$$\frac{53}{\sin 59^o 30'} = \frac{b}{\sin 88^o 36'}$$

$$b = \frac{53}{\sin 59^o 30'} \times \sin 88^o 36'$$

$$\boxed{b = 61.49}$$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\frac{61.49}{\sin 88^o 36'} = \frac{c}{\sin 31^o 54'}$$

$$c = \frac{61.49}{\sin 88^o 36'} \times \sin 31^o 54'$$

$$\boxed{c = 32.5}$$

EXERCISE 12.5

Solve the triangle ABC, in which

Q.1 b = 59 , c = 34 , and α = 52^o (Gujranwala Board 2007)

Solution:

Using law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$a^2 = (95)^2 + (34)^2 - 2(95)(34)\cos 52^o$$

$$= 9025 + 1156 - 3977$$

$$a^2 = 6204$$

$$\boxed{a = 78.76}$$

$$\because\ b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$2ac\cos\beta = a^2 + c^2 - b^2$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\because\ \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Now}\quad \beta = \cos^{-1}\left[\frac{a^2 + c^2 - b^2}{2ac}\right]$$

$$\beta = \cos^{-1}\left[\frac{(78.76)^2 + (34)^2 - (95)^2}{2(78.76)(34)}\right]$$

$$\boxed{\beta = 71^o 53'}$$

so $\gamma = 180^o - \beta - \alpha$

$$= 180^o - 71^o 53' - 52^o$$

$$\boxed{\gamma = 56^o 7'}$$

Q.2 $b = 12.5$, $c = 23$, & $\alpha = 38^o\,20'$

Solution:

By law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$= (12.5)^2 + (23)^2 - 2(12.5)(23)\cos 38^o\,20'$$
$$a^2 = 234.21$$
$$\boxed{a = 15.3}$$
$$\because \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\text{Now}\quad \beta = \cos^{-1}\left[\frac{a^2 + c^2 - b^2}{2ac}\right]$$
$$\cos^{-1}\left[\frac{(15.3)^2 + (23)^2 - (12.5)^2}{2(15.3)(23)}\right]$$
$$\boxed{\beta = 30^o\,26'}$$

so
$$\gamma = 180^o - \alpha - \beta$$
$$= 180^o - 38^o\,20' - 30^o\,26'$$
$$\boxed{\gamma = 111^o\,14'}$$

Q.3 $a = \sqrt{3} - 1$, $b = \sqrt{3} + 1$, $\gamma = 60^o$

Solution:

$$a = \sqrt{3} - 1 = 0.7320$$
$$b = \sqrt{3} + 1 = 2.7320$$

By law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$c^2 = (0.7320)^2 + (2.7320)^2 - 2(0.7320)(2.7320)\cos 60^o$$
$$c^2 = 6$$
$$\boxed{c = \sqrt{6}}$$

$$\because a^2 = b^2 + c^2 - 2ac\cos\alpha$$
$$2ac\cos\alpha = b^2 + c^2 - a^2$$
$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2ac}$$

$$\because \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\alpha = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right]$$
$$= \cos^{-1}\left[\frac{(2.7320)^2 + (6)^2 - (0.7320)^2}{2(2.7320)(\sqrt{6})}\right]$$
$$\boxed{\alpha = 15^o}$$

since
$$\alpha + \beta + \gamma = 180^o$$
$$\beta = 180^o - 15^o - 60^o$$
$$\boxed{\beta = 105^o}$$

Q.4 $\alpha = 3$, $c = 6$, $\beta = 36^o\ 20'$

Solution:

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$= (3)^2 + (6)^2 - 2\,(3)\,(6)\cos 36^o\ 20'$

$b^2 = 16$

$\boxed{b = 4}$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\alpha = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right] \Rightarrow \cos^{-1}\left[\frac{16 + 36 - 9}{2\,(4)\,(6)}\right]$$

