Taking square of (i) & (ii) & adding

$r^2(\cos^2\phi + \sin^2\phi) = 1 + 1$

$r^2 = 2 \Rightarrow r = \sqrt{2}$

Dividing (ii) by (i)

$\dfrac{r\sin\phi}{r\cos\phi} = 1$

$\tan\phi = 1$

$\boxed{r\sin(\theta+\phi) = \sqrt{2}\,\sin(\theta+\phi)\ ,\ \tan\phi = 1}$ Ans.

**(vi) $3\sin\theta - 5\cos\theta$**

$r\sin(\theta-\phi) = r\sin\theta\cos\phi - r\cos\theta\sin\phi$

Let $r\cos\phi = 3$ ……… (i)

$r\sin\phi = 5$ ……… (ii)

Taking square & adding of (i) and (ii)

$r^2(\cos^2\phi + \sin^2\phi) = (5)^2 + (3)^2$

$r^2 = 25 + 9$

$r^2 = 34 \Rightarrow r = \sqrt{34}$

$\tan\phi = \dfrac{5}{3}$ (dividing (ii) by (i))

$\boxed{r\sin(\theta-\phi) = \sqrt{34}\sin(\theta-\phi)\ ,\ \tan\phi = \dfrac{5}{3}}$ Ans.

## EXERCISE 10.3

**Q.1 Find the values of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ when**

**(i) $\sin\alpha = \dfrac{12}{13}$ (ii) $\tan\alpha = \dfrac{12}{13}$**

**(iii) $\cos\alpha = \dfrac{3}{5}$ where $0 < \alpha < \dfrac{\pi}{2}$**

**Solution**

**(i) $\sin\alpha = \dfrac{12}{13}$ $0 < \alpha < \dfrac{\pi}{2}$**

$\sin\alpha = \dfrac{12}{13}$

$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\dfrac{144}{169}} = \sqrt{\dfrac{169-144}{169}} = \sqrt{\dfrac{25}{169}}$

$\cos\alpha = \dfrac{5}{13}$ (since $\alpha$ in I Quadrant)

(i) $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \frac{12}{13} \times \frac{5}{13}$

$\boxed{\sin 2\alpha = \frac{120}{169}}$ Ans.

(ii) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= (\cos\alpha)^2 - (\sin\alpha)^2$

$= \frac{25}{169} - \frac{144}{169} = \frac{25-144}{169} = \frac{-119}{169}$ Ans.

$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$

$= \frac{120}{169} \div \frac{-119}{169} = \frac{120}{169} \times \frac{169}{-119}$

$\boxed{\tan 2\alpha = \frac{-120}{119}}$ Ans.

**(ii)** $\mathbf{\cos\alpha = \frac{3}{5}}$

$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\sin\alpha = \frac{4}{5}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\sin 2\alpha = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$ Ans.

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = \frac{9}{25} - \frac{16}{25} = \frac{9-16}{25} = \frac{-7}{25}$ Ans.

$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{24}{25} \div \frac{-7}{25} = \frac{24}{25} \times \frac{-25}{7} = \frac{-24}{7}$ Ans.

Prove the following identities

**Q.2** $\cot\alpha - \tan\alpha = 2\cot 2\alpha.$

**Solution:**

$$\text{R.H.S.} = 2\cot 2\alpha$$

$$= 2\frac{\cos 2\alpha}{\sin 2\alpha} = \frac{2(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha}$$

$$= \frac{\cos^2\alpha}{\sin\alpha\cos\alpha} - \frac{\sin^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \cot\alpha - \tan\alpha = \text{L.H.S.}$$

Hence proved.

**Alternative Method:**

$$\text{L.H.S} = \cot\alpha - \tan\alpha$$

$$= \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{2\cos 2\alpha}{2\sin\alpha\cos\alpha} = \frac{2\cos 2\alpha}{\sin 2\alpha}$$

$$= 2\cot 2\alpha$$

$$= \text{R.H.S}$$

**Q.3** $\dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$ **(Lahore Board 2007, 2010)**

**Solution:**

$$\text{L.H.S.} = \frac{\sin 2\alpha}{1+\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha + \sin^2\alpha + \cos^2\alpha - \sin^2\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{R.H.S.}$$

**Q.4** $\dfrac{1-\cos\alpha}{\sin\alpha} = \tan\dfrac{\alpha}{2}.$

**Solution:**

$$\text{L.H.S.} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$= \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \text{R.H.S.}$$

Hence proved.

