By using chain rule

$$\frac{dx}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dx}{du} = \frac{-4x}{(x^2-1)^2}\times\frac{1}{3x^2}$$

$$\frac{dx}{du} = \frac{-4x}{3x(x^2-1)^2} \qquad \text{Ans.}$$

## EXERCISE 2.5

**Q.1: Differentiate the following trigonometric functions from the first principles.**

**(i)** $\sin 2x$ **(ii)** $\tan 3x$ **(iii)** $\sin 2x + \cos 2x$

**(iv)** $\cos x^2$ **(v)** $\tan^2 x$ **(vi)** $\sqrt{\tan x}$

**(vii)** $\cos\sqrt{x}$

**Solution:**

**(i) sin 2x** ***(L.B 2003)***

Let $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) - y$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x \qquad \because y = \sin 2x$$

$$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right).\ \sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$\left[\because \sin p - \sin q = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)\right]$$

$$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \sin(\delta x)$$

Dividing both sides by $\delta x$

$$\frac{\delta y}{\delta x} = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \frac{\sin\delta x}{\delta x}$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \frac{\sin\delta x}{\delta x}$$

$$\frac{dy}{dx} = 2\cos\left(\frac{4x}{2}\right).1$$

$$\boxed{\frac{dy}{dx}(\sin 2x) = 2\cos 2x}$$ Ans.

**(ii)** **tan 3x**

Let $y = \tan 3x$

$y + \delta y = \tan 3(x + \delta x)$

$\delta y = \tan(3x + 3\delta x) - y$

$\delta y = \tan(3x + 3\delta x) - \tan 3x \qquad \because \quad y = \tan 3x$

$$\delta y = \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$\delta y = \frac{\sin(3x + 3\delta x)\cos 3x - \cos(3x + 3\delta x)\sin 3x}{\cos(3x + 3\delta x).\cos 3x}$$

$$\delta y = \frac{\sin(3x + 3\delta x - 3x)}{\cos(3x + 3\delta x)\cos 3x} \qquad [\because \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta]$$

$$\delta y = \frac{\sin 3\delta x}{\cos(3x + 3\delta x)\cos 3x}$$

Dividing both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{3}{\cos(3x + 3\delta x)\cos 3x}.\frac{\sin 3\delta x}{3\delta x}$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{3}{\cos(3x + 3\delta x)\cos 3x}.\frac{\sin 3\delta x}{3\delta x}$$

$$\frac{dy}{dx} = \frac{3}{\cos 3x\,.\,\cos 3x}.1$$

$$\frac{dy}{dx} = \frac{3}{\cos^2 3x}$$

$$\boxed{\frac{d}{dx}(\tan 3x) = 3\sec^2 3x}$$ Ans.

**(iii)** **sin 2x + cos 2x**

Let $y = \sin 2x + \cos 2x$

$y + \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x)$

$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - y$

$\delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - (\sin 2x + \cos 2x)$

$\because \quad y = \sin 2x + \cos 2x$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x + \cos(2x + 2\delta x) - \cos 2x$$

$$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right).\sin\left(\frac{2x + 2\delta x - 2x}{2}\right) - 2\sin\left(\frac{2x + 2\delta x + 2x}{2}\right)$$
$$\sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\sin\left(\frac{2\delta x}{2}\right) - 2\sin\left(\frac{4x + 2\delta x}{2}\right)\sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2\sin\delta x\left[\cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right)\right]$$

Dividing both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{2\sin\delta x}{\delta x}\left[\cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right)\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 2\frac{\sin\delta x}{\delta x}\left[\cos\left(\frac{4x + 2\delta x}{2}\right) - \sin\left(\frac{4x + 2\delta x}{2}\right)\right]$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \left[\cos\left(\frac{4x}{2}\right) - \sin\left(\frac{4x}{2}\right)\right]$$

$$\boxed{\frac{d}{dx}(\sin 2x + \cos 2x) = 2(\cos 2x - \sin 2x)} \qquad \text{Ans.}$$

