(ix) $\dfrac{(n+1)(n)(n-1)}{3.2.1} = \dfrac{(n+1)\,n\,(n-1)(n-2)!}{3.2.1.(n-2)!} = \dfrac{(n+1)!}{3!\,(n-2)!}$

(x) $n(n-1)(n-2)\ldots\ldots(n-r+1) = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)!}{(n-r)!} = \dfrac{n!}{(n-r)!}$

PERMUTATION

An ordering (arrangement) of n objects is called a permutation of the objects.

EXPLANATION

Think of three places as shown $\boxed{}\ \boxed{}\ \boxed{}$. Since we can write any one of three vertices A, B, C at first place, so it is written in 3 different ways as shown $\boxed{3}\ \boxed{}\ \boxed{}$.

Now two vertices are left. So, corresponding to each way of writing at first place, there are two ways of writing at second place as shown $\boxed{3}\ \boxed{2}\ \boxed{}$.

Now just one vertex is left. So, we can write at third place only one vertex in one way as shown $\boxed{3}\ \boxed{2}\ \boxed{1}$.

$\Rightarrow$ The total number of possible ways is the product $3.2.1. = 6$.

FUNDAMENTAL PRINCIPLE OF COUNTING

Suppose A and B are two events. The first event A can occur in P different ways. After A has occurred, B can occur in q different ways. The number of ways that the two events can occur is the product p.q.

THEOREM

A permutation of n different objects taken $r\ (\le n)$ at a time is an arrangement of the r objects. Generally it is denoted by ${}^nP_r$ or $P(n, r)$ where

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

EXERCISE 7.2

Q.1 Evaluate the following:

(i) ${}^{20}P_3$ (ii) ${}^{16}P_4$ (iii) ${}^{12}P_5$ (Lahore Board 2006) (iv) ${}^{10}P_7$ (v) ${}^9P_8$

Solution:

Using formula ${}^nP_r = \dfrac{n!}{(n-r)!}$

(i) ${}^{20}P_3 = \dfrac{20!}{(20-3)!} = \dfrac{20.19.18.17!}{17!} = 6840$

(ii) $^{16}P_4 = \dfrac{16!}{(16-4)!} = \dfrac{16.15.14.13.12!}{12!} = 43680$

(iii) $^{12}P_5 = \dfrac{12!}{(12-5)!} = \dfrac{12.11.10.9.8.7!}{7!} = 95040$

(iv) $^{10}P_7 = \dfrac{10!}{(10-7)!} = \dfrac{10.9.8.7.6.5.4.3!}{3!} = 604800$

(v) $^{9}P_8 = \dfrac{9!}{(9-8)!} = \dfrac{9.8.7.6.5.4.3.2.1}{1!} = 362880$

Q.2 Find the value of n when:

(i) $^nP_2 = 30$ (Gujranwala Board 2004, 2005, 2007)

(ii) $^{11}P_n = 11.10.9$ (Lahore Board 2008, 2009) (iii) $^nP_4 : {}^{n-1}P_3 = 9:1$

Solution:

(i) $^nP_2 = 30$

$\Rightarrow \dfrac{n!}{(n-2)!} = 30$

$\Rightarrow \dfrac{n(n-1)(n-2)!}{(n-2)!} = 30$

$\Rightarrow n(n-1) = 30$

$\Rightarrow n^2 - n - 30 = 0$

$\Rightarrow n^2 - 6n + 5n - 30 = 0$

$\Rightarrow n(n-6) + 5(n-6) = 0$

$\Rightarrow (n+5)(n-6) = 0$

$\Rightarrow n = -5$ (not possible)

$\Rightarrow \boxed{n = 6}$

(ii) Solution

$^{11}P_n = 11.10.9$

$\Rightarrow \dfrac{11!}{(11-n)!} = 11 . 10 . 9$

$\Rightarrow \dfrac{11!}{11.10.9} = (11-n)!$

$\Rightarrow \dfrac{11!}{11.10.9} = (11-n)!$

$\Rightarrow \quad \dfrac{11!\,8!}{11.10.9.8!} = (11-n)!$

$\Rightarrow \quad \dfrac{11!\,8!}{11!} = (11-n)!$

$\Rightarrow \quad 8! = (11-n)!$

$\Rightarrow \quad (11-3)! = (11-n)!$

$\Rightarrow \quad \boxed{n = 3}$

(iii) Solution

$^nP_4 : {}^{n-1}P_3 = 9 : 1$

$\Rightarrow \quad \dfrac{^nP_4}{^{n-1}P_3} = \dfrac{9}{1}$

$\Rightarrow \quad \dfrac{\dfrac{n!}{(n-4)!}}{\dfrac{(n-1)!}{(n-1-3)!}} = \dfrac{9}{1}$

