NASA TECHNICAL NOTE

NASA TN D-2716

c.1

NASA TN D-2716

EXPANSION OF THE FOKKER-PLANCK EQUATION IN SPHERICAL HARMONICS

by Frederic A. Lyman

Lewis Research Center

Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1965

Completed 21021 6/21/65

ERRATA

NASA Technical Note D-2716

EXPANSION OF THE FOKKER-PLANCK EQUATION
IN SPHERICAL HARMONICS

By Frederic A. Lyman
March 1965

Page 15, equation (51): The factor v should precede the integral $\int_v^\infty \vec{F}_f^1(v')dv'$.

Issued 5-24-65

NASA TN D-2716

EXPANSION OF THE FOKKER-PLANCK EQUATION IN SPHERICAL HARMONICS

By Frederic A. Lyman

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $2.00

EXPANSION OF THE FOKKER-PLANCK EQUATION

IN SPHERICAL HARMONICS

by Frederic A. Lyman

Lewis Research Center

SUMMARY

The first two terms in the general spherical harmonics expansion of the Fokker-Planck equation are derived by a simplified method. Some remarks are made on the feasibility of obtaining the general term of the expansion. Errors in previously published results are pointed out.

INTRODUCTION

The Fokker-Planck equation is extensively used in the analysis of nonequilibrium phenomena in plasmas. The basic assumption underlying this equation is that collisions which result in scattering through small angles are the most probable and therefore have the predominant effect on the distribution function. This assumption is well founded for electrically charged particles interacting according to an inverse-square force.

If the validity of the Fokker-Planck equation is accepted, a difficult nonlinear integro-partial differential equation for the distribution function still remains. The usual method of simplifying it involves the expansion of the distribution function in spherical harmonics in velocity space. The result of this procedure is a system of coupled nonlinear integro-differential equations whose independent variables are the magnitude of the velocity and time.

Rosenbluth, et al. (ref. 1) described the spherical harmonics expansion for the case of axial symmetry, but the resulting system of equations was not presented. In a subsequent paper (ref. 2), however, the equation for an isotropic distribution function, which is the zero-order equation of the system, was given. Dreicer (ref. 3) obtained the zero- and first-order equations of the system. Other authors (refs. 4 and 5) have also derived the zero- and first-order equations. These two equations usually suffice for most practical purposes, since the zero-order term in the spherical harmonics expansion gives the energy distribution of the particles, while the first-order term describes the manner in which the current is distributed over particles of various energies.

The Fokker-Planck equation for electrons may be combined with the usual electric and magnetic field terms of the Boltzmann equation and the appropriate collision terms for elastic and inelastic interactions between electrons and molecules to yield a description of phenomena in a partially ionized gas (ref. 3). Even in a partially ionized gas, electron-electron collisions have a significant effect on the electron distribution function, except when the degree of ionization is extremely low. The inclusion of this effect entails considerable analytical difficulty unless simplifications of the type described previously can be made. Because of the importance of electron-electron collisions in many situations, the correct formulation of the collision term is vital.

Errors of a nontrivial nature were discovered by the present author in the results of references 3 to 5. Dreicer's results (ref. 3) are the most nearly correct, but much more serious errors occur in references 4 and 5. The analysis leading to these results is not simple, but it could not be included in detail in the previously cited papers because of space limitations. Therefore, it is worthwhile to present in full the derivation of the first two equations in the spherical harmonics expansion of the Fokker-Planck equation and to point out the errors in the results of references 3 to 5. Although the derivation follows the general outline of references 1 and 3, parts of the derivation presented herein are believed to be somewhat simpler than the analyses described in those references.

FOKKER-PLANCK EQUATION

The Fokker-Planck equation may be written (refs. 1 and 3)

$$\left(\frac{\partial F}{\partial t}\right)_c = -\frac{\partial}{\partial v_i}\left(\langle \Delta v_i \rangle F\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(\langle \Delta v_i \Delta v_j \rangle F\right) \tag{1a}$$

where $F(\vec{v},\vec{r},t)$ is the distribution function, which is defined such that Fd^3v is the number of test particles in the element d^3v in velocity space centered about the velocity $\vec{v}$. The components of $\vec{v}$ on a fixed system of rectangular Cartesian coordinate axes are denoted by $v_i (i = 1, 2, 3)$. Repeated subscripts imply summation. (Symbols are defined in appendix A.)

Unfortunately, the terminology applied to the various kinetic equations is by no means universal. Equation (1a) will be referred to as the Fokker-Planck equation in this report, following the usage of references 1 and 3. Actually, equation (1a) is simply the definition of $(\partial F/\partial t)_c$, the rate of change of F due to collisions. In the general case where there are spatial gradients and external forces acting on the test particles, expression (1a) is equated to the total derivative following the motion of the test particles

$$\frac{\partial F}{\partial t} + v_i \frac{\partial F}{\partial x_i} + a_i \frac{\partial F}{\partial v_i} = \left(\frac{\partial F}{\partial t}\right)_c \tag{1b}$$

and equation (1b) is sometimes called the Boltzmann-Fokker-Planck equation.

The vector $\langle \Delta v_i \rangle$ is the rate of change in the i^{th} component of velocity of

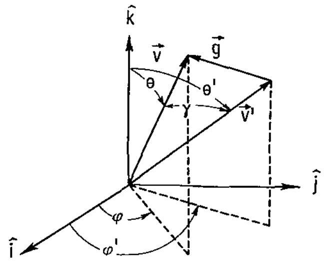

Figure 1. - Orientations of velocities of test and field particles with respect to coordinate system.

a test particle of velocity v_i due to its simultaneous interaction with all other particles, which are called field particles. If the distribution function of the field particles is denoted by F_f, $\langle \Delta v_i \rangle$ is defined as (refs. 1 and 3)

$$\langle \Delta v_i \rangle \equiv \sum_f \int_{v'} F_f(\vec{v}')d^3v' \int_\Omega \sigma(g,\chi)g \, \Delta v_i \, d\Omega \tag{2}$$

where $\sigma(g,\chi)$ is the differential cross section for scattering through the angle χ, g is the relative speed, that is,

$$g = \left|\vec{v} - \vec{v}'\right| \tag{3}$$

Δv_i is the increment in velocity of a test particle of initial velocity $\vec{v}$ due to a collision with a field particle of velocity $\vec{v}'$, and $d\Omega$ is the element of scattering solid angle. The velocity vectors $\vec{v}$, $\vec{v}'$, and $\vec{g}$ are shown in figure 1. The summation in equation (2) is carried out over all species of field particles. The meaning of equation (2) becomes clearer if it is noted that $\sum_f \int_{v'} F_f \, d^3v' \int_\Omega \sigma g \, d\Omega$ is the instantaneous collision rate of a test particle of velocity $\vec{v}$ with all of the field particles. The quantity $\langle \Delta v_i \rangle$ is frequently called the coefficient of dynamical friction, since it corresponds to the slowing down of the test particles along their direction of motion (ref. 6). Note also that $\langle \Delta v_i \rangle$ has units of velocity per unit time.

The quantity $\langle \Delta v_i \Delta v_j \rangle$ is defined similarly as

$$\langle \Delta v_i \Delta v_j \rangle \equiv \sum_f \int_{v'} F_f(\vec{v}')d^3v' \int_\Omega \sigma(g,\chi)g \, \Delta v_i \Delta v_j \, d\Omega \tag{4}$$

and is referred to as the dispersion coefficient, since only this term in the Fokker-Planck equation contributes to the spreading of a beam of test particles (ref. 6).

The Fokker-Planck equation (eq. (1a)) may be derived from the Boltzmann equation by means of an expansion of the kernel of the Boltzmann collision integral in powers of the velocity increments (ref. 6). The Fokker-Planck equation is obtained when only terms through the second order in the velocity increments are retained. Thus the Fokker-Planck equation is valid when collisions resulting in small angle scattering are predominant. This is true for Coulomb interactions because of the long range nature of the inverse-square force.

The differential cross section for Coulomb scattering is the familiar Rutherford cross section

$$\sigma(g,\chi) = \frac{1}{4}\left(\frac{e_t e_f}{4\pi\epsilon_o}\right)^2 \frac{1}{m_{tf}^2 g^4} \frac{1}{\sin^4 \frac{\chi}{2}} \tag{5}$$[1]

In this equation the subscripts t and f refer to the test and field particles, respectively. The reduced mass m_{tf} is

$$m_{tf} \equiv \frac{m_t m_f}{m_t + m_f} \tag{6}$$

(Mks units are used throughout this report. Thus the electric charges e_t and e_f are measured in coulombs (e.g., $e_t = \pm Z_t e$ where Z_t is the charge number and $e = 1.602\times10^{-19}$ C is the absolute value of the electron charge). All the results herein may be converted to cgs electrostatic units by simply deleting the $4\pi\epsilon_o$.)

