CSC304 Lecture 3
Guest Lecture: Prof. Allan Borodin

Game Theory
(More examples, PoA, PoS)

Recap

- Normal form games
- Domination among strategies
  - A strategy weakly/strictly dominating another
  - A strategy being weakly/strictly dominant
  - Iterated elimination of dominated strategies
- Nash equilibria
  - Pure – may be none, unique, or multiple
    - Identified using best response diagrams
  - Mixed – at least one!
    - Identified using the indifference principle

This Lecture

- More examples of games
  - Identifying pure and mixed Nash equilibria
  - More careful analysis

- Price of Anarchy
  - How bad it is for the players to play a Nash equilibrium compared to playing the best outcome (if they could coordinate)?

Revisiting Cunning Airlines

- Two travelers, both lose identical luggage
- Airline asks them to individually report the value between 2 and 99 (inclusive)
- If they report (s, t), the airline pays them
  - (s, s) if $s = t$
  - $(s + 2, s - 2)$ if $s < t$
  - $(t - 2, t + 2)$ if $t < s$
- How do you formally derive equilibria?

Revisiting Cunning Airlines

- Pure Nash Equilibria: When can (s, t) be a NE?
  - Case 1: $s < t$
    - Player 2 is currently rewarded $s - 2$.
    - Switching to (s, s) will increase his reward to s.
    - Not stable
  - Case 2: $s > t \;\rightarrow$ symmetric.
  - Case 3: $s = t = x$ (say)
    - Each player currently gets x.
    - Each player wants to switch to $x - 1$, if possible, and increase his reward to $x - 1 + 2 = x + 1$.
    - For stability, $x - 1$ must be disallowed $\Rightarrow x = 2$.
- (2,2) is the only pure Nash equilibrium.

Revisiting Cunning Airlines

- Additional mixed strategy Nash equilibria?

- Hint:
  - Say player 1 fully randomizes over a set of strategies T.
  - Let M be the highest value in T.
  - Would player 2 ever report any number that is M or higher with a positive probability?

Revisiting Rock-Paper-Scissor

- No pure strategy Nash equilibria
  - Why? Because "there's always an action that makes a given player win".

- Suppose row and column players play (a_r, a_s)
  - If one player is losing, he can change his strategy to win.
    - If the other player is playing Rock, change to Paper; if the other player is playing Paper, change to Scissor; …
  - If it's a tie $(a_r = a_s)$, both want to deviate and win!
  - Cannot be stable.

Revisiting Rock-Paper-Scissor

- Mixed strategy Nash equilibria

- Suppose the column player plays (R,P,S) with probabilities (p,q,1-p-q).

- Row player:
  - Calculate $\mathbb{E}[R], \mathbb{E}[P], \mathbb{E}[S]$ for the row player strategies.
  - Say expected rewards are 3, 2, 1. Would the row player randomize?
  - What if they were 3, 3, 1?
  - When would he fully randomize over all three strategies?

Revisiting Rock-Paper-Scissor

- Solving a special case
  - Fully mixed: Both randomize over all three strategies.
  - Symmetric: Both use the same randomization (p,q,1-p-q).

1. Assume column player plays (p,q,1-p-q).
2. For the row player, write $\mathbb{E}[R] = \mathbb{E}[P] = \mathbb{E}[S]$.

- All cases?
  - 4 possibilities of randomization for each player
  - Asymmetric strategies (need to write equal rewards for column players too)

Revisiting Stag-Hunt

Hunter 1 / Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Game
  - Stag requires both hunters, food is good for 4 days for each hunter.
  - Hare requires a single hunter, food is good for 2 days
  - If they both catch the same hare, they share.

- Two pure Nash equilibria: (Stag,Stag), (Hare,Hare)

Revisiting Stag-Hunt

Hunter 2 \ Hunter 1	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Two pure Nash equilibria: (Stag,Stag), (Hare,Hare)
  - Other hunter plays "Stag" → "Stag" is best response
  - Other hunter plays "Hare" → "Hare" is best reponse

- What about mixed Nash equilibria?

Revisiting Stag-Hunt

Hunter 2 \ Hunter 1	Stag	Hare
Stag	**(4 , 4)**	**(0 , 2)**
Hare	**(2 , 0)**	**(1 , 1)**

- Symmetric: $s \rightarrow \{\text{Stag w.p. } p, \text{Hare w.p. } 1-p\}$
- Indifference principle:
  - *Given the other hunter plays s, equal $\mathbb{E}$[reward] for Stag and Hare*
  - $\mathbb{E}[\text{Stag}] = p * 4 + (1-p) * 0$
  - $\mathbb{E}[\text{Hare}] = p * 2 + (1-p) * 1$
  - Equate the two $\Rightarrow p = 1/3$

Nash Equilibria: Critique

- Noncooperative game theory provides a framework for analyzing rational behavior.

