Complex Numbers

The numbers of the form $x + iy$, where $x, y, \in R$ and $i = \sqrt{-1}$, are called complex numbers, here 'x' is called real part and 'y' is called imaginary part of the complex number. For example $3 + 4i$, $2 - \frac{5}{7}i$ etc. are complex numbers.

A complex number can be written in the form of an ordered pair i.e. $x + iy = (x, y)$. the set 'C' of complex numbers does not satisfy the order axioms. Infact there is no sense in saying that one complex number is greater or less than another.

EXERCISE 1.2

Q.1 Verify the addition properties of complex numbers.

Solution:

Addition properties of complex numbers are:

(i) Closure property

$\forall\ (a, b),\ (c, d) \in C$

$(a, b) + (c, d) = a + ib + c + id = a + c + i(b + d) = (a + c, b + d) \in C$

(ii) Associative Property

$\forall\ (a, b),\ (c, d),\ (e, f) \in C$

$[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$

L.H.S.

$$\begin{aligned}[(a, b) + (c, d)] + (e, f) &= [a + ib + c + id] + (e + if) \\ &= a + ib + c + id + e + if \\ &= a + c + e + i(b + d + f) \\ &= (a + c + e\ ,\ b + d + f)\end{aligned}$$

R.H.S.

$(a, b) + [(c, d) + (e, f)]$

$= (a + ib) + [c + id + e + if]$

$= a + ib + c + id + e + if$

$= a + c + e + i(b + d + f)$

$= (a + c + e\ ,\ b + d + f)$

$\Rightarrow$ L.H.S. = R.H.S.

(iii) Additive Identity

$\forall\ (a, b) \in C \quad \exists\ (0, 0) \in C$

such that $(a, b) + (0, 0) = (0, 0) + (a, b) = (a, b)$

(iv) Additive Inverse

$\forall$ (a, b) $\in$C $\exists$ (– a, – b) $\in$ C

such that (a, b) + (– a, – b) = (0, 0) = (– a, – b) + (a, b)

(v) Commutative Property

$\forall$ (a, b) , (c, d) $\in$ C

$$(a, b) + (c, d) = a + ib + c + id$$
$$= a + c + i (b + d)$$
$$= c + a + i (d + b) = (c + id) + (a + ib) = (c, d) + (a, b)$$

Q.2 Verify the multiplication properties of complex numbers.

Solution:

Multiplication Properties of complex numbers are:

Closure property

$\forall$ (a, b) , (c, d) $\in$ C

$$(a, b) . (c, d) = (a + ib) (c + id) = ac + aid + ibc + i^2bd$$
$$= ac – bd + i (ad + bc) = (ac – bd, ad + bc) \in C$$

Associative Property

$\forall$ (a, b) , (c, d) (e, f) $\in$ C

[(a, b) . (c, d)] . (e, f) = (a, b) [(c, d) . (e, f)]

L.H.S.

$$[(a, b) . (c, d)] . (e, f) = [(a + ib) (c + id)] . (e + if)$$
$$= [ac + iad + ibc + i^2bd] . (e + if)$$
$$= [ac – bd + iad + ibc] . (e + if)$$
$$= ace + iacf – bde – ibdf + iade + i^2adf + ibce + i^2bcf$$
$$= ace – adf – bcf – bde + i (acf – bdf + ade + bce)$$

Now R.H.S.

$$(a, b) [(c, d) . (e, f)] = (a + ib) [(c + id) (e + if)]$$
$$= (a + ib) [ce + icf + ide + i^2df]$$
$$= (a + ib) [ce – df + icf + ide]$$
$$= ace – adf + iacf + iade + ibce – ibdf + i^2bcf + i^2bde$$
$$= ace – adf – bcf – bde + i (acf – bdf + ade + bce)$$

$\Rightarrow$ L.H.S. = R.H. S

Multiplicative Identity

$\forall\ (a, b) \in C \quad \exists\ (1, 0) \in C$ such that

$(a, b) . (1, 0) = (a + ib) . (1 + 0i) = a + 0i + ib + 0 = a + ib = (a, b)$

and

$(1, 0)(a, b) = (1 + 0i) . (a + ib) = a + ib + 0i + 0 = a + ib = (a, b)$

$\Rightarrow$ (1,0) is multiplicative identity in C.

