$c = 22.24$ sq. units.

Q.6 One side of a triangular garden is 30m. If its two corner angles are $22^o\frac{1}{2}$ and $112^o\frac{1}{2}$. Find the cost of planting the grass at the rate of Rs. 5 per square meter.

Solution:

$a = 30$

$\beta = 22^o\frac{1}{2} = 22.5° = 22^o\,30'$

$\gamma = 112^o\frac{1}{2} = 112^o 30'$

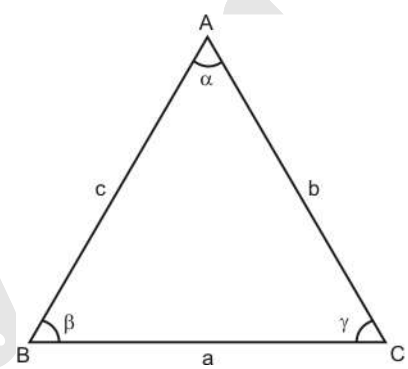
$\alpha = ?$

$\because \alpha + \beta + \gamma = 180°$

$\alpha = 180^o - \beta - \gamma$

$= 180^o - 22^o\,30' - 112^o\,30'$

$\alpha = 45^o$



$$\text{Area of triangle} = \frac{1}{2}a^2\frac{\sin\beta\,\sin\gamma}{\sin\alpha}$$

$$= \frac{1}{2}(30)^2\frac{\sin 22^o\,30'\,\sin 112^o\,30'}{\sin 45^o}$$

$\Delta = 225$ sq. m

Grass planting @ Rs. 5/sq. m $= 225 \times 5$

$=$ Rs. 1125 Ans.

EXERCISE 12.8

Q.1 Show that

(i) $\mathbf{r = 4\,R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$

(ii) $\mathbf{S = 4\,R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

Solution:

(i) $\mathbf{r = 4\,R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4\,R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\,\Delta}\sqrt{\frac{(S-b)\,(S-c)}{bc}}\sqrt{\frac{(S-a)\,(S-c)}{ac}}\sqrt{\frac{(S-a)\,(S-b)}{ab}}$$

$$= \frac{a\,b\,c}{\Delta} \sqrt{\frac{(S-b)^2 (S-c)^2 (S-a)^2}{a^2 b^2 c^2}}$$

$$= \frac{a\,b\,c}{\Delta} \frac{(S-a)(S-b)(S-c)}{a\,b\,c}$$

$$= \frac{1}{\Delta} \frac{S(S-a)(S-b)(S-c)}{S} \quad \text{(multiply and dividing by } S)$$

$$= \frac{1}{\Delta} \times \frac{\Delta^2}{S}$$

$$= \frac{\Delta}{S} = r$$

= L.H.S. Hence proved.

(ii) $\mathbf{S = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\frac{a\,b\,c}{4\,\Delta} \sqrt{\frac{S(S-a)}{bc}} \sqrt{\frac{S(S-b)}{ac}} \sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{a\,b\,c}{\Delta} \sqrt{\frac{S^3 (S-a)(S-b)(S-c)}{a^2 b^2 c^2}}$$

$$= \frac{S\,abc}{\Delta\,abc} \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \frac{S}{\Delta} \times \Delta$$

$$= S = \text{L.H.S.}$$

Hence proved.

Q.2 Show that $\mathbf{r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}}$

Solution:

$$\text{R.H.S.} = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$$

$$= a\sqrt{\frac{(S-a)(S-c)}{ac}} \sqrt{\frac{(S-a)(S-b)}{ab}} \sqrt{\frac{bc}{S(S-a)}}$$

$$= a\sqrt{\frac{(S-a)(S-c)(S-a)(S-b)\,bc}{a^2\,bc\,S(S-a)}}$$

$$= \frac{a^2}{a}\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}$$

Next R.H.S. $= b \sin\frac{\gamma}{2} \sin\frac{\alpha}{2} \sec\frac{\beta}{2}$

$$= b\sqrt{\frac{(S-a)(S-b)}{ab}}\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{ac}{S(S-b)}}$$

$$= b\sqrt{\frac{(S-a)(S-b)(S-b)(S-c)\,ac}{a\,b^2\,c\,S^2\,(S-b)}}$$

$$= \frac{b}{b}\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}$$

Next R.H.S. $= c \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sec\frac{\gamma}{2}$

$$= c\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{ab}{S(S-c)}}$$

$$= c\sqrt{\frac{(S-b)(S-c)(S-c)(S-a)\,ab}{ba\,c^2\,S\,(S-c)}}$$

$$= \frac{c}{c}\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved.}$$

Q.3 Prove that

(i) $\mathbf{r_1 = 4\,R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}}$

(ii) $\mathbf{r_2 = 4\,R \cos\frac{\alpha}{2} \sin\frac{\beta}{2} \cos\frac{\gamma}{2}}$

