

# An efficient kinetic modeling in plasmas by using the AWBS transport equation

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## Abstract

The AWBS Boltzmann transport equation for electrons equipped with a simplified e-e collision operator [?] provides an efficient, yet physically relevant, kinetic extension compared the classical Spitzer-Harm heat flux (SH) based on local approximation and flux-limiting. This classical approach is widely used in plasma kinetics models coupled to hydrodynamics. Even though SH reflects the electron-electron collision effect, the essential physical properties of the electron transport cannot be modeled with an explicitly local model. A simple form of the AWBS model opens a way to couple kinetics to hydrodynamic codes while describing the important physics correctly. Since the electron-electron collision effect becomes especially important in the case of low ion potential, we focus on presenting results related to low Z plasmas.

*Keywords:* kinetics; hydrodynamics; nonlocal electron transport; laser-heated plasmas.

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## 17 1. Introduction

## 18 2. The AWBS nonlocal transport model

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \tilde{\mathbf{E}} \cdot \nabla_{\mathbf{v}} f = v \frac{\nu_e}{2} \frac{\partial}{\partial v} (f - f_M) + \frac{\nu_{ei} + \frac{\nu_e}{2}}{2} \frac{\partial^2 f}{\partial \mathbf{n}^2}, \quad (1)$$

19 [1]

## 20 3. BGK, AWBS, and Fokker-Planck models in diffusive regime

21 We can try to find an approximate solution while using the first term of  
22 expansion in  $\lambda_e$  and  $muas$

$$\tilde{f}(z, v, \mu) = f^0(z, v) + f^1(z, v) \lambda_{ei} \mu. \quad (2)$$

### 23 3.1. The BGK diffusive electron transport

$$\begin{aligned} \mathbf{n} \cdot \nabla f + \frac{1}{v} \left[ \tilde{\mathbf{E}} \cdot \mathbf{n} \frac{\partial f}{\partial v} + \frac{\tilde{\mathbf{E}} \cdot \mathbf{e}_\phi - v \tilde{\mathbf{B}} \cdot \mathbf{e}_\theta}{v} \frac{\partial f}{\partial \phi} + \frac{\tilde{\mathbf{E}} \cdot \mathbf{e}_\theta + v \tilde{\mathbf{B}} \cdot \mathbf{e}_\phi}{v \sin(\phi)} \frac{\partial f}{\partial \theta} \right] = \\ \frac{(f_M - f)}{\lambda_e} + \frac{1}{2\lambda_{ei}} \left( \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{1}{\sin^2(\phi)} \frac{\partial^2 f}{\partial \theta^2} \right), \quad (3) \end{aligned}$$

24 where  $\mu = \cos(\phi)$ ,  $\lambda_e$  is the electron-electron mean free path, and  $\lambda_{ei}$  is  
25 the electron-ion mean free path. We also approximate  $\lambda_e = \bar{Z} \lambda_{ei}$ .

Clearly,  $\frac{\partial \tilde{f}}{\partial \theta} = 0$ , and if  $\tilde{\mathbf{B}} = \tilde{B}_z \mathbf{e}_z$ , there is no effect of magnetic field. We also assume, that  $\nabla f = \frac{\partial f}{\partial z} \mathbf{e}_z$  and appropriately  $\tilde{\mathbf{E}} = \tilde{E}_z \mathbf{e}_z$ . From the orientation of the Cartesian basis vectors and spherical basis vectors, one can find  $\tilde{\mathbf{E}} \cdot \mathbf{n} = \tilde{E}_z \cos(\phi) = \mu$  and  $\tilde{\mathbf{E}} \cdot \mathbf{e}_\phi = -\tilde{E}_z \sin(\phi)$ . As a result, the analyzed BGK equation reads

$$\begin{aligned} \mu \frac{\partial}{\partial z} (f^0 + f^1 \lambda_{ei} \mu) + \frac{1}{v} \left[ \tilde{E}_z \mu \frac{\partial}{\partial v} (f^0 + f^1 \lambda_{ei} \mu) - \frac{\tilde{E}_z \sin(\phi)}{v} \frac{\partial}{\partial \phi} (f^0 + f^1 \lambda_{ei} \mu) \right] = \\ \frac{(f_M - (f^0 + f^1 \lambda_{ei} \mu))}{\lambda_e} + \frac{1}{2\lambda_{ei}} \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial}{\partial \mu} (f^0 + f^1 \lambda_{ei} \mu) \right), \quad (4) \end{aligned}$$

$$\mu \frac{\partial f^0}{\partial z} + \mu^2 \frac{\partial}{\partial z} (f^1 \lambda_{ei}) + \frac{\tilde{E}_z}{v} \left[ \mu \frac{\partial f^0}{\partial v} + \mu^2 \frac{\partial}{\partial v} (f^1 \lambda_{ei}) + \frac{1 - \mu^2}{v} f^1 \lambda_{ei} \right] = \frac{f_M - f^0}{\bar{Z} \lambda_{ei}} - \mu \frac{1}{\bar{Z}} f^1 - \mu f^1, \quad (5)$$

consequently, we have the following anisotropy expansion  $\mu^0, \mu^1, \mu^2, \dots$  equations

$$\begin{aligned} \frac{f_M - f^0}{\bar{Z} \lambda_{ei}} &= \frac{\tilde{E}_z}{v^2} f^1 \lambda_{ei}, \\ \frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v} \frac{\partial f^0}{\partial v} &= -\frac{1}{\bar{Z}} f^1 - f^1, \\ \frac{\partial}{\partial z} (f^1 \lambda_{ei}) + \frac{\tilde{E}_z}{v} \left[ \frac{\partial}{\partial v} (f^1 \lambda_{ei}) - \frac{1}{v} f^1 \lambda_{ei} \right] &= 0, \end{aligned}$$

which lead to the definitions

$$f^0 = f_M + \frac{1}{v} f^1 \bar{Z} \lambda_{ei}^2, \quad (6)$$

$$\begin{aligned} f^1 &= -\frac{\bar{Z}}{\bar{Z} + 1} \left[ \frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v} \frac{\partial f^0}{\partial v} \right] \\ &= -\frac{\bar{Z}}{\bar{Z} + 1} \left[ \frac{1}{\rho} \frac{\partial \rho}{\partial z} + \left( \frac{v^2}{2v_{th}^2} - \frac{3}{2} \right) \frac{1}{T} \frac{\partial T}{\partial z} - \frac{\tilde{E}_z}{v_{th}^2} \right] f_M \end{aligned} \quad (7)$$

In order to ensure the plasma to be quasi-neutral, the zero-current condition

$$\mathbf{j} = \int_0^\infty \int_{4\pi} q_e v \mathbf{n} f \, d\mathbf{n} \, v^2 \, dv = \mathbf{0}, \quad (8)$$

can be achieved by providing a consistent electric field in (15), i.e.

$$\tilde{\mathbf{E}} = \frac{v_{th}^2 \int_{4\pi} \mathbf{n} \otimes \mathbf{n} \cdot \int_0^\infty v f_M \frac{\lambda}{\alpha} \left( \frac{\nabla \rho}{\rho} + \left( \frac{v^2}{2v_{th}^2} - \frac{3}{2} \right) \frac{\nabla T}{T} \right) v^2 \, dv \, d\mathbf{n}}{\int_{4\pi} \mathbf{n} \otimes \mathbf{n} \cdot \int_0^\infty v f_M \frac{\lambda}{\alpha} v^2 \, dv \, d\mathbf{n}}, \quad (9)$$

which may be further simplified as

$$\tilde{\mathbf{E}} = \frac{\int_0^\infty f_M \frac{1}{2} \frac{\nabla T}{T} v^9 \, dv}{\int_0^\infty f_M v^7 \, dv} + v_{th}^2 \left( \frac{\nabla \rho}{\rho} - \frac{3}{2} \frac{\nabla T}{T} \right) = v_{th}^2 \left( \frac{\nabla \rho}{\rho} + \frac{5}{2} \frac{\nabla T}{T} \right), \quad (10)$$

where it is worth mentioning, that the part  $f_M + \frac{v\lambda}{\alpha} \frac{\partial f_1}{\partial v}$  of the distribution does not contribute to the current since it is isotropic. One can write the quasi-neutral distribution function explicitly distinguishing between original part (blue color) and E field correction (red color) as

$$f \approx f_M \left( 1 - \frac{\lambda}{\alpha} \mathbf{n} \cdot \left( \frac{v^2}{2v_{th}^2} - \frac{3}{2} - \frac{5}{2} \right) \frac{\nabla T}{T} \right) + \frac{v\lambda}{\alpha} \frac{\partial f_1}{\partial v}. \quad (11)$$

which leads to the resulting heat flux

$$\mathbf{q}_H = \int_{4\pi} \int_0^\infty \frac{m_e v^2}{2} v \mathbf{n} f v^2 dv d\mathbf{n} = \frac{4\pi}{3} \frac{m_e}{2} \frac{1}{\alpha \sigma \rho} \int_0^\infty \left( \frac{v^2}{2v_{th}^2} - \frac{3}{2} - \frac{5}{2} \right) v^9 f_M dv \frac{\nabla T}{T}.$$

Based on the Gauss integral formula

$$\int v^{2s+1} \exp \left( -\frac{v^2}{2v_{th}^2} \right) dv = \frac{s! (2v_{th}^2)^{s+1}}{2}$$

and Maxwell-Boltzmann distribution (??) the heat flux can be written as

$$\mathbf{q}_H = \frac{4\pi}{3} \frac{m_e}{2} \frac{1}{\alpha \sigma \rho} \frac{\rho}{v_{th}^3} \frac{1}{(2\pi)^{3/2}} \frac{4!}{T} \frac{2^4 v_{th}^{10}}{T} \left( \frac{5}{2} - \frac{3}{2} - \frac{5}{2} \right) \nabla T = \frac{m_e}{\alpha \sigma} \frac{128}{\sqrt{2\pi}} \left( \frac{k_B}{m_e} \right)^{\frac{7}{2}} T^{\frac{5}{2}} \nabla T. \quad (12)$$

In conclusion, equation (12) provides nothing else than the well known Lorentz approximation heat flux and its nonlinearity 2.5 in temperature. What is worth mentioning is the effect of E field (quasi-neutrality), which reduces the flux of about 71.4% (also assuming constant density).

Finally, one can find the approximate solution

$$\tilde{f} = f_M - \lambda_{ei} \frac{\bar{Z}}{\bar{Z} + 1} \left( \frac{v^2}{2v_{th}^2} - \frac{3}{2} - \alpha \right) \frac{\mathbf{n} \cdot \nabla T}{T} f_M. \quad (13)$$

### 3.2. The AWBS diffusive electron transport

The AWBS electron transport equation in 6D reads

$$\mathbf{n} \cdot \nabla f + \frac{1}{v} \left[ \tilde{\mathbf{E}} \cdot \mathbf{n} \frac{\partial f}{\partial v} + \frac{\tilde{\mathbf{E}} \cdot \mathbf{e}_\phi - v \tilde{\mathbf{B}} \cdot \mathbf{e}_\theta}{v} \frac{\partial f}{\partial \phi} + \frac{\tilde{\mathbf{E}} \cdot \mathbf{e}_\theta + v \tilde{\mathbf{B}} \cdot \mathbf{e}_\phi}{v \sin(\phi)} \frac{\partial f}{\partial \theta} \right] = \frac{v}{\lambda^e} \frac{\partial}{\partial v} (f - f_M) + \left( \frac{1}{\lambda_{ei}} + \frac{1}{\lambda_e} \right) \frac{1}{2} \left( \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{1}{\sin^2(\phi)} \frac{\partial^2 f}{\partial \theta^2} \right), \quad (14)$$

46 where  $\mu = \cos(\phi)$ ,  $\lambda_e$  is the electron-electron mean free path, and  $\lambda_{ei}$  is  
 47 the electron-ion mean free path, and  $\lambda_e = \bar{Z}\lambda_{ei}$ .

48 We can try to find an approximate solution while using the first term of  
 49 expansion in  $\lambda_e$  and  $\mu$  as

$$\tilde{f}(z, v, \mu) = f^0(z, v) + f^1(z, v)\lambda_{ei}\mu. \quad (15)$$

Clearly,  $\frac{\partial \tilde{f}}{\partial \theta} = 0$ , and if  $\tilde{\mathbf{B}} = \tilde{B}_z \mathbf{e}_z$ , there is no effect of magnetic field. We also assume, that  $\nabla f = \frac{\partial f}{\partial z} \mathbf{e}_z$  and appropriately  $\tilde{\mathbf{E}} = \tilde{E}_z \mathbf{e}_z$ . From the orientation of the Cartesian basis vectors and spherical basis vectors, one can find  $\tilde{\mathbf{E}} \cdot \mathbf{n} = \tilde{E}_z \cos(\phi) = \mu$  and  $\tilde{\mathbf{E}} \cdot \mathbf{e}_\phi = -\tilde{E}_z \sin(\phi)$ . As a result, the analyzed AWBS equation reads

$$\begin{aligned} \mu \frac{\partial}{\partial z} (f^0 + f^1 \lambda_{ei} \mu) + \frac{1}{v} \left[ \tilde{E}_z \mu \frac{\partial}{\partial v} (f^0 + f^1 \lambda_{ei} \mu) - \frac{\tilde{E}_z \sin(\phi)}{v} \frac{\partial}{\partial \phi} (f^0 + f^1 \lambda_{ei} \mu) \right] = \\ \frac{v}{\lambda_e} \frac{\partial}{\partial v} ((f^0 + f^1 \lambda_{ei} \mu) - f_M) + \frac{\bar{Z} + 1}{2\lambda_{ei}\bar{Z}} \frac{\partial}{\partial \mu} \left( (1 - \mu^2) \frac{\partial}{\partial \mu} (f^0 + f^1 \lambda_{ei} \mu) \right), \end{aligned} \quad (16)$$

$$\begin{aligned} \mu \frac{\partial f^0}{\partial z} + \mu^2 \frac{\partial}{\partial z} (f^1 \lambda_{ei}) + \frac{\tilde{E}_z}{v} \left[ \mu \frac{\partial f^0}{\partial v} + \mu^2 \frac{\partial}{\partial v} (f^1 \lambda_{ei}) + \frac{1 - \mu^2}{v} f^1 \lambda_{ei} \right] = \\ \frac{v}{\bar{Z}\lambda_{ei}} \frac{\partial}{\partial v} (f^0 - f_M) + \mu \frac{v}{\bar{Z}\lambda_{ei}} \frac{\partial (f^1 \lambda_{ei})}{\partial v} - \mu \frac{\bar{Z} + 1}{\bar{Z}} f^1, \end{aligned} \quad (17)$$

50 consequently, we have the following anisotropy expansion  $\mu^0, \mu^1, \mu^2, \dots$  equa-  
 51 tions

$$\begin{aligned} \frac{v}{\bar{Z}\lambda_{ei}} \frac{\partial}{\partial v} (f^0 - f_M) &= \frac{\tilde{E}_z}{v^2} f^1 \lambda_{ei}, \\ \frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v} \frac{\partial f^0}{\partial v} &= \frac{v}{\bar{Z}\lambda_{ei}} \frac{\partial (f^1 \lambda_{ei})}{\partial v} - \frac{\bar{Z} + 1}{\bar{Z}} f^1, \\ \frac{\partial}{\partial z} (f^1 \lambda_{ei}) + \frac{\tilde{E}_z}{v} \left[ \frac{\partial}{\partial v} (f^1 \lambda_{ei}) - \frac{1}{v} f^1 \lambda_{ei} \right] &= 0, \end{aligned}$$

52 which lead to the definitions

$$\begin{aligned}
\frac{\partial}{\partial v} (f^0 - f_M) &= \frac{1}{v^2} f^1 \bar{Z} \lambda_{ei}^2, \\
\frac{v}{\bar{Z} \lambda_{ei}} \frac{\partial(f^1 \lambda_{ei})}{\partial v} - \frac{\bar{Z} + 1}{\bar{Z}} f^1 &= \frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v} \frac{\partial f^0}{\partial v} \\
\frac{v}{\bar{Z}} \frac{\partial f^1}{\partial v} + \frac{4}{\bar{Z}} f^1 - \frac{\bar{Z} + 1}{\bar{Z}} f^1 &= \frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v} \frac{\partial f^0}{\partial v} \\
\frac{\partial f^1}{\partial v} + \frac{1}{v} (3 - \bar{Z}) f^1 &= \frac{\bar{Z}}{v} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial z} + \left( \frac{v^2}{2v_{th}^2} - \frac{3}{2} \right) \frac{1}{T} \frac{\partial T}{\partial z} - \frac{\tilde{E}_z}{v_{th}^2} \right) f^1
\end{aligned} \tag{18}$$

53 3.3. The Fokker-Planck diffusive electron transport

$$v_{2th} = \sqrt{\frac{2k_B T}{m_e}} = 1/j,$$

54

$$\begin{aligned}
A &= -\frac{m_e^2 v_{2th}^2 \tilde{\mathbf{E}}}{2\pi e^4 n_e \ln \Lambda} = -\frac{mE}{2\pi j^2 e^3 n_e \ln \Lambda}, \\
B &= \frac{m_e^2 v_{2th}^4 |\nabla T|}{2\pi e^4 n_e \ln \Lambda T} = \frac{2k_B^2 T |\nabla T|}{\pi e^4 n_e \ln \Lambda},
\end{aligned}$$

55

$$\frac{A}{B} = -\frac{|\tilde{\mathbf{E}}|T}{v_{2th}^2 |\nabla T|},$$

56

$$\tilde{\mathbf{E}} = -\frac{3}{2} \frac{v_{2th}^2}{2} \frac{\gamma_T}{\gamma_E} \frac{\nabla T}{T},$$

57 From Eq. (24) CSR, we can write the form of  $f_1$  including both  $\nabla T$  and  $\tilde{\mathbf{E}}$   
58 effects as

$$f_1(v, \theta) = \cos(\theta) \frac{B}{\bar{Z}} \left( d_T(v/v_{2th}) + \frac{A}{B} d_E(v/v_{2th}) \right) f_M(v),$$

59 where in the case of vanishing current one gets

$$\frac{A}{B} = \frac{3}{2} \frac{\gamma_T}{2\gamma_E},$$

60 i.e.

$$f_1(v, \theta) = \cos(\theta) \frac{m_e^2}{4\pi e^4 \ln \Lambda} \frac{v_{2th}^4}{\bar{Z}} \left( 2d_T(v/v_{2th}) + \frac{3}{2} \frac{\gamma_T}{\gamma_E} d_E(v/v_{2th}) \right) \frac{f_M(v)}{n_e} \frac{\nabla T}{T}, \quad (20)$$

61 where  $d_T(x) = \bar{Z}D_T(x)/B$  and  $d_E(x) = \bar{Z}D_E(x)/A$  are represented by nu-  
 62 merical values in TABLE I and TABLE II in [5], respectively. In the case of  
 63 high  $\bar{Z}$  limit,  $\gamma_T \rightarrow 1$ ,  $\gamma_E \rightarrow 1$ ,  $d_E(x) = x^4$ , and  $d_T(x) = x^4(2.5 - x^2)/2$  [5],  
 64 which leads to the standard Lorentz gas model

$$f_1(v, \theta) = \cos(\theta) \frac{m_e^2}{4\pi e^4 \ln \Lambda} \frac{v^4}{\bar{Z}} \left( 4 - \frac{v^2}{v_{2th}^2} \right) \frac{f_M(v)}{n_e} \frac{\nabla T}{T}, \quad (21)$$

65 [2], [3], [4]