**Q.5** $\dfrac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha.$

**Solution:**

$$\begin{aligned}
\text{R.H.S.} &= \sec 2\alpha - \tan 2\alpha \\
&= \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha} \\
&= \frac{1-\sin 2\alpha}{\cos 2\alpha} = \frac{1-(2\sin\alpha\cos\alpha)}{\cos^2\alpha - \sin^2\alpha} \\
&= \frac{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} \\
&= \frac{(\cos\alpha - \sin\alpha)^2}{(\cos\alpha-\sin\alpha)(\cos\alpha+\sin\alpha)} \\
&= \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha} = \text{L.H.S. Hence proved.}
\end{aligned}$$

**Q.6** $\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}} = \dfrac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$

**Solution:**

$$\text{L.H.S.} = \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$$

$$\boxed{\begin{aligned} \because\ \sin 2\alpha &= 2\sin\alpha\cos\alpha \\ \sin\alpha &= 2\sin\alpha/2\cos\alpha/2 \end{aligned}} \quad \boxed{\because\ \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1}$$

$$\begin{aligned}
&= \sqrt{\frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}} \\
&= \sqrt{\frac{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2}} \\
&= \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}} \ \text{R.H.S.} \quad \text{Hence proved.}
\end{aligned}$$

**Q.7** $$\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta} = \cot\frac{\theta}{2}$$

**Solution:**

$$\text{L.H.S.} = \frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta}$$

$$= \left(\frac{1}{\sin\theta} + 2\frac{1}{\sin 2\theta}\right)\cos\theta$$

$$= \left(\frac{1}{\sin\theta} + \frac{2}{2\sin\cos\theta}\right)\cos\theta$$

$$= \left(\frac{\cos\theta + 1}{\sin\theta\cos\theta}\right)\cos\theta$$

$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} = \text{R.H.S.}$$

Hence proved.

**Q.8** **$1 + \tan\alpha\tan 2\alpha = \sec 2\alpha$. (Gujranwala Board 2006, Lahore Board 2006)**

**Solution:**

$$\text{L.H.S.} = 1 + \tan\alpha \,.\, \tan 2\alpha$$

$$= 1 + \frac{\sin\alpha}{\cos\alpha}\,\frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= \frac{\cos2\alpha\cos\alpha + \sin2\alpha\sin\alpha}{\cos\alpha\cos2\alpha}$$

$$= \frac{\cos(2\alpha - \alpha)}{\cos\alpha\cos2\alpha}$$

$$= \frac{\cos\alpha}{\cos\alpha\cos2\alpha}$$

$$= \frac{1}{\cos2\alpha}$$

$$= \sec2\alpha$$

$$= \text{R.H.S}$$

Hence proved.

**Q.9 Prove that $\dfrac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} = \tan 2\theta\tan\theta$**

**(Gujranwala Board 2007, Lahore Board 2006)**

**Solution:**

$$\text{L.H.S.} = \frac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta} = \frac{2\sin\theta\sin2\theta}{\cos\theta+4\cos^3\theta-3\cos\theta}$$

$$= \frac{2\sin\theta\sin2\theta}{4\cos^3\theta-2\cos\theta}$$

$$= \frac{2\sin\theta\sin2\theta}{2\cos\theta\,(2\cos^2\theta-1)}$$

$$= \frac{2\sin\theta\sin2\theta}{2\cos\theta\cos2\theta} = \tan\theta\tan2\theta$$

$$= \text{R.H.S}$$

**Q.10 $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2.$**

**Solution:**

$$\text{L.H.S.} = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{R.H.S.}$$

Hence proved.

**Q.11 $\dfrac{\cos 3\theta}{\cos\theta} + \dfrac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta.$**

**Solution:**

$$\text{L.H.S.} = \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} =$$

$$= \frac{\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\sin(3\theta+\theta)}{\cos\theta\sin\theta} = \frac{\sin 4\theta}{\cos\theta\sin\theta} = \frac{2\sin 2\theta\cos 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\,.\,2\sin\theta\cos\theta\cos 2\theta}{\sin\theta\cos\theta} \qquad (\because\ \sin 2\theta = 2\sin\theta\cos\theta)$$

$$= 4\cos 2\theta = \text{R.H.S.}$$

Hence proved.