(iii) $\mathbf{r_3 = 4\,R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \sin\frac{\gamma}{2}}$

Solution:

(i) $\mathbf{r_1 = 4\,R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}}$

R.H.S. $= 4\,R \sin\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

$$= 4\,\frac{a\,b\,c}{4\,\Delta}\sqrt{\frac{(S-b)\,(S-c)}{bc}}\;\sqrt{\frac{S\,(S-b)}{ac}}\;\sqrt{\frac{S\,(S-c)}{ab}}$$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{(S-b)^2\,(S-c)^2\,S^2}{a^2\,b^2\,c^2}}$$

$$= \frac{abc}{\Delta\,abc}\;\frac{S(S-b)(S-c)(S-a)}{S-a}$$

$$= \frac{\Delta^2}{\Delta\,(S-a)}$$

$$= \frac{\Delta}{S-a} = r_1 \quad \text{L.H.S.} \quad \text{Hence proved}$$

(ii) $\mathbf{r_2 = 4\,R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4\,R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\,\Delta}\right)\sqrt{\frac{S\,(S-a)}{bc}}\sqrt{\frac{(S-a)\,(S-c)}{a\,c}}\;\sqrt{\frac{S(S-c)}{ac}}$$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{S^2\,(S-a)^2\,(S-c)^2}{a^2\,b^2\,c^2}}$$

$$= \frac{a\,b\,c}{\Delta} \times \frac{S(S-a)\,(S-c)}{a\,b\,c}$$

$$= \frac{S\,(S-a)\,(S-b)\,(S-c)}{(S-b)\,\Delta}$$

$$= \frac{\Delta^2}{\Delta\,(S-b)}$$

$$= \frac{\Delta}{S-b} = r_2 = \text{L.H.S.} \quad \text{Hence proved}$$

Hence proved.

(iii) $\mathbf{r_3 = 4\,R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4\,R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\,\Delta}\right)\sqrt{\frac{S\,(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ab}}\sqrt{\frac{(S-a)\,(S-b)}{a\,b}}$$

$$= \frac{a\,b\,c}{\Delta}\sqrt{\frac{S^2\,(S-a)^2\,(S-b)^2}{a^2\,b^2\,c^2}}$$

$$= \frac{a\,b\,c}{\Delta}\;\frac{S(S-a)\,(S-b)\,(S-c)}{a\,b\,c\,(S-c)}$$

$$= \frac{\Delta^2}{\Delta\,(S-c)}$$

$$= \frac{\Delta}{S-c} = r_3 = \text{L.H.S.}\ \text{Hence proved.}$$

Q.4 Show that

(i) $\mathbf{r_1 = S\tan\frac{\alpha}{2}}$

(ii) $\mathbf{r_2 = S\tan\frac{\beta}{2}}$ **(Lahore Board 2007)**

(iii) $\mathbf{r_3 = S\tan\frac{\gamma}{2}}$ **(Gujranwala Board 2005)**

Solution:

(i) $\mathbf{r_1 = S\tan\frac{\alpha}{2}}$

$$\text{R.H.S.} = S\tan\frac{\alpha}{2}$$

$$= S\sqrt{\frac{(S-b)\,(S-c)}{S\,(S-a)}}$$

$$= S\sqrt{\frac{S\,(S-a)\,(S-b)\,(S-c)}{S^2\,(S-a)^2}}$$

$$= S\,\frac{\sqrt{S\,(S-a)\,(S-b)\,(S-c)}}{S\,(S-a)}$$

$$= \frac{\Delta}{S-a} = r_1 = \text{L.H.S.}\ \text{Hence proved}$$

(ii) $\mathbf{r_2 = S \tan \frac{\beta}{2}}$

$$\text{R.H.S.} = S \tan \frac{\alpha}{2}$$

$$= S\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}$$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-b)^2}}$$

$$= S\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-b)}$$

$$= \frac{\Delta}{S-b} = r_2 = \text{L.H.S. Hence proved}$$

(iii) $\mathbf{r_3 = S \tan \frac{\gamma}{2}}$

$$\text{R.H.S.} = S \tan \frac{\gamma}{2}$$

$$= S\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

$$= S\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}}$$

$$= S\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-c)}$$

$$= S \times \frac{\Delta}{S(S-b)} = r_3 = \text{L.H.S. Hence proved}$$

Q.5 Prove that

(i) $\mathbf{r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2}$ **(Lahore Board 2009)**