The calculation of $\langle\Delta v_i\rangle$ and $\langle\Delta v_i \Delta v_j\rangle$ for Coulomb collisions was carried out by Rosenbluth, et al. (ref. 1), who showed that these quantities could be expressed by means of the following simple formulas:

$$\langle\Delta v_i\rangle = \frac{\partial H_t}{\partial v_i} \tag{7}$$

$$\langle\Delta v_i \Delta v_j\rangle = \frac{\partial^2 G_t}{\partial v_i \partial v_j} \tag{8}$$

where the functions H_t and G_t, which are now generally known as the Rosenbluth potentials, are defined as[2]

---

[1]Dreicer's expression for σ (following eq. (4) of ref. 3) erroneously has $(e_t^2 e_f^2)^2$ in the numerator. This rather trivial mistake is not one of the errors referred to in the INTRODUCTION.

[2]Dreicer's notation is used in equations (9) to (11) to facilitate comparison with his final results. His definition of H_t and G_t differs from that of Rosenbluth, et al. (ref. 1) by the presence of Γ_{tf}. Dreicer's equation for Γ_{tf} (eq. (8) of ref. 3) contains the same error as noted in footnote 1, in addition to incorrectly having the reduced mass instead of m_t. The latter error also appears in reference 1.

$$H_t(\vec{v}) \equiv \sum_f H_f(\vec{v}) \equiv \sum_f \frac{m_t + m_f}{m_f} \Gamma_{tf} \int \frac{F_f(\vec{v}\,')d^3v'}{g} \tag{9}$$

$$G_t(\vec{v}) \equiv \sum_f G_f(\vec{v}) \equiv \sum_f \Gamma_{tf} \int F_f(\vec{v}\,')g\ d^3v' \tag{10}$$

and the velocity-independent factor Γ_{tf} is

$$\Gamma_{tf} \equiv 4\pi \left(\frac{e_t e_f}{4\pi\epsilon_o}\right)^2 \frac{1}{m_t^2} \ln \sqrt{1 + \Lambda_{tf}^2} \approx 4\pi \left(\frac{e_t e_f}{4\pi\epsilon_o}\right)^2 \frac{1}{m_t^2} \ln \Lambda_{tf} \tag{11}$$

The quantity Λ_{tf}, which is usually much larger than unity, is defined as the dimensionless ratio

$$\Lambda_{tf} \equiv \frac{d}{b_o} \tag{12}$$

where d is the Debye radius and b_o the impact parameter for 90° scattering. When the charged species are in thermodynamic equilibrium at the temperature T, the Debye radius is given by (ref. 6, p. 21)

$$\frac{1}{d^2} = \sum_s \frac{4\pi N_s}{kT} \frac{e_s^2}{4\pi\epsilon_o} \tag{13}$$

where the summation is over all charged species. The average impact parameter for 90° scattering is (ref. 6, pp. 16 and 21)

$$b_o = \frac{|e_t e_f|}{4\pi\epsilon_o} \frac{1}{m_{tf}\langle g^2\rangle} = \frac{|e_t e_f|}{4\pi\epsilon_o} \frac{1}{3kT} \tag{14}$$

since for thermodynamic equilibrium, $m_{tf}\langle g^2\rangle = 3kT$, as may be easily verified by integrating g^2 over Maxwellian distributions of field and test particles. Therefore, equation (12) becomes

$$\Lambda_{tf} = 3kT \frac{4\pi\epsilon_o}{|e_t e_f|} \left(\frac{kT}{\sum_s 4\pi N_s \frac{e_s^2}{4\pi\epsilon_o}} \right)^{1/2} \tag{15}$$

By direct differentiation of equations (9) and (10), the following useful relations can be derived[3] (refs. 1 and 3)

$$\nabla_v^2 H_t \equiv \frac{\partial^2 H_t}{\partial v_i \partial v_i} = -4\pi \sum_f \frac{m_t + m_f}{m_f} \Gamma_{tf} F_f(\vec{v}) \tag{16}$$

$$\nabla_v^2 G_t = 2 \sum_f \frac{m_f}{m_t + m_f} H_f \tag{17}$$

$$\nabla_v^4 G_t \equiv \frac{\partial^4 G_t}{\partial v_i \partial v_i \partial v_j \partial v_j} = -8\pi \sum_f \Gamma_{tf} F_f(\vec{v}) \tag{18}$$

where H_f and H_t are related by the first identity in equation (9).

With the use of equations (7) and (8), the Fokker-Planck equation (eq. (1a)) for Coulomb collisions may be written in terms of the Rosenbluth potentials H_t and G_t as follows:

$$\left(\frac{\partial F}{\partial t} \right)_{cc} = -\frac{\partial}{\partial v_i} \left(F \frac{\partial H_t}{\partial v_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(F \frac{\partial^2 G_t}{\partial v_i \partial v_j} \right) \tag{19}$$

Rosenbluth (ref. 1) and Dreicer (ref. 3) express equation (19) in spherical coordinates without further reduction. For the present purposes, it is more convenient to first reduce equation (19) to a simpler form. Expansion of the derivatives of the products leads to

$$\left(\frac{\partial F}{\partial t} \right)_{cc} = -F \frac{\partial^2 H_t}{\partial v_i \partial v_i} - \frac{\partial F}{\partial v_i} \frac{\partial H_t}{\partial v_i} + \frac{1}{2} F \frac{\partial^4 G_t}{\partial v_i \partial v_i \partial v_j \partial v_j} + \frac{\partial F}{\partial v_i} \frac{\partial}{\partial v_i} \left(\frac{\partial^2 G_t}{\partial v_j \partial v_j} \right) + \frac{1}{2} \frac{\partial^2 F}{\partial v_i \partial v_j} \frac{\partial^2 G_t}{\partial v_i \partial v_j} \tag{20}$$

[3] The factor Γ_{tf} is missing from equations (9a) and (9c) of reference 3.

Substitution of equations (16) to (18) into equation (20) yields

$$\left(\frac{\partial F}{\partial t}\right)_{cc} = 4\pi \sum_f \frac{m_t}{m_f} \Gamma_{tf} F_f(\vec{v}) F(\vec{v}) - \frac{\partial F}{\partial v_i} \frac{\partial}{\partial v_i} \sum_f \frac{m_t - m_f}{m_t + m_f} H_f + \frac{1}{2} \frac{\partial^2 F}{\partial v_i \partial v_j} \frac{\partial^2 G_t}{\partial v_i \partial v_j} \tag{21}$$

This form of the Fokker-Planck collision term is not new, since it was given by Longmire (ref. 7), for example. Equation (21) is converted to an equation in spherical coordinates most simply by first writing it in terms of general covariant derivatives valid for any coordinate system and then specializing the result to spherical coordinates. This procedure is carried out in appendix B. The result, equation (B17), could be used as the starting point of a general spherical harmonic expansion of the Fokker-Planck equation. Because of the nonlinear form of the equation, however, the general term of the spherical harmonics expansion would be exceedingly cumbersome and not too useful for practical purposes. This report will therefore confine itself to the derivation of only the first two terms of the spherical harmonics expansion, and for this limited purpose a much less cumbersome procedure will suffice.

EXPANSION IN SPHERICAL HARMONICS AND THE LORENTZ APPROXIMATION

A considerable simplification of the Fokker-Planck equation is obtained if it is assumed that the distribution function is nearly spherically symmetric in velocity space. The distribution function is written as a spherically symmetric distribution $F^0(v)$ (not necessarily Maxwellian) plus a small perturbation $\xi(\vec{v})$; that is

$$F(\vec{v}) = F^0(v) + \xi(\vec{v}) \tag{22}$$

This is equivalent to retaining the first few terms in the general spherical harmonic expansion of F

$$F(\vec{v}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F^{lm}(v) Y_{lm}(\theta,\varphi) \tag{23}$$

In general, F^{lm} is a function of the position vector $\vec{r}$ in physical space and time, as well as v. This space and time dependence is understood in equation (23) and all subsequent expressions. The surface harmonics $Y_{lm}(\theta,\varphi)$ are defined as (ref. 8, sec. 3.4)

$$Y_{lm}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\varphi} \tag{24a}$$

where P_l^m are the associated Legendre polynomials, defined as

$$P_l^m(\cos\theta) \equiv \frac{1}{2^l l!} (-\sin\theta)^m \frac{d^{l+m}(\cos^2\theta - 1)^l}{(d\cos\theta)^{l+m}} \tag{24b}$$

The angles θ and φ are the polar and azimuthal angles, respectively, measured with respect to an arbitrary but fixed orthogonal triad of unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ (fig. 1). Since $Y_{l,-m} = (-1)^m Y^*_{lm}$ where the asterisk denotes the complex conjugate, one way to ensure that F is real is to assume that $F^{l,-m} = (-1)^m (F^{lm})^*$.

