Binary Operation

A binary operation denoted as $*$ (read as star) on a non–empty G is a function associates with each ordered pair (a, b), of elements of G, a unique element, denoted as $a * b$ of G.

Properties of Binary Operation

Let 'S' be a non–empty set and $*$ a binary operation on it. Then $*$ may possess one or more of the following properties.

(1) Commutativity: $*$ is a said to be commutative if

$a * b = b * a \ \forall \ a. b \in S.$

(2) Associativity: $*$ is said to be associative on S

$a * (b * c) = (a * b) * c \ \forall \ a, b, c, \in S$

(3) Existence of an identity element: An element $e \in S$ is called an identity element w.r.t. $*$ if

$a * e = e * a = a \ \forall \ a \in S.$

(4) Existence of inverse of each element: For any element $a \in S$, $\exists$ an element $a' \in S$ such that

$a * a' = a' * a = e$ (the identity element)

EXERCISE 2.7

Q.1 Complete the table, indicating by tick mark those properties which are satisfied by the specified set of number.

Property	Set of numbers	Natural	Whole	Integers	Rationale	Reals
Closure	$\oplus$	✓	✓	✓	✓	✓
	$\boxtimes$	✓	✓	✓	✓	✓
Associative	$\oplus$	✓	✓	✓	✓	✓
	$\boxtimes$	✓	✓	✓	✓	✓
Identity	$\oplus$	×	✓	✓	✓	✓
	$\boxtimes$	✓	✓	✓	✓	✓
Inverse	$\oplus$	×	×	✓	✓	✓
	$\boxtimes$	×	×	×	✓	×
Commutative	$\oplus$	✓	✓	✓	✓	✓
	$\boxtimes$	✓	✓	✓	✓	✓

Q.2 What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers.

Solution:

A non empty set F is said to be field if

(i) F is an abelian group w.r.t. '+'.

(ii) $F-\{0\}$ is an abelian group w.r.t. '×'.

(iii) Left and Right Distributive laws hold in F.

Field of real numbers is different from the field of complex numbers because in the field of real numbers we can compare two real numbers such that one number is less than the other, but in the field of complex numbers, we can not compare two complex numbers such that one number is less than the other.

Q.3 Show that the adjoining table is that of multiplication of the elements of the set of residue classes modulo 5.

*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Solution:

The above table shows the multiplication of the elements of the set of residue classes modulo 5 because in table

$2 \times 3 = 6 = 1$ (remainder after dividing 6 by 5)

$2 \times 4 = 8 = 3$ (remainder after dividing 8 by 5)

$3 \times 3 = 9 = 4$ (remainder after dividing 9 by 5)

$3 \times 4 = 12 = 2$ (remainder after dividing 12 by 5)

$4 \times 4 = 16 = 1$ (remainder after dividing 16 by 5)

Q.4 Prepare a table of addition of the elements of the set of residue classes modulo 4.

Solution:

Required table is given below.

Consider the set $\{0, 1, 2, 3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Q.5 Which of the following binary operations shown in table (a) or (b) is commutative?

*	a	b	c	d
a	a	c	b	d
b	b	c	b	a
c	c	d	b	c
d	a	a	b	b

(a)

$*$ in table (a) is not commutative because

$a * b = c$ and $b * a = b$

$\Rightarrow$ $a * b \neq b * a$

*	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

(b)

$*$ in table (b) is commutative

because

$a * b = c$ and $b * a = c$

$\Rightarrow$ $a * b = b * a$

Similarly other elements can be checked.

Q.6 Supply the missing elements of the third row of the given table. So that the operation may be associative.

*	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	—	—	—	—
d	d	c	c	d

Solution:

Suppose the required elements are p, q, r, s then

$p = c * a = (d * b) * a = d * (b * a) = d * b = c$

$q = c * b = (d * b) * b = d * (b * b) = d * a = d$

$r = c * c = (d * b) * c = d * (b * c) = d * c = c$

$s = c * d = (d * b) * d = d * (b * d) = d * d = d$

Q.7 What operation is represented by the adjoining table? Name the identity element of relevant set, if it exists. Is the operation associative? Find inverse of 0, 1, 2, 3 if they exist.

Solution:

$*$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(i) Operation is '+' modulo 4.

(ii) '0' is the identity element.

(iii) yes, operation is associative.

(iv) Inverse of $0 = 0$

Inverse of $1 = 3$

Inverse of $2 = 2$

Inverse of $3 = 1$

GROUPS

Groupiod

A closed set with respect to an operation $*$ is called a groupoid.

Semi Group

A non–empty set S is semi group if

(i) It is closed with respect to an operation $*$

(ii) the operation $*$ is associative.

Monoid

A semi group having an identity in it is called monoid.

Group

A monoid having inverse of each of its elements under $*$ is called a group.

Abelian Group

If a group G satisfies an additional condition i.e.

for every $a, b \in G$, $a * b = b * a$

then G is called an Abelian or commutative group under $*$.