Multiplicative Inverse

$\forall\ (a + ib) \in C \quad \exists\ (a + ib)^{-1} \in C$

where

$$(a + ib)^{-1} = \frac{1}{a + ib} = \frac{1}{a + ib} . \frac{a - ib}{a - ib}$$

$$= \frac{a - ib}{(a)^2 - (ib)^2} = \frac{a - ib}{a^2 - i^2b^2} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

$$= \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$

Commutative Property

$\forall\ (a, b), (c, d) \in C$

$(a, b) . (c, d) = (c, d) . (a, b)$

L.H.S.

$(a, b) . (c, d) = (a + ib)(c + id) = ac + iad + ibc + i^2bd$

$= (ac - bd) + i(ad + bc)$

R.H.S.

$(c, d) . (a, b) = (c + id)(a + ib) = ca + icb + ida + i^2db$

$= (ac - bd) + i(bc + ad)$

$\Rightarrow$ L.H.S. = R.H.S

Q.3 Verify the distributive law of complex numbers.

(a, b) [(c, d) + (e, f)] = (a, b) (c, d) + (a, b) (e, f)

Solution:

To show

$(a, b)[(c, d) + e, f)] = (a, b)(c, d) + (a, b)(e, f)$

L.H.S.

$(a, b)[(c, d) + (e, f)]$

$= (a + ib)[c + id + e + if] = ac + iad + ae + iaf + ibc + i^2bd + ibe + i^2bf \quad \because i^2 = -1$

$= ac + ae - bd - bf + i(ad + af + bc + be)$

R.H.S.

$(a, b)(c, d) + (a, b)(e, f)$

$= (a + ib)(c + id) + (a + ib)(e + if)$

$= ac + iad + ibc + i^2bd + ae + iaf + ibe + i^2bf \qquad \because \ i^2 = -1$

$= ac + ae - bd - bf + i(ad + bc + af + be)$

L. H. S. = R.H.S.

Hence Proved.

Q.4 Simplify the following:

(i) i^9

Solution:

$i^9 = i^8 . i = (i^2)^4 . i = (-1)^4 . i = (1)(i) = i \qquad \because \ i^2 = -1$

(ii) i^{14}

Solution:

$i^{14} = (i^2)^7 = (-1)^7 = -1 \qquad \because \ i^2 = -1$

(iii) $(-i)^{19}$ **(Lahore Board 2004)**

Solution:

$(-i)^{19} = (-1)\, i^{19} = -(1)\, i^{18} . i = -(i^2)^9 . i = -(-1)^9\, i = (-1) . i = i \qquad \because \ i^2 = -1$

(iv) $(-1)^{-21/2}$ **(Lahore Board 2007)**

Solution:

$$(-1)^{-21/2} = [(-1)^{1/2}]^{-21} = [(i^2)^{1/2}]^{-21} = i^{-21}$$

$$= \frac{1}{i^{21}} = \frac{1}{i^{20} . i} = \frac{1}{(i^2)^{10} . i}$$

$$= \frac{1}{(-1)^{10}\, i} = \frac{1}{i} . \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i \qquad \because \ i^2 = -1$$

Q.5 Write in terms of i

Solution:

(i) $\sqrt{-1}\, b$

Solution:

$\sqrt{-1}\, b = ib \qquad \because \sqrt{-1} = i$

(ii) $\sqrt{-5}$

Solution:

$= \sqrt{-1}\,.\sqrt{5} = i\sqrt{5} \because \sqrt{-1} = i$

(iii) $\sqrt{\dfrac{-16}{25}}$

Solution:

$$\sqrt{\frac{-16}{25}} = \sqrt{-1}\sqrt{\frac{16}{25}} = i\left(\frac{4}{5}\right) = \frac{4}{5}i \qquad \because \quad \sqrt{-1} = i$$

(iv) $\sqrt{\dfrac{1}{-4}}$

Solution:

$$\sqrt{\frac{1}{-4}} = \frac{1}{\sqrt{-1}}\sqrt{\frac{1}{4}} = \frac{1}{i}\cdot\frac{1}{2} = \frac{i}{i\,.\,i^2}\frac{1}{2} = \frac{i}{i^2\,.\,2} = \frac{+i}{-2} = \frac{i}{2} \qquad \because \quad i^2 = -1$$

