EXERCISE 2.4

Q.1 Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i) $y = \sqrt{\frac{1-x}{1+x}}$ (ii) $y = \sqrt{x+\sqrt{x}}$

(iii) $y = \sqrt{\frac{a+x}{a-x}}$ (iv) $y = (3x^2 - 2x + 7)^6$

(v) $y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}$

Solution:

(i) $y = \sqrt{\frac{1-x}{1+x}}$

Let $u = \frac{1-x}{1+x}$ So $y = \sqrt{u} = u^{1/2}$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$\frac{du}{dx} = \frac{(1+x)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{(1+x)(0-1) - (1-x)(0+1)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-2}{(1+x)^2}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du}(u^{1/2})$$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\frac{dy}{du} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}\,(1+x)^{2-\frac{1}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}\,(1+x)^{3/2}}} \quad \text{Ans.}$$

(ii) $\mathbf{y = \sqrt{x+\sqrt{x}}}$ ***(G.B 2007)***

Let

$$u = x + \sqrt{x} \qquad \text{So} \quad y = \sqrt{u} = u^{1/2}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{1/2})$$

$$\frac{du}{dx} = 1 + \frac{1}{2}x^{-1/2}$$

$$\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'

$$\frac{dy}{du} = \frac{d}{du}(u^{1/2})$$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \cdot 1$$

$$\frac{dy}{du} = \frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{x+\sqrt{x}}}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x}+1}{4\sqrt{x}\ \sqrt{x+\sqrt{x}}} \qquad \text{Ans.}$$

(iii) $\mathbf{y = x\sqrt{\frac{a+x}{a-x}}}$

Put,

$$u = \frac{a+x}{a-x} \qquad \text{So} \qquad y = x\sqrt{u} = xu^{1/2}$$

Now,

$$u = xu^{1/2}$$

Diff. w.r.t. 'x'.

$$\frac{dy}{dx} = \frac{d}{dx}(xu^{1/2})$$

$$\frac{dy}{dx} = x\frac{d}{dx}(u^{1/2}) + u^{1/2} \cdot \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = \frac{x}{2}\ u^{-1/2}\frac{du}{dx} + u^{1/2} \qquad \ldots\ldots\ldots (1)$$

$$u = \frac{a+x}{a-x}$$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$\frac{du}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{(a-x)(0+1)-(a+x)(0-1)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{a-x+a+x}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{2a}{(a-x)^2}$$

$\therefore$ From equation (1).

$$\frac{dy}{dx} = \frac{x}{2}\left(\frac{a+x}{a-x}\right)^{-1/2} \cdot \frac{2a}{(a-x)^2} + \left(\frac{a+x}{a-x}\right)^{1/2}$$

$$\frac{dy}{dx} = ax\frac{(a+x)^{-1/2}}{(a-x)^{-1/2}} \cdot \frac{1}{(a-x)^2} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x}\,(a-x)^{-1/2+2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x}\,(a-x)^{3/2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax+(a+x)(a-x)}{\sqrt{a+x}\,(a-x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{ax+a^2-x^2}{\sqrt{a+x}\,(a-x)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{a^2-x^2+ax}{\sqrt{a+x}\,(a-x)^{3/2}}} \quad \text{Ans.}$$

(iv) $\mathbf{y = (3x^2 - 2x + 7)^6}$ ***(L.B 2009 (s))***

Let $u = 3x^2 - 2x + 7$ So $y = u^6$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = 3\frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) + \frac{d}{dx}(7)$$

$$= 3(2x) - 2(1) + 0$$

$$\frac{du}{dx} = 6x - 2$$

$$y = u^6$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du}(u^6)$$

$$\frac{dy}{du} = 6u^5$$

$$\frac{dy}{du} = 6(3x^2 - 2x + 7)^5$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 6(3x^2 - 2x + 7)^5 \, . \, (6x - 2)$$

$$\boxed{\frac{dy}{dx} = 12(3x^2 - 2x + 7)^5 \, (3x - 1)} \qquad \text{Ans.}$$

(v) $\mathbf{y = \sqrt{\dfrac{a^2 + x^2}{a^2 - x^2}}}$ ***(G.B 2004)***

Let $u = \dfrac{a^2 + x^2}{a^2 - x^2}$ So $y = \sqrt{u} = u^{1/2}$

Diff. w.r.t. 'x'.

