Chapter 2

DIFFERENTIATION

EXERCISE 2.1

Q.1 Find by definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $2x^2 + 1$ **(ii)** $2 - \sqrt{x}$ **(iii)** $\dfrac{1}{\sqrt{x}}$ **(iv)** $\dfrac{1}{x^3}$

(v) $\dfrac{1}{x - a}$ **(vi)** $x(x - 3)$ **(vii)** $\dfrac{2}{x^4}$ **(viii)** $x^2 + \dfrac{1}{x^2}$

(ix) $(x + 4)^{\frac{1}{3}}$ **(x)** $x^{\frac{3}{2}}$ **(xi)** $x^{\frac{5}{2}}$ **(xii)** x^m

(xiii) $\dfrac{1}{x^m}$, $m \in N$ **(xiv)** x^{40} **(xv)** x^{-100}

Solution:

(i) $2x^2 + 1$ ***(Lahore Board 2011)***

Let

$$y = 2x^2 + 1$$

$$y + \delta y = 2(x + \delta x)^2 + 1$$

$$\delta y = 2(x + \delta x)^2 + 1 - y$$

$$\delta y = 2(x^2 + \delta x^2 + 2x\,\delta x) + 1 - (2x^2 + 1) \qquad \because\ y = 2x^2 + 1$$

$$\delta y = 2x^2 + 2\delta x^2 + 4x\delta x + 1 - 2x^2 - 1$$

$$\delta y = 2\delta x^2 + 4x\delta x$$

$$\delta y = 2\delta x\,(\delta x + 2x)$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{2\delta x(\delta x + 2x)}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2(\delta x + 2x)$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} [2(\delta x + 2x)]$$

$$\frac{dy}{dx} = 2(0 + 2x)$$

$$\frac{dy}{dx} = 4x$$

$$\boxed{\frac{d}{dx}(2x^2 + 1) = 4x} \qquad \text{Ans.}$$

(ii) $\mathbf{2 - \sqrt{x}}$

Let

$$y = 2 - \sqrt{x}$$

$$y + \delta y = 2 - \sqrt{x + \delta x}$$

$$\delta y = 2 - \sqrt{x + \delta x} - y$$

$$\delta y = 2 - \sqrt{x + \delta x} - \left(2 - \sqrt{x}\right) \qquad \because\ y = 2 - \sqrt{x}$$

$$\delta y = 2 - \sqrt{x + \delta x} - 2 + \sqrt{x}$$

$$\delta y = \sqrt{x} - \sqrt{x + \delta x}$$

$$\delta y = x^{\frac{1}{2}} - \left[x\left(1 + \frac{\delta x}{x}\right)\right]^{\frac{1}{2}}$$

$$\delta y = x^{\frac{1}{2}} - x^{\frac{1}{2}}\left(1 + \frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\delta y = x^{\frac{1}{2}}\left[1 - \left(1 + \frac{\delta x}{x}\right)^{\frac{1}{2}}\right]$$

$$\delta y = x^{\frac{1}{2}}\left[1 - \left\{1 + \frac{1}{2}\left(\frac{\delta x}{x}\right) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots\right\}\right]$$

$$\delta y = x^{\frac{1}{2}}\left[1 - 1 - \frac{1}{2}\left(\frac{\delta x}{x}\right) - \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} \cdot \left(\frac{\delta x}{x}\right)^2 - \ldots\ldots\ldots\right]$$

$$\delta y = x^{\frac{1}{2}}\left[-\frac{1}{2}\left(\frac{\delta x}{x}\right) - \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!} \cdot \left(\frac{\delta x}{x}\right)^2 - \ldots\ldots\ldots\right]$$

$$\delta y = x^{\frac{1}{2}} \frac{\delta x}{x}\left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x \;\, x^{1-\frac{1}{2}}}\left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^{\frac{1}{2}}}\left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{x^{1/2}}\left[\frac{-1}{2} - \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{\delta x}{x} - \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx}\,(2-\sqrt{x}) = \frac{-1}{2\sqrt{x}}} \qquad \text{Ans.}$$

