EXERCISE 2.2

Q. Find from first principles, the derivatives of the following expressions w.r.t. their respective independent variables.

(i) $(ax+b)^3$ ***(L.B 2004)*** **(ii)** $(2x+3)^5$

(iii) $(3t+2)^{-2}$ **(iv)** $(ax+b)^{-5}$

(v) $\dfrac{1}{(az-b)^7}$ ***(L.B 2010)***

Solution:

(i) $(ax+b)^3$

Let
$$y = (ax+b)^3$$
$$y+\delta y = [a(x+\delta x)+b]^3$$
$$\delta y = (ax+a\delta x+b)^3 - y$$
$$\delta y = (ax+b+a\delta x)^3 - (ax+b)^3 \qquad \because\ y=(ax+b)^3$$
$$\delta y = \left[(ax+b)\left(1+\frac{a\delta x}{ax+b}\right)\right]^3 - (ax+b)^3$$
$$\delta y = (ax+b)^3\left(1+\frac{a\delta x}{ax+b}\right)^3 - (ax+b)^3$$
$$\delta y = (ax+b)^3\left[\left(1+\frac{a\delta x}{ax+b}\right)^3 - 1\right]$$
$$\delta y = (ax+b)^3\left[1+3\left(\frac{a\delta x}{ax+b}\right)+\frac{3(3-1)}{2!}\cdot\left(\frac{a\delta x}{ax+b}\right)^2+\dots\dots\dots -1\right]$$
$$\delta y = (ax+b)^3\cdot\frac{a\delta x}{ax+b}\left[3+\frac{3(3-1)}{2!}\cdot\frac{a\delta x}{ax+b}+\dots\dots\dots\right]$$

Dividing both sides by δx.
$$\frac{\delta y}{\delta x} = \frac{(ax+b)^{3-1}\cdot a\delta x}{\delta x}\left[3+\frac{3(3-1)}{2!}\cdot\frac{a\delta x}{ax+b}+\dots\dots\dots\right]$$
$$\frac{\delta y}{\delta x} = a(ax+b)^2\left[3+\frac{3(3-1)}{2!}\cdot\frac{a\delta x}{ax+b}+\dots\dots\dots\right]$$

Taking limit $\delta x \to 0$
$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} a(ax+b)^2\left[3+\frac{3(3-1)}{2!}\cdot\frac{a\delta x}{ax+b}+\dots\dots\dots\right]$$
$$\frac{dy}{dx} = 3a(ax+b)^2$$

$$\boxed{\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2} \qquad \text{Ans.}$$

(ii) $\mathbf{(2x+3)^5}$

Let
$$
\begin{aligned}
y &= (2x+3)^5 \\
y + \delta y &= [2(x+\delta x)+3]^5 \\
\delta y &= (2x+2\delta x+3)^5 - y \\
\delta y &= (2x+3+2\delta x)^5 - (2x+3)^5 \qquad \because\ y = (2x+3)^5 \\
\delta y &= \left[(2x+3)\left(1+\frac{2\delta x}{2x+3}\right)\right]^5 - (2x+3)^5 \\
\delta y &= (2x+3)^5\left(1+\frac{2\delta x}{2x+3}\right)^5 - (2x+3)^5 \\
\delta y &= (2x+3)^5\left[\left(1+\frac{2\delta x}{2x+3}\right)^5 - 1\right] \\
\delta y &= (2x+3)^5\left[1+5\left(\frac{2\delta x}{2x+3}\right)+\frac{5(5-1)}{2!}\cdot\left(\frac{2\delta x}{2x+3}\right)^2+\dots\dots\dots -1\right] \\
\delta y &= (2x+3)^5\,\frac{2\delta x}{2x+3}\left[5+\frac{5(5-1)}{2!}\cdot\frac{2\delta x}{2x+3}+\dots\dots\dots\right]
\end{aligned}
$$

Dividing both sides by δx.
$$
\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{2\delta x\,(2x+3)^{5-1}}{\delta x}\left[5+\frac{5(5-1)}{2!}\cdot\frac{2\delta x}{2x+3}+\dots\dots\dots\right] \\
\frac{\delta y}{\delta x} &= 2(2x+3)^4\left[5+\frac{5(5-1)}{2!}\cdot\frac{2\delta x}{2x+3}+\dots\dots\dots\right]
\end{aligned}
$$

