Introduction to CAP: Constructive category theory and applications

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University of Siegen

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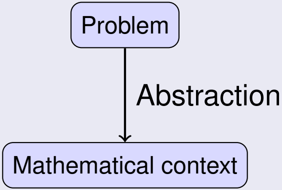
Part I

Constructive category theory

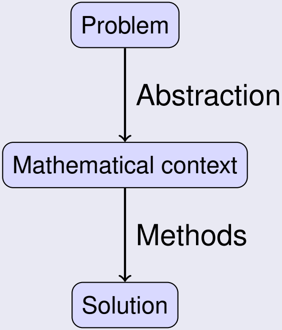
Mathematics

Problem

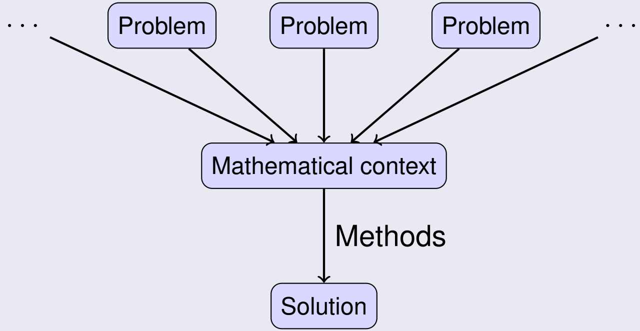
Mathematics



Mathematics



Mathematics



Mathematics

$$\begin{tikzcd}
\cdots \arrow[drrrr] & \simeq & \boxed{\text{Problem}} \arrow[drr] & \simeq & \boxed{\text{Problem}} \arrow[d] & \simeq & \boxed{\text{Problem}} \arrow[dll] & \simeq & \cdots \arrow[dllll] \\
 & & & & \boxed{\text{Mathematical context}} \arrow[d, "\text{Methods}"] & & & & \\
 & & & & \boxed{\text{Solution}} & & & &
\end{tikzcd}$$

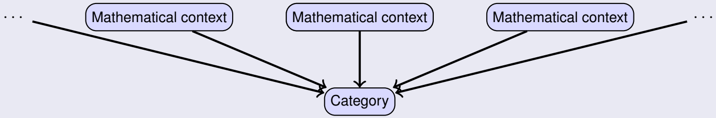
Constructive category theory

Mathematical context

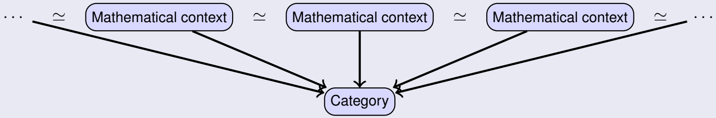
Constructive category theory

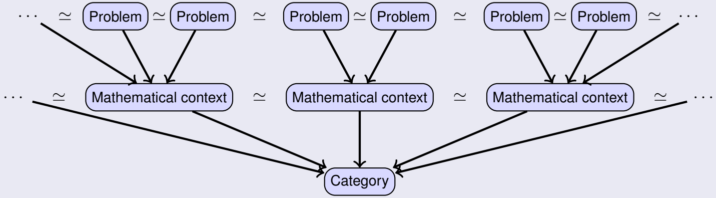
$$\begin{tikzcd} \text{Mathematical context} \arrow[d, "\text{Abstraction}"] \\ \text{Category} \end{tikzcd}$$

Constructive category theory

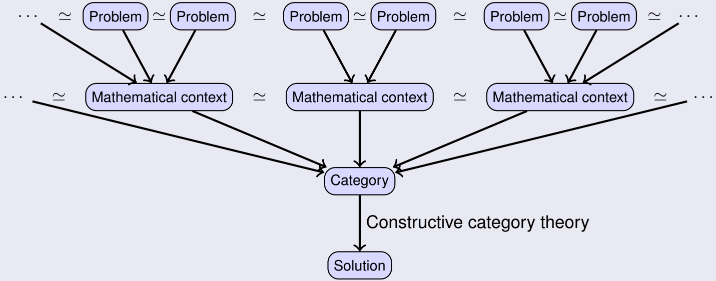


Constructive category theory




· · · ≃ Problem ≃ Problem ≃ Problem ≃ Problem ≃ Problem ≃ Problem ≃ · · ·
· · · ≃ Mathematical context ≃ Mathematical context ≃ Mathematical context ≃ · · ·
Category

Constructive category theory



Abstraction of language

Abstraction of language

Addition of two numbers:

Data type: `int`

Data type: `float`

Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

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Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

```
addi:
movl   %edi, -4(%rsp)
movl   %esi, -8(%rsp)
movl   -4(%rsp), %esi
addl   -8(%rsp), %esi
movl   %esi, %eax
ret
```

Data type: `float`

Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

```
addi:
movl  %edi, -4(%rsp)
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movl  -4(%rsp), %esi
addl  -8(%rsp), %esi
movl  %esi, %eax
ret
```

Data type: `float`

```
addf:
movss %xmm0, -4(%rsp)
movss %xmm1, -8(%rsp)
movss -4(%rsp), %xmm0
addss -8(%rsp), %xmm0
ret
```

Abstraction of language

Addition of two numbers: C

Data type: `int`

Data type: `float`

Abstraction of language

Addition of two numbers: C

Data type: `int`

```
int addi( int a,
          int b )
{
  return a + b;
}
```

Data type: `float`

Abstraction of language

Addition of two numbers: C

Data type: `int`

```
int addi( int a,
          int b )
{
  return a + b;
}
```

Data type: `float`

```
float addf( float a,
            float b )
{
  return a + b;
}
```

Abstraction of language

Addition of two numbers: GAP or Julia

Data type: `int`

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Abstraction of language

Addition of two numbers: GAP or Julia

Data type: `int`

```
function( a, b )
    return a + b;
end;
```

Data type: `float`

Abstraction of language

Addition of two numbers: GAP or Julia

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High language leads to generic code!

Abstraction of language

Computing the intersection of two subobjects

Abstraction of language

Computing the intersection of two subobjects

Vector spaces

$\langle v_1, v_2 \rangle, \langle w_1, w_2 \rangle \leq V$:

Abstraction of language

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Solution of

$$\begin{aligned} x_1 v_1 &+ x_2 v_2 \\ = y_1 w_1 &+ y_2 w_2 \end{aligned}$$

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Ideals of $\mathbb{Z}$

Abstraction of language

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$\langle x\rangle, \langle y\rangle \leq \mathbb{Z}$:
Euclidean algorithm:

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Generic algorithm for both cases?

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Generic algorithm for both cases? **Category theory!**

Category theory as programming language

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- abstracts mathematical structures

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- defines a *language* to formulate theorems and algorithms for different structures *at the same time*

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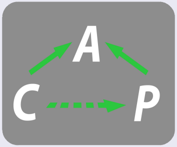

CAP implements a
categorical programming language

Categories

Definition

A category $\mathcal{A}$ contains the following data:

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- $\mathrm{Obj}_{\mathcal{A}}$

$A \qquad\qquad B \qquad\qquad C$

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$$A \longrightarrow B \longrightarrow C$$

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$$

Finite dimensional vector spaces

Let k be a field.

