$$= -\int (\sec^2 x - 1)\,dx$$

$$= -\int \sec^2 x\,dx + \int dx$$

$$= x - \tan x + c \qquad \text{Ans.}$$

(xiv) $\int \tan^2 x\,dx$ **(Guj. Board 2005, 2007) (Lhr. Board 2011)**

$$= \int (\sec^2 x - 1)\,dx$$

$$= \int \sec^2 x\,dx - \int dx$$

$$= \tan x - x + c \qquad \text{Ans.}$$

EXERCISE 3.3

Evaluate the following integrals.

Q.1 $\int \frac{-2x}{\sqrt{4 - x^2}}$

Solution:

$$\int \frac{-2x}{\sqrt{4 - x^2}}\,dx$$

$$= \int (4 - x^2)^{-\frac{1}{2}} - 2x\,dx$$

$$= \frac{(4 - x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{4 - x^2} + c \qquad \text{Ans.}$$

Q.2 $\int \frac{dx}{x^2 + 4x + 13}$

Solution:

$$\int \frac{dx}{x^2 + 4x + 13}$$

$= \int \frac{dx}{x^2 + 4x + 4 - 4 + 13}$ $\quad \because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$= \int \frac{dx}{(x+2)^2 + 9}$

$= \int \frac{dx}{(x+2)^2 + (3)^2}$

$= \frac{1}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + c$ Ans.

Q.3 $\int \frac{x^2}{4+x^2}\, dx$

Solution:

$\int \frac{x^2}{4+x^2}\, dx$

$= \int \left(1 - \frac{4}{4+x^2}\right) dx$

$$\begin{array}{r|l} & \ \ 1 \\ 4+x^2 & \ \ x^2 \\ & \underline{-\ x^2 \pm 4} \\ & \ \ -4 \end{array}$$

$= \int dx - 4 \int \frac{4x\, dx}{(2)^2 + x^2}$ $\quad = x - 4 \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$

$= x - 2 \tan^{-1}\left(\frac{x}{2}\right) + c$ Ans.

Q.4 $\int \frac{1}{x\ln x}\, dx$

Solution:

$\int \frac{1}{x\ln x}\, dx$ $\quad \because \int [f(x)]^{-1} . f'(x) dx = \boldsymbol{ln}\, [f(x)] + c$

$= \int \frac{1/x}{\ln x}\, dx$

$$\boxed{\begin{array}{l} f(x) = \ell nx \\ f'(x) = \frac{1}{x} \end{array}}$$

$= \ln(\ln x) + c$ Ans.

Q.5 $\int \frac{e^x}{e^x + 3}\, dx$

Solution:

$\int \frac{e^x}{e^x + 3}\, dx$

$= \ln(e^x + 3) + c$ Ans.

Q.6 $\int \frac{x+b}{(x^2+2bx+c)^{1/2}}\, dx$

Solution:

$$\int \frac{x+b}{(x^2+2bx+c)^{1/2}}\, dx$$

$f(x) = x^2 + 2bx + x$
$f'(x) = 2x + 2b$
$f'(x) = 2(x+b)$

$$= \frac{1}{2}\int (x^2+2bxc+c)^{-1/2} \,.\, 2\,(x+b)\, dx$$

$$= \frac{1}{2}\int (x^2+2bx+c)^{-1/2}\,(2x+2b)\, dx$$

$$= \frac{1}{2}\,\frac{(x^2+2bx+c)^{1/2}}{\frac{1}{2}} + c \qquad \because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= \sqrt{x^2+2bx+c} + c \qquad \text{Ans.}$$

Q.7 $\int \frac{Sec^2x}{\sqrt{tanx}}\, dx$ **(Lhr. Board 2005, 2008)**

Solution:

$$\int \frac{Sec^2x}{\sqrt{tanx}}\, dx$$

$f(x) = \tan x$
$f'(x) = \sec^2x$

$$= \int (tanx)^{-1/2} \,.\, Sec^2x\, dx$$

$$= \frac{(tanx)^{1/2}}{\frac{1}{2}} + c \qquad \because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$= 2\sqrt{tanx} + c \qquad \text{Ans.}$$