**Q.12** $$\frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}} = \sec\theta.$$

**Solution:**

$$\text{L.H.S.} = \frac{\tan\frac{\theta}{2}+\cot\frac{\theta}{2}}{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}$$

$$= \frac{\frac{\sin\theta/2}{\cos\theta/2}+\frac{\cos\theta/2}{\sin\theta/2}}{\frac{\cos\theta/2}{\sin\theta/2}-\frac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\frac{\sin^2\theta/2+\cos^2\theta/2}{\sin\theta/2\cos\theta/2}}{\frac{\cos^2\theta/2-\sin^2\theta/2}{\sin\theta/2\cos\theta/2}}$$

$$= \frac{1}{\cos^2\theta/2-\sin^2\theta/2} = \frac{1}{\cos\theta}$$

$$= \sec\theta$$

$$= \text{R.H.S}$$

**Q.13** $$\frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$$ **(Gujranwala Board 2006)**

**Solution:**

$$\text{L.H.S.} = \frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta}$$

$$= \frac{\sin 3\theta\sin\theta+\cos 3\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos(3\theta-\theta)}{\cos\theta\sin\theta} = \frac{\cos 2\theta}{\cos\theta\sin\theta}$$

multiplying & dividing by 2

$$= 2\frac{\cos 2\theta}{2\sin\theta\cos\theta}$$

$$= 2\frac{\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta = \text{R.H.S.}$$

Hence proved.

**Q.14 Reduce $\sin^4\theta$ to an expression involving only function of multiples of $\theta$, raised to first power. (Lahore Board 2004, 2010)**

**Solution:**

$$\text{Sin}^4\theta = (\sin^2\theta)^2 \qquad \left(\because\ \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \left(\frac{1-\cos 2\theta}{2}\right)^2 \qquad \left(\because\ \cos^2 2\theta = \frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}$$

$$= \frac{1}{4}\left[1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4}\left[\frac{2+4\cos 2\theta+1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4}\left[\frac{3-4\cos 2\theta+\cos 4\theta}{2}\right]$$

$$= \frac{3-4\cos 2\theta+\cos 4\theta}{8}$$

**Q.15 Find values of $\sin\theta$, $\cos\theta$ when $\theta$ is**

**(i) $18^o$ (ii) $36^o$ (iii) $54^o$ (iv) $72^o$ (v) $140^o$**

**Hence prove that $\cos 36^o \cos 72^o \cos 108^o \cos 144^o = \frac{1}{16}$**

**Solution:**

**(i) $\theta = 18^o$**

$5\theta = 5 \times 18^o = 90^o$

Now $3\theta + 2\theta = 90^o \Rightarrow 3\theta = 90^o - 2\theta$

taking cos on both sides

$\cos 3\theta = \cos(90^o - 2\theta) = \cos 90^o \cos 2\theta + \sin 90^o \sin 2\theta$

$\cos 3\theta = \sin 2\theta$

$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta$

$\cos\theta\,[4\cos^2\theta - 3] = 2\sin\theta\cos\theta$

$4(1-\sin^2\theta) - 3 - 2\sin\theta = 0$

$4 - 4\sin^2\theta - 3 - 2\sin\theta = 0$

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$a = 4,\ b = 2,\ c = -1$ By Quadratic formula

$$\sin\theta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4\,(4)\,(-1)}}{2\,(4)} = \frac{-2 \pm \sqrt{4 + 16}}{8}$$

$$\sin\theta = \frac{-2 \pm \sqrt{20}}{8}$$

$$\sin\theta = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

As $18^o$ is in I quadrant so

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

$$\cos 18^o = \sqrt{\frac{10 + 2\sqrt{5}}{16}} = \sqrt{\frac{10 + 2\sqrt{5}}{16}} = \cos 18^o$$

**(ii)** $\boldsymbol{\theta = 36^o}$

$$5 \times \theta = 5 \times 36^o$$

$$5\theta = 180^o$$

$$3\theta + 2\theta = 180^o \Rightarrow 3\theta = 180^o - 2\theta$$

$$\sin 3\theta = \sin(180^o - 2\theta)$$

$$3\sin\theta - 4\sin^3\theta = \sin 2\theta$$

$$\sin\theta\,(3 - 4\sin^2\theta) = 2\sin\theta\cos\theta$$

$$3 - 4\sin^2\theta - 2\cos\theta = 0$$

$$3 - 4\,(1 - \cos^2\theta) - 2\cos\theta = 0$$

$$3 - 4 + 4\cos^2\theta - 2\cos\theta = 0$$

$$4\cos^2\theta - 2\cos\theta - 1 = 0$$

$a = 4$ , $b = -2$ , $c = -1$ By Quadratic formula

$$\cos\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos\theta = \frac{2 \pm \sqrt{4 - 4\,(4)\,(-1)}}{2\,(4)}$$

$$= \frac{2 \pm \sqrt{4 + 16}}{8} = \frac{2 \pm \sqrt{20}}{8} = \frac{2 \pm 2\sqrt{5}}{8}$$