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Example: k-vec

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- $\mathrm{Obj} := \mathbb{N}_0$

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$$\simeq$$

Example: matrices (computerfriendly model)

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Computable categories

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$$1 \xrightarrow[\begin{pmatrix} 1 & 2 \end{pmatrix}]{} 2 \xrightarrow[\begin{pmatrix} 3 \\ 4 \end{pmatrix}]{} 1$$

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Equivalences

Example

$$\mathrm{Rep}_k(G) \xrightarrow{\quad\quad\quad\sim\quad\quad\quad} \bigoplus_{i \in \mathrm{Irr}(G)} k\text{-vec}$$

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Equivalences

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$$\begin{array}{ccc} \mathrm{Rep}_k(G) & \xrightarrow{\quad\sim\quad} & \bigoplus_{i \in \mathrm{Irr}(G)} k\text{-vec} \\ V \simeq \bigoplus_{i \in \mathrm{Irr}(G)} a_i V^i & \longmapsto & (a_i)_i \end{array}$$

Equivalences: $\mathrm{Rep}_k(G)$

Example: S_3, irreducible representations: $V^{\mathbb{1}}$, V^{sgn}, V^{χ}

$$\begin{tikzcd}
V \arrow[rr, "{\begin{pmatrix} -102 & 5824 & -96 & 20 & 1444 & 584 \\ 58 & -2366 & 60 & 8 & -590 & -240 \\ 83 & -5366 & 75 & -28 & -1328 & -536 \\ -25 & 1354 & -24 & 3 & 336 & 136 \\ -377 & 17200 & -384 & -28 & 4279 & 1736 \\ 351 & -18877 & 348 & -12 & -4682 & -1893 \end{pmatrix}}"] \arrow[d, "\wr"'] & & V \arrow[d, "\wr"'] \\
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Some categorical operations in abelian categories

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$$\dashrightarrow = \mathrm{KernelLift}(\varphi, \alpha \circ \kappa')$$

Features of CAP

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Compute their intersection $\gamma : M_1 \cap M_2 \hookrightarrow N$.

$$\begin{tikzcd}
 & M_1 \arrow[dr, hook, "\iota_1"] & \\
M_1 \oplus M_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & N \\
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 & M_2 \arrow[ur, hook, "\iota_2"'] &
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M_1 \oplus M_2 \arrow[ur, two heads, "\pi_1"] \arrow[dr, two heads, "\pi_2"'] \arrow[rr, "\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2"] & & N \\
 & M_2 \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & & M_1 \arrow[dr, hook, "\iota_1"] & \\
M_1 \cap M_2 \arrow[r, hook, "\kappa"] & M_1 \oplus M_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, "\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2"] & & N \\
 & & M_2 \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

- $\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left((M_1, M_2), i\right)$, $i = 1, 2$
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 & & & M_2 \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

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Translation to CAP

$\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left(\left(M_1, M_2\right), i\right)$, $i = 1, 2$

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$\pi_i := \text{ProjectionInFactorOfDirectSum}\left(\left(M_1, M_2\right), i\right),\ i = 1, 2$

```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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```

$\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2$

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$\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2$

```
lambda := PostCompose( iota1, pi1 );
phi := lambda - PostCompose( iota2, pi2 );
```

$\kappa := \mathrm{KernelEmbedding}\left(\varphi\right)$

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```

$\kappa := \text{KernelEmbedding}\left(\varphi \right)$

```
kappa := KernelEmbedding( phi );
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```

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```
kappa := KernelEmbedding( phi );
```

$\gamma := \iota_1 \circ \pi_1 \circ \kappa$

```
gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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lambda := PostCompose( iota1, pi1 );
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Translation to CAP

```
IntersectionOfSubobject := function( iota1, iota2 )

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  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
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```

Translation to CAP

```
IntersectionOfSubobject := function( iota1, iota2 )

  M1 := Source( iota1 );
  M2 := Source( iota2 );

  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );

  lambda := PostCompose( iota1, pi1 );
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  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Translation to CAP

```
IntersectionOfSubobject := function( iota1, iota2 )
  local M1, M2, pi1, pi2, lambda, phi, kappa, gamma;
  M1 := Source( iota1 );
  M2 := Source( iota2 );
  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Computing the intersection: $\mathbb{Q}$-vec

Compute the intersection of

$$
\begin{tikzcd}
M_1 \arrow[rr, hook, "\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[d, phantom, "\cong"] & & N \arrow[d, phantom, "\cong"] & & M_2 \arrow[ll, hook', "\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}"'] \arrow[d, phantom, "\cong"] \\
2 & & 3 & & 2
\end{tikzcd}
$$

Computing the intersection: $\mathbb{Q}$-vec

Compute the intersection of

$$
\begin{tikzcd}[column sep=huge]
M_1 \arrow[r, hook, "\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[d, phantom, sloped, "\cong"] & N \arrow[d, phantom, sloped, "\cong"] & M_2 \arrow[l, hook', "\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}"'] \arrow[d, phantom, sloped, "\cong"] \\
2 & 3 & 2
\end{tikzcd}
$$

```
gap> gamma := IntersectionOfSubobject( iota1, iota2 );
<A morphism in the category of matrices over Q>
```

Computing the intersection: $\mathbb{Q}$-vec

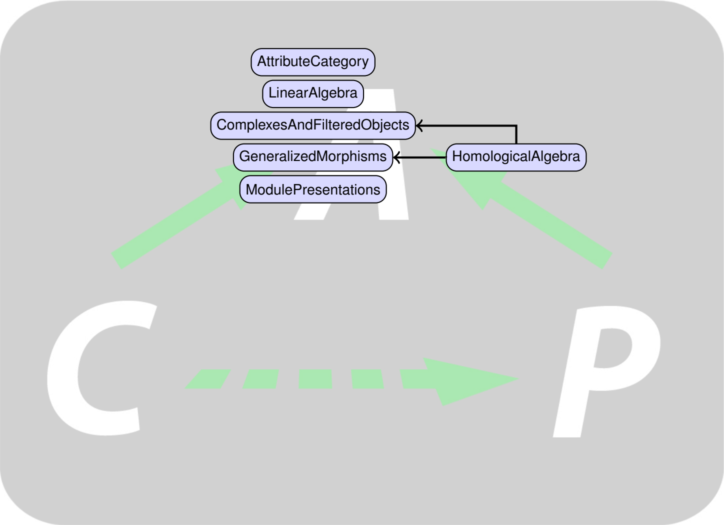
Compute the intersection of

$$\begin{tikzcd} M_1 \arrow[r, hook, "\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[d, phantom, "\sqcap"] & N \arrow[d, phantom, "\sqcap"] & M_2 \arrow[l, hook', "\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}"'] \arrow[d, phantom, "\sqcap"] \\ 2 & 3 & 2 \end{tikzcd}$$

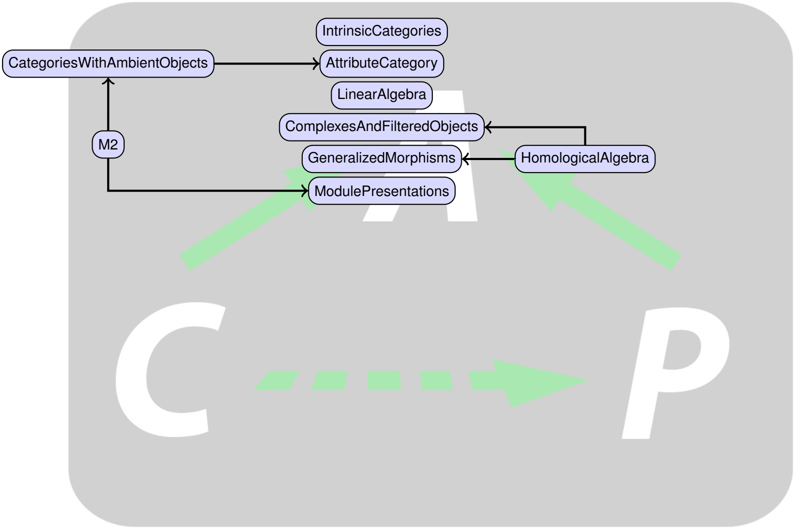
```
gap> gamma := IntersectionOfSubobject( iota1, iota2 );
<A morphism in the category of matrices over Q>
gap> Display( gamma );
[ [  1,  1,  0 ] ]