Q.8 (a) **Show that** $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x + \sqrt{x^2-a^2}) + c$

(b) **Show that** $\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}x + \frac{x}{a}\sqrt{a^2-x^2} + c$

Solution:

(a) Taking

$$\int \frac{dx}{\sqrt{(x^2-a^2)}}$$

Put

$$x = a\sec\theta \quad \Rightarrow \quad \sec\theta = \frac{x}{a}$$

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$$

$$= \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\sec^2\theta - 1)}}\, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{a\sqrt{\tan^2\theta}}\, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln(\sec\theta + \tan\theta) + c_1$$

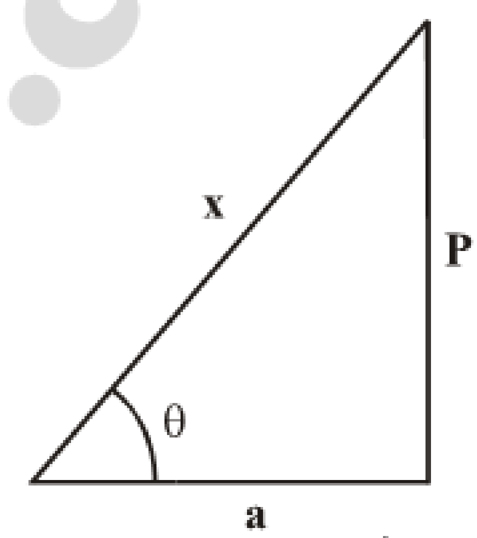
$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right) + c_1$$

$$= \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c_1$$

$$= \ln(x + \sqrt{x^2 - a^2} - \ln(a) + c_1$$

$$= \ln(x + \sqrt{x^2 - a^2}) + c \quad \text{where } c = -\ln a + c_1$$

Hence proved.



$$x^2 = a^2 + P^2$$
$$P^2 = x^2 - a^2$$
$$P = \sqrt{x^2 - a^2}$$

$$\tan\theta = \frac{p}{a} = \frac{\sqrt{x^2 - a^2}}{a}$$

(b) Taking

$$\int \sqrt{a^2 - x^2}\; dx$$

Put $x = a\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{x}{a} \quad \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$dx = a\cos\theta\, d\theta \quad \Rightarrow \quad \theta = \sin^{-1}\frac{x}{a}$$

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\; .\; a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2(1 - \sin^2\theta)}\; .\; a\cos\theta\, d\theta$$

$$= \int a\sqrt{\cos^2\theta}\ .\ a\cos\theta\ d\theta$$

$$= a^2\int \cos\theta\ .\ \cos\ d\theta$$

$$= a^2\int \cos\theta\ d\theta$$

$$= a^2\int \frac{1+\cos 2\theta}{2}\ d\theta \qquad \left(\begin{array}{l} \because\ \cos 2\theta = 2\cos^2\theta - 1 \\ \quad 2\cos^2\theta = 1+\cos 2\theta \\ \quad \cos^2\theta = \dfrac{1+\cos 2\theta}{2} \end{array}\right)$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\ d\theta$$

$$= \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int \cos 2\theta\ d\theta$$

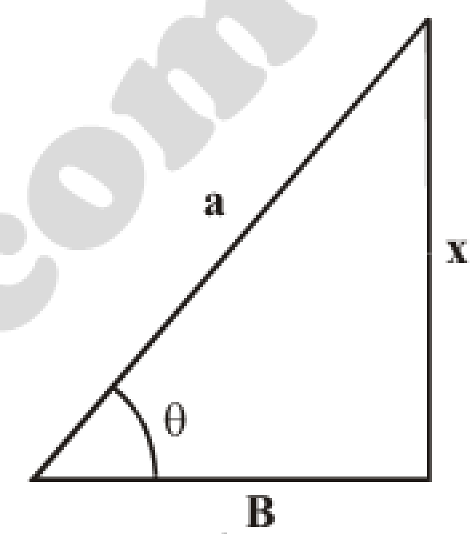
$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\ .\ \frac{\sin 2\theta}{2} + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{4}\ .\ 2\sin\theta\cos\theta + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\ .\ \frac{x}{a}\ .\ \frac{\sqrt{a^2-x^2}}{a} + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Hence proved



