$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 30 & 54 \\ 0 & 0 & 6 & 11 & 16 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 & -9 & -16 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 6 & 11 & 16 \end{bmatrix} \text{ By } \left(\frac{1}{6}\right)R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1+1 & -9+5 & -16+9 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 6-6(1) & 11-6(5) & 16-6(9) \end{bmatrix} \text{ By } \begin{matrix} R_1+R_3 \\ R_4-6R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & -19 & -38 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4 & -7 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ By } \left(-\frac{1}{19}\right)R_4$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & -4+4(1) & -7+4(2) \\ 0 & 1 & 0 & 3-3(1) & 7-3(2) \\ 0 & 0 & 1 & 5-5(1) & 9-5(2) \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ By } \begin{matrix} R_1+4R_4 \\ R_2-3R_4 \\ R_3-5R_4 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

There are four non–zero rows

$\Rightarrow$ Rank = 4

EXERCISE 3.5

Q.1 Solve the following systems of linear equations by Cramer rule.

(i) $2x + 2y + z = 13$
$3x - 2y - 2z = 1$
$5x + y - 3z = 2$

(ii) $2x_1 - x_2 + x_3 = 5$
$4x_1 + 2x_2 + 3x_3 = 8$
$3x_1 - 4x_2 - x_3 = 3$

(iii) $2x_1 - x_2 + x_3 = 8$
$x_1 + 2x_2 + 2x_3 = 6$
$x_1 - 2x_2 - x_3 = 1$

Solution:

(i) Given system of linear equations is

$$2x + 2y + z = 13$$
$$3x - 2y - 2z = 1$$
$$5x + y - 3z = 2$$

In matrix form

Let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix} = 2(6+2) - 2(-9+10) - 1(3+10)$$

$$= 16 - 2 + 13) = 27 \neq 0$$

As $|A| \neq 0 \Rightarrow$ solution exists.

Now by Cramer's rule.

$$x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{27}$$

$$= \frac{3(6+2) - 2(-3+4) + 1(1+4)}{27}$$

$$= \frac{24 - 2 + 5}{27} = \frac{27}{27} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{27}$$

$$= \frac{2(-3+4) - 3(-9+10) + 1(6-5)}{27}$$

$$= \frac{2 - 3 + 1}{27} = \frac{0}{27} = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{27}$$

$$= \frac{2(-4-1) - 2(6-5) + 3(3+10)}{27}$$

$$= \frac{-10 - 2 + 39}{27} = \frac{27}{27} = 1$$

Thus $x = 1$, $y = 0$, $z = 1$

(ii) Given system of linear equations is

$2x_1 - x_2 + x_3 = 5$

$4x_1 + 2x_2 + 3x_3 = 8$

$3x_1 - 4x_2 - x_3 = 3$

In matrix form

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix} = 2(-2+12) + 1(-4-9) + 1(16-6)$$

$= 20 - 13 - 22 = -15 \neq 0$

As $|A| \neq 0$

$\Rightarrow$ solution exists.

Now by Cramer's rule.

$$x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{-15}$$

$$= \frac{5(-2+12) + 1(-8-9) + 1(-32-6)}{-15}$$

$$= \frac{50 - 17 - 38}{-15} = \frac{-5}{-15} = \frac{1}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{-15}$$

$$= \frac{2(-8-9) - 5(-4-9) + 1(12-24)}{-15}$$

$$= \frac{-34 + 65 - 12}{-15} = \frac{19}{-15} = -\frac{19}{15}$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{-15}$$

$$= \frac{2(6+32) + 1(12-24) + 5(-16-6)}{-15}$$

$$= \frac{76 - 12 - 110}{-15} = \frac{46}{15}$$

So $x_1 = \frac{1}{3}$, $x_2 = -\frac{19}{15}$, $x_3 = \frac{46}{15}$

(iii) Given system of linear equations is

$2x_1 - x_2 + x_3 = 8$

$x_1 + 2x_2 + 2x_3 = 6$

$x_1 - 2x_2 - x_3 = 1$

In matrix form

Let

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 2(-2+4) + (-1-2) + (-2-2) = 4-3-4 = -3 \neq 0$$

As $|A| \neq 0 \Rightarrow$ solution exists.

