$$\sin\theta = \frac{|\underline{u} \times \underline{v}|}{|\underline{u}|\ |\underline{v}|}$$

Important Points;

(i) $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$

(ii) $\underline{i} \times \underline{j} = \underline{k}$, $\underline{j} \times \underline{k} = \underline{i}$, $\underline{k} \times \underline{i} = \underline{j}$

(iii) $\underline{i} \times \underline{j} \neq \underline{j} \times \underline{i}$ i.e., Cross product is not commutative

(vi) Area of parallelogram = $|\underline{u} \times \underline{v}|$

(v) Area of triangle = $\frac{1}{2}|\underline{u} \times \underline{v}|$

Parallel vectors:

If $\underline{u}$ & $\underline{v}$ area parallel vectors then $\underline{u} \times \underline{v} = 0$

EXERCISE 7.4

Q.1 Compute the cross product $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$, check your answer by showing that each $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$.

(i) $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \underline{i}\ (1 - 1) - \underline{j}\ (2 + 1) + \underline{k}\ (-2 - 1)$$

$$\underline{a} \times \underline{b} = 0\underline{i} - 3\underline{j} - 3\underline{k}$$

We will show that $\underline{a}$ is perpendicular to $\underline{a} \times \underline{b}$, for this we have $\underline{a} \cdot (\underline{a} \times \underline{b})$

$$= (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$$= 0 - 3 + 3 = 0$$

$\therefore$ $\underline{a}$ and $\underline{a} \times \underline{b}$ are perpendicular.

Next, we will show that $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$. For this we have $\underline{b} \cdot (\underline{a} \times \underline{b})$

$$(\underline{i} - \underline{j} + \underline{k}) \cdot (0\underline{i} - 3\underline{j} - 3\underline{k})$$

$= \quad 0 + 3 - 3 = 0$

Hence $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$.

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= \underline{i}\,(1 - 1) - \underline{j}\,(-1 - 2) + \underline{k}\,(1 + 2)$

$\underline{b} \times \underline{a} = 0\underline{i} + 3\underline{j} + 3\underline{k}$

We will show that $\underline{a}$ is perpendicular to $\underline{b} \times \underline{a}$.

$\underline{a} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} - \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$

$= \quad 0 + 3 - 3$

$= \quad 0$

Hence $\underline{a}$ and $\underline{b} \times \underline{a}$ are perpendicular to each other

Next, we will show that $\underline{b}$ is perpendicular to $\underline{b} \times \underline{a}$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + \underline{k}) \cdot (0\underline{i} + 3\underline{j} + 3\underline{k})$

$= \quad 0 - 3 + 3 = 0$

Hence $\underline{b}$ and $\underline{b} \times \underline{a}$ are perpendicular to each other.

(ii) $\underline{a} = \underline{i} + \underline{j} + 0\underline{k}$, $\quad \underline{b} = \underline{i} - \underline{j} + 0\underline{k}$ **(Lahore Board 2009)**

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$= \quad 0\underline{i} - 0\underline{j} + \underline{k}\,(-1 - 1)$

$= \quad -2\underline{k}$

We will show that $\underline{a}$ is perpendicular to $\underline{a} \times \underline{b}$.

For this $\underline{a} \cdot (\underline{a} \times \underline{b}) = (\underline{i} + \underline{j} + 0\underline{k}) \cdot (0\underline{i} + 0\underline{j} - 2\underline{k})$

$= 0 + 0 + 0 = 0$

Next, we will show that $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$

For this $\underline{b} \cdot (\underline{a} \times \underline{b})$

$= \quad (\underline{i} - \underline{j} + 0\underline{k}) \cdot (0\underline{i} + 0\underline{j} - 2\underline{k})$

$= \quad 0 + 0 + 0 = 0$

Next, we will show that $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$

For this

$$\underline{b} \,.\, (\underline{a} \times \underline{b})$$
$$= (\underline{i} - \underline{j} + 0\underline{k}) \,.\, (0\underline{i} + 0\underline{j} - 2\underline{k})$$
$$= 0 + 0 + 0 = 0$$

Hence proved

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$
$$= \underline{i}\begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$
$$= 0\underline{i} - 0\underline{j} + \underline{k}\ (1 + 1)$$
$$\underline{b} \times \underline{a} = 0\underline{i} - 0\underline{j} + 2\underline{k}$$

We will show that $\underline{a}$ is perpendicular to $\underline{b} \times \underline{a}$.

