**Q.14 Show that**

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix} = rI_3$$

**Solution:**

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\phi\,.\cos\phi + 0.0 + (-\sin\phi)(-r\sin\phi) & r\cos\phi\,.\,0 + 0.1 + (-\sin\phi) & r\cos\phi\,.\sin\phi + 0.0 + (-\sin\phi)(r\cos\phi) \\ 0\,.\cos\phi + r\,.\,0 + 0\,.\,(-r\sin\phi) & 0.0 + r.1 + 0.0 & 0\,.\sin\phi + r.0 + 0.r\cos\phi \\ r\sin\phi\,.\cos\phi + 0.0 + \cos\phi\,(-r\sin\phi) & r\sin\phi\,.\,0 + 0.1 + \cos\phi\,.\,0 & r\sin\phi\,.\sin\phi + 0.0 + \cos\phi\,.\,r\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} r\cos^2\phi + 0 + r\sin^2\phi & 0+0+0 & r\cos\phi\sin\phi + 0 - r\cos\phi\sin\phi \\ 0+0-0 & 0+r+0 & 0+0+0 \\ r\sin\phi\cos\phi + 0 - r\sin\phi\cos\phi & 0+0+0 & r\sin^2\phi + 0 + r\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos^2\phi + \sin^2\phi) & 0 & -0 \\ 0 & r & 0 \\ 0 & 0 & r(\sin^2\phi + \cos^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$= r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = rI_3 = \text{R.H.S.}$$

Hence proved.

## EXERCISE 3.2

**Q.1 If $A = [a_{ij}]_{3\times4}$ then show that**
**(i) $I_3A = A$ (ii) $AI_4 = A$**

**Solution:**

Given

$$A = [a_{ij}]_{3\times4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

(i) To show $I_3A = A$ where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Take L.H.S.

$$I_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} 1.a_{11}+0.a_{21}+0.a_{31} & 1.a_{12}+0.a_{22}+0.a_{32} & 1.a_{13}+0.a_{23}+0.a_{33} & 1.a_{14}+0.a_{24}+0.a_{34} \\ 0.a_{11}+1.a_{21}+0.a_{31} & 0.a_{12}+1.a_{22}+0.a_{32} & 0.a_{13}+1.a_{23}+0.a_{33} & 0.a_{14}+1.a_{24}+0.a_{34} \\ 0.a_{11}+0.a_{21}+1.a_{31} & 0.a_{12}+0.a_{22}+1.a_{32} & 0.a_{13}+0.a_{23}+1.a_{33} & 0.a_{14}+0.a_{24}+1.a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 & a_{14}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 & 0+a_{24}+0 \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} & 0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A = \text{R.H.S.}$$

**(ii)** $\mathbf{AI_4 = A}$

To show that

$$AI_4 = A \quad \text{where} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

L.H.S.

$$A\,I_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}+0+0+0 & 0+a_{32}+0+0 & 0+0+a_{33}+0 & 0+0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A \;\; \text{R.H.S.}$$

**Q.2 Find inverse of the following matrices**

**(i)** $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{-2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{5} \end{bmatrix}$

**(iii)** $\begin{bmatrix} \mathbf{2i} & \mathbf{i} \\ \mathbf{i} & \mathbf{-i} \end{bmatrix}$ **(iv)** $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{6} & \mathbf{3} \end{bmatrix}$

**Solution:**

(i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

(i) Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-1)(2) = 3 + 2 = 5 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

**(ii)** $\begin{bmatrix} \mathbf{-2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{5} \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix} = (-2)(5) - (-4)(3) = -10 + 12 = 2 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ \frac{4}{2} & -\frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

**(iii)** $\begin{bmatrix} \mathbf{2i} & \mathbf{i} \\ \mathbf{i} & \mathbf{-i} \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix} = (2i)(-i) - (i)(i) = -2i^2 - i^2 = -2(-1) - (-1) = 2 + 1 = 3$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$$

As $$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{3}\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3}i & -\frac{1}{3}i \\ -\frac{1}{3}i & \frac{2}{3}i \end{bmatrix}$$

**(iv)** $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{6} & \mathbf{3} \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$|A| = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = (2)(3) - (6)(1) = 6 - 6 = 0$

As $|A| = 0 \Rightarrow$ inverse does not exists.

