Chapter 5

PARTIAL FRACTIONS

We know that how to add two or more rational fractions into a single rational fraction. For example,

(i) $\dfrac{1}{x-1} + \dfrac{2}{x+2} = \dfrac{3x}{(x-1)(x+2)}$

(ii) $\dfrac{2}{x+1} + \dfrac{1}{(x+1)^2} + \dfrac{3}{x-2} = \dfrac{5x^2+5x-3}{(x+1)^2(x-2)}$

Here we shall learn how to reverse the order (i) and (ii)

Partial Fractions

To express a single rational function as a sum of two or more single rational functions which are called Partial Fractions.

Partial Fraction Resolution

Expressing a rational function as a sum of partial fractions is called partial fraction resolution.

Rational Fraction

The quotient of two polynomials $\dfrac{P(x)}{Q(x)}$ where $Q(x) \neq 0$, with no common factors is called a Rational Fraction. A rational fraction is of two types.

(1) Proper Rational Fraction

A rational fraction $\dfrac{P(x)}{Q(x)}$ is called a proper rational fraction if the degree of the polynomial $P(x)$ in the numerator is less than the degree of the polynomial $Q(x)$ in the denominator. For example,

$\dfrac{3}{x+1}$, $\dfrac{2x-5}{x^2+4}$ and $\dfrac{9x^2}{x^3-1}$ are proper rational fractions.

(2) Improper Rational Fraction

A rational fraction $\dfrac{P(x)}{Q(x)}$ is called an improper rational fraction if the degree of the polynomial $P(x)$ in the numerator is equal to or greater than the degree of the polynomial $Q(x)$ in the denominator. For example

$$\frac{x}{2x-3}, \quad \frac{(x-2)(x+1)}{(x-1)(x+4)} \quad \text{and} \quad \frac{x^3-x^2+x+1}{x^2+5}$$

are improper rational fractions.

EXERCISE 5.1

Resolve into Partial Fractions.

Q.1 $\mathbf{\frac{1}{x^2-1}}$ **(Gujranwala Board 2007, Lahore Board 2011)**

Solution:

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$$

Let

$$\frac{1}{(x-1)(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} \qquad (1)$$

$$\frac{1}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$1 = A(x-1) + B(x+1) \qquad (2)$$

put $x = 1$ in equation (2), we get

$$1 = A(1-1) + B(1+1)$$

$$1 = A(0) + 2B$$

$$1 = 2B \quad \Rightarrow \quad \boxed{B = \frac{1}{2}}$$

Put $x = -1$ in equation (2), we get

$$1 = A(-1-1) + B(-1+1)$$

$$1 = -2A + B(0)$$

$$-2A = 1 \quad \Rightarrow \quad \boxed{A = -\frac{1}{2}}$$

Put values of A and B in equation (1)

$$\frac{1}{(x+1)(x-1)} = \frac{-\frac{1}{2}}{(x+1)} + \frac{\frac{1}{2}}{(x-1)}$$

$$= \frac{-1}{2(x+1)} + \frac{1}{2(x-1)}$$ are required partial fractions.

Q.2 $\dfrac{\mathbf{x^2 (x^2 + 1)}}{\mathbf{(x + 1) (x - 1)}}$ **(Lahore Board 2007, 2008)**

Solution:

$$\frac{x^2 (x^2 + 1)}{(x + 1) (x - 1)} = \frac{x^4 + x^2}{x^2 - 1}$$

Dividing

$$\begin{array}{r|l}
 & \quad x^2 + 2 \\
x^2 - 1 & \overline{x^4 + x^2} \\
 & \underline{\begin{array}{l} +x^4 - x^2 \\ - \quad + \end{array}} \\
 & \qquad 2x^2 \\
 & \underline{\begin{array}{l} \quad 2x^2 - 2 \\ \quad - \quad + \end{array}} \\
 & \qquad\quad 2
\end{array}$$

