Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called symmetric if $A^t = A$.

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called skew symmetric matrix if $A^t = A$

Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ with complex entries is called Hermitian matrix if $(\bar{A})^t = A$

Skew Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ with complex enteries is called skew Hermitian if $(\bar{A})^t = -A$

Echelon Form of a Matrix

A matrix A is called in (row) echelon form if

(i) in each successive non–zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row,

(ii) the first non–zero entry (or leading entry) in each row is 1.

For example

$$\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in echelon form.}$$

Reduced Echelon Form of a Matrix

A matrix A is in reduced echelon form if it is in echelon form and if the first non–zero entry in R_i lies in C_j then all other entries of C_j are zero.

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in reduced echelon form.}$$

Rank of a Matrix

If a matrix A is in reduced echelon form then the number of non–zero rows of matrix A is called the rank of the matrix A.

EXERCISE 3.4

Q.1 If $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$

then show that (A + B) is symmetric.

Solution:

$$A + B = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & -2+1 & 5-2 \\ -2+1 & 3+0 & -1-1 \\ 5-2 & -1-1 & 0+2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

$$(A+B)^t = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}^t$$

$$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix} = A+B$$

$\Rightarrow$ $A+B$ is a symmetric matrix.

Q.2 If $\mathbf{A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}}$ **show that**

(i) $\mathbf{A + A^t}$ **is symmetric** **(ii)** $\mathbf{A - A^t}$ **is skew symmetric**

Solution:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+3 & 0-1 \\ 3+2 & 2+2 & -1+3 \\ -1+0 & 3-1 & 2+2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix} = A + A^t$$

$\Rightarrow$ $(A + A^t)^t = (A + A^t)$

$\Rightarrow$ $A + A^t$ is symmetric.

(ii) $\mathbf{A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}}$

$$A^t = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$(A - A^t) = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} - \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-3 & 0+1 \\ 3-2 & 2-2 & -1-3 \\ -1-0 & 3-(-1) & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$(A - A^t)^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} = -(A - A^t)$$

$\Rightarrow$ $(A - A^t)$ is a skew symmetric.

Q.3 If A is any square matrix of order 3, show that

(i) $A + A^t$ is symmetric

(ii) $(A - A^t)$ is skew–symmetric.

Solution:

(i) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is a square matrix of order 3, then}$$

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{13} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11}+a_{11} & a_{21}+a_{12} & a_{31}+a_{13} \\ a_{12}+a_{21} & a_{22}+a_{22} & a_{32}+a_{23} \\ a_{13}+a_{31} & a_{23}+a_{32} & a_{33}+a_{33} \end{bmatrix} = A + A^t$$

$\Rightarrow$ $(A + A^t)^t = A + A^t$

$\Rightarrow$ $A + A^t$ is symmetric.

(ii) **Let**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is a square matrix of order } 3, \text{ then}$$

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & a_{22}-a_{22} & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & a_{33}-a_{33} \end{bmatrix}$$

$$(A - A^t)^t = \begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & a_{22}-a_{22} & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & a_{33}-a_{33} \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11}-a_{11} & a_{21}-a_{12} & a_{31}-a_{13} \\ a_{12}-a_{21} & a_{22}-a_{22} & a_{32}-a_{23} \\ a_{13}-a_{31} & a_{23}-a_{32} & a_{33}-a_{33} \end{bmatrix}$$

$$= -\begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & a_{22}-a_{22} & a_{23}-a_{32} \\ a_{31}-a_{13} & a_{32}-a_{23} & a_{33}-a_{33} \end{bmatrix}$$

$\Rightarrow$ $(A - A^t)^t = -(A - A^t)$

$\Rightarrow$ $A - A^t$ is a skew symmetric matrix.

Q.4 If the matrices A and B are symmetric and $AB = BA$, show that AB is symmetric.

Solution:

If the matrices A and B are symmetric then

$A^t = A$ .... (1)

and $B^t = B$ .... (2)

also $AB = BA$ .... (3)

To show that AB is symmetric

We will prove that $(AB)^t = AB$

Now take $(AB)^t = B^t A^t$ by definition

$= BA$ from (1) and (2)

$= AB$ from (3)

$\Rightarrow$ $(AB)^t = AB$

$\Rightarrow$ AB is symmetric.