**Q.3 Solve the following system of linear equations.**

**(i)** $\left.\begin{aligned} \mathbf{2x_1 - 3x_2} &= \mathbf{5} \\ \mathbf{5x_1 + x_2} &= \mathbf{4} \end{aligned}\right\}$ **(ii)** $\left.\begin{aligned} \mathbf{4x_1 + 3x_2} &= \mathbf{5} \\ \mathbf{3x_1 - x_2} &= \mathbf{7} \end{aligned}\right\}$ **(iii)** $\left.\begin{aligned} \mathbf{3x - 5x} &= \mathbf{1} \\ \mathbf{-2x + y} &= \mathbf{-3} \end{aligned}\right\}$

**Solution:**

(i) Given

$2x_1 - 3x_2 = 5$

$5x_1 + x_2 = 4$

In matrix form

$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$ ……… (1)

where

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = (2)(1) - (5)(-3) = 2 + 15 = 17 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists

$$\text{Adj } A = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

As

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{17} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$$

Put values in (1)

$$X = \frac{1}{17} \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} (1)(5)+(3)(4) \\ (-5)(5)+(2)(4) \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 5+12 \\ -25+8 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 17 \\ -17 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17}{17} \\ -\frac{17}{17} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow x_1 = 1, \quad x_2 = -1$$

(ii) Given

$4x_1 + 3x_2 = 5$

$3x_1 - x_2 = 7$

In matrix form

$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$ (1)

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; A = \begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = (4)(-1) - (3)(3) = -4 - 9 = -13 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } A = \begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

As, $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

$$X = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} (-1)(5) + (-3)(7) \\ (-3)(5) \ + (4)(7) \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} -5 - 21 \\ -15 + 28 \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} -26 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-26}{-13} \\ -\frac{13}{-13} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \boxed{x_1 = 2} \ , \ \boxed{x_2 = -1}$$

(iii) Given

$3x - 5y = 1$

$-2x + y = -3$

In matrix form

$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$A\,X = B$ (say)

$X = A^{-1}B$ ……… (1)

where

$$A = \begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \ x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} = (3)(1) - (-2)(-5) = 3 - 10 = -7 \neq 0$$

$|A| \neq 0 \Rightarrow$ its inverse exists.

Now

$$\text{Adj } A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

As

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= -\frac{1}{7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

Putting values in equation (1)

$$X = -\frac{1}{7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} (1)(1)+(5)(-3) \\ (2)(1)+(3)(-3) \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} 1 & -15 \\ 2 & -9 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-14}{-7} \\[2mm] \dfrac{-7}{-7} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \boxed{x = 2} \quad , \quad \boxed{y = 1}$$

**Q.4 If** $\mathbf{A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}}$, $\mathbf{B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}}$

**and** $\mathbf{C = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}}$ **then find**

**(i) A – B (ii) B – A (iii) (A – B) – C (iv) A – (B – C)**

**Solution:**

**(i) Given**

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

Then $$A - B = \begin{bmatrix} 1-2 & -1-1 & 2-(-1) \\ 3-1 & 2-3 & 5-4 \\ -1-(-1) & 0-2 & 4-1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

**(ii) B – A**

$$B - A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1-(-1) & -1-2 \\ 1-3 & 3-2 & 4-5 \\ -1-(-1) & 2-0 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$$

**(iii) (A – B) – C**

$$(A-B)-C = \left(\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}\right) - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -1-1 & 2-(-1) \\ 3-1 & 2-3 & 5-4 \\ -1-(-1) & 0-2 & 4-1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-1 & -2-3 & 3-(-2) \\ 2-(-1) & -1-2 & 1-0 \\ 0-3 & -2-4 & 3-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 & 5 \\ 3 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix}$$

**(iv) A – (B – C)**

$$A-(B-C) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \left(\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2-1 & 1-3 & -1-(-2) \\ 1-(-1) & 3-2 & 4-0 \\ -1-3 & 2-4 & 1-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 4 \\ -4 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1-(-2) & 2-1 \\ 3-2 & 2-1 & 5-4 \\ -1-(-4) & 0-(-2) & 4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

**Q.5** **If** $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ **and** $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$ **then show that**

**(i)** $\mathbf{(AB)\,C = A\,(BC)}$ **(ii)** $\mathbf{(A + B)\,C = AC + BC.}$

**Solution:**

**(i)** **To show (AB) C = A (BC)**

L.H.S. = (AB) C

$$(AB)\,C = \left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\right)\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (i)(-i) + (2i)(2i) & (i)(1) + (2i)(i) \\ (1)(-i) + (-i)(2i) & (1)(1) + (-i)(i) \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -i^2 + 4i^2 & i + 2i^2 \\ -i - 2i^2 & 1 - i^2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -(-1) + 4(-1) & i + 2(-1) \\ i - 2(-1) & 1 - (-1) \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} -3 & i - 2 \\ 2 - i & 2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} -3 & i - 2 \\ -i + 2 & 2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (-3)(2i) + (i - 2)(-1) & -(3)(-1) + i\,(i - 2)(i) \\ (2 - i)(2i) + 2(-i) & (2 - i)(-1) + (2)(i) \end{bmatrix}$$