**(iv)** **$\cos x^2$** ***(L.B 2003)***

Let $y = \cos x^2$

$$y + \delta y = \cos(x + \delta x)^2$$

$$\delta y = \cos(x + \delta x)^2 - y$$

$$\delta y = \cos(x + \delta x)^2 - \cos x^2 \qquad \because (y = \cos x^2)$$

$$\delta y = -2\sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right].\sin\left[\frac{(x + \delta x)^2 - x^2}{2}\right]$$

$$\delta y = -2\sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right].\sin\left(\frac{x^2 + \delta x^2 + 2x\delta x - x^2}{2}\right)$$

$$\delta y = -2\sin\left[\frac{(x + \delta x)^2 + x^2}{2}\right].\sin\left[\frac{\delta x(\delta x + 2x)}{2}\right]$$

Dividing both sides by $\delta x$

$$\frac{\delta y}{\delta x} = -2\sin\left[\frac{(x+\delta x)^2+x^2}{2}\right]\frac{\sin\left[\frac{\delta x(\delta x+2x)}{2}\right]}{\left(\frac{2}{\delta x+2x}\right).\,\delta x\left(\frac{\delta x+2x}{2}\right)}$$

Taking limit $\delta x \to 0$

$$\frac{dy}{dx} = -2\sin\left(\frac{x^2+x^2}{2}\right).\frac{1}{\frac{2}{2x}}\;.\;1$$

$$= -2\sin\left(\frac{2x^2}{2}\right).\,x$$

$$\boxed{\frac{d}{dx}(\cos x^2) = -2x\sin x^2}$$ Ans.

**(v) $\tan^2 x$**

Let

$$y = \tan^2 x$$

$$y+\delta y = \tan^2(x+\delta x)$$

$$\delta y = \tan^2(x+\delta x) - y$$

$$\delta y = \tan^2(x+\delta x) - \tan^2 x \qquad \because \quad y = \tan^2 x$$

$$\delta y = [\tan(x+\delta x)+\tan x\,]\,[\,\tan(x+\delta x) - \tan\,]$$

$$\delta y = [\tan(x+\delta x)+\tan x\,]\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$$

$$\delta y = [\tan(x+\delta x)+\tan x\,]\left[\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\right]$$

$$\delta y = \left[\frac{\tan(x+\delta x)+\tan x}{\cos(x+\delta x)\,.\,\cos x}\right]\sin(x+\delta x - x)$$

Dividing both sides by $\delta x$.

$$\frac{\delta y}{\delta x} = \frac{[\tan(x+\delta x)+\tan x\,]}{\cos(x+\delta x).\,\cos x}\,.\,\frac{\sin\delta x}{\delta x}$$

Taking limit $\delta x \to 0$

$$\frac{dy}{dx} = \frac{\tan x+\tan x}{\cos x\,.\,\cos x}\;.\;1$$

$$\frac{dy}{dx} = \frac{2\tan x}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}(\tan^2 x) = 2\tan x\sec^2 x}$$ Ans.

**(vi)** $\sqrt{\tan x}$ ***(L.B 2003, 2004)***

Let

$$y = \sqrt{\tan x}$$

$$y + \delta y = \sqrt{\tan(x+\delta x)}$$

$$\delta y = \sqrt{\tan(x+\delta x)} - y$$

$$\delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x} \qquad (\because \ y = \sqrt{\tan x}\,)$$

$$\delta y = [\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\,] \left[\frac{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\right]$$

$$\delta y = \frac{(\sqrt{\tan(x+\delta x)}\,)^2 - (\sqrt{\tan x}\,)^2}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{\tan(x+\delta x) - \tan x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x).\sin x}{\cos(x+\delta x).\cos x}\right]$$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \times \frac{\sin(x+\delta x - x)}{\cos(x+\delta x).\cos x}$$

$$\delta y = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \times \frac{\sin \delta x}{\cos(x+\delta x).\cos x}$$

Dividing both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \cdot \frac{1}{\cos(x+\delta x).\cos x} \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit $\delta x \to 0$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} \cdot \frac{1}{\cos x . \cos x} \cdot 1$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x}} \cdot \frac{1}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}(\sqrt{\tan x}) = \frac{\sec^2 x}{2\sqrt{\tan x}}} \qquad \text{Ans.}$$