Now by Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{-3}$$

$$= \frac{8(-2+4) + 1(-6-2) + 1(-12-2)}{-3}$$

$$= \frac{16-8-14}{-3} = \frac{-6}{-3} = 2$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{-3}$$

$$= \frac{2(-6-2) - 8(-1-2) + 1(1-6)}{-3}$$

$$= \frac{-16+24-5}{-3} = \frac{3}{-3} = -1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{-3}$$

$$= \frac{2(2+12) + 1(1-6) + 8(-2-2)}{-3}$$

$$= \frac{28-5-32}{-3} = \frac{-9}{-3} = 3$$

So $x_1 = 2$, $x_2 = -1$, $x_3 = 3$.

Q.2 Use matrices to solve the following system:

(i) $x - 2y + z = -1$ **(ii)** $2x_1 + x_2 + 3x_3 = 3$

$3x + y - 2z = 4$ $x_1 + x_2 - 2x_3 = 0$

$y - z = 1$ $-3x_1 - x_2 + 2x_3 = -4$

(iii) $x + y = 2$

$2x - z = 1$

$2y - 3z = -1$

Solution:

(i) Given system is

$x - 2y + z = -1$

$3x + y - 2z = 4$

$y - z = 1$

In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$ ……… (1)

where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 1(-1+2) + 2(-3-0) + 1(3-0)$

$= 1 - 6 + 3 = -2 \neq 0$

$|A| \neq 0 \Rightarrow$ inverse of A exists.

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = (-1)^2(-1+2) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = (-1)^3(-3-0) = 3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = (-1)^4(3-0) = 3$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = (-1)^3(2-1) = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-1-0) = -1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = (-1)^5(-1-0) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-1)^4(4-1) = 3$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = (-1)^5(-2-3) = 5$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = (-1)^6(1+6) = 7$$

As $A^{-1} = \dfrac{\text{adj } A}{|A|}$

$$= \frac{1}{-2}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{-2}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -3 & 7 \end{bmatrix}$$

Put values in (1)

$$X = -\frac{1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -3 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-2}{-2} \\ \frac{-2}{-2} \\ \frac{0}{-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x = 1, \ y = 1, \ z = 0$$

(ii) Given system is

$$2x_1 + x_2 + 3x_3 = 3$$
$$x_1 + x_2 - 2x_3 = 0$$
$$-3x_1 - x_2 + 2x_3 = -4$$

In matrix form

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$A\ X = B \qquad \text{(say)}$$
$$X = A^{-1}B$$

where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix}$$

$$= 2(2-2) - 1(2-6) + 3(-1+3) = 0 + 4 + 6 = 10 \neq 0$$

$$|A| \neq 0 \Rightarrow A^{-1} \text{ exists.}$$

Now

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = (-1)^2(2-2) = 0$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = (-1)^3(2-6) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^4(-1+3) = 2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = (-1)^3(2+3) = -5$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = (-1)^4(4+9) = 13$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = (-1)^5(-2+3) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^4(-2-3) = -5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = (-1)^5(-4-3) = 7$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^6(2-1) = 1$$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$$

Put values in (1)

$$X = \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0-0+20 \\ 12+0-28 \\ 6+0-4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -16 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{20}{10} \\ \frac{-16}{10} \\ \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{8}{5} \\ \frac{1}{5} \end{bmatrix}$$

$\Rightarrow \quad x_1 = 2 \;,\; x_2 = -\frac{8}{5} \;,\; x_3 = \frac{1}{5}$

(iii) Given system is

$$x + y = 2$$
$$2x - z = 1$$
$$2y - 3z = -1$$

In matrix form

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= 1(0+2) - 1(-6-0) \neq 0(4-0) = 2 + 6 + 0 = 8 \neq 0$$

$|A| \neq 0 \Rightarrow A^{-1}$ exists.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = (-1)^2(0+2) = 2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = (-1)^3(-6) = 6$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (-1)^4(4-0) = 4$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = (-1)^3(-3-0) = 3$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = (-1)^4(-3-0) = -3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = (-1)^5(2-0) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = (-1)^4(-1-0) = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = (-1)^5(-1-0) = 1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = (-1)^6(0-2) = -2$$

$$A^{-1} = \frac{\text{adJ } A}{|A|}$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

Put values in (1)

$$X = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{8}{8} \\ \frac{8}{8} \\ \frac{8}{8} \end{bmatrix}$$