Although the expansion of F (eq. (23)) is convenient for many purposes because of the orthogonality properties of the spherical harmonics, it still must be interpreted with reference to some set of axes, and this is sometimes inconvenient. It is often desirable to recast equation (23) in an invariant form, which is independent of the particular choice of axes. This was done by Johnston (ref. 9), who has shown that the expansion of F (eq. (23)) is equivalent to a single sum of contracted tensors of the form

$$F(\vec{v}) = \sum_{l=0}^{\infty} F^l_{i_1 i_2 \cdots i_l}(v)\, \frac{v_{i_1} v_{i_2} \cdots v_{i_l}}{v^l} \tag{25}$$

where $F^l_{i_1 i_2 \ldots i_l}$ is a tensor of order l. The general expression relating this tensor and the coefficients F^{lm} of equation (23) is given in reference 9. This somewhat complicated expression will not be required in the present report because only the first two terms ($l = 0, 1$) of equation (23) will be considered. When these terms are written out explicitly by using the expressions for Y_{lm} obtained from equations (24),

$$F(\vec{v}) = F^{00}(v) Y_{00}(\theta,\varphi) + \sum_{m=-1}^{1} F^{1m}(v) Y_{1m}(\theta,\varphi) + \ldots$$

$$= \sqrt{\frac{1}{4\pi}}\, F^{00}(v) - \sqrt{\frac{3}{8\pi}} \sin\theta \left\{ 2\, \mathrm{Re}\left[F^{11}(v)\right] \cos\varphi - 2\, \mathrm{Im}\left[F^{11}(v)\right] \sin\varphi \right\} + \sqrt{\frac{3}{4\pi}}\, F^{10}(v) \cos\theta + \ldots \tag{26}$$

Equation (26) may be cast into the desired form

$$F(\vec{v}) = F^0(v) + \vec{F}^1(v) \cdot \frac{\vec{v}}{v} + \ldots \tag{27}$$

by making the definitions[4]

$$F^0(v) \equiv \sqrt{\frac{1}{4\pi}}\, F^{00}(v) \tag{28a}$$

$$\vec{F}^1(v) \equiv \sqrt{\frac{3}{4\pi}} \left\{ -\sqrt{2}\ \mathrm{Re}\left[F^{11}(v)\right]\hat{i} + \sqrt{2}\ \mathrm{Im}\left[F^{11}(v)\right]\hat{j} + F^{10}(v)\hat{k} \right\} \tag{28b}$$

When there is rotational symmetry about the polar axis $\hat{k}$, only the terms for $m = 0$ in equation (23) are retained and the expansion reduces to an expansion in ordinary Legendre polynomials

$$F(\vec{v}) = \sum_{l=0}^{\infty} F^l(v)P_l(\cos\ \theta) \tag{29}$$

where

$$F^l(v) = \sqrt{\frac{2l + 1}{4\pi}}\, F^{l0}(v) \tag{30}$$

(see eq. (3.57) of ref. 8). In this case, the vector $\vec{F}^1$ is directed along the polar axis, as is clear from equation (28b). For axial symmetry, the polar axis is the direction of the force on the test particles, for example, an electric field or concentration gradient (see ref. 10, pp. 404-408). In the presence of electric and magnetic fields, however, there is in general no axial symmetry in velocity space because, for arbitrary directions of $\vec{E}$ and $\vec{B}$, the force $(\vec{E} + \vec{v} \times \vec{B})$ is not rotationally symmetric about any axis and the direction of $\vec{F}^1(v)$ is unknown a priori.

The Lorentz approximation (see refs. 3 and 11) consists of retaining only the first two terms in equation (27) under the assumptions that

$$|\vec{F}^1| \ll F^0 \tag{31}$$

and that the remaining terms are of negligible importance. As Johnston (ref. 9) has pointed out, F^0 alone is needed to calculate the average value of any scalar function $S(v)$, such as the kinetic energy, while the average value of any vector quantity of the form $Q(v)\vec{v}$, such as the velocity vector itself or the heat flux vector, depends only upon $\vec{F}^1$. Terms of higher order than those written out in equation (27) account for higher order tensor properties, such as the nondiagonal elements of the pressure tensor or the energy flux tensor (ref. 9).

The physical meaning of condition (31) can be established as follows. The average drift velocity of the test particles is (ref. 9, eq. (10b))

[4]Equations (28a) and (28b) differ from equations (9a) and (9b) of reference 9 by numerical factors because the spherical harmonics are defined differently in reference 9.

$$\vec{v}_d \equiv \langle\vec{v}\rangle = \frac{1}{N}\int \vec{v}F\ d^3v = \frac{4\pi}{3N}\int_0^\infty v^3\vec{F}^1(v)dv \tag{32}$$

The average random speed $\langle v\rangle$ is (ref. 9, eq. (10a))

$$\langle v\rangle = \frac{1}{N}\int vF\ d^3v = \frac{4\pi}{N}\int_0^\infty v^3F^0(v)dv \tag{33}$$

Therefore, condition (31) implies that the magnitude of the drift velocity is much smaller than the random speed.

It should be emphasized that in the present analysis, as well as in the analyses of references 1 to 3, neither the spherically symmetric part $F^0(v)$ of the distribution function of the test particles, nor the corresponding function $F_f^0(v)$ for the field particles, is assumed to be Maxwellian. Furthermore, the effect of collisions among the test particles (self-collisions) is included in the analysis, inasmuch as the summations over the various species f of the field particles include the case $f = t$, where the field and test particles are the same. In general, the Fokker-Planck equation is nonlinear due to the presence of the self-collision terms.

On the other hand, some authors (refs. 12 and 13) linearize the Fokker-Planck equation by assuming that the distribution functions of both the test and field particles may be represented by a Maxwellian distribution plus a small perturbation. In the notation of this report, such an assumption corresponds to taking $F^0(v)$ and $F_f^0(v)$ to be Maxwellian distributions, for which the rate of change of $F^0(v)$ due to collisions vanishes. Linear ordinary integro-differential equations for the perturbation terms, such as $\vec{F}^1(v)$ (in the notation of this report), are obtained in references 12 and 13 as a result of this assumption. Balescu (ref. 14) uses a somewhat different method of linearizing the Fokker-Planck equation. He assumes that the test particles do not interact among themselves but only with the field particles, which are assumed to be in thermal equilibrium and, therefore, have a Maxwellian distribution. In Balescu's linearization, the distribution function of the test particles is not assumed to be close to equilibrium, however. Neither of the aforementioned linearization methods is used herein, inasmuch as the present analysis is intended to include those cases where the distribution function, although it is nearly spherically symmetric, is not close to a Maxwellian distribution, and self-collisions are important.

The simplified expansion method to be used herein proceeds in the following manner. The Lorentz approximation (eq. (27)) is substituted into the Fokker-Planck equation (eq. (21)), which is then reduced to two equations, describing the rates of change due to collisions of F^0 and $\vec{F}^1$, respectively. In order to do this, it is necessary to determine at least the first few terms of the expansions for the functions H and G in spherical harmonics. It turns out that it is rather easy to expand functions H and G in full.

Rosenbluth, et al. (ref. 1), did this previously for the case of axial symmetry. In the next section, the expansions of H and G in terms of general spherical harmonics are obtained by a method that is simpler than that of reference 1.

EXPANSION OF FUNCTIONS H AND G

The function $H_t(\vec{v})$ defined in equation (9) involves the integration of g^{-1} over the distribution function of the field particles according to

$$\int_{v'} \frac{F_f(\vec{v}')d^3v'}{g} = \int_0^\infty (v')^2 dv' \int_0^\pi \sin\theta' \, d\theta' \int_0^{2\pi} d\varphi' \, \frac{F_f(\vec{v}')}{g} \tag{34}$$

where the angles θ' and φ' are shown in figure 1. In order to expand H_t, it is convenient to first expand g^{-1} as follows (ref. 8, p. 62, eq. (3.41)):

$$\frac{1}{g} = \frac{1}{|\vec{v} - \vec{v}'|} = \sum_{l=0}^{\infty} \frac{v_<^l}{v_>^{l+1}} P_l(\cos\gamma) \tag{35}$$

where $v_<(v_>)$ is the smaller (larger) of (v,v') and γ is the angle between $\vec{v}$ and $\vec{v}'$, as shown in figure 1. The relation between γ and the angles θ, φ, θ', φ' measured in the fixed coordinate system $\hat{i}$, $\hat{j}$, $\hat{k}$ is

$$\cos\gamma = \frac{\vec{v} \cdot \vec{v}'}{vv'} = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi') \tag{36}$$

The addition theorem for spherical harmonics (ref. 8, p. 68, eq. (3.62)) is used to express $P_l(\cos\gamma)$ as a function of θ, φ, θ', φ'

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \tag{37}$$

When equations (23), (35), and (37) are substituted into equation (34), it becomes

$$\int_{v'} \frac{F_f(\vec{v}')\,d^3v'}{g} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \frac{4\pi}{2l' + 1}$$

$$\times \int_0^{\infty} (v')^2\, dv'\, \frac{v_<^{l'}}{v_>^{l'+1}} F_f^{lm}(v') Y_{l'm'}(\theta,\varphi)$$

$$\times \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\theta'\, d\theta'\, Y^*_{l'm'}(\theta',\varphi') Y_{lm}(\theta',\varphi') \tag{38}$$

Because of the orthogonality property of the spherical harmonics (ref. 8, p. 65, eq. (3.55))

$$\int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\theta'\, d\theta'\, Y^*_{l'm'}(\theta',\varphi') Y_{lm}(\theta',\varphi') = \delta_{l'l}\delta_{m'm} \tag{39}$$

(where $\delta_{l'l} = 1$ if $l = l'$ and is zero if $l \neq l'$), all terms in the m' and l' summations are seen to vanish when $m' \neq m$ and $l' \neq l$, while the terms for $m' = m$ and $l' = l$ integrate to unity. The resulting expansion for H_t can be written

$$H_t(\vec{v}) = \sum_f H_f(\vec{v}) = \sum_f \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_f^{lm}(v) Y_{lm}(\theta,\varphi) \tag{40}$$

where

$$H_f^{lm}(v) = \frac{4\pi}{2l + 1} \frac{m_t + m_f}{m_f} \Gamma_{tf} \left[\int_0^{v} \frac{(v')^{l+2}}{v^{l+1}} F_f^{lm}(v')dv' + \int_v^{\infty} \frac{v^l}{(v')^{l-1}} F_f^{lm}(v')dv' \right] \tag{41}$$

Note that $H_f(\vec{v})$ is related to $F_f(v',\theta,\varphi)$ by means of a one-dimensional linear integral operator in v'.

The expansion of $G_t(\vec{v})$ is carried out by first expanding g in Legendre polynomials. Both sides of equation (35) are multiplied by $g^2 = v^2 + (v')^2 - 2vv' \cos\gamma$, yielding

$$g = \sum_{l=0}^{\infty} \frac{v_<^l}{v_>^{l+1}} \left\{\left[v^2 + (v')^2\right] P_l - 2vv' \cos\gamma \, P_l\right\} \tag{42}$$

If the recursion formula (ref. 8, p. 59, eq. (3.29))

$$(l + 1)P_{l+1} - (2l + 1) \cos\gamma \, P_l + lP_{l-1} = 0 \tag{43}$$

is used to eliminate $\cos\gamma \, P_l$ from equation (42) and the numbering of the terms in the series is changed, the following expansion of g is immediately obtained[5]

$$g = \sum_{l=0}^{\infty} \frac{v_<^l}{v_>^{l+1}} \left[v^2 + (v')^2 - 2vv' \left(\frac{l}{2l - 1} \frac{v_>}{v_<} + \frac{l + 1}{2l + 3} \frac{v_<}{v_>}\right)\right] P_l(\cos\gamma) \tag{44}$$

Since $v^2 + (v')^2$ and vv' are unchanged by an interchange of v and v', they can be replaced by $(v_<^2 + v_>^2)$ and $v_<v_>$, respectively, and equation (44) can be written more simply as

$$g = -\sum_{l=0}^{\infty} \frac{1}{2l - 1} \frac{v_<^l}{v_>^{l-1}} \left(1 - \frac{2l - 1}{2l + 3} \frac{v_<^2}{v_>^2}\right) P_l(\cos\gamma) \tag{45}$$

The argument used in deriving the expansion for H_t is repeated for G_t with equation (45) replacing equation (35). The result can be written down immediately

$$G_t(\vec{v}) = \sum_f G_f(\vec{v}) = \sum_f \sum_{l=0}^{\infty} \sum_{m=-l}^{l} G_f^{lm}(v) Y_{lm}(\theta,\varphi) \tag{46}$$

---

[5]This method of deriving the Legendre polynomial expansion of g was suggested by Willis H. Braun of Lewis.

where

$$G_f^{lm}(v) = -\frac{4\pi}{4l^2 - 1}\Gamma_{tf}\left[\int_0^v \frac{(v')^{l+2}}{v^{l-1}}\left(1 - \frac{2l-1}{2l+3}\frac{(v')^2}{v^2}\right)F_f^{lm}(v')dv'\right.$$

$$\left. + \int_v^\infty \frac{v^l}{(v')^{l-3}}\left(1 - \frac{2l-1}{2l+3}\frac{v^2}{(v')^2}\right)F_f^{lm}(v')dv'\right] \qquad (47)$$

Note that $G_f(\vec{v})$, like $H_f(\vec{v})$, is related to $F_f(v',\theta,\varphi)$ by means of a one-dimensional linear integral operator in v'. Equations (40), (41), (46), and (47) represent the extensions of equations (40), (41), (45), and (46) of reference 1 to general spherical harmonics.[6] In the case $m = 0$, the two sets of results become identical. In reference 1, the expansions H_t and G_t are derived by first expanding g^{-1} and g in three-dimensional Fourier integrals. The subsequent integrations in that paper involve integrals of products of Bessel functions. The present derivation seems to be considerably simpler. In reference 3 the general terms of the expansions for H_t and G_t are not given, so comparison is impossible.

For subsequent purposes, only the terms for $l = 0$ and $l = 1$ in the expansions of H_t and G_t (eqs. (40) and (46)) are required. These terms may be written in a form exactly analogous to equation (27)

$$H_t(\vec{v}) = H_t^0(v) + \vec{H}_t^1(v) \cdot \frac{\vec{v}}{v} + \cdot \cdot \cdot \qquad (48)$$

$$G_t(\vec{v}) = G_t^0(v) + \vec{G}_t^1(v) \cdot \frac{\vec{v}}{v} + \cdot \cdot \cdot \qquad (49)$$

Since H_f and G_f are related to $F_f(v',\theta,\varphi)$ by linear integral operators in v', it is clear that H_f^0, G_f^0, and $\vec{H}_f^1$, $\vec{G}_f^1$ are related to F_f^0 and $\vec{F}_f^1$, respectively, by the same integral operators, namely

$$H_t^0(v) = \sum_f H_f^0(v) = 4\pi \sum_f \frac{m_t + m_f}{m_f}\Gamma_{tf}\left[\frac{1}{v}\int_0^v (v')^2 F_f^0(v')dv' + \int_v^\infty v' F_f^0(v')dv'\right]$$

(50)

[6]In equation (40) of reference 1, $a_n^{(a)}(v,\mu)$ should be replaced by $A_n^{(a)}(v,\mu)$ on the left side; a similar change should be made in equation (41).

$$\vec{H}_t^1(v) = \sum_f \vec{H}_f^1(v) = \frac{4\pi}{3}\sum_f \frac{m_t + m_f}{m_f}\Gamma_{tf}\left[\frac{1}{v^2}\int_0^v (v')^3\vec{F}_f^1(v')dv' + \int_v^\infty \vec{F}_f^1(v')dv'\right]$$ see errata

$$(51)$$

$$G_t^0(v) = \sum_f G_f^0(v) = 4\pi\sum_f \Gamma_{tf}\left[v\int_0^v (v')^2F_f^0(v')dv' + \frac{1}{3v}\int_0^v (v')^4F_f^0(v')dv'\right.$$

$$\left. + \int_v^\infty (v')^3F_f^0(v')dv' + \frac{v^2}{3}\int_v^\infty v'F_f^0(v')dv'\right] \qquad (52)$$

$$\vec{G}_t^1(v) = \sum_f \vec{G}_f^1(v)$$

$$= \frac{4\pi}{15}\sum_f \Gamma_{tf}\left[\frac{1}{v^2}\int_0^v (v')^5\vec{F}_f^1(v')dv' - 5\int_0^v (v')^3\vec{F}_f^1(v')dv'\right.$$

$$\left. + v^3\int_v^\infty \vec{F}_f^1(v')dv' - 5v\int_v^\infty (v')^2\vec{F}_f^1(v')dv'\right] \qquad (53)$$

Equations (52) and (53) agree with equations (16) and (17) of reference 3, when equation (17) is written in vector form and the incorrect upper limit ∞ in the first term of that equation is replaced by v. No expressions for H_t^0 and $\vec{H}_t^1$ are given in reference 3.