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{a^2 + x^2}{a^2 - x^2}\right)$$

$$\frac{du}{dx} = \frac{(a^2 - x^2)\dfrac{d}{dx}(a^2 + x^2) - (a^2 + x^2)\dfrac{d}{dx}(a^2 - x^2)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{(a^2 - x^2)2x - (a^2 + x^2) \, . \, (-2x)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(a^2 - x^2 + a^2 + x^2)}{(a^2 - x^2)^2}$$

$$\frac{du}{dx} = \frac{2x(2a^2)}{(a^2 - x^2)^2} = \frac{4a^2x}{(a^2 - x^2)^2}$$

$$y = u^{1/2}$$

Diff. w.r.t. 'u'.

$$\frac{dy}{du} = \frac{d}{du}(u^{-1/2})$$

$$\frac{dy}{du} = \frac{1}{2} \, u^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{2}\left(\frac{a^2 + x^2}{a^2 - x^2}\right)^{-1/2}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \, . \, \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 + x^2}{a^2 - x^2}\right)^{-1/2} \cdot \frac{4a^2x}{(a^2 - x^2)^2}$$

$$\frac{dy}{dx} = \frac{(a^2 + x^2)^{-1/2}}{(a^2 - x^2)^{-1/2}} \cdot \frac{2a^2x}{(a^2 - x^2)^2}$$

$$\frac{dy}{dx} = \frac{2a^2x}{(a^2+x^2)^{1/2}(a^2-x^2)^{-1/2+2}}$$

$$\boxed{\frac{dy}{dx} = \frac{2a^2x}{\sqrt{a^2+x^2}\,(a^2-x^2)^{3/2}}} \quad \text{Ans.}$$

Q.2 Find $\frac{dy}{dx}$ if:

(i) $3x + 4y + 7 = 0$ **(ii)** $xy + y^2 = 2$

(iii) $x^2 - 4xy - 5y = 0$ **(iv)** $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ **(vi)** $y(x^2-1) = x\sqrt{x^2+4}$

Solution:

(i) $3x + 4y + 7 = 0$

Diff. w.r.t 'x'.

$$3\frac{d}{dx}(x) + 4\frac{dy}{dx} + \frac{d}{dx}(7) = 0$$

$$3.1 + 4\frac{dy}{dx} = 0$$

$$4\frac{dy}{dx} = -3$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{4}} \quad \text{Ans.}$$

(ii) $xy + y^2 = 2$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2)$$

$$x\frac{dy}{dx} + y\frac{d}{dx}(x) + 2y\frac{dy}{dx} = 0$$

$$(x+2y)\frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x+2y}} \quad \text{Ans.}$$

(iii) $x^2 - 4xy - 5y = 0$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) - 5\frac{dy}{dx} = 0$$

$$2x - 4\left[x\frac{dy}{dx} + y\,.\,1\right] - 5\frac{dy}{dx} = 0$$

$$2x - 4x\frac{dy}{dx} - 4y - 5\frac{dy}{dx} = 0$$

$$-(4x+5)\frac{dy}{dx} = -2x + 4y$$

$$\frac{dy}{dx} = \frac{-2(x-2y)}{-(4x+5)}$$

$$\boxed{\frac{dy}{dx} = \frac{2(x-2y)}{4x+5}} \qquad \text{Ans.}$$

(iv) $\mathbf{4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$

Diff. w.r.t 'x'

$$4\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2g\frac{d}{dx}(x) + 2f\frac{dy}{dx} + \frac{d}{dx}(c) = 0$$

$$4(2x) + 2h\left[x\frac{dy}{dx} + y\,.\,1\right] + b\,.\,2y\frac{dy}{dx} + 2g\,.\,1 + 2f\frac{dy}{dx} + 0 = 0$$

$$8x + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2(hx + by + f)\frac{dy}{dx} = -8x - 2hy - 2g$$

$$\frac{dy}{dx} = \frac{-2(4x + hy + g)}{2(hx + by + f)}$$

$$\boxed{\frac{dy}{dx} = -\frac{4x + hy + g}{hx + by + f}} \qquad \text{Ans.}$$

(v) $\mathbf{x\sqrt{1+y} + y\sqrt{1+x} = 0}$ ***(L.B 2007) (G.B 2007)***

Diff. w.r.t 'x'.