$$= \begin{bmatrix} -6i - i^2 + 2i & 3 + i^2 - 2i \\ 4i + 2i^2 - 2i & -2 + i + 2i \end{bmatrix}$$

$$= \begin{bmatrix} -4i - (-1) & 3 + (-1) - 2i \\ 2i - 2(-1) & -2 + 3i \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 4i & 2 - 2i \\ 2 + 2i & 3i - 2 \end{bmatrix}$$

R.H.S.

$$A\,(BC) = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\left(\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} - \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}\right)$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} (-i)(2i) + (1)(-i) & (-i)(-1) + (1)(i) \\ (2i)(2i) + (i)(-i) & (2i)(-1) + (i)(i) \end{bmatrix}$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -2i^2 - i & i + i \\ 4i^2 - i^2 & -2i + i^2 \end{bmatrix}$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -2(-1) - i & 2i \\ 4(-1) - (-1) & -2i + (-1) \end{bmatrix}$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}$$

$$= \begin{bmatrix} (i)(2-i)+(2i)(-3) & (i)(2i)+(2i)(-2i-1) \\ (1)(2-i)+(-3)(-i) & (1)(2i)+(-i)(-2i-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2i-i^2-6i & 2i^2-4i^2-2i \\ 2-i+3i & 2i+2i^2+i \end{bmatrix}$$

$$= \begin{bmatrix} -4i-(-1) & 2(-1)-4(-1)-2i \\ 2+2i & 2i+2(-1)+i \end{bmatrix}$$

$$= \begin{bmatrix} -4i+1 & 2-2i \\ 2+2i & 3i-2 \end{bmatrix}$$

$\Rightarrow \quad (AB)\,C = A\,(BC)$

**(ii) To show (A + B) C = AC + BC**

L.H.S.

$$(A+B)\,C = \left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\right) \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} i-i & 2i+1 \\ 1+2i & -i+i \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2i+1 \\ 2i+1 & 0 \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (0)(2i)+(1+2i)(-i) & (0)(-1)+(1+2i)\,i \\ (1+2i)(2i)+(0)(-i) & (1+2i)(-1)+(0)(i) \end{bmatrix}$$

$$= \begin{bmatrix} 0-2i^2-i & 0+2i^2+i \\ 4i^2+2i+0 & -2i-1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2(-1)-i & 2(-1)+i \\ 4(-1)+2i & 2i-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-i & -2+i \\ -4+2i & -2i-1 \end{bmatrix}$$

Now R.H.S

$$(AC)+BC = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} (i)(2i)+(2i)(-i) & (i)(-1)+(2i)(i) \\ (1)(2i)+(-i)(-i) & (1)(-1)(-1)+(-i)(i) \end{bmatrix} + \begin{bmatrix} (-i)(2i)+(1)(-i) & (-i)(-1)+(1)(i) \\ (2i)(2i)+(i)(-i) & (2)(i)(-1)+(i)(i) \end{bmatrix}$$

$$= \begin{bmatrix} +2i^2 - 2i^2 & -i + 2i^2 \\ 2i + i^2 & -1 + i^2 \end{bmatrix} + \begin{bmatrix} -2i^2 - i & i + i \\ 4i^2 - i^2 & -2i + i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i + 2(-1) \\ 2i + (-1) & -1 - (-1) \end{bmatrix} + \begin{bmatrix} -2(-1) - i & 2i \\ 4(-1) - (-1) & -2i + (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i - 2 \\ 2i - 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 - i & 2i \\ -3 & -2i - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 2 - i & -i - 2 + 2i \\ 2i - 1 - 3 & 0 - 2i - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - i & i - 2 \\ -4 + 2i & -2i - 1 \end{bmatrix}$$

$\Rightarrow$ $(A + B)C = AC + BC$

Hence proved.

**Q.6 If A and B are square matrices of same order, then explain why in general.**

**(i)** $(A + B)^2 \neq A^2 + 2AB + B^2$

**(ii)** $(A - B)^2 \neq A^2 - 2AB + B^2$

**(iii)** $(A + B)(A - B) \neq A^2 - B^2$

**Solution:**

Let A and B are square matrices of the same order.

