e.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ are in H.P

since

$1, 3, 5, 7$ are in A.P.

nth term or general term of H.P is given by

$$\frac{1}{a_1 + (n-1)d}$$

## Harmonic Mean

A number H is said to be the harmonic mean (H.M) between two numbers a and b if a, H, b are in H.P

Also $H = \frac{2ab}{a+b}$

## EXERCISE 6.10

**Q.1 Find the 9th term of the harmonic sequence**

**(i)** $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ ....... **(ii)** $\frac{-1}{5}, \frac{-1}{3}, -1$ ......

**Solution:**

**(i)** $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ .......

Given sequence

$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ ....... which is H.P

$3, 5, 7,$ ....... is A.P

$a_1 = 3, \quad d = 5 - 3 = 2,$

As

$a_n = a + (n-1)d$

$a_9 = a_1 + 8d$

$= 3 + 8(2) = 3 + 16 = 19$

$\Rightarrow$ 9th term of A.P $= 19$

$\Rightarrow$ 9th term of H.P $= \frac{1}{19}$

**(ii)** $\frac{-1}{5}, \frac{-1}{3}, -1$ ......

Given sequence

$\frac{-1}{5}, \frac{-1}{3}, -1$ ...... which is H.P

$-5, -3, -1, \ldots\ldots$ is A.P

$a_1 = -5, \quad d = -3(-5) = 2, \quad n = 9, \quad a_9 = ?$

As

$$a_n = a_1 + (n-1)d$$
$$a_9 = a_1 + 8d$$
$$= -5 + 8(2) = -5 + 16 = 11$$

$\Rightarrow$

$$9^{th} \text{ term of H.P} = \frac{1}{11}$$

**Q.2 Find the 12th term of the following Harmonic sequence.**

**(i)** $\mathbf{\frac{1}{2}, \frac{1}{5}, \frac{1}{8} \ldots\ldots}$ **(ii)** $\mathbf{\frac{1}{3}, \frac{2}{9}, \frac{1}{6} \ldots\ldots}$

**Solution:**

**(i)** $\mathbf{\frac{1}{2}, \frac{1}{5}, \frac{1}{8} \ldots\ldots}$

Given sequence

$\frac{1}{2}, \frac{1}{5}, \frac{1}{8} \ldots\ldots$ which is H.P

$2, 5, 8, \ldots\ldots$ A.P

$a_1 = 2, \quad d = 5 - 2 = 3, \quad n = 12, \quad a_{12} = ?$

As

$$a_n = a + (n-1)d$$
$$a_{12} = a_1 + 11d$$
$$= 2 + (12-1)$$
$$= 2 + 11(3) = 2 + 33 = 35$$

$\Rightarrow$ 12th term of H.P $= \frac{1}{35}$

**(ii)** $\mathbf{\frac{1}{3}, \frac{2}{9}, \frac{1}{6} \ldots\ldots}$

Given sequence

$\frac{1}{3}, \frac{2}{9}, \frac{1}{6} \ldots\ldots$ which is H.P

$3, \frac{9}{2}, 6 \ldots\ldots$ A.P

$a_1 = 3, \quad d = \frac{9}{2} - 3 = \frac{3}{2}, \quad n = 12, \quad a_{12} = ?$

As

$$a_n = a + (n-1)\,d$$

$$a_{12} = a_1 + 11d$$

$$= 3 + 11\left(\frac{3}{2}\right) = 3 + \frac{33}{2} = \frac{39}{2}$$

so

the 12th term of H.P $= \dfrac{2}{39}$

**Q.3 Insert five harmonic means between the following given numbers.**

**(i)** $-\dfrac{2}{5}$ **and** $\dfrac{2}{13}$ **(ii)** $\dfrac{1}{4}$ **and** $\dfrac{1}{24}$

**Solution:**

**(i)** $-\dfrac{2}{5}$ **and** $\dfrac{2}{13}$

Let $A_1, A_2, A_3, A_4, A_5$ are arithmetic means between $-\dfrac{2}{5}$ and $\dfrac{2}{13}$