A morphism in the category of matrices over Q
```

CAP packages

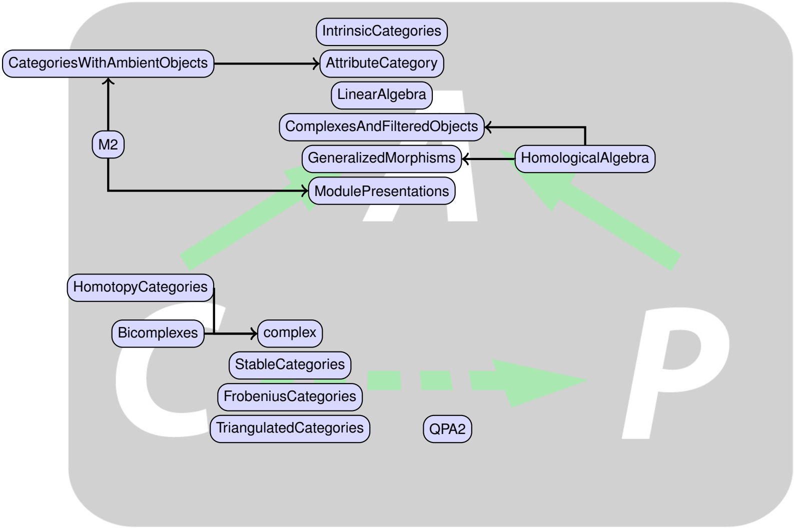
CAP packages



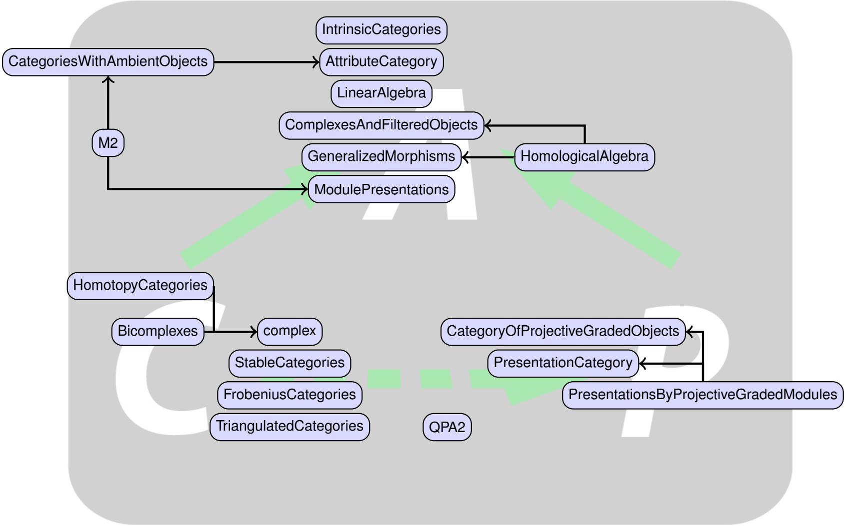
CAP packages



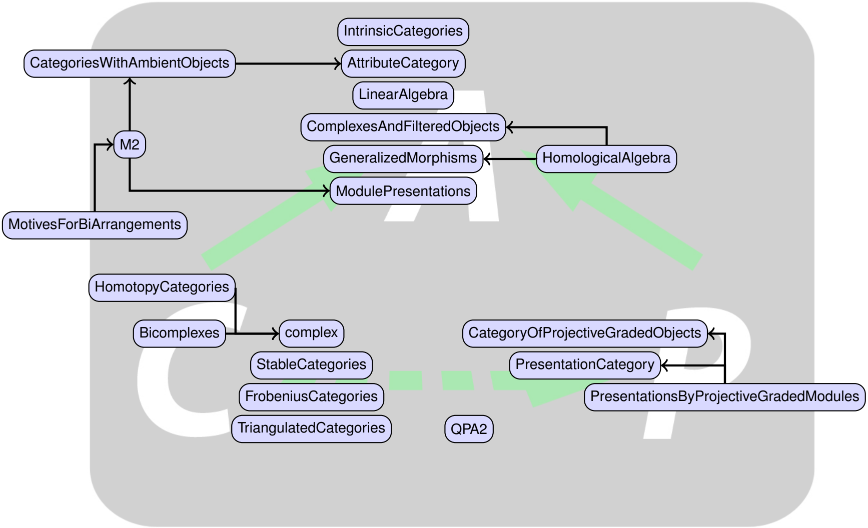
CAP packages



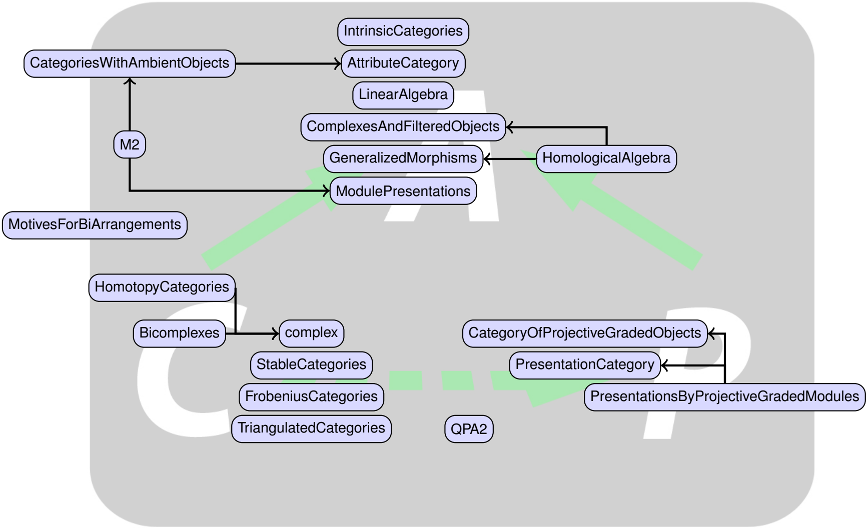
CAP packages



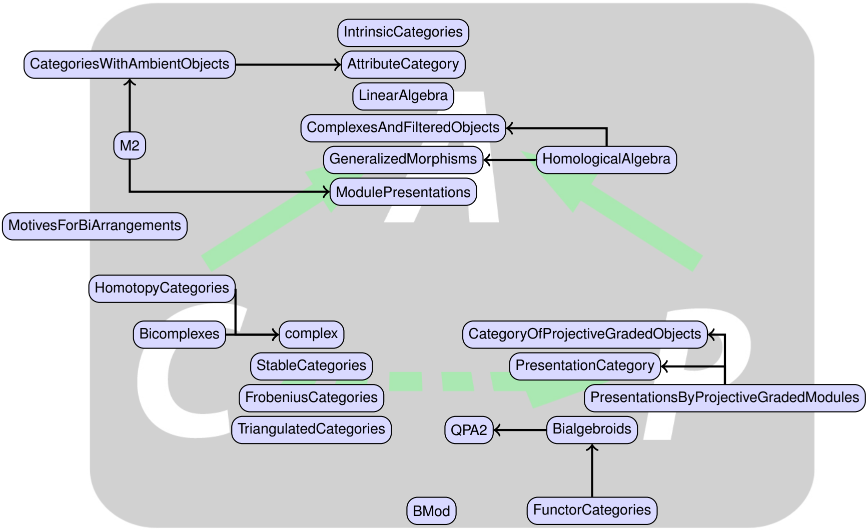
CAP packages



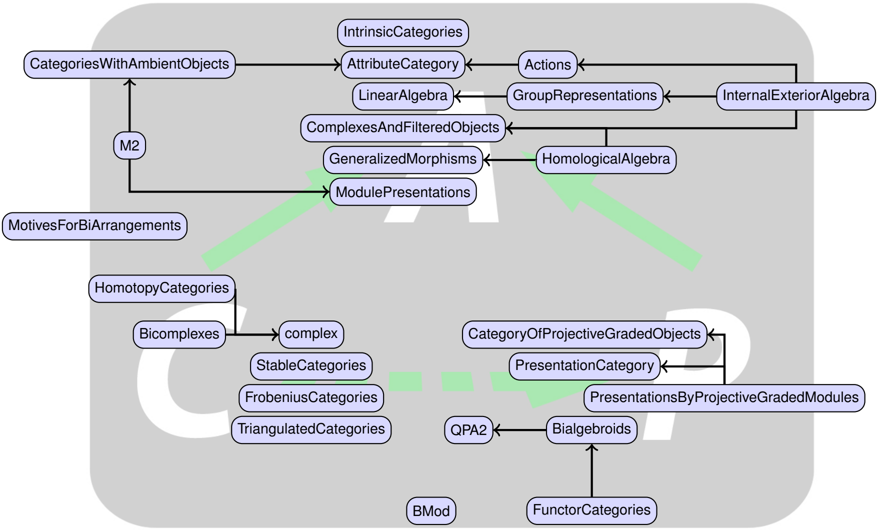
CAP packages



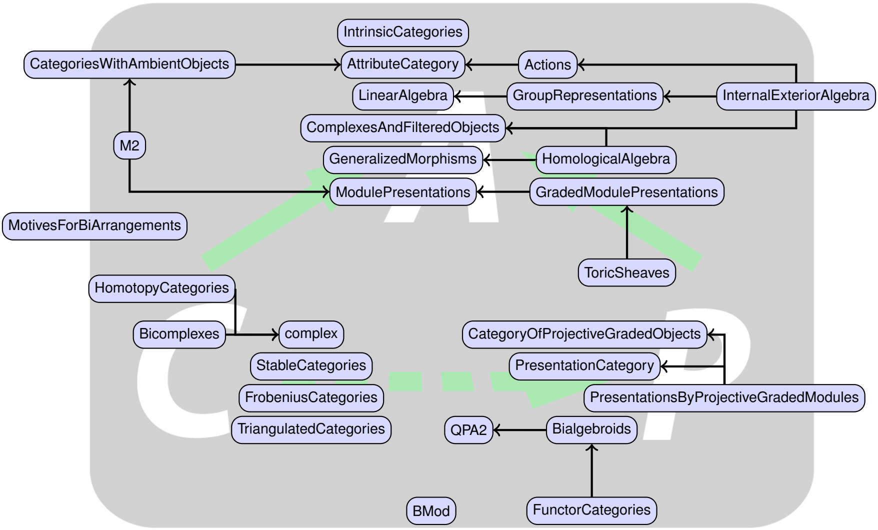
CAP packages



CAP packages



CAP packages



Snake lemma

$$\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \operatorname{coker}(\alpha) & & &
\end{tikzcd}$$

Snake lemma

$$
\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d] \arrow[dddll, bend left, blue] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

Part II

Generalized morphisms

1. Classical diagram chases
2. Additive relations
3. Generalized morphisms
4. Applications of generalized morphisms
   - An algorithm for spectral sequences
   - The purity filtration

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2. Additive relations
3. Generalized morphisms
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   - An algorithm for spectral sequences
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Connecting homomorphism in the snake lemma

$$
\begin{tikzcd}
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0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

Wanted: $\ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha)$.

Connecting homomorphism in the snake lemma

$$\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Start: $c \in \ker(\gamma)$.

Connecting homomorphism in the snake lemma

$$
\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \operatorname{coker}(\alpha) & & &
\end{tikzcd}
$$

This lies in C.

Connecting homomorphism in the snake lemma

$$\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & {\color{red}b} \in B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \operatorname{coker}(\alpha) & & &
\end{tikzcd}$$

Choose: $b \in \epsilon^{-1}(\{c\})$.

Connecting homomorphism in the snake lemma

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0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & {\color{red}b'} \in B' \arrow[r] & C' & \\
 & \operatorname{coker}(\alpha) & & &
\end{tikzcd}$$

Map: $b \overset{\beta}{\mapsto} b'$.

Connecting homomorphism in the snake lemma

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0 \arrow[r] & {\color{red}a'} \in A' \arrow[r, "\mu"'] \arrow[d] & b' \in B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Compute: $a' \in \mu^{-1}(b')$.

Connecting homomorphism in the snake lemma

$$\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] & \\
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0 \arrow[r] & a' \in A' \arrow[r, "\mu"'] \arrow[d] & b' \in B' \arrow[r] & C' & \\
 & {\color{red} a' + \mathrm{im}(\alpha)} \in \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Map: $a' \mapsto a' + \mathrm{im}(\alpha)$.

Connecting homomorphism in the snake lemma

$$
\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] \arrow[dddll, blue, rounded corners] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & b \in B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & a' \in A' \arrow[r, "\mu"'] \arrow[d] & b' \in B' \arrow[r] & C' & \\
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\end{tikzcd}
$$

Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$.

Connecting homomorphism in the snake lemma

$$
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 & a' + \mathrm{im}(\alpha) \in \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$. **Context**: modules

Classical solutions: embedding theorems

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Freyd-Mitchell embedding theorem

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Freyd-Mitchell embedding theorem

Any small abelian category $\mathbf{A}$ admits an exact fully faithful covariant embedding

$$F : \mathbf{A} \hookrightarrow R - \mathbf{mod}$$

into the category of R-modules for some ring R.

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Application: existence of morphisms

$$\mathrm{Hom}_{\mathbf{A}}(A, B) \quad \cong \quad \mathrm{Hom}_{R-\mathbf{mod}}(FA, FB)$$

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$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{A}}(A, B) & \cong & \mathrm{Hom}_{R-\mathbf{mod}}(FA, FB) \\ & & \uplus \\ & & \varphi \end{array}$$

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$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{A}}(A, B) & \cong & \mathrm{Hom}_{R-\mathbf{mod}}(FA, FB) \\ \Cup & & \Cup \\ F^{-1}\varphi & \leftrightarrow & \varphi \end{array}$$

Classical solutions: embedding theorems

Freyd-Mitchell embedding theorem

Any small abelian category $\mathbf{A}$ admits an exact fully faithful covariant embedding

$$F : \mathbf{A} \hookrightarrow R - \mathbf{mod}$$

into the category of R-modules for some ring R.

Application: existence of morphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{A}}(A, B) & \cong & \mathrm{Hom}_{R-\mathbf{mod}}(FA, FB) \\ \Cup & & \Cup \\ F^{-1}\varphi & \leftrightarrow & \varphi \end{array}$$

Problem: this isomorphism between Hom-sets is **not constructive.**

Back to the snake lemma

$$\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] \arrow[dddll, blue, bend left] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & b \in B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & a' \in A' \arrow[r, "\mu"'] \arrow[d] & b' \in B' \arrow[r] & C' & \\
 & a' + \mathrm{im}(\alpha) \in \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$.

Back to the snake lemma

$$