$$\frac{d}{dx}\left(x\sqrt{1+y}\right) + \frac{d}{dx}\left(y\sqrt{1+x}\right) = 0$$

$$x\frac{d}{dx}\left(\sqrt{1+y}\right) + \sqrt{1+y}\frac{d}{dx}(x) + y\frac{d}{dx}\left(\sqrt{1+x}\right) + \sqrt{1+x}\frac{dy}{dx} = 0$$

$$x\,.\,\frac{1}{2}(1+y)^{-1/2}\frac{dy}{dx} + \sqrt{1+y}\,.\,1 + y\,.\,\frac{1}{2}(1+x)^{-1/2}\,.\,1 + \sqrt{1+x}\frac{dy}{dx} = 0$$

$$\left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right)\frac{dy}{dx} + \sqrt{1+y} + \frac{y}{2\sqrt{1+x}} = 0$$

$$\left(\frac{x + 2\sqrt{1+x}\sqrt{1+y}}{2\sqrt{1+y}}\right)\frac{dy}{dx} = -\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{1+y}}{x+2\sqrt{1+x}\sqrt{1+y}}\left[\frac{-2\sqrt{1+x}\sqrt{1+y}-y}{2\sqrt{1+x}}\right]$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y}\left(y+2\sqrt{1+x}\sqrt{1+y}\right)}{\sqrt{1+x}\left(x+2\sqrt{1+x}\sqrt{1+y}\right)} \text{ Ans.}$$

(vi) $\mathbf{y(x^2-1) = x\sqrt{x^2+4}}$

Diff. w.r.t 'x'.

$$\frac{d}{dx}[y(x^2-1)] = \frac{d}{dx}\left(x\sqrt{x^2+4}\right)$$

$$y\frac{d}{dx}(x^2-1)+(x^2-1)\frac{dy}{dx} = x\frac{d}{dx}\left(\sqrt{x^2+4}\right)+\sqrt{x^2+4}\frac{d}{dx}(x)$$

$$y\,.\,2x+(x^2-1)\frac{dy}{dx} = x\frac{1}{2}(x^2+4)^{-1/2}\,.\,2x+\sqrt{x^2+4}$$

$$(x^2-1)\frac{dy}{dx} = \frac{x^2}{\sqrt{x^2+4}}+\sqrt{x^2+4}-2xy$$

$$\frac{dy}{dx} = \frac{1}{x^2-1}\left[\frac{x^2+x^2+4-2xy\sqrt{x^2+4}}{\sqrt{x^2+4}}\right]$$

$$\frac{dy}{dx} = \frac{2x^2+4-2xy\sqrt{x^2+4}}{(x^2-1)\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{2x^2+4-2x\sqrt{x^2+4}\,.\,\frac{x\sqrt{x^2+4}}{x^2-1}}{(x^2-1)\sqrt{x^2+4}} \qquad \because\ y=\frac{x\sqrt{x^2+4}}{x^2-1}$$

$$\frac{dy}{dx} = \frac{2x^2+4-\frac{2x^2(x^2+4)}{x^2-1}}{(x^2-1)\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{\frac{2x^2(x^2-1)+4(x^2-1)-2x^2(x^2+4)}{x^2-1}}{(x^2-1)\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{2x^4-2x^2+4x^2-4-2x^4-8x^2}{(x^2-1)^2\sqrt{x^2+4}}$$

$$= \frac{-6x^2-4}{(x^2-1)^2\sqrt{x^2+4}}$$

$$\boxed{\frac{dy}{dx} = \frac{-2(3x^2+2)}{(x^2-1)^2\sqrt{x^2+4}}} \qquad \text{Ans.}$$

Q.3 Find $\frac{dy}{dx}$ of the following parametric functions.

(i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$ (ii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

Solution:

(i) $x = \theta + \frac{1}{\theta}$ and $y = \theta + 1$

$$x = \theta + \theta^{-1}$$

Diff. w.r.t. 'θ'

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1})$$

$$\frac{dx}{d\theta} = 1 + (-1)\theta^{-2}$$

$$\frac{dx}{d\theta} = 1 - \frac{1}{\theta^2}$$

$$\frac{dx}{d\theta} = \frac{\theta^2 - 1}{\theta^2}$$

$$y = \theta + 1$$

Diff. w.r.t. 'θ'

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(1)$$

$$\frac{dy}{d\theta} = 1 + 0 = 1$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \Rightarrow \frac{dy}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1}$$

$$\boxed{\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}} \quad \text{Ans.}$$