Q.5 Show that AA^t and A^tA are symmetric for any matrix of order 2×3.

Solution:

Consider a matrix A of order 2×3 such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$A\,A^t = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}.a_{11} + a_{12}.a_{12} + a_{13}.a_{13} & a_{11}.a_{21} + a_{12}.a_{22} + a_{13}.a_{23} \\ a_{21}.a_{11} + a_{22}.a_{12} + a_{23}.a_{13} & a_{21}.a_{21} + a_{22}.a_{22} + a_{23}.a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & a_{21}^2 + a_{22}^2 + a_{23}^2 \end{bmatrix}$$

$$(A\,A^t)^t = \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{21}^2 + a_{22}^2 + a_{23}^2 \end{bmatrix}$$

$$(A\,A^t)^t = A\,A^t$$

$\Rightarrow$ $A\,A^t$ is symmetric.

Now $$A^t A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}a_{11}+a_{21}a_{21} & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23} \\ a_{12}a_{11}+a_{22}a_{21} & a_{12}a_{12}+a_{22}a_{22} & a_{12}a_{13}+a_{22}a_{23} \\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}a_{13}+a_{23}a_{23} \end{bmatrix}$$

$$A^t A = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23} \\ a_{12}a_{11}+a_{22}a_{21} & a_{12}^2 + a_{22}^2 & a_{12}a_{13}+a_{22}a_{23} \\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}^2 + a_{23}^2 \end{bmatrix}$$

$$(A^t A)^t = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{12}a_{11} + a_{22}a_{21} & a_{13}a_{11} + a_{23}a_{21} \\ a_{11}a_{12} + a_{21}a_{22} & a_{12}^2 + a_{22}^2 & a_{13}a_{12} + a_{23}a_{22} \\ a_{11}a_{13}+a_{21}a_{23} & a_{12}a_{13} + a_{22}a_{23} & a_{13}^2 + a_{23}^2 \end{bmatrix}$$

$$(A^t A)^t = (A^t A)$$

$\Rightarrow$ $A^t A)$ is symmetric.

Q.6 If

$$\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{1+i} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}, \text{ show that}$$

(i) $\mathbf{A + (\bar{A})^t}$ **is Hermitian** **(ii)** $\mathbf{A - (\bar{A})^t}$ **is skew Hermitian.**

Solution:

(i) $$\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{1+i} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & -1 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -1-1 \end{bmatrix}$$

$$A + (\bar{A})^t = \begin{bmatrix} 0 & 2+i \\ 2-i & -2 \end{bmatrix}$$

$$\overline{A + (\bar{A})^t} = \begin{bmatrix} 0 & 2-i \\ 2+i & -2 \end{bmatrix}$$

$$\left(\overline{A + (\bar{A})^t}\right)^t = \begin{bmatrix} 0 & 2+i \\ 2-i & -2 \end{bmatrix}$$

$$\left(\overline{A + (\bar{A})^t}\right)^t = A + (\bar{A})^t$$

$\Rightarrow$ $A + (\bar{A})^t$ is Hermitian.

(ii) $$\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{1+i} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & -1 \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$A-(\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} i-(-i) & 1+i-1 \\ 1-(1-i) & -1-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} i+i & i \\ 1-1+i & -1+1 \end{bmatrix} \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$A-(\bar{A})^t = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$\overline{A+(\bar{A})^t} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})^t}\right)^t = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$\left(\overline{A+(\bar{A})^t}\right)^t = -(A-(\bar{A})^t)$$

$\Rightarrow$ $A-(\bar{A})^t$ is skew Hermitian.

Q.7 If A is symmetric or skew symmetric show that A^2 is symmetric.

Solution:

Let A is symmetric

$\Rightarrow$ $A^t = A$ ……… (1)

To show that A^2 is symmetric, we will show that $(A^2)^t = A^2$

$\Rightarrow$ Take

$(A^2)^t = (A.\,A)^t = A^t.\,A^t = A\,.\,A = A^2$ from (1)

$(A^2)^t = A^2$

$\Rightarrow$ A^2 is symmetric.