\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] \arrow[dddll, blue] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & {\color{red} b \in B} \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & a' \in A' \arrow[r, "\mu"'] \arrow[d] & b' \in B' \arrow[r] & C' & \\
 & a' + \mathrm{im}(\alpha) \in \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

Crucial step: the **uncanonical** choice $b \in \epsilon^{-1}(\{c\})$.

Back to the snake lemma

$$
\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] \arrow[dddll, blue, rounded corners] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & {\color{red} b \in B} \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
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 & a' + \mathrm{im}(\alpha) \in \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

Make this step canonical: **relations** instead of maps: $c \mapsto \epsilon^{-1}(\{c\})$

Relations

Let A, B be abelian groups.

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Definition

A subgroup $f \subseteq A \oplus B$ is called a **relation from A to B.**

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is a relation from B to A, called **pseudo-inverse of** ϵ.

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$$g \circ f := \{(a, c) \in A \oplus C \mid \exists b \in B : (a, b) \in f, (b, c) \in g\}$$

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Given $f \subseteq A \oplus B$ and $g \subseteq B \oplus C$, define

$$g \circ f := \{(a, c) \in A \oplus C \mid \exists b \in B : (a, b) \in f, (b, c) \in g\}$$

If f and g correspond to maps, this describes their usual composition.

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Q: When does an additive relation $f \subseteq A \oplus B$ defines an honest map (a group homomorphism)?

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Domain

$\mathrm{dom}(f) := \left\{ a \in A \mid \exists b \in B : (a, b) \in f \right\}$

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Q: When does an additive relation $f \subseteq A \oplus B$ defines an honest map (a group homomorphism)?

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$$\mathrm{dom}(f) := \left\{ a \in A \mid \exists b \in B : (a, b) \in f \right\}$$

Defect

$$\mathrm{def}(f) := \left\{ b \in B \mid (0, b) \in f \right\}$$

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A: When it has a full domain

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$\mathrm{dom}(f) := \left\{ a \in A \mid \exists b \in B : (a, b) \in f \right\} = A$

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The snake lemma for a last time

$$\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d, "\iota"] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, "\pi"'] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Wanted: $\ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha)$.

The snake lemma for a last time

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$$

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0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, "\pi"'] & B' \arrow[r] & C' & \\
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\end{tikzcd}
$$

$$\epsilon^{-1} \circ \iota$$

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0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, "\pi"'] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

$$\beta \circ \epsilon^{-1} \circ \iota$$

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 & & & \ker(\gamma) \arrow[d, "\iota"] & \\
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\end{tikzcd}$$

$$\mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

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& \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

$$\pi \circ \mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

The snake lemma for a last time

$$
\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d, "\iota"] \arrow[dddll, rounded corners] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] \arrow[l, bend right, "\epsilon^{-1}"'] & 0 \\
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∂ is an honest map given by a composition of relations!

From relations to generalized morphisms

- **Wanted**: a categorical framework for relations.

From relations to generalized morphisms

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- **Solution**: generalized morphisms.

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

Relation

$$
\begin{tikzcd}
A \oplus B \\
D \arrow[u, hook]
\end{tikzcd}
$$

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

Relation

$$\begin{tikzcd} A \oplus B \\ D \arrow[u, hook, "\begin{pmatrix} \alpha & \beta \end{pmatrix}"] \end{tikzcd}$$

From relations to generalized morphisms

Let A, B be objects in an abelian category **A**.

Relation

$$\begin{tikzcd} A & & B \\ & D \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

Relation $\rightsquigarrow$ generalized morphism

$$\begin{tikzcd} A \arrow[rr, dashed] & & B \\ & D \arrow[ul, "\alpha", red] \arrow[ur, "\beta"', blue] & \end{tikzcd}$$

From relations to generalized morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

Relation $\rightsquigarrow$ generalized morphism (data structure: span)

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A \arrow[rr, dashed] & & B \\
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Equality

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Two spans (α, β) and (α', β') are **equal as generalized morphisms** if

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Equality

Two spans (α, β) and (α', β') are **equal as generalized morphisms** if

