EXERCISE 2.8

Q.1 Operation $\oplus$ performed on the two member set $G = \{0, 1\}$ is shown in the adjoining table.

$\oplus$	0	1
0	0	1
1	1	0

Answer these question

(i) Name identity element if it exists?

(ii) What is the inverse of 1?

(iii) Is the set G under the given operation a group.

(iv) Abelian or non–abelian.

Solution:

Let $G = \{0, 1\}$

(i) '0' is the identity element

(ii) Inverse of 1 is 1

(iii) Yes, it is a group because it satisfied all conditions for a group.

Because $1 + 0 = 0 + 1$

Q.2 The operation $\oplus$ as performed on the set $\{0, 1, 2, 3\}$ is shown in adjoining table. Show that the set is an abelian group.

Solution:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Let $G = \{0, 1, 2, 3\}$

(i) G is closed w.r.t. $\oplus$ because each element in the table belongs to G.

(ii) $\oplus$ is associative in G because $\forall$ $a, b, c \in G$ $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

(iii) '0' is the identity element in G. Because $\forall$ $a \in G$ $a + 0 = a = 0 + a$.

(iv) As inverse of each element of G exists in G, w.r.t. $\oplus$

inverse of $0 = 0$

inverse of $1 = 3$

inverse of $2 = 2$

inverse of $3 = 1$

(v) G satisfies commutative property w.r.t. $\oplus$

i.e. $\forall$ $a, b \in G$

$a \oplus b = b \oplus a$

$\Rightarrow$ 'G' is an Abelian group w.r.t. $\oplus$.

Q.3 For each of the following sets, determine whether or not the set forms a group w.r.t. indicated operation

(i) The set of rational numbers w.r.t. '×'.

(ii) The set of rational numbers w.r.t. '+'.

(iii) The set of positive rational numbers w.r.t. '×'.

(iv) The set of integers w.r.t. '+'.

(v) The set of integers w.r.t. '×'.

Solution:

(i) The set of rational numbers w.r.t. '×'.

The set of rational numbers w.r.t. multiplication is not a group because inverse of 0 does not exists.

(ii) The set of rational numbers w.r.t. '+'.

The set of rational numbers w.r.t. $+$ is a group.

because

c – 1 $\forall\ a, b, \in Q$

$a + b \in Q$

$\Rightarrow$ Q is closed w.r.t. $+$

c – 2 $a, b, c \in Q$

$(a + b) + c = a + (b + c)$

$\Rightarrow$ '+' is associative in Q

c – 3 '0' is the identity element in Q

such that $\forall\ a \in Q\ ,\ a + 0 = 0 + a = a$

c – 4 Inverse of each element in Q belongs to Q.

i.e. $\forall\ a \in Q\ \exists - a \in Q$ such that

$a + (-a) = 0 = (-a) + a$

As all conditions for a group are satisfied so $(Q, +)$ is a group.

(iii) The set of positive rational numbers w.r.t. '×'

The set of positive rational numbers $Q+$ is a group under multiplication. Because

c – 1 $\forall\ a, b \in Q^+$

$a\ .\ b \in Q^+$

$\Rightarrow$ $Q+$ is closed under '•'

$(a + b) + c = a + (b + c)$

$\Rightarrow$ '•' is associative in Q^+.

c – 3 $1 \in Q^+$ and 1 is an identity element in Q^+.

such that $\forall \ a \in Q^+ \ , \ a . 1 = 1 . 1 = a$

c – 4 As $0 \notin Q^+$ and inverse of each element of Q^+ exist in Q^+ i.e.

$\forall \ a \in Q^+ \ , \ \frac{1}{a} \in Q^+$ such that

$$a . \frac{1}{a} = \frac{1}{a} . a = 1$$

As all the conditions for a group are satisfied so Q^+ is a group under multiplication.

(iv) The set of integers w.r.t. '+'.

The set of integers Z w.r.t. + is a group because

c – 1 $\forall \ a, b \in Z$,

'Z' is closed under +.

$(a + b) + c = a + (b + c)$

$\Rightarrow$ '+' is associative.

C – 3 '0' is the identity element in Z.

i.e. $\forall \ a \in Z$

$a + 0 = 0 + a = a$

c – 4 Inverse of each element in Z exists in Z.

i.e. $\forall \ a \in Z \ \exists \ -a \in Z$ such that

$a + (-a) = (-a) + a = 0$

$\Rightarrow$ (Z, +) is a group.