$\Rightarrow$

$$\frac{x^4 + x^2}{x^2 - 1} = x^2 + 2 + \frac{2}{x^2 - 1} \qquad (1)$$

Here

$$\frac{2}{x^2 - 1} = \frac{2}{(x + 1) (x - 1)} \qquad (2)$$

Take $\dfrac{2}{(x + 1) (x - 1)} = \dfrac{A}{x + 1} + \dfrac{B}{x - 1}$

$$\frac{2}{(x + 1) (x - 1)} = \frac{A (x - 1) + B (x + 1)}{(x + 1) (x - 1)}$$

$$2 = A (x - 1) + B (x + 1) \qquad (3)$$

Put $x = 1$ in equation (3), we get

$2 = A (1 - 1) + B (1 + 1)$

$2 = A (0) + 2B$

3 $2B = 2 \Rightarrow \boxed{B = 1}$

Put $x = -1$ in equation (3), we get

$2 = A (-1 - 1) + B (-1 + 1)$

$2 = -2A + (0) B$

$\Rightarrow \quad -2A = 2 \Rightarrow \boxed{A = -1}$

Put values of A, B in equation (2)

$$\frac{2}{(x+1)(x-1)} = \frac{-1}{x-1} + \frac{1}{(x-1)}$$

equation (1) becomes

$$\frac{x^4 + x^2}{x^2 - 1} = x^2 + 2 - \frac{1}{x+1} + \frac{1}{x-1}$$

Hence $x^2 + 2 - \frac{1}{x+1} + \frac{1}{x-1}$ are required partial fractions.

Q.3 $\dfrac{\mathbf{2x + 1}}{\mathbf{(x-1)(x+2)(x+3)}}$

Solution:

Let

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3} \qquad (1)$$

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)}{(x-1)(x+2)(x+3)}$$

$$2x + 1 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2) \qquad (2)$$

Put $x = 1$ in equation (2)

$$2(1) + 1 = A(1+2)(1+3) + B(1-1)(1+3) + C(1-1)(1+2)$$

$$3 = 12A + 0 + 0$$

$$12A = 3 \Rightarrow A = \frac{3}{12} \Rightarrow \boxed{A = \frac{1}{4}}$$

Put $x = -2$ in equation (2)

$$2(-2) + 1 = A(-2+2)(-2+3) + B(-2-1)(-2+3) + C(-2-1)(-2+2)$$

$$-4 + 1 = A(0) + B(-3)(1) + C(0)$$

$$-3 = -3B \Rightarrow \boxed{B = 1}$$

Put $x = -3$ in equation (2)

$$2(-3) + 1 = A(-3+2)(-3+3) + B(-3-1)(-3+3) + C(-3-1)(-3+2)$$

$$-6 + 1 = A(0) + B(0) + C(-4)(-1)$$

$$-5 = 0 + 0 + 4C$$

$$-5 = 4C \quad \Rightarrow \quad \boxed{C = \frac{-5}{4}}$$

Put these values in equation (1)

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{\frac{1}{4}}{x-1} + \frac{1}{x+2} - \frac{\frac{5}{4}}{x+3}$$

$$= \frac{1}{4(x-1)} + \frac{1}{x+2} - \frac{5}{4(x+3)}$$

are required partial fractions.

Q.4 $\mathbf{\dfrac{3x^2 - 4x - 5}{(x-2)(x^2+7x+10)}}$

Solution:

As $\dfrac{3x^2 - 4x - 5}{(x-2)(x^2+7x+10)} = \dfrac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)}$

Let

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+5)} \qquad (1)$$

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{A(x+2)(x+5) + B(x-2)(x+5) + C(x-2)(x+2)}{(x-2)(x+2)(x+5)}$$

$$3x^2 - 4x - 5 = A(x+2)(x+5) + B(x-2)(x+5) + C(x+2)(x-2) \qquad (2)$$

Put $x = 2$ in equation (2)

$$3(2)^2 - 4(2) - 5 = A(2+2)(2+5) + B(2-2)(2+5) + C(2-2)(2+2)$$

$$3(4) - 8 - 5 = A(4)(7) + 0 + 0$$

$$12 - 8 - 5 = 28A$$

$$4 - 5 = 28A \quad \Rightarrow \quad \boxed{A = -\frac{1}{28}}$$

Put $x = -2$ in equation (2), we get

$$3(-2)^2 - 4(-2) - 5 = A(-2+2)(-2+5) + B(-2-2)(-2+5) + C(-2-2)(-2+2)$$

$$3(4) + 8 - 5 = A(0) + B(-4)(3) + C(0)$$

$$12 + 8 - 5 = -12B$$

$$15 = -12B \Rightarrow B = -\frac{15}{12} \Rightarrow \boxed{B = -\frac{5}{4}}$$