$a^2 = B^2 + x^2$

$B^2 = a^2 - x^2$

$B = \sqrt{a^2 - x^2}$

$\cos\theta = \dfrac{B}{a}$

$= \dfrac{\sqrt{a^2-x^2}}{a}$

$\sin\theta = \dfrac{x}{a}$

Evaluate the following integrals

Q.9 $\displaystyle\int \frac{dx}{(1+x^2)^{3/2}}$

Solution:

$$\int \frac{dx}{(1+x^2)^{3/2}}$$

Put $\quad x = \tan\theta$

$\quad dx = \sec^2\theta d\theta$

$$= \int \frac{\sec^2\theta d\theta}{(1+\tan^2\theta)^{3/2}} \quad = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\ d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

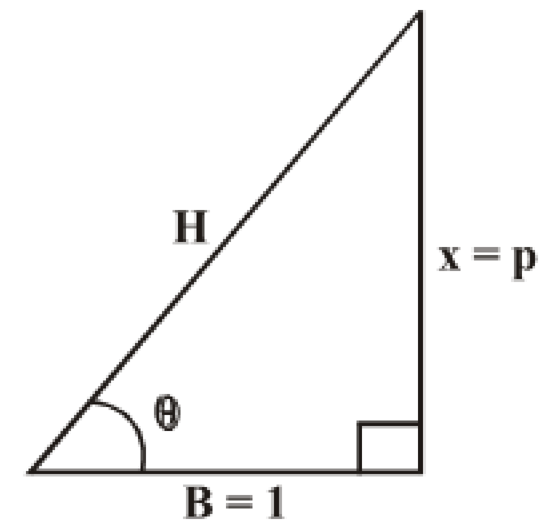
$$= \int \frac{d\theta}{\sec\theta}\, d\theta$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + c$$

$$\because \quad \sin\theta = \frac{P}{H}$$

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$



$$H^2 = 1 + x^2$$

$$H = \sqrt{1 + x^2}$$

$$= \frac{x}{\sqrt{1 + x^2}} + c \qquad \text{Ans.}$$

Q.10 $\displaystyle \int \frac{1}{(1 + x^2)\tan^{-1}x}\, dx$

Solution:

$$\int \frac{1}{(1 + x^2)\tan^{-1}x}\, dx$$

$$= \int \frac{\frac{1}{1+x^2}}{\tan^{-1}x}\, dx$$

$$\because \quad f(x) = \tan^{-1}x$$

$$\because \quad f'(x) = \frac{1}{1 + x^2}$$

$$= \ln(\tan^{-1}x) + c \qquad \text{Ans.}$$

Q.11 $\displaystyle \int \sqrt{\frac{1 + x}{1 - x}}\, dx$

Solution:

$$\int \sqrt{\frac{1 + x}{1 - x}}\, dx$$

$$= \int \sqrt{\frac{1 + x}{1 - x} \times \frac{1 + x}{1 + x}}\, dx$$

$$= \int \sqrt{\frac{(1 + x)^2}{1 - x^2}}\, dx$$

$$= \int \frac{1 + x}{\sqrt{1 - x^2}}\, dx \qquad \because \quad \text{L.C.M Breaking}$$

$$= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right) dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} . -2x\, dx$$

$$= \sin^{-1}x - \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \sin^{-1}x - \sqrt{1-x^2} + c \qquad \text{Ans.}$$

Q.12 $\displaystyle\int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$

Solution:

$$\int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$$

Put

$\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$$d\theta = \frac{dt}{-\text{Sin}\theta}$$

$$= \int \frac{\sin\theta}{1+t^2} \times \frac{dt}{-\sin\theta}$$

$$= -\int \frac{dt}{1+t^2}$$

$$= -\tan^{-1}(t) + c$$

$$= -\tan^{-1}(\cos\theta) + c \qquad \text{Ans.} \qquad \because \quad t = \cos\theta$$