DERIVATION OF EQUATIONS FOR F^0 AND $\vec{F}^1$

Equation (21) will now be decomposed into equations for F^0 and $\vec{F}^1$. The distribution functions F and F_f of test and field particles, and the potentials H_f and G_t are written

$$F(\vec{v}) = F^0(v) + \xi(\vec{v}) \qquad (22)$$

$$F_f(\vec{v}) = F_f^0(v) + \xi_f(\vec{v}) \qquad (54)$$

$$H_f(\vec{v}) = H_f^0(v) + \eta(\vec{v}) \tag{55}$$

$$G_t(\vec{v}) = G_t^0(v) + \zeta(\vec{v}) \tag{56}$$

where in the spirit of the Lorentz approximation ξ, ξ_f, η, ζ are considered to be small perturbations. When products of ξ, ξ_f, η, and ζ are neglected in comparison with the first powers of these quantities, the various terms in equation (21) become

$$F_f F = F_f^0 F^0 + F_f^0 \xi + F^0 \xi_f \tag{57}$$

$$\frac{\partial F}{\partial v_i} \frac{\partial H_f}{\partial v_i} = \frac{\partial F^0}{\partial v_i} \frac{\partial H_f^0}{\partial v_i} + \frac{\partial F^0}{\partial v_i} \frac{\partial \eta}{\partial v_i} + \frac{\partial H_f^0}{\partial v_i} \frac{\partial \xi}{\partial v_i} \tag{58}$$

$$\frac{\partial^2 F}{\partial v_i \partial v_j} \frac{\partial^2 G_t}{\partial v_i \partial v_j} = \frac{\partial^2 F^0}{\partial v_i \partial v_j} \frac{\partial^2 G_t^0}{\partial v_i \partial v_j} + \frac{\partial^2 F^0}{\partial v_i \partial v_j} \frac{\partial^2 \zeta}{\partial v_i \partial v_j} + \frac{\partial^2 G_t^0}{\partial v_i \partial v_j} \frac{\partial^2 \xi}{\partial v_i \partial v_j} \tag{59}$$

Since all the quantities with superscript 0 are functions of v, the following reductions can be made in the derivative terms:

$$\frac{\partial F^0}{\partial v_i} = \frac{\partial F^0}{\partial v} \frac{\partial v}{\partial v_i} = \frac{v_i}{v} \frac{\partial F^0}{\partial v} \tag{60}$$

$$\frac{\partial^2 F^0}{\partial v_i \partial v_j} = \frac{\partial}{\partial v_j}\left(\frac{v_i}{v} \frac{\partial F^0}{\partial v}\right) = \frac{\delta_{ij}}{v} \frac{\partial F^0}{\partial v} + \frac{v_i v_j}{v} \frac{\partial}{\partial v}\left(\frac{1}{v} \frac{\partial F^0}{\partial v}\right) \tag{61}$$

Similar expressions are of course obtained for H_f^0 and G_t^0. If the gradient ∇_v is written in spherical coordinates, it follows immediately that

$$v_i \frac{\partial}{\partial v_i} = \vec{v} \cdot \nabla_v = v \frac{\partial}{\partial v} \tag{62}$$

Equation (62) is then used to show that

$$v_i v_j \frac{\partial^2}{\partial v_i \partial v_j} = v_i \frac{\partial}{\partial v_i}\left(v_j \frac{\partial}{\partial v_j}\right) - v_i \delta_{ij} \frac{\partial}{\partial v_j} = v \frac{\partial}{\partial v}\left(v \frac{\partial}{\partial v}\right) - v \frac{\partial}{\partial v} = v^2 \frac{\partial^2}{\partial v^2} \tag{63}$$

Expressions such as equations (60) and (61) are substituted for the derivatives of F^0, H_f^0, and G_t^0 in equations (58) and (59), and use is made of equations

(62) and (63). The results may be written

$$\frac{\partial F}{\partial v_i}\frac{\partial H_f}{\partial v_i} = \frac{\partial F^0}{\partial v}\frac{\partial H_f^0}{\partial v} + \frac{\partial F^0}{\partial v}\frac{\partial \eta}{\partial v} + \frac{\partial H_f^0}{\partial v}\frac{\partial \xi}{\partial v} \tag{64}$$

$$\frac{\partial^2 F}{\partial v_i \partial v_j}\frac{\partial^2 G_t}{\partial v_i \partial v_j} = \frac{\partial^2 F^0}{\partial v^2}\frac{\partial^2 G_t^0}{\partial v^2} + \frac{2}{v^2}\frac{\partial G_t^0}{\partial v}\frac{\partial F^0}{\partial v}$$

$$+ \frac{1}{v}\frac{\partial F^0}{\partial v}\nabla_v^2 \zeta + v\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial F^0}{\partial v}\right)\frac{\partial^2 \zeta}{\partial v^2} + \frac{1}{v}\frac{\partial G_t^0}{\partial v}\nabla_v^2 \xi + v\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial G_t^0}{\partial v}\right)\frac{\partial^2 \xi}{\partial v^2} \tag{65}$$

If the quantities ξ, η, ζ are identified as the terms for $l = 1$ in the spherical harmonics expansion of F, H_f, and G_t, it follows from equations (27), (48), and (49) that

$$\xi(\vec{v}) = \vec{F}^1(v) \cdot \frac{\vec{v}}{v} \tag{66}$$

$$\eta(\vec{v}) = \vec{H}_f^1(v) \cdot \frac{\vec{v}}{v} \tag{67}$$

$$\zeta(\vec{v}) = \vec{G}_t^1(v) \cdot \frac{\vec{v}}{v} \tag{68}$$

The derivatives of ξ, η, ζ with respect to v can be calculated immediately, since the unit vector $\vec{v}/v$ remains fixed in magnitude and direction as v changes but θ and φ remain fixed. Thus, for example

$$\frac{\partial \xi}{\partial v} = \frac{\vec{v}}{v} \cdot \frac{\partial \vec{F}^1(v)}{\partial v} \tag{69}$$

$$\frac{\partial^2 \xi}{\partial v^2} = \frac{\vec{v}}{v} \cdot \frac{\partial^2 \vec{F}^1(v)}{\partial v^2} \tag{70}$$

The simplest method of calculating $\nabla_v^2 \xi$ and $\nabla_v^2 \zeta$ is to note that, for any function $\psi(v,\theta,\varphi)$ expandable in spherical surface harmonics as

$$\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \psi^{lm}(v) Y_{lm}(\theta,\varphi) \tag{71}$$

the Laplacian is

$$\nabla_v^2 \psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ Y_{lm} \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{\partial \psi^{lm}}{\partial v} \right) \right.$$

$$\left. + \frac{\psi^{lm}}{v^2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{lm} \right\} \tag{72}$$

But Y_{lm} satisfies the equation (ref. 15, p. 53, eq. (5))

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{lm} = - l(l + 1) Y_{lm} \tag{73}$$

hence,

$$\nabla_v^2 \psi = \sum_{l=0}^{\infty} \left[\frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{\partial}{\partial v} \right) - \frac{l(l + 1)}{v^2} \right] \sum_{m=-l}^{l} \psi^{lm} Y_{lm} \tag{74}$$

Then, because ξ can be written (see eq. (26))

$$\xi = \sum_{m=-1}^{1} F^{1m}(v) Y_{1m}(\theta,\varphi) \tag{75}$$

it follows from equation (74) that

$$\nabla_v^2 \xi = \left[\frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{\partial}{\partial v} \right) - \frac{2}{v^2} \right] \sum_{m=-1}^{1} F^{1m}(v) Y_{1m}(\theta,\varphi)$$

$$= \left[\frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{\partial}{\partial v} \right) - \frac{2}{v^2} \right] \left[\vec{F}^1(v) \cdot \frac{\vec{v}}{v} \right]$$

$$= \frac{\vec{v}}{v} \cdot \left[\frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{\partial}{\partial v} \right) - \frac{2}{v^2} \right] \vec{F}^1(v) \tag{76}$$

