Homework 4

Ex 6.1

We define a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Compute its determinant using the `MATLAB` function `det`. Explain why the result is not an integer.

In [63]:

```
A = [1 2 3; 4 5 6; 7 8 9];
det(A)
```

```
ans =

  -9.5162e-16
```

In [67]:

```
[L U] = lu(A)
```

```
L =

    0.1429    1.0000         0
    0.5714    0.5000    1.0000
    1.0000         0         0


U =

    7.0000    8.0000    9.0000
         0    0.8571    1.7143
         0         0   -0.0000
```

This error is due to round-off errors appearing in LU decomposition.

Ex 6.2

The goal of this exercise is to compare the performances of the LU and Cholesky methods.

1. Write a function `LUFacto` returning the matrices L and U determined via Algorithm 6.1. If the algorithm cannot be executed (division by 0), return an error message.

In [129…

```
%%file LUFacto.m

function [L U] = LUFacto(A)
    n = size(A,1);

    for k=1:(n-1)
        if (A(k,k)==0)
            disp('Error! Pivot must be nonzero')
            L = NaN;
            U = NaN;
            return
        else
            for i=(k+1):n
                A(i,k) = A(i,k)/A(k,k);
                for j=(k+1):n
                    A(i,j) = A(i,j) - A(i,k)*A(k,j);
                end
            end
        end
    end
    L = eye(n) + tril(A,-1);
    U = triu(A);
end
```

```
Created file 'D:\GitHub\NLA-IMA\hw4\LUFacto.m'.
```

1. Write a function `Cholesky` returning the matrix B computed by Algorithm 6.2. If the algorithm cannot be executed (nonsymmetric matrix, division by 0, negative square root), return an error message. Compare with the `MATLAB` function `chol`.

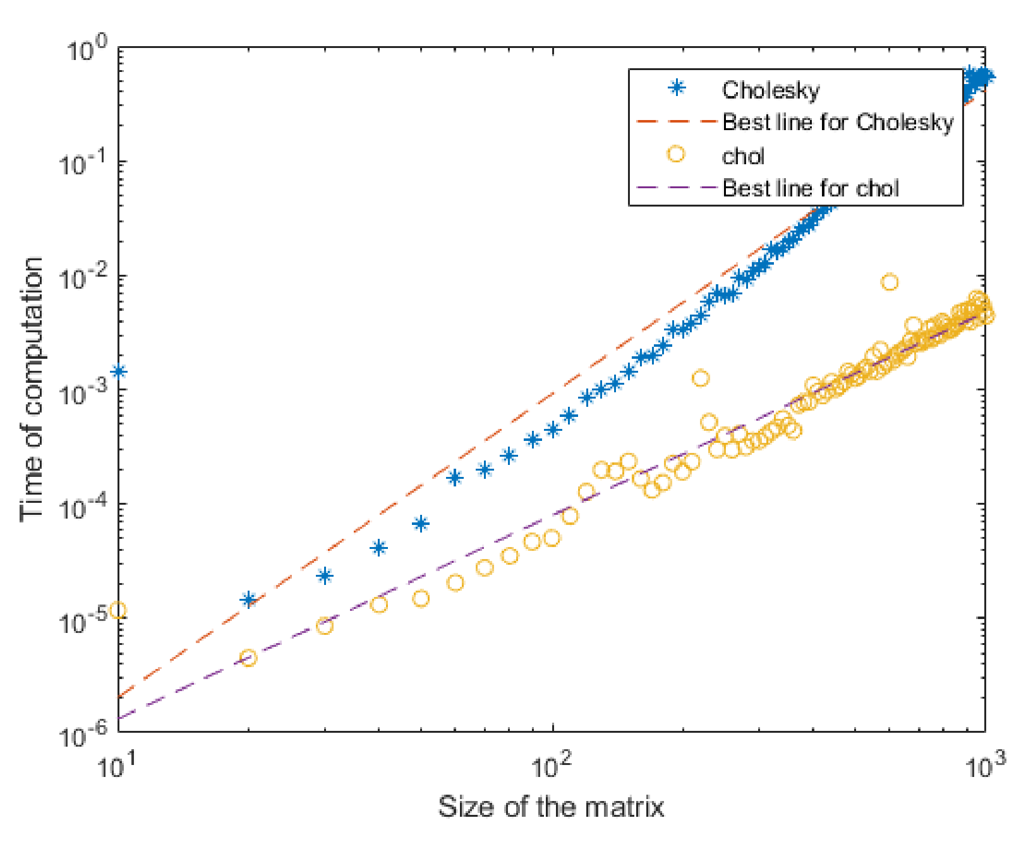
In [123…

```
%%file Cholesky.m

function B = Cholesky(A)
    if (A~=A')
        disp('Input must be symmetric!')
        B = NaN;
        return
    else
        B = tril(A);
        n = size(A,1);
        for j=1:n
            for k=1:(j-1)
                B(j,j) = B(j,j) - (B(j,k))^2;
            end
            if (B(j,j)<=0)
                disp('Error! Pivot must be positive')
                B = NaN;
                return
            else
                B(j,j) = sqrt(B(j,j));
                for i=(j+1):n
                    for k=1:(j-1)
                        B(i,j) = B(i,j) - B(j,k)*B(i,k);
                    end
                    B(i,j) = B(i,j)/B(j,j);
                end
            end
        end
    end
end
```

```
Created file 'D:\GitHub\NLA-IMA\hw4\Cholesky.m'.
```