$\Rightarrow \quad \dfrac{n!}{(n-1)!} = \dfrac{9}{1}$

$\Rightarrow \quad \dfrac{n\,(n-1)!}{(n-1)!} = 9$

$\Rightarrow \quad \boxed{n = 9}$

Q.3 Prove from the first principle that:

(i) $^nP_r = n\,.\,{}^{n-1}P_{r-1}$

(ii) $^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$ **(Lahore Board 2009, 2011)**

Solution:

(i) $^nP_r = n\,.\,{}^{n-1}P_{r-1}$

L.H.S. $= {}^nP_r = \dfrac{n!}{(n-r)!}$

R.H.S. $= n\,{}^{n-1}P_{r-1}$

$= n\,.\,\dfrac{(n-1)!}{(n-1-r+1)!}$

$= \dfrac{n\,(n-1)!}{(n-r)!} = \dfrac{n!}{(n-r)!} = \text{L.H.S.}$

$\Rightarrow \quad {}^nP_r = n\,.\,{}^{n-1}P_{r-1}$

Proved

(ii) Solution

$^{n}P_{r} = {}^{n-1}P_{r} + r\,.\,{}^{n-1}P_{r-1}$

$$\text{L.H.S.} = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\text{R.H.S.} = {}^{n-1}P_{r} + r\,.\,{}^{n-1}P_{r-1}$$

$$= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-1-r+1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + \frac{r\,.\,(n-1)!}{(n-r)\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left(1 + \frac{r}{n-r}\right)$$

$$= \frac{(n-1)!}{(n-r-1)!}\left(\frac{n-r+r}{n-r}\right)$$

$$= \frac{n\,(n-1)!}{(n-r)\,(n-r-1)!}$$

$$= \frac{n!}{(n-r)!} = \text{L.H.S.}$$

$\Rightarrow$ $^{n}P_{r} = {}^{n-1}P_{r} + r\,.\,{}^{n-1}P_{r-1}$

Proved.

Q.4 How many signals can be given by 5 flags of different colours, using 3 at a time?

Solution:

Total number of flags $= n = 5$

Used number of flags $= r = 3$

Required numbers of signals $= {}^{n}P_{r} = {}^{5}P_{3} = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 60$

Q.5 How many signals can be given by 6 flags of different colours when any number of flags can be used at a time?

(Lahore Board 2004, Gujranwala Board 2007)

Solution:

Total number of flags $= 6$

Number of signals using 1 flag $= {}^{6}P_{1} = 6$

Number of signals using 2 flags $= {}^{6}P_{2} = 30$

Number of signals using 3 flags $= {}^6P_3 = 120$

Number of signals using 4 flags $= {}^6P_4 = 360$

Number of signals using 5 flags $= {}^6P_5 = 720$

Number of signals using 6 flags $= {}^6P_6 = 720$

Total number of signals $= 6 + 30 + 120 + 360 + 720 + 720 = 1956$

Q.6 How many words can be formed by the letters of the following words using all letters when no letter is to be repeated?

(i) PLANE (ii) OBJECT (Lahore Board 2010) (iii) FASTING

Solution:

(i) PLANE

Permutation of 5 letters using all at a time $= {}^5P_5 = \dfrac{5!}{(5-5)!} = \dfrac{5!}{6!} = 120$

(ii) OBJECT

Permutation of 6 letters using all at a time $= {}^6P_6 = \dfrac{6!}{0!} = 6! = 720$

(iii) FASTING

Permutation of 7 letters using all at a time $= {}^7P_7 = \dfrac{7!}{0!} = 7! = 5040$

Q.7 How many 3 – digit numbers can be formed by using each one of the digits 2, 3, 5, 7, 9 only once? (Gujranwala Board 2005)

Solution:

Total number of digits $n = 5$

Used number of digits $r = 3$

Required no. of permutations $= {}^nP_r = {}^5P_3 = 60$

Q.8 Find the numbers greater than 23000 that can be formed from the digits 1, 2, 3, 5, 6 without repeating any digit. (Lahore Board 2005)

Solution:

Numbers greater than 23000 are of the form

$23\square\;\square\;\square = {}^3P_3 = 3! = 6$

$25\square\;\square\;\square = {}^3P_3 = 3! = 6$

$26\square\;\square\;\square = {}^3P_3 = 3! = 6$

$3\square\;\square\;\square\;\square = {}^4P_4 = 4! = 24$

$5\square\;\square\;\square\;\square = {}^4P_4 = 4! = 24$

$6\square\;\square\;\square\;\square = {}^4P_4 = 4! = 24$

Total numbers $= 6 + 6 + 6 + 24 + 24 + 24 = 90$

Q.9 Find the number of 5 – digit numbers that can be formed from the digits 1, 2, 4, 6, 8 (when no digit is repeated), but

(i) the digits 2 and 8 are next to each other.

