NATURE OF THE ROOTS OF A QUADRATIC EQUATION

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The nature of the roots of an equation depends on the value of the expression $b^2 - 4ac$, which is called its Discriminate.

Case 1: If $b^2 - 4ac = 0$ then the roots will be $-\frac{b}{2a}$ and $\frac{-b}{2a}$. So the roots are real and repeated equal.

Case 2: If $b^2 - 4ac < 0$ then $\sqrt{b^2 - 4ac}$ will be imaginary so, the roots are complex and distinct/unequal.

Case 3: If $b^2 - 4ac > 0$ then $\sqrt{b^2 - 4ac}$ will be real. So, the roots are real and distinct/unequal.

NOTE: If $b^2 - 4ac$ is a perfect square then $\sqrt{b^2 - 4ac}$ will be rational, and the roots are rational otherwise irrational.

EXERCISE 4.7

Q.1 Discuss the nature of the roots of the following equations:

Solution:

(i) $\mathbf{4x^2 + 6x + 1 = 0}$

Comparing with $ax^2 + bx + c = 0$

We have $a = 4, \quad b = 6, \quad c = 1$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac = (6)^2 - 4(4)(1) = 36 - 16 = 20 > 0$

$\Rightarrow$ $D > 0$

and 'D' is not a perfect square.

$\Rightarrow$ roots are real unequal and irrational.

(ii) $\mathbf{x^2 - 5x + 6 = 0}$ **(Lahore Board 2005, Gujranwala Board 2007)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 \quad b = -5 \quad c = 6$

$\Rightarrow$ Discriminant $= b^2 - 4ac = (-5)^2 - 4(1)(6) = 25 - 24 = 1 > 0$

$\Rightarrow$ $D > 0$

and D is a perfect square also

$\Rightarrow$ roots are real unequal and rational.

(iii) $\mathbf{2x^2 - 5x + 1 = 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 2 \quad b = -5 \quad c = 1$

$\Rightarrow$ Discriminant = D = $b^2 - 4ac = (-5)^2 - 4(2)(1) = 25 - 8 = 17 > 0$

$\Rightarrow$ $D > 0$ and 'D' is not a perfect square.

$\Rightarrow$ roots are real, unequal and irrational.

(iv) $\mathbf{25x^2 - 30x + 9 = 0}$ **(Lahore Board 2007)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = 25 \quad b = -30 \quad c = 9$

$\Rightarrow$ Discriminant = D = $b^2 - 4ac = (-30)^2 - 4(25)(9) = 900 - 900 = 0$

$\Rightarrow$ $D > 0$ and D is a perfect square also

$\Rightarrow$ roots are real, equal and rational.

Q.2 Show that the roots of the following equations will be real.

Solution:

(i) $\mathbf{x^2 - 2\left(m + \frac{1}{m}\right) + 3 = 0, \quad m \neq 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 \quad b = -2\left(m + \frac{1}{m}\right), \quad c = 3$

Now Discriminant = D = $b^2 - 4ac$

$$= \left[-2\left(m + \frac{1}{m}\right)\right]^2 - 4(1)(3)$$
$$= \left[4\left(m + \frac{1}{m}\right)^2\right] - 12$$
$$= \left[4\left(m^2 + \frac{1}{m^2} + 2\right)\right] - 12$$
$$= 4\left[m^2 + \frac{1}{m^2} + 2 - 3\right]$$
$$= 4\left[m^2 + \frac{1}{m^2} - 1\right]$$
$$= 4\left[m^2 + \frac{1}{m^2} - 2 + 1\right]$$
$$= 4\left[\left(m - \frac{1}{m}\right)^2 + 1\right] > 0$$

As 'D' is always positive because $\left(m - \frac{1}{m}\right)^2 > 0 \quad \forall\ m \in R$ and $m \neq 0$.

$\Rightarrow$ roots are real.