$\Rightarrow$ $-\dfrac{5}{2}, A_1, A_2, A_3, A_4, A_5, \dfrac{13}{2}$ are in A.P

$\Rightarrow$ $a_1 = -\dfrac{5}{2}, \quad a_7 = \dfrac{13}{2}$

$\Rightarrow$ $a_1 + 6d = \dfrac{13}{2}$ $\qquad \because a_1 = -\dfrac{5}{2}$

$-\dfrac{5}{2} + 6d = \dfrac{13}{2}$

$\Rightarrow$ $6d = \dfrac{13}{2} + \dfrac{5}{2}$

$6d = \dfrac{18}{2} = 9 \Rightarrow d = \dfrac{3}{2}$

$\Rightarrow$ $A_1 = a_2 = a_1 + d = \dfrac{-5}{2} + \dfrac{3}{2} = \dfrac{-2}{2} = -1 \Rightarrow H_1 = -1$

$A_2 = a_3 = a_1 + 2d = \dfrac{-5}{2} + 2\left(\dfrac{3}{2}\right) = \dfrac{-5}{2} + 3 = -\dfrac{1}{2} \Rightarrow H_2 = -2$

$A_3 = a_4 = a_1 + 3d = \dfrac{-5}{2} + 3\left(\dfrac{3}{2}\right) = \dfrac{-5}{2} + \dfrac{9}{2} = 2 \Rightarrow H_3 = \dfrac{1}{2}$

$A_4 = a_5 = a_1 + 4d = \dfrac{-5}{2} + 4\left(\dfrac{3}{2}\right) = \dfrac{-5}{2} + 6 = -\dfrac{7}{2} \Rightarrow H_4 = \dfrac{2}{7}$

$A_5 = a_6 = a_1 + 5d = \dfrac{-5}{2} + 5\left(\dfrac{3}{2}\right) = \dfrac{-5}{2} + \dfrac{15}{2} = 5 \Rightarrow H_5 = \dfrac{1}{5}$

$\Rightarrow$ Required H.Ms are $-1, -2, \dfrac{1}{2}, \dfrac{2}{7}$ and $\dfrac{1}{5}$

**(ii)** $\frac{1}{4}$ **and** $\frac{1}{24}$

Let $A_1, A_2, A_3, A_4, A_5$ are A.Ms between 4 and 24

$\Rightarrow$ $4, A_1, A_2, A_3, A_4, A_5, 24$ are in A.P

$a_1 = 4, \quad a_7 = 24,$

$a_1 + 6d = 24$

$4 + 6d = 24$

$6d = 24 - 4$

$\Rightarrow$ $6d = 20 \Rightarrow d = \frac{20}{6} = \frac{10}{3}$

$$A_1 = a_2 = a_1 + d = 4 + \frac{10}{3} = \frac{22}{3} \Rightarrow H_1 = \frac{3}{22}$$

$$A_2 = a_3 = a_1 + 2d = 4 + 2\left(\frac{10}{3}\right) = 4 + \frac{20}{3} = \frac{32}{3} \Rightarrow H_2 = \frac{3}{32}$$

$$A_3 = a_4 = a_1 + 3d = 4 + 3\left(\frac{10}{3}\right) = 14 \Rightarrow H_3 = \frac{1}{14}$$

$$A_4 = a_5 = a_1 + 4d = 4 + 4\left(\frac{10}{3}\right) = 4 + \frac{40}{3} = \frac{52}{3} \Rightarrow H_4 = \frac{3}{52}$$

$$A_5 = a_6 = a_1 + 5d = 4 + 5\left(\frac{10}{3}\right) = 4 + \frac{50}{3} = \frac{62}{3} \Rightarrow H_5 = \frac{3}{62}$$

so the required five H.Ms are

$\frac{3}{22}, \frac{3}{32}, \frac{1}{14}, \frac{3}{52},$ and $\frac{3}{62}$

**Q.4 Insert four H.Ms between the given numbers:**

**(i)** $\frac{1}{3}$ **and** $\frac{1}{23}$ **(ii)** $\frac{7}{3}$ **and** $\frac{7}{11}$

**(iii)** **4 and 20**

**Solution:**

**(i)** $\frac{1}{3}$ **and** $\frac{1}{23}$

Let $A_1, A_2, A_3, A_4$ are four A.M's between 3 and 23.