For this

$$\underline{a} \,.\, (\underline{b} \times \underline{a}) = (\underline{i} + \underline{j} + 0\underline{k}) \,.\, (0\underline{i} + 0\underline{j} + 2\underline{k})$$
$$= 0 + 0 + 0$$
$$= 0$$

Hence a and $\underline{b} \times \underline{a}$ are perpendicular to each other. Next, we will show that $\underline{b}$ is perpendicular to $\underline{b} \times \underline{a}$

For this

$$\underline{b} \,.\, (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + 0\underline{k}) \,.\, (0\underline{i} + 0\underline{j} + 2\underline{k})$$
$$= 0 + 0 + 0$$
$$= 0$$

Hence $\underline{b}$ and $\underline{b} \times \underline{a}$ are perpendicular to each other.

(iii) $\quad \underline{\mathbf{a}} = 3\underline{i} - 2\underline{j} + \underline{k} \quad , \quad \underline{\mathbf{b}} = \underline{i} + \underline{j} + 0\underline{k}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$
$$= \underline{i}\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}$$
$$= \underline{i}\,(0 - 1) - \underline{j}\,(0 - 1) + \underline{k}\,(3 + 2)$$

$\underline{a} \times \underline{b} = -\underline{i} + \underline{j} + 5\underline{k}$

We will show that $\underline{a}$ is perpendicular to $\underline{a} \times \underline{b}$.

For this $\underline{a} \cdot (\underline{a} \times \underline{b})$

$= (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (-\underline{i} + \underline{j} + 5\underline{k})$

$= -3 -2 + 5 = 0$

$\therefore$ $\underline{a}$ and $\underline{a} \times \underline{b}$ are perpendicular to each other.

Next $\underline{b} \cdot (\underline{a} \times \underline{b})$

$= (\underline{i} + \underline{j} + 0\underline{k}) \cdot (-\underline{i} + \underline{j} + 5\underline{k})$

$= -1 + 1 + 0 = 0$

Hence $\underline{b}$ and $\underline{a} \times \underline{b}$ are perpendicular to each other.

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$$

$= \underline{i}(1 - 0) - \underline{j}(1 - 0) + \underline{k}(-2 - 3)$

$\underline{b} \times \underline{a} = \underline{i} - \underline{j} - 5\underline{k}$

We will show that $\underline{a}$ and $\underline{b} \times \underline{a}$ are perpendicular to each other.

For this $\underline{a} \cdot (\underline{b} \times \underline{a})$

$= (3\underline{i} - 2\underline{j} + \underline{k}) \cdot (\underline{i} - \underline{j} - 5\underline{k})$

$= 3 + 2 - 5 = 0$

$\therefore$ $\underline{a}$ and $\underline{b} \times \underline{a}$ are perpendicular to each other.

Next,

We will show that $\underline{b}$ is perpendicular to $\underline{b} \times \underline{a}$

For this

$\underline{b} \cdot (\underline{b} \times \underline{a})$

$= (\underline{i} + \underline{j} + 0\underline{k}) \cdot (\underline{i} - \underline{j} - 5\underline{k})$

$= 1 - 1 + 0 = 0$

Hence $\underline{b}$ is perpendicular to $\underline{b} \times \underline{a}$

(iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

$= \underline{i}\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} -4 & -2 \\ 2 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix}$

$= \underline{i}(1+2) - \underline{j}(-4+4) + \underline{k}(-4-2)$

$\underline{a} \times \underline{b} = 3\underline{i} + 0\underline{j} - 6\underline{k}$

We will show that $\underline{a}$ is perpendicular to $\underline{a} \times \underline{b}$.

For this $\underline{a} \cdot (\underline{a} \times \underline{b})$

$= (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (3\underline{i} + 0\underline{j} - 6\underline{k})$

$= -12 + 0 + 12 = 0$

Hence $\underline{a}$ is perpendicular to $\underline{a} \times \underline{b}$.