(iii) $\mathbf{\dfrac{1}{\sqrt{x}}}$ ***(L.B 2007)***

Let

$$y = \frac{1}{\sqrt{x}}$$

$$y = x^{-1/2}$$

$$y + \delta y = (x + \delta x)^{-1/2}$$

$$\delta y = (x + \delta x)^{-1/2} - y$$

$$\delta y = (x + \delta x)^{-1/2} - x^{-1/2} \qquad \because\; y = x^{-1/2}$$

$$\delta y = \left[x\left(1 + \frac{\delta x}{x}\right)\right]^{-1/2} - x^{-1/2}$$

$$\delta y = x^{-1/2}\left(1 + \frac{\delta x}{x}\right)^{-1/2} - x^{-1/2}$$

$$\delta y = x^{-1/2}\left[\left(1 + \frac{\delta x}{x}\right)^{-1/2} - 1\right]$$

$$\delta y = x^{-1/2}\left[1 + \left(\frac{-1}{2}\right)\left(\frac{\delta x}{x}\right) + \frac{-1/2\,(-1/2-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots - 1\right]$$

$$\delta y = x^{-1/2}\,\frac{\delta x}{x}\left[\frac{-1}{2} + \frac{-1/2\,(-1/2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x\ \ x^{1+\frac{1}{2}}}\left[\frac{-1}{2} + \frac{-1/2\,(-1/2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^{3/2}}\left[\frac{-1}{2} + \frac{-1/2\,(-1/2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{x^{3/2}}\left[\frac{-1}{2} + \frac{-1/2\,(-1/2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{-1}{2\ x^{3/2}}$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2\,x^{3/2}}} \qquad \text{Ans.}$$

(iv) $\mathbf{\dfrac{1}{x^3}}$

Let

$$y = \frac{1}{x^3}$$

$$y = x^{-3}$$

$$y + \delta y = (x+\delta x)^{-3}$$

$$\delta y = (x+\delta x)^{-3} - y$$

$$\delta y = (x+\delta x)^{-3} - x^{-3} \qquad \because\ y = x^{-3}$$

$$\delta y = \left[x\left(1+\frac{\delta x}{x}\right)\right]^{-3} - x^{-3}$$

$$\delta y = x^{-3}\left(1+\frac{\delta x}{x}\right)^{-3} - x^{-3}$$

$$\delta y = x^{-3}\left[\left(1+\frac{\delta x}{x}\right)^{-3} - 1\right]$$

$$\delta y = x^{-3}\left[1 + (-3)\left(\frac{\delta x}{x}\right) + \frac{(-3)(-3-1)}{2!}\cdot\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots -1\right]$$

$$\delta y = x^{-3} \cdot \frac{\delta x}{x}\left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x \;\; x^{1+3}}\left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^4}\left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{x^4}\left[-3 + \frac{(-3)(-3-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{-3}{x^4}$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{-3}{x^4}} \qquad \text{Ans.}$$

(v) $\mathbf{\dfrac{1}{x-a}}$

Let

$$y = \frac{1}{x-a}$$

$$y = (x-a)^{-1}$$

$$y + \delta y = (x + \delta x - a)^{-1}$$

$$\delta y = (x - a + \delta x)^{-1} - y$$

$$\delta y = \left[(x-a)\left(1 + \frac{\delta x}{x-a}\right)\right]^{-1} - (x-a)^{-1} \qquad \because \;\; y = (x-a)^{-1}$$

$$\delta y = (x-a)^{-1}\left(1 + \frac{\delta x}{x-a}\right)^{-1} - (x-a)^{-1}$$

$$\delta y = (x-a)^{-1}\left[\left(1 + \frac{\delta x}{x-a}\right)^{-1} - 1\right]$$

$$\delta y = (x-a)^{-1}\left[1 + (-1)\left(\frac{\delta x}{x-a}\right) + \frac{(-1)(-1-1)}{2!} \cdot \left(\frac{\delta x}{x-a}\right)^2 + \ldots\ldots\ldots - 1\right]$$

