$\Rightarrow \quad d^2 + 14d - 51 + 36 = 0$

$\Rightarrow \quad d^2 + 14d - 15 = 0$

$\Rightarrow \quad d^2 + 15d - d - 15 = 0$

$\Rightarrow \quad d(d+15) - 1(d+15) = 0$

$\Rightarrow \quad (d-1)(d+15) = 0$

$\Rightarrow \quad d = 1 \quad \text{or} \quad d = -15$

when $a = 2, \quad d = 1$

$a - d = 2 - 1 = 1$

$a = 2$

$a + d = 2 + 1 = 3$

when $a = 2, \quad d = -15$

$a - d = 2 - (-15) = 2 + 15 = 17$

$a = 2$

$a + d = 2 + (-15) = 2 - 15 = -13$

so the required numbers are 1, 2, 3 or 17, 2, – 13

GEOMETRIC MEANS

A number G is said to be a geometric means (G.M) between two numbers a and b if a, G, b are in G.P. therefore

$$\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \pm\sqrt{ab}$$

EXERCISE 6.7

Q.1 Find G.M. between

(i) – 2 and 8 **(Lahore Board 2007)**

(ii) – 2i and 8i **(Gujranwala Board 2007, Lahore Board 2008)**

Solution:

(i) – 2 and 8

Let $a = -2$ and $b = 8$

as $\quad G.M. = \pm\sqrt{ab}$

$$= \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16} = \pm\sqrt{-1}\sqrt{16} = \pm 4i$$

(ii) – 2i and 8i

Let $a = -2i, \; b = 8i$

as $\quad$ $G.M. = \pm\sqrt{ab}$

$$= \pm\sqrt{(-2i)(8i)} = \pm\sqrt{-16i^2} = \pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$$

Q.2 Insert two G.M's between

(i) 1 and 8 **(Gujranwala Board 2007)**

(ii) 2 and 16 **(Lahore Board 2011)**

Solution:

(i) 1 and 8

Let the required G.Ms are G_1, G_2

$\Rightarrow$ $1, G_1, G_2, 8$ are in G.P.

Here

$a = 1, \quad a_4 = 8$

$\Rightarrow \; a_1r^3 = 8$

$\Rightarrow \; (1)\,r^3 = 8 \qquad$ ¤ $\; r_1 = 1$

$\Rightarrow \; r^3 = 8 \;\Rightarrow\; r = 2$

$G_1 = a_2 = a_1r = (1)(2) = 2$

$G_2 = a_3 = a_1r^2 = (1)(2)^2 = 4$

$\Rightarrow$ 2, 4 are required G.M's between 1 and 8

(ii) 2 and 16

Let G_1, G_2 are required G.Ms

then $2, G_1, G_2, 16$ are in G.P.

Here $a_1 = 2, \quad$ and $\quad a_4 = 16$

$\Rightarrow \; a_1r^3 = 16$

$\Rightarrow \; 2\,r^3 = 16 \qquad$ ¤ $\; a_1 = 2$

$\Rightarrow \; r^3 = 8$

$\Rightarrow \; r = 2$

Now $G_1 = a_2 = a_1r = (2)(2) = 4$

$G_2 = a_3 = a_1r^2 = (2)(2)^2 = 8$

$\Rightarrow$ 4, 8 are required G.M's between 2 and 16

Q.3 Insert three G.M's between

(i) 1 and 16 **(Lahore Board 2004)**

(ii) 2 and 32

Solution:

(i) 1 and 16

Let the required G.Ms are G_1, G_2, G_3

then

$1, G_1, G_2, G_3, 16$ are in G.P

$a = 1, \quad a_5 = 16,$

$a_1\,r^4 = 16$

$(1)\, r^4 = 16$ ¤ $a_1 = 1$

$r^4 = 16$

$r^2 = \pm 4$

when $r^2 = 4 \Rightarrow r = \pm\sqrt{4} = \pm 2$

when $r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = \pm 2i$

when $r = 2,\quad a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(2) = 2$

$G_2 = a_3 = a_1 r^2 = (1)(2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (1)(2)^3 = 8$

when $r = -2,\quad a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(-2) = -2$

$G_2 = a_3 = a_1 r^2 = (1)(-2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (1)(-2)^3 = -8$

when $r = 2i,\quad a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(2i) = 2i$