$$\operatorname{im}\left((\alpha, \beta) : D \to A \oplus B\right) = \operatorname{im}\left((\alpha', \beta') : D' \to A \oplus B\right).$$

Composition of generalized morphisms

Composition

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$$\begin{tikzcd} A \arrow[rr, dashed] & & B \arrow[rr, dashed] & & C \\ & X \arrow[ul] \arrow[ur] & & Y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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\end{tikzcd}$$

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\begin{tikzcd}
A \arrow[rr, dashed] & & B \arrow[rr, dashed] & & C \\
& X \arrow[ul] \arrow[ur] & & Y \arrow[ul] \arrow[ur] & \\
& & X \times_B Y \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

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& & X \times_B Y \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

$\rightsquigarrow$ Category of generalized morphisms $\mathrm{G}(\mathbf{A})$

Pseudo-inverses

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$$\begin{tikzcd} A \arrow[rr, dashed] & & B \\ & D \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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$$\begin{tikzcd} A \arrow[rr, dashed] & & B \\ & D \arrow[ul, red] \arrow[ur, blue] & \end{tikzcd}$$

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Pseudo-inverses

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$$\begin{tikzcd} A \arrow[rr, dashed] & & B \\ & D \arrow[ul, red] \arrow[ur, blue] & \end{tikzcd} \quad \rightsquigarrow$$

Pseudo-inverses

Pseudo-inverses

$$
\begin{tikzcd}
A \arrow[rr, dashed] & & B \\
& D \arrow[ul, red] \arrow[ur, blue] \arrow[u, phantom, "\leftrightarrow"] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
B \arrow[rr, dashed] & & A \\
& D \arrow[ul, blue] \arrow[ur, red] &
\end{tikzcd}
$$

Honest morphisms

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A embeds into G(**A**):

Honest morphisms

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$$A \longrightarrow B$$

Honest morphisms

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$$A \xrightarrow{\quad\quad\quad} B \quad \mapsto$$

Honest morphisms

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A embeds into G(**A**):

$$
\begin{tikzcd}
 & & & A \arrow[rr, dashed] & & B \\
A \arrow[r] & B & \mapsto & & & \\
 & & & & A \arrow[uul, "\mathrm{id}_A"] \arrow[uur] &
\end{tikzcd}
$$

Honest morphisms

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A embeds into G(**A**):

$$
A \longrightarrow B \quad \mapsto \quad
\begin{tikzcd}
A \arrow[rr, dashed] & & B \\
& A \arrow[ul, "\mathrm{id}_A"] \arrow[ur] &
\end{tikzcd}
$$

Generalized morphisms with such a representation are called **honest**.

Honest morphisms

Q: When does $A \xleftarrow{\alpha} D \xrightarrow{\beta} B$ define an honest morphism?

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Computing representatives

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Given an honest generalized morphism in $G(\mathbf{A})$, compute the corresponding morphism in $\mathbf{A}$.

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Given an honest generalized morphism in G($\mathbf{A}$), compute the corresponding morphism in $\mathbf{A}$.

$$\begin{tikzcd} A & & B \\ & D \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$
\begin{tikzcd}
 & A \sqcup_D B & \\
A \arrow[ur] & & B \arrow[ul] \\
 & D \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$\begin{tikzcd} & A \sqcup_D B & \\ A \arrow[ur] & & B \arrow[ul] \\ & A \times_{A \sqcup_D B} B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$\begin{tikzcd}
A & & B \\
& A \times_{A \sqcup_D B} B \arrow[ul, "\sim"'] \arrow[ur] &
\end{tikzcd}$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$
\begin{tikzcd}
A \arrow[dr, "\sim"] & & B \\
& A \times_{A \sqcup_D B} B \arrow[ur] &
\end{tikzcd}
$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$A \longrightarrow B$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$
\begin{tikzcd}
\mathbb{Q} & & \mathbb{Q} \\
& \mathbb{Q}^2 \arrow[ul, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] \arrow[ur, "\begin{pmatrix} 2 \\ -2 \end{pmatrix}"'] &
\end{tikzcd}
$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$
\begin{tikzcd}
 & \mathbb{Q} & \\
\mathbb{Q} \arrow[ur, "2"] & & \mathbb{Q} \arrow[ul, "1"'] \\
 & \mathbb{Q}^2 \arrow[ul, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] \arrow[ur, "\begin{pmatrix} 2 \\ -2 \end{pmatrix}"'] &
\end{tikzcd}
$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$\begin{tikzcd}
& \mathbb{Q} & \\
\mathbb{Q} \arrow[ur, "2"] & & \mathbb{Q} \arrow[ul, "1"'] \\
& \mathbb{Q} \arrow[ul, "1"] \arrow[ur, "2"'] &
\end{tikzcd}$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$
\begin{tikzcd}
\mathbb{Q} & & \mathbb{Q} \\
& \mathbb{Q} \arrow[ul, "1", "\sim"'] \arrow[ur, "2"'] &
\end{tikzcd}
$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$\begin{tikzcd}
\mathbb{Q} \arrow[dr, "1"', "\sim"] & & \mathbb{Q} \\
& \mathbb{Q} \arrow[ur, "2"'] &
\end{tikzcd}$$

Computing representatives

Given an honest generalized morphism in G(**A**), compute the corresponding morphism in **A**.

$$\mathbb{Q} \xrightarrow{2} \mathbb{Q}$$

Constructive diagram chases

Constructive diagram chases

Strategy for constructive diagram chases

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1. Compute in $\mathrm{G}(\mathbf{A})$ using pseudo-inverses and compositions.

Constructive diagram chases

Strategy for constructive diagram chases

1. Compute in $\mathrm{G}(\mathbf{A})$ using pseudo-inverses and compositions.
2. Compute the honest representative of the resulting generalized morphism.

Example: functoriality of homology