Q.13 $\displaystyle\int \frac{ax}{\sqrt{a^2-x^4}}\, dx$

Solution:

$$\int \frac{ax}{\sqrt{a^2-x^4}}\, dx$$

$$= \int \frac{ax}{\sqrt{a^2 - (x^2)^2}}\,dx$$

Put

$$x^2 = t$$

$$2xdx = dt$$

$$dx = \frac{dt}{2x}$$

$$= \int \frac{ax}{\sqrt{a^2 - t^2}} \times \frac{dt}{2x}$$

$$= \frac{a}{2}\int \frac{dt}{\sqrt{a^2 - t^2}}$$

$$= \frac{a}{2}\sin^{-1}\left(\frac{t}{a}\right) + c$$

$$= \frac{a}{2}\sin^{-1}\left(\frac{x^2}{a}\right) + c \quad \text{Ans.} \qquad \because \quad t = x^2$$

Q.14 $\displaystyle\int \frac{dx}{\sqrt{7 - 6x - x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{7 - 6x - x^2}}$$

$$= \int \frac{dx}{\sqrt{-(x^2 + 6x - 7)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2 + 6x + 9 - 9 - 7)}}$$

$$= \int \frac{dx}{\sqrt{-[(x + 3)^2 - 16]}}$$

$$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$$

$$= \int \frac{dx}{\sqrt{(4)^2-(x+3)^2}} \qquad \because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C \qquad \text{Ans.}$$

Q.15 $\int \frac{\cos x}{\sin x \, l n \sin x}\,dx$ **(Guj. Board 2008)**

Solution:

$$\int \frac{\cos x}{\sin x \, l n \sin x}\,dx$$

Put

$$\ln \sin x = t$$

$$\frac{1}{\sin x}\cos x dx = dt$$

$$dx = \frac{\sin x}{\cos x}\,dt$$

$$= \int \frac{\cos x}{\sin x \; t} \times \frac{\sin x}{\cos x}\,dt$$

$$= \int \frac{dt}{t}$$

$$= l n t + c$$

$$= l n\,(l n \sin x) + c \qquad \text{Ans.} \qquad \because t = \ell n \sin x$$

Q.16 $\int \cos x \left(\frac{l n \sin x}{\sin x}\right) dx$ **(Lhr. Board 2005)**

Solution:

$$\int \cos x \left(\frac{l n \sin x}{\sin x}\right) dx$$

Put

$$l n \sin x = t$$

$$\frac{1}{\sin x}\cos x \,.\, dx = dt$$

$$dx = \frac{\sin x}{\cos x}$$

$$= \int \cos x\left(\frac{t}{\sin x}\right) . \frac{\sin x}{\cos x}\, dt$$

$$= \int t\, dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{(\ln \sin x)^2}{2} + c \quad \text{Ans.} \qquad \because t = \ell n \sin x$$

Q.17 $\displaystyle\int \frac{\mathbf{xdx}}{\mathbf{4 + 2x + x^2}}$

Solution:

Formula used

$$\because \int [f(x)^{-1}]\, f'(x)\, dx = \ell n\, f(x) + c$$

$$\because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{xdx}{4 + 2x + x^2}$$

$$= \frac{1}{2}\int \frac{2x + 2 - 2}{4 + 2x + x^2}\, dx$$

$$= \frac{1}{2}\int \left(\frac{2x + 2}{4 + 2x + x^2} - \frac{2}{4 + 2x + x^2}\right) dx$$

$$= \frac{1}{2}\int \frac{2x + 2}{4 + 2x + x^2}\, dx - \frac{2}{2}\int \frac{dx}{4 + 2x + x^2}$$

$$= \frac{1}{2}\ \ln (x^2 + 2x + 4) - \int \frac{dx}{x^2 + 2x + 1 - 1 + 4}$$

$$= \frac{1}{2}\ \ln (x^2 + 2x + 4) - \int \frac{dx}{(x + 1)^2 + 3}$$

$$= \frac{1}{2}\ln (x^2 + 2x + 4) - \int \frac{dx}{(x + 1)^2 + (\sqrt{3})^2}$$

$$= \frac{1}{2}\ \ln (x^2 + 2x + 4) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x + 1}{\sqrt{3}}\right) + c \quad \text{Ans.}$$