Q.6 Simplify the following:

(7, 9) + (3, – 5)

Solution:

$(7, 9) + (3, -5) = 7 + 9i + 3 - 5i = 10 + 4i = (10, 4)$

Q.7 (8, – 5) – (– 7, 4)

Solution:

$(8, -5) - (-7, 4) = 8 - 5i - (-7 + 4i) = 8 - 5i + 7 - 4i = 15 - 9i = (15, -9)$

Q.8 (2, 6) (3, 7)

Solution:

$$\begin{aligned}(2, 6)(3, 7) &= (2 + 6i)\,.\,(3 + 7i) = 6 + 14i + 18i + 42\,i^2 \\ &= 6 + 32i - 42 = -36 + 32i = (-36, 32) \qquad \because\ i^2 = -1\end{aligned}$$

Q.9 (5, – 4) (– 3, – 2)

Solution:

$$\begin{aligned}(5, -4)(-3, -2) &= (5 - 4i)(-3 - 2i) \\ &= -15 - 10i + 12i + 8i^2 = -15 + 2i - 8 = -23 + 2i = (-23, 2)\end{aligned}$$

Q.10 (0, 3) (0, 5)

Solution:

$$\begin{aligned}(0, 3)(0, 5) &= (0 + 3i)(0 + 5i) \\ &= 0 + 0 + 0 + 15i^2 = 0 + 15(-1) = -15 = -15 + 0i = (-15, 0)\end{aligned}$$

Q.11 (2, 6) ÷ (3, 7)

Solution:

$$\begin{aligned}(2, 6) \div (3, 7) &= (2 + 6i) \div (3 + 7i) = \frac{(2 + 6i)}{(3 + 7i)} \times \frac{(3 - 7i)}{(3 - 7i)} \qquad \text{Rationalizing} \\ &= \frac{(2 + 6i)(3 - 7i)}{(3 + 7i)(3 - 7i)} = \frac{6 - 14i + 18i - 42\,i^2}{(3)^2 - (7i)^2} \qquad \because\ i^2 = -1 \\ &= \frac{6 + 4i + 42}{9 - 49i^2} = \frac{48 + 4i}{9 + 49} = \frac{48 + 4i}{58}\end{aligned}$$

$$= \left(\frac{48}{58}, \frac{4i}{58}\right) = \left(\frac{24}{29}, \frac{2i}{29}\right)$$

Q.12 (5, – 4) ÷ (– 3, – 8)

Solution:

$$(5, -4) \div (-3, -8) = (5 - 4i) \div (-3 - 8i)$$

$$= \frac{5 - 4i}{-3 - 8i} \times \frac{-3 + 8i}{-3 + 8i} \quad \text{Rationalizing}$$

$$= \frac{(5 - 4i)(-3 + 8i)}{(-3 - 8i)(-3 + 8i)} = \frac{-15 + 40i + 12i - 32i^2}{(-3)^2 - (8i)^2}$$

$$= \frac{-15 + 52i + 32}{9 - 64i^2} = \frac{17 + 52i}{9 + 64} = \frac{17 + 52i}{73}$$

$$= \frac{17}{73} + \frac{52}{73}i = \left(\frac{17}{73}, \frac{52}{73}\right)$$

Q.13 Prove that the sum as well as product of any two conjugate complex numbers is a real number.

Solution:

Let $z = a + bi$ is a complex number then its conjugate is $\bar{z} = \overline{a + ib} = a - ib$

Sum $= z + \bar{z} = a + ib + a - ib = 2a$ (real number)

Product $= z \, . \, \bar{z} = (a + ib)(a - ib) = (a)^2 - (ib)^2 = a^2 - i^2 b^2 \qquad \because i^2 = -1$

$= a^2 + b^2$ (real number)

Hence the sum as well as the product of any two conjugate complex numbers is a real number.

Q.14 Find the multiplicative inverse of each of the following numbers.