$\cos 36^\circ = \dfrac{1 \pm \sqrt{5}}{4}$

As $36^o$ is in I quadrant so

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\dfrac{1+\sqrt{5}}{4}\right)^2}$

$\sin\theta = \sqrt{\dfrac{10-2\sqrt{5}}{16}}$

$\boxed{\sin 36^o = \sqrt{\dfrac{10-2\sqrt{5}}{16}}}$

**(iii) sin 54$^o$**

$\sin 54^o = \sin(90^o - 36^o)$

$= \cos 36^o \quad \left(\because \sin\left(\dfrac{\pi}{2}-\theta\right) = \cos\theta\right)$

$\boxed{\sin 54^o = \dfrac{\sqrt{5}+1}{4}}$ Ans.

**cos 54$^o$**

$= \cos(90^o - 36^o) = \sin 36^o \quad \left(\because \cos\left(\dfrac{\pi}{2}-\theta\right) = \sin\theta\right)$

$\boxed{\cos 54^o = \sqrt{\dfrac{10-2\sqrt{5}}{16}}}$

**(iv) 72$^o$**

$\sin 72^o = \sin(90^o - 18^o) = \cos 18^o \quad \left(\because \sin\left(\dfrac{\pi}{2}-\theta\right) = \cos\theta\right)$

$= \cos 18^o$

$\boxed{\sin 72^o = \sqrt{\dfrac{10+2\sqrt{5}}{16}}}$

$\cos 72^o = \cos(90^o - 18^o) = \sin 18^o$

$\boxed{\cos 72^o = \dfrac{\sqrt{5}-1}{4}}$

**(v) sin 144$^o$**

$\sin 144^o = \sin((180^o - 36^o) \quad (\because \sin(\pi-\theta) = \sin\theta)$

$$= \sin 36^o$$

$$\boxed{\sin 144^o = \sqrt{\frac{10 - 2\sqrt{5}}{16}}} \quad \text{Ans.}$$

$$\cos 144^o = \cos (180^o - 36^o) = -\cos 36^o$$

$$\boxed{\cos 144^o = -\left(\frac{1 + \sqrt{5}}{4}\right)} \quad \text{Ans.}$$

**Next** $\mathbf{\cos 36^o \,.\, \cos 72^o \,.\, \cos 108^o \,.\, \cos 144^o = \frac{1}{16}}$

$$\begin{aligned}
\text{L.H.S.} &= \cos 36^o \cos 72^o \cos (180^o - 72^o) \cos 144 \\
&= \cos 36^o \cos 72^o (-\cos 72^o) \cos 144^o \\
&= \left(\frac{1 + \sqrt{5}}{4}\right)\left(\frac{\sqrt{5} - 1}{4}\right)\left(-\left(\frac{\sqrt{5} - 1}{4}\right)\right)\left(\frac{1 + \sqrt{5}}{4}\right) \\
&= \left(\frac{1 + \sqrt{5}}{4}\right)^2 \left(\frac{\sqrt{5} - 1}{4}\right)^2 \\
&= \left[\frac{(1 + \sqrt{5})(\sqrt{5} - 1)}{16}\right]^2 \\
&= \left[\frac{(\sqrt{5})^2 - (1)^2}{16}\right]^2 = \left(\frac{5 - 1}{16}\right)^2 \\
&= \left(\frac{4}{16}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{R.H.S.}
\end{aligned}$$

Hence proved.

## EXERCISE 10.4

**Q.1 Express the following product as sums and differences**

**(i)** $\mathbf{2 \sin 3\theta \cos \theta}$ **(Lahore Board 2006)**

**(ii)** $\mathbf{2 \cos 5\theta \sin 3\theta}$

**(iii)** $\mathbf{\sin 5\theta \cos 2\theta}$ **(Gujranwala Board 2004)**

**(iv)** $\mathbf{2 \sin 7\theta \sin 2\theta}$

**(v)** $\mathbf{\cos (x + y) \sin (x - y)}$ **(vi)** $\mathbf{\cos (2x + 30) \cos (2x - 30)}$

**(vii)** $\mathbf{\sin 12^o \sin 46^o}$ **(viii)** $\mathbf{\sin (x + 45^o) \sin (x - 45^o)}$