$$\delta y = (x-a)^{-1} \cdot \frac{\delta x}{x-a}\left[-1 + \frac{(-1)(-1-1)}{2!} \cdot \frac{\delta x}{x-a} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x\ (x-a)^{1+1}}\left[-1+\frac{(-1)(-1-1)}{2!}\cdot\frac{\delta x}{x-a}+\ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x-a)^2}\left[-1+\frac{(-1)(-1-1)}{2!}\cdot\frac{\delta x}{x-a}+\ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{(x-a)^2}\left[-1+\frac{(-1)(-1-1)}{2!}\cdot\frac{\delta x}{x-a}+\ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{-1}{(x-a)^2}$$

$$\boxed{\frac{d}{dx}\left(\frac{1}{x-a}\right) = \frac{-1}{(x-a)^2}}$$ Ans.

(vi) $\mathbf{x(x-3)}$

Let $y = x(x-3)$

$$y = x^2 - 3x$$

$$y+\delta y = (x+\delta x)^2 - 3(x+\delta x)$$

$$\delta y = x^2+\delta x^2+2x\delta x-3x-3\delta x-y$$

$$\delta y = x^2+\delta x^2+2x\delta x-3x-3\delta x-x^2+3x$$

$$\delta y = \delta x(\delta x+2x-3)$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x(\delta x+2x-3)}{\delta x} = \delta x+2x-3$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(\delta x+2x-3)$$

$$\frac{dy}{dx} = 2x-3$$

$$\boxed{\frac{dy}{dx}[x(x-3)] = 2x-3}$$ Ans.

(vii) $\mathbf{\frac{2}{x^4}}$

Let $y = \frac{2}{x^4}$

$$y = 2x^{-4}$$

$$y + \delta y = 2(x + \delta x)^{-4}$$
$$\delta y = 2(x + \delta x)^{-4} - y$$
$$\delta y = 2\left[x\left(1 + \frac{\delta x}{x}\right)\right]^{-4} - 2x^{-4} \qquad \because\ y = 2x^{-4}$$
$$\delta y = 2x^{-4}\left(1 + \frac{\delta x}{x}\right)^{-4} - 2x^{-4}$$
$$\delta y = 2x^{-4}\left[\left(1 + \frac{\delta x}{x}\right)^{-4} - 1\right]$$
$$\delta y = 2x^{-4}\left[1 + (-4)\left(\frac{\delta x}{x}\right) + \frac{(-4)(-4-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots -1\right]$$
$$\delta y = 2x^{-4}\frac{\delta x}{x}\left[-4 + \frac{(-4)(-4-1)}{2!}.\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{2\delta x}{\delta x\,.\,x^{1+4}}\left[-4 + \frac{(-4)(-4-1)}{2!}.\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$
$$\frac{\delta y}{\delta x} = \frac{2}{x^5}\left[-4 + \frac{(-4)(-4-1)}{2!}.\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{2}{x^5}\left[-4 + \frac{(-4)(-4-1)}{2!}.\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$
$$\frac{dy}{dx} = \frac{2}{x^5}(-4) = \frac{-8}{x^5}$$
$$\boxed{\frac{d}{dx}\left(\frac{2}{x^4}\right) = \frac{-8}{x^5}} \qquad \text{Ans.}$$

(viii) $\mathbf{x^2 + \frac{1}{x^2}}$

Let $y = x^2 + \dfrac{1}{x^2}$

$$y = x^2 + x^{-2}$$
$$y + \delta y = (x + \delta x)^2 + (x + \delta x)^{-2}$$
$$\delta y = x^2 + \delta x^2 + 2x\delta x + \left[x\left(1 + \frac{\delta x}{x}\right) - y\right]^{-2} - y$$
$$\delta y = x^2 + \delta x^2 + 2x\delta x + x^{-2}\left(1 + \frac{\delta x}{x}\right)^{-2} - (x^2 + x^{-2}) \qquad \because\ y = x^2 + x^{-2}$$

$$\delta y = x^2 + \delta x^2 + 2x\delta x + x^{-2}\left(1 + \frac{\delta x}{x}\right)^{-2} - x^2 - x^{-2}$$

$$\delta y = \delta x^2 + 2x\delta x + x^{-2}\left[\left(1 + \frac{\delta x}{x}\right)^{-2} - 1\right]$$