**(vii)** $\cos\sqrt{x}$ ***(L.B 2004)***

Let

$$y = \cos\sqrt{x}$$

$$y + \delta y = \cos\sqrt{x + \delta x}$$

$$\delta y = \cos\sqrt{x + \delta x} - y$$

$$\delta y = \cos\sqrt{x + \delta x} - \cos\sqrt{x} \qquad \because \quad y = \cos\sqrt{x}$$

$$\delta y = -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\ \sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

$$\delta y = -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\ \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$\delta y = -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\ \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{2\left(\sqrt{x+\delta x}+\sqrt{x}\right)\frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)}{2}}$$

$$\because \ \delta x = (\sqrt{x+\delta x}+\sqrt{x})\ (\sqrt{x+\delta x}-\sqrt{x})$$

Taking limit $\delta x \to 0$

$$\frac{dy}{dx} = \frac{-\sin\left(\frac{\sqrt{x}+\sqrt{x}}{2}\right)}{\sqrt{x}+\sqrt{x}}\ .\ 1$$

$$\frac{dy}{dx} = \frac{-\sin\left(\frac{2\sqrt{x}}{2}\right)}{2\sqrt{x}}$$

$$\boxed{\frac{d}{dx}(\cos\sqrt{x}) = \frac{-\sin\sqrt{x}}{2\sqrt{x}}} \quad \text{Ans.}$$

**Q.2: Differentiate the following w.r.t. the variable involved.**

**(i)** $\mathbf{x^2 \sec 4x}$ **(ii)** $\mathbf{\tan^3\theta \sec^2\theta}$

**(iii)** $\mathbf{(\sin 2\theta - \cos 3\theta)^2}$ **(iv)** $\mathbf{\cos\sqrt{x} + \sqrt{\sin x}}$

**Solution:**

**(i)** $\mathbf{x^2 \sec 4x}$ ***(G.B 2005)***

Let $y = x^2 \sec 4x$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sec 4x)$$

$$= x^2 \frac{d}{dx}(\sec 4x) + \sec 4x \frac{d}{dx}(x^2)$$

$$= x^2 \sec 4x \tan 4x \,.\, 4 + \sec 4x \,.\, 2x$$

$$= 2x \sec 4x\,(1 + 2x \tan 4x) \qquad \text{Ans.}$$

**(ii)** $\tan^3\theta \sec^2\theta$

Let $y = \tan^3\theta \sec^2\theta$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan^3\theta \sec^2\theta)$$

$$= \tan^3\theta \frac{d}{d\theta}(\sec^2\theta) + \sec^2\theta \frac{d}{d\theta}(\tan^3\theta)$$

$$= \tan^3\theta \,.\, 2\sec\theta \,.\, \sec\theta \tan\theta + \sec^2\theta\, 3\tan^2\theta \,.\, \sec^2\theta$$

$$= \tan^2\theta \sec^2\theta\,(2\tan^2\theta + 3\sec^2\theta) \qquad \text{Ans.}$$

**(iii)** $(\sin 2\theta - \cos 3\theta)^2$

Let $y = (\sin 2\theta - \cos 3\theta)^2$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)^2$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta) \,.\, \frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)$$

$$\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta) \qquad \text{Ans.}$$

**(iv)** $\cos\sqrt{x} + \sqrt{\sin x}$ ***(L.B 2008)***

Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(\cos\sqrt{x}) + \frac{d}{dx}(\sqrt{\sin x})$$

$$= -\sin\sqrt{x}\, \frac{d}{dx}(\sqrt{x}) + \frac{1}{2}(\sin x)^{\frac{-1}{2}} \,.\, \cos x$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

$$= \frac{1}{2}\left(\frac{-\sin\sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}}\right) \qquad \text{Ans.}$$

**Q.3: Find $\frac{dy}{dx}$ if**

**(i)** $y = x\cos y$ ***(L.B 2009)*** **(ii)** $x = y\sin y$ ***(L.B 2009)***

**Solution:**

**(i)** $y = x\cos y$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos y)$$

$$\frac{dy}{dx} = x\frac{d}{dx}(\cos y) + \cos y\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x \times \left(-\sin y\frac{dy}{dx}\right) + \cos y \,.\, 1$$

$$\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx}(1 + x\sin y) = \cos y$$

$$\boxed{\frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}} \quad \text{Ans.}$$