In [23]:

```
times = zeros(2,length(1:100));

for i = 1:100
    n = i*10;
    A = PdSMat(n);

    tic
    B = Cholesky(A);
    times(1,i) = toc;

    tic
    B = chol(A);
    times(2,i) = toc;
end
x = 10:10:1000;
p1 = polyfit(log(x),log(times(1,:)),1);
p2 = polyfit(log(x),log(times(2,:)),1);

figure
loglog(x,times(1,:),'*')
hold on
loglog(x,exp(p1(2)).*x.^p1(1),'--')
loglog(x,times(2,:),'o')
loglog(x,exp(p2(2)).*x.^p2(1),'--')
hold off
xlabel('Size of the matrix')
ylabel('Time of computation')
legend('Cholesky','Best line for Cholesky','chol','Best line for chol')
```



In [24]:

```
p1
p2
```

```
p1 =

   2.653036013266808 -19.212901563663547


p2 =

   1.777763598656285 -17.626797864320473
```

We can see that $\text{time}_{\text{Cholesky}}(n) \approx \exp(-19.2129)n^{2.6530}$ and $\text{time}_{\text{chol}}(n) \approx \exp(-17.6267)n^{1.7777}$, under the assumption that computation time behaves like $C_1 \cdot n^{C_2}$.

1. For $n = 10, 20, \ldots, 100$, we define a matrix `A=MatSdp(n)` (see Exercise 2.20) and a vector `b=ones(n,1)`. Compare:

- On the one hand, the running time for computing the matrices L and U given by the function `LUFacto`, then the solution x of the system $Ax = b$. Use the functions `BackSub` and `ForwSub` defined in Exercise 5.2.
- On the other hand, the running time for computing the matrix B given by the function `Cholesky`, then the solution x of the system $Ax = b$. Use the functions `BackSub` and `ForwSub`. Plot on the same graph the curves representing the running times in terms of n. Comment.

In [15]:

```
%%file BackSub.m

function x = BackSub(U,y)
    n = size(U,1);
    x = zeros(n,1);
    for i = n:-1:1
        s = 0;
        for j = n:-1:(i+1)
            s = s + U(i,j)*x(j);
        end
        x(i) = (y(i)-s)/U(i,i);
    end
end
```

```
Created file 'D:\GitHub\NLA-IMA\hw4\BackSub.m'.
```

In [16]:

```
%%file ForwSub.m

function x = ForwSub(L,y)
    n = size(L,1);
    x = zeros(n,1);
    for i = 1:n
        s = 0;
        for j = 1:(i-1)
            s = s + L(i,j)*x(j);
        end
        x(i) = (y(i)-s)/L(i,i);
    end
end
```

```
Created file 'D:\GitHub\NLA-IMA\hw4\ForwSub.m'.
```

In [17]:

```
%%file SymmetricMat.m

function S = SymmetricMat(n)
    S = rand(n);
    S = S + S';
end
```

```
Created file 'D:\GitHub\NLA-IMA\hw4\SymmetricMat.m'.
```