$$\delta y = \delta x(\delta x + 2x) + x^{-2}\left[1 + (-2)\left(\frac{\delta x}{x}\right) + \frac{(-2)(-2-1)}{2!}\cdot\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots -1\right]$$

$$\delta y = \delta x(\delta x + 2x) + x^{-2}\frac{\delta x}{x}\left[-2 + \frac{(-2)(-2-1)}{2!}\ .\ \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\delta y = \delta x\left[\delta x + 2x + \frac{1}{x^{1+2}}\left\{-2 + \frac{(-2)(-2-1)}{2!}\frac{\delta x}{x} + \ldots\ldots\ldots\right\}\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}\left[\delta x + 2x + \frac{1}{x^3}\left\{-2 + \frac{(-2)(-2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right\}\right]$$

$$\frac{\delta y}{\delta x} = \delta x + 2x + \frac{1}{x^3}\left[-2 + \frac{(-2)(-2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\delta x + 2x + \frac{1}{x^3}\left\{-2 + \frac{(-2)(-2-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right\}\right]$$

$$\frac{dy}{dx} = 2x - \frac{2}{x^3}$$

$$\boxed{\frac{d}{dx}\left(x^2 + \frac{1}{x^2}\right) = 2x - \frac{2}{x^3}} \quad \text{Ans.}$$

(ix) $\mathbf{(x + 4)^{1/3}}$ ***(Guj. Board 2003)***

Let $y = (x + 4)^{1/3}$

$$y + \delta y = (x + \delta x + 4)^{1/3}$$

$$\delta y = (x + 4 + \delta x)^{1/3} - y$$

$$\delta y = \left[(x+4)\left(1 + \frac{\delta x}{x+4}\right)\right]^{1/3} - (x+4)^{1/3} \qquad \because\ y = (x+4)^{1/3}$$

$$\delta y = (x+4)^{1/3}\left(1 + \frac{\delta x}{x+4}\right)^{1/3} - (x+4)^{1/3}$$

$$\delta y = (x+4)^{1/3}\left[\left(1 + \frac{\delta x}{x+4}\right)^{1/3} - 1\right]$$

$$\delta y = (x+4)^{1/3}\left[1 + \frac{1}{3}\left(\frac{\delta x}{x+4}\right) + \frac{1/3\ (1/3 - 1)}{2!}\left(\frac{\delta x}{x+4}\right)^2 + \ldots\ldots\ldots -1\right]$$

$$\delta y = (x+4)^{1/3} \cdot \frac{\delta x}{x+4}\left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots\dots\dots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x(x+4)\, x^{1-\frac{1}{3}}}\left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots\dots\dots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x+4)^{2/3}}\left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots\dots\dots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{(x+4)^{2/3}}\left[\frac{1}{3} + \frac{1/3(1/3-1)}{2!} \cdot \frac{\delta x}{x+4} + \dots\dots\dots\right]$$

$$\frac{dy}{dx} = \frac{1}{3(x+4)^{2/3}}$$

$$\boxed{\frac{d}{dx}[(x+4)^{1/3}] = \frac{1}{3(x+4)^{2/3}}} \qquad \text{Ans.}$$

(x) $\mathbf{x^{3/2}}$

Let $y = x^{3/2}$

$$y + \delta y = (x+\delta x)^{3/2}$$

$$\delta y = (x+\delta x)^{3/2} - y$$

$$\delta y = \left[x\left(1+\frac{\delta x}{x}\right)\right]^{3/2} - x^{3/2} \qquad \because\ y = x^{3/2}$$

$$\delta y = x^{3/2}\left(1+\frac{\delta x}{x}\right)^{3/2} - x^{3/2}$$

$$\delta y = x^{3/2}\left[\left(1+\frac{\delta x}{x}\right)^{3/2} - 1\right]$$

$$\delta y = x^{3/2}\left[1 + \frac{3}{2}\left(\frac{\delta x}{x}\right) + \frac{3/2\,(3/2-1)}{2!} \cdot \left(\frac{\delta x}{x}\right)^2 + \dots\dots\dots -1\right]$$