$\Rightarrow$ $3, A_1, A_2, A_3, A_4, 23$ are A.P.

$\boxed{a_1 = 3}, \quad a_6 = 23$

$$a_1 + 5d = 23$$
$$3 + 5d = 23$$
$$5d = 20$$
$$\boxed{d = 4}$$

$$A_1 = a_2 = a_1 + d = 3 + 4 = 7 \Rightarrow H_1 = \frac{1}{7}$$

$$A_2 = a_3 = a_1 + 2d = 3 + 2(4) = 11 \Rightarrow H_2 = \frac{1}{11}$$

$$A_3 = a_4 = a_1 + 3d = 3 + 3(4) = 15 \Rightarrow H_3 = \frac{1}{15}$$

$$A_4 = a_5 = a_1 + 4d = 3 + 4(4) = 19 \Rightarrow H_4 = \frac{1}{19}$$

so required 4 H.Ms are $\frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}$

**(ii)** $\mathbf{\frac{7}{3}}$ **and** $\mathbf{\frac{7}{11}}$

Let $A_1, A_2, A_3, A_4$ are four A.Ms between $\frac{7}{3}$ and $\frac{11}{7}$

$\Rightarrow$ $\frac{3}{7}, A_1, A_2, A_3, A_4, \frac{11}{7}$ are in A.P

$a_1 = \frac{3}{7}$ and $a_6 = \frac{11}{7}$

$$a_1 + 5d = \frac{11}{7}$$
$$\frac{3}{7} + 5d = \frac{11}{7}$$
$$5d = \frac{11}{7} - \frac{3}{7} = \frac{8}{7}$$
$$d = \frac{8}{35}$$

$$A_1 = a_2 = a_1 + d = \frac{3}{7} + \frac{8}{35} = \frac{23}{35} \Rightarrow H_1 = \frac{35}{23}$$

$$A_2 = a_3 = a_1 + 2d = \frac{3}{7} + 2\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{16}{35} = \frac{31}{35} \Rightarrow H_2 = \frac{35}{31}$$

$$A_3 = a_4 = a_1 + 3d = \frac{3}{7} + 3\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{24}{35} = \frac{39}{35} \Rightarrow H_3 = \frac{35}{39}$$

$$A_4 = a_5 = a_1 + 4d = \frac{3}{7} + 4\left(\frac{8}{35}\right) = \frac{3}{7} + \frac{32}{35} = \frac{47}{35} \Rightarrow H_4 = \frac{35}{47}$$

so the required 4 H.Ms are

$$\frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}$$

**(iii) 4 and 20**

Let $A_1, A_2, A_3, A_4$ are four A.Ms between $\frac{1}{4}$ and $\frac{1}{2}$

$\Rightarrow$ $\frac{1}{4}, A_1, A_2, A_3, A_4, \frac{1}{20}$ are in A.P

$a_1 = \frac{1}{4}$ and $a_6 = \frac{1}{20}$

$$a_1 + 5d = \frac{1}{20}$$

$$\frac{1}{4} + 5d = \frac{1}{20}$$

$$5d = \frac{1}{20} - \frac{1}{4} = -\frac{4}{20} = -\frac{2}{10}$$

$$d = -\frac{2}{50}$$

$$\Rightarrow A_1 = a_2 = a_1 + d = \frac{1}{4} - \frac{2}{50} = \frac{21}{100} \Rightarrow H_1 = \frac{100}{21}$$

$$A_2 = a_3 = a_1 + 2d = \frac{1}{4} + 2\left(-\frac{2}{50}\right) = \frac{1}{4} - \frac{4}{50} = \frac{17}{100} = \Rightarrow H_2 = \frac{100}{17}$$

$$A_3 = a_4 = a_1 + 3d = \frac{1}{4} + 3\left(-\frac{2}{50}\right) = \frac{1}{4} - \frac{6}{50} = \frac{13}{100} \Rightarrow H_3 = \frac{100}{3}$$

$$A_4 = a_5 = a_1 + 4d = \frac{1}{4} + 4\left(-\frac{2}{50}\right) = \frac{1}{4} - \frac{8}{50} = \frac{9}{100} \Rightarrow H_4 = \frac{100}{9}$$

so the required four H.Ms are

$$\frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}$$

**Q.5 If the 7$^{th}$ and 10$^{th}$ terms of an H.P are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14$^{th}$ term.**