$G_2 = a_3 = a_1 r^2 = (1)(2i)^2 = 4i^2 = -4$

$G_3 = a_4 = a_1 r^3 = (1)(2i)^3 = 8i^3 = 8\, i^2\, i = -8i$

when $r = -2i,\quad a_1 = 1$

$G_1 = a_2 = a_1 r = (1)(-2i) = -2i$

$G_2 = a_3 = a_1 r^2 = (1)(-2i)^2 = 4i^2 = -4$

$G_3 = a_4 = a_1 r^3 = (1)(-2i)^3 = -8i^3 = -8\, i^2\, i = 8i$

(ii) 2 and 32

Let G_1, G_2, G_3 are required G.Ms,

then $2, G, G_2, G_3, 32$ are in G.P

Here $a_1 = 2,\quad a_5 = 32,$

$a_1 r^4 = 32$

$2 r^4 = 32$ ¤ $a_1 = 2$

$r^4 = 16$

$r^2 = \pm 4$

when $r^2 = 4 \Rightarrow r = \pm\sqrt{4} = \pm 2$

when $r^2 = -4 \Rightarrow r = \pm\sqrt{-4} = \pm 2i$

when $r = 2,\quad a_1 = 2$

$G_1 = a_2 = a_1 r = (2)(2) = 4$

$G_2 = a_3 = a_1r^2 = (2)(2)^2 = 4$

$G_3 = a_4 = a_1 r^3 = (2)(2)^3 = 16$

when $r = -2, \quad a_1 = 2$

$G_1 = a_2 = a_1r = (2)(-2) = -4$

$G_2 = a_3 = a_1r^2 = (2)(-2)^2 = 8$

$G_3 = a_4 = a_1 r^3 = (2)(-2)^3 = -16$

when $r = 2i, \quad a_1 = 2$

$G_1 = a_2 = a_1r = (2)(2i) = 4i$

$G_2 = a_3 = a_1r^2 = (2)(2i)^2 = 8i^2 = -8$

$G_3 = a_4 = a_1 r^3 = (2)(2i)^3 = 16i^3 = 16\, i^2\, i = -16i$

when $r = -2i, \quad a_1 = 2$

$G_1 = a_2 = a_1r = (2)(-2i) = -4i$

$G_2 = a_3 = a_1r^2 = (2)(-2i)^2 = 8i^2 = -8$

$G_3 = a_4 = a_1 r^3 = (2)(-2i)^3 = -16i^3 = -16\, i^2\, i = 16$

Q.4 Insert four real G.M's between 3 and 96. (Gujranwala Board 2006)

Solution:

Let G_1, G_2, G_3, G_4, are G.M's

then $3, G_1, G_2, G_3, G_4, 96$ are in G.P.

Here $a_1 = 3, \quad a_6 = 96$

$$a_1 r^5 = 96$$

$$3r^5 = 96 \qquad ¤\ a_1 = 3$$

$$r^5 = \frac{96}{3} = 32$$

$$\boxed{r = 2}$$

Now $G_1 = a_2 = a_1r = (3)(2) = 4$

$G_2 = a_3 = a_1r^2 = (3)(2)^2 = 12$

$G_3 = a_4 = a_1 r^3 = (3)(2)^3 = 24$

$G_4 = a_5 = a_1 r^4 = (3)(2)^4 = 48$

$\Rightarrow$ 6, 12, 24, 48 are required G.Ms.

Q.5 If both x and y positive distinct real numbers, show that geometric mean between x and y is less than their arithmetic's mean.

Solution:

Let 'G' be the geometric mean between x and y and 'A' be the arithmetic mean between x and y.

$G = \pm\sqrt{xy}$ and $A = \dfrac{x+y}{2}$

We will show that

$G < A$

i.e. $\pm\sqrt{xy} < \dfrac{x+y}{2}$

or $\pm 2\sqrt{xy} < x+y$

or $0 < x+y \mp 2\sqrt{xy}$

or $x+y \mp 2\sqrt{xy} > 0$

or $(\sqrt{x})^2 + (\sqrt{y})^2 \mp 2\sqrt{xy} > 0$

or $\left(\sqrt{x} \mp \sqrt{y}\right)^2 > 0$

which is true as square is always positive.

Hence proved.

Q.6 For what value of n $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b? (Gujranwala Board 2007, Lahore Board 2006)

Solution:

The positive G.M between a and b is given by

$G.M. = \sqrt{ab} = a^{1/2}\, b^{1/2}$

It is given that

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = a^{1/2}\, b^{1/2}$$

$\Rightarrow$ $a^n + b^n = a^{1/2}\, b^{1/2}\,(a^{n-1} + b^{n-1})$

$\Rightarrow$ $a^n + b^n = a^{n-1/2}\, b^{1/2} + a^{1/2}\, b^{n-1/2}$

$\Rightarrow$ $a^n - a^{n-1/2} \,.\, b^{1/2} = a^{1/2}\, b^{n-1/2} - b^n$

$\Rightarrow$ $a^{n-1/2}\,(a^{1/2} - b^{1/2}) = b^{n-1/2}\,(a^{1/2} - b^{1/2})$

$\Rightarrow \quad a^{n-1/2} = b^{n-1/2}$

$\Rightarrow \quad \dfrac{a^{n-1/2}}{b^{n-1/2}} = 1$

$\Rightarrow \quad \left(\dfrac{a}{b}\right)^{n-1/2} = \left(\dfrac{a}{b}\right)^{0}$

$\Rightarrow \quad n - \dfrac{1}{2} = 0 \quad \Rightarrow \quad \boxed{n = \dfrac{1}{2}}$

Q.7 A.M. of two positive integral numbers exceeds their (positive) G.M. by 2 and their sum is 20, find the numbers.