Put $x = -5$ in equation (2), we get

$$3(-5)^2 - 4(-5) - 5 = A(-5+2)(-5+5) + B(-5-2)(-5+5) + C(-5-2)(-5+2)$$

$$3(25) + 20 - 5 = A(0) + B(0) + C(-7)(-3)$$

$$75 + 15 = 21C$$

$$90 = 21C \Rightarrow C = \frac{90}{21} \Rightarrow \boxed{C = \frac{30}{7}}$$

Put these values in equation (1)

$$\frac{3x^2 - 4x - 5}{(x-2)(x+2)(x+5)} = \frac{-\frac{1}{28}}{x-2} + \frac{-\frac{5}{4}}{x+2} + \frac{\frac{30}{7}}{x+5}$$

$$= \frac{1}{28(x-2)} - \frac{5}{4(x+2)} + \frac{30}{7(x+5)}$$

are required partial fractions.

Q.5 $\dfrac{\mathbf{1}}{\mathbf{(x-1)(2x-1)(3x-1)}}$

Solution:

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{3x-1} \qquad (1)$$

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1)}{(x-1)(2x-1)(3x-1)}$$

$$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1) \qquad (2)$$

Put $x = 1$ in equation (2), we get

$$1 = A(2(1)-1)(3(1)-1) + B(1-1)(3(1)-1) + C(1-1)(2(1)-1)$$

$$1 = A(2-1)(3-1) + B(0) + C(0)$$

$$1 = A(1)(2) \Rightarrow \boxed{A = \frac{1}{2}}$$

Put $x = \frac{1}{2}$ in equation (2), we get

$$1 = A\left(2\left(\frac{1}{2}\right)-1\right)\left(3\left(\frac{1}{2}\right)-1\right)+B\left(\frac{1}{2}-1\right)\left(3\left(\frac{1}{2}\right)-1\right)$$
$$+C\left(\frac{1}{2}-1\right)\left(2\left(\frac{1}{2}\right)-1\right)$$

$$1 = A\,(1-1)\left(\frac{3}{2}-1\right)+B\left(-\frac{1}{2}\right)\left(\frac{3}{2}-1\right)+C\left(-\frac{1}{2}\right)(1-1)$$

$$1 = A\,(0)+B\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)+C\,(0)$$

$$1 = -\frac{1}{4}B \quad \Rightarrow \quad \boxed{B = -4}$$

Put $x = \frac{1}{3}$ in equation (2), we get

$$1 = A\left(2\left(\frac{1}{3}\right)-1\right)\left(3\left(\frac{1}{3}\right)-1\right)+B\left(\frac{1}{3}-1\right)\left(3\left(\frac{1}{3}\right)-1\right)$$
$$+C\left(\frac{1}{3}-1\right)\left(2\left(\frac{1}{3}\right)-1\right)$$

$$1 = A\left(\frac{2}{3}-1\right)(1-1)+B\left(\frac{-2}{3}\right)(1-1)+C\left(-\frac{2}{3}\right)\left(\frac{2}{3}-1\right)$$

$$1 = A\,(0)+B\,(0)+C\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$

$$1 = \frac{2}{9}C \quad \Rightarrow \quad \boxed{C = \frac{9}{2}}$$

Put these values in equation (1), we get

$$\frac{1}{(x-1)\,(2x-1)\,(3x-1)} = \frac{\frac{1}{2}}{x-1}+\frac{-4}{2x-1}+\frac{\frac{9}{2}}{3x-1}$$
$$= \frac{1}{2\,(x-1)}-\frac{4}{(2x-1)}+\frac{9}{2\,(3x-1)}$$

are required partial fractions.