**Solution:**

7th term of H.P $= \frac{1}{3}$

7th term of A.P $= 3 = a_7$

10th term of H.P $= \frac{5}{21}$

10th term of A.P $= \frac{21}{5} = a_{10}$

$\Rightarrow$ $a_1 + 6d = 3$ ......... (1)

$a_1 + 9d = \frac{21}{5}$ ......... (2)

subtracting (1) from (2), we get

$3d = \frac{21}{5} - 3 = \frac{6}{5}$

$\boxed{d = \frac{2}{5}}$

Put $d = \frac{2}{5}$ in (1), we get

$a_1 + 6 . \frac{2}{5} = 3$

$a_1 + \frac{12}{5} = 3$

$a_1 = 3 - \frac{12}{5} = \frac{15-12}{5}$

$\boxed{a_1 = \frac{3}{5}}$

As in A.P

$$\begin{aligned} a_{14} &= a_1 + 13d \\ &= \frac{3}{5} + 13\left(\frac{2}{5}\right) \\ &= \frac{3}{5} + \frac{26}{5} = \frac{29}{5} \end{aligned}$$

$\Rightarrow$ 14$^{th}$ term of the given H.P $= \frac{5}{29}$

**Q.6 If first term of an H.P. is $-\frac{1}{3}$ and fifth term is $\frac{1}{5}$. Find its 9th term.**

**Solution:**

$1^{st}$ term of H.P $= -\frac{1}{3}$

$1^{st}$ term of A.P $= -3 = a_1$

$5^{th}$ term of H.P $= \frac{1}{5}$

$5^{th}$ term of A.P $= 5 = a_5$

As $a_5 = a_1 + 4d$

$5 = -3 + 4d$

$5 + 3 = 4d \Rightarrow 8 = 4d = 8 \Rightarrow \boxed{d = 2}$

so $9^{th}$ term of A.P

$$\begin{aligned} a_9 &= a_1 + 8d \\ &= -3 + 8(2) \\ &= 13 \end{aligned}$$

and 9th term of H.P $= \frac{1}{13}$

**Q.7 If 5 is the HM between 2 and b. Find b. (Lahore Board 2007, 2010)**

**Solution:**

If H is H.M between a and b

then

$$\text{H.M.} = \frac{2ab}{a+b}$$

but here 5 is the H.M between 2 and b.

$\Rightarrow \quad 5 = \frac{2(2)b}{2+b}$

$\Rightarrow \quad 5 = \frac{4b}{2+b}$

$\Rightarrow \quad 5(2+b) = 4b$

$\Rightarrow \quad 10 + 5b - 4b = 0$

$\Rightarrow \quad 10 + b = 0 \Rightarrow \boxed{b = -10}$

**Q.8 If the numbers $\frac{1}{k}, \frac{1}{2k+1}, \frac{1}{4k-1}$ are in harmonic sequence. Find k.**

**(Lahore Board 2003, Gujranwala Board 2007)**

**Solution:**

As $\frac{1}{k}, \frac{1}{2k+1}, \frac{1}{4k-1}$ are in H.P

$\Rightarrow$ $k,\ 2k+1,\ 4k-1$ are in A.P

$\Rightarrow$ $2k+1-k = 4k-1-(2k+1)$ common difference should same

$\Rightarrow$ $k+1 = 4k-1-2k-1$

$\Rightarrow$ $k+1 = 2k-2$ (B)