#### 66 3.4. Summary of BGK, AWBS, and Fokker-Planck diffusion

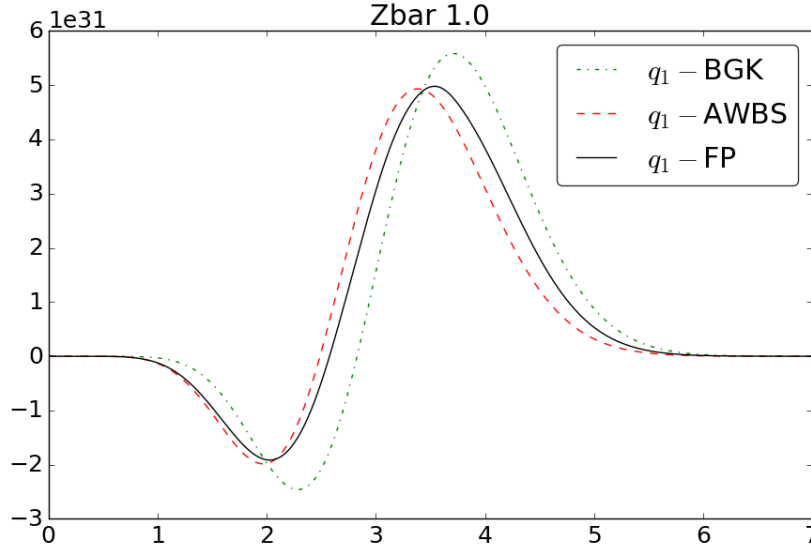


Figure 1: The flux velocity moment of the anisotropic part of the electron distribution function in low Z plasmas in diffusive regime.



	$\bar{Z} = 1$	$\bar{Z} = 2$	$\bar{Z} = 4$	$\bar{Z} = 16$	$\bar{Z} \rightarrow \infty$
$error(\mathbf{q}_{AWBS})$	0.057	0.004	0.038	0.049	0.000

Table 1: Relative  $error(\mathbf{q}_{AWBS}) = |\mathbf{q}_{AWBS} - \mathbf{q}_{SH}|/|\mathbf{q}_{SH}|$  of the AWBS kinetic model equation (1) showing the discrepancy (maximum around 5%) with respect to the original solution of the heat flux given by Spitzer and Harm [5].

## 67 4. Benchmarking the AWBS nonlocal transport model

### 68 4.1. Review of simulation codes

#### 69 4.1.1. C7

#### 70 4.1.2. ALADIN

#### 71 4.1.3. IMPACT

#### 72 4.1.4. CALDER

### 73 4.2. Simulation results

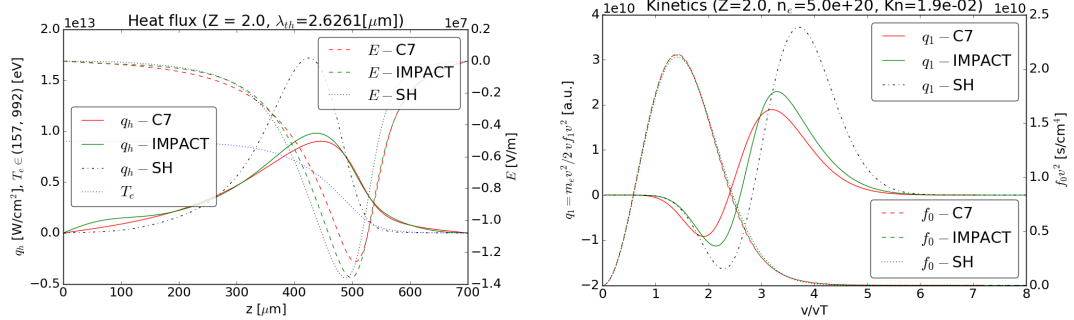


Figure 2: Snapshot 12 ps. Left: correct steady solution. Right: correct comparison to kinetic profiles at point 437  $\mu\text{m}$  by IMPACT.

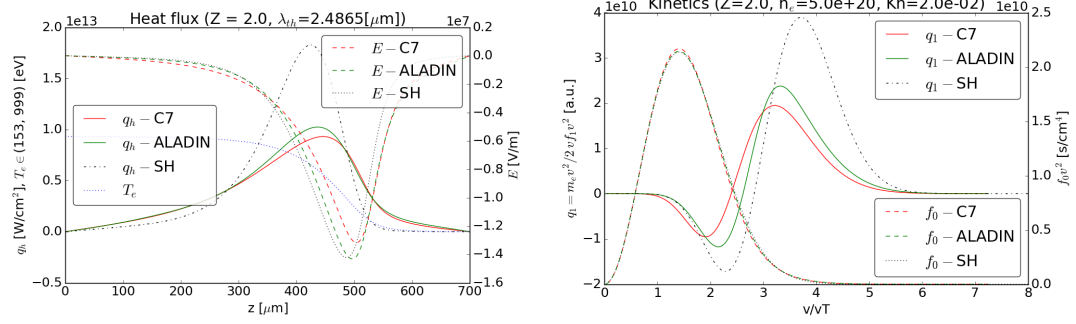


Figure 3: Snapshot 12 ps. Left: correct steady solution. Right: correct comparison to kinetic profiles at point 442 μm by ALADIN.