**(ii)** $x = y\sin y$

Diff. w.r.t. 'x'

$$\frac{d}{dx}(x) = \frac{d}{dx}(y\sin y)$$

$$1 = y\frac{d}{dx}(\sin y) + \sin y\frac{dy}{dx}$$

$$1 = y\cos y\frac{dy}{dx} + \sin y\frac{dy}{dx}$$

$$1 = (y\cos y + \sin y)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y\cos y + \sin y}} \quad \text{Ans.}$$

**Q.4: Find the derivative w.r.t. 'x'**

**(i)** $\cos\sqrt{\frac{1+x}{1+2x}}$ **(ii)** $\sin\sqrt{\frac{1+2x}{1+x}}$

**Solution:**

**(i)** $\cos\sqrt{\frac{1+x}{1+2x}}$ ***(G.B 2005)***

Let

$$y = \cos\sqrt{\frac{1+x}{1+2x}}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos\sqrt{\frac{1+x}{1+2x}}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}} \cdot \frac{d}{dx}\left(\sqrt{\frac{1+x}{1+2x}}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{\frac{-1}{2}} \cdot \frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2}\ \frac{(1+x)^{\frac{-1}{2}}}{(1+2x)^{\frac{-1}{2}}}\left[\frac{(1+2x)\frac{d}{dx}(1+x) - (1+x)\cdot\frac{d}{dx}(1+2x)}{(1+2x)^2}\right]$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2}\ \frac{[\,1+2x-(1+x)\,.\,2]}{(1+x)^{\frac{1}{2}}\ \text{eq}\ (1+2x)^{2-\frac{1}{2}}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\ \frac{(1+2x-2-2x)}{2\sqrt{1+x}\,(1+2x)^{\frac{3}{2}}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\ \frac{-1}{2\sqrt{1+x}\,(1+2x)^{\frac{3}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{\sin\sqrt{\frac{1+x}{1+2x}}}{2\sqrt{1+x}\,(1+2x)^{\frac{3}{2}}}} \qquad \text{Ans.}$$

**(ii)** $\mathbf{\sin\sqrt{\frac{1+2x}{1+x}}}$ ***(G.B 2004)***

Let $y = \sin\sqrt{\frac{1+2x}{1+x}}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \left[\sin\sqrt{\frac{1+2x}{1+x}}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}}\ \cdot \frac{d}{dx}\left(\sqrt{\frac{1+2x}{1+x}}\right)$$

$$= \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{-1}{2}} \cdot \frac{d}{dx}\left(\frac{1+2x}{1+x}\right)$$

$$= \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \frac{(1+2x)^{-\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}}\left[\frac{(1+x)\frac{d}{dx}(1+2x)-(1+2x)\cdot\frac{d}{dx}(1+x)}{(1+x)^2}\right]$$

$$= \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{[\,(1+x)2-(1+2x)\,]}{2(1+2x)^{\frac{1}{2}}(1+2x)^{\frac{-1}{2}+2}}$$

$$= \cos\sqrt{\frac{1+2x}{1+x}} \frac{(2+2x-1-2x)}{2\sqrt{1+2x}\,(1+x)^{\frac{3}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{\cos\sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}\,(1+x)^{\frac{3}{2}}}} \qquad \text{Ans.}$$

**Q.5: Differentiate**

**(i) sin x w.r.t. cot x (ii) $\sin^2 x$ w.r.t. $\cos^4 x$**

**Solution:**

sin x w.r.t. cot x ***(L.B 2009)***

Let

$$y = \sin x \quad , \quad u = \cot x$$

| | |
|---|---|
| $y = \sin x$ | $u = \cot x$ |
| Diff. w.r.t. 'x' | Diff. w.r.t. 'x' |
| $\frac{dy}{dx} = \cos x$ | $\frac{du}{dx} = -\operatorname{cosec}^2 x$ |