(ii) $\mathbf{r\, r_1 r_2 r_3 = \Delta^2}$

(iii) $\mathbf{r_1 + r_2 + r_3 - r = 4R}$

(iv) $\mathbf{r_1 r_2 r_3 = rS^2}$

Solution:

(i) $\mathbf{r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2}$

$$\text{L.H.S.} = r_1 r_2 + r_2 r_3 + r_3 r_1$$

$$= \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} + \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} + \frac{\Delta}{S-c} \times \frac{\Delta}{S-a}$$

$$= \frac{\Delta^2}{(S-a)(S-b)} + \frac{\Delta^2}{(S-b)(S-c)} + \frac{\Delta^2}{(S-c)(S-a)}$$

$$= \Delta^2\left[\frac{S-c+S-a+S-b}{(S-a)(S-b)(S-c)}\right]$$

$$= \Delta^2\left[\frac{3S-(a+b+c)}{(S-a)(S-b)(S-c)}\right]$$

$$= \Delta^2\left[\frac{3S-2S}{(S-a)(S-b)(S-c)}\right]$$

$$= \frac{\Delta^2 S \cdot S}{S(S-a)(S-b)(S-c)}$$

$$= \frac{\Delta^2 S^2}{\Delta^2} = S^2 = \text{R.H.S.} \quad \text{Hence proved}$$

(ii) $\mathbf{r\, r_1\, r_2\, r_3 = \Delta^2}$

L.H.S. $= r\, r_1\, r_2\, r_3$

$$= \frac{\Delta}{S} \times \frac{\Delta}{S-a} \times \frac{\Delta}{S-b} \times \frac{\Delta}{S-c}$$

$$= \frac{\Delta^4}{S(S-a)(S-b)(S-c)}$$

$$= \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{R.H.S.} \quad \text{Hence proved}$$

(iii) $\mathbf{r_1 + r_2 + r_3 - r = 4R}$

L.H.S. $= r_1 + r_2 + r_3 - r$

$$= \frac{\Delta}{S-a} + \frac{\Delta}{S-b} + \frac{\Delta}{S-c} - \frac{\Delta}{S} = \Delta\left[\frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c} - \frac{1}{S}\right]$$

$$= \Delta\left[\frac{S-b+S-a}{(S-a)(S-b)} + \frac{S-(S-c)}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-a-b}{(S-a)(S-b)} + \frac{S-S+c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-a-b-c+c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-(a+b+c)+c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta\left[\frac{2S-2S+c}{(S-a)(S-b)} + \frac{c}{S(S-c)}\right]$$

$$= \Delta.C\left[\frac{1}{(S-a)(S-b)} + \frac{1}{S(S-c)}\right]$$

$$= C\Delta\left[\frac{S(S-C)+(S-a)(S-b)}{S(S-a)(S-b)(S-c)}\right]$$

$$= C\Delta\left[\frac{S^2-CS+S^2-bS-aS+ab}{\Delta^2}\right]$$

$$= \frac{C}{\Delta}[2S^2 - S(a+b+C)+ab]$$

$$= \frac{C}{\Delta}[2S^2 - S(2S)+ab]$$

$$= \frac{C}{\Delta}[2S^2 - 2S^2+ab]$$

$$= \frac{abc}{\Delta} = \frac{4abc}{4\Delta} = 4R$$

= R.H.S Hence proved

(iv) $\mathbf{r_1 r_2 r_3 = rS^2}$

L.H.S. $= r_1 r_2 r_3$

$$\frac{\Delta}{S-a} \times \frac{\Delta}{S-b} \times \frac{\Delta}{S-c} = \frac{\Delta^3}{(S-a)(S-b)(S-c)}$$

$$= \frac{S\,\Delta^3}{S(S-a)(S-b)(S-c)}$$

$$= \frac{S\,\Delta^3}{\Delta^2}$$

$$= S\,\Delta$$

$$= S\Delta \times \frac{S}{S}$$

$$= S^2 r$$

$$= \text{R.H.S.}$$

Hence proved.