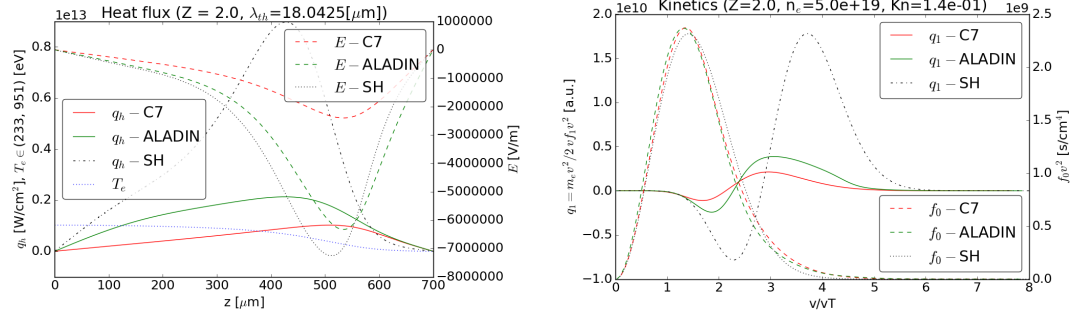


Figure 4: Snapshot 12 ps. Left: Does not look as steady solution. Right: correct comparison to kinetic profiles at point 480 μm by ALADIN.

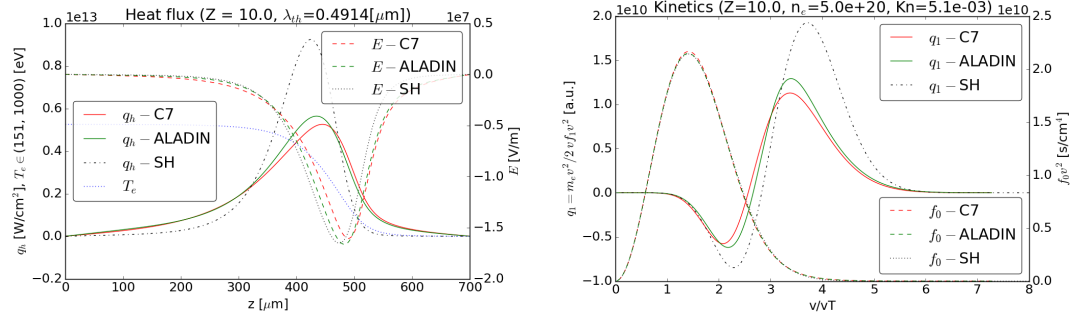


Figure 5: Snapshot 12 ps. Left: correct steady solution. Right: correct comparison to kinetic profiles at point 442 μm by ALADIN.

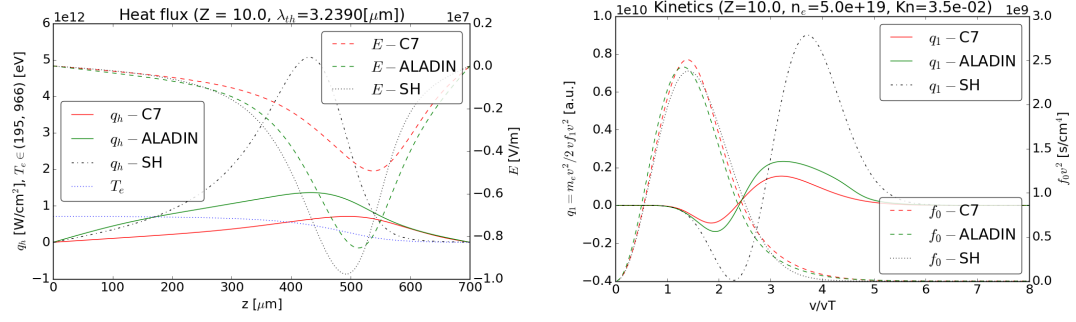


Figure 6: Snapshot 12 ps. Left: Does not look as steady solution. Right: correct comparison to kinetic profiles at point  $480 \mu\text{m}$  by ALADIN.

74 **5. Conclusions**

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