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$= \cos x \cdot \frac{-1}{\operatorname{cosec}^2 x}$$

$$\boxed{\frac{dy}{du} = -\cos x \sin^2 x} \qquad \text{Ans.}$$

**(ii) $\sin^2 x$ w.r.t. $\cos^4 x$** ***(G.B 2003, L.B 2009, L.B 2004, L.B 2008)***

Let

$$y = \sin^2 x \quad , \quad u = \cos^4 x$$

$$y = \sin^2 x$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = 2 \sin x \cos x$$

$$u = \cos^4 x$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = 4 \cos^3 x - \sin x$$

$$= -4 \sin x \cos^3 x$$

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$= 2 \sin x \cos x \cdot \frac{-1}{4 \sin x \cos^3 x}$$

$$= \frac{-1}{2 \cos^2 x} = \frac{-1}{2} \sec^2 x \quad \text{Ans.}$$

**Q.6: If tan y (1 + tan x) = 1 – tan x, show that $\frac{dy}{dx} = -1$** ***(G.B 2009)***

**Solution:**

$$\tan y (1 + \tan x) = 1 - \tan x$$

$$\tan y = \frac{1 - \tan x}{1 + \tan x}$$

$$\tan y = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x}$$

$$\tan y = \tan\left(\frac{\pi}{4} - x\right)$$

$$y = \frac{\pi}{4} - x$$

Diff. w.r.t. 'x'

$$\boxed{\frac{dy}{dx} = -1} \quad \text{Hence proved.}$$

**Q.7: If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \ldots\ldots\ldots \infty$, prove that $(2y - 1)\frac{dy}{dx} = \sec^2 x$.**

**Solution:**

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \ldots\ldots\ldots \infty$$

Squaring on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x}} + \ldots\ldots\ldots \infty$$

$$y^2 = \tan x + y$$

Diff. w.r.t. 'x'

$$2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\boxed{(2y-1)\frac{dy}{dx} = \sec^2 x}$$ Hence proved.

**Q.8: If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that $a\frac{dy}{dx} + b\tan\theta = 0$**

***(G.B 2004, G.B 2011, G.B 2007)***

**Solution:**

$$x = a\cos^3\theta$$

Diff. w.r.t. 'θ'

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$= -3a\sin\theta\cos^2\theta$$

$$y = b\sin^3\theta$$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= 3b\sin^2\theta\cos\theta \cdot \frac{-1}{3a\sin\theta\cos^2\theta}$$

$$= \frac{-b}{a}\tan\theta$$

$$a\frac{dy}{dx} = -b\tan\theta$$

$$\boxed{a\frac{dy}{dx} + b\tan\theta = 0}$$ Hence proved.

**Q.9: Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$, $y = a(\sin t - t \text{ cost } t)$**

**Solution:**

$$x = a(\cos t + \sin t)$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = a(-\sin t + \cos t)$$

$$y = a(\sin t - t\cos t)$$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = a[\cos t - \{t \cdot -\sin t + \cos t \cdot 1)$$

$$\frac{dy}{dt} = a\cos t + a\,t\,.\,\sin t - a\cos t$$

$$\frac{dy}{dt} = a\,t\,.\,\sin t$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= at\sin t\;.\;\frac{1}{a(-\sin t + \cos t)}$$

$$\boxed{\frac{dy}{dx} = \frac{t\sin t}{\cos t - \sin t}} \text{ Ans.}$$

**Q.10: Differentiate w.r.t. 'x'**

**(i)** $\cos^{-1}\frac{x}{a}$ **(ii)** $\cot^{-1}\frac{x}{a}$ **(iii)** $\frac{1}{a}\sin^{-1}\frac{a}{x}$

**(iv)** $\sin^{-1}\sqrt{1-x^2}$ **(v)** $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ **(vi)** $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

**(vii)** $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

**Solution:**

**(i)** $\cos^{-1}\frac{x}{a}$

Let $y = \cos^{-1}\frac{x}{a}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\frac{\sqrt{a^2-x^2}}{a}} \cdot \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2-x^2}} \qquad \text{Ans.}$$

**(ii)** $\mathbf{\cot^{-1}\frac{x}{a}}$ ***(L.B 2006)***

Let $y = \cot^{-1}\frac{x}{a}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{-1}{\frac{a^2+x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{-a}{a^2+x^2} \qquad \text{Ans.}$$

**(iii)** $\mathbf{\frac{1}{a}\sin^{-1}\frac{a}{x}}$ ***(L.B 2010)***

Let $y = \frac{1}{a}\sin^{-1}\frac{a}{x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{a}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\frac{a^2}{x^2}}} \cdot a\frac{d}{dx}(x^{-1})$$

$$= \frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}} \cdot -1\,x^{-2}$$

$$= \frac{1}{\frac{\sqrt{x^2 - a^2}}{x^2}} \cdot -\frac{1}{x^2}$$

$$= \frac{-1}{x\sqrt{x^2 - a^2}} \quad \text{Ans.}$$

**(iv)** $\mathbf{\sin^{-1}\sqrt{1 - x^2}}$

Let $y = \sin^{-1}\sqrt{1 - x^2}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \cdot \frac{d}{dx}(\sqrt{1 - x^2})$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - 1 + x^2}} \cdot \frac{1}{2}(1 - x^2)^{\frac{-1}{2}} \cdot \frac{d}{dx}(1 - x^2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1 - x^2}} \cdot (-2x)$$

$$= \frac{-x}{x\sqrt{1 - x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}} \quad \text{Ans.}$$

**(v)** $\mathbf{\sec^{-1}\left(\frac{x^2 + 1}{x^2 - 1}\right)}$ $\quad \left(\because \mathbf{\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2 - 1}}}\right)$

Let

$$y = \sec^{-1}\left(\frac{x^2 + 1}{x^2 - 1}\right)$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2 + 1}{x^2 - 1}\right)\sqrt{\left(\frac{x^2 + 1}{x^2 - 1}\right)^2 - 1}} \cdot \frac{d}{dx}\left(\frac{x^2 + 1}{x^2 - 1}\right)$$

$$= \frac{x^2 - 1}{(x^2 + 1)\sqrt{\frac{(x^2 + 1)^2}{(x^2 - 1)^2} - 1}} \left[\frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}\right]$$

$$= \frac{(x^2-1)\,2x-(x^2+1).2x}{(x^2+1)(x^2-1)\sqrt{\dfrac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}}$$

$$= \frac{2x\,(x^2-1-x^2-1)}{(x^2+1)(x^2-1)\ \dfrac{\sqrt{x^4+1+2x^2-(x^4+1-2x^2)}}{x^2-1}}$$

$$= \frac{2x\,(-2)}{(x^2+1)\sqrt{x^4+1+2x^2-x^4-1+2x^2}}$$

$$= \frac{-4x}{(x^2+1)\sqrt{4x^2}}$$

$$= \frac{-4x}{(x^2+1).2x} \qquad = \qquad \frac{-2}{x^2+1} \qquad \text{Ans.}$$

**(vi)** $\mathbf{\cot^{-1}\left(\dfrac{2x}{1-x^2}\right)}$

Let $y = \cot^{-1}\left(\dfrac{2x}{1-x^2}\right)$

Diff. w.r.t. 'x' $\left(\dfrac{d}{dx}\cot^{-1}x = \dfrac{-1}{1+x^2}\right)$

$$\frac{dy}{dx} = \frac{-1}{1+\left(\dfrac{2x}{1-x^2}\right)^2}.\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$= \frac{-2}{1+\dfrac{4x^2}{(1-x^2)^2}}\left[\frac{(1-x^2)\dfrac{d}{dx}(x)-x\dfrac{d}{dx}(1-x^2)}{(1-x^2)^2}\right]$$

$$= \frac{-2}{\dfrac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\left[\frac{(1-x^2)-x(-2x)}{(1-x^2)^2}\right]$$

$$= \frac{-2\,[1-x^2+2x^2]}{1+x^4-2x^2+4x^2}$$

$$= \frac{-2\,(1+x^2)}{1+x^4+2x^2}$$

$$= \frac{-2\,(1+x^2)}{(1+x^2)^2}$$

$$= \frac{-2}{1+x^2} \qquad \text{Ans.}$$

**(vii)** $\mathbf{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}$ ***(L.B 2004)***

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \left(\frac{d}{dx}\cos^{-1}x = \frac{-1}{x\sqrt{1-x^2}}\right)$$

$$= \frac{-1}{\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}}\left[\frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$$