Taking limit $\delta x \to 0$
$$
\begin{aligned}
\lim_{\delta x\to 0}\frac{\delta y}{\delta x} &= \lim_{\delta x\to 0} 2(2x+3)^4\left[5+\frac{5(5-1)}{2!}\cdot\frac{2\delta x}{2x+3}+\dots\dots\dots\right] \\
\frac{dy}{dx} &= 2(2x+3)^4 \cdot 5
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(2x+3)^5 = 10(2x+3)^4} \qquad \text{Ans.}$$

(iii) $\mathbf{(3t+2)^{-2}}$

Let
$$
\begin{aligned}
y &= (3t+2)^{-2} \\
y + \delta y &= [3(t+\delta t)+2]^{-2} \\
\delta y &= (3t+3\delta t+2)^{-2} - y \\
\delta y &= (3t+2+3\delta t)^{-2} - (3t+2)^{-2} \qquad \because\ y = (3t+2)^{-2}
\end{aligned}
$$

$$\delta y = \left[(3t+2)\left(1+\frac{3\delta t}{3t+2}\right)\right]^{-2} - (3t+2)^{-2}$$

$$\delta y = (3t+2)^{-2}\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2}$$

$$\delta y = (3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - 1\right]$$

$$\delta y = (3t+2)^{-2}\left[1+(-2)\left(\frac{3\delta t}{3t+2}\right)+\frac{(-2)(-2-1)}{2!}\cdot\left(\frac{3\delta t}{3t+2}\right)^{2}+\ldots\ldots\ldots -1\right]$$

$$\delta y = (3t+2)^{-2}\ \frac{3\delta t}{3t+2}\left[-2+\frac{(-2)(-2-1)}{2!}\cdot\frac{3\delta t}{3t+2}+\ldots\ldots\ldots\right]$$

Dividing both sides by δt.

$$\frac{\delta y}{\delta t} = \frac{3\delta t}{\delta t(3t+2)^{1+2}}\left[-2+\frac{(-2)(-2-1)}{2!}\cdot\frac{3\delta t}{3t+2}+\ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta t} = \frac{3}{(3t+2)^{3}}\left[-2+\frac{(-2)(-2-1)}{2!}\cdot\frac{3\delta t}{3t+2}+\ldots\ldots\ldots\right]$$

Taking limit $\delta t \to 0$

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta t} = \lim_{\delta t\to 0}\frac{3}{(3t+2)^{3}}\left[-2+\frac{(-2)(-2-1)}{2!}\cdot\frac{3\delta t}{3t+2}+\ldots\ldots\ldots\right]$$

$$\frac{dy}{dt} = \frac{3}{(3t+2)^{3}}\ (-2)$$

$$\boxed{\frac{d}{dt}(3t+2)^{-2} = \frac{-6}{(3t+2)^{3}}}$$ Ans.

(iv) $\mathbf{(ax+b)^{-5}}$

Let $y = (ax+b)^{-5}$

$$y+\delta y = [a(x+\delta x)+b]^{-5}$$

$$\delta y = (ax+a\delta x+b)^{-5} - y$$

$$\delta y = (ax+b+a\delta x)^{-5} - (ax+b)^{-5} \qquad \because\ y=(ax+b)^{-5}$$

$$\delta y = \left[(ax+b)\left(1+\frac{a\delta x}{ax+b}\right)\right]^{-5} - (ax+b)^{-5}$$

$$\delta y = (ax+b)^{-5}\left(1+\frac{a\delta x}{ax+b}\right)^{-5} - (ax+b)^{-5}$$

$$\delta y = (ax+b)^{-5}\left[\left(1+\frac{a\delta x}{ax+b}\right)^{-5} - 1\right]$$

$$\delta y = (ax+b)^{-5}\left[1 + (-5)\left(\frac{a\delta x}{ax+b}\right) + \frac{(-5)(-5-1)}{2!}\cdot\left(\frac{a\delta x}{ax+b}\right)^2 + \ldots\ldots -1\right]$$

$$\delta y = (ax+b)^{-5}\ \frac{a\delta x}{ax+b}\left[-5 + \frac{(-5)(-5-1)}{2!}\cdot\frac{a\delta x}{ax+b} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{a\delta x}{\delta x(ax+b)^{1+5}}\left[-5 + \frac{(-5)(-5-1)}{2!}\cdot\frac{a\delta x}{ax+b} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{a}{(ax+b)^6}\left[-5 + \frac{(-5)(-5-1)}{2!}\cdot\frac{a\delta x}{ax+b} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{a}{(ax+b)^6}\left[-5 + \frac{(-5)(-5-1)}{2!}\cdot\frac{a\delta x}{ax+b} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{a}{(ax+b)^6}\ (-5)$$

$$\boxed{\frac{d}{dt}(ax+b)^{-5} = \frac{-5a}{(ax+b)^6}}$$ Ans.

(v) $\mathbf{\dfrac{1}{(az-b)^7}}$

Let

$$y = \frac{1}{(az-b)^7}$$

$$y = (az-b)^{-7}$$

$$y + \delta y = [a(z+\delta z) - b]^{-7}$$

$$\delta y = (az + a\delta z - b)^{-7} - y$$

$$\delta y = (az - b + a\delta z)^{-7} - (az-b)^{-7} \qquad \because\ y = (az-b)^{-7}$$

$$\delta y = \left[(az-b)\left(1 + \frac{a\delta z}{az-b}\right)\right]^{-7} - (az-b)^{-7}$$

$$\delta y = (az-b)^{-7}\left(1 + \frac{a\delta z}{az-b}\right)^{-7} - (az-b)^{-7}$$

$$\delta y = (az-b)^{-7}\left[\left(1 + \frac{a\delta z}{az-b}\right)^{-7} - 1\right]$$

$$\delta y = (az-b)^{-7}\left[1 + (-7)\left(\frac{a\delta z}{az-b}\right) + \frac{(-7)(-7-1)}{2!}\cdot\left(\frac{a\delta z}{az-b}\right)^2 + \ldots\ldots -1\right]$$

$$\delta y = (az-b)^{-7}\ \frac{a\delta z}{az-b}\left[-7 + \frac{(-7)(-7-1)}{2!}\cdot\frac{a\delta z}{az-b} + \ldots\ldots\ldots\right]$$

Dividing both sides by δz.

$$\frac{\delta y}{\delta z} = \frac{a\delta z}{\delta z(az-b)^{1+7}}\left[-7+\frac{(-7)(-7-1)}{2!}\cdot\frac{a\delta z}{az-b}+\ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta z} = \frac{a}{(az-b)^{8}}\left[-7+\frac{(-7)(-7-1)}{2!}\cdot\frac{a\delta z}{az-b}+\ldots\ldots\ldots\right]$$

Taking limit $\delta z \to 0$

$$\lim_{\delta z\to 0}\frac{\delta y}{\delta z} = \lim_{\delta z\to 0}\frac{a}{(az-b)^{8}}\left[-7+\frac{(-7)(-7-1)}{2!}\cdot\frac{a\delta z}{az-b}+\ldots\ldots\ldots\right]$$

$$\frac{dy}{dz} = \frac{a}{(az-b)^{8}}(-7)$$

$$\boxed{\frac{d}{dz}\left[\frac{1}{(az-b)^{7}}\right] = \frac{-7a}{(az-b)^{8}}}$$ Ans.

EXERCISE 2.3

Q.1 Differentiate w.r.t. 'x'
$x^4 + 2x^3 + x^2$

Solution:

Let $y = x^4 + 2x^3 + x^2$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 4x^{4-1}\cdot\frac{d}{dx}(x) + 2.3\,x^{3-1}\cdot\frac{d}{dx}(x) + 2x^{2-1}\cdot\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = 4x^3 \cdot 1 + 6x^2 \cdot 1 + 2x \cdot 1$$

$$\boxed{\frac{dy}{dx} = 4x^3 + 6x^2 + 2x}$$ Ans.

Q.2 $x^{-3} + 2x^{-3/2} + 2$

Solution:

Let $y = x^{-3} + 2x^{-3/2} + 2$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-3}) + 2\frac{d}{dx}(x^{-3/2}) + \frac{d}{dx}(2)$$