SYNTHETIC DIVISION

There is a shortcut method for long division of a polynomial f (x) by a polynomial of the form x – a. This process of division is called synthetic division.

EXERCISE 4.5

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial:

Q.1 $x^2 + 3x + 7, \quad x + 1$ **(Gujranwala Board 2005)**

Solution:

$x^2 + 3x + 7, \quad x + 1$

Let $f(x) = x^2 + 3x + 7$ ……… (1) $\qquad x + 1 = 0$

$\Rightarrow$ $f(-1) = (-1)^2 + 3(-1) + 7$ $\qquad x = -1$

$= 1 - 3 + 7$ $\qquad$ Put in equation (1)

$= -2 + 7 = 5 = R$

$\Rightarrow$ Remainder $= R = 5$

Q.2 $x^3 - x^2 + 5x + 4, \quad x - 2$

Solution:

Let $f(x) = x^3 - x^2 + 5x + 4$ ……… (1) $\qquad x - 2 = 0$

$f(2) = (2)^3 - (2)^2 + 5(2) + 4$ $\qquad x = 2$

$= 8 - 4 + 10 + 4$ $\qquad$ Put in (1)

$= 18 = R$

$\Rightarrow$ Remainder $= R = 18$

Q.3 $3x^4 + 4x^3 + x - 5, \quad x + 1$

Solution:

Let $f(x) = 3x^4 + 4x^3 + x - 5$ ……… (1) $\qquad x + 1 = 0$

$f(-1) = 3(-1)^4 + 4(-1)^3 + (-1) - 5$ $\qquad x = -1$

$= 3 - 4 - 1 - 5$ $\qquad$ Put in (1)

$= -7 = R$

$\Rightarrow$ Remainder $= R = -7$

Q.4 $x^3 - 2x^2 + 3x + 3, \quad x - 3$ **(Gujranwala Board 2007)**

Solution:

Let $f(x) = x^3 - 2x^2 + 3x + 3$ (1) $\qquad x - 3 = 0$

$f(3) = (3)^3 - 2(3)^2 + 3(3) + 3$	$x = 3$
$= 27 - 2(9) + 9 + 3$	Put in (1)
$= 27 - 28 + 9 + 3 = 21 = R$	

$\Rightarrow$ Remainder $= R = 21$

Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

Q.5 $x - 1$, $x^2 + 4x - 5$

Solution:

Let $f(x) = x^2 + 4x - 5$ ……… (1)	$x - 1 = 0$
$f(1) = (1)^2 + 4(1) - 5$	$x = 1$
$= 1 + 4 - 5$	Put in (1)
$= 0 =$ Remainder	

$\Rightarrow$ $x - 1$ is a factor of $x^2 + 4x - 5$.

Q.6 $(x - 2)$, $(x^3 + x^2 - 7x - 1)$ (Lahore Board 2006)

Solution:

Let $f(x) = x^3 + x^2 - 7x + 1$ ……… (1)	$x - 2 = 0$
$f(2) = (2)^3 + (2)^2 - 7(2) + 1$	$x = 2$
$= 8 + 4 - 14 + 1$	Put in (1)
$= -1 = 0$	

$\Rightarrow$ $(x - 2)$ is not a factor of $x^3 + x^2 - 7x - 1$.

Q.7 $\omega + 2$, $2\omega^3 + \omega^2 - 4\omega + 7$

Solution:

Let $f(x) = 2\omega^3 + \omega^2 - 4\omega + 7$ ……… (1)	$\omega + 2 = 0$
$f(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 7$	$\omega = -2$
$= 2(-8) + 4 + 8 + 7$	Put in (1)
$= 3 \neq 0$	

$\Rightarrow$ $(\omega + 2)$ is not a factor of $2\omega^3 + \omega^2 - 4\omega + 7$.

Q.8 $x - a$, $x^n - a^n$, where ‘n’ is a positive integer. (Lahore Board 2007)

Solution:

Let $f(x) = x^n - a^n$ ……… (1)	$x - a = 0$
$f(a) = (a)^n - (a)^n$	$x = a$
$= 0 = R$	Put in (1)

$\Rightarrow$ $x - a$ is a factor of $x^n - a^n$.

Q.9 **$x + a$, $x^n + a^n$, where 'n' is an odd integer.** **(Lahore Board 2009)**

Solution:

$x + a = 0$
$x = -a$
Put in (1)

Let $f(x) = x^n + a^n$ ……… (1)

$f(-a) = (-a)^n + a^n$

$= -a^n + a^n$ $\because$ 'n' is odd

$= 0 = R$

$\Rightarrow$ $x + a$ is a factor of $x^n + a^n$.