Next,

We will show that $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$.

For this

$\underline{b} \cdot (\underline{a} \times \underline{b})$

$\underline{b} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (3\underline{i} + 0\underline{j} - 6\underline{k})$

$= 6 + 0 - 6 = 0$

Hence $\underline{b}$ is perpendicular to $\underline{a} \times \underline{b}$.

Now

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix}$$

$= \underline{i}(-2-1) - \underline{j}(-4+4) + \underline{k}(2+4)$

$= -3\underline{i} + 0\underline{j} + 6\underline{k}$

We will show that $\underline{a}$ is perpendicular to $\underline{b} \times \underline{a}$.

For this $\underline{a} \cdot (\underline{b} \times \underline{a})$

$= (-4\underline{i} + \underline{j} - 2\underline{k}) \cdot (-3\underline{i} + 0\underline{j} + 6\underline{k})$

$= 12 + 0 - 12 = 0$

Hence $\underline{a}$ is perpendicular to $\underline{b} \times \underline{a}$

Next,

We will show that $\underline{b}$ is perpendicular to $\underline{b} \times \underline{a}$

$\underline{b} \cdot (\underline{b} \times \underline{a}) = (2\underline{i} + \underline{j} + \underline{k}) \cdot (-3\underline{i} + 0\underline{j} + 6\underline{k})$

$= \quad -6 + 6 = 0$

Hence $\underline{b}$ & $\underline{b} \times \underline{a}$ are perpendicular to each other.

Q.2 Find the unit vector perpendicular to the plane containing a & b. Also find Sine of angle between them.

(i) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ $\quad \underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$ **(Lahore Board 2009)**

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$= \quad \underline{i}\,(6 + 9) \; -\underline{j}\,(-2 + 12) + \underline{k}\,(6 + 24)$

$\underline{a} \times \underline{b} \quad = \quad 15\underline{i} - 10\underline{j} + 30\underline{k}$

$|\underline{a} \times \underline{b}| \quad = \quad \sqrt{(15)^2 + (-10)^2 + (30)^2} \quad = \sqrt{225 + 100 + 900} \quad = \sqrt{1225}$

$|\underline{a} \times \underline{b}| \quad = \quad 35$

Required unit vector $= \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35}$

$= \quad \dfrac{15}{35}\underline{i} - \dfrac{10}{35}\underline{j} + \dfrac{30}{35}\underline{k}$

$= \quad \dfrac{3}{7}\underline{i} - \dfrac{2}{7}\underline{j} - \dfrac{2}{7}\underline{k} + \dfrac{6}{7}\,\underline{k}$

$|\underline{a}| \quad = \quad \sqrt{(2)^2 + (-6)^2 + (-3)^2} \quad = \sqrt{4 + 36 + 9} = \sqrt{49} \quad = 7$

$|\underline{b}| \quad = \quad \sqrt{(4)^2 + (3)^2 + (-1)^2} \quad = \sqrt{16 + 9 + 1} = \sqrt{26}$

$\text{Sin}\theta \quad = \quad \dfrac{|\underline{a} \times \underline{b}|}{|\underline{a}|\;|\underline{b}|} = \dfrac{35}{7\sqrt{26}}$

$\text{Sin}\theta \quad = \quad \dfrac{5}{\sqrt{26}}$ Ans.

(ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}\,,$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$= \quad \underline{i}\,(-4 - 3) \; -\underline{j}\,(-4 + 2) + \underline{k}\,(3 + 2)$

$\underline{a} \times \underline{b} \quad = \quad -7\underline{i} + 2\underline{j} + 5\underline{k}$

$|\underline{a} \times \underline{b}| \quad = \quad \sqrt{(-7)^2 + (2)^2 + (5)^2} \; = \sqrt{49 + 4 + 25} \; = \sqrt{78}$

required unit vector

$$\hat{n} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$$

$$\hat{n} = \frac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$$

$$= \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}i + \frac{5}{\sqrt{78}}\underline{k}$$

$$|\underline{a}| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2}$$

$$= \sqrt{1+1+1} = \sqrt{3}$$

$$|\underline{b}| = \sqrt{(2)^2 + (-3)^2 + (4)^2} = \sqrt{4+9+16} = \sqrt{29}$$

$$\text{Sin}\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\ |\underline{b}|} = \frac{\sqrt{78}}{\sqrt{3}\sqrt{29}}$$

$$= \frac{\sqrt{3 \times 78}}{\sqrt{3} \times \sqrt{29}} = \sqrt{\frac{26}{29}} \quad \text{Ans}$$

(iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}, \quad \underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix}$$

$$= \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2)$$

$$= 0\underline{i} + 0\underline{j} + 0\underline{k}$$

$$|\underline{a}| = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{4+4+16} = \sqrt{24}$$

$$|\underline{b}| = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$|\underline{a} \times \underline{b}| = \sqrt{0} = 0$$

It is not possible to find out the required unit vector.

$$\text{Sin}\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\ |\underline{b}|}$$

$$= \frac{0}{\sqrt{24}\sqrt{6}}$$

$$\text{Sin}\theta = 0 \quad \text{Ans.}$$

(iv) $\underline{a} = \underline{i} + \underline{j}$, $\underline{b} = \underline{i} - \underline{j}$

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$= \underline{i}\,(0-0) - \underline{j}\,(0-0) + \underline{k}\,(-1-1)$

$\underline{a} \times \underline{b} = 0\underline{i} - 0\underline{j} - 2\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(0)^2 + (0)^2 + (-2)^2} = \sqrt{4} = 2$

Required unit vector

$$\hat{n} = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\ |\underline{b}|} = \frac{-2k}{2} = -\underline{k}$$

$|\underline{a}| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$|\underline{b}| = \sqrt{1+1} = \sqrt{2}$

$$\text{Sin}\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\ |\underline{b}|} = \frac{2}{\sqrt{2}\sqrt{2}} = \frac{2}{2}$$

$\text{Sin}\theta = 1$ Ans.

Q.3 Find the area of triangle, determined by the point P,Q and R.

(i) P (0, 0, 0); Q (2, 3, 2) ; R (–1, 1, 4)

Solution:

P (0, 0, 0) ; Q (2, 3, 2); R (– 1, 1, 4)

Area of triangle having P, Q, R as its vertices $= \frac{1}{2}\ |\overrightarrow{PQ} \times \overrightarrow{PR}|$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (2-0)\underline{i} + (3-0)\underline{j} + (2-0)\underline{k}$

$\overrightarrow{PQ} = 2\underline{i} + 3\underline{j} + 2\underline{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= (-1-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k}$

$\overrightarrow{PR} = -\underline{i} - \underline{j} + 4\underline{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & -1 & 4 \end{vmatrix}$$

$$= \underline{i}\,(12-2) - \underline{j}\,(8+2) + \underline{k}\,(2+3)$$
$$= 10\underline{i} - 10\underline{j} + 5\underline{k}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(10)^2 + (-10)^2 + (5)^2} = \sqrt{100+100+25} = \sqrt{225} = 15$$

$$\text{Area of triangle} = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$$
$$= \frac{1}{2}(15)$$
$$= \frac{15}{2} \text{ sq. units} \qquad \text{Ans.}$$

(ii) P (1, –1, –1); Q (2, 0, –1) ; R (0, 2, 1)

Solution:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (2-1)\underline{i} + (0+1)\underline{j} + (-1+1)\underline{k}$$
$$\overrightarrow{PQ} = \underline{i} + \underline{j} + 0\underline{k}$$
$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$$
$$= (0-1)\underline{i} + (2-1)\underline{j} + (1+1)\underline{k}$$
$$= -\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$
$$= \underline{i}\,(2-0) - \underline{j}\,(2-0) + \underline{k}\,(3+1)$$
$$= 2i - 2\underline{j} + 4\underline{k}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{4+4+16} = \sqrt{24}$$

$$\text{Area of triangle} = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{24} = \frac{1}{2}(2\sqrt{6}) = \sqrt{6} \text{ sq. units} \qquad \text{Ans.}$$

Q.4 Find the area of parallelogram, whose vertices are

(i) A (0, 0, 0), B (1, 2, 3); C (2, –1,1); D (3, 1, 4)

Solution:

A (0, 0, 0), B (1, 2, 3); C (2, –1,1); D (3, 1, 4)

Area of parallelogram $|\overrightarrow{AB} \times \overrightarrow{AC}|$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 2\underline{i} - \underline{j} + \underline{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$= \underline{i}\,(2+3) - \underline{j}\,(1-6) + \underline{k}\,(-1-4)$

$= 5\underline{i} + 5\underline{j} - 5\underline{k}$

Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \sqrt{(5)^2 + (5)^2 + (-5)^2} = \sqrt{25 + 25 + 25} = \sqrt{75}$ sq. units

$= 5\sqrt{3}$ sq. units Ans.