$\Rightarrow$ $2k-2-k-1 = 0$

$\Rightarrow$ $k-3 = 0$

$\Rightarrow$ $\boxed{k = 3}$

**Q.9 Find n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be HM between a and b.**

**(Lahore Board 2003)**

**Solution:**

As harmonic mean between a and b is given by

$$H = \frac{2ab}{a+b}$$

therefore

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$$

$\Rightarrow$ $(a+b)(a^{n+1}+b^{n+1}) = 2ab(a^n+b^n)$

$\Rightarrow$ $a^{n+2}+ab^{n+1}+b\,a^{n+1}+b^{n+2} = 2a^{n+1}b+2ab^{n+1}$

$\Rightarrow$ $a^{n+2}+b^{n+2} = a^{n+1}b+a\,b^{n+1}$

$\Rightarrow$ $a^{n+2}-a^{n+1}b = a\,b^{n+1}-b^{n+2}$

$\Rightarrow$ $a^{n+1}(a-b) = b^{n+1}(a-b)$

$\Rightarrow$ $a^{n+1} = b^{n+1}$

$\Rightarrow$ $\frac{a^{n+1}}{b^{n+1}} = 1$

$\Rightarrow$ $\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0$

$\Rightarrow$ $n+1 = 0$

$\Rightarrow$ $\boxed{n = -1}$

**Q.10 $a^2, b^2, c^2$ are in A.P. Show that $b + c, c + a, a + b$ are in H.P.**

**(Gujranwala Board 2004)**

**Solution:**

Given that $a^2, b^2, c^2$ in A.P.

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2 \quad \ldots\ldots\ldots (1)$

To show that $b + c,\ c + a,$ are in H.P we have to show that

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

For this we will show that

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

or $$\frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\frac{b-a}{b+c} = \frac{c-b}{a+b}$$

$$(b-a)(b+a) = (c-b)(c+b)$$

$$b^2 - a^2 = c^2 - b^2$$

which is true from (1)

$\Rightarrow \quad \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

$\Rightarrow \quad b + c,\ c + a,\ a + b$ are in H.P

**Q.11 The sum of the first and fifth terms of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$, find the sequence.**

**Solution:**

Given that

$$\frac{1}{a_1} + \frac{1}{a_5} = \frac{4}{7}$$

or $$\frac{1}{a_1} + \frac{1}{a_1 + 4d} = \frac{4}{7} \quad \ldots\ldots\ldots (1)$$

also $$\frac{1}{a_1} + \frac{1}{2 + 4d} = \frac{4}{7}$$

Put this value of a1 in equation (1), we get

$$\frac{1}{2} + \frac{1}{2+4d} = \frac{4}{7}$$

$\Rightarrow \quad \dfrac{1}{2+4d} = \dfrac{4}{7} - \dfrac{1}{2} = \dfrac{8-7}{14}$

$\Rightarrow \quad \dfrac{1}{2+4d} = \dfrac{1}{14} \quad \Rightarrow \quad 2+4d = 14$

$$4d = 12$$

$$\boxed{d = 3}$$

so required H.P is

$\dfrac{1}{a_1}, \dfrac{1}{a_1+d}, \dfrac{1}{a_1+2d}, \ldots\ldots$

$\Rightarrow \quad \dfrac{1}{2}, \dfrac{1}{2+3}, \dfrac{1}{2+2(3)}, \ldots\ldots$

$\Rightarrow \quad \dfrac{1}{2}, \dfrac{1}{5}, \dfrac{1}{8}, \ldots\ldots$

**Q.12 If, A, G and H are arithmetic geometric and harmonic means between a and b respectively. Show that $G^2 = AH$.**

**Solution:**

As A, G, H are arithmetic, geometric and harmonic means between a and b.

$\Rightarrow \quad A = \dfrac{a+b}{2}, \; G = \pm\sqrt{ab}, \; H = \dfrac{2ab}{a+b}$

and $\quad AH = \dfrac{a+b}{2} \cdot \dfrac{2ab}{a+b} = ab = G^2$

$\Rightarrow \quad AH = G^2$ or $G^2 = AH$.

Hence proved.

**Q.13 Find A, G and H and show that $G^2 = A.H.$ if**

**(i) $a = -2, \; b = -6$ (ii) $a = 2i, \; b = 4i$**

**(iii) $a = 9, \; b = 4$**

**Solution:**

**(i) $a = -2, \; b = -6$**

$A.M. = \dfrac{a+b}{2} = \dfrac{-2+-6}{2} = -\dfrac{8}{2} = -4$

$G = \sqrt{ab} = \pm\sqrt{(-2)(-6)} = \pm\sqrt{12}$

$H.M. = \dfrac{2ab}{a+b} = \dfrac{2(-2)(-6)}{-2+(-6)} = \dfrac{24}{-8} = -3$

Now $AH = (-4)(-3) = 12 = G^2$

$\Rightarrow \quad G^2 = AH$

**(ii)** $\mathbf{a = 2i, \; b = 4i}$

$$A = \frac{a+b}{2} = \frac{2i+4i}{2} = \frac{6i}{2} = 3i$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(2i)\,(4i)} = \pm\sqrt{8i^2} = \pm\sqrt{-8}$$