$$= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}\left[\frac{(1+x^2)(-2x)-(1-x^2).2x}{(1+x^2)^2}\right]$$

$$= \frac{-2x[-1-x^2-1+x^2]}{\frac{\sqrt{1+x^4+2x^2-(1+x^4-2x^2)}}{1+x^2}\ (1+x^2)^2}$$

$$= \frac{-2x(-2)}{\sqrt{1+x^4+2x^2-1-x^4+2x^2}\ (1+x^2)}$$

$$= \frac{4x}{\sqrt{4x^2}\ (1+x^2)}$$

$$= \frac{4x}{2x\ (1+x^2)}$$

$$= \frac{2}{1+x^2} \qquad \text{Ans.}$$

**Q.11: Show that** $\mathbf{\frac{dy}{dx} = \frac{y}{x}}$ **if** $\mathbf{\frac{y}{x} = \tan^{-1}\frac{x}{y}}$ ***(L.B 2003, G.B 2007, L.B 2007, G.B 2008)***

**Solution:**

$$\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

$$y = x\tan^{-1}\frac{x}{y}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x\frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{x}{1+\frac{x^2}{y^2}}\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{1+\frac{x^2}{y^2}}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{\frac{y^2+x^2}{y^2}}\left[\frac{\left(y-\frac{dy}{dx}\right)}{y^2}\right] + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2}\frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx} + \frac{x^2}{x^2+y^2}\frac{dy}{dx} = y\left[\frac{x}{x^2+y^2} + \frac{1}{x}\right]$$

$$\frac{dy}{dx}\left[1 + \frac{x}{x^2+y^2}\right] = y\left[\frac{x^2+x^2+y^2}{x(x^2+y^2)}\right]$$

$$\frac{dy}{dx}\left(\frac{x^2+y^2+x^2}{x^2+y^2}\right) = y\left[\frac{2x^2+y^2}{x(x^2+y^2)}\right] = \frac{y(x^2+y^2)(2x^2+y^2)}{x(x^2+y^2)(2x^2+y^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$ Hence proved.

**Q.12: If $y = \tan(P\tan^{-1}x)$, show that $(1+x^2)y_1 - P(1+y^2) = 0$**

***(L.B 2006) (G.B 2006)***

**Solution:**

$$y = \tan(P\tan^{-1}x)$$
$$\tan^{-1}y = P\tan^{-1}x$$
Diff. w.r.t. 'x'
$$\frac{1}{1+y^2}\cdot y_1 = P\frac{1}{1+x^2}$$
$$(1+x^2)y_1 = P(1+y^2)$$
$$\boxed{(1+x^2)y_1 - P(1+y^2) = 0}$$ Hence proved.

## EXERCISE 2.6

**Q.1: Find $f'(x)$ if**

**(i)** $f(x) = e^{\sqrt{x}-1}$ **(ii)** $f(x) = x^3 e^{\frac{1}{x}} \ (x \neq 0)$

**(iii)** $f(x) = ex(1+\ell nx)$ **(iv)** $f(x) = \dfrac{e^x}{e^{-x}+1}$

**(v)** $f(x) = \ell n(e^x + e^{-x})$ **(vi)** $f(x) = \dfrac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

**(vii)** $f(x) = \sqrt{\ell n(e^{2x}+e^{-2x})}$ **(viii)** $f(x) = \ell n\left(\sqrt{e^{2x}+e^{-2x}}\right)$

**Solution:**

**(i)** $f(x) = e^{\sqrt{x}-1}$

Diff. w.r.t. 'x'
$$f'(x) = e^{\sqrt{x}-1}\cdot\frac{d}{dx}(\sqrt{x}-1)$$
$$f'(x) = e^{\sqrt{x}-1}\cdot\frac{1}{2}x^{\frac{-1}{2}}$$
$$\boxed{f'(x) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}}}$$ Ans.

**(ii)** $f(x) = x^3 e^{\frac{1}{x}} \ (x \neq 0)$

Diff. w.r.t. 'x'
$$f'(x) = x^3\frac{d}{dx}\left(e^{\frac{1}{x}}\right) + e^{\frac{1}{x}}\frac{d}{dx}(x^3)$$