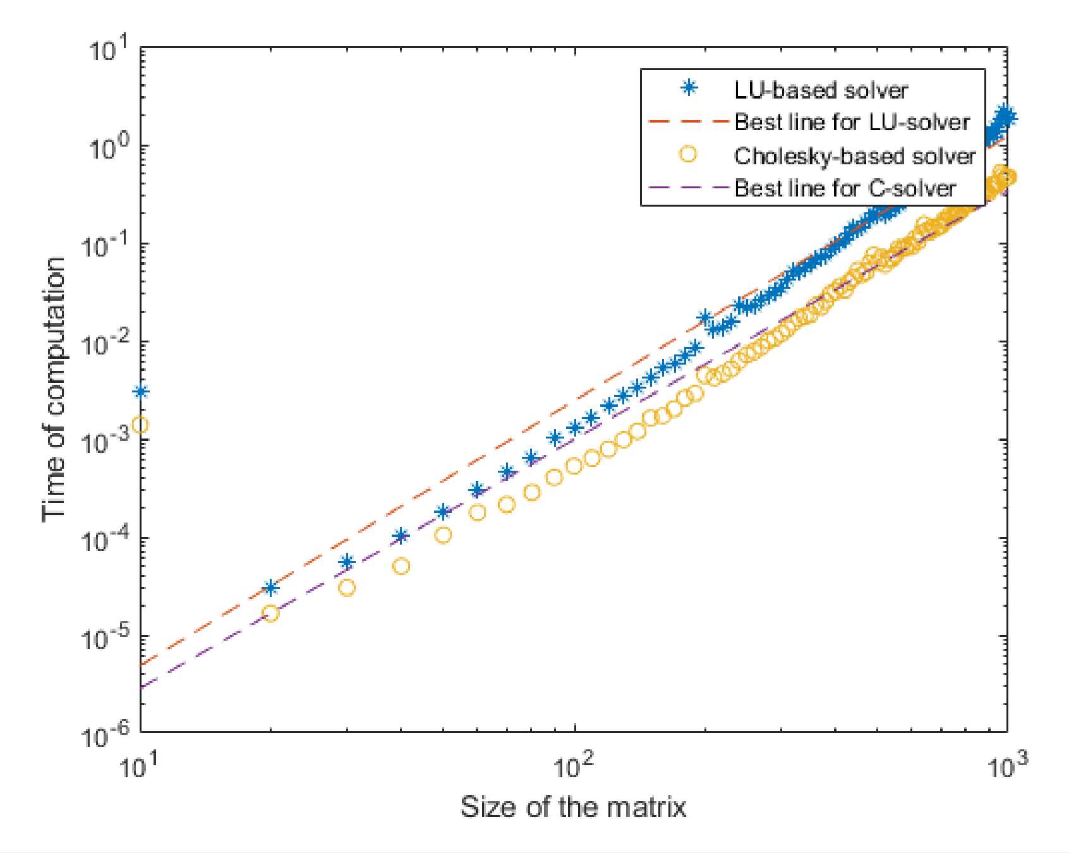
In [18]:

```
%%file PdSMat.m

function S = PdSMat(n)
    S = SymmetricMat(n);
    [P, D] = eig(S);
    D = abs(D);
    D = D + norm(D)*eye(size(D));
    S = P*D*inv(P);
end
```

```
Created file 'D:\GitHub\NLA-IMA\hw4\PdSMat.m'.
```

In [25]:

```
times = zeros(2,length(1:50));

for i = 1:100
    n = i*10;
    A = PdSMat(n);
    b = ones(n,1);

    tic;
    [L U] = LUFacto(A);
    x1 = BackSub(U,ForwSub(L,b));
    times(1,i) = toc;

    tic;
    B = Cholesky(A);
    x2 = BackSub(B',ForwSub(B,b));
    times(2,i) = toc;
end

x = 10:10:1000;
p1 = polyfit(log(x),log(times(1,:)),1);
p2 = polyfit(log(x),log(times(2,:)),1);

figure
loglog(x,times(1,:),'*')
hold on
loglog(x,exp(p1(2)).*x.^p1(1),'--')
loglog(x,times(2,:),'o')
loglog(x,exp(p2(2)).*x.^p2(1),'--')
hold off
xlabel('Size of the matrix')
ylabel('Time of computation')
legend('LU-based solver','Best line for LU-solver','Cholesky-based solver','Best line for C-solver')
```



In [26]:

```
p1
p2
```

```
p1 =

   2.700435568617929 -18.452550099850395


p2 =

   2.538527947687385 -18.613953300022562
```

We see that $\text{time}_{\text{LU}-\text{Solver}}(n) \approx \exp(-18.4525)n^{2.7004}$ and $\text{time}_{\text{C}-\text{Solver}}(n) \approx \exp(-18.6139)n^{2.5385}$, under the assumption that computation time behaves like $C_1 \cdot n^{C_2}$.

Ex 6.4

The goal of this exercise is to evaluate the influence of row permutation in Gaussian elimination. Let A and b be defined by `e=1.E-15`, `A=[e 1 1;1 -1 1; 1 0 1]` and `b=[2 0 1]’`.