- But it relies on many assumptions that are often violated in the real world.

- Due to this, human actors are observed to play Nash equilibria in some settings, but play something far different in other settings.

Nash Equilibria: Critique

- Assumptions:
  - Rationality is common knowledge.
    - All players are rational.
    - All players know that all players are rational.
    - All players know that all players know that all players are rational.
    - … [Aumann, 1976]
    - Behavioral economics
  - Rationality is perfect = “infinite wisdom”
    - Computationally bounded agents
  - Full information about what other players are doing.
    - Bayes-Nash equilibria

Nash Equilibria: Critique

- Assumptions:
  - No binding contracts.
    - Cooperative game theory
  - No player can commit first.
    - Stackelberg games (will study this in a few lectures)
  - No external help.
    - Correlated equilibria
  - Humans reason about randomization using expectations.
    - Prospect theory

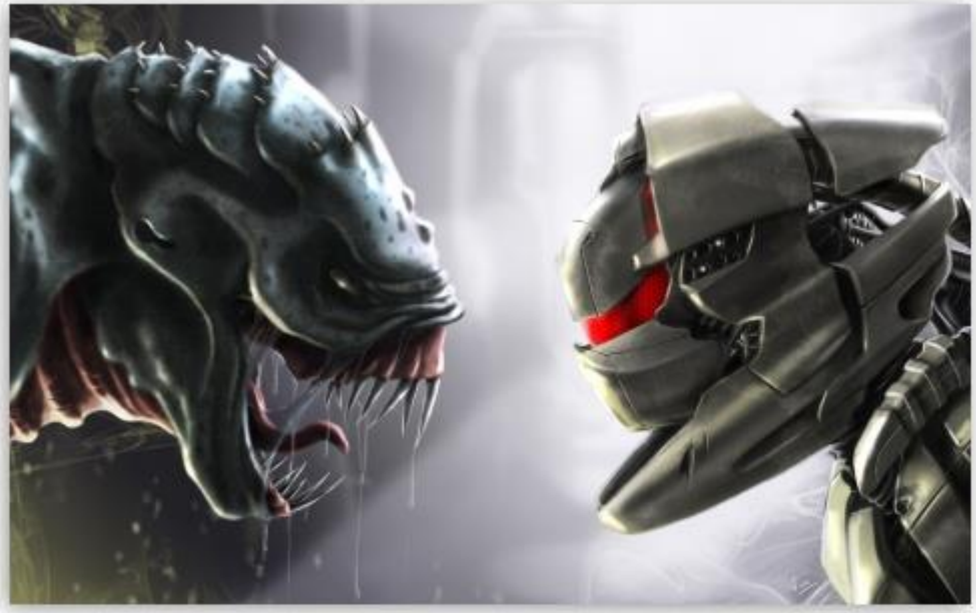
Nash Equilibria: Critique

- Also, there are often multiple equilibria, and no clear way of “choosing” one over another.

- For many classes of games, finding a single equilibrium is provably hard.
  - Cannot expect humans to find it if your computer cannot.

Nash Equilibria: Critique

- Conclusion:
  - For human agents, take it with a grain of salt.
  - For AI agents playing against AI agents, perfect!



Price of Anarchy and Stability

- If players play a Nash equilibrium instead of "socially optimum", how bad will it be?

- Objective function: e.g., sum of utilities

- Price of Anarchy (PoA): compare the optimum to the worst Nash equilibrium

- Price of Stability (PoS): compare the optimum to the best Nash equilibrium

Price of Anarchy and Stability

- Price of Anarchy (PoA)

$$\frac{\text{Maximum social utility}}{\text{Minimum social utility in any Nash equilibrium}}$$

- Price of Stability (PoS)

Costs $\rightarrow$ flip:
Nash equilibrium
divided by optimum

$$\frac{\text{Maximum social utility}}{\text{Maximum social utility in any Nash equilibrium}}$$

Revisiting Stag-Hunt

Hunter 1 / Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Optimum social utility = 4+4 = 8
- Three equilibria:
  - (Stag, Stag) : Social utility = 8
  - (Hare, Hare) : Social utility = 2
  - (Stag:1/3 - Hare:2/3, Stag:1/3 - Hare:2/3)
    - Social utility = (1/3)*(1/3)*8 + (1-(1/3)*(1/3))*2 = Btw 2 and 8
- Price of stability? Price of anarchy?

Revisiting Prisoner's Dilemma

Sam \ John	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

- Optimum social cost = 1+1 = 2
- Only equilibrium:
  - (Betray, Betray) : Social cost = 2+2 = 4

- Price of stability? Price of anarchy?