$$H = \frac{2ab}{a+b} = \frac{2\,(2i)\,(4i)}{2i+4i} = \frac{16i^2}{61} = \frac{8}{3}\,i$$

Now $\quad A.H = 3i \times \frac{8}{3}\,i = 8i^2 = -8 = G^2$

$\Rightarrow \quad G^2 = A\,H$

**(iii)** $\mathbf{a = 9, \quad b = 4}$

$$A = \frac{a+b}{2} = \frac{9+4}{2} = \frac{13}{2}$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(9)\,(4)} = \pm\sqrt{36} = \pm 6$$

$$H = \frac{2ab}{a+b} = \frac{2\,(9)\,(4)}{9+4} = \frac{72}{13}$$

Now $\quad A\,H = \frac{13}{2} \cdot \frac{72}{13} = 36 = G^2$

$\Rightarrow \quad G^2 = A\,H.$

**Q.14 Find A. G, H and verify that $A < G < H \;(G > 0)$ if**

**(i)** $\mathbf{a = 2, \; b = 8}$ $\qquad$ **(ii)** $\mathbf{a = -\frac{2}{5}, \; b = -\frac{8}{5}}$

**(Lahore Board 2008)**

**Solution:**

**(i)** $\mathbf{a = 2, \; b = 8}$

As $\quad A = \frac{a+b}{2} = \frac{2+8}{2} = \frac{10}{2} = 5$

$$G = \pm\sqrt{ab} = \pm\sqrt{(2)\,(8)} = \pm\sqrt{16} = \pm 4$$

$\Rightarrow \quad G = 4 \quad$ ¤ $\quad G > 0$

$$H = \frac{2ab}{a+b} = \frac{2\,(2)\,(8)}{2+8} = \frac{32}{10} = \frac{16}{5}$$

Here $\;5 > 4 < \frac{16}{5} \quad \Rightarrow \quad A > G > H$

**(ii)** $\mathbf{a = \dfrac{-2}{5}, \quad \dfrac{-8}{5}}$

As $A = \dfrac{a+b}{2} = \dfrac{\frac{2}{5}+\frac{8}{5}}{2} = \dfrac{\frac{2+8}{5}}{2} = \dfrac{\frac{10}{5}}{2} = \dfrac{2}{2} = 1$

$G = \sqrt{ab} = \sqrt{\left(\dfrac{2}{5}\right)\left(\dfrac{8}{5}\right)} = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$ ¤ $G > 0$

$G = \dfrac{4}{5}$

$H = \dfrac{2ab}{a+b} = \dfrac{2\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)}{\frac{2}{5}+\frac{8}{5}} = \dfrac{\frac{32}{25}}{\frac{10}{5}} = \dfrac{32}{25} \cdot \dfrac{5}{10} = \dfrac{16}{25}$

Here $+1 > \dfrac{4}{5} > \dfrac{16}{25} \Rightarrow A > G > H.$

**Q.15 Find A, G, H and verify that A < G < H (G < 0) if**

**(i)** $\mathbf{a = -2, \; b = -8}$ **(Lahore Board 2009)**

**(ii)** $\mathbf{a = \dfrac{-2}{5}, \; b = \dfrac{-8}{5}}$

**Solution:**

**(i)** $\mathbf{a = -2, \; b = -8}$

As $A = \dfrac{a+b}{2} = \dfrac{-2-8}{2} = \dfrac{-10}{2} = -5$

$G = -\sqrt{ab} = -\sqrt{(-2)(-8)} = -\sqrt{16} = -4$ ¤ $G < 0$

$H = \dfrac{2ab}{a+b} = \dfrac{2(-2)(-8)}{-2-8} = \dfrac{32}{-10} = -\dfrac{16}{5}$