(ii) A (1, 2,–1); B (4, 2, –3) ; C (6, –5, 2) ; D (9, –5, 0)

Solution:

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (4-1)\underline{i} + (2-2)\underline{j} + (-3+1)\underline{k}$

$= 3\underline{i} + 0\underline{j} - 2\underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (6-1)\underline{i} + (-5-2)\underline{j} + (2+1)\underline{k}$

$\overrightarrow{AC} = 5\underline{i} - 7\underline{j} + 3\underline{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix}$$

$= \underline{i}\,(0-14) - \underline{j}\,(9+10) + \underline{k}\,(-21+0)$

$= -14\,\underline{i} - 19\underline{j} - 21\underline{k}$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-14)^2 + (-19)^2 + (-21)^2} = \sqrt{196 + 361 + 441} = \sqrt{998}$

Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{998}$ sq. units Ans.

(iii) A (–1, 1, 1); B (–1, 2, 2) ; C (–3, 4, –5) ; D (–3, 5, –4)

Solution:

A (–1, 1, 1); B (–1, 2, 2) ; C (–3, 4, –5) ; D (–3, 5, –4)

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (-1+1)\underline{i} + (2-1)\underline{j} + (2-1)\underline{k}$$
$$\overrightarrow{AB} = 0\underline{i} + \underline{j} + \underline{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= (-3+1)\underline{i} + (4-1)\underline{j} + (-5-1)\underline{k}$$
$$= -2\underline{i} + 3\underline{j} - 6\underline{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix}$$
$$= \underline{i}(-6-3) - \underline{j}(0+2) + \underline{k}(0+2)$$
$$= -9\underline{i} - 2\underline{j} + 2\underline{k}$$
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-9)^2 + (-2)^2 + (2)^2} = \sqrt{81+4+4} = \sqrt{89} \text{ sq. units}$$

Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{89}$ sq. units Ans.

Q.5 Which vectors if any, are perpendicular or parallel.

(i) $\underline{\mathbf{u}} = 5\underline{i} - \underline{\mathbf{j}} + \underline{\mathbf{k}}$, $\underline{\mathbf{v}} = 0\underline{i} + \underline{\mathbf{j}} - 5\underline{\mathbf{k}}$; $\underline{\mathbf{w}} = -15\underline{i} + 3\underline{\mathbf{j}} - 3\underline{\mathbf{k}}$

Solution:

$$\underline{u} . \underline{v} = (5\underline{i} - \underline{j} + \underline{k}) \;.\; (0\underline{i} + \underline{j} - 5\underline{k})$$
$$= 0 - 1 - 5 = -6 \neq 0$$

So $\underline{u}$ & $\underline{v}$ are not perpendicular to each other.

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix}$$
$$= \underline{i}(5-1) - \underline{j}(-25-0) + \underline{k}(5-0)$$
$$= 4\underline{i} + 25\underline{j} + 5\underline{k}$$
$$\neq 0$$

So $\underline{u}$ and $\underline{v}$ are not parallel

$$\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$$

$$\underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$$
$$\underline{w} = -3\underline{u} \Rightarrow \underline{w} = \lambda\underline{u},\ \lambda \in R$$ Hence $\underline{u}$ & $\underline{w}$ are parallel
$$\underline{v}.\underline{w} = (0\underline{i} + \underline{j} - 5\underline{k}) . (-15\underline{i} + 3\underline{j} - 3\underline{k})$$
$$= 0 + 3 + 15 = 18 \neq 0$$

Hence $\underline{v}$ & $\underline{w}$ are not perpendicular.