(ii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

$$x = \frac{a(1-t^2)}{1+t^2}$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{a\,2t(-1-t^2-1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2at(-2)}{(1+t^2)^2}$$

$$y = \frac{2bt}{1+t^2}$$

Diff. w.r.t. 't'

$$\frac{dy}{dt} = 2b\frac{d}{dt}\left[\frac{t}{1+t^2}\right]$$

$$\frac{dy}{dt} = 2b\left[\frac{(1+t^2)\frac{d}{dt}(t) - t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = 2b\left[\frac{(1+t^2) - t.2t}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \frac{2b(1+t^2-2t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

By using chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at}$$

$$\frac{dy}{dx} = -\frac{b(1-t^2)}{2at} \qquad \text{Ans.}$$

Q.4: Prove that $y\frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ ***(Guj. Board 2005, 2008)***

Solution:

$$x = \frac{1-t^2}{1+t^2} \quad , \quad y = \frac{2t}{1+t^2}$$

$$x = \frac{1-t^2}{1+t^2}$$

Diff. w.r.t. 't'

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$\frac{dx}{dt} = \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2t(-1-t^2-1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2t(-2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$$y = \frac{2t}{1+t^2}$$

Diff. w.r.t 't'

$$\frac{dy}{dt} = 2\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

$$\frac{dy}{dt} = 2\left[\frac{(1+t^2)\frac{d}{dt}(t) - t\frac{d}{dt}(+t^2)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = 2\left[\frac{(1+t^2).1 - t.2t}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \frac{2(1+t^2-2t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

By using chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)}{-4t}$$

$$\frac{dy}{dx} = \frac{-(1-t^2)}{2t} = \frac{t^2-1}{2t}$$

Taking

$$y\frac{dy}{dx} + x = \frac{2t}{1+t^2}\left(\frac{t^2-1}{2t}\right) + \frac{1-t^2}{1+t^2}$$

$$= \frac{t^2-1}{1+t^2} + \frac{1-t^2}{1+t^2}$$

$$= \frac{t^2-1+1-t^2}{1+t^2}$$

$$= \frac{0}{1+t^2} = 0$$ Hence proved.

Q.5: Differentiate

(i) $x^2 - \frac{1}{x^2}$ **w.r.t.** x^4 **(ii)** $(1+x^2)^n$ **w.r.t.** x^2

(iii) $\frac{x^2+1}{x^2-1}$ **w.r.t.** $\frac{x-1}{x+1}$ **(iv)** $\frac{ax+b}{cx+d}$ **w.r.t.** $\frac{ax^2+b}{ax^2+d}$

(v) $\frac{x^2+1}{x^2-1}$ **w.r.t.** x^3

Solution:

(i) $x^2 - \frac{1}{x^2}$ **w.r.t.** x^4 ***(L.B 2006)***

Let $y = x^2 - \frac{1}{x^2}$, $u = x^4$

$y = x^2 - x^{-2}$

Diff. w.r.t. 'x'

$u = x^4$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = 2x - (-2)x^{-3}$$
$$= 2x + \frac{2}{x^3}$$
$$= \frac{2x^4 + 2}{x^3}$$
$$= \frac{2(x^4 + 1)}{x^3}$$

$$\frac{du}{dx} = 4x^3$$

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$
$$\frac{dy}{du} = \frac{2(x^4 + 1)}{x^3} \times \frac{1}{4x^3}$$
$$\frac{dy}{du} = \frac{x^4 + 1}{2x^6} \quad \text{Ans}$$

(ii) $\mathbf{(1 + x^2)^n}$ **w.r.t.** $\mathbf{x^2}$

Let

$$y = (1 + x^2)^n \quad , \quad u = x^2$$

$$y = (1 + x^2)^n$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(1 + x^2)^n$$
$$= n(1 + x^2)^{n-1} . 2x$$
$$= 2nx(1 + x^2)^{n-1}$$

$$u = x^4$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}x^2$$
$$\frac{du}{dx} = 2x$$

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dy}{du}$$
$$= 2nx(1 + x^2)^{n-1} \cdot \frac{1}{2x}$$

$$\boxed{\frac{dy}{du} = n(1 + x^2)^{n-1}} \quad \text{Ans.}$$

(iii) $\mathbf{\frac{x^2 + 1}{x^2 - 1}}$ **w.r.t.** $\mathbf{\frac{x - 1}{x + 1}}$