The expression for $\nabla_v^2 \zeta$ is similar to equation (76), with $\vec{F}^1$ replaced by $\vec{G}_t^1$. Substitution of equations (69), (70), and (76) and their counterparts for the derivatives of η and ζ into equations (64) and (65) yields after some rearrangement and cancellation of terms

$$\frac{\partial F}{\partial v_i}\frac{\partial H_f}{\partial v_i} = \frac{\partial F^0}{\partial v}\frac{\partial H_f^0}{\partial v} + \frac{\vec{v}}{v}\cdot\left(\frac{\partial F^0}{\partial v}\frac{\partial \vec{H}_f^1}{\partial v} + \frac{\partial H_f^0}{\partial v}\frac{\partial \vec{F}^1}{\partial v}\right) \tag{77}$$

$$\frac{\partial^2 F}{\partial v_i \partial v_j}\frac{\partial^2 G_t}{\partial v_i \partial v_j} = \frac{\partial^2 F^0}{\partial v^2}\frac{\partial^2 G_t^0}{\partial v^2} + \frac{2}{v^2}\frac{\partial G_t^0}{\partial v}\frac{\partial F^0}{\partial v} + \frac{\vec{v}}{v}\cdot\left[\frac{2}{v^2}\frac{\partial F^0}{\partial v}\left(\frac{\partial \vec{G}_t^1}{\partial v} - \frac{1}{v}\vec{G}_t^1\right)\right.$$

$$\left. + \frac{2}{v^2}\frac{\partial G_t^0}{\partial v}\left(\frac{\partial \vec{F}^1}{\partial v} - \frac{1}{v}\vec{F}^1\right) + \frac{\partial^2 F^0}{\partial v^2}\frac{\partial^2 \vec{G}_t^1}{\partial v^2} + \frac{\partial^2 G_t^0}{\partial v^2}\frac{\partial^2 \vec{F}^1}{\partial v^2}\right] \tag{78}$$

When equations (57), (77), and (78) are substituted into equation (21), the Fokker-Planck equation can finally be decomposed into the following two component equations for F^0 and $\vec{F}^1$:

$$\left(\frac{\partial F^0}{\partial t}\right)_{cc} = \sum_f\left[4\pi\frac{m_t}{m_f}\Gamma_{tf}F_f^0F^0 + \frac{\partial F^0}{\partial v}\left(\frac{m_f - m_t}{m_f + m_t}\frac{\partial H_f^0}{\partial v} + \frac{1}{v^2}\frac{\partial G_f^0}{\partial v}\right) + \frac{1}{2}\frac{\partial^2 G_f^0}{\partial v^2}\frac{\partial^2 F^0}{\partial v^2}\right] \tag{79}$$

$$\left(\frac{\partial \vec{F}^1}{\partial t}\right)_{cc} = \sum_f\left[\frac{1}{2}\frac{\partial^2 G_f^0}{\partial v^2}\frac{\partial^2 \vec{F}^1}{\partial v^2} + \frac{\partial \vec{F}^1}{\partial v}\left(\frac{m_f - m_t}{m_f + m_t}\frac{\partial H_f^0}{\partial v} + \frac{1}{v^2}\frac{\partial G_f^0}{\partial v}\right)\right.$$

$$+ \vec{F}^1\left(4\pi\frac{m_t}{m_f}\Gamma_{tf}F_f^0 - \frac{1}{v^3}\frac{\partial G_f^0}{\partial v}\right) + \frac{1}{2}\frac{\partial^2 F^0}{\partial v^2}\frac{\partial^2 \vec{G}_f^1}{\partial v^2}$$

$$\left. + \frac{\partial F^0}{\partial v}\left(\frac{m_f - m_t}{m_f + m_t}\frac{\partial \vec{H}_f^1}{\partial v} + \frac{1}{v^2}\frac{\partial \vec{G}_f^1}{\partial v} - \frac{1}{v^3}\vec{G}_f^1\right) + 4\pi\frac{m_t}{m_f}\Gamma_{tf}F^0\vec{F}_f^1\right] \tag{80}$$

COMPARISON WITH OTHER WORK

Equations (79) and (80) are not exactly in the form given by Dreicer (compare eqs. (18) and (19) of ref. 3). Dreicer evidently eliminated the potential H by means of relations (17) and (18). To compare the results obtained herein with Dreicer's, it is necessary to do the same thing. From equation (17)

$$H_f = \frac{m_t + m_f}{2m_f}\nabla_v^2 G_f \tag{81}$$

The expansions for H_f (eq. (40)) and for G_f (eq. (46)) are substituted into equation (81), and the Laplacian of G_f is calculated. When equation (74) is used and the coefficients of Y_{lm} on both sides of equation (81) are equated, the result is

$$H_f^{lm}(v) = \frac{m_t + m_f}{2m_f}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right) - \frac{l(l+1)}{v^2}\right] G_f^{lm}(v) \tag{82}$$

In particular,

$$H_f^0(v) = \left(\frac{m_t + m_f}{2m_f}\right)\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right) G_f^0(v) \tag{83}$$

$$\vec{H}_f^1(v) = \frac{m_t + m_f}{2m_f}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right) - \frac{2}{v^2}\right]\vec{G}_f^1(v) \tag{84}$$

Equation (83) is substituted for H_f^0 in equation (79), which, after some rearrangement, can be written as

$$\left(\frac{\partial F^0}{\partial t}\right)_{cc} = \sum_f\left[4\pi b\Gamma_{tf}F_f^0F^0 + \frac{\partial F^0}{\partial v}\left(\frac{b}{v^2}\frac{\partial G_f^0}{\partial v} + \frac{2c}{v}\frac{\partial^2 G_f^0}{\partial v^2} + c\,\frac{\partial^3 G_f^0}{\partial v^3}\right) + \frac{1}{2}\frac{\partial^2 G_f^0}{\partial v^2}\frac{\partial^2 F^0}{\partial v^2}\right] \tag{85}$$

where

$$b = \frac{m_t}{m_f} \tag{86}$$

$$c = \frac{m_f - m_t}{2m_f} \tag{87}$$

Equation (85) is exactly the same as Dreicer's equation (18). It is not clear why he chooses to eliminate H_f in favor of G_f, since higher derivatives are obtained in this manner.

In order to compare equation (80) with Dreicer's equation (19), the following relation must be used:

$$-8\pi\Gamma_{tf}F_f(\vec{v}) = \nabla_v^4 G_f \tag{88}$$

which follows from equation (18). When the expansions of F_f and G_f (eqs. (23) and (46), respectively) are substituted in equation (88), the Laplacian operation performed on G_f twice, with the use of equation (74) each time, and finally the coefficients of Y_{lm} on each side of equation (88) are set equal to each other; the following result is obtained:

$$- 8\pi\Gamma_{tf} F_f^{lm}(v) = \left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right) - \frac{l(l+1)}{v^2}\right]^2 G_f^{lm}(v) \tag{89}$$

In particular, the relations that are needed are

$$- 8\pi\Gamma_{tf} F_f^{0}(v) = \left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right)\right]^2 G_f^{0}(v) \tag{90}$$

$$- 8\pi\Gamma_{tf} \vec{F}_f^{1}(v) = \left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right) - \frac{2}{v^2}\right]^2 \vec{G}_f^{1}(v) \tag{91}$$

Equations (90) and (91) are substituted for F_f^0 and $\vec{F}_f^1$ in equation (80). Also, H_f^0 and $\vec{H}_f^1$ are eliminated by means of equations (83) and (84). After some manipulation, equation (80) becomes

$$\begin{aligned}\left(\frac{\partial\vec{F}^1}{\partial t}\right)_{cc} = \sum_f \Bigg[&\frac{1}{2}\frac{\partial^2 G_f^0}{\partial v^2}\frac{\partial^2\vec{F}^1}{\partial v^2} + \frac{\partial\vec{F}^1}{\partial v}\left(c\frac{\partial^3 G_f^0}{\partial v^3} + \frac{2c}{v}\frac{\partial^2 G_f^0}{\partial v^2} + \frac{b}{v^2}\frac{\partial G_f^0}{\partial v}\right) \\ &- \vec{F}^1\left(\frac{b}{2}\frac{\partial^4 G_f^0}{\partial v^4} + \frac{2b}{v}\frac{\partial^3 G_f^0}{\partial v^3} + \frac{1}{v^3}\frac{\partial G_f^0}{\partial v}\right) + \frac{1}{2}\frac{\partial^2 F^0}{\partial v^2}\frac{\partial^2\vec{G}_f^1}{\partial v^2} \\ &+ \frac{\partial F^0}{\partial v}\left(c\frac{\partial^3\vec{G}_f^1}{\partial v^3} + \frac{2c}{v}\frac{\partial^2\vec{G}_f^1}{\partial v^2} + \frac{2b-1}{v^2}\frac{\partial\vec{G}_f^1}{\partial v} + \frac{1-2b}{v^3}\vec{G}_f^1\right) \\ &- F^0\left(\frac{b}{2}\frac{\partial^4\vec{G}_f^1}{\partial v^4} + \frac{2b}{v}\frac{\partial^3\vec{G}_f^1}{\partial v^3} - \frac{2b}{v^2}\frac{\partial^2\vec{G}_f^1}{\partial v^2}\right)\Bigg]\end{aligned} \tag{92}$$

which is in the form of equation (19) of reference 3. It would appear that equation (80) is more convenient than equation (92), since the latter contains higher derivatives of the G functions.