$\boxed{\alpha = 26^o\ 23'}$

$\gamma = 180^o - \alpha - \beta$

$= 180^o - 26^o\ 23' - 36^o\ 20'$

$\boxed{\gamma = 117^o\ 17'}$

Q.5 $a = 7$, $b = 3$ **and** $\gamma = 38^o\ 13'$

Solution:

By law of cosines

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = (7)^2 + (3)^2 - 2\,(7)\,(3)\cos 38^o\ 13'$

$c^2 = 25 \Rightarrow \boxed{c = 5}$

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Now $\alpha = \cos^{-1}\left[\frac{b^2 + c^2 - a^2}{2bc}\right]$

$$\alpha = \cos^{-1}\left[\frac{9 + 25 - 49}{2\,(3)\,(5)}\right]$$

$$\alpha = \cos{-1}\left(\frac{-1}{2}\right)$$

$\boxed{\alpha = 120^o}$

$\beta = 180^o - \alpha - \gamma$

$= 180^o - 120^o - 38^o\ 13'$

$\boxed{\beta = 21^o\ 47'}$

Solve the following triangles, using first law of tangents, and then law of sines.

Q.6 $a = 36.21$, $b = 42.09$, $\gamma = 44^o\, 29'$

Solution:

We know that $\alpha + \beta + \gamma = 180^o$

$$\beta + \alpha = 180^o - \gamma$$
$$= 180^o - 44^o\, 29'$$
$$\beta + \alpha = 135^o\, 31' \quad \ldots\ldots\ldots (1)$$

$$\frac{b-a}{b+a} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{\tan\left(\frac{\beta+\alpha}{2}\right)}$$

$$\frac{42.09 - 36.21}{42.09 + 36.21} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{\tan 67° \, 45'}$$

$$\frac{5.88}{78.3} = \frac{\tan\left(\frac{\beta-\alpha}{2}\right)}{2.4443}$$

$$(0.0750)(2.4443) = \tan\left(\frac{\beta-\alpha}{2}\right)$$

$$\tan^{-1}(0.18355) = \frac{\beta-\alpha}{2}$$

$$2(10°24') = \beta - \alpha$$

$$\beta - \alpha = 20°48' \quad \ldots\ldots\ldots (2)$$

Adding (1) to (2).

$$\beta + \alpha = 135^o\, 31'$$

adding
$$\beta - \alpha = 20^o\, 48'$$
$$2\beta = 156^o\, 19'$$

$$\boxed{\beta = 78^o\, 10'}$$

Put in (2)

$$78^o\, 10' - \alpha = 20^o\, 48'$$
$$78^o\, 10' - 20^o\, 48' = \alpha$$

$$\boxed{\alpha = 57^o\, 22'}$$

We find 'c' using law of sines

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

$$c = \frac{42.09}{\sin 78^o\, 10'} \times \sin 44^o\, 29'$$

$$\boxed{c = 30.13}$$

Q.7 $a = 93$, $c = 101$, $\beta = 80^o$

Solution:

We know that $\alpha + \beta + \gamma = 180^o$

$\alpha + \gamma = 180^o - \beta$

$\alpha + \gamma = 180^o - 80^o$

$\alpha + \gamma = 100^o$

By law of tangents.

$$\frac{c-a}{c+a} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

$$\frac{101-93}{101+93} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

$$\frac{8}{194} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{100^\circ}{2}\right)}$$

$$0.04124 = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{1.1918}$$

$$\tan\left(\frac{\gamma-\alpha}{2}\right) = (0.04124)(1.1918)$$

$$\frac{\gamma-\alpha}{2} = \tan^{-1}(0.04915)$$

$$\gamma - \alpha = 2(2^\circ 48')$$

$$\gamma - \alpha = 5^\circ 37'$$

$\gamma - \alpha = 5^o\,37'$

By adding.