Q.6 Find R, r, r_1, r_2, r_3 if measures of the sides of triangle ΔABC are
(Gujranwala Board 2005, 2007) (Lahore Board, 2004, 2005, 2009)

(i) a = 13, b = 14, c = 15

(ii) a = 34, b = 20, c = 42

Solution:

(i) a = 13, b = 14, c = 15

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{21(21-13)(21-14)(21-15)}$$

$$= \sqrt{21 \times 8 \times 7 \times 6} = 84$$

$$R = \frac{abc}{4\Delta} = \frac{13 \times 14 \times 15}{4(84)} = \frac{2730}{336} = 8.125$$

$$r = \frac{\Delta}{S} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{S-a} = \frac{84}{8} = 10.5$$

$$r_2 = \frac{\Delta}{S-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{84}{6} = 14$$

(ii) $\mathbf{a = 34, \quad b = 20, \quad c = 42}$

$$S = \frac{a+b+c}{2} = \frac{34+20+42}{2} = \frac{96}{2} = 48$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{48(48-34)(48-20)(48-42)}$$

$$= \sqrt{48(14)(28)(6)} = 336$$

$$R = \frac{abc}{4\Delta} = \frac{34 \times 20 \times 42}{4 \times 336} = \frac{28560}{1344} = 21.25$$

$$r = \frac{\Delta}{S} = \frac{336}{48} = 7$$

$$r_1 = \frac{\Delta}{S-a} = \frac{336}{14} = 24$$

$$r_2 = \frac{\Delta}{S-b} = \frac{336}{28} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{336}{6} = 56$$

Q.7 Prove that in an equilateral triangle

(i) $\mathbf{r : R : r_1 = 1 : 2 : 3}$

(ii) $\mathbf{r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3}$

Solution:

(i) $\mathbf{r : R : r_1 = 1 : 2 : 3}$

We know that in an equilateral triangle

$$a = b = c$$

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$S-a = \frac{3a}{2} - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$S-b = S-a = \frac{a}{2}$$

$$S-c = S-a = \frac{a}{2}$$

$$\Delta = \sqrt{S(S-a)(S-a)(S-a)} = \sqrt{\frac{3a}{2} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}\,a^2}{4}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3}\,a^2}{4} \div \frac{3a}{2}$$

$$r = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{3a} = \frac{\sqrt{3}\,a}{6}$$

$$R = \frac{abc}{4\Delta} = \frac{a^3}{4 \times \frac{\sqrt{3}\,a^2}{4}} = \frac{a^3}{\sqrt{3}\,a^2} = \frac{a}{\sqrt{3}}$$

$$r_1 = \frac{\Delta}{S-a} = \frac{\sqrt{3}\,a^2}{4} \div \frac{a}{2}$$

$$r_1 = \frac{\sqrt{3}a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}a}{2}$$

L.H.S. $= r : R : r_1$

$$\frac{\sqrt{3}\,a}{6} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}\,a}{2}$$

Dividing throughout by $\frac{\sqrt{3}\,a}{6}$

$$\frac{\sqrt{3}\,a}{6} \times \frac{6}{\sqrt{3}\,a} : \frac{a}{\sqrt{3}} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a}$$

$$= 1 : \frac{6}{3} : 3$$

$$= 1 : 2 : 3$$

$=$ R.H.S.

Hence proved.

(ii) $\mathbf{r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3}$

We know that in an equilateral triangle

$$a = b = c$$

$$S = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$S - a = \frac{3a}{2} - a = \frac{3a-2a}{2} = \frac{a}{2}$$

$$S - b = \frac{3a}{2} - a = \frac{a}{2}$$

$$S - c = \frac{3a}{2} - a = \frac{a}{2}$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{\frac{3a}{2} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}\,a^2}{4}$$

$$r = \frac{\Delta}{S} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{3a} = \frac{\sqrt{3}\,a}{6}$$

$$r_1 = \frac{\Delta}{S-a} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}\,a}{2} \qquad R = \frac{a\,b\,c}{4\Delta} = \frac{a^3}{4 \times \frac{\sqrt{3}\,a^2}{4}} = \frac{a}{\sqrt{3}}$$

$$r_2 = \frac{\Delta}{S-b} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}\,a}{2}$$

$$r_3 = \frac{\Delta}{S-c} = \frac{\sqrt{3}\,a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}\,a}{2}$$

$$\text{L.H.S.} = r : R : r_1 : r_2 : r_3$$

$$= \frac{\sqrt{3}\,a}{6} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}\,a}{2} : \frac{\sqrt{3}\,a}{2} : \frac{\sqrt{3}\,a}{2}$$