$$\delta y = x^{3/2} \cdot \frac{\delta x}{x}\left[\frac{3}{2} + \frac{3/2\,(3/2-1)}{2!} \cdot \frac{\delta x}{x}\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{x^{\frac{3}{2}-1} \cdot \delta x}{\delta x}\left[\frac{3}{2} + \frac{3/2\,(3/2-1)}{2!} \cdot \frac{\delta x}{x} + \dots\dots\dots\right]$$

$$\frac{\delta y}{\delta x} = x^{1/2}\left[\frac{3}{2} + \frac{3/2\,(3/2-1)}{2!} \cdot \frac{\delta x}{x} + \dots\dots\dots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{1/2} \left[\frac{3}{2} + \frac{3/2\,(3/2-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = x^{1/2} \cdot \frac{3}{2}$$

$$\boxed{\frac{d}{dx}(x^{1/3}) = \frac{3}{2}\sqrt{x}} \qquad \text{Ans.}$$

(xi) $\mathbf{x^{5/2}}$

Let

$$y = x^{5/2}$$

$$y + \delta y = (x + \delta x)^{5/2}$$

$$\delta y = (x + \delta x)^{5/2} - y$$

$$\delta y = \left[x\left(1 + \frac{\delta x}{x}\right)\right]^{5/2} - x^{5/2} \qquad \because\ y = x^{5/2}$$

$$\delta y = x^{5/2}\left(1 + \frac{\delta x}{x}\right)^{5/2} - x^{5/2}$$

$$\delta y = x^{5/2}\left[\left(1 + \frac{\delta x}{x}\right)^{5/2} - 1\right]$$

$$\delta y = \left[1 + \frac{5}{2}\left(\frac{\delta x}{x}\right) + \frac{5/2(5/2-1)}{2!} \cdot \left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots - 1\right]$$

$$\delta y = x^{5/2} \cdot \frac{\delta x}{x}\left[\frac{5}{2} + \frac{5/2(5/2-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{x^{\frac{5}{2}-1} \cdot \delta x}{\delta x}\left[\frac{5}{2} + \frac{5/2(5/2-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = x^{3/2}\left[\frac{5}{2} + \frac{5/2(5/2-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{3/2}\left[\frac{5}{2} + \frac{5/2\,(5/2-1)}{2!} \cdot \frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{5}{2}\,x^{3/2}$$

$$\boxed{\frac{d}{dx}(x^{5/2}) = \frac{5}{2}x^{3/2}} \qquad \text{Ans.}$$

(xii) $\mathbf{x^m}$

Let
$$y = x^m$$
$$y + \delta y = (x+\delta x)^m$$
$$\delta y = \left[x\left(1+\frac{\delta x}{x}\right)\right]^m - y$$
$$\delta y = x^m\left(1+\frac{\delta x}{x}\right)^m - x^m \qquad \because\ y = x^m$$
$$\delta y = x^m\left[\left(1+\frac{\delta x}{x}\right)^m - 1\right]$$
$$\delta y = x^m\left[1 + m\left(\frac{\delta x}{x}\right) + \frac{m(m-1)}{2!}\cdot\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots -1\right]$$
$$\delta y = x^m\cdot\frac{\delta x}{x}\left[m + \frac{m(m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.
$$\frac{\delta y}{\delta x} = \frac{x^{m-1}\,\delta x}{\delta x}\left[m + \frac{m(m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$
$$\frac{\delta y}{\delta x} = x^{m-1}\left[m + \frac{m(m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$
$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{m-1}\left[m + \frac{m(m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$
$$\frac{dy}{dx} = mx^{m-1}$$
$$\boxed{\frac{d}{dx}(x^m) = mx^{m-1}} \qquad \text{Ans.}$$