Now let A is skew symmetric

$\Rightarrow$ $A^t = -A$ ……… (2)

To show that $(A^2)^t = A^2$

Take

$(A^2)^t = (A\,A)^t = A^t.\,A^t = (-A)(-A) = A^2$ from (2)

$\Rightarrow$ $(A^2)^t = A^2$

$\Rightarrow$ A^2 is symmetric.

Q.8 If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$ find $A\left(\bar{A}\right)^t$.

Solution:

$$A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$$

$$\left(\bar{A}\right)^t = [1 \quad 1-i \quad -i]$$

$$A\left(\bar{A}\right)^t = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}[1 \quad 1-i \quad -i]$$

$$= \begin{bmatrix} (1)(1) & (1)(1-i) & (1)(-i) \\ (1+i)(1) & (1+i)(1-i) & (1+i)(-i) \\ (i)(1) & (i)(1-i) & (i)(-i) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-(-1) & -i-(-1) \\ i & i-(-1) & -(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & i+1 & 1 \end{bmatrix}$$

Q.9 Find the inverse of the following matrices. Also find their inverses by using row and column operations.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution:

(i) By Adjoint Method

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1\left((-2)(2) - 0\right) - 2\left((0)(2) - (-2)(0)\right) - 3\left((0)(-2) - (-2)(-2)\right)$$

$$= -4 - 2(0) - 3(0-4)$$

$$= -4-3(-4) = -4+12 = 8 \neq 0$$

$$|A| = \neq 0 \Rightarrow A^{-1} \text{ exists.}$$

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} = (-1)^2\big((-2)(2)-(0)(-2)\big) = (1)(-4+0) = -4$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = (-1)^3(0-0) = 0$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} = (-1)^4\big(0-(-2)(-2)\big) = -4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = (-1)^3(4-6) = -1(-2) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = (-1)^4(2-6) = -4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = (-1)^5(-2+4) = -(2) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = (-1)^4(0-6) = 1(-6) -6$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = (-1)^5(0-0) = 0$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = (-1)^6(-2-0) = 1(-2) = -2$$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{8}\begin{bmatrix} -4 & 0 & -4 \\ 2 & -4 & -2 \\ -6 & 0 & -2 \end{bmatrix}^t$$

$$= \frac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{8} & \frac{2}{8} & -\frac{6}{8} \\ \frac{0}{8} & -\frac{4}{8} & \frac{0}{8} \\ -\frac{4}{8} & -\frac{2}{8} & -\frac{2}{8} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

To find A^{-1} by row operations:

$$\left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ -2+2(1) & -2+2(2) & -2+2(-3) & 0+2(1) & 0+2(0) & 1+2(0) \end{array}\right] R_3+2R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & -4 & 2 & 0 & 1 \end{array}\right] \text{By } -\frac{1}{2}R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1+(-2)(0) & 2+(-2)(1) & -3+(-2)(0) & 1+(-2)(0) & 0+(-2)-\frac{1}{2} & 0+(-2)(0) \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0+(-2)(0) & 2+(-2)(1) & -4+(-2)(0) & 2+(-2)(0) & 0+(-2)\left(-\frac{1}{2}\right) & 1+(-2)(0) \end{array}\right] \begin{array}{l} R_1+(-2)R_2 \\ R_3+(-2)R_2 \end{array}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -4 & 2 & 1 & 1 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \text{By } -\frac{1}{4}R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc:ccc} 1 & 0 & -3+3(1) & 1+3\left(-\frac{1}{2}\right) & 1+3\left(\frac{1}{4}\right) & 0+3\left(-\frac{1}{4}\right) \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \text{By } R_1+3R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

To find A^{-1} by using column operations.