Solution:

Let a and b two positive integral numbers then their G.M and A.M is given by

A.M. $= \dfrac{a+b}{2}$, G.M $= \sqrt{ab}$

By given condition

$\dfrac{a+b}{2} - 2 = \sqrt{ab}$ ……… (1)

also it is given that

$a + b = 20$ ……… (2)

from equation (2)

$a = 20 - b$ ……… (3)

Put (3) in (1), we get

$\dfrac{20 - b + b}{2} - 2 = \sqrt{(20-b)\,b}$

$10 - 2 = \sqrt{20b - b^2}$

$8 = \sqrt{20b - b^2}$

squaring both sides we get

$64 = 20b - b^2$

$\Rightarrow \quad b^2 - 20b + 64 = 0$

$\Rightarrow \quad b^2 - 16b - 4b + 64 = 0$

$\Rightarrow \quad b\,(b - 16) - 4\,(b - 16) = 0$

$\Rightarrow \quad (b - 4)\,(b - 16) = 0$

$\Rightarrow$ $b = 4$ $\quad$ $b = 16$

Put $b = 4$ in equation (3), we get

$a = 20 - 4 = 16$

Put $b = 16$ in equation (3), we get

$a = 20 - 16 = 4$

so the required numbers are

4, 16 or 16, 4

Q.8 The A.M. between two numbers is 5 and their (positive) G.M is 4. Find the numbers.

Solution:

Let a, b are required numbers

then by given conditions

$$\frac{a+b}{2} = 5$$

$\Rightarrow$ $a + b = 10$ ……… (1)

and $\sqrt{ab} = 4$

$\Rightarrow$ $ab = 16$ ……… (2)

from equation (1)

$a = 10 - b$ ……… (3)

Put (3) in (2), we get

$(10 - b)\,b = 16$

$\Rightarrow$ $10b - b^2 = 16$

$\Rightarrow$ $b^2 - 10b + 16 = 0$

$\Rightarrow$ $b^2 - 8b - 2b + 16 = 0$

$\Rightarrow$ $b(b - 8) - 2(b - 8) = 0$

$\Rightarrow$ $(b - 2)(b - 8) = 0$

$\Rightarrow$ $b = 2$ or $b = 8$

Put $b = 2$ in equation (3), we get

$a = 10 - 2 = 8$

Put $b = 8$ in equation (3), we get

$a = 10 - 8 = 2$

so the required numbers are

2, 8 or 8, 2

sum of n terms of a geometric Series

The formulas to find the sum of n terms of a geometric series is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{if } |r| < 1$$

and

$$S_n = \frac{a_1(r^n-1)}{r-1} \quad \text{if } |r| > 1$$

Infinite Geometric Series

The geometric series which has infinite number of terms is called infinite geometric series. For example,

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots\ldots + a_1r^{n-1} + \ldots\ldots$$

is an infinite geometric series.

The formula to find the sum of infinite terms of a geometric series is given by

$$S_\infty = \frac{a_1}{1-r} \quad \text{if } |r| < 1$$

EXERCISE 6.8

Q.1 Find sum of first 15 terms of geometric sequence, $1, \frac{1}{3}, \frac{1}{9}, \ldots\ldots$

Solution:

Given sequence

$$1, \frac{1}{3}, \frac{1}{9}, \ldots\ldots$$

Here $a_1 = 1, \quad r = \dfrac{\frac{1}{3}}{1} = \dfrac{1}{3}, \quad n = 15$

As $S_n = \dfrac{a(1-r^n)}{1-r}$

$$S_n = \frac{1\left[1-\left(\frac{1}{3}\right)^{15}\right]}{1-\frac{1}{3}} = \frac{1-\left(\frac{1}{3}\right)^{15}}{\frac{2}{3}}$$

$$= \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{15}\right] = \frac{3}{2}\left[1-\frac{1}{14348907}\right]$$

$$= \frac{3}{2}\left[\frac{14348907-1}{14348907}\right] = \frac{3}{2}\left[\frac{14348906}{14348907}\right] = \frac{7174453}{4782969}$$