Q.10 **When $x^4 + 2x^3 + kx^2 + 3$ is divided by $x - 2$, the remainder is 1. Find the value of K.**

Solution:

$x - 2 = 0$
$x = 2$
Put in (1)

Let $f(x) = x^4 + 2x^3 + kx^2 + 3$ ……… (1)

$f(2) = (2)^4 + 2(2)^3 + k(2)^2 + 3$

$= 16 + 16 + 4k + 3$

$= 35 + 4k =$ Remainder

$\Rightarrow$ Remainder $= 35 + 4k$

But given that remainder $= 1$.

$\Rightarrow$ $35 + 4k = 1$

$\Rightarrow$ $4k = 1 - 35$

$\Rightarrow$ $4k = -34$

$\Rightarrow$ $k = \dfrac{-34}{4}$

$\Rightarrow$ $k = \dfrac{-17}{2}$

Q.11 **When the polynomial $x^3 + 2x^2 + kx + 4$ is divided by $x - 2$ the remainder is 14? Find the value of k.** **(Lahore Board 2006)**

Solution:

$x - 2 = 0$
$x = 2$
Put in (1)

Let $f(x) = x^3 + 2x^2 + kx + 4$ ……… (1)

$f(2) = (2)^3 + 2(2)^2 + k(2) + 4$

$= 8 + 8 + 2k + 4$

$= 20 + 2k =$ Remainder

$\Rightarrow$ Remainder $= 20 + 2k$

but given that remainder $= 14$

$\Rightarrow$ $14 = 20 + 2k$

$\Rightarrow$ $14 - 20 = 2k$

$\Rightarrow$ $-6 = 2k$

$\Rightarrow$ $k = -3$

Use synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

Q.12 $\mathbf{x^3 - 7x + 6 = 0 \quad x = 2}$ **(Lahore Board 2005)**

Solution:

Let $f(x) = x^3 - 7x + 6, \quad x = 2$

By synthetic division

$$\begin{array}{c|cccc}
 & 1 & 0 & -7 & 6 \\
2 & 0 & 2 & 4 & -6 \\
\hline
 & 1 & 2 & -3 & \boxed{0}
\end{array} \rightarrow \text{Remainder}$$

As remainder $= 0$

$\Rightarrow$ $x = 2$ is the solution of the polynomial $x^3 - 7x + 6 = 0$

Now

$$\begin{aligned}
\text{Quotient} &= x^2 + 2x - 3 \\
&= x^2 + 3x - x - 3 \\
&= x(x + 3) - 1(x + 3) \\
&= (x + 3)(x - 1)
\end{aligned}$$

$\Rightarrow$ other factors are $x + 3$ and $x - 1$.

Q.13 $\mathbf{x^3 - 28x - 48 = 0, \; x = -4}$

Solution:

Let $f(x) = x^3 - 28x - 48, \quad x = -4$

By synthetic division

$$\begin{array}{c|cccc}
 & 1 & 0 & -28 & -48 \\
-4 & 0 & -4 & 16 & 48 \\
\hline
 & 1 & -4 & -12 & \boxed{0}
\end{array} \rightarrow \text{Remainder}$$

As remainder $= 0$

$\Rightarrow$ $x = -4$ is the solution of the polynomial $x^3 - 28x - 48$.

Now

$$\begin{aligned}
\text{Quotient} &= x^2 - 4x - 12 \\
&= x^2 - 6x + 2x - 12 \\
&= x(x - 6) + 2(x - 6) \\
&= (x - 6)(x + 2)
\end{aligned}$$

$\Rightarrow$ other factors are $x - 6$ and $x + 2$.

Q.14 $\mathbf{2x^4 + 7x^3 - 4x^2 - 27x - 18, \quad x = 2, \; x = -3.}$

Solution:

Let $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18, \quad x = 2, \; x = -3$

By synthetic division

$$
\begin{array}{r|rrrrr}
 & 2 & 7 & -4 & -27 & -18 \\
2 & 0 & 4 & 22 & 36 & 18 \\
\hline
 & 2 & 11 & 18 & 9 & \boxed{0} \\
-3 & 0 & -6 & -15 & -9 & \\
\hline
 & 2 & 5 & 3 & \boxed{0} &
\end{array}
\quad
\begin{array}{l}
\\ \\ \rightarrow \text{Remainder} \\ \\ \rightarrow \text{Remainder}
\end{array}
$$

As remainder $= 0$

$\Rightarrow$ $x = 2, \; x = -3$ are solutions of the polynomial $2x^4 + 7x^3 - 4x^2 - 27x - 18$.