Here $-5 < -4 < -\dfrac{16}{5}$

$\Rightarrow$ $A < G < H.$

**(ii)** $\mathbf{a = \dfrac{-2}{5}, \; b = \dfrac{-8}{5}}$

As $A = \dfrac{a+b}{2} = \dfrac{-\frac{2}{5}-\frac{8}{5}}{2} = \dfrac{\frac{-2-8}{5}}{2} = \dfrac{\frac{-10}{5}}{2} = \dfrac{-2}{2} = 1$

$$G = \sqrt{ab} = \sqrt{\left(\frac{-2}{5}\right)\left(\frac{-8}{5}\right)} = -\sqrt{\frac{16}{25}} = -\frac{4}{5} \quad ¤ \; G > 0$$

$$H = \frac{2ab}{a+b} = \frac{2\left(-\frac{2}{5}\right)\left(-\frac{8}{5}\right)}{-\frac{2}{5}-\frac{8}{5}} = \frac{\frac{32}{25}}{\frac{-10}{5}} = -\frac{32}{25} \times \frac{5}{10} = \frac{-16}{25}$$

Here $\quad -1 < -\frac{4}{5} < -\frac{16}{25} \Rightarrow A < G < H.$

**Q.16 If the H.M. and A. M. between two numbers are 4 and $\frac{9}{2}$ respectively. Find the numbers. (Lahore Board 2003, 2007)**

**Solution:**

Let a and b are required numbers

then by given conditions

$$\frac{a+b}{2} = \frac{9}{2} \Rightarrow a + b = 9 \qquad \text{......... (1)}$$

and $\quad \frac{2ab}{a+b} = 4$

$2ab = 4(a+b)$

$2ab = 4a + 4b \qquad \text{......... (2)}$

from (1) $a = 9 - b \qquad \text{......... (3)}$

Put $a = 9 - b$ in (2), we get

$\Rightarrow \quad 2(9-b)b = 4(9-b) + 4b$

$\Rightarrow \quad 18b - 2b^2 = 36 - 4b + 4b$

$\Rightarrow \quad 18b - 2b^2 = 36$

$\Rightarrow \quad 9b - b^2 = 18$

$\Rightarrow \quad b^2 - 9b + 18 = 0$

$\Rightarrow \quad b^2 - 6b - 3b + 18 = 0$

$\Rightarrow \quad b(b-6) - 3(b-6) = 0$

$\Rightarrow \quad (b-6)(b-3) = 0$

$\Rightarrow \quad b = 6 \quad \text{or} \quad b = 3$

Put $b = 6$ in equation (3), we get

$a = 9 - 6 = 3$

Put $b = 3$ in equation (3), we get

$a = 9 - 3 = 6$

so the required numbers are 3, 6, or 6, 3

**Q.17 If the (positive) G.M. and H.M. between two numbers 4 and $\frac{16}{5}$. Find numbers.**

**Solution:**

Let a and b are required numbers

then by given conditions

$$\sqrt{ab} = 4 \quad \Rightarrow \quad ab = 16 \qquad ¤ \; G > 0 \qquad \text{……… (1)}$$

and $$\frac{2ab}{a+b} = \frac{16}{5}$$

$$10ab = 16(a+b) \qquad \text{……… (2)}$$

From equation (1)

$$a = \frac{16}{b} \qquad \text{……… (3)}$$

Put (3) in (2), we get

$$10\,\frac{16}{b}\,b = 16\left(\frac{16}{b} + b\right)$$

$$10 = \frac{16}{b} + b$$

Multiplying by b, we get

$\Rightarrow \quad 10b = 16 + b^2$

$\Rightarrow \quad b^2 - 10b + 16 = 0$

$\Rightarrow \quad b^2 - 8b - 2b + 16 = 0$

$\Rightarrow \quad b(b-8) - 2(b-8) = 0$

$\Rightarrow \quad (b-2)(b-8) = 0$

$\Rightarrow \quad b = 2 \quad \text{or} \quad b = 8$

Put $b = 2$ in equation (3), we get

$$a = \frac{16}{2} = 8$$

Put $b = 8$ in equation (3), we get

$$a = \frac{16}{8} = 2$$

So the required numbers are 2, 8 or 8, 2.