$\alpha + \gamma = 100^o$

$-\alpha + \gamma = 5^o\,37'$

$2\gamma = 105^o\,37'$

$\boxed{\gamma = 52^o\,49'}$ Put in (2)

$52^o\,49' - \alpha = 5^o\,37'$

$52^o\,49' - 5^o\,37' = \alpha$

$\boxed{\alpha = 47^o\,11'}$

To find 'b' using law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$b = \frac{93}{\sin 47^o\, 11'} \times \sin 80^o$$

$$\boxed{b = 125}$$

Q.8 b = 14.8 , c = 16.1 , α = 42° 45′

Solution:

$$\because \quad \alpha + \beta + \gamma = 180°$$

$$42°45' + \beta + \gamma = 180°$$

$$\beta + \gamma = 180° - 42°45'$$

$$\beta + \gamma = 137°\,15' \quad \ldots\ldots\ldots (1)$$

By law of tangents.

$$\frac{c-b}{c+b} = \frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{\tan\left(\frac{\gamma+\beta}{2}\right)}$$

$$\frac{16.1 - 14.8}{16.1 + 14.8} = \frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{\tan\left(\frac{137°15'}{2}\right)}$$

$$\frac{1.3}{30.9} = \frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{2.555}$$

$$\tan\left(\frac{\gamma-\beta}{2}\right) = (0.0420)\,(2.555)$$

$$\frac{\gamma-\beta}{2} = \tan^{-1}(0.1075)$$

$$\gamma - \beta = 2(6°8')$$

$$\gamma - \beta = 12°16' \quad \ldots\ldots\ldots (2)$$

Adding (1) to (2).

$$\beta + \gamma = 137°15'$$

$$\gamma - \beta = 12°16'$$

$$2\gamma = 149°31'$$

$$\boxed{\gamma = 74°45'}$$

$$\boxed{\beta = 62°29'}$$

To find 'α' using law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$a = \frac{b}{\sin\beta} \times \sin\alpha$$

$$= \frac{14.8}{\sin 62^o 29'} \times \sin 42^o 45'$$

$$\boxed{a = 11.33}$$

Q.9 $\mathbf{a = 319 \ , \ b = 168 \ , \ \gamma = 110^o 22'}$

Solution:

Since $\alpha + \beta + \gamma = 180^o$

$$\alpha + \beta = 180^o - \gamma$$

$$\alpha + \beta = 180^o - 110^o 22'$$

$$\alpha + \beta = 69^o 38' \qquad \ldots\ldots\ldots (1)$$

By law of tangent.

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{319-168}{319+168} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{69° 38'}{2}\right)}$$

$$\frac{151}{487} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{0.695}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.3101)(0.695)$$

$$\frac{\alpha-\beta}{2} = \tan^{-1}(0.2155)$$

$$\alpha - \beta = 2(12°9')$$

$$\alpha - \beta = 24° 20' \qquad \ldots\ldots\ldots (2)$$

Adding (1) and (2).

$$\alpha+\beta = 69^{o}\,38'$$
$$\underline{\alpha-\beta = 24^{o}\,20'}$$
$$2\alpha = 93^{o}\,58'$$

$\boxed{\alpha = 46^{o}\,58'}$ Put in (ii)

$\boxed{\beta = 22^{o}\,39'}$

to find ‘c’ using law of sines

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$c = \frac{319}{\sin 46^{o}\,60'} \times \sin 110^{o}\,22'$$

$\boxed{c = 408.9}$

Q.10 $b = 61$, $c = 32$, $\alpha = 59^{o}\,30'$

Solution:

since $\alpha+\beta+\gamma = 180^{o}$

$$\beta+\gamma = 180^{o} - 59^{o}\,30'$$
$$\beta+\gamma = 120^{o}\,30' \qquad \text{………}(1)$$

By law of tangent.