Multiplying throughout by $\frac{6}{\sqrt{3}\,a}$

$$= \frac{\sqrt{3}\,a}{6} \times \frac{6}{\sqrt{3}\,a} : \frac{a}{\sqrt{3}} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a} : \frac{\sqrt{3}\,a}{2} \times \frac{6}{\sqrt{3}\,a}$$

$$= 1 : 2 : 3 : 3 : 3$$

= R.H.S. Hence proved.

Q.8 Prove that

(i) $\boldsymbol{\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}}$

(ii) $\boldsymbol{r = S \tan\frac{\alpha}{2} \tan\frac{\beta}{2} \tan\frac{\gamma}{2}}$ **(Lahore Board 2010)**

(iii) $\boldsymbol{\Delta = 4\,R\,r \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}}$

Solution:

(i) $\boldsymbol{\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}}$

$$\text{R.H.S.} = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{S}\right)^2 \sqrt{\frac{S\,(S-a)}{(S-b)\,(S-c)}}\,\sqrt{\frac{S\,(S-b)}{(S-a)\,(S-c)}}\,\sqrt{\frac{S\,(S-c)}{(S-a)\,(S-b)}}$$

$$= \frac{\Delta^2}{S^2}\sqrt{\frac{S^3(S-a)(S-b)(S-c)}{(S-b)^2(S-c)^2(S-a)^2}}$$

$$= \frac{\Delta^2}{S^2}S\sqrt{\frac{S(S-a)(S-b)(S-c)}{(S-a)^2(S-b)^2(S-c)^2}}$$

$$= \frac{\Delta^2}{S}\sqrt{\frac{S\,.\,S}{S(S-a)(S-b)(S-c)}}$$

$$= \frac{\Delta^2}{S}\frac{S}{\sqrt{S(S-a)(S-b)(S-c)}} = \frac{\Delta^2}{\Delta} = \Delta = \text{L.H.S. Hence proved}$$

(ii) $\mathbf{r = S\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}$

$$\text{R.H.S.} = S\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$= S\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

$$= S\sqrt{\frac{(S-b)(S-c)(S-a)(S-c)(S-a)(S-b)}{S^3(S-a)(S-b)(S-c)}}$$

$$= \frac{S}{S}\sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{S(S-a)(S-b)(S-c)}}$$

$$= \sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2}}$$

$$= \frac{\sqrt{S(S-a)(S-b)(S-c)}}{S} = \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

Hence proved.

(iii) $\mathbf{\Delta = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

$$\text{R.H.S.} = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\left(\frac{\Delta}{S}\right)\sqrt{\frac{S(S-a)}{bc}}\sqrt{\frac{S(S-b)}{ac}}\sqrt{\frac{S(S-c)}{ab}}$$

$$= \frac{abc}{S}\sqrt{\frac{S^3(S-a)(S-b)(S-c)}{a^2b^2c^2}}$$

$$= \frac{abc\,S}{S}\sqrt{\frac{S(S-a)(S-b)(S-c)}{a^2b^2c^2}}$$

$$= \frac{abc}{abc}\sqrt{S(S-a)(S-b)(S-c)} = \Delta = \text{L.H.S. Hence proved.}$$

Q.9 Show that

(i) $\dfrac{1}{2rR} = \dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}$

(ii) $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

Solution:

(i) $\dfrac{1}{2rR} = \dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca}$

$$\text{L.H.S.} = \frac{1}{2rR}$$

$$= \frac{1}{2\left(\frac{\Delta}{S}\right)\left(\frac{abc}{4\Delta}\right)} = \frac{2S}{abc} = \frac{a+b+c}{abc}$$

$$= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc} = \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab} = \text{R.H.S. Hence proved}$$

(ii) $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$

$$\text{R.H.S.} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$= \frac{S-a}{\Delta} + \frac{S-b}{\Delta} + \frac{S-c}{\Delta}$$

$$= \frac{S-a+S-b+S-c}{\Delta} = \frac{3S-(a+b+c)}{\Delta}$$

$$= \frac{3S-2S}{\Delta} = \frac{S}{\Delta} = \frac{1}{r} = \text{L.H.S. Hence proved}$$