**Q.18 If the numbers $\frac{1}{2}, \frac{4}{21}$ and $\frac{1}{36}$ are subtracting from three consecutive terms of a G.P., the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$.**

**Solution:**

Let $\frac{a}{r}$, a, ar are required term of G.P. then by given condition

$$\frac{a}{r} a \cdot ar = \frac{1}{27} \Rightarrow a^3 = \frac{1}{27} \Rightarrow \boxed{a = \frac{1}{3}}$$

and $\frac{a}{r} - \frac{1}{2},\ a - \frac{4}{21},\ ar - \frac{1}{36}$ are in H.P.

Put $a = \frac{1}{3}$

$\frac{1}{3r} - \frac{1}{2}, \frac{1}{3} - \frac{4}{21}, \frac{r}{3} - \frac{1}{36}$ are in H.P.

$\frac{2-3r}{6r}, \frac{3}{21}, \frac{12r-1}{36}$ are in H.P.

$\frac{6r}{2-3r}, 7, \frac{36}{12r-1}$ are in A.P.

$$\Rightarrow \quad 7 = \frac{\frac{6r}{2-3r} + \frac{36}{12r-1}}{2}$$

$$\Rightarrow \quad 14 = \frac{6r}{2-3r} + \frac{36}{12r-1}$$

$$\Rightarrow \quad 7 = \frac{3r}{2-3r} + \frac{18}{12r-1}$$

$$\Rightarrow \quad 7 = \frac{36r^2 - 3r + 36 - 54r}{(2-3r)(12r-1)}$$

$$\Rightarrow \quad 7 = \frac{36r^2 - 57r + 36}{(2-3r)(12r-1)}$$

$\Rightarrow \quad 7(2-3r)(12r-1) = 36r^2 - 57r + 36$

$\Rightarrow \quad 7(-36r^2 + 27r - 2) = 36r^2 - 57r + 36$

$\Rightarrow \quad -252r^2 + 189r - 14 = 36r^2 - 57r + 36$

$\Rightarrow \quad 288r^2 - 246r + 50 = 0$

$\Rightarrow \quad 2(144r^2 - 123r + 25) = 0$

$\Rightarrow \quad 144r^2 - 123r + 25 = 0$

$\Rightarrow \quad 144r^2 - 48r - 75r + 25 = 0$

$\Rightarrow \quad 48r(3r-1) - 25(3r-1) = 0$

$\Rightarrow \quad (3r-1)(48r-25) = 0$

$\Rightarrow \quad r = \frac{1}{3} \quad \text{or} \quad r = \frac{25}{48}$

When $a = \frac{1}{3}$, $r = \frac{1}{3}$ then

$$\frac{a}{r} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1, \quad a = \frac{1}{3}, \quad ar = \frac{1}{3}\,\frac{1}{3} = \frac{1}{9}$$

When $a = \frac{1}{3}$, $r = \frac{25}{48}$

$$\frac{a}{r} = \frac{\frac{1}{3}}{\frac{25}{48}} = \frac{16}{25}, \quad a = \frac{1}{3}, \quad ar = \frac{1}{3}\,\frac{25}{48} = \frac{25}{144}$$

So the required numbers are

$$1, \frac{1}{3}, \frac{1}{9} \quad \text{or} \quad \frac{16}{25}, \frac{1}{3}, \frac{25}{144}$$

**FORMULAE FOR THE SUMS**

$$\sum_{k=1}^{n} 1 = n$$

$$1 + 2 + 3 + \ldots\ldots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

## EXERCISE 6.11

**Sum the following series upto n terms.**

**Q.1** $\mathbf{1 \times 1 + 2 \times 4 + 3 \times 7 + \ldots\ldots}$

**Solution:**

$a_n$ of $1, 2, 3, \ldots\ldots$ is $n$ ¤ $a_n = a_1 + (n-1)d$

and $a_n$ of $1, 4, 7, \ldots\ldots$ is $1 + (n-1)(+3) = 3n - 2$ so nth term of the given series is

$T_n = n(3n-2) = 3n^2 - 2n \quad \Rightarrow \quad T_k = 3k^2 - 2k$