(xiii) $\mathbf{\dfrac{1}{x^m}}$ **, m ∈ N**

Let
$$y = \frac{1}{x^m}$$
$$y = x^{-m}$$
$$y + \delta y = (x+\delta x)^{-m}$$
$$\delta y = \left[x\left(1+\frac{\delta x}{x}\right)\right]^{-m} - y$$
$$\delta y = x^{-m}\left(1+\frac{\delta x}{x}\right)^{-m} - x^{-m} \qquad \because\ y = x^{-m}$$

$$\delta y = x^{-m}\left[\left(1+\frac{\delta x}{x}\right)^{-m} - 1\right]$$

$$\delta y = x^{-m}\left[1 + (-m)\left(\frac{\delta x}{x}\right) + \frac{(-m)(-m-1)}{2!}\cdot\left(\frac{\delta x}{x}\right)^{2} + \ldots\ldots\ldots -1\right]$$

$$= x^{m}\cdot\frac{\delta x}{x}\left[-m + \frac{(-m)(-m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{x^{-m-1}\,\delta x}{\delta x}\left[-m + \frac{(-m)(-m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = x^{-m-1}\left[-m + \frac{(-m)(-m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{-m-1}\left[-m + \frac{(-m)(-m-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = -mx^{-m-1}$$

$$\boxed{\frac{dy}{dx}\left(\frac{1}{x^{m}}\right) = -mx^{-m-1}} \qquad \text{Ans.}$$

(xiv) $\mathbf{x^{40}}$

Let $\quad y = x^{40}$

$$y + \delta y = (x+\delta x)^{40}$$

$$\delta y = \left[x\left(1+\frac{\delta x}{x}\right)\right]^{40} - y$$

$$\delta y = x^{40}\left(1+\frac{\delta x}{x}\right)^{40} - x^{40} \qquad \because\ y = x^{40}$$

$$\delta y = x^{40}\left[\left(1+\frac{\delta x}{x}\right)^{40} - 1\right]$$

$$\delta y = x^{40}\left[1 + 40\left(\frac{\delta x}{x}\right) + \frac{40(40-1)}{2!}\cdot\left(\frac{\delta x}{x}\right)^{2} + \ldots\ldots\ldots -1\right]$$

$$\delta y = x^{40}\cdot\frac{\delta x}{x}\left[40 + \frac{40(40-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{x^{40-1}\,\delta x}{\delta x}\left[40 + \frac{40(40-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = x^{39}\left[40 + \frac{40(40-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{39}\left[40 + \frac{40(40-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = 40x^{39}$$

$$\boxed{\frac{dy}{dx}(x^{40}) = 40x^{39}} \qquad \text{Ans.}$$

(xv) $\mathbf{x^{-100}}$

Let $y = x^{-100}$

$$y + \delta y = (x+\delta x)^{-100}$$

$$\delta y = \left[x\left(1+\frac{\delta x}{x}\right)\right]^{-100} - y$$

$$\delta y = x^{-100}\left(1+\frac{\delta x}{x}\right)^{-100} - x^{-100} \qquad \because\ y = x^{-100}$$

$$\delta y = x^{-100}\left[\left(1+\frac{\delta x}{x}\right)^{-100} - 1\right]$$

$$\delta y = x^{-100}\left[1 + (-100)\left(\frac{\delta x}{x}\right) + \frac{(-100)(-100-1)}{2!}\cdot\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\ldots -1\right]$$

$$\delta y = x^{-100}\cdot\frac{\delta x}{x}\left[-100 + \frac{(-100)(-100-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x\ x^{1+100}}\left[-100 + \frac{(-100)(-100-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x^{101}}\left[-100 + \frac{(-100)(-100-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{1}{x^{101}}\left[-100 + \frac{(-100)(-100-1)}{2!}\cdot\frac{\delta x}{x} + \ldots\ldots\ldots\right]$$

$$\frac{dy}{dx} = \frac{-100}{x^{101}}$$

$$\boxed{\frac{d}{dx}(x^{-100}) = \frac{-100}{x^{101}}} \qquad \text{Ans.}$$

Q.2 Find $\frac{dy}{dx}$ from first principles if

(i) $\sqrt{x+2}$ **(ii)** $\frac{1}{\sqrt{x+a}}$ ***(L.B 2004, 2010)***

Solution:

(i) $\sqrt{x+2}$

Let

$$y = \sqrt{x+2}$$

$$y = (x+2)^{1/2}$$

$$y + \delta y = (x + \delta x + 2)^{1/2}$$

$$\delta y = (x + 2 + \delta x)^{1/2} - y$$

$$\delta y = \left[(x+2)\left(1 + \frac{\delta x}{x+2}\right)\right]^{1/2} - (x+2)^{1/2} \quad \because\ y = \sqrt{x+2}$$

$$\delta y = (x+2)^{1/2}\left(1 + \frac{\delta x}{x+2}\right)^{1/2} - (x+2)^{1/2}$$

$$\delta y = (x+2)^{1/2}\left[\left(1 + \frac{\delta x}{x+2}\right)^{1/2} - 1\right]$$

$$\delta y = (x+2)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\delta x}{x+2}\right) + \frac{1/2(1/2-1)}{2!}\cdot\left(\frac{\delta x}{x+2}\right)^2 + \ldots\ldots - 1\right]$$

$$\delta y = (x+2)^{1/2}\cdot\frac{\delta x}{x+2}\left[\frac{1}{2} + \frac{1/2\ (1/2-1)}{2!}\cdot\frac{\delta x}{x+2} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x\ \ (x+2)\ x^{1-\frac{1}{2}}}\left[\frac{1}{2} + \frac{1/2\ (1/2-1)}{2!}\cdot\frac{\delta x}{x+2} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{(x+2)^{1/2}}\left[\frac{1}{2} + \frac{1/2\ (1/2-1)}{2!}\cdot\frac{\delta x}{x+2} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{1}{\sqrt{x+2}}\left[\frac{1}{2} + \frac{1/2\ (1/2-1)}{2!}\cdot\frac{\delta x}{x+2} + \ldots\ldots\ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{2\sqrt{x+2}}$$

$$\boxed{\frac{d}{dx}\left(\sqrt{x+2}\right) = \frac{1}{2\sqrt{x+2}}}$$ Ans.

(ii) $\mathbf{\dfrac{1}{\sqrt{x + a}}}$

Let
$$y = \frac{1}{\sqrt{x+a}}$$
$$y = (x + a)^{-1/2}$$
$$y + \delta y = (x + \delta x + a)^{-1/2}$$
$$\delta y = (x + a + \delta x)^{-1/2} - y$$
$$\delta y = \left[(x + a)\left(1 + \frac{\delta x}{x + a}\right)\right]^{-1/2} - (x + a)^{-1/2} \qquad \because\ y = (x + a)^{-1/2}$$
$$\delta y = (x + a)^{-1/2}\left(1 + \frac{\delta x}{x + a}\right)^{-1/2} - (x + a)^{-1/2}$$
$$\delta y = (x + a)^{-1/2}\left[\left(1 + \frac{\delta x}{x + a}\right)^{-1/2} - 1\right]$$
$$\delta y = (x{+}a)^{-1/2}\left[1 + \left(\frac{1}{2}\right)\left(\frac{\delta x}{x + a}\right) + \frac{(-1/2)(-1/2 - 1)}{2!} \cdot \left(\frac{\delta x}{x + 2}\right)^2 + \ldots\ldots - 1\right]$$
$$\delta y = (x + a)^{-1/2} \cdot \frac{\delta x}{x + a}\left[\frac{-1}{2} + \frac{(-1/2)\ (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x + a} + \ldots\ldots\ldots\right]$$

Dividing both sides by δx.
$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x\ \ (x + a)\ x^{1+\frac{1}{2}}}\left[\frac{-1}{2} + \frac{-1/2\ (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x + a} + \ldots\ldots\ldots\right]$$
$$\frac{\delta y}{\delta x} = \frac{1}{(x + a)^{3/2}}\left[\frac{-1}{2} + \frac{-1/2\ (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x + a} + \ldots\ldots\ldots\right]$$

Taking limit $\delta x \to 0$
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{(x + a)^{3/2}}\left[\frac{-1}{2} + \frac{-1/2\ (-1/2 - 1)}{2!} \cdot \frac{\delta x}{x + a} + \ldots\ldots\ldots\right]$$
$$\frac{dy}{dx} = \frac{-1}{2(x + a)^{3/2}}$$
$$\boxed{\frac{d}{dx}\left(\frac{1}{\sqrt{x + a}}\right) = \frac{-1}{2(x + a)^{3/2}}} \qquad \text{Ans.}$$