Q.6 $\boldsymbol{\dfrac{x}{(x-a)\,(x-b)\,(x-c)}}$

Solution:

Let

$$\frac{x}{(x-a)\,(x-b)\,(x-c)} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c} \qquad \ldots\ldots\ldots (1)$$

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$$

$$x = A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b) \quad \ldots\ldots\ldots (2)$$

Put $x = a$ in equation (2), we get

$$a = A(a-b)(a-c)+B(a-a)+C(a-a)(a-b)$$

$$a = A(a-b)(a-c)+0+0$$

$$a = A(a-b)(a-c)$$

$$\boxed{A = \frac{a}{(a-b)(a-c)}}$$

Put $x = b$ in equation (2), we get

$$b = A(b-b)(b-c)+B(b-a)(b-c)+C(b-a)(b-b)$$

$$b = 0+B(b-a)(b-c)+0$$

$$\boxed{B = \frac{b}{(b-a)(b-c)}}$$

Put $x = c$ in equation (2), we get

$$c = A(c-b)(c-c)+B(c-a)(c-c)+C(c-a)(c-b)$$

$$c = 0+0+C(c-a)(c-b)$$

$$\boxed{C = \frac{c}{(c-a)(c-b)}}$$

put these values in equation (1), we get

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a}{(a+b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)}$$

are required partial fractions.

Q.7 $\dfrac{\mathbf{6x^3+5x^2-7}}{\mathbf{2x^2-x-1}}$

Solution:

First dividing

$$\begin{array}{r} 3x+4 \\ 2x^2-x-1\ \overline{\big)\,6x^3+5x^2-7} \end{array}$$

$$\begin{array}{r} 6x^3 - 3x^2 \quad - 3x \\ - \quad + \quad\quad + \\ \hline 8x^2 + 3x - 7 \\ 8x^2 - 4x - 4 \\ - \quad + \quad + \\ \hline 7x - 3 \end{array}$$

$$\Rightarrow \quad \frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1} = 3x + 4 + \frac{7x - 3}{2x^2 - x - 1} \quad \text{......... (1)}$$

Take

$$\frac{7x - 3}{2x^2 - x - 1} = \frac{7x - 3}{(x - 1)(2x + 1)}$$

Let

$$\frac{7x - 3}{(x - 1)(2x + 1)} = \frac{A}{(x - 1)} + \frac{B}{(2x + 1)} \quad \text{......... (2)}$$

$$\frac{7x - 3}{(x - 1)(2x + 1)} = \frac{A(2x + 1) + B(x - 1)}{(x - 1)(2x + 1)}$$

$$7x - 3 = A(2x + 1) + B(x - 1) \quad \text{......... (3)}$$

Put $x = 1$ in equation (3), we get

$$7(1) - 3 = A(2(1) + 1) + B(1 - 1)$$

$$7 - 3 = A(2 + 1) + 0$$

$$\Rightarrow \quad 4 = 3A \quad \Rightarrow \quad \boxed{A = \frac{4}{3}}$$

Put $x = -\frac{1}{2}$ in equation (3), we get

$$7\left(-\frac{1}{2}\right) - 3 = A\left(2\left(-\frac{1}{2}\right) + 1\right) + B\left(-\frac{1}{2} - 1\right)$$

$$\frac{7}{2} - 3 = A(-1 + 1) + B\left(-\frac{3}{2}\right)$$

$$-\frac{13}{2} = 0 - \frac{3}{2}B$$

$$13 = 3b \quad \Rightarrow \quad \boxed{B = \frac{13}{3}}$$

Put these values in equation (2), we get

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{\frac{4}{3}}{x-1} + \frac{\frac{13}{3}}{2x+1}$$

$$= \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}$$

equation (1) becomes

$$\frac{6x^3+5x^2-7}{2x^2-x-1} = (3x+4) + \frac{4}{3(x-1)} + \frac{13}{3(2x+1)}$$

are required partial fractions.