Let $(P_\bullet, \partial)$ be a complex in an abelian category $\mathcal{A}$.

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Let $(P_\bullet, \partial)$ be a complex in an abelian category $\mathcal{A}$. Then we can compute the generalized embedding of the i-th homology.

$$P_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} P_{i-1}$$

Example: functoriality of homology

Let $(P_\bullet, \partial)$ be a complex in an abelian category $\mathcal{A}$. Then we can compute the generalized embedding of the i-th homology.

$$
\begin{tikzcd}
P_{i+1} \arrow[rr, "\partial_{i+1}"] & & P_i \arrow[rr, "\partial_i"] & & P_{i-1} \\
& \operatorname{im}(\partial_{i+1}) \arrow[ur, hook] & & &
\end{tikzcd}
$$

Example: functoriality of homology

Let $(P_\bullet, \partial)$ be a complex in an abelian category $\mathcal{A}$. Then we can compute the generalized embedding of the i-th homology.

$$\begin{tikzcd}
P_{i+1} \arrow[rr, "\partial_{i+1}"] & & P_i \arrow[rr, "\partial_i"] & & P_{i-1} \\
& \operatorname{im}(\partial_{i+1}) \arrow[ur, hook] & & \ker(\partial_i) \arrow[ul, hook'] &
\end{tikzcd}$$

Example: functoriality of homology

Let $(P_\bullet, \partial)$ be a complex in an abelian category $\mathcal{A}$. Then we can compute the generalized embedding of the i-th homology.

$$\begin{tikzcd}
P_{i+1} \arrow[rr, "\partial_{i+1}"] & & P_i \arrow[rr, "\partial_i"] & & P_{i-1} \\
& \operatorname{im}(\partial_{i+1}) \arrow[ur, hook] \arrow[rr, hook] & & \ker(\partial_i) \arrow[ul, hook'] &
\end{tikzcd}$$

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\end{tikzcd}$$

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\end{tikzcd}$$

Example: functoriality of homology

Theorem

Let $\mathcal{A}$ be an abelian category and $\varepsilon : P_\bullet \to Q_\bullet$ a chain morphism.

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Let $\mathcal{A}$ be an abelian category and $\varepsilon : P_\bullet \to Q_\bullet$ a chain morphism. Then the morphism $\mathrm{H}_i(P_\bullet) \to \mathrm{H}_i(Q_\bullet)$ can be computed using generalized morphisms:

$$\begin{tikzcd} \mathrm{H}_i(P_\bullet) \arrow[r, dashed] \arrow[rrr, bend right] & P_i \arrow[r, "\varepsilon_i"] & Q_i \arrow[r, dashed, red] & \mathrm{H}_i(Q_\bullet) \end{tikzcd}$$

1. Classical diagram chases
2. Additive relations
3. Generalized morphisms
4. Applications of generalized morphisms
   - An algorithm for spectral sequences
   - The purity filtration

Spectral sequences via generalized morphisms

Given: an excerpt of a filtered chain complex.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A_{i+1} & \longrightarrow & B_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A_{i} & \longrightarrow & B_{i} & \longrightarrow & C_{i} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Spectral sequences via generalized morphisms

We pass to its graded parts.

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \frac{A_{i+1}}{A_i} \arrow[r, "\overline{\partial}"] & \frac{B_{i+1}}{B_i} \arrow[r, "\overline{\partial}"] & \frac{C_{i+1}}{C_i} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_i}{A_{i-1}} \arrow[r] & \frac{B_i}{B_{i-1}} \arrow[r] & \frac{C_i}{C_{i-1}} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_{i-1}}{A_{i-2}} \arrow[r] & \frac{B_{i-1}}{B_{i-2}} \arrow[r] & \frac{C_{i-1}}{C_{i-2}} \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$\overline{\partial}$:

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} : \frac{A_{i+1}}{A_i} \qquad\qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$\overline{\partial} \; : \; \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \qquad\qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

We can compute the differentials via generalized morphisms.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
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 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$\overline{\partial} \; : \; \frac{A_{i+1}}{A_i} \twoheadleftarrow A_{i+1} \hookrightarrow A \qquad\qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

This is a generalized **subquotient embedding**.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$\overline{\partial} \; : \; \begin{tikzcd} \frac{A_{i+1}}{A_i} \arrow[rr, dashed, bend left] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \end{tikzcd} \qquad \frac{B_{i+1}}{B_i}$$

Spectral sequences via generalized morphisms

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \frac{A_{i+1}}{A_i} \arrow[r, "\overline{\partial}"] & \frac{B_{i+1}}{B_i} \arrow[r, "\overline{\partial}"] & \frac{C_{i+1}}{C_i} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_i}{A_{i-1}} \arrow[r] & \frac{B_i}{B_{i-1}} \arrow[r] & \frac{C_i}{C_{i-1}} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_{i-1}}{A_{i-2}} \arrow[r] & \frac{B_{i-1}}{B_{i-2}} \arrow[r] & \frac{C_{i-1}}{C_{i-2}} \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\overline{\partial} : \begin{tikzcd}
\frac{A_{i+1}}{A_i} \arrow[rr, bend left, dashed] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B
\end{tikzcd} \qquad \frac{B_{i+1}}{B_i}
$$

Spectral sequences via generalized morphisms

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i}}{A_{i-1}} & \longrightarrow & \frac{B_{i}}{B_{i-1}} & \longrightarrow & \frac{C_{i}}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} : \quad
\begin{tikzcd}
\frac{A_{i+1}}{A_i} & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[ll, bend right, dashed] \arrow[r, "\partial"] & B & B_{i+1} \arrow[l, hook'] & & \frac{B_{i+1}}{B_i}
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} : \begin{tikzcd} \frac{A_{i+1}}{A_i} \arrow[rr, dashed, bend left] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B & B_{i+1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i+1}}{B_i} \end{tikzcd}
