Tarea 4

jueves, 20 de mayo de 2021 15:01

Homework 4

Ex 6.1

We define a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Compute its determinant using the `MATLAB` function `det`. Explain why the result is not an integer.

obs

$$(-1)\begin{bmatrix}1\\2\\3\end{bmatrix}^T + (2)\begin{bmatrix}4\\5\\6\end{bmatrix}^T = \begin{bmatrix}7\\8\\9\end{bmatrix}^T$$

$\therefore \det(A) = 0.$

In [63]:

```
A = [1 2 3; 4 5 6; 7 8 9];
det(A)
```

```
ans =

  -9.5162e-16
```

In [67]:

```
[L U] = lu(A)
```

```
L =

    0.1429    1.0000         0
    0.5714    0.5000    1.0000
    1.0000         0         0

U =

    7.0000    8.0000    9.0000
         0    0.8571    1.7143
         0         0   -0.0000
```

MATLAB hace eliminación Gaussiana sobre el pivote más grande.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 6/7 & 12/7 & 1 & 0 & 1/7 \\ 0 & 3/7 & 6/7 & 0 & 1 & 4/7 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right]$$

$\frac{6}{7} \approx 0.8571$, $\frac{12}{7} \approx 1.7142$

$\frac{3}{7} \approx 0.4285$, $\frac{6}{7} \approx 0.8571$

This error is due to round-off errors appearing in LU decomposition.

`det` uses the LU decomposition to calculate the determinant, which is susceptible to floating-point round-off errors.

Ver mathworks.com

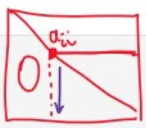
Ex 6.2

The goal of this exercise is to compare the performances of the LU and Cholesky methods.

1. Write a function `LUFacto` returning the matrices L and U determined via Algorithm 6.1. If the algorithm cannot be executed (division by 0), return an error message.

In [129…

idea del algoritmo:



- operaciones fila
- usar "bien" la memoria

```
%%file LUFacto.m

function [L U] = LUFacto(A)
    n = size(A,1);

    for k=1:(n-1)
        if (A(k,k)==0)
            disp('Error! Pivot must be nonzero')
            L = NaN;
            U = NaN;
            return
        else
            for i=(k+1):n
                A(i,k) = A(i,k)/A(k,k);
                for j=(k+1):n
                    A(i,j) = A(i,j) - A(i,k)*A(k,j);
                end
            end
        end
    end
    L = eye(n) + tril(A,-1);
    U = triu(A);
end
```

Descartar pivote nulo. Retornar NaN.

Operación fila: Restar, a la k-ésima fila, $\frac{a_{ik}}{a_{kk}}$ veces la i-ésima fila.

Recuperar L y U.

```
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```

obs: Queremos $A = LU$. Notar que L y U solo usan "media" matriz! El algoritmo las almacena en UNA matriz.

$A = \square \rightarrow \underbrace{\Delta}_{L-I}\nabla\ U.$

Más aún, desprecia la diagonal de L, pues será conocida.

2. Write a function `Cholesky` returning the matrix B computed by Algorithm 6.2. If the algorithm cannot be executed (nonsymmetric matrix, division by 0, negative square root), return an error message. Compare with the `MATLAB` function `chol`.

In [123…

```
%%file Cholesky.m

function B = Cholesky(A)
    if (A~=A')
        disp('Input must be symmetric!')
        B = NaN;
        return
    else
        B = tril(A);
        n = size(A,1);
        for j=1:n
            for k=1:(j-1)
                B(j,j) = B(j,j) - (B(j,k))^2;
```

descartar A no simetrica.

La info. de A se concentra en una mitad.

obs: En mi opinión es más fácil ver Cholesky como un método LDU:

```
        B = tril(A);
        n = size(A,1);
        for j=1:n
            for k=1:(j-1)
                B(j,j) = B(j,j) - (B(j,k))^2;
            end
            if (B(j,j)<=0)
                disp('Error! Pivot must be positive')
                B = NaN;
                return
            else
                B(j,j) = sqrt(B(j,j));
                for i=(j+1):n
                    for k=1:(j-1)
                        B(i,j) = B(i,j) - B(j,k)*B(i,k);
                    end
                    B(i,j) = B(i,j)/B(j,j);
                end
            end
        end
    end
end
```

La info. de A se concentra en una mitad.

Cálculo pivote (diagonal)

Operaciones fila