(ii) $\mathbf{(b-c)\,x^2 + (c-a)\,x + (a-b) = 0; \quad a, b, c \in Q}$

$(b-c)\,x^2 + (c-a)\,x + (a-b) = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = b - c, \quad b = c - a, \quad c = a - b$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac$

$$= (c-a)^2 - 4\,(b-c)\,(a-b)$$
$$= c^2 + a^2 - 2ac - 4\,(ab - b^2 - ac + bc)$$
$$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$$
$$= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac$$
$$= (a - 2b + c)^2 > 0$$

As square is always positive.

$\Rightarrow$ 'D' > 0 $\Rightarrow$ roots are real.

Q.3 Show that the roots of the following equations will be rational.

Solution:

(i) $\mathbf{(p+q)\,x^2 - px - q = 0}$ **(Lahore Board 2008)**

Comparing with $ax^2 + bx + c = 0$, we have

$a = p + q, \quad b = -p, \quad c = -q$

$\Rightarrow$ Discriminant $= D = b^2 - 4ac$

$$= (-p)^2 - 4\,(p+q)\,(-q)$$
$$= p^2 - 4q\,(p+q)$$
$$= p^2 + 4pq + 4q^2$$
$$= (p + 2q)^2$$

As 'D' is a perfect square.

$\Rightarrow$ roots are rational.

(ii) $\mathbf{px^2 - (p-q)\,x - q = 0}$

Comparing with $ax^2 + bx + c = 0$, we have

$a = p, \quad b = -(p-q), \quad c = -q$

Now Discriminant $= D = b^2 - 4ac$

$$= \left[(-p-q)\right]^2 - 4\,(p)\,(-q)$$
$$= (p-q)^2 + 4pq$$
$$= p^2 + q^2 - 2pq + 4pq$$
$$= p^2 + q^2 + 2pq$$
$$= (p+q)^2$$

As D is a perfect square

$\Rightarrow$ roots are rational.

Q.4 For what values of m will the roots of the following equations be equal.

Solution:

(i) $(m + 1)x^2 + 2(m + 3)x + m + 8 = 0$ (Lahore Board 2011)

Comparing with $ax^2 + bx + c = 0$, we have

$a = m + 1 \quad b = 2(m + 3) \quad c = m + 8$

$$\begin{aligned}
\text{Discriminant} = D &= b^2 - 4ac \\
&= [2(m + 3)]^2 - 4(m + 1)(m + 8) \\
&= 4(m + 3)^2 - 4(m^2 + 8m + m + 8) \\
&= 4(m^2 + 9 + 6m)^2 - 4(m^2 + 9m + 8) \\
&= 4m^2 + 36 + 24m - 4m^2 - 36m - 32 \\
&= -12m + 4
\end{aligned}$$

But it is given that the roots of the equation are equal.

$\Rightarrow \quad D = 0$

$\Rightarrow \quad -12m + 4 = 0 \quad \Rightarrow \quad 12m = 4 \quad \Rightarrow \quad m = \frac{1}{3}$

(ii) $x^2 - 2(1 + 3m)x + 7(3 + 2m) = 0$ (Lahore Board 2009)

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1, \quad b = -2(1 + 3m), \quad c = 7(3 + 2m)$

$$\begin{aligned}
\text{Discriminant} = D &= b^2 - 4ac \\
&= [-2(1 + 3m)]^2 - 4(1)7(3 + 2m) \\
&= 4(1 + 3m)^2 - 28(3 + 2m) \\
&= 4(1 + 9m^2 + 6m) - 84 - 56m \\
&= 4 + 36m^2 + 24m - 84 - 56m \\
&= 36m^2 - 32m - 80
\end{aligned}$$