1. Compute the matrices `L` and `U` given by the function `LUFacto` of Exercise 6.2.

In [11]:

```
format long
e=1.E-16;
A=[e 1 1;1 -1 1; 1 0 1];
b=[2 0 1]';
[L, U] = LUFacto(A)
det(A)
```

```
L =

   1.0e+16 *

   0.000000000000000                   0                   0
   1.000000000000000   0.000000000000000                   0
   1.000000000000000   0.000000000000000   0.000000000000000


U =

   1.0e+16 *

   0.000000000000000   0.000000000000000   0.000000000000000
                   0  -1.000000000000000  -1.000000000000000
                   0                   0                   0


ans =

   1.000000000000000
```

1. We define two matrices `l` and `u` by `[l u]=LUFacto(p*A)`, where `p` is the permutation matrix defined by the instruction `[w z p]=lu(A)`. Display the matrices `l` and `u`. What do you observe?

In [14]:

```
[w z p]=lu(A);
p
[l u] = LUFacto(p*A)
```

```
p =

     0     1     0
     1     0     0
     0     0     1


l =

   1.000000000000000                   0                   0
   0.000000000000000   1.000000000000000                   0
   1.000000000000000   1.000000000000000   1.000000000000000


u =

   1.000000000000000  -1.000000000000000   1.000000000000000
                   0   1.000000000000000   1.000000000000000
                   0                   0  -1.000000000000000
```

First, notice that in this computation `e=1.E-16`. Notice that by permuting the rows of `A` we obtain a better-posed problem: the floating-point arithmetic neglects the small perturbation, avoiding losing the linear independence of the row of `p*A` while doing Gaussian elimination. (See the notes attached to this document)

1. Determine the solution of the system $Ax = b$ computed by the instruction `BackSub (U,ForwSub(L,b))`, then the solution computed by the instruction `BackSub (u,ForwSub(l,p*b))`. Compare with the exact solution $x = [0, 1, 1]^{\mathrm{T}}$. Conclude.

In [13]:

```
x1 = BackSub(U,ForwSub(L,b))
x2 = BackSub(u,ForwSub(l,p*b))
```

```
x1 =

   NaN
  -Inf
   Inf


x2 =

  -0.000000000000000
   1.000000000000000
   1.000000000000000
```

The issue here is the floating-point arithmetic. Employing Gaussian elimination, we can see that we obtain a singular matrix `U` due to the round-off of the FP arithmetic. On the other hand, matrix `U` is non-singular due to the reorder of the rows of `A`. This reorder ensures longer pivots while doing Gaussian elimination. In conclusion, one way of precondition a matrix is to ensure pivots with a larger magnitude. In this manner, we avoid a magnification of tiny perturbations errors and undesired round-off errors of FP arithmetic.

Ex 6.9

Let A be a band matrix of order n and half bandwidth p. For $n >> p >> 1$ compute the number of operations $N_{\text{op}}(n, p)$ required for the LU factorization (having in mind Proposition 6.2.1).

Consider the Algorithm 6.1:

In [ ]:

```
for k=1:(n-1)
    for i=(k+1):n
        A(i,k) = A(i,k)/A(k,k);
        for j=(k+1):n
            A(i,j) = A(i,j) - A(i,k)*A(k,j);
        end
    end
end
```

This algorithm is a compact form of Gaussian elimination. For each step k, we pick a row i below the $k-$th row and choose α to delete the first term of the $k-$th row by subtracting the $i-$th row. This α must be $a_{i,k}/a_{k,k}$. Then we apply the elementary row operation.

Due to the additional structure of the matrix A, we can truncate the inner loops until the half bandwidth instead of studying the whole row vector. The associated Algorithm is:

In [ ]:

```
for k=1:(n-1)
    for i=(k+1):min([n k+p])
        A(i,k) = A(i,k)/A(k,k);
        for j=(k+1):min([n k+p])
            A(i,j) = A(i,j) - A(i,k)*A(k,j);
        end
    end
end
```

Due to the previous algorithm, we find that the number of operations is described by

$$N_{\text{op}}(n, p) = \sum_{k=1}^{n-1} \sum_{i=k+1}^{\min\{n,k+p\}} \left(1 + \sum_{j=k+1}^{\min\{n,k+p\}} 1 \right).$$

Notice that $k + p \leq n$ if and only if $k \leq n - p$. Thus,

$$\begin{aligned} N_{\text{op}}(n, p) &= \textstyle\sum_{k=1}^{n-1} (\min\{n, k+p\} - k)(\min\{n, k+p\} - k - 1) \\ &= \textstyle\sum_{k=1}^{n-p} p(p-1) + \sum_{k=n-p+1}^{n-1} (n-k)(n-k-1) \\ &= p(p-1)(n-p) + \textstyle\sum_{k=0}^{p-2} k(k+1) \\ &= p(p-1)(n-p) + \dfrac{1}{3} p(p-1)(p-2). \end{aligned}$$

We conclude that $N_{\text{op}} = O(n)$ **if** p is fixed.