```
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```

In [23]:

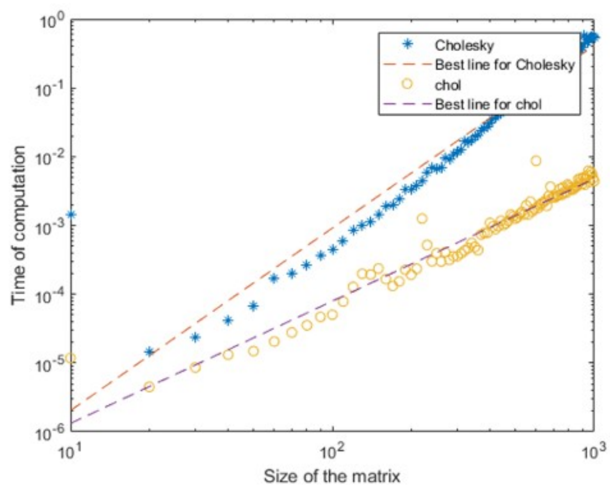
```
times = zeros(2,length(1:100));

for i = 1:100
    n = i*10;
    A = PdSMat(n);

    tic
    B = Cholesky(A);
    times(1,i) = toc;

    tic
    B = chol(A);
    times(2,i) = toc;
end
x = 10:10:1000;
p1 = polyfit(log(x),log(times(1,:)),1);
p2 = polyfit(log(x),log(times(2,:)),1);

figure
loglog(x,times(1,:),'*')
hold on
loglog(x,exp(p1(2)).*x.^p1(1),'--')
loglog(x,times(2,:),'o')
loglog(x,exp(p2(2)).*x.^p2(1),'--')
hold off
xlabel('Size of the matrix')
ylabel('Time of computation')
legend('Cholesky','Best line for Cholesky','chol','Best line for chol')
```

→ Matriz Simétrica, def. positiva

Algoritmo propuesto

Rutina de MATLAB

Calzar la mejor curva $f(n) = p_2 \cdot n^{p_1}$



In [24]:

```
p1
p2
```

```
p1 =

   2.653036013266808 -19.212901563663547


p2 =

   1.777763598656285 -17.626797864320473
```

un método LDU:

$A = LDU = U^*DL^* = A^*$ y $L = U^*$.

$\therefore A = (L\sqrt{D})(L\sqrt{D})^*$.

i.e.: Triangularizar y normalizar.

Obs: asintóticamente es útil ver este resultado, pues pese a que time(n) es p2 =

```
   1.777763598656285 -17.626797864320473
```

pese a que time(n) es polinomial, para $n >> 1$ dominará el término de mayor grado.

We can see that $\text{time}_{\text{Cholesky}}(n) \approx \exp(-19.2129)n^{2.6530}$ and $\text{time}_{\text{chol}}(n) \approx \exp(-17.6267)n^{1.7777}$, under the assumption that computation time behaves like $C_1 \cdot n^{C_2}$.

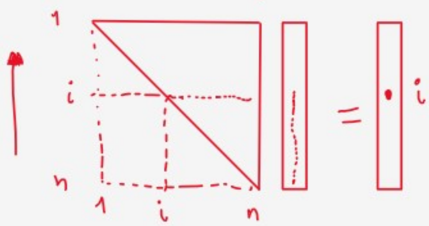
3. For $n = 10, 20, \ldots, 100$, we define a matrix `A=MatSdp(n)` (see Exercise 2.20) and a vector `b=ones(n,1)`. Compare:

- On the one hand, the running time for computing the matrices L and U given by the function `LUFacto`, then the solution x of the system $Ax = b$. Use the functions `BackSub` and `ForwSub` defined in Exercise 5.2.
- On the other hand, the running time for computing the matrix B given by the function `Cholesky`, then the solution x of the system $Ax = b$. Use the functions `BackSub` and `ForwSub`. Plot on the same graph the curves representing the running times in terms of n. Comment.

In [15]:

```
%%file BackSub.m

function x = BackSub(U,y)
    n = size(U,1);
    x = zeros(n,1);
    for i = n:-1:1
        s = 0;
        for j = n:-1:(i+1)
            s = s + U(i,j)*x(j);
        end
        x(i) = (y(i)-s)/U(i,i);
    end
end
```

$Ux = y$



$u_{ii}x_i + s_i = y_i$

$s_i = \sum_{j=(i+1)}^{n} u_{ij}x_j.$

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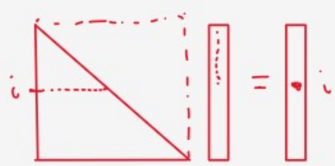
In [16]:

```
%%file ForwSub.m

function x = ForwSub(L,y)
    n = size(L,1);
    x = zeros(n,1);
    for i = 1:n
        s = 0;
        for j = 1:(i-1)
            s = s + L(i,j)*x(j);
        end
        x(i) = (y(i)-s)/L(i,i);
    end
end
```

$Lx = y$



$l_{ii}\, x_i + s_i = y_i$

$s_i = \sum_{j=1}^{i-1} l_{ij}x_j.$

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In [17]:

```
%%file SymmetricMat.m

function S = SymmetricMat(n)
    S = rand(n);
    S = S + S';
end
```

Afi: $\forall A \in \mathcal{M}_n(\mathbb{K})$. $A^T + A$ es simétrica.

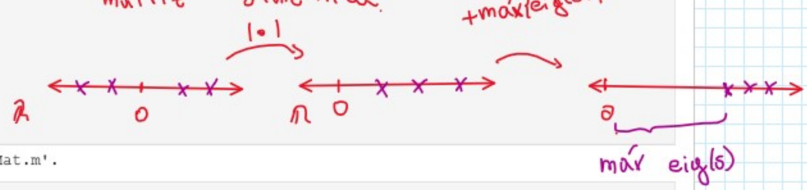
Created file 'D:\GitHub\NLA-IMA\hw4\SymmetricMat.m'.

In [18]:

```
%%file PdSMat.m

function S = PdSMat(n)
    S = SymmetricMat(n);
    [P, D] = eig(S);
    D = abs(D);
    D = D + norm(D)*eye(size(D));
    S = P*D*inv(P);
end
```

idea: perturbar valores propios de una matriz simétrica



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In [25]:

```
times = zeros(2,length(1:50));

for i = 1:100
    n = i*10;
    A = PdSMat(n);
    b = ones(n,1);

    tic;
    [L U] = LUFacto(A);
    x1 = BackSub(U,ForwSub(L,b));
    times(1,i) = toc;

    tic;
    B = Cholesky(A);
    x2 = BackSub(B',ForwSub(B,b));
    times(2,i) = toc;
end

x = 10:10:1000;
p1 = polyfit(log(x),log(times(1,:)),1);
p2 = polyfit(log(x),log(times(2,:)),1);
```

Resolver vía LU.

" vía Cholesky.

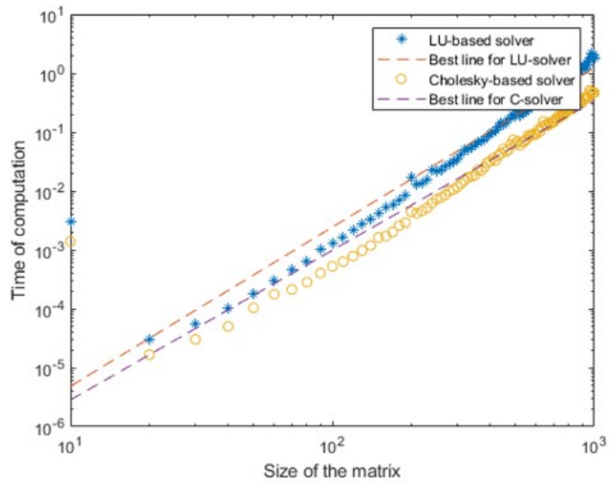
Encajar $f(n) = p_2 n^A$.

```
end

x = 10:10:1000;
p1 = polyfit(log(x),log(times(1,:)),1);
p2 = polyfit(log(x),log(times(2,:)),1);

figure
loglog(x,times(1,:),'*')
hold on
loglog(x,exp(p1(2)).*x.^p1(1),'--')
loglog(x,times(2,:),'o')
loglog(x,exp(p2(2)).*x.^p2(1),'--')
hold off
xlabel('Size of the matrix')
ylabel('Time of computation')
legend('LU-based solver','Best line for LU-solver','Cholesky-based solver','Best line for C-solver')
```

Encajar $f(n) = p_2 n^A$.



In [26]:

```
p1
p2
```

```
p1 =

   2.700435568617929 -18.452550099850395


p2 =

   2.538527947687385 -18.613953300022562
```

We see that $\mathrm{time}_{\mathrm{LU-Solver}}(n) \approx \exp(-18.4525)n^{2.7004}$ and $\mathrm{time}_{\mathrm{C-Solver}}(n) \approx \exp(-18.6139)n^{2.5385}$, under the assumption that computation time behaves like $C_1 \cdot n^{C_2}$.

Ex 6.4

The goal of this exercise is to evaluate the influence of row permutation in Gaussian elimination. Let A and b be defined by `e=1.E-15`, `A=[e 1 1;1 -1 1; 1 0 1]` and `b=[2 0 1]'`.

1. Compute the matrices `L` and `U` given by the function `LUFacto` of Exercise 6.2.

In [11]:

```
format long
e=1.E-16;
A=[e 1 1;1 -1 1; 1 0 1];
b=[2 0 1]';
[L, U] = LUFacto(A)
det(A)
```

$$\begin{bmatrix} 10^{-15} & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{o.f.}} \begin{bmatrix} 10^{-15} & 1 & 1 \\ 0 & -1-10^{15} & 1-10^{15} \\ 0 & -10^{15} & 1-10^{15} \end{bmatrix} \approx \begin{bmatrix} 10^{-15} & 1 & 1 \\ 0 & -10^{15} & -10^{15} \\ 0 & -10^{15} & -10^{15} \end{bmatrix}$$

Obs: Un mal pivote incide en la aritmética de pto. flotante.

```
L =

   1.0e+16 *

   0.000000000000000                   0                   0
   1.000000000000000   0.000000000000000                   0
   1.000000000000000   0.000000000000000   0.000000000000000


U =

   1.0e+16 *

   0.000000000000000   0.000000000000000   0.000000000000000
                   0  -1.000000000000000  -1.000000000000000
                   0                   0                   0


ans =

   1.000000000000000
```

Notar que A es no-singular

0 0 0

```
ans =

   1.000000000000000
```

Notar que A es no-singular.

2. We define two matrices `l` and `u` by `[l u]=LUFacto(p*A)`, where `p` is the permutation matrix defined by the instruction `[w z p]=lu(A)`. Display the matrices `l` and `u`. What do you observe?

In [14]:
```
[w z p]=lu(A);
p
[l u] = LUFacto(p*A)
```

```
p =

     0     1     0
     1     0     0
     0     0     1
```

Precondicionamos. vía permutaciones (asegurando pivotes más grandes)

```
l =

   1.000000000000000                   0                   0
   0.000000000000000   1.000000000000000                   0
   1.000000000000000   1.000000000000000   1.000000000000000


u =

   1.000000000000000  -1.000000000000000   1.000000000000000
                   0   1.000000000000000   1.000000000000000
                   0                   0  -1.000000000000000
```

First, notice that in this computation `e=1.E-16`. Notice that by permuting the rows of `A` we obtain a better-posed problem: the floating-point arithmetic neglects the small perturbation, avoiding losing the linear independence of the row of `p*A` while doing Gaussian elimination. (See the notes attached to this document)

3. Determine the solution of the system $Ax = b$ computed by the instruction `BackSub (U,ForwSub(L,b))`, then the solution computed by the instruction `BackSub (u,ForwSub(l,p*b))`. Compare with the exact solution $x = [0, 1, 1]^{\mathrm{T}}$. Conclude.