Comparison of equation (92) with Dreicer's equation (19) discloses two differences. First, and more important, the last term in Dreicer's equation, $-(4/v^3)\partial G_f^1/\partial v$, was not obtained herein. Since the calculations were checked by a number of different methods and the term in question was unobtainable, it is believed to be incorrect. The second difference is evidently a misprint in the term $(1/2)(\partial^2 F^0/\partial v^2)(\partial^2\vec{G}_f^1/\partial v^2)$, where Dreicer has F_f^0 instead of F^0.

Shkarofsky (ref. 4) quotes Fokker-Planck collision terms similar to equations (79) and (80), but specialized to the case of electron-electron collisions. A comparison of the present results with those of reference 4 in that special case showed exact agreement for $(\partial\vec{F}^1/\partial t)_{ee}$ (ref. 4, eq. (5b)), but wide differences in the expression for $(\partial F^0/\partial t)_{ee}$ (ref. 4, eq. (5a)). Shkarofsky's equation (5a) is in fact dimensionally incorrect, and the term containing the second derivative of F^0 is missing from it.

Finally, some comments on an early paper by Cahn (ref. 5) may be of interest, because he apparently managed to obtain an electron-electron collision term in a form that enabled him to integrate the Boltzmann equation and express the distribution function by means of a single integral. If this result were correct, it would be quite significant due to its simplicity; however, it is believed that Cahn's result is not correct. The reason has to do with the way in which Cahn carried out integrations over the velocities $\vec{v}'$ of the field particles. In reference 5 (p. 298), the lower limit of integration on the relative speed $|\vec{v}' - \vec{v}|$ is quoted as $v' - v$. Actually this should be $|v' - v|$; otherwise, it is incorrect in the case where $v' < v$, because the relative speed must be positive. The result of the angular integrations is therefore different, depending on whether $v' < v$ or $v' > v$. The integration over v' from 0 to ∞ then breaks into two integrals with different integrands, one integral from 0 to v and the other from v to ∞, as obtained in this report (see eqs. (41) and (47)) and references 1 and 3. Cahn has only one integral, from 0 to ∞. For instance, the following result is given in reference 5 (p. 299):

$$J_i^{00} = \frac{8\pi}{3} v_i \mathcal{L} \int_0^\infty U^{00} v' dv'$$

where Cahn's notation need not be explained in detail here, except to note that $\mathcal{L}$ is a constant related to Γ_{tf}, while U^{00} involves the distribution functions of the field and test particles and their derivatives. The result should be

$$J_i^{00} = \frac{8\pi}{3} v_i \mathcal{L} \left[\int_0^v U^{00} \frac{v'^4}{v^3} dv' + \int_v^\infty U^{00} v' dv' \right]$$

This difference is important, because Cahn's electron-electron collision term is the divergence of the vector J_i, and a number of extra terms would appear from the differentiation of the integrals having variable limits. These extra terms complicate the result considerably, and there is no longer a possibility of obtaining the distribution function F^0 by simple quadratures, as claimed by Cahn.

CONCLUDING REMARKS

The first two terms in the spherical harmonics expansion of the Fokker-Planck equation have been obtained. The results are nonlinear integro-differential equations whose variables are the speed v of the test particle and the time t. Although neither these results nor the general method of deriving them is entirely new, the fairly complete and straightforward derivations presented herein resolves the errors and differences that exist in previously published results. Although the Fokker-Planck equation is the best presently available means of including the effect of electron-electron interactions on the distribution function, it is still rather intractable to all but numerical means. A simpler electron-electron collision model, which would bear the same relation to the Fokker-Planck collision term as the Krook model does to the Boltzmann collision integral, would be invaluable for workers in the field of plasma physics.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, January 8, 1965.

APPENDIX A

SYMBOLS

The mks system of units is used throughout this report

a_i	i^{th} component of acceleration
B	magnetic field
b	m_t/m_f
b_o	impact parameter for 90^o scattering, eq. (14)
c	$(m_f - m_t)/2m_f$
d	Debye radius, eq. (13)
E	electric field
e	electron charge $(e = 1.602 \times 10^{-19}$ C)
e_f	electric charge of field particle, $\pm Z_f e$
e_t	electric charge of test particle, $\pm Z_t e$
F	distribution function of test particles
F_f	distribution function of field particles
G	Rosenbluth potential, defined in eq. (10)
g	relative speed of test and field particles, eq. (3)
$\vec{g}$	relative velocity, $\vec{v} - \vec{v}'$
g^{ij}	conjugate of metric tensor, eq. (B5)
g_{ij}	metric tensor
H	Rosenbluth potential, defined in eq. (9)
$\left\{ {h \atop ij} \right\}$	Christoffel symbol of second kind, eq. (B2)
Im[ ]	imaginary part of
$\hat{i},\hat{j},\hat{k}$	unit vectors along Cartesian coordinate axes

$[ij,k]$	Christoffel symbol of first kind, eq. (B3)
k	Boltzmann constant, 1.3804×10^{-23} J/°K
m_f	mass of field particle
m_t	mass of test particle
m_{tf}	reduced mass, $m_t m_f/(m_t + m_f)$
N	particle number density, m^{-3}
P_l	Legendre polynomial of order l
P_l^m	associated Legendre polynomial, defined by eq. (24b)
Re[ ]	real part of
$\vec{r}$	position vector
s	arc length in velocity space
T	temperature
t	time
u^i	general curvilinear coordinate in velocity space
v	magnitude of velocity of test particle, $(v_i v_i)^{1/2}$
v'	magnitude of velocity of field particle, $(v_i' v_i')^{1/2}$
$\vec{v}$	velocity vector of test particle
$\vec{v}'$	velocity of field particle
$\vec{v}_d$	drift velocity of test particles
v_i	i^{th} component of velocity of test particle
Δv_i	increment in v_i due to collision
v_i'	i^{th} component of velocity of field particle
$v_>$	larger of (v,v')
$v_<$	smaller of (v,v')
x_i	i^{th} component of position vector $\vec{r}$
Y_{lm}	spherical harmonic function defined by eq. (24a)

Z	charge number
Γ_{tf}	defined by eq. (11)
γ	angle between $\vec{v}$ and $\vec{v}'$
δ_{ij}, δ^i_j	Kronecker deltas in Cartesian and general coordinates, respectively
ϵ_o	dielectric permittivity of vacuum, 8.854×10^{-12} F/m
ζ	perturbation in G_t, eq. (56)
η	perturbation in H_f, eq. (55)
θ	polar angle in velocity space
Λ_{tf}	defined by eq. (12)
μ	$\cos\theta$
ξ	perturbation in F, eq. (22)
ξ_f	perturbation in F_f, eq. (54)
σ	differential scattering cross section
φ	azimuthal angle in velocity space
χ	scattering angle in center of mass coordinates
Ω	scattering solid angle
$\langle\ \rangle$	average value
∇_v	gradient operator in velocity space
∇^2_v	Laplacian operator in velocity space

Subscripts:

c	collision
cc	Coulomb collision
ee	electron-electron collision
f	field particle
h,i,j,k	tensor indices of covariant vectors and tensors
s	species s

t	test particle
,	denotes covariant differentiation

Superscripts:

h,i,j,k	tensor indices of contravariant vectors and tensors
l	index denoting term in Legendre polynomial expansion
lm	indices denoting term in spherical harmonics expansion
m	index denoting term in azimuthal angle expansion
0	zero-order terms
1	first-order terms
*	complex conjugate
$\rightarrow$	vector
$\wedge$	unit vector

APPENDIX B

TRANSFORMATION OF THE FOKKER-PLANCK EQUATION TO SPHERICAL COORDINATES

In this appendix, equation (21) is transformed to the spherical coordinates (v,θ,φ) in velocity space. This transformation was performed on the more complicated equation (19) in reference 1 for the special case of axial symmetry. Basically the same method is used, namely, transformation to a general curvilinear coordinate system and the specialization of the result to spherical coordinates. Axial symmetry is not assumed, however.

Let the general curvilinear coordinates be denoted as u^1, u^2, u^3. First the term $(\partial^2 F/\partial v_i \partial v_j)$, which was written as a Cartesian tensor in equation (21), will be transformed into a second-order covariant tensor. Since F is a scalar, the proper extension of $(\partial^2 F/\partial v_i \partial v_j)$ to general coordinates is the second covariant derivative of F (ref. 16, p. 32, eq. (22.7)); that is,

$$F_{,ij} = \frac{\partial^2 F}{\partial u^i \partial u^j} - \begin{Bmatrix} h \\ ij \end{Bmatrix} \frac{\partial F}{\partial u^h} \tag{B1}$$

where (ref. 16, p. 26)

$$\begin{Bmatrix} h \\ ij \end{Bmatrix} = g^{hk}[ij,k] \tag{B2}$$

and

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \tag{B3}$$

are the Christoffel symbols of the second and first kinds, respectively. The quantities g_{ij} are the elements of the metric tensor; that is, the element of distance ds between two points whose coordinates differ by du^1, du^2, du^3 is

$$ds^2 = g_{ij} du^i du^j \tag{B4}$$

and g^{ij} is the contravariant tensor conjugate to g_{ij}, defined such that

$$g^{ij} g_{jk} = \delta^i_k \tag{B5}$$

where δ^i_k is the Kronecker delta.