$\Rightarrow \quad x = 1 \;,\; y = 1 \;,\; z = 1.$

Q.3 Solve the following systems by reducing their augmented matrices into echelon or reduced echelon from

(i)
$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \\ 2x + 3y - z &= 9 \end{aligned}$$

(iii)
$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 9 \\ 3x_1 + 2x_2 - 2x_3 &= 12 \end{aligned}$$

Solution:

(i) Given system is

$$\begin{aligned} x_1 - 2x_2 - 2x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}$$

Augmented matrix is

$$A_b = \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & -4 & -3 & : & 1 \end{bmatrix}$$

Method 1: By Echelon Form:

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2-2(1) & 3-2(-2) & 1-2(-2) & : & 1-2(-1) \\ 5-5(1) & -4-5(-2) & -3-5(-2) & : & 1-5(-1) \end{bmatrix} \text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 7 & 5 & : & 3 \\ 0 & 6 & 7 & : & 6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 6 & 7 & : & 6 \end{bmatrix} \text{By } \left(\frac{1}{7}\right) R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 6-6(1) & 7-6\left(\frac{5}{7}\right) & : & 6-6\left(\frac{3}{7}\right) \end{bmatrix} \text{By } R_3 - 6R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & \frac{19}{7} & : & \frac{24}{7} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \text{By } \left(\frac{7}{19}\right) R_3$$

Now the Augumented matrix is in row echelon form

It can be written as

$x_1 - 2x_2 - 2x_3 = -1$ .... (1)

$x_2 + \frac{5}{7}x_3 = \frac{3}{7}$ .... (2)

$x_3 = \frac{24}{19}$ .... (3)

Put $x_3 = \frac{24}{19}$ in (2)

$$x_2 + \frac{5}{7}\left(\frac{24}{19}\right) = \frac{3}{7}$$

$$x_2 + \frac{120}{133} = \frac{3}{7}$$

$$x_2 = \frac{3}{7} - \frac{120}{133} = \frac{57-120}{133} = \frac{-63}{133} = \frac{-9}{19}$$

Put $x_2 = \frac{-9}{19}$ and $x_3 = \frac{24}{19}$ in (1)

$$x_1 - 2\left(-\frac{9}{19}\right) - 2\left(\frac{24}{19}\right) = -1$$

$$\Rightarrow \quad x_1 + \frac{18}{19} - \frac{48}{19} = -1$$

$$\Rightarrow \quad x_1 + \frac{18-48}{19} = -1$$

$$\Rightarrow \quad x_1 - \frac{30}{19} = -1$$

$$\Rightarrow \quad x_1 = \frac{30}{19} - 1 = \frac{30-19}{19}$$

$$\Rightarrow \quad x_1 = \frac{11}{19}$$

Thus required solution is

$$x_1 = \frac{11}{19}, \quad x_2 = -\frac{9}{19}, \quad x_3 = \frac{24}{19}$$

Method 2: By Reduced Echelon Form

Here we reduce the matrix.

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$ to reduced echelon form

$\Rightarrow$

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2+2(1) & -2+2\left(\frac{5}{7}\right) & : & -1+2\left(\frac{3}{7}\right) \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \text{By } R_1 + 2R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -\frac{4}{7} & : & -\frac{1}{7} \\ 0 & 1 & \frac{5}{7} & : & \frac{3}{7} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -\frac{4}{7}+\frac{4}{7}(1) & : & -\frac{1}{7}+\frac{4}{7}\left(\frac{24}{19}\right) \\ 0 & 1 & \frac{5}{7}-\frac{5}{7}(1) & : & \frac{3}{7}-\frac{5}{7}\left(\frac{24}{19}\right) \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \text{By } \begin{matrix} R_1 + \frac{4}{7}R_3 \\ R_2 - \frac{5}{7}R_3 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & : & \frac{11}{19} \\ 0 & 1 & 0 & : & -\frac{9}{19} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$$

It can be written as

$x_1 = \frac{11}{19}$, $x_2 = -\frac{9}{19}$, $x_3 = \frac{24}{19}$

Which is required solution by reduced echelon form.