But it is given that the roots of the equation are equal

$\Rightarrow \quad D = 0$

$\Rightarrow \quad 36m^2 - 32m - 80 = 0$

$\Rightarrow \quad 4(9m^2 - 8m - 20) = 0$

$\Rightarrow \quad 9m^2 - 8m - 20 = 0$

$\Rightarrow \quad 9m^2 - 18m + 10m - 20 = 0$

$\Rightarrow \quad 9m(m-2) + 10(m-2) = 0$

$\Rightarrow \quad (m-2)(9m+10) = 0$

$\Rightarrow$ Either $m - 2 = 0$ or $9m + 10 = 0$

$\Rightarrow \quad m = 2$ or $m = \dfrac{-10}{9}$

(iii) $(1+m)x^2 - 2(1+3m)x + (1+8m) = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 + m, \quad b = -2(1+3m), \quad c = 1 + 8m$

$$\begin{aligned}
\text{Discriminant} = D &= b^2 - 4ac \\
&= [-2(1+3m)]^2 - 4(1+m)(1+8m) \\
&= 4(1+3m)^2 - 4(1+8m+m+8m^2) \\
&= 4(1+9m^2+6m) - 4(1+9m+8m^2) \\
&= 4 + 36m^2 + 24m - 4 - 36m - 32m^2 \\
&= 4m^2 - 12m
\end{aligned}$$

But it is given that the roots of the equation are equal.

$\Rightarrow \quad D = 0$

$\Rightarrow \quad 4m^2 - 12m = 0$

$\Rightarrow \quad 4m(m-3) = 0$

$\Rightarrow \quad m(m-3) = 0$

$\Rightarrow$ Either $m = 0$ or $m - 3 = 0$

or $m = 3$

Q.5 Show that the roots of $x^2 + (mx+c)^2 = a^2$ will be equal, if $c^2 = a^2(1+m^2)$.

(G.B 2004, 2006, L.B. 2007)

Solution:

Given equation

$x^2 + (mx+c)^2 = a^2$

$\Rightarrow \quad x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$

$\Rightarrow \quad (1+m^2)x^2 + 2mcx + c^2 - a^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = 1 + m^2, \quad b = 2mc, \quad c = c^2 - a^2$

$$\text{Discriminant} = D = b^2 - 4ac$$
$$= (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$$
$$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2)$$
$$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2$$
$$= 4a^2 + 4m^2a^2 - 4c^2$$

Now roots of the given equation will be equal if $D = 0$, i.e.

$$4a^2 + 4m^2a^2 - 4c^2 = 0$$

or $\quad 4(a^2 + m^2a^2 - c^2) = 0$

or $\quad a^2 + m^2a^2 - c^2 = 0$

or $\quad c^2 = a^2 + m^2a^2$

or $\quad c^2 = a^2(1 + m^2)$

Hence proved.

Q.6 Show that the roots of the $(mx + c)^2 = 4ax$ will be equal if $C = \dfrac{a}{m}$; $m \neq 0$.

(Lahore Board 2010)

Solution:

Given equation

$$(mx + c)^2 = 4ax$$

$\Rightarrow \quad m^2x^2 + c^2 + 2mcx - 4ax = 0$

$\Rightarrow \quad m^2x^2 + 2mcx - 4ax + c^2 = 0$

$\Rightarrow \quad m^2x^2 + (2mc - 4a)x + c^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$$a = m^2, \quad b = 2mc - 4a, \quad c = c^2$$

$$\text{Discriminant} = D = b^2 - 4ac$$
$$= (2mc - 4a)^2 - 4(m^2)(c^2)$$
$$= 4m^2c^2 + 16a^2 - 16amc - 4m^2c^2$$
$$= 16a^2 - 16amc$$

roots of the given equation will be equal if $D = 0$ i.e.

$$16a^2 - 16amc = 0$$

or $\quad 16a(a - mc) = 0$

or $\quad a - mc = 0$

or $\quad a = mc$

or $\quad c = \dfrac{a}{m}$

Hence proved.