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{61-32}{61+32} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{120^\circ\,30'}{2}\right)}$$

$$\frac{29}{93} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan(60^\circ 15')}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.3118)\,(1.7496)$$

$$\frac{\beta-\gamma}{2} = \tan^{-1}(0.5455)$$

$$\beta-\gamma = 2(28^\circ 36')$$

$$\beta-\gamma = 57^\circ\,14' \qquad \text{………}(2)$$

Adding eq. (1) and eq. (2).

$$\beta + \gamma = 120^\circ 30'$$

Adding
$$\begin{aligned}\beta - \gamma &= 57^\circ 14' \\ \hline 2\beta &= 177^\circ 44'\end{aligned}$$

$\boxed{\beta = 88^\circ 51'}$ Put in (3)

$\boxed{\gamma = 31^\circ 38'}$

To find 'a' we will use law of sines

$$\frac{a}{\sin\alpha} = \frac{B}{\sin\beta}$$

$$a = \frac{b}{\sin\beta} \times \sin\alpha$$

$$a = \frac{61}{\sin 88^\circ 52'} \times \sin 59^\circ 30'$$

$\boxed{a = 53}$

Q.11 Measures of two sides of a triangle are in the ratio 3 : 2 and they include an angle of measure 57°. Find remaining two angles.

Solution:

Let a and b the two sides of a triangle are in the ratio 3 : 2 and include angle is $\gamma = 57^\circ$.

Then $a = 3$, $b = 2$, and $\gamma = 57^\circ$

since $\alpha + \beta + \gamma = 180^\circ$

$$\beta + \gamma = 180^\circ - \alpha$$

$$\alpha + \beta = 180^\circ - \gamma$$

$$\alpha + \beta = 123^\circ \qquad \text{......... (1)}$$

By law of tangent.

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{3-2}{3+2} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{123^\circ}{2}\right)}$$

$$\frac{1}{5} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan(61°30')}$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = (0.2)\,(1.8418)$$

$$\frac{\alpha-\beta}{2} = \tan^{-1}(0.3684)$$

$$\alpha-\beta = 2(20°13')$$

$$\alpha-\beta = 24°\,27' \qquad \ldots\ldots\ldots (1)$$

Adding eq. (1) and (2).

$$\alpha+\beta = 123^{o}$$

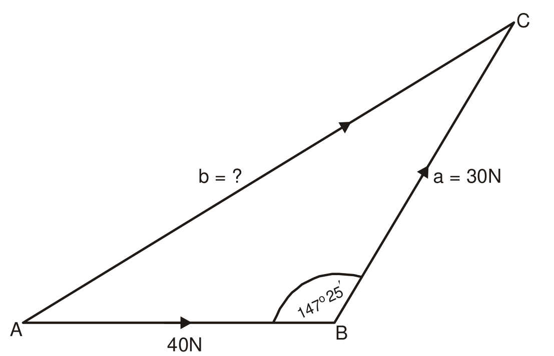
adding
$$\begin{array}{l} \alpha-\beta = 40^{o}\,27' \\ \hline 2\alpha = 163^{o}\,27' \end{array}$$

$\boxed{\alpha = 81^{o}\,44'}$ Put in (2)

$\boxed{\beta = 41^{o}16'}$

Q.12 Two forces of 40N and 30N are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ which are inclined at an angle of $147^{o}\,25'$. Find $\overrightarrow{AC}$, the resultant of $\overrightarrow{AB}$ & $\overrightarrow{BC}$.

Solution:



Given $\overrightarrow{AB} = 40$ N $\quad\therefore\quad c = 40$

$\overrightarrow{BC} = 30$ N $\quad\therefore\quad a = 30$

Now $m\angle ABC = 147o\,25'$

$\beta = 147^{o}\,25'$

By law of cosines

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$= (30)^2 + (40)^2 - 2\,(30)\,(40)\cos 147^{o}\,25'$$

$$b^2 = 4522.262$$

$$b = 67.25$$

$$\overrightarrow{AC} = 67.25\text{ N}$$