(v) The set of integers w.r.t. 'x'.

The set of integers is not a group under multiplication because inverse of 0 does not exists in Z.

Q.4 Show that the adjoining table represents the sum of the elements of the set {E, 0}. What is an identity element of this set? Show that this set is an abelian group.

$\oplus$	E	0
E	E	0
0	0	E

Solution:

As $E + E = E$

$E + O = O$

$O + E = O$

$O + O = E$

'E' is the identity element in {E, O} {E, O} is an abelian group under +

because

c – 1 all elements in table belongs to $\{E, O\}$

$\Rightarrow$ $\{E, O\}$ is closed under '+'.

c – 2 As associative law holds in $\{E, O\}$

$\Rightarrow$ '+' is associative.

c – 3 'E' is the identity element in $\{E, O\}$

c – 4 inverse of $E = E$

inverse of $O = O$

$\Rightarrow$ inverse of each element in $\{E, O\}$ belongs to $\{E, O\}$

c – 5 As

$E \oplus O = O \oplus E$

$\Rightarrow$ Commutative law holds under '+'.

$\Rightarrow$ $\{E, O\}$ is an abelian group.

Q.5 Show that the set $\{1, w, w^2\}$ when $w^3 = 1$ is an abelian group w.r.t. ordinary multiplication. (Gujranwala Board 2007)

Solution:

Let $G = \{1, \omega, \omega^3\}$

First we construct multiplication table

$\bullet$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Now

c – 1 As all elements in table belong to G.

$\Rightarrow$ 'G' is closed under multiplication.

c – 2 operation '$\bullet$' is associative because

$\forall\ a, b, c, \in G$, $(a . b) . c = a . (b . c)$

c – 3 $1 \in G$ is an identity element

c – 4 inverse of $1 = 1$

inverse of $\omega = \omega^2$

inverse of $\omega^2 = \omega$

$\Rightarrow$ inverse of each element of G exist in G.

c – 5 commutative law holds in G w.r.t. '$\bullet$'

because $\forall\ a, b \in G$ $a . b = b . a$

$\Rightarrow$ $G = \{1, \omega, \omega^2\}$ is an abelian group under multiplication.

Q.6 If G is a group under the operation $*$ and a, b $\in$ G, find the solution of equations: $a * x = b$, $x * a = b$. (Lahore Board 2010)

Solution:

Given

$$a * x = b$$

$$a^{-1} * (a * x) = a^{-1} * b$$

$$(a^{-1} * a) * x = a^{-1} * b \qquad \text{by associative law}$$

$$e * x = a^{-1} * b$$

$$x = a^{-1} * b$$

Now

$$x * a = b$$

$$(x * a) * a^{-1} = b * a^{-1}$$

$$x * (a * a^{-1}) = b * a^{-1} \qquad \text{by associative law}$$

$$x * e = b * a^{-1}$$

$$x = b * a^{-1}$$

which is required solution.

Q.7 Show that set consisting of elements of form $a + \sqrt{3}\, b$ (a, b being rational), is an abelian group w.r.t. addition. (Gujranwala Board, Lahore Board 2007)

Solution:

Let $S = \left\{ a + \sqrt{3}\, b , a , b \in Q \right\}$

c – 1 $\forall \ a + \sqrt{3}\, b ,\ c + \sqrt{3}\, d \in G$

$$\left(a + \sqrt{3}\, b \right) + \left(c + \sqrt{3}\, d \right)$$

$$= a + \sqrt{3} + c + \sqrt{3}\, d = (a + c) + \sqrt{3}\,(b + d) \in G$$

$\Rightarrow$ G is closed w.r.t. addition.

c – 2 '+' is associative in G because

$$\forall \ a + \sqrt{3}\, b,\ c + \sqrt{3}\, d,\ e + \sqrt{3}\, f \in G$$

$$\left[\left(a + \sqrt{3}\, b \right) + \left(c + \sqrt{3}\, d \right) \right] + \left(e + \sqrt{3}\, f \right)$$

$$= \left(a + \sqrt{3}\, b \right) + \left[\left(c + \sqrt{3}\, d \right) + \left(e + \sqrt{3}\, f \right) \right]$$

c – 3 $0 + \sqrt{3}\, 0 \in G$ such that

$\forall \ a+\sqrt{3}\,b \in G$

$\left(0+\sqrt{3}\,0\right)+\left(a+\sqrt{3}\,b\right)$

$= a+\sqrt{3}\,b$

$= \left(a+\sqrt{3}\,b\right)+\left(0+\sqrt{3}\,0\right)$

identity element exists in G.