Q.8 $\mathbf{\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x}}$

Solution:

First divide $2x^3+x^2-5x+3$ by $2x^3+x^2-3x$

i.e.

$$\begin{array}{r|l} & \quad 1 \\ 2x^3+x^2-3x & 2x^3+x^2-5x+3 \\ & 2x^3+x^2-3x \\ & - \quad - \quad + \\ \hline & \qquad -2x+3 \end{array}$$

$\Rightarrow$

$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x} = 1 + \frac{-2x+3}{2x^3+x^2-3x}$$

$$= 1 + \frac{-2x+3}{x(x-1)(2x+3)} \qquad \text{......... (1)}$$

Let

$$\frac{-2x+3}{x(x-1)(2x+3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+3} \qquad \text{......... (2)}$$

$$\frac{-2x+3}{x(x-1)(2x+3)} = \frac{A(x-1)(2x+3) + Bx(2x+3) + Cx(x-1)}{x(x-1)(2x+3)}$$

$$-2x + 3 = A(x-1)(2x+3) + Bx(2x+3) + Cx(x-1) \quad \text{.........} (3)$$

Put $x = 0$ in equation (3), we get

$$-2(0) + 3 = A(0-1)\big(2(0)+3\big) + B(0)\big(2(0)+3\big) + C(0)(0-1$$

$$3 = A(-1)(3) + 0 + 0$$

$$3 = -3A \quad \Rightarrow \quad \boxed{A = -1}$$

Put $x = 1$ in equation (3), we get

$$-2(1) + 3 = A(1-1)\big(2(1)+3\big) + B(1)\big(2(1)+3\big) + C(1)(1-1)$$

$$-2 + 3 = A(0) + B(2+3) + C(0)$$

$$1 = 5B \quad \Rightarrow \quad \boxed{B = \frac{1}{5}}$$

Put $x = -\frac{3}{2}$ in equation (3), we get

$$-2\left(-\frac{3}{2}\right) + 3 = A\left(-\frac{3}{2}-1\right)\left[2\left(-\frac{3}{2}\right)+3\right] B\left(-\frac{3}{2}\right)\left[2\left(-\frac{3}{2}\right)+3\right] + C\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)$$

$$3 + 3 = A(0) + B(0) + C\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)$$

$$6 = \frac{15}{4}C \quad \Rightarrow \quad C = \frac{6 \text{ x } 4}{15} = \frac{8}{5}$$

$$\Rightarrow \quad \boxed{C = \frac{8}{5}}$$

Put these values in equation (2), we get

$$\frac{-2x+3}{x(x-1)(2x+3)} = \frac{-1}{x} + \frac{\frac{1}{5}}{x-1} + \frac{\frac{8}{5}}{(2x+3)}$$

$$= -\frac{1}{x} + \frac{1}{5(x-1)} + \frac{8}{5(2x+3)}$$

equation (1) becomes

$$\frac{2x^3 + x^2 - 5x + 3}{2x^3 + x^2 - 3x} = 1 - \frac{1}{x} + \frac{1}{5(x-1)} + \frac{8}{5(2x+3)}$$

are required Partial formula.

Q.9 $\dfrac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

Solution:

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = \frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48}$$

By division

$$\begin{array}{r|l} & \quad 1 \\ x^3-12x^2+44x-48 & x^3-9x^2+23x-15 \\ & x^3-12x^2+44x-48 \\ & -\ \ +\qquad -\qquad + \\ \hline & \quad 3x^2-21x+33 \end{array}$$

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = 1+\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}$$

$$= 1+\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} \quad \text{......... (1)}$$

Now Let

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A}{x-2}+\frac{B}{x-4}+\frac{C}{x-6} \quad \text{......... (2)}$$

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A(x-4)(x-6)+B(x-2)(x-6)+C(x-2)(x-4)}{(x-2)(x-4)(x-6)}$$