(ii) Given system is

$x + 2y + z = 2$

$2x + y + 2z = -1$

$2x + 3y - z = 9$

Augmented matrix is given by

$$A_b = \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$$

Method I: By Echelon form

$$\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2-2(1) & 1-2(2) & 2-2(1) & : & -1-2(2) \\ 2-2(1) & 3-2(2) & -1-2(1) & : & 9-2(2) \end{bmatrix} \text{By } \begin{matrix} R_2-2R_1 \\ R_3-2R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -3 & 0 & : & -5 \\ 0 & -1 & -3 & : & 5 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & -1 & -3 & : & 5 \end{bmatrix} \text{By } \left(-\frac{1}{3}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & -1+1 & -3+0 & : & 5+\frac{5}{3} \end{bmatrix} \text{By } R_3+R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & -3 & : & -\frac{20}{3} \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \text{By } \left(-\frac{1}{3}\right)R_3$$

Now it is in echelon form so it can be written as

$x + 2y + z = 2$ .... (1)

$y = \frac{5}{3}$ .... (2)

$z = -\frac{20}{9}$ .... (3)

Put (2) and (3) in equation (1)

$$x + 2\left(\frac{5}{3}\right) + \left(-\frac{20}{9}\right) = 2$$

$$x + \frac{10}{3} - \frac{20}{9} = 2$$

$$x = 2 - \frac{10}{3} + \frac{20}{9} = \frac{18 - 30 + 20}{9}$$

$$x = \frac{8}{9}$$

Thus solution is $x = \frac{8}{9}$, $y = \frac{5}{3}$, $z = -\frac{20}{9}$

Method 2: By Reduced Echelon Form:

Here

We reduce the matrix $\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$ in reduced echelon form.

so $\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2-2(1) & 1-2(0) & : & 2-2\left(\frac{5}{3}\right) \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \text{ By } R_1 - 2R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1 & : & -\frac{4}{3} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 1-1 & : & -\frac{4}{3}+\frac{20}{9} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \text{ By } R_1 - R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{8}{9} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix}$$

Now

It can be written as

$$x = \frac{8}{9}, \quad y = \frac{5}{3}, \quad z = -\frac{20}{9}$$

which is the required solution.

(iii) Given system is

$x_1 + 4x_2 + 2x_3 = 2$

$2x_1 + x_2 - 2x_3 = 9$

$3x_1 + 2x_2 - 2x_3 = 12$

Augmented matrix is given by

$$A_b = \begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 2 & -2 & : & 12 \end{bmatrix}$$

Method 1 : By Echelon form

$$\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 2 & -2 & : & 12 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2-2(1) & 1-2(4) & -2-(2) & : & 9-2(2) \\ 3-3(1) & 2-3(4) & -2-3(2) & : & 12-3(2) \end{bmatrix} \text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & -7 & -6 & : & 5 \\ 0 & -10 & -8 & : & 6 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & \frac{5}{7} \\ 0 & -10 & -8 & : & 6 \end{bmatrix} \text{By } -\left(-\frac{1}{7}\right)R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & -10+10(1) & -8+10\left(\frac{6}{7}\right) & : & 6+10\left(-\frac{5}{7}\right) \end{bmatrix} \text{By } R_3 + 10R_2$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & \frac{4}{7} & : & -\frac{8}{7} \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \text{ By } \left(\frac{7}{4}\right)R_3$$

The Augumented matrix is n echelon form it can be written as

$$x_1 + 4x_2 + 2x_3 = 2 \quad \ldots\ldots\ldots (1)$$

$$x_2 + \frac{6}{7}x_3 = -\frac{5}{7} \quad \ldots\ldots\ldots (2)$$

$$x_3 = -2 \quad \ldots\ldots\ldots (3)$$

Put $x_3 = -2$ in equation (2)

$$x_2 + \frac{6}{7}(-2) = -\frac{5}{7}$$

$$x_2 - \frac{12}{7} = -\frac{5}{7}$$

$$x_2 = \frac{12}{7} - \frac{5}{7} = \frac{12-5}{7} = \frac{7}{7} = 1$$

Put $x_2 = 1$ and $x_3 = -2$ in equation (1)

$$x_1 + 4(1) + 2(-2) = 2$$

$$x_1 = 2 + 4 - 4 = 2$$

$$x_1 = 2$$

so required solution is

$$x_1 = 2 \quad , \quad x_2 = 1 \quad , \quad x_3 = -2$$

Method 2 By Reduced Echelon Form:

Now we reduce $\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$ in reduced form

So
$$\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 4-4(1) & 2-4\left(\frac{6}{7}\right) & : & 2-4\left(-\frac{5}{7}\right) \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{10}{7} & : & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -\frac{10}{7}+\frac{10}{7}(1) & : & \frac{34}{7}+\frac{10}{7}(-2) \\ 0 & 1 & \frac{6}{7}-\frac{6}{7}(1) & : & -\frac{5}{7}-\frac{6}{7}(-2) \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \text{By } \begin{matrix} R_1+\frac{10}{7}R_3 \\ R_2-\frac{6}{7}R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

It can be written as

$x_1 = 2$, $x_2 = 1$, $x_3 = -2$

which is the required solution by Reduced Echelon form.

Q.4 Solve the following systems of homogeneous linear equations.

(i) $\mathbf{x + 2y - 2z = 0}$
$\mathbf{2x + y + 5z = 0}$
$\mathbf{5x + 4y + 8z = 0}$

(ii) $\mathbf{x_1 + 4x_2 + 2x_3 = 0}$
$\mathbf{2x_1 + x_2 - 3x_3 = 0}$
$\mathbf{3x_1 + 2x_2 - 4x_3 = 0}$

(iii) $\mathbf{x_1 - 2x_2 - x_3 = 0}$
$\mathbf{x_1 + x_2 + 5x_3 = 0}$
$\mathbf{2x_1 - x_2 + 4x_3 = 0}$

Solution:

(i) Given system is

$x + 2y - 2z = 0$ ……… (1)

$2x + y + 5z = 0$ ……… (2)

$5x + 4y + 8z = 0$ ……… (3)

In matrix form, it can be written as

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$

where $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$$= 1(8-20) - 2(16-25) - 2(8-5)$$

$$= -12 + 18 - 6 = 0$$

$\Rightarrow$ $|A| = 0$

$\Rightarrow$ we cannot find A^{-1}

$\Rightarrow$ given system has a non–trivial solution.

Now we sole the system such that from equation (1) and (2)

$x_1 + 4x_2 + 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}$$

$\Rightarrow$ $\dfrac{x_1}{-12-2} = \dfrac{-x_2}{-3-4} = \dfrac{x_3}{1-8}$

$\Rightarrow$ $\dfrac{x_1}{-14} = \dfrac{-x_2}{-7} = \dfrac{x_3}{-7}$

$\Rightarrow$ $\dfrac{x_1}{2} = \dfrac{-x_2}{1} = \dfrac{x_3}{1} = t$ (say)

$\Rightarrow$ $\dfrac{x_1}{2} = t \Rightarrow x_1 = 2t$

$\Rightarrow$ $x_2 = -t$, $x_3 = t$

The system has infinite many solutions depending upon the value of t.

(ii) Given system is

$x_1 + 2y - 2z = 0$ ……… (1)

$2x + y + 5z = 0$ ……… (2)

$5x + 4y + 8z = 0$ ……… (3)

In matrix form, we have

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A X = B$ (say)

$X = A^{-1} B$

Where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{vmatrix}$$

$$= 1(8-20) - 2(16-25) - 2(8-5)$$

$$= -12 + 18 - 6 = 0$$

$\Rightarrow$ $|A| = 0$

$\Rightarrow$ we cannot find A^{-1}

$\Rightarrow$ given system has a non–trivial solution.

Now we solve the system such that from equation (1) and (2)

$$x + 2y - 2z = 0$$

$$2x + y + 5z = 0$$

$$\frac{x}{\begin{vmatrix} 2 & -2 \\ 1 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x}{10+2} = \frac{-y}{5+4} = \frac{z}{1-4}$$

$$\frac{x}{12} = \frac{-y}{9} = \frac{z}{-3}$$

multiplying by (-3)

$$\frac{x}{-4} = \frac{-y}{3} = \frac{z}{1} = t \quad \text{(say)}$$

$\Rightarrow$ $\frac{x}{-4} = t \Rightarrow x = -4t$

$\Rightarrow$ $\frac{y}{3} = t \Rightarrow y = 3t$

and $z = t$

$\Rightarrow$ $x = -4t$, $y = 3t$, $z = t$ is the solution of the system.

The system has infinite many solutions depending upon the value of t.