Taking I_3 as

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \\ -- & -- & -- \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 2+(-2)(1) & -3+3(1) \\ 0 & -2+(-2)(0) & 0+3(0) \\ -2 & -2+(-2)(-2) & 2+3(0) \\ -- & -- & -- \\ 1 & 0+(-2)(1) & 0+3(1) \\ 0 & 1+(-2)(0) & 0+3(0) \\ 0 & 0+(-2)(0) & 1+3(0) \end{bmatrix} \quad \begin{matrix}\text{By} \\ C_2+(-2)C_1 \\ C_3+3C_1\end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & -4 \\ -- & -- & -- \\ 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -4 \\ -- & -- & -- \\ 1 & 1 & 3 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By } \left(-\frac{1}{2}\right)C_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \\ -- & -- & -- \\ 1 & 1 & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \text{By } \left(-\frac{1}{4}\right) C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2+2(1) & -1+1 & 1 \\ -- & -- & -- \\ 1+2\left(-\frac{3}{4}\right) & 1-\frac{3}{4} & -\frac{3}{4} \\ 0+2(0) & -\frac{1}{2}+0 & 0 \\ 0+2\left(-\frac{1}{4}\right) & 0-\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{matrix} \text{By } C_1 + 2C_3 \\ C_2 + C_3 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -- & -- & -- \\ -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(ii) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \quad |A| = 1(-2-0) + 1(6-1) = -2+5 = 3 \neq 0$

$\Rightarrow \quad |A| = \neq 0 \Rightarrow$ inverse of A exists.

Now $A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = (-1)^2(-2+0) = -2$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^3(0-3) = -(3) = 3$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = (-1)^4(0+1) = 1$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = (-1)^3(4-0) = -4$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = (-1)^4(2+1) = 3$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = (-1)^5(0-2) = 2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = (-1)^4(6-1) = 1(5) = 5$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = (-1)^5(3-0) = -3$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = (-1)^6(-1-0) = -1$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{3}\begin{bmatrix} -2 & 3 & 1 \\ -4 & 3 & 2 \\ 5 & -3 & -1 \end{bmatrix}^t = \frac{1}{3}\begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

To find A^{-1} by using row operations. Take I_3 as

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1-1 & 0-2 & 2+1 & 0-1 & 0-0 & 1-0 \end{array}\right] \quad \text{By } R_3 - R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array}\right] \text{ By } (-1)R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2-2(1) & -1-2(-3) & 1-2(0) & 0-2(-1) & 0-2(0) \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & -2+2(1) & 3+2(-3) & -1+2(0) & 0+2(-1) & 1+2(0) \end{array}\right] \text{ By } \begin{array}{l} R_1-2R_2 \\ R_3+2R_2 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right]$$

$$\underset{\sim}{R} = \left[\begin{array}{ccccccc} 1 & 0 & 5 & : & 1 & 2 & 0 \\ 0 & 1 & -3 & : & 0 & -1 & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \text{ By } \left(-\frac{1}{3}\right)R_3$$

$$\underset{\sim}{R} = \left[\begin{array}{ccccccc} 1 & 0 & 5-5(1) & : & 1-5\left(\frac{1}{3}\right) & 2-5\left(\frac{2}{3}\right) & 0-5\left(-\frac{1}{3}\right) \\ 0 & 1 & -3+3(1) & : & 0+3\left(\frac{1}{3}\right) & -1+3\left(\frac{2}{3}\right) & 0+3\left(-\frac{1}{3}\right) \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \text{By } \begin{array}{l} R_1-5R_3 \\ R_2+3R_3 \end{array}$$

$$\underset{\sim}{R} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & : & -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 0 & : & 1 & 1 & -1 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \left[\begin{array}{ccc} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

To find A^{-1} by using column operations.