$\underline{v}$ & $\underline{w}$ cannot be written $\underline{v} = \lambda\underline{w}$, $\lambda \in R$ so they are not parallel.

(ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$; $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$; $\underline{w} = \dfrac{-\pi}{2}\underline{i} - \pi\underline{j} + \dfrac{\pi}{2}\underline{k}$

Solution:

$$\underline{u}.\underline{v} = (\underline{i} + 2\underline{j} - \underline{k}) . (-\underline{i} + \underline{j} + \underline{k})$$
$$= -1 + 2 - 1 = 0$$

Therefore $\underline{u}$ and $\underline{v}$ are perpendicular to each other.

$$\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$$
$$= \frac{-\pi\underline{i} - 2\pi\underline{j} + \pi\underline{k}}{2}$$
$$= \frac{1}{2}[-\pi\underline{i} - 2\pi\underline{j} + \pi\underline{k}]$$
$$= \frac{-\pi}{2}[\underline{i} + 2\underline{j} - \underline{k}]$$
$$\underline{w} = -\frac{\pi}{2}\underline{u} \Rightarrow \underline{w} = \lambda\underline{u},\ \lambda \in R$$

Hence $\underline{u}$ & $\underline{w}$ are parallel

$$\underline{v}.\underline{w} = (-\underline{i} + \underline{j} + \underline{k}) . (-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k})$$
$$= \frac{\pi}{2} - \pi + \frac{\pi}{2}$$
$$= \pi - \pi = 0$$

$\therefore$ $\underline{v}$ & $\underline{w}$ are perpendicular

Q.6 Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$ (Lahore Board 2005)

Solution:

L.H.S $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$
$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$$
$$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c}$$
$$= 0 = \text{R.H.S}$$ Hence proved.

Hence proved

Q.7 If $\underline{a} + \underline{b} + \underline{c} = 0$, then prove that
$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$ **(Gujranwala Board 2005)**

Solution:

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{a} = -\underline{b} - \underline{c}$

$\underline{a} = -(\underline{b} + \underline{c})$ Taking cross product with $\underline{b}$

$$\underline{a} \times \underline{b} = -(\underline{b} + \underline{c}) \times \underline{b}$$
$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{b} - \underline{c} \times \underline{b}$$
$$\underline{a} \times \underline{b} = 0 - \underline{c} \times \underline{b}$$
$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} \quad \text{........ (i)}$$

Again

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c}$

Taking cross product with $\underline{c}$

$$\underline{b} \times \underline{c} = -(\underline{a} + \underline{c}) \times \underline{c}$$
$$= -\underline{a} \times \underline{c} - \underline{c} \times \underline{c}$$
$$\underline{b} \times \underline{c} = \underline{c} \times \underline{a} \quad \text{........ (ii)}$$

from (i) & (ii) we have

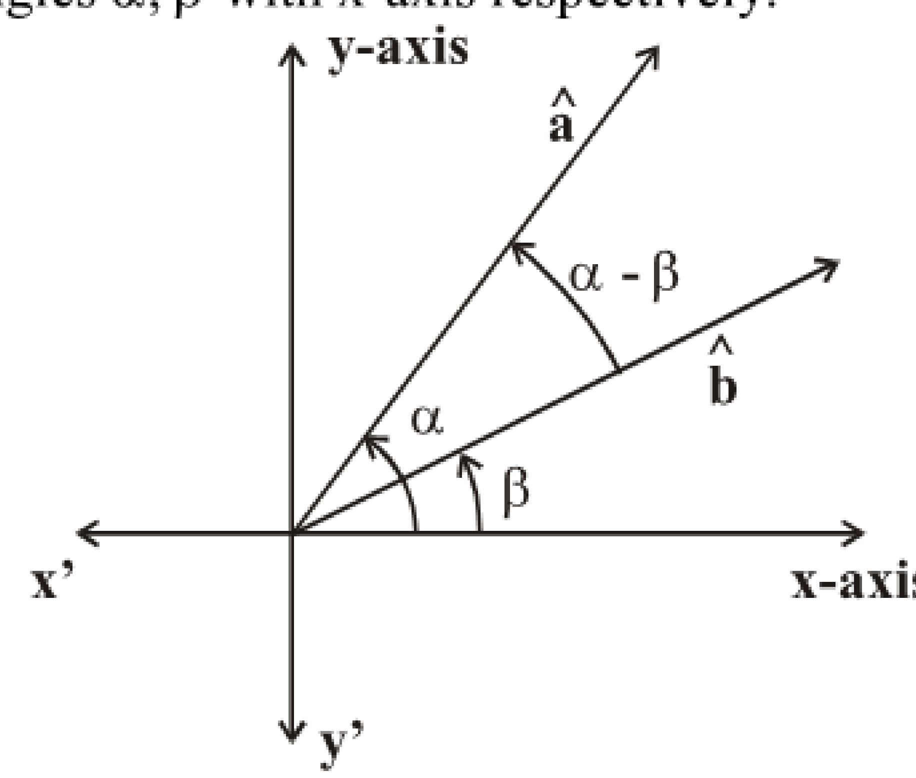
$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