Q.18 $\int \frac{x}{x^4 + 2x^2 + 5} \; dx$

Solution:

$$\int \frac{x}{x^4 + 2x^2 + 5} \; dx$$

$$= \int \frac{x}{(x^2)^2 + 2x^2 + 5} \; dx$$

Put

$x^2 = t$

$2xdx = dt$

$dx = \frac{dt}{2x}$

$$= \int \frac{x}{t^2 + 2t + 5} \times \frac{dt}{2x}$$

$$= \frac{1}{2} \int \frac{dt}{t^2 + 2t + 1 - 1 + 5}$$

$$= \frac{1}{2} \int \frac{dt}{(t+1)^2 + 4}$$

$$= \frac{1}{2} \int \frac{dt}{(t+1)^2 + (2)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \left(\frac{t+1}{2}\right) + c$$

$$= \frac{1}{4} \tan^{-1} \left(\frac{x^2+1}{2}\right) + c \qquad \text{Ans.} \qquad \because t = x^2$$

Q.19 $\int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right)\right] \times \left(\frac{1}{\sqrt{x}} - 1\right) dx$

$$\because f(x) = \sqrt{x} - \frac{x}{2}$$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2}$$

$$f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right)$$

Solution:

$$\int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right)\right] \times \left(\frac{1}{\sqrt{x}} - 1\right) dx$$

$$= 2 \int \cos\left(\sqrt{x} - \frac{x}{2}\right) \times \frac{1}{2} \left(\frac{1}{\sqrt{x}} - 1\right) dx$$

$= \quad 2\int \cos\left(\sqrt{x} - \frac{x}{2}\right) \times \left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\right) dx$

$= \quad 2\sin\left(\sqrt{x} - \frac{x}{2}\right) + c \qquad$ Ans.

Q.20 $\int \frac{x+2}{\sqrt{x+3}}\, dx$

Solution:

$\int \frac{x+2}{\sqrt{x+3}}\, dx$

Put

$\sqrt{x+3} = t$

$x + 3 = t^2 \quad \Rightarrow \quad x = t^2 - 3$

$dx = 2t\, dt$

$= \quad \int \frac{t^2 - 3 + 2}{t} \times 2t dt$

$= \quad 2\int (t^2 - 1)\, dt$

$= \quad 2\int t^2\, dt - 2\int dt$

$= \quad \frac{2t^3}{3} - 2t + c$

$= \quad \frac{2}{3}(x+3)^{\frac{3}{2}} - 2\sqrt{x+3} + c \qquad$ Ans. $\qquad \because t = \sqrt{x+3}$

Q.21 $\int \frac{\sqrt{2}}{\sin x + \cos x}\, dx$ **(Lhr. Board 2008)**

Solution:

$\int \frac{\sqrt{2}}{\sin x + \cos x}\, dx$

$= \quad \int \frac{1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)}\, dx$

$$= \int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \int \frac{dx}{\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}}$$

$\because \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \int \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)}$$

$$= \int \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \ln\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right| + c \qquad \text{Ans.}$$

Q.22 $\displaystyle\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$

Solution:

$$\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

$$= \int \frac{dx}{\sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3}}$$

$\because \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \int \frac{dx}{\sin\left(x + \frac{\pi}{3}\right)}$$

$$= \int \operatorname{cosec}\left(x + \frac{\pi}{3}\right) dx$$

$$= \ln\left|\operatorname{cosec}\left(x + \frac{\pi}{3}\right) - \cot\left(x + \frac{\pi}{3}\right)\right| + c \qquad \text{Ans.}$$