CSC304 Fall'17

Assignment 1

Due Date: October 11, 2017, by 3pm

Be sure to include your name and student number with your assignment. Typed assignments are preferred (e.g., PDFs created using LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Instructions for online submission will be provided.

Remember our citation policy. You are free to read online material (though, if you find the exact homework problem, do not read the solution until you have tried solving the problem yourself), and you are free to discuss any problem with your peers. There are two rules you must follow: You have to write the solution in your own words (it helps not to take pictures or notes of discussions), and you have to cite any peers or online sources from where you obtained a full or a partial solution to the problem.

The problems are (roughly) in the increasing order of difficulty. I do *not* expect everyone to be able to solve all the problems. Solve the problems that you can, give your best effort for the remaining, and remember the 20% rule: If you just state that you do not know how to approach a problem (or subproblem), you get 20% of its points. The bonus question is optional; if you solve all other questions, you will still get 100% credit for this homework.

Notation

Let us fix some notation. For an n-player normal-form game, let S_i denote the set of actions (pure strategies) of player i. Let $S = S_1 \times S_2 \times \ldots \times S_n$. We use s_i to denote the strategy (pure or mixed) of player i. We use $\vec{s} = (s_1, \ldots, s_n)$ to denote the strategy profile. Given the strategy profile, we can define the reward $\pi_i(\vec{s})$ or the cost $c_i(\vec{s})$ to player i. The social welfare $\Pi(\vec{s})$ or the social cost $C(\vec{s})$ is the sum of the individual rewards or costs, respectively. Lastly, in case you are not familiar with this notation, $\vec{s}_{-i}$ denotes all components of vector $\vec{s}$ except component i (corresponding to player i).

Q1 [5 Points] Russell Crowe Was Wrong

Watch the scene from the move *A Beautiful Mind* that purports to explain what a Nash equilibrium is (it's easy to find this on Youtube). This can be modeled as a game with four players (the men) and each player having five actions (the women). Explain why the solution proposed by Russell Crowe (playing John Nash) is *not* a Nash equilibrium.

Q2 [15 Points] Nash Equilibria

Consider the following normal-form game between two players. Payoffs for the row player Sheila are indicated first in each cell, and payoffs for the column player Adam are second.

	L	R
U	(1,1)	(4,0)
D	(4,0)	(3,3)

(a) [5 Points] Show that this game has no pure strategy Nash equilibria.

(b) [5 Points] Find a mixed strategy Nash equilibrium of the game.

(c) [5 Points] Notice that in the equilibrium you found in part (b), Sheila plays U more often than D. Your friend claims that your calculations must have an error because D is clearly a better strategy than U for Sheila. Both U and D give her a reward of 4 in the off-diagonal cells of the matrix, but D gives her a reward of 3 on the diagonal cell while U only gives her a reward of 1 on the diagonal cell. Explain what is wrong with your friend's reasoning.

Q3 [10 Points] Iterated Elimination of Dominated Strategies

Recall the definition of weak/strict domination. We say that a pure strategy $s_i \in S_i$ is *weakly dominated* (by some other strategy) if there exists another strategy $s_i' \in S_i$ such that $\pi_i(s_i, \vec{s}_{-i}) \leq \pi_i(s_i', \vec{s}_{-i})$ for all $\vec{s}_{-i}$, and the inequality is strict for some $\vec{s}_{-i}$. We say that s_i is *strictly dominated* if the inequality is strict for every $\vec{s}_{-i}$.

(a) [4 Points] Show that eliminating a single strictly dominated strategy of any player from the game neither adds new nor removes existing Nash equilibria, i.e., the set of Nash equilibria is preserved. The argument for iterated elimination follows trivially (you do not need to argue this).

(b) [4 Points] Show that eliminating a single weakly dominated strategy of any player from the game cannot add new Nash equilibria, but can remove existing Nash equilibria (give an example where this happens). Once again, the argument for iterated elimination follows trivially.

(c) [2 Points] Produce an example game in which the order of elimination of weakly dominated strategies can affect which Nash equilibria remain in the resulting game.

Q4 [10 Points] Simultaneous Zero-Sum Games

There are three players $P1$, $P2$, and $P3$. $P1$ is simultaneously playing two zero-sum games: one with $P2$ in which $P1$'s reward matrix is A_{12}, and one with $P3$ in which $P1$'s reward matrix is A_{13}. The total reward for $P1$ is the sum of his rewards from the two games. Suppose all three players have the same set of actions S. The twist is that $P1$ is required to play the same (possibly mixed) strategy in both games.

(a) [3 Points] Argue that this can be modeled as a single three-player zero-sum game.

(b) [7 Points] Using Nash's theorem for the game derived in part (a), argue that an extension of the minimax theorem holds in this situation. Specifically, let x_1, x_2, and x_3 denote mixed strategies of $P1$, $P2$, and $P3$, respectively. Then show that

$$\max_{x_1} \min_{x_2,x_3} (x_1)^T(A_{12}x_2 + A_{13}x_3) = \min_{x_2,x_3} \max_{x_1} (x_1)^T(A_{12}x_2 + A_{13}x_3). \tag{1}$$

[Hint: Proceed along the line of the proof of minimax theorem we did in the class. To prove the two quantities are equal, we are going to show an inequality in each direction. First, find the "obvious inequality". Then, take a Nash equilibrium $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Write the three "best response" equations that describe each $\tilde{x}_i$ as the the best response of player i given the strategies of the other players. Combine the three equations to get the inequality in the reverse direction.]

A few more questions might be added later...

Bonus Question

Q5 [10 Points] Potential Games

Recall the theory of potential games: We are given an n-player game, in which each player i has a set of pure strategies S_i. Given a pure strategy profile $\vec{s} \in S = S_1 \times \ldots \times S_n$, the cost to player i is $c_i(\vec{s})$, and the social cost is $C(\vec{s}) = \sum_{i=1}^{n} c_i(\vec{s})$. A function $\Phi : S \to \mathbb{R}$ is called a potential function if for every $\vec{s} \in S$, player i, and his strategy $s'_i \in S_i$,

$$c_i(s'_i, \vec{s}_{-i}) - c_i(\vec{s}) = \Phi(s'_i, \vec{s}_{-i}) - \Phi(\vec{s}).$$

(a) [5 Points] Prove that if a game admits two potential functions Φ_1 and Φ_2, then they must differ by a constant. That is, there must exist $c \in \mathbb{R}$ such that $\Phi_1(\vec{s}) = \Phi_2(\vec{s}) + c$ for all $\vec{s} \in S$.

(b) [5 Points] An *agreement game* is a game where all players always have the same cost, i.e., $c_i(\vec{s}) = c_j(\vec{s})$ for all $i, j, \vec{s}$. A *faux game* is a game where a player's cost is independent of his strategy, i.e., $c_i(s_i, \vec{s}_{-i}) = c_i(s'_i, \vec{s}_{-i})$ for all $i, s_i, s'_i, \vec{s}_{-i}$.

Prove that a game with cost functions $c_1, \ldots, c_n$ is a potential game (i.e., admits a potential function Φ) if and only if it is the sum of an agreement game with costs $c_1^a, \ldots, c_n^a$ and a faux game with costs $c_1^f, \ldots, c_n^f$ (i.e., $c_i(\vec{s}) = c_i^a(\vec{s}) + c_i^f(\vec{s})$ for every i and $\vec{s}$).