(iii) Given system is

$x_1 - 2x_2 - x_3 = 0$ .... (1)

$x_1 + x_2 + 5x_3 = 0$ .... (2)

$2x_1 - x_2 + 4x_3 = 0$ .... (3)

In matrix form, we have

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\ X = B$ (say)

$X = A^{-1} B$

where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

$= 1(4 + 5) + 2(4 - 10) - 1(-1 - 2)$

$= 9 - 12 + 3 = 0$

$|A| = 0$

$\Rightarrow$ we cannot find A^{-1}

$\Rightarrow$ system has non–trivial solution.

Now we find the solution such that from equation (1) and (2)

$x_1 - 2x_2 - x_3 = 0$

$x_1 + x_2 + 5x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-10+1} = \frac{-x_2}{5+1} = \frac{x_3}{1+2}$$

$$\frac{x_1}{-9} = \frac{-x_2}{6} = \frac{x_3}{3}$$

Multiplying by $+3$

$$\frac{x_1}{-3} = \frac{x_2}{-2} = \frac{x_3}{1} = t \quad \text{(say)}$$

$\Rightarrow \quad \frac{x_1}{-3} = t \Rightarrow x_1 = -3t$

$\Rightarrow \quad \frac{x_2}{-2} = t \Rightarrow x_2 = -2t$

$\Rightarrow \quad x_3 = t$

$\Rightarrow \quad x_1 = -3t \;,\; x2 = -2t \;,\; x_3 = t$ is the solution of the system.

The system has infinite many solutions depending upon the value of t.

Q.5 Find the value of λ for which the following systems has a non–trivial solution. Also solve the system for the value of a

(i) $x + y + z = 0$, $2x + y - \lambda z = 0$, $x + 2y - 2z = 0$

(ii) $x_1 + 4x_2 + \lambda x_3 = 0$, $2x_1 + x_2 - 3x_3 = 0$, $3x_1 + \lambda x_2 - 4x_3 = 0$

Solution:

(i) The given system is

$x + y + z = 0$ ……… (1)

$2x + y - \lambda z = 0$ ……… (2)

$x + 2y - 2z = 0$ ……… (3)

In matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\ X = B$ (say)

It is given that the system has non–trivial solution.

$\Rightarrow \quad |A| = 0$

$\Rightarrow$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$1(-2 + 2\lambda) - 1(-4 + \lambda) + 1(4 - 1) = 0$

$2\lambda - 2 + 4 - \lambda + 3 = 0$

$\lambda + 5 = 0 \Rightarrow \lambda = -5$

Take equation(1) and (2)

$X + y + z = 0$

$2x + y + 5z = 0$ ¤ $\lambda = 5$

$$\frac{x}{\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x_1}{5-1} = \frac{-y}{5-2} = \frac{z}{1-2}$$

$$\frac{x_1}{4} = \frac{-y}{3} = \frac{z}{-1}$$

multiplying by (-1)

$$\frac{x}{-4} = \frac{y}{3} = \frac{z}{1} = t \quad \text{(say)}$$

$\Rightarrow \quad \dfrac{x}{-4} = t \Rightarrow x = -4t$

$\Rightarrow \quad \dfrac{y}{3} = t \Rightarrow y = 3t$

$\Rightarrow$ and $z = t$

$\Rightarrow$ $x = -4t$, $y = 3t$, $z = t$ is the solution of the system.

(ii) Given system is

$x_1 + 4x_2 + \lambda x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$3x_1 + \lambda x_2 - 4x_3 = 0$

In matrix form, we have

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \ X = B$ (say)

As it is given that the system has non–trivial solution.

so $|A| = 0$

i.e. $\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$

$\Rightarrow \quad 1(-4 + 3\lambda) - 4(-8 + 9) + \lambda(2\lambda - 3) = 0$

$\Rightarrow \quad 3\lambda - 4 - 4 + 2\lambda^2 - 3\lambda = 0$

$\Rightarrow \quad 2\lambda^2 - 8 = 0$

$\Rightarrow \quad \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda^2 = 4 \Rightarrow \lambda = \pm 2$