(i) As $(A + B)^2 = (A + B).(A + B)$

$= A.A + A.B + B.A + B.B$

$= A^2 + A.B + B.A + B^2$

but in general $AB \neq BA$

$\Rightarrow$ $A.B + B.A \neq 2AB$

$\Rightarrow$ $(A + B)^2 \neq A^2 + 2AB + B^2$

(ii) $(A - B)^2 = (A - B)(A - B)$

$= A.A - A.B - B.A + B.B$

$= A^2 - AB - B.A + B^2$

but in general $AB \neq BA$

$\Rightarrow$ $-BA - BA \neq 2AB$

$\Rightarrow$ $(A - B)^2 \neq A^2 - 2AB + B^2$

(iii) $(A + B)(A - B) = A.A - A.B + B.A - B^2$

$= A^2 - AB + B.A - B^2$

but in general $AB \neq BA \Rightarrow -AB + BA \neq 0$

$\Rightarrow$ $(A + B)(A - B) \neq A^2 - B^2$

**Q.7 If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$, then find $AA^t$ and $A^tA$.**

**Solution:**

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

Then $$A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$A.A^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(2)+(-1)(-1)+(3)(3)+(0)(0) & (2)(1)+(-1)(0)+(3)(4)+(0)(-2) & 2(-3)+(-1)(5)+(3)(2)+(0)(-1) \\ (1)(2)+(0)(-1)+(4)(3)+(-2)(0) & (1)(1)+(0)(0)+(4)(4)+(-2)(-2) & (1)(-3)+(0)(5)+(4)(2)+(-2)(-1) \\ (-3)(2)+(5)(-1)+(2)(3)+(-1)(0) & (-3)(1)+(5)(0)+(2)(4)+(-1)(-2) & (-3)(-3)+(5)(5)+(2)(2)+(-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9+0 & 2+0+12-0 & -6-5+6+0 \\ 2-0+12-0 & 1+0+16+4 & -3+0+8+2 \\ 6-5+6-0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9 & -2+0-15 & 6+4-6 & 0-2+3 \\ -2+0-15 & 1+0+25 & -3+0+10 & 0-0-5 \\ 6+4-6 & -3+0+10 & 9+16+4 & -0-8-2 \\ 0-2+3 & 0+0-5 & 0-8-2 & 0+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -17 & 4 & 1 \\ -17 & 26 & 7 & -5 \\ 4 & 7 & 29 & -10 \\ 1 & -5 & -10 & 5 \end{bmatrix}$$

**Q.8 Solve the following matrix equations for X:**

**(i)** $3x - 2A = B$ **if** $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

**(ii)** $2x - 3A = B$ **if** $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

**Solution:**

**(i) Given**

$$A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$$

$$3X - 2A = B$$

$$3X = B + 2A$$

$$X = \frac{1}{3}(B + 2A)$$

Put values of A and B

$$X = \frac{1}{3}\left(\begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 2+4 & -3+6 & 1-4 \\ 5-2 & 4+2 & -1+10 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{3} & \frac{3}{3} & -\frac{3}{3} \\ \frac{3}{3} & \frac{6}{3} & \frac{9}{3} \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

(ii) Given $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

and

$$2X - 3A = B$$

$$2X = B + 3A$$

$$X = \frac{1}{2}(B + 3A)$$

Put values of A and B

$$X = \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3+3 & -1-3 & 0+6 \\ 4-6 & 2+12 & 1+15 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{2} & -\frac{4}{2} & \frac{6}{2} \\ -\frac{2}{2} & \frac{14}{2} & \frac{16}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 3 \\ -1 & 7 & 8 \end{bmatrix}$$

**Q.9 Solve the following matrix equation for A.**

**(i)** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

**(ii)** $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

**Solution:**

**(i) Given**

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1+2 & -4+3 \\ 3-1 & 6-2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$BA = C$ (say)

$B^{-1}BA = B^{-1}C$

$IA = B^{-1}C$

$A = B^{-1}C$ ……… (1)

Where

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = (4)(2) - (3)(2) = 8 - 6 = 2$$

As $|B| \neq 0 \Rightarrow$ its inverse exists.

$$\text{Adj } B = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

¤ $$B^{-1} = \frac{\text{Adj } B}{|B|} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

Putting this values in (1)

$$A = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (2)(1) + (-3)(2) & (2)(-1) + (-3)(4) \\ (-2)(1) + (4)(2) & (-2)(-1) + (4)(4) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2-6 & -2-12 \\ -2+8 & 2+16 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4 & -14 \\ 6 & 18 \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{4}{2} & -\frac{14}{2} \\ \frac{6}{3} & \frac{18}{2} \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ 3 & 9 \end{bmatrix}$$

is the required matrix.