Now

$$
\begin{aligned}
\text{Quotient} &= 2x^2 + 5x + 3 \\
&= 2x^2 + 3x + 2x + 3 \\
&= x(2x + 3) + 1(2x + 3) \\
&= (2x + 3)(x + 1)
\end{aligned}
$$

$\Rightarrow$ other factors are $2x + 3$ and $x + 1$.

Q.15 Use synthetic division to find the values of p and q if $\mathbf{x + 1}$ and $\mathbf{x - 2}$ are the factors of the polynomial $\mathbf{x^3 + px^2 + qx + 6}$. (Gujranwala Board 2006)

Solution:

Let $f(x) = x^3 + px^2 + qx + 6, \quad x + 1 = 0 \Rightarrow x = -1$

$x - 2 = 0 \Rightarrow x = 2$

By synthetic division

$$
\begin{array}{r|rrrr}
 & 1 & p & q & 6 \\
-1 & 0 & -1 & -p + 1 & -q + p - 1 \\
\hline
 & 1 & p - 1 & q - p + 1 & \boxed{p - q + 5} \\
2 & 0 & 2 & 2p + 2 & \\
\hline
 & 1 & p + 1 & \boxed{p + q + 3} &
\end{array}
\quad
\begin{array}{l}
\\ \\ \rightarrow \text{Remainder} \\ \\ \rightarrow \text{Remainder}
\end{array}
$$

As $x = -1$ and $x = 2$ are factors of the given polynomial.

$\Rightarrow$ $p - q + 5 = 0$ .... (1)

and $\quad p + q + 3 = 0 \qquad \ldots\ldots\ldots (2)$

Adding (1) and (2)

$$2p + 8 = 0$$

$$2p = -8 \quad \Rightarrow \quad p = -4$$

Put $p = -4$ in equation (1)

$$-4 - q + 5 = 0$$

$\Rightarrow \quad -q + 1 = 0$

$\Rightarrow \quad q = 1$

Q.16 Find the values of 'a' and 'b' if -2 and 2 are the roots of the polynomial $x^3 - 4x^2 + ax + b$. (Gujranwala Board 2007)

Solution:

Let $f(x) = x^3 - 4x^2 + ax + b$

As -2 and 2 are the roots of the given polynomial.

$\Rightarrow \quad f(-2) = 0$ $\qquad$ and $\quad f(2) = 0$

$\Rightarrow \quad (-2)^3 - 4(-2)^2 + a(-2) + b = 0$ $\qquad \Rightarrow \quad (2)^3 - 4(2)^2 + a(2) + b = 0$

$\Rightarrow \quad -8 - 4(4) - 2a + b = 0$ $\qquad \Rightarrow \quad 8 - 4(4) + 2a + b = 0$

$\Rightarrow \quad -24 - 2a + b = 0 \quad \ldots\ldots\ldots (1)$ $\qquad \Rightarrow \quad -8 + 2a + b = 0 \quad \ldots\ldots\ldots (2)$

Adding (1) and (2)

$$-24 - 2a + b = 0$$

$$-8 - 2a + b = 0$$

$$\overline{-32 + 2b = 0}$$

$\Rightarrow \quad 2b = 32$

$\Rightarrow \quad b = 16$

Put $b = 16$ in equation (1)

$$-24 - 2a + 16 = 0$$

$\Rightarrow \quad -8 - 2a = 0$

$\Rightarrow \quad +2a = -8$

$\Rightarrow \quad a = -4$

$\Rightarrow \quad a = -4$ and $b = 16$

RELATION BETWEEN THE ROOTS AND THE COEFFICIENTS OF A QUADRATIC EQUATION

Let α, β be the roots of $ax^2 + bx + c = 0, \; a \neq 0$ such that

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

then $$\alpha + \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a}$$

and $$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 + (\sqrt{b^2 - 4ac})}{(2a)^2} = \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$\Rightarrow$ $$\text{Sum of roots} = S = \frac{-b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

$$\text{Product of roots} = P = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

FORMATION OF AN EQUATION WHOSE ROOTS ARE GIVEN

As $(x - \alpha)(x - \beta) = 0$ has roots 'α' and 'β'.

$\Rightarrow$ $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ has the roots α and β

$\Rightarrow$ $x^2 - Sx + p = 0$ has the roots α and β

where

$$S = \text{Sum of the roots} = -\frac{b}{a}$$

$$P = \text{Product of the roots} = \frac{c}{a}$$