$$

Spectral sequences via generalized morphisms

This is a generalized **subquotient projection**.

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \frac{A_{i+1}}{A_i} \arrow[r, "\overline{\partial}"] & \frac{B_{i+1}}{B_i} \arrow[r, "\overline{\partial}"] & \frac{C_{i+1}}{C_i} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_i}{A_{i-1}} \arrow[r] & \frac{B_i}{B_{i-1}} \arrow[r] & \frac{C_i}{C_{i-1}} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_{i-1}}{A_{i-2}} \arrow[r] & \frac{B_{i-1}}{B_{i-2}} \arrow[r] & \frac{C_{i-1}}{C_{i-2}} \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$\overline{\partial} : \begin{tikzcd}
\frac{A_{i+1}}{A_i} \arrow[rr, dashed, bend left] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B \arrow[rr, dashed, bend left] & B_{i+1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i+1}}{B_i}
\end{tikzcd}$$

Spectral sequences via generalized morphisms

We can compose the arrows.

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \frac{A_{i+1}}{A_i} \arrow[r, "\overline{\partial}"] & \frac{B_{i+1}}{B_i} \arrow[r, "\overline{\partial}"] & \frac{C_{i+1}}{C_i} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_i}{A_{i-1}} \arrow[r] & \frac{B_i}{B_{i-1}} \arrow[r] & \frac{C_i}{C_{i-1}} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_{i-1}}{A_{i-2}} \arrow[r] & \frac{B_{i-1}}{B_{i-2}} \arrow[r] & \frac{C_{i-1}}{C_{i-2}} \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overline{\partial} \ : \ \frac{A_{i+1}}{A_i} \arrow[rr, bend left, dashed, red] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B \arrow[rr, bend left, dashed] & B_{i+1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i+1}}{B_i}
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

We can compose the arrows.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_i}{A_{i-1}} & \longrightarrow & \frac{B_i}{B_{i-1}} & \longrightarrow & \frac{C_i}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} : \begin{tikzcd} \frac{A_{i+1}}{A_i} \arrow[rr, bend left, dashed] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] \arrow[rr, bend left, dashed] & B & B_{i+1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i+1}}{B_i} \end{tikzcd}
$$

Spectral sequences via generalized morphisms

We can compose the arrows.

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \frac{A_{i+1}}{A_i} \arrow[r, "\overline{\partial}"] & \frac{B_{i+1}}{B_i} \arrow[r, "\overline{\partial}"] & \frac{C_{i+1}}{C_i} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_i}{A_{i-1}} \arrow[r] & \frac{B_i}{B_{i-1}} \arrow[r] & \frac{C_i}{C_{i-1}} \arrow[r] & \cdots \\
\cdots \arrow[r] & \frac{A_{i-1}}{A_{i-2}} \arrow[r] & \frac{B_{i-1}}{B_{i-2}} \arrow[r] & \frac{C_{i-1}}{C_{i-2}} \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\overline{\partial} : \begin{tikzcd}
\frac{A_{i+1}}{A_i} \arrow[rr, bend left, dashed] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] \arrow[rr, bend left, dashed] & B & B_{i+1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i+1}}{B_i}
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

This formula still makes sense if we map 1 step down.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\partial} & B & \xrightarrow{\partial} & C & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \frac{A_{i+1}}{A_i} & \xrightarrow{\overline{\partial}} & \frac{B_{i+1}}{B_i} & \xrightarrow{\overline{\partial}} & \frac{C_{i+1}}{C_i} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i}}{A_{i-1}} & \longrightarrow & \frac{B_{i}}{B_{i-1}} & \longrightarrow & \frac{C_{i}}{C_{i-1}} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \frac{A_{i-1}}{A_{i-2}} & \longrightarrow & \frac{B_{i-1}}{B_{i-2}} & \longrightarrow & \frac{C_{i-1}}{C_{i-2}} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$$
\overline{\partial} : \begin{tikzcd} \frac{A_{i+1}}{A_i} \arrow[rr, dashed, bend left] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B \arrow[rr, dashed, bend left] & B_{i+1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i+1}}{B_i} \end{tikzcd}
$$

Spectral sequences via generalized morphisms

This formula still makes sense if we map 1 step down.

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[dr, dashed] & \frac{A_{i+1}}{A_i} \arrow[dr, dashed, "\overline{\partial}^1"] & \frac{B_{i+1}}{B_i} \arrow[dr, dashed] & \frac{C_{i+1}}{C_i} \arrow[dr, dashed] & \cdots \\
\cdots \arrow[dr, dashed] & \frac{A_i}{A_{i-1}} \arrow[dr, dashed] & \frac{B_i}{B_{i-1}} \arrow[dr, dashed, "\overline{\partial}^1"] & \frac{C_i}{C_{i-1}} \arrow[dr, dashed] & \cdots \\
\cdots & \frac{A_{i-1}}{A_{i-2}} & \frac{B_{i-1}}{B_{i-2}} & \frac{C_{i-1}}{C_{i-2}} & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overline{\partial}^1 : \frac{A_{i+1}}{A_i} \arrow[rr, bend left, dashed, red] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, red, "\partial"] & B \arrow[rr, bend left, dashed, red] & B_i \arrow[l, hook'] \arrow[r, two heads] & \frac{B_i}{B_{i-1}}
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

One more step . . .

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
& \vdots & \vdots & \vdots & \\
\cdots \arrow[dr, dashed] & \frac{A_{i+1}}{A_i} \arrow[dr, dashed, "\overline{\partial}^1"] & \frac{B_{i+1}}{B_i} \arrow[dr, dashed] & \frac{C_{i+1}}{C_i} \arrow[dr, dashed] & \cdots \\
\cdots \arrow[dr, dashed] & \frac{A_i}{A_{i-1}} \arrow[dr, dashed] & \frac{B_i}{B_{i-1}} \arrow[dr, dashed, "\overline{\partial}^1"] & \frac{C_i}{C_{i-1}} \arrow[dr, dashed] & \cdots \\
\cdots & \frac{A_{i-1}}{A_{i-2}} & \frac{B_{i-1}}{B_{i-2}} & \frac{C_{i-1}}{C_{i-2}} & \cdots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\overline{\partial}^1 : \begin{tikzcd}