Let $y = \frac{x^2 + 1}{x^2 - 1}$, $u = \frac{x - 1}{x + 1}$

$$y = \frac{x^2+1}{x^2-1}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{2x\,(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$= \frac{2x\,(-2)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

$$u = \frac{x-1}{x+1}$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$= \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{x+1-x+1}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{2x}{(x+1)^2}$$

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$= \frac{-4x}{(x^2-1)^2} \times \frac{(x+1)^2}{2}$$

$$= \frac{-2x\,(x+1)^2}{[(x+1)(x-1)]^2}$$

$$= \frac{-2x\,(x+1)^2}{(x+1)^2(x-1)^2}$$

$$\boxed{\frac{dy}{du} = \frac{-2x}{(x-1)^2}} \quad \text{Ans.}$$

(iv) $\mathbf{\dfrac{ax+b}{cx+d}}$ **w.r.t.** $\mathbf{\dfrac{ax^2+b}{ax^2+d}}$

Let $y = \dfrac{ax+b}{cx+d}$, $u = \dfrac{ax^2+b}{ax^2+d}$

$$y = \frac{ax+b}{cx+d}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)$$

$$u = \frac{ax^2+b}{ax^2+d}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{ax^2+b}{ax^2+d}\right)$$

$$= \frac{(cx + d)\frac{d}{dx}(ax + b) - (ax + b)\frac{d}{dx}(cx + d)}{(cx + d)^2}$$

$$= \frac{(cx + d)a - (ax + b).c}{(cx + d)^2}$$

$$= \frac{acx + ad - acx - bc}{(cx + d)^2}$$

$$\frac{dy}{dx} = \frac{ad - bc}{(cx + d)^2}$$

$$= \frac{(ax^2 + d)\frac{d}{dx}(ax^2 + b) - (ax^2 + b)\frac{d}{dx}(ax^2 + d)}{(ax^2 + d)^2}$$

$$= \frac{(ax^2 + d)(2ax) - (ax^2 + b)(2ax)}{(ax^2 + d)^2}$$

$$\frac{du}{dx} = \frac{2ax\,(ax^2 + d - ax^2 - b)}{(ax^2 + d)^2}$$

$$\frac{du}{dx} = \frac{2ax\,(d - b)}{(ax^2 + d)^2}$$

By using chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$= \frac{ad - bc}{(cx + d)^2} \cdot \frac{(ax^2 + d)^2}{2ax\,(d - b)}$$

$$\boxed{\frac{dy}{du} = \frac{(ad - bc)\,(ax^2 + d)^2}{2ax\,(cx + d)^2(d - b)}} \qquad \text{Ans.}$$

(v) $\quad \mathbf{\dfrac{x^2 + 1}{x^2 - 1}}$ **w.r.t.** $\mathbf{x^3}$ ***(G.B 2003)***

Let

$$y = \frac{x^2 + 1}{x^2 - 1} \quad , \quad u = x^3$$

$$y = \frac{x^2 + 1}{x^2 - 1}$$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2 + 1}{x^2 - 1}\right)$$

$$= \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{2x\,(-2)}{(x^2 - 1)^2}$$

$$= \frac{-4x}{(x^2 - 1)^2}$$

$$u = x^3$$

Diff. w.r.t. 'x'

$$\frac{du}{dx} = \frac{d}{dx}\,(x^3)$$

$$= 3x^2$$

By using chain rule

$$\frac{dx}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dx}{du} = \frac{-4x}{(x^2-1)^2}\times\frac{1}{3x^2}$$

$$\frac{dx}{du} = \frac{-4x}{3x(x^2-1)^2} \qquad \text{Ans.}$$

EXERCISE 2.5

Q.1: Differentiate the following trigonometric functions from the first principles.

(i) $\sin 2x$ **(ii)** $\tan 3x$ **(iii)** $\sin 2x + \cos 2x$

(iv) $\cos x^2$ **(v)** $\tan^2 x$ **(vi)** $\sqrt{\tan x}$

(vii) $\cos\sqrt{x}$

Solution:

(i) sin 2x ***(L.B 2003)***

Let $y = \sin 2x$

$$y + \delta y = \sin 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) - y$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x \qquad \because y = \sin 2x$$

$$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right).\ \sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$\left[\because \sin p - \sin q = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)\right]$$

$$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \sin\left(\frac{2\delta x}{2}\right)$$

$$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \sin(\delta x)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \frac{\sin \delta x}{\delta x}$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 2\cos\left(\frac{4x + 2\delta x}{2}\right).\ \frac{\sin \delta x}{\delta x}$$