When $\lambda = 2$

equation(1) and (2) becomes

$x_1 + 4x_2 + 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-12-2} = \frac{-x_2}{-3-4} = \frac{x_3}{1-8}$$

$$\frac{x_1}{-14} = \frac{-x_2}{-7} = \frac{x_3}{-7}$$

multiplying by (-7)

$$\Rightarrow \quad \frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{1} = t \quad \text{(say)}$$

$$\Rightarrow \quad \frac{x_1}{2} = t \Rightarrow x_1 = 2t$$

$$\Rightarrow \quad \frac{-x_2}{1} = t \Rightarrow x_2 = -t$$

and $\quad x_3 = t$

When $\lambda = -2$

equation(1) and (2) becomes

$x_1 + 4x_2 - 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$\Rightarrow$

$$\frac{x_1}{\begin{vmatrix} 4 & -2 \\ 1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-12+2} = \frac{-x_2}{-3+4} = \frac{x_3}{1-8}$$

$$\frac{x_1}{-10} = \frac{-x_2}{1} = \frac{x_3}{-7}$$

Multiplying by (-7)

$$\frac{7x_1}{10} = \frac{7x_2}{1} = \frac{x_3}{1} = t \quad \text{(say)}$$

$$\Rightarrow \quad \frac{7x_1}{10} = t \Rightarrow x_1 = \frac{10}{7}t$$

$$\Rightarrow \quad \frac{7x_2}{1} = t \Rightarrow x_2 = \frac{1}{7}t$$

and $\quad x_3 = t$

is required solution.

Q.6 Find the value of λ for which the following system does not possess a unique solution. Also solve the system for value of λ.

$$\mathbf{x_1 + 4x_2 + \lambda x_3 = 2}$$

$$\mathbf{2x_1 + x_2 - 2x_3 = 11}$$

$$\mathbf{3x_1 + 2x_2 - 2x_3 = 16}$$

Solution:

Given system is

$$x_1 + 4x_2 + \lambda x_3 = 2$$

$$2x_1 + x_2 - 2x_3 = 11$$

$$3x_1 + 2x_2 - 2x_3 = 16$$

In matrix form, we have

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 16 \end{bmatrix}$$

$$A\ X = B \qquad \text{(say)}$$

It is given that the system does not possess a unique solution.

$$\Rightarrow \quad |A| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$$

$\Rightarrow \quad 2 - 8 + \lambda = 0$

$\Rightarrow \quad \lambda - 6 = 0 \Rightarrow \lambda = 6$

Put $\lambda = 6$ in the given system it becomes

$x_1 + 4x_2 + 6x_3 = 2$

$2x_1 + x_2 - 2x_3 = 11$

$3x_1 + 2x_2 - 2x_3 = 16$

The augumented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 2 & 1 & -2 & : & 11 \\ 3 & 2 & -2 & : & 16 \end{bmatrix}$$

we reduce this matrix to reduced echelon form, i.e.

$$\begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 2 & 1 & -2 & : & 11 \\ 3 & 2 & -2 & : & 16 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 2-2(1) & 1-2(4) & -2-2(6) & : & 11-2(2) \\ 3-3(1) & 2-3(4) & -2-3(6) & : & 16-3(2) \end{bmatrix} \text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & -7 & -14 & : & 7 \\ 0 & -10 & -20 & : & 10 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & 2 & : & -1 \\ 0 & -10 & -20 & : & 10 \end{bmatrix} \text{By } \left(-\frac{1}{7}\right) R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4-4(1) & 6-4(2) & : & 2-4(-1) \\ 0 & 1 & 2 & : & -1 \\ 0 & -10+10(1) & -20+10(2) & : & 10+10(-1) \end{bmatrix} \text{By } \begin{matrix} R_1 - 4R_2 \\ R_3 - 10R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -2 & : & 6 \\ 0 & 1 & 2 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

It can be written as

$x_1 + 0x_2 - 2x_3 = 6 \qquad \ldots\ldots\ldots (1)$

$x_2 + 2x_3 = -1 \qquad \ldots\ldots\ldots (2)$

$0x_3 = 0$

Let $x_3 = t$ (arbitrary value)

Put in (2)

$x_2 + 2t = -1$

$x_2 = -2t - 1$

Put $x_3 = t$ in (1)

$x_1 - 2t = 6$

$\Rightarrow \quad x_1 = 2t + 6$

$\Rightarrow \quad x_1 = 2t + 6 \ , \ x_2 = -2t - 1 \ , \ x_3 = t$

This system has infinitely many solutions depending upon the value of t.