In [13]:
```
x1 = BackSub(U,ForwSub(L,b))
x2 = BackSub(u,ForwSub(l,p*b))
```

```
x1 =

   NaN
  -Inf
   Inf
```

notar que: • L está bien def.
• U es singular. → No tiene sentido hacer sustitución!

```
x2 =

  -0.000000000000000
   1.000000000000000
   1.000000000000000
```

The issue here is the floating-point arithmetic. Employing Gaussian elimination, we can see that we obtain a singular matrix `U` due to the round-off of the FP arithmetic. On the other hand, matrix `U` is non-singular due to the reorder of the rows of `A`. This reorder ensures longer pivots while doing Gaussian elimination. In conclusion, one way of precondition a matrix is to ensure pivots with a larger magnitude. In this manner, we avoid a magnification of tiny perturbations errors and undesired round-off errors of FP arithmetic.

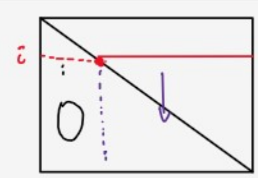
Ex 6.9

Let A be a band matrix of order n and half bandwidth p. For $n >> p >> 1$ compute the number of operations $N_{\mathrm{op}}(n, p)$ required for the LU factorization (having in mind Proposition 6.2.1).

Consider the Algorithm 6.1:

In [ ]:
```
for k=1:(n-1)
    for i=(k+1):n
        A(i,k) = A(i,k)/A(k,k);
        for j=(k+1):n
            A(i,j) = A(i,j) - A(i,k)*A(k,j);
        end
    end
end
```

Operaciones fila!



$a_{kk} \quad \overbrace{\underbrace{a_{k,k+1} \quad \cdots \quad a_{k,k+p}}_{\downarrow}}^{p}$

This algorithm is a compact form of Gaussian elimination. For each step k, we pick a row i below the $k-$th row and choose α to delete the first term of the $k-$th row by subtracting the $i-$th row. This α must be $a_{i,k}/a_{k,k}$. Then we apply the elementary row operation.

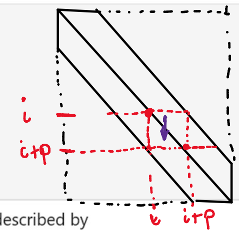
Due to the additional structure of the matrix A, we can truncate the inner loops until the half bandwidth instead of studying the whole row vector. The associated Algorithm is:

In [ ]:
```
for k=1:(n-1)
    for i=(k+1):min([n k+p])
        A(i,k) = A(i,k)/A(k,k);
```

Las operaciones fila se realizan a lo más p veces, en renglones

In [ ]:

```
for k=1:(n-1)
    for i=(k+1):min([n k+p])
        A(i,k) = A(i,k)/A(k,k);
        for j=(k+1):min([n k+p])
            A(i,j) = A(i,j) - A(i,k)*A(k,j);
        end
    end
end
```



Las operaciones fila se realizan a lo más p veces, en renglones de a lo más p elementos.

Due to the previous algorithm, we find that the number of operations is described by

$$N_{\mathrm{op}}(n,p) = \sum_{k=1}^{n-1} \sum_{i=k+1}^{\min\{n,k+p\}} \left(1 + \sum_{j=k+1}^{\min\{n,k+p\}} 1 \right).$$

Notice that $k+p \leq n$ if and only if $k \leq n-p$. Thus,

$$\begin{aligned}
N_{\mathrm{op}}(n,p) &= \textstyle\sum_{k=1}^{n-1} (\min\{n,k+p\} - k)(\min\{n,k+p\} - k - 1) \\
&= \textstyle\sum_{k=1}^{n-p} p(p-1) + \sum_{k=n-p+1}^{n-1} (n-k)(n-k-1) \\
&= p(p-1)(n-p) + \textstyle\sum_{k=0}^{p-2} k(k+1) \\
&= p(p-1)(n-p) + \frac{1}{3} p(p-1)(p-2). = np^2 - \frac{2}{3}p^3 + O(np).
\end{aligned}$$

obs: We conclude that $N_{\mathrm{op}} = O(n)$ **if** p is fixed.