Q.7 Prove that $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$ will have equal roots,

if $c^2 = a^2m^2 + b^2$; $\quad a \neq b \neq 0$. **(Lahore Board 2008)**

Solution:

Given that

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$\Rightarrow \quad \dfrac{b^2x^2 + a^2(mx+c)^2}{a^2b^2} = 1$

$\Rightarrow \quad b^2x^2 + a^2(m^2x^2 + c^2 + 2mcx) = a^2b^2$

$\Rightarrow \quad b^2x^2 + a^2m^2x^2 + a^2c^2 + 2mca^2x - a^2b^2 = 0$

$\Rightarrow \quad b^2x^2 + a^2m^2x^2 + a^2c^2 + 2mca^2x - a^2b^2 = 0$

$\Rightarrow \quad (b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$

Comparing with $ax^2 + bx + c = 0$, we have

$a = b^2 + a^2m^2, \quad b = 2mca^2, \quad c = a^2c^2 - a^2b^2$

$$\begin{aligned}
\text{Discriminant} = D &= b^2 - 4ac \\
&= (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) \\
&= 4m^2c^2a^4 - 4(a^2b^2c^2 - a^2b^4 + a^4m^2c^2 - a^4b^2m^2) \\
&= 4m^2c^2a^4 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4b^2m^2 \\
&= 4(a^2b^4 - 4a^4b^2m^2 - 4a^2b^2m^2)
\end{aligned}$$

As the roots of the given equation will be equal if $D = 0$

or $\quad 4a^2b^4 + 4a^4b^2m^2 - 4a^2b^2c^2 = 0$

or $\quad 4a^2b^2(b^2 + a^2m^2 - c^2) = 0$

or $\quad b^2 + a^2m^2 - c^2 = 0$

or $\quad c^2 = c^2m^2 + b^2$

Hence proved.

Q.8 Show that the roots of the equation
$(a^2 - bc)\,x^2 + 2\,(b^2 - ca)\,x + c^2 - ab = 0$ will be equal, if either
$a^3 + b^3 + c^3 = 3abc$ or $b = 0$.

Solution:

Given equation

$$(a^2 - bc)\,x^2 + 2\,(b^2 - ca)\,x + c^2 - ab = 0$$

Comparing with $ax^2 + bx + c = 0$, we have

$$a = a^2 - bc, \quad b = 2\,(b^2 - ac), \quad c = c^2 - ab$$

$$
\begin{aligned}
\text{Discriminant} = D &= b^2 - 4ac \\
&= \left[2\,(b^2 - ac)\right]^2 - 4\,(a^2 - bc)\,(c^2 - ab) \\
&= 4\,(b^2 - ac)^2 - 4\,(a^2c^2 - a^3b - bc^3 + ab^2c) \\
&= 4\,(b^4 + a^2c^2 - 2ab^2c) - 4a^2c^2 + 4a^3b + 4bc^3 - 4ab^2c \\
&= 4b^4 + 4a^2c^2 - 8ab^2c - 4a^2c^2 + 4a^3b + 4bc^3 - 4ab^2c \\
&= 4b^4 - 12ab^2c + 4a^3b + 4bc^3
\end{aligned}
$$

As, the roots of the given equation will be equal if $D = 0$. i.e.

$$4b^4 - 12ab^2c + 4a^3b + 4bc^3 = 0$$

or $\quad 4b\,(b^3 - 3abc + a^3 + c^3) = 0$

or $\quad 4b\,(b^3 - 3abc + a^3 + c^3) = 0$

or $\quad b\,(b^3 - 3abc + a^3 + c^3) = 0$

$\Rightarrow \quad$ Either $b = 0$ $\quad$ or $\quad b^3 - 3abc + a^3 + c^3 = 0$

$\quad$ or $\quad a^3 + b^3 + c^3 = 3abc$

Hence proved.

SYSTEM OF TWO EQUATIONS INVOLVING TWO VARIABLES

Here we shall solve the equations in two variables when at least one of them is quadratic.

To determine the value of two variables, we need a pair of equations. Such a pair of equations is called a system of Simultaneous Equations.

To solve One Linear Equation and One Quadratic Equation

If one of the equations is linear, we can find the value of one variable in terms of the other variable from linear equation. Substituting this value of one variable in the quadratic equation, we can solve it.

NOTE: Two quadratic equations in which xy term is missing and the coefficients of x^2 and y^2 are equal, give a linear equation by subtraction.