Taking I_3 as

$$\left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \\ -- & -- & -- \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \underset{\sim}{C} \left[\begin{array}{ccc} 1 & 2-2(1) & -1+1 \\ 0 & -1-2(0) & 3+0 \\ 1 & 0-2(1) & 2+1 \\ -- & --- & --- \\ 1 & 0-2(1) & 0+1 \\ 0 & 1-2(0) & 0+0 \\ 1 & 0-2(0) & 1+(0) \end{array}\right] \quad \begin{array}{c} \text{By} \\ C_2+(-2)C_1 \\ C_3+C_1 \end{array}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 1 & -2 & 3 \\ -- & -- & -- \\ 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \\ -- & -- & -- \\ 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ By } (-1)\,C_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3-3(1) \\ 1 & 2 & 3-3(2) \\ -- & -- & -- \\ 1 & 2 & 1-3(2) \\ 0 & -1 & 0-3(-1) \\ 0 & 0 & 1-3(0) \end{bmatrix} \text{ By } C_3-3\,C_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -3 \\ -- & -- & -- \\ 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \\ -- & -- & -- \\ 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \qquad \text{By } \left(-\frac{1}{3}\right)C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-1 & 2-2(1) & 1 \\ -- & -- & -- \\ 1-\frac{5}{3} & 2-2\left(\frac{5}{3}\right) & \frac{5}{3} \\ 0+1 & -1-2(-1) & -1 \\ 0+\frac{1}{3} & 0-2\left(-\frac{1}{3}\right) & -\frac{1}{3} \end{bmatrix} \text{ By } \begin{matrix} C_1 - C_3 \\ C_2 - 2C_3 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -- & -- & -- \\ -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

(iii) $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

To find inverse by adjoint method

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 1(1-0) + 3(2-0) + 2(-2-0) = 1 + 6 - 4 = 3 \neq 0$$

$\Rightarrow$ $|A| \neq 0 \Rightarrow$ inverse of A exists.

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = (-1)^2(1-0) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = (-1)^3(2-0) = -2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-2-0) = -2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 2 \\ -1 & 1 \end{vmatrix} = (-1)^3(-3+2) = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)^4(1-0) = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix} = (-1)^5(-1-0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 2 \\ 1 & 0 \end{vmatrix} = (-1)^4(0-2) = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = (-1)^5(0-4) = 4$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = (-1)^6(1+6) = 7$$

As

$$A^{-1} = \frac{\text{AdJ A}}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ 1 & 1 & 1 \\ -2 & 4 & 7 \end{bmatrix}^t$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ -2 & 1 & 4 \\ -2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

To find A^{-1} by using row operations.

Take I_3 as

$$\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 2 & 1 & 0 & : & 0 & 1 & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 2-2(1) & 1-2(-3) & 0-2(2) & : & 0-2(1) & 1-2(0) & 0-2(0) \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \text{By } R_2 - 2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 0 & 7 & -4 & : & -2 & 1 & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & \frac{-4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & -1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \text{By } \left(\frac{1}{7}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3+3(1) & 2+3\left(-\frac{4}{7}\right) & : & 1+3\left(-\frac{2}{7}\right) & 0+3\left(\frac{1}{7}\right) & 0+3(0) \\ 0 & 1 & \frac{-4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & -1+1 & 1-\frac{4}{7} & : & 0-\frac{2}{7} & 0+\frac{1}{7} & 1+0 \end{bmatrix} \text{By } \begin{matrix} R_1+3R_2 \\ R_3+R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{2}{7} & : & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{3}{7} & : & \frac{-2}{7} & \frac{1}{7} & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{2}{7} & : & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & : & \frac{-2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & : & \frac{-2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \text{By } \left(\frac{7}{3}\right)R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{2}{7}-\frac{2}{7}(1) & : & \frac{1}{7}-\frac{2}{7}\left(-\frac{2}{3}\right) & \frac{3}{7}-\frac{2}{7}\left(\frac{1}{3}\right) & 0-\frac{2}{7}\left(\frac{7}{3}\right) \\ 0 & 1 & -\frac{4}{7}+\frac{4}{7}(1) & : & -\frac{2}{7}+\frac{4}{7}\left(-\frac{2}{3}\right) & \frac{1}{7}+\frac{4}{7}\left(\frac{1}{3}\right) & 0+\frac{4}{7}\left(\frac{7}{3}\right) \\ 0 & 0 & 1 & : & -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \text{By } \begin{matrix} R_1-\left(\frac{2}{7}\right)R_3 \\ R_2+\left(\frac{4}{7}\right)R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & : & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & : & -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$

To find A^{-1} by using column operations:

Take I_3 as

$$\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \\ -- & -- & -- \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & -3+3(1) & 2-2(1) \\ 2 & 1+3(2) & 0-2(2) \\ 0 & -1+3(0) & 1-2(0) \\ -- & --- & --- \\ 1 & 0+3(1) & 0-2(1) \\ 0 & 1+3(0) & 0-2(0) \\ 0 & 0+3(0) & 1-2(0) \end{bmatrix} \text{By } \begin{matrix} C_2 + 3C_1 \\ C_3 - 2C_1 \end{matrix}$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \\ -- & -- & -- \\ 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 0 & -\frac{1}{7} & 1 \\ -- & -- & -- \\ 1 & \frac{3}{7} & -2 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{By } \left(\frac{1}{7}\right)C_2$$

$$\underset{\sim}{C}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 2-2\,(1) & 1 & -4+4\,(1) \\ 0-2\left(-\frac{1}{7}\right) & -\frac{1}{7} & 1+4\left(-\frac{1}{7}\right) \\ \hline 1-2\left(+\frac{3}{7}\right) & \frac{3}{7} & -2+4\left(\frac{3}{7}\right) \\ 0-2\left(\frac{1}{7}\right) & \frac{1}{7} & 0+4\left(\frac{1}{7}\right) \\ 0-2\,(0) & 0 & 1+4\,(0) \end{array}\right] \text{By } \begin{array}{l} C_1-2C_2 \\ C_3+4C_2 \end{array}$$

$$\underset{\sim}{C}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \\ \hline \frac{1}{7} & \frac{3}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{4}{7} \\ 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{C}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{7} & -\frac{1}{7} & 1 \\ \hline \frac{1}{7} & \frac{3}{7} & -\frac{2}{3} \\ -\frac{2}{7} & \frac{1}{7} & \frac{4}{3} \\ 0 & 0 & \frac{7}{3} \end{array}\right] \text{By } \left(\frac{7}{3}\right)C_3$$

$$\underset{\sim}{C}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{7}-\frac{2}{7}\,(1) & -\frac{1}{7}+\frac{1}{7}\,(1) & 1 \\ \hline \frac{1}{7}-\frac{2}{7}\left(-\frac{2}{3}\right) & \frac{3}{7}+\frac{1}{7}\left(-\frac{2}{3}\right) & -\frac{2}{3} \\ -\frac{2}{7}-\frac{2}{7}\left(\frac{4}{3}\right) & \frac{1}{7}+\frac{1}{7}\left(\frac{4}{3}\right) & \frac{4}{3} \\ 0-\frac{2}{7}\left(\frac{7}{3}\right) & 0+\frac{1}{7}\left(\frac{7}{3}\right) & \frac{7}{3} \end{array}\right] \text{By } \begin{array}{l} C_1-\left(\frac{2}{7}\right)C_3 \\ C_2+\left(\frac{1}{7}\right)C_3 \end{array}$$

$$\underset{\sim}{C}\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ \hline \frac{1}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3}\end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3}\end{bmatrix}$$

Q.10 Find the rank of the following matrices

(i) $\begin{bmatrix}\mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{1}\\ \mathbf{2} & \mathbf{-6} & \mathbf{5} & \mathbf{1}\\ \mathbf{3} & \mathbf{5} & \mathbf{4} & \mathbf{-3}\end{bmatrix}$ **(ii)** $\begin{bmatrix}\mathbf{1} & \mathbf{-4} & \mathbf{-7}\\ \mathbf{2} & \mathbf{-5} & \mathbf{1}\\ \mathbf{1} & \mathbf{-2} & \mathbf{3}\\ \mathbf{3} & \mathbf{-7} & \mathbf{4}\end{bmatrix}$

(iii) $\begin{bmatrix}\mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{0} & \mathbf{-1}\\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-2}\\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{5}\\ \mathbf{2} & \mathbf{5} & \mathbf{-2} & \mathbf{-3} & \mathbf{3}\end{bmatrix}$

Solution:

(i) $\begin{bmatrix}\mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{1}\\ \mathbf{2} & \mathbf{-6} & \mathbf{5} & \mathbf{1}\\ \mathbf{3} & \mathbf{5} & \mathbf{4} & \mathbf{-3}\end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & 1\\ 2-2(1) & -6-2(-1) & 5-2(2) & 1-2(1)\\ 3-3(1) & 5-3(-6) & 4-3(5) & -3-3(1)\end{bmatrix} \text{By } \begin{matrix}R_2-2R_1\\ R_3-3R_1\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & 1\\ 0 & -4 & 1 & -1\\ 0 & 8 & -2 & -6\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & -1 & 2 & 1\\ 0 & 1 & -\frac{1}{4} & \frac{1}{4}\\ 0 & 8 & -2 & -6\end{bmatrix} \text{ By } \left(\frac{-1}{4}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 8-8(1) & -2-8\left(-\frac{1}{4}\right) & -6-8\left(\frac{1}{4}\right) \end{bmatrix} \text{By } R_3 - 8R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1+1 & 2-\frac{1}{4} & 1+\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix} \text{By } R_1 + R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

since There are three non–zero rows

$\Rightarrow$ Rank = 3

(ii) $\begin{bmatrix} \mathbf{1} & \mathbf{-4} & \mathbf{-7} \\ \mathbf{2} & \mathbf{-5} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-7} & \mathbf{4} \end{bmatrix}$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 2-2(1) & -5-2(-4) & 1-2(-7) \\ 1-1 & -2+4 & 3+7 \\ 3-3(1) & -7-3(-4) & 4-3(-7) \end{bmatrix} \text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \text{By } \left(\frac{1}{3}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4+4(1) & -7+4(5) \\ 0 & 1 & 5 \\ 0 & 2-2(1) & 10-2(5) \\ 0 & 5-5(1) & 25-5(5) \end{bmatrix} \text{ By } \begin{matrix} R_1+4R_2 \\ R_3-2R_2 \\ R_4-5R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are two non–zero rows

$\Rightarrow$ Rank = 2

(iii) $$\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{5} & \mathbf{-2} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & -1 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} \text{ interchanging } R_1 \text{ \& } R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3-3(1) & -1-3(2) & 3-3(-1) & 0-3(-3) & -1-3(-2) \\ 2-2(1) & 3-2(2) & 4-2(-1) & 2-2(-3) & 5-2(-2) \\ 2-2(1) & 5-2(2) & -2-2(-1) & -3-2(-3) & 3-2(-2) \end{bmatrix} \text{ By } \begin{matrix} R_2-3R_1 \\ R_3-2R_1 \\ R_4-2R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5 \end{bmatrix} \text{ Interchanging } R_2 \text{ \& } R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2-2(1) & -1-2(0) & -3-2(3) & -2-2(7) \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1+1 & 6+0 & 8+3 & 9+7 \\ 0 & -7+7(1) & 6+7(0) & 9+7(3) & 5+7(7) \end{bmatrix} \text{ By } \begin{matrix} R_1-2R_2 \\ R_3+R_2 \\ R_4+7R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 6 & 30 & 54 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 30 & 54 \\ 0 & 0 & 6 & 11 & 16 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 6 & 11 & 16 \end{bmatrix} \text{By } \left(\frac{1}{6}\right)R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1+1 & -9+5 & -16+9 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 6-6(1) & 11-6(5) & 16-6(9) \end{bmatrix} \text{By } \begin{matrix} R_1 + R_3 \\ R_4 - 6R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & -19 & -38 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{By } \left(-\frac{1}{19}\right)R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4+4(1) & -7+4(2) \\ 0 & 1 & 0 & 3-3(1) & 7-3(2) \\ 0 & 0 & 1 & 5-5(1) & 9-5(2) \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{By } \begin{matrix} R_1 + 4R_4 \\ R_2 - 3R_4 \\ R_3 - 5R_4 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

There are four non–zero rows

$\Rightarrow$ Rank = 4

EXERCISE 3.5

Q.1 Solve the following systems of linear equations by Cramer rule.

(i)
$$2x + 2y + z = 13$$
$$3x - 2y - 2z = 1$$
$$5x + y - 3z = 2$$

(ii)
$$2x_1 - x_2 + x_3 = 5$$
$$4x_1 + 2x_2 + 3x_3 = 8$$
$$3x_1 - 4x_2 - x_3 = 3$$

(iii)
$$2x_1 - x_2 + x_3 = 8$$
$$x_1 + 2x_2 + 2x_3 = 6$$
$$x_1 - 2x_2 - x_3 = 1$$