(i) (– 4, 7) **(Lahore Board 2007)**

Solution:

(i) (– 4, 7)

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(-4, 7) = \left(\frac{-4}{(-4)^2 + (7)^2}, \frac{-7}{(-4)^2 + (7)^2}\right)$

$$= \left(\frac{-4}{16 + 49}, \frac{-7}{16 + 49}\right) = \left(\frac{-4}{65}, \frac{-7}{65}\right)$$

(ii) $(\sqrt{2}, -\sqrt{5})$

Solution:

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(\sqrt{2}, -\sqrt{5}) = \left(\frac{\sqrt{2}}{(\sqrt{2})^2 + (-\sqrt{5})^2}, \frac{+\sqrt{5}}{(\sqrt{2})^2 + (-\sqrt{5})^2}\right)$

$$= \left(\frac{\sqrt{2}}{2+5}, \frac{\sqrt{5}}{2+5}\right) = \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$$

(iii) (1, 0)

As multiplicative inverse of (a, b) $= \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

So multiplicative inverse of (1, 0) $= \left(\dfrac{1}{(1)^2+(0)^2}, \dfrac{-0}{(1)^2+(0)^2}\right)$

$$= \left(\frac{1}{1+0}, 0\right) = (1, 0)$$

Q.15 Factorize the following:

(i) $a^2 + 4b^2$

Solution:

$a^2 + 4b^2$

$= a^2 - (-4b^2) = (a^2) - (i^2 4b^2)$ ¤ $i^2 = -1$

$= (a)^2 - (2bi)^2 = (a + 2bi)(a - 2bi)$

(ii) $9a^2 + 16b^2$ **(Lahore Board 2006)**

Solution:

$9a^2 + 16b^2 = 9a^2 - (-16b^2)$

$= 9a^2 - (i^2 16b^2)$ $\because i^2 = -1$

$= (3a)^2 - (4bi)^2 = (3a + 4bi)(3a - 4bi)$

(iii) $3x^2 + 3y^2$ **(Gujranwala Board 2007)**

Solution:

$3x^2 + 3y^2 = 3[x^2 - (-y)^2] = 3[x^2 - (i^2y^2)]$

$= 3[(x)^2 - (iy)^2] = 3(x + iy)(x - iy)$

Q.16 Separate into real and imaginary parts (write as a simple complex number)

(i) $\dfrac{2-7i}{4+5i}$ **(Lahore Board 2011)**

Solution:

$\dfrac{2-7i}{4+5i} = \dfrac{2-7i}{4+5i} \times \dfrac{4-5i}{4-5i}$ Rationalizing

$= \dfrac{(2-7i)(4-5i)}{(4+5i)(4-5i)} = \dfrac{8-10i-28i+35i^2}{(4)^2-(5i)^2}$ $\because i^2 = -1$

$$= \frac{8 - 38i - 35}{16 - 25i^2} = \frac{-27 - 38i}{16 + 25}$$

$$= \frac{-27 - 38i}{41} = \frac{-27}{41} - \frac{38}{41}i$$

(ii) $\mathbf{\dfrac{(-2 + 3i)^2}{(1 + i)}}$ **(Lahore Board 2003)**

Solution:

$$\frac{(-2 + 3i)^2}{(1 + i)} = \frac{(-2)^2 + (3i)^2 + 2(-2)(3i)}{1 + i}$$

$$= \frac{4 + 9i^2 - 12i}{1 + i} = \frac{4 - 9 - 12i}{1 + i} = \frac{-5 - 12i}{1 + i}$$

$$= \frac{-5 - 12i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{(-5 - 12i)(1 - i)}{(1 + i)(1 - i)} = \frac{-5 + 5i - 12i + 12i^2}{(1)^2 - (i)^2}$$

$$= \frac{-5 + 5i - 12i - 12}{1 - i^2} = \frac{-5 - 7i - 12}{1 - (-1)}$$

$$= \frac{-17 - 7i}{2} = -\frac{17}{2} - \frac{7}{2}i$$

(iii) $\mathbf{\dfrac{i}{1 + i}}$ **(Gujranwala Board 2007)**

Solution:

$$\frac{i}{1 + i}$$

$$= \frac{i}{1 + i} \times \frac{1 - i}{1 - i} = \frac{i(1 - i)}{(1 + i)(1 - i)} = \frac{i - i^2}{(1)^2 - i^2} = \frac{i - (-1)}{1 - (-1)}$$

$$= \frac{i + 1}{1 + 1} = \frac{i + 1}{2}$$

$$= \frac{i}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}i$$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

The Complex Plane: