(xi) $\lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$, $x > 0$ $\left(\dfrac{\infty}{\infty}\right)$

$$\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \to 0} \frac{e^{1/x}\left(1 - \frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1 + \frac{1}{e^{1/x}}\right)}$$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}}$$

$$= \frac{1 - \frac{1}{e^{\infty}}}{1 + \frac{1}{e^{\infty}}}$$

$$= \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1 \quad \text{Ans.}$$

Continuous Function

A function f is said to be continuous at a number "c" if and only if the following three conditions are satisfied.

(i) $f(c)$ is defined.

(ii) $\lim\limits_{x \to c} f(x)$ exists.

(iii) $\lim\limits_{x \to c} f(x) = f(c)$

EXERCISE 1.4

Q.1 Determine the left hand limit and right hand limit and then find limits of the following functions at x = c.

(i) $f(x) = 2x^2 + x - 5$, $c = 1$

(ii) $f(x) = \dfrac{x^2 - 9}{x - 3}$, $c = -3$

(iii) $f(x) = |x - 5|$, $c = 5$

Solution:

(i) $f(x) = 2x^2 + x - 5$, $c = 1$

Left hand limit

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x^2 + x - 5)$$
$$= 2(1)^2 + 1 - 5$$
$$= 2 - 4 = -2 \quad \text{Ans.}$$

Right hand limit

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x^2 + x - 5)$$
$$= 2(1)^2 + 1 - 5$$
$$= 2 + 1 - 5$$
$$= -2 \quad \text{Ans.}$$

(ii) $\mathbf{f(x) = \dfrac{x^2 - 9}{x - 3} \quad , \quad c = -3}$

Left hand limit

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{x^2 - 9}{x - 3}$$
$$= \lim_{x\to -3^-} \frac{(x+3)(x-3)}{x - 3}$$
$$= \lim_{x\to -3^-} (x + 3)$$
$$= -3 + 3 = 0 \quad \text{Ans.}$$

Right hand limit

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{x^2 - 9}{x - 3}$$
$$= \lim_{x\to -3^+} \frac{(x+3)(x-3)}{x - 3}$$
$$= \lim_{x\to -3^+} (x + 3)$$
$$= -3 + 3 = 0 \quad \text{Ans.}$$

(iii) $\mathbf{f(x) = |x - 5| \quad , \quad c = 5}$

Left hand limit

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} |x - 5|$$
$$= \lim_{x\to 5^-} -(x - 5)$$
$$= -(5 - 5) = 0 \quad \text{Ans.}$$

Right hand limit

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} |x - 5|$$

$$= \lim_{x \to 5^+} (x - 5)$$

$$= 5 - 5$$

$$= 0 \quad \text{Ans.}$$

Q.2 Discuss the continuity of f(x) at x = c:

(i) $f(x) = \quad , \quad c = 2$ ***(G.B 2007, L.B 2008)***

(ii) $f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}, \quad c = 1$ ***(L.B 2009, L.B 2006)*** ***(L.B 2009, G.B 2007)***

Solution:

(i) $f(x) = \begin{cases} 2x + 5 & \text{if } x \le 2 \\ 4x + 1 & \text{if } x > 2 \end{cases}, \quad c = 2$

$$f(2) = 2(2) + 5$$

$$= 4 + 5$$

$$= 9$$

Left hand limit

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 5)$$

$$= 2(2) + 5$$

$$= 4 + 5 = 9$$

Right hand limit

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4x + 1)$$

$$= 4(2) + 1$$

$$= 8 + 1$$

$$= 9$$

$\therefore$ Left hand limit = Right hand limit

So $\lim_{x \to 2} f(x)$ exists

$\therefore$ $f(2) = \lim_{x \to 2} f(x) = 9$

So the function is continuous at x = 2.

(ii) $\quad f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}$, $\quad c = 1$ ***(L.B 2006, 2007)***

$f(1) = 4$

Left hand limit

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 1)$$
$$= 3(1) - 1$$
$$= 3 - 1$$
$$= 2$$

Right hand limit

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x)$$
$$= 2(1) = 2$$

$\therefore$ Left hand limit = Right hand limit

So $\lim_{x \to 1} f(x)$ exists

$\therefore$ $f(1) \neq \lim_{x \to 1} f(x)$

So the function is discontinuous at $x = 1$.

Q.3 If $\quad f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$ ***(L.B 2011)***

Discuss continuity at $x = 2$ and $x = -2$.

Solution:

At $x = 2$

$f(2) = 3$

Left hand limit

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 1)$$
$$= 2^2 - 1 = 4 - 1 = 3$$

Right hand limit

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3$$
$$= 3$$

$\therefore$ Left hand limit = Right hand limit

So, $\lim_{x \to 2} f(x)$ exists

$\therefore \quad f(2) = \lim_{x \to 2} f(x) = 3$

So the function is continuous at $x = 2$.

At $\quad x = -2$

$f(-2) = 3(-2) = -6$

Left hand limit.

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (3x)$$
$$= 3(-2) = -6$$

Right hand limit.

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 1)$$
$$= (-2)^2 - 1$$
$$= 4 - 1$$
$$= 3$$

$\therefore$ Left hand limit $\neq$ Right hand limit

So, $\lim_{x \to -2} f(x)$ does not exists.

$\therefore \quad f(-2) \neq \lim_{x \to -2} f(x)$

So the function is discontinuous at $x = -2$.

Q.4 If $f(x) = \begin{cases} x + 2 & , \ x \leq -1 \\ c + 2 & , \ x > -1 \end{cases}$ **find 'c' so that** $\lim_{x \to -1} f(x)$ **exists.** ***(L.B 2009 Supply) (G.B 2008)***

Solution:

Left hand limit

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x + 2)$$
$$= -1 + 2 = 1$$

Right hand limit

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (c + 2)$$
$$= c + 2$$

Since $\lim_{x \to -1} f(x)$ exists.

$\therefore$ Left hand limit $=$ Right hand limit

$$1 = c + 2$$
$$c = 1 - 2$$
$$\boxed{c = -1} \quad \text{Ans.}$$

Q.5 Find the values m and n, So that given function f is continuous at x = 3:

(i) $f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases}$ **(ii)** $f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$

Solution:

(i) $f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases}$ ***(L.B 2004, 2005) (G.B 2006, 2009)***

$f(3) = n$

Left hand limit

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (mx)$$
$$= 3m$$

Right hand limit

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-2x + 9)$$
$$= -2(3) + 9$$
$$= -6 + 9$$
$$= 3$$

Since f(x) is continuous at x = 3

$\therefore$ Left hand limit = Right hand limit = f(3)

$$3m = 3 = n$$
$$3m = 3 \quad , \quad 3 = n$$
$$m = \frac{3}{3} \qquad n = 3$$
$$m = 1$$
$$\boxed{\therefore \ m = 1 \ , \ n = 3} \qquad \text{Ans.}$$

(ii) $f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$ ***(L.B 2007)***

$f(3) = (3)^2 = 9$

Left hand limit

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (mx)$$
$$= 3m$$

Right hand limit

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2)$$
$$= 3^2 = 9$$

Since f(x) is continuous at x = 3

$\therefore$ Left hand limit = Right hand limit = f(3)

$$3m = 9 = 9$$
$$3m = 9$$
$$m = \frac{9}{3} = 3 \quad \text{Ans.}$$

Q.6: If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

(G.B 2004)
(L.B 2009 (s) 2004)
(G.B 2006)
(L.B 2008)
(G.B 2008)

Find value of k so that f is continuous at x = 2.

Solution:

$$f(2) = k$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{x \to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \lim_{x \to 2} \frac{\left(\sqrt{2x+5}\right)^2-\left(\sqrt{x+7}\right)^2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x \to 2} \frac{(2x+5)-(x+7)}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x \to 2} \frac{x-2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x \to 2} \frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}}$$

$$= \frac{1}{\sqrt{4+5}+\sqrt{9}}$$

$$= \frac{1}{3+3} = \frac{1}{6}$$

Since f(x) is continuous at x = 2

$$\therefore \quad f(2) = \lim_{x \to 2} f(x)$$

$$\boxed{k = \frac{1}{6}} \quad \text{Ans.}$$