Q.10 Prove that

$$r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

Solution:

$$r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$\text{R.H.S.} = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{a\sqrt{\frac{(S-a)(S-c)}{ac}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-a)}{bc}}}$$

$$= a\sqrt{\frac{\frac{(S-a)^2(S-c)(S-b)}{a^2 bc}}{\frac{S(S-a)}{bc}}}$$

$$= a\sqrt{\frac{(S-a)^2(S-c)(S-b)}{a^2 bc} \times \frac{abc}{S(S-a)}}$$

$$= a\sqrt{\frac{(S-a)(S-b)(S-c)S}{S^2 a^2}}$$

$$= \frac{a}{a} \times \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

$$\mathbf{r = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}}$$

$$\text{R.H.S.} = \frac{b \sin\frac{\alpha}{2} \sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$

$$= b\frac{\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-b)}{ab}}}{\sqrt{\frac{S(S-b)}{ac}}}$$

$$= b\sqrt{\frac{(S-b)^2(S-c)(S-a)}{a b^2 c} \times \frac{ac}{S(S-b)}}$$

$$= b\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2 b^2}}$$

$$= \frac{b}{b} \times \frac{\Delta}{S} = \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

$$\mathbf{r = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}}$$

$$\text{R.H.S.} = \frac{c \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

$$= \frac{c\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{(S-a)(S-c)}{ac}}}{\sqrt{\frac{S(S-c)}{ab}}}$$

$$= c\sqrt{\frac{(S-b)(S-c)^2(S-a)}{a\,b\,c^2} \times \frac{ba}{S(S-c)}}$$

$$= c\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2 c^2}}$$

$$= \frac{\Delta}{S} = r = \text{L.H.S. Hence proved}$$

Q.11 Prove that abc (sin α + sin β + sin γ) = 4 Δ S (Lahore Board 2007, 2011)

Solution:

$$\text{L.H.S.} = abc(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= abc\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right) \qquad \because R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{\sin\gamma}$$

$$= abc\left(\frac{a+b+c}{2R}\right) = abc\left(\frac{2S}{2\left(\frac{abc}{4\Delta}\right)}\right)$$

$$= abc\left(\frac{4\Delta S}{abc}\right) = 4\Delta S = \text{R.H.S. Hence proved}$$

Q.12 Prove that

(i) $(r_1 + r_2)\tan\frac{\gamma}{2} = C$ **(Lahore Board 2006) (Gujranwala Board 2007)**

(ii) $(r^3 - r)\cot\frac{\gamma}{2} = C$

Solution:

(i) $(r_1 + r_2)\tan\frac{\gamma}{2} = C$

$$\text{L.H.S.} = (r_1 + r_2)\tan\frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{S-a} + \frac{\Delta}{S-b}\right)\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$$

$$= \Delta\left(\frac{S-b+S-a}{(S-a)(S-b)}\right)\sqrt{\frac{S(S-a)(S-b)(S-c)}{S^2(S-c)^2}}$$

$$= \Delta\left(\frac{2S-b-a}{(S-a)(S-b)}\right)\frac{\Delta}{S(S-c)}$$

$$= \frac{\Delta^2(a+b+c-b-a)}{S(S-a)(S-b)(S-c)} \qquad (\because\ 2S = a+b+c)$$

$$= \frac{\Delta^2 C}{\Delta^2} = C = \text{R.H.S. Hence proved.}$$

(ii) $\mathbf{(r^3 - r)\cot\frac{\gamma}{2} = C}$

$$\text{L.H.S.} = (r^3 - r)\cot\frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{S-c} - \frac{\Delta}{S}\right)\sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$$

$$= \Delta\left(\frac{1}{S-c} - \frac{1}{S}\right)\sqrt{\frac{S(S-a)(S-b)(S-c)}{(S-a)^2(S-b)^2}}$$

$$= \Delta\left(\frac{S-S+c}{S(S-c)}\right)\frac{\sqrt{S(S-a)(S-b)(S-c)}}{(S-a)(S-b)}$$

$$= \frac{\Delta C}{S(S-c)} \times \frac{\Delta}{(S-a)(S-b)}$$

$$= \frac{\Delta^2 C}{S(S-a)(S-b)(S-c)} = \frac{\Delta^2 C}{\Delta^2} = C = \text{R.H.S.}$$

Hence proved.