Hence proved

Q.8 Proved that $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$
(Gujranwala Board 2003, Lahore Board, 2009)

Solution:

Let $\hat{a}$, $\hat{b}$ be two unit vectors making angles α, β with x-axis respectively.



$$\hat{a} = \cos\alpha\, \underline{i} + \sin\alpha\, \underline{j} + 0\underline{k}$$
$$\hat{b} = \cos\beta\, \underline{i} + \sin\beta\, \underline{j} + 0\underline{k}$$
$$\hat{b} \times \hat{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$
$$= \underline{i}\,(0-0) - \underline{j}\,(0-0) + \underline{k}\,(\sin\alpha \cos\beta - \cos\alpha \sin\beta)$$
$$\hat{b} \times \hat{a} = \underline{k}\,(\sin\alpha \cos\beta - \cos\alpha \sin\beta)$$
$$|\hat{b}|\,|\hat{a}|\,\sin(\alpha - \beta)\,\hat{n} = \underline{k}\,(\sin\alpha \cos\beta - \cos\alpha \sin\beta)$$
$$\sin(\alpha - \beta)\,\underline{k} = \underline{k}\,(\sin\alpha \cos\beta - \cos\alpha \sin\beta)$$

$(\because \quad |\hat{b}| = 1, \quad |\hat{a}| = 1)$

$\text{Sin}(\alpha - \beta) = \text{Sin}\alpha \text{ Cos}\beta - \text{Cos}\alpha\text{Sin}\beta$ Hence proved

Q.9 If $\underline{a} \times \underline{b} = 0$ and $\underline{a} \cdot \underline{b} = 0$. What conclusion can be drawn about $\underline{a}$ or $\underline{b}$?

(Gujranwala Board 2004, 2007, Lahore Board 2009 (Supply)

Solution:

If $\underline{a} \times \underline{b} = 0 \quad \Rightarrow$ (i) $\underline{a}$ and $\underline{b}$ are parallel (ii) Either $\underline{a} = 0$ or $\underline{b} = \underline{0}$

If $\underline{a} \cdot \underline{b} = 0 \quad \Rightarrow$ (iii) $\underline{a}$ and $\underline{b}$ are perpendicular (iv) Either $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$

This is not possible that $\underline{a}$ and $\underline{b}$ are parallel and perpendicular at the same time

So either $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$

$\therefore$ $\underline{a}$ and $\underline{b}$ are null vectors.

EXERCISE 7.5

Q.1 Find the volume of parallelepiped for which the given vectors are three edges.

(i) $\underline{u} = 3\underline{i} + 0\underline{j} + 2\underline{k}\ ; \quad \underline{v} = \underline{i} + 2\underline{j} + \underline{k} \quad ; \quad \underline{w} = 0\underline{i} - \underline{j} + 4\underline{k}$

Solution:

Formula

Volume of parallelepiped $= \underline{u} \cdot (\underline{v} \times \underline{w})$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$$= \quad 3\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$$

$= \quad 3(8+1) - 0 + 2(-1) \quad = \quad 27 - 2 = 25$ cubic units Ans.

(ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}\ ; \quad \underline{v} = \underline{i} - \underline{j} - 2\underline{k} \quad ; \quad \underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Solution:

Volume of parallelepiped $= \underline{u} \cdot (\underline{v} \times \underline{w})$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$