$$3x^2-21x+33 = A(x-4)(x-6)+B(x-2)(x-6)+C(x-2)(x-4) \quad \text{......... (3)}$$

Put $x = 2$ in equation (3), we get

$$3(2)^2-21(2)+33 = A(2-4)(2-6)+B(0)+C(0)$$

$$3(4)-42+33 = A(-2)(-4)$$

$$12-42+33 = 8A$$

$$3 = 8A \quad \Rightarrow \quad \boxed{A = \frac{3}{8}}$$

Put $x = 4$ in equation (3), we get

$$3(4)^2 - 21(4) + 33 = A(0) + B(4-2)(4-6) + C(0)$$

$$3(16) - 84 + 33 = B(2)(-2)$$

$$48 - 84 + 33 = -4B$$

$$-3 = -4B \Rightarrow \boxed{B = \frac{3}{4}}$$

Put $x = 6$ in equation (3), we get

$$3(6)^2 - 21(6) + 33 = 0 + 0 + C(6-2)(6-4)$$

$$3(36) - 126 + 33 = C(4)(2)$$

$$108 - 126 + 33 = 8C$$

$$15 = 8C \Rightarrow \boxed{C = \frac{15}{8}}$$

Put these values in equation (2), we get

$$\frac{3x^2 - 21x + 33}{(x-2)(x-4)(x-6)} = \frac{\frac{3}{8}}{(x-2)} + \frac{\frac{3}{4}}{(x-4)} + \frac{\frac{15}{8}}{(x-6)}$$

$$= \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

$\Rightarrow$ equation (1) becomes

$$\frac{x^3 - 9x^2 + 23x - 15}{x^3 - 12x^2 + 44x - 48} = 1 + \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

are required Partial fractions.

Q.10 $\mathbf{\dfrac{1}{(1-ax)(1-bx)(1-cx)}}$ **(Lahore Board 2007)**

Solution:

Let

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{(1-ax)} + \frac{B}{(1-bx)} + \frac{C}{(1-cx)} \quad \text{.........} (1)$$

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A(1-bx)(1-cx) + B(1-ax)(1-cx) + C(1-ax)(1-bx)}{(1-ax)(1-bx)(1-cx)}$$

$$1 = A(1-bx)(1-cx) + B(1-ax)(1-cx) + C(1-ax)(1-bx) \quad \text{.........} (2)$$

Put $x = \frac{1}{a}$ in equation (2), we get

$$1 = A\left(1 - b\,.\,\frac{1}{a}\right)\left(1 - c\,.\,\frac{1}{a}\right) + B\,(0) + C\,(0)$$

$$1 = \left(1 - \frac{b}{a}\right)\left(1 - \frac{c}{a}\right)$$

$$1 = A\left(\frac{a-b}{a}\right)\left(\frac{a-c}{a}\right)$$

$$1 = \frac{A\,(a-b)\,(a-c)}{a^2} \quad \Rightarrow \quad \boxed{A = \frac{a^2}{(a-b)\,(a-c)}}$$

Put $x = \frac{1}{b}$ in equation (2), we get

$$1 = A\,(0) + B\left(1 - a\,.\,\frac{1}{b}\right)\left(1 - c\,.\,\frac{1}{b}\right)C\,(0)$$

$$1 = 0 + B\left(1 - a\,.\,\frac{1}{b}\right)\left(1 - c\,.\,\frac{1}{b}\right) + 0$$

$$1 = B\,\frac{(b-a)\,(b-c)}{b^2} \quad \Rightarrow \quad \boxed{B = \frac{b^2}{(b-a)\,(b-c)}}$$

Put $x = \frac{1}{c}$ in equation (2), we get

$$1 = 0 + 0 + C\left(1 - a\,.\,\frac{1}{c}\right)\left(1 - b\,.\,\frac{1}{c}\right)$$

$$1 = C\left(1 - \frac{a}{c}\right)\left(1 - \frac{b}{c}\right)$$

$$1 = C\left(\frac{c-a}{c}\right)\left(\frac{c-b}{c}\right)$$

$$1 = \frac{C\,(c-a)\,(c-b)}{c^2} \quad \Rightarrow \quad \boxed{C = \frac{c^2}{(c-a)\,(c-b)}}$$

Put these values in equation (1), we get

$$\frac{1}{(1-ax)\,(1-bx)\,(1-cx)} = \frac{\frac{a^2}{(a-b)\,(a-c)}}{(1-ax)} + \frac{\frac{b^2}{(b-a)\,(b-c)}}{(1-bx)} + \frac{\frac{c^2}{(c-a)\,(c-b)}}{(1-cx)}$$

$$= \frac{a^2}{(a-b)\,(a-c)\,(1-ax)} + \frac{b^2}{(b-a)\,(b-c)\,(1-bx)} + \frac{c^2}{(c-a)\,(c-b)\,(1-bx)}$$

are required Partial fractions.

