^2} = \frac{3}{18}$$

$$\frac{n-1}{18n} = \frac{3}{18}$$

$$\frac{n-1}{n} = 3 \qquad (\because \text{ multiplying both sides by } 18)$$

$$n - 1 = 3n$$

$$\Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the value of n in (3)

$$x = \frac{1}{3\left(\frac{-1}{2}\right)} = \frac{1}{\frac{-3}{2}} = \frac{-2}{3}$$

$$\Rightarrow \quad \boxed{x = \frac{-2}{3}}$$

Now, putting the values of x and n in,

$$(1 + x)^n = \left(1 + \left(\frac{-2}{3}\right)\right)^{-1/2} = \left(1 - \frac{2}{3}\right)^{-1/2}$$

$$(1 + y) = \left(\frac{1}{3}\right)^{-1/2} = \sqrt{3}$$

Taking square on both sides,

$$(1 + y)^2 = (\sqrt{3})^2$$

$$1 + y^2 + 2y = 3$$

$$y^2 + 2y + 1 = 3$$

$$\Rightarrow \quad y^2 + 2y + 1 - 3 = 0$$

$$\Rightarrow \quad y^2 + 2y - 2 = 0$$

Hence proved.

Q.12 If $2y = \frac{1}{2^2} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3.5}{3!}\left(\frac{1}{2^6}\right) + \ldots\ldots\ldots$ then prove that $4y^2 + 4y - 1 = 0.$

(Lahore Board 2006)

Solution:

By adding '1' on both sides

$$1 + 2y = 1 + \frac{1}{2^2} + \frac{1\,.\,3}{2!}\ \frac{1}{2^4} + \frac{1\,.\,3\,.\,5}{3!}\ \frac{1}{2^6} + \ldots\ldots.$$

$$\text{Let } (1 + x)^n = 1 + \frac{1}{2^2} + \frac{1\,.\,3}{2!}\ \frac{1}{2^4} + \ldots\ldots.$$

Also, $(1 + x)^n = 1 + nx + \frac{n\,(n - 1)}{2!}x^2 + \ldots\ldots.$

Now, comparing term by term of the above two equations, we have

$$nx = \frac{1}{2^2} = \frac{1}{4} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1\,.\,3}{2!}\,\frac{1}{2^4} = \frac{3}{2}\times\frac{1}{16} = \frac{3}{32} \qquad \ldots\ldots\ldots (2)$$

By (1) we have,

$$x = \frac{1}{4n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2 = \frac{3}{32}$$

$$\frac{n(n-1)}{2!}\times\frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{2}\,\frac{1}{16n} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\frac{n-1}{n} = 3 \qquad (\because \text{ multiplying both sides by } 32)$$

$$\frac{n-1}{n} = 3$$

$$\Rightarrow \quad n-1 = 3n$$

$$\Rightarrow \quad \boxed{n = -\frac{1}{2}}$$

Putting the value of n in (3)

$$x = \frac{1}{2^{4(-1/2)}}$$

$$\Rightarrow \quad \boxed{x = \frac{-1}{2}}$$

Now putting the values of x and n in

$$(1+x)^n = \left(1+\left(\frac{-1}{2}\right)\right)^{-1/2}$$

$$(1 + 2y) = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2}$$

$$(1 + 2y) = \sqrt{2}$$

Taking square on both sides

$$(1 + 2y)^2 = (\sqrt{2})^2$$

$$1 + 4y^2 + 4y = 2$$

$$\Rightarrow \quad 4y^2 + 4y + 1 - 2 = 0$$

$$\Rightarrow \quad 4y^2 + 4y - 1 = 0$$

Hence proved.

Q.13 If $y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots.$ then prove that $y^2 + 2y - 4 = 0$.

(Gujranwala Board 2003)

Solution:

By adding '1' on both sides,

$$1 + y = 1 + \frac{2}{5} + \frac{1\,.\,3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1\,.\,3\,.\,5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots.$$

$$\text{Let } (1 + x)^n = 1 + \frac{2}{5} + \frac{1\,.\,3}{2!}\left(\frac{2}{5}\right)^2 + \ldots\ldots.$$

$$\text{Also, } (1 + x)^n = 1 + nx + \frac{n\,(n-1)}{2!}x^2 + \ldots\ldots..$$

Now, comparing term by term of the above two equations we have

$$nx = \frac{2}{5} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n\,(n-1)}{2!}x^2 = \frac{1\,.\,3}{2!}\left(\frac{2}{5}\right)^5 = \frac{3}{2} \times \frac{4}{25} = \frac{6}{25} \qquad \ldots\ldots\ldots (2)$$

By (1) we have,

$$x = \frac{2}{5n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n\,(n-1)}{2} \times \left(\frac{2}{5n}\right)^2 = \frac{6}{25}$$

$$\frac{n(n-1)}{2} \times \frac{4}{25n^2} = \frac{6}{25}$$

$$\frac{n-1}{2} \times \frac{4}{25} = \frac{6}{25}$$

$$\frac{2(n-1)}{25n} = \frac{6}{25}$$

$$\frac{2(n-1)}{n} = 6 \qquad (\because \text{ multiplying both sides by } 25)$$

$$2(n-1) = 6n$$

$$n-1 = 3n \quad \Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the value of n in (3)

$$x = \frac{2}{5\left(\frac{-1}{2}\right)} = \frac{2}{\frac{-5}{2}}$$

$$\boxed{x = \frac{-4}{5}}$$

Now, putting the values of x and n in,

$$(1+x)^n = \left(1 - \frac{4}{5}\right)^{-1/2}$$

$$(1+y) = \left(\frac{1}{5}\right)^{-1/2}$$

$$(1+y) = \sqrt{5}$$

Taking square on both sides,

$$(1+y)^2 = \left(\sqrt{5}\right)^2$$

$$1 + y^2 + 2y = 5$$

$$\Rightarrow \quad y^2 + 2y - 4 = 0$$

Hence proved.