(ii) the digits 2 and 8 are not next to each other.

Solution:

(i) Let (2, 8) as one digit

Now the digits are 1, 28, 4, 6 or 1, 82, 4, 6

Permutation containing (28) $= {}^4P_4 = 4! = 24$

Permutation containing (82) $= {}^4P_4 = 4! = 24$

Total permutation in which 2 and 8 are next to each other $= 24 + 24 = 48$

(ii) Total permutations of 5 digits $= {}^5P_5 = 5! = 120$

Numbers in which (2, 8) are together $= 48$

Numbers in which (2, 8) are not together $= 120 - 48 = 72$

Q.10 How many 6–digit numbers can be formed without repeating any digit from the digits 0, 1, 2, 3, 4, 5? In how many of them will 0 be at the tens place?

(Gujranwala Board 2007)

Solution:

'0' cannot be at first place from left so this place can be filled by 5 ways then second place can be filled by 5 ways, third place by 4 ways, fourth place by 3 ways, fifth place by 2 ways six place by 1 way.

So required numbers $= 5.5.4.3.2.1 = 600$

If we fix 0 at tens place then remaining places are 5 and digits are also 5.

So required numbers $= {}^5P_5 = 5! = 120$

Q.11 How many 5 – digit multiples of 5 can be formed from the digits 2, 3, 5, 7, 9 when no digit is repeated?

Solution:

As multiples of 5 will have the digit 5 at unit place.

So fixing 5 at unit place we have to find permutations of remaining four digits using all at a time.

$\Rightarrow$ Multiples of 5 are $= {}^4P_4 = 4! = 24$

Q.12 In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together?

Solution:

Consider the two English books E_1 and E_2 as one book.

Now we find permutation of 7 books taken all at a time.

Permutation containing $E_1 E_2 = {}^7P_7 = 7! = 5040$

Permutation containing $E_2 E_1 = {}^7P_7 = 7! = 5040$

Permutation in which English book are together $= 5040 + 5040 = 10080$

Total permutation of 8 books $= {}^8P_8 = 8! = 40320$

Permutation in which English books are not together $= 40320 - 10080 = 30240$

Q.13 Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subjects are together?

Solution:

Let E, U denote English and Urdu books.

Permutation in the form EEEEEUUU $= 5!\,3! = 720$

Permutation in the form UUUEEEEE $= 3!\,5! = 720$

Total permutation $= 720 + 720 = 1440$

Q.14 In how many ways can 5 boys and 4 girls be seated on a bench so that the girls & the boys occupy alternate seats?

Solution:

Let B, G denote boys and girls then

Permutation in the form BGBGBGBGB $= {}^5P_5\,{}^4P_4 = 5!\,4! = 120 \times 24 = 2880$

PERMUTATION OF THINGS NOT ALL DIFFERENT

If there are n_1 alike things of one kind, n_2 alike things of second kind and n_3 alike things of third kind, then the number of permutation of n things taken all at a time is given by

$$\frac{n!}{(n_1)! \times (n_2)! \times (n_3)!} = \binom{n}{n_1, n_2, n_3}$$

CIRCULAR PERMUTATION

The permutation of things which can be represented by the points on a circle are called circular permutation.

EXERCISE 7.3

Q.1 How many arrangements of the letters of the following words, taken all together can be made?

(i) PAKPATTAN (Lahore Board 2007) **(ii) PAKISTAN**

(iii) MATHEMATICS (Lahore Board 2004) **(iv) ASSASSINATION**

Solution:

(i) PAKPATTAN

$n = 9, \quad n_1 = 3\,(A), \quad n_2 = 2\,(P), \quad n_3 = 2\,(T)$

so using formula

$$\text{Permutation} = \binom{n}{n_1, n_2, n_3} = \frac{n!}{n_1!\,n_2!\,n_3!} = \frac{9!}{3!\,2!\,2!} = \frac{9.8.7.6.5.3!}{3!\,2.1\,.\,2.1} = 15120$$