\frac{A_{i+1}}{A_i} \arrow[rr, dashed, bend left] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B \arrow[rr, dashed, bend left] & B_i \arrow[l, hook'] \arrow[r, two heads] & \frac{B_i}{B_{i-1}}
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

One more step ...

$$
\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "\partial"] & B \arrow[r, "\partial"] & C \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[ddr, dashed] & \frac{A_{i+1}}{A_i} \arrow[ddr, dashed, "\overline{\partial}^2"'] & \frac{B_{i+1}}{B_i} \arrow[ddr, dashed] & \frac{C_{i+1}}{C_i} \arrow[ddr, dashed] & \cdots \\
\cdots & \frac{A_i}{A_{i-1}} & \frac{B_i}{B_{i-1}} & \frac{C_i}{C_{i-1}} & \cdots \\
\cdots & \frac{A_{i-1}}{A_{i-2}} & \frac{B_{i-1}}{B_{i-2}} & \frac{C_{i-1}}{C_{i-2}} & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\overline{\partial}^2 : \begin{tikzcd}
\frac{A_{i+1}}{A_i} \arrow[rr, dashed, bend left] & A_{i+1} \arrow[l, two heads] \arrow[r, hook] & A \arrow[r, "\partial"] & B \arrow[rr, dashed, bend left] & B_{i-1} \arrow[l, hook'] \arrow[r, two heads] & \frac{B_{i-1}}{B_{i-2}}
\end{tikzcd}
$$

Spectral sequences via generalized morphisms

For all $r \geq 0$, we get so-called generalized chain complexes.

$$\cdots \dashrightarrow \frac{A_{i+1}}{A_i} \xrightarrow{\overline{\partial_A^r}} \frac{B_{i+1-r}}{B_{i-r}} \xrightarrow{\overline{\partial_B^r}} \frac{C_{i+1-2r}}{C_{i-2r}} \dashrightarrow \cdots$$

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$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & \frac{A_{i+1}}{A_i} \arrow[r, dashed, "\overline{\partial^r_A}"] & \frac{B_{i+1-r}}{B_{i-r}} \arrow[r, dashed, "\overline{\partial^r_B}"] & \frac{C_{i+1-2r}}{C_{i-2r}} \arrow[r, dashed] & \cdots \\
& & \frac{\mathrm{dom}(\overline{\partial^r_B})}{\mathrm{def}(\overline{\partial^r_A})} \arrow[u, dashed] & &
\end{tikzcd}
$$

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\cdots \arrow[r, red] & \color{red}{\frac{\mathrm{dom}}{\mathrm{def}}} \arrow[u, dashed] \arrow[r, red] & \color{red}{\frac{\mathrm{dom}(\overline{\partial^r_B})}{\mathrm{def}(\overline{\partial^r_A})}} \arrow[u, dashed] \arrow[r, red] & \color{red}{\frac{\mathrm{dom}}{\mathrm{def}}} \arrow[u, dashed] \arrow[r, red] & \cdots
\end{tikzcd}$$

- These are the chain complexes on the r-th page of the associated **spectral sequence.**

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For all $r \geq 0$, we get so-called generalized chain complexes.

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\cdots \arrow[r] & C_{j+1} \arrow[r] & C_j \arrow[r] & C_{j-1} \arrow[r] & \cdots \\
\cdots \arrow[r, dashed] & \frac{F_{i+1}C_{j+1}}{F_iC_{j+1}} \arrow[u, dashed] \arrow[r, dashed] & \frac{F_{i+1-r}C_j}{F_{i-r}C_j} \arrow[u, dashed] \arrow[r, dashed] & \frac{F_{i+1-2r}C_{j-1}}{F_{i-2r}C_{j-1}} \arrow[u, dashed] \arrow[r, dashed] & \cdots \\
\cdots \arrow[r] & E^r_{i+1,j-i} \arrow[u, dashed] \arrow[r] & E^r_{i+1-r,j-i+(r-1)} \arrow[u, dashed] \arrow[r] & E^r_{i+1-2r,j-i+2(r-1)} \arrow[u, dashed] \arrow[r] & \cdots
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- These are the chain complexes on the r-th page of the associated **spectral sequence.**
- We just computed them **without a recursive strategy**.

1. Classical diagram chases
2. Additive relations
3. Generalized morphisms
4. Applications of generalized morphisms
   - An algorithm for spectral sequences
   - The purity filtration

Spectral sequences

Convergence

Let $C_\bullet := 0 = F_{-n-1}C_\bullet \leq F_{-n}C_\bullet \leq \cdots \leq F_0C_\bullet$ be a finitely filtered complex

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$$\begin{aligned} F_pH/F_{p-1}H &\cong E^{\infty}_{p,q} \\ F_pH &\cong \mathrm{im}\left(\beta\right). \end{aligned}$$

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Theorem

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which yields the purity filtration of M, i.e., a finite filtration where all graded parts $F_{-i}M/F_{-(i+1)}M$ are pure of codimension i.

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\end{tikzcd}
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$$\begin{aligned} F_{-p}M/F_{-p-1}M &\cong E^{\infty}_{-p,p} \\ F_{-p}M &\cong \operatorname{im}(\beta)\,. \end{aligned}$$

Presentations from filtrations

Let $F_{-n}M \leq F_{-n+1}M \leq \cdots \leq F_0M := M$ be a finitely presented filtered module.

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Presentations from filtrations

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If $\mathtt{M}_i$ is a presentation matrix for $F_i M / F_{i-1} M$, then M can be presented by an upper block triangular matrix

$$\begin{pmatrix} \mathtt{M}_0 & * & \dots & \dots & * \\ & \mathtt{M}_{-1} & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & \mathtt{M}_{-n+1} & * \\ & & & & \mathtt{M}_{-n} \end{pmatrix}.$$

Example: filtered presentation

Consider the module with relations

$$\begin{pmatrix} 0 & 0 & 0 & 0 & xz & -z^2 \\ 0 & 0 & 0 & 0 & xy & -yz \\ 0 & -x^2z+xyz+xz^2 & y^2z & -xz+yz & x-y & 0 \\ 0 & 0 & 0 & 0 & x^2 & -xz \\ -xy & -x^3+x^2y+x^2z & xy^2 & -x^2+xy & 0 & x-y \\ z & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Computing the purity filtration by using the bidualizing spectral sequence yields

$$\left(\begin{array}{ccccc|c|c} x & -z & 0 & 0 & 0 & 0 & 1 \\ -y & z & y^2z & -yz^2 & -xz+yz & 0 & -1 \\ 0 & x-y & xy^2 & -xyz & -x^2+xy & xy & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & z & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & x \end{array}\right)$$

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