Q.11 $\dfrac{\mathbf{x^2 + a^2}}{\mathbf{(x^2 + b^2)\,(x^2 + c^2)\,(x^2 + d^2)}}$

Solution:

$$\frac{x^2 + a^2}{(x^2 + b^2)\,(x^2 + c^2)\,(x^2 + d^2)}$$

Let $x^2 = y$ and neglecting the square of each term

We have

$$\frac{x^2 + a^2}{(x^2 + b^2)\,(x^2 + c^2)\,(x^2 + d^2)} = \frac{y + a}{(y + b)\,(y + c)\,(y + d)}$$

Let

$$\frac{y + a}{(y + b)\,(y + c)\,(y + d)} = \frac{A}{(y + b)} + \frac{B}{(y + c)} + \frac{C}{(y + d)} \quad \text{......... (1)}$$

$$\frac{y + a}{(y + b)\,(y + c)\,(y + d)} = \frac{A\,(y + c)\,(y + d) + B\,(y + b)\,(y + d) + C\,(y + b)\,(y + c)}{(y + b)\,(y + c)\,(y + d)}$$

$$y + a = A\,(y + c)\,(y + d) + B\,(y + b)\,(y + d) + C\,(y + b)\,(y + c) \quad \text{......... (2)}$$

Put $y = -b$ in equation (2), we get

$$-b + a = A\,(-b + c)\,(-b + d) + 0 + 0$$

$$a - b = A\,(c - b)\,(d - b)$$

$$\boxed{A = \frac{(a - b)}{(c - b)\,(d - b)}}$$

Put $y = -c$ in equation (2), we get

$$-c + a = 0 + B\,(-c + b)\,(-c + d) + 0$$

$$a - c = B\,(b - c)\,(d - c)$$

$$\boxed{B = \frac{(a - c)}{(b - c)\,(d - c)}}$$

Put $y = -d$ in equation (2), we get

$$-d + a = 0 + 0 + C\,(-d + b)\,(-d + c)$$

$$a - d = C\,(b - d)\,(c - d)$$

$$\boxed{C = \frac{(a-d)}{(b-d)(c-d)}}$$

Put these values in equation (1), we get

$$\frac{y+a}{(y+b)(y+c)(y+d)} = \frac{\frac{(a-b)}{(c-b)(d-b)}}{y+b} + \frac{\frac{(a-c)}{(b-c)(d-c)}}{y+c} + \frac{\frac{(a-d)}{(b-b)(c-d)}}{y+d}$$

$$\frac{a}{b+c+d} = \frac{a-b}{(c-b)(d-b)(y+b)} + \frac{a-c}{(b-c)(d-c)(y+c)} + \frac{a-d}{(b-d)(c-d)(y+d)}$$

Replacing the neglecting squares, we get

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x^2+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)}$$
$$+ \frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)(x^2+d^2)}$$

are required Partial fractions.

EXERCISE 5.2

Resolve the following into Partial fraction.

Q.1 $\mathbf{\dfrac{2x^2-3x+4}{(x-1)^3}}$

Solution:

Let

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \qquad \ldots\ldots\ldots (1)$$

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$2x^2-3x+4 = A(x-1)^2 + B(x-1) + C \qquad \ldots\ldots\ldots (2)$$

$$2x^2-3x+4 = A(x^2+1-2x) + Bx - B + C$$

$$2x^2-3x+4 = Ax^2 + A - 2Ax + Bx - B + C \qquad \ldots\ldots\ldots (3)$$

Put $x = 1$ in equation (2), we get

$$2(1)^2 - 3(1) + 4 = A(1-1)^2 + B(1-1) + C$$

$$2-3+4 = 0+0+C$$

$$3 = C \quad \Rightarrow \quad \boxed{C = 3}$$

Equating coefficients of x^2, x in equation (3), we get