**(ii)** **Given**

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2-1 & 0+2 \\ -1+3 & 5+1 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$AB = C$ (say)

$ABB^{-1} = CB^{-1}$

$AI = CB^{-1}$

$A = CB^{-1}$ ……… (1)

where

$$B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = (3)(2) - (4)(1) = 6 - 4 = 2$$

$|B| \neq 0 \Rightarrow$ inverse of $B$ exists.

$$\text{Adj } B = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

As

$$B^{-1} = \frac{\text{Adj } B}{|B|} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

Putting values in equation (1)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (1)(2) + (2)(-4) & (1)(-1) + (2)(3) \\ (2)(2) + (6)(-4) & (2)(-1) + (6)(3) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 - 8 & -1 + 6 \\ 4 - 24 & -2 + 18 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -6 & 5 \\ -20 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{2} & \frac{5}{2} \\ -\frac{20}{2} & \frac{16}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & \frac{5}{2} \\ -10 & 8 \end{bmatrix}$$

is the required matrix

## DETERMINANT OF ORDER $n \geq 3$

Consider a square matrix of order 3.

such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{then}$$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

we can expand this determinant by $R_1$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

we can expand $|A|$ by $C_1$ such that

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

## Minor and Cofactor of an Element of A Matrix or Its Determinant

Consider a square matrix $A$ of order 3. then the minor of an element $a_{ij}$, denoted by $M_{ij}$ is the determinant of the $(3 - 1) \times (3 - 1)$ matrix formed by deleting the ith row and jth column of $A$ or $|A|$.

For example if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{then}$$

$$\text{Minor of } a_{11} = M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$\text{Minor of } a_{12} = M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \text{ and so on.}$$

Cofactor of an element $a_{ij}$ denoted by $A_{ij}$ is defined by

$A_{ij} = (-1)^{i+j} . M_{ij}$ where $M_{ij}$ is the minor of $a_{ij}$.

So in above matrix

Cofactor of $a_{11} = A_{11} = (-1)^{1+1} M_{11}$

Cofactor of $a_{13} = A_{13} = (-1)^{1+3} M_{13}$ and so on.

**Note:** In above matrix $A$

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

## Properties of Determinants

(1) For a square matrix $A$, $|A| = |A^t|$.

(2) If in a square matrix $A$, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.

(3) If a square matrix $A$ has two identical rows or two identical columns, then $|A| = 0$.

(4) If all the entries of a row (or a column) of a square matrix $A$ are zero, then $|A| = 0$.

(5) If the entries of a row (or a column) in a square matrix $A$ are multiplied by a number $k \in R$, then the determinant of the resulting matrix is $k|A|$.

(6) If each entry of a row (or a column) of a square matrix consists of two terms then its determinant can be written as the sum of two determinants i.e. if

$$A = \begin{bmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{bmatrix} \text{ then}$$

$$|A| = \begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}.$$

(7) If to each entry of a row (or a column) of a square matrix $A$ is added a non–zero multiple of the corresponding entry of another row (or column) then the determinant of the resulting matrix is $|A|$.

(8) If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

## Adjoint and Inverse of a Square Matrix of Order $n \geq 3$

If $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the matrix of cofactors of $A$ is

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and $$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

## Inverse of Square Matrix of Order $n \geq 3$

Let $A$ be a non singular matrix of order $n$. If there exists a matrix $B$ such that

$AB = BA = I_n$ then $B$ is called the multiplicative inverse of $A$ and is denoted by $A^{-1}$, and order of $A^{-1}$ is $n \times n$. Thus $AA^{-1} = I_n$ and $A^{-1}A = I_n$. If $A$ is a non singular matrix, then

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

# EXERCISE 3.3

**Q.1 Evaluate the following determinants.**

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$ (iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ (vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

**Solution:**

**(i)** $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Expanding the determinant by $R_1$.

$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5\,(2 \times (-1) - 1 \times (-3)) + 2\,(3 \times 2 - (-3) \times (-2)) - 4\,(3 \times 1 - (-2) \times (-1))$

$= 5\,(-2+3) + 2\,(6-6) - 4\,(3-2)$

$= 5\,(1) + 2\,(0) - 4\,(1)$

$= 5 + 0 - 4 = 1$

**(ii)** $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding the determinant by $R_1$.

$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} -2\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5\,(2 \times (-1) - 1 \times (1)) - 2\,((3) \times (-2) - (-2) \times (1)) - 3\,((3) \times (1) - (-2) \times (-1))$

$= 5\,(2-1) - 2\,(-6+2) - 3\,(3-2)$

$= 5\,(1) - 2\,(-4) - 3\,(1)$

$= 5 + 8 - 3 = 10$