c – 4 $\forall \ \left(a+\sqrt{3}\,b\right) \in G \ \ \exists \ \left(-a-\sqrt{3}\,b\right) \in G$

such that $\left(a+\sqrt{3}\,b\right) + \left(-a-\sqrt{3}\,b\right) = 0+\sqrt{3}\,0$

$\Rightarrow$ inverse of each element exists in G.

c – 5 $\forall \ \left(a+\sqrt{3}\,b\right), \left(c+\sqrt{3}\,d\right) \in G$

$$\begin{aligned}\left(a+\sqrt{3}\,b\right)+\left(c+\sqrt{3}\,d\right) &= a+\sqrt{3}\,b+c+\sqrt{3}\,d \\ &= a+c+\sqrt{3}\,(b+d) \\ &= \left(c+\sqrt{3}\,d\right)+\left(a+\sqrt{3}\,b\right)\end{aligned}$$

$\Rightarrow$ Commutative law holds in G.

$\Rightarrow$ G is an abelian group w.r.t. addition.

Q.8 Determine whether (P (S), ∗), where ∗ stands for intersection is a semi–group, a monoid or neither. If it is monoid, specify its identity.

Solution:

$(P(S), *)$ is a monoid because

c – 1 $\forall \ A, B \in P(S)$, $A * B \in P(S)$

i.e. $A \cap B \in P(S)$

$\Rightarrow$ $P(S)$ is closed under $*$

c – 2 $\forall \ A, B, C \in P(S)$

$A * (B * C) = (A * B) * C$

i.e. $A \cap (B \cap C) = (A \cap B) \cap C$

$\Rightarrow$ $*$ is associative in $P(S)$

c – 3 'S' is the identity element in $P(S)$

i.e. $\forall \ A \in P(S)$

$A * S = A \cap S = A$

and $S * A = S \cap A = A$

As inverse of each element in $P(S)$ does not exist in $P(S)$.

$\Rightarrow$ $(P(S), *)$ is a semi group and monoid.

Q.9 Complete the following table to obtain a semi–group under $*$.

*	a	b	c
a	**c**	**a**	**b**
b	**a**	**b**	**c**
c	—	—	**a**

Solution:

Let l, m are required elements.

then to obtain a semi group associative law must be satisfy

i..e $(a * a) * a = a * (a * a)$

$c * a = a * c$

$l = b$

also $(a * a) * b = a * (a * b)$

$c * b = a * a$

$m = c$

$\Rightarrow$ b, c are required elements.

Q.10 Prove that all 2 x 2 non–singular matrices over the real field form a non–abelian group under multiplication.

Solution:

$$\text{Let } G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ a, b, c, d \in R,\ ad - bc \neq 0 \right\}$$

c– 1 Let $A, B \in G$

such that

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix},\ B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$A . B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} . \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \in G$$

$\Rightarrow$ G is closed under multiplication.

c – 2 '$\bullet$' is associative in G.

because in matrices, $\forall\ A, B, C \in G$

$(A . B) . C = A . (B . C)$

c – 3 $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ which is an identity element in G such that $\forall\ A \in G$

$A I_2 = A = I_2A.$

$\Rightarrow$ Identity element exists in G.

c – 4 $\forall\ A \in G\ \exists\ A^{-1} \in G$ such that

$A. A^{-1} = I_2 = A^{-1} A$

We can check it as

if

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

then

$$A^{-1} = \frac{\text{adJ } A}{|A|} = \frac{1}{a_1d_1 - b_1c_1}\begin{bmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{d_1}{a_1d_1 - b_1c_1} & \dfrac{-b_1}{a_1d_1 - b_1c_1} \\ \dfrac{-c_1}{a_1d_1 - b_1c_1} & \dfrac{a}{a_1d_1 - b_1c_1} \end{bmatrix} \in G$$

$\Rightarrow$ Iverse of each element in G exist in G.

c – 5 In matrices, we know that

$\forall \quad A, B \in G$

$A \,.\, B \neq B \,.\, A$

$\Rightarrow$ Commutative law does not hold in G.

$\Rightarrow$ 'G' form a non–Abelian group under multiplication.