In Cartesian coordinates, $(\partial^2 G_t/\partial v_i \partial v_j)$ is defined such that

$$\langle \Delta v_i \Delta v_j \rangle = \frac{\partial^2 G_t}{\partial v_i \partial v_j} \tag{B6}$$

but, in general curvilinear coordinates, this must be written as a second-order contravariant tensor, because the velocity u^i and velocity increment Δu^i are contravariant vectors. Thus the proper extension of equation (B6) to curvilinear coordinates is

$$\langle \Delta u^i \Delta u^j \rangle = g^{ih} g^{jk} G_{t,hk} \tag{B7}$$

where the nontensor subscript t on G should not be confused with the tensor indices h,i,j,k.

The term $(\partial F/\partial v_i)(\partial H_f/\partial v_i)$ in equation (21) can be transformed easily, since the first covariant derivative of F is simply

$$F_{,i} = \frac{\partial F}{\partial u^i} \tag{B8}$$

and the quantity $\langle \Delta v_i \rangle = \partial H_t/\partial v_i$ defined in equation (7), when written as a contravariant tensor, becomes

$$\langle \Delta u^i \rangle = g^{ij} H_{t,j} \tag{B9}$$

In general curvilinear coordinates, equation (21) becomes

$$\left(\frac{\partial F}{\partial t}\right)_{cc} = 4\pi \sum_f \frac{m_t}{m_f} \Gamma_{tf} F_f(\vec{v}) F(\vec{v})$$

$$- g^{ij} \frac{\partial F}{\partial u^i} \frac{\partial}{\partial u^j} \sum_f \frac{m_t - m_f}{m_t + m_f} H_f + \frac{1}{2} g^{ih} g^{jk} F_{,ij} G_{t,hk} \tag{B10}$$

For spherical coordinates the element of distance ds is given by

$$ds^2 = dv^2 + v^2\, d\theta^2 + v^2 \sin^2\theta\, d\varphi^2 = dv^2 + \frac{v^2}{1 - \mu^2} d\mu^2 + v^2(1 - \mu^2) d\varphi^2 \tag{B11}$$

where for convenience the coordinate

$$\mu \equiv \cos \theta \tag{B12}$$

will be used in place of θ, as in references 1 and 3. Hence, if

$$u^1 \equiv v,\ u^2 \equiv \mu,\ u^3 \equiv \varphi \tag{B13}$$

it follows from equations (B4), (B5), and (B11) that

$$g_{11} = \frac{1}{g^{11}} = 1 \tag{B14a}$$

$$g_{22} = \frac{1}{g^{22}} = \frac{v^2}{1 - \mu^2} \tag{B14b}$$

$$g_{33} = \frac{1}{g^{33}} = v^2(1 - \mu^2) \tag{B14c}$$

$$g_{ij} = g^{ij} = 0 \qquad (i \neq j) \tag{B14d}$$

Equation (B14d), which is true for any orthogonal coordinates, may be used to considerably simplify expressions (B2) and (B3) for the Christoffel symbols (see ref. 16, p. 28, Ex. 2). For spherical coordinates, it then follows from equations (B2), (B3), and (B14) that

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\frac{v}{1 - \mu^2} \tag{B15a}$$

$$\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = -\begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = -\begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \frac{\mu}{1 - \mu^2} \tag{B15b}$$

$$\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -v(1 - \mu^2) \tag{B15c}$$

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \mu(1 - \mu^2) \tag{B15d}$$

$$\begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{v} \tag{B15e}$$

all other Christoffel symbols being zero. Substitution of equations (B15) into equation (B1) yields

$$F_{,11} = \frac{\partial^2 F}{\partial v^2} \tag{B16a}$$

$$F_{,22} = \frac{\partial^2 F}{\partial\mu^2} + \frac{v}{1 - \mu^2}\frac{\partial F}{\partial v} - \frac{\mu}{1 - \mu^2}\frac{\partial F}{\partial\mu} \tag{B16b}$$

$$F_{,33} = \frac{\partial^2 F}{\partial\varphi^2} + v(1 - \mu^2)\frac{\partial F}{\partial v} - \mu(1 - \mu^2)\frac{\partial F}{\partial\mu} \tag{B16c}$$

$$F_{,12} = F_{,21} = \frac{\partial^2 F}{\partial v\partial\mu} - \frac{1}{v}\frac{\partial F}{\partial\mu} \tag{B16d}$$

$$F_{,13} = F_{,31} = \frac{\partial^2 F}{\partial v\partial\varphi} - \frac{1}{v}\frac{\partial F}{\partial\varphi} \tag{B16e}$$

$$F_{,23} = F_{,32} = \frac{\partial^2 F}{\partial\mu\partial\varphi} + \frac{\mu}{1 - \mu^2}\frac{\partial F}{\partial\varphi} \tag{B16f}$$

The expressions for the components of $G_{t,ij}$ are of course analogous to the formulas of equation (B16). Substituting equations (B14) and (B16) into equation (B10) results in

$$\begin{aligned}
\left(\frac{\partial F}{\partial t}\right)_{cc} = \sum_f \Bigg\{ & 4\pi\frac{m_t}{m_f}\Gamma_{tf}F_f(\vec{v})F(\vec{v}) + \frac{m_f - m_t}{m_t + m_f}\left[\frac{\partial F}{\partial v}\frac{\partial H_f}{\partial v} + \frac{1 - \mu^2}{v^2}\frac{\partial F}{\partial\mu}\frac{\partial H_f}{\partial\mu} + \frac{1}{v^2(1 - \mu^2)}\frac{\partial F}{\partial\varphi}\frac{\partial H_f}{\partial\varphi}\right] \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}\frac{\partial^2 G_f}{\partial v^2} + \frac{(1 - \mu^2)^2}{2v^4}\left(\frac{\partial^2 F}{\partial\mu^2} + \frac{v}{1 - \mu^2}\frac{\partial F}{\partial v} - \frac{\mu}{1 - \mu^2}\frac{\partial F}{\partial\mu}\right)\left(\frac{\partial^2 G_f}{\partial\mu^2} + \frac{v}{1 - \mu^2}\frac{\partial G_f}{\partial v} - \frac{\mu}{1 - \mu^2}\frac{\partial G_f}{\partial\mu}\right) \\
& + \frac{1}{2v^4(1 - \mu^2)^2}\left[\frac{\partial^2 F}{\partial\varphi^2} + v(1 - \mu^2)\frac{\partial F}{\partial v} - \mu(1 - \mu^2)\frac{\partial F}{\partial\mu}\right]\left[\frac{\partial^2 G_f}{\partial\varphi^2} + v(1 - \mu^2)\frac{\partial G_f}{\partial v} - \mu(1 - \mu^2)\frac{\partial G_f}{\partial\mu}\right] \\
& + \frac{1 - \mu^2}{v^2}\left(\frac{\partial^2 F}{\partial v\partial\mu} - \frac{1}{v}\frac{\partial F}{\partial\mu}\right)\left(\frac{\partial^2 G_f}{\partial v\partial\mu} - \frac{1}{v}\frac{\partial G_f}{\partial\mu}\right) + \frac{1}{v^2(1 - \mu^2)}\left(\frac{\partial^2 F}{\partial v\partial\varphi} - \frac{1}{v}\frac{\partial F}{\partial\varphi}\right)\left(\frac{\partial^2 G_f}{\partial v\partial\varphi} - \frac{1}{v}\frac{\partial G_f}{\partial\varphi}\right) \\
& + \frac{1}{v^4}\left(\frac{\partial^2 F}{\partial\mu\partial\varphi} + \frac{\mu}{1 - \mu^2}\frac{\partial F}{\partial\varphi}\right)\left(\frac{\partial^2 G_f}{\partial\mu\partial\varphi} + \frac{\mu}{1 - \mu^2}\frac{\partial G_f}{\partial\varphi}\right)\Bigg\}
\end{aligned} \tag{B17}$$

Equation (B17) is the Fokker-Planck equation in spherical coordinates. It is possible to utilize it to obtain the general term in the spherical harmonics expansion of $(\partial F/\partial t)_{cc}$. This expansion could in principle be obtained by substituting the following expansion in equation (B17):

$$F(\vec{v}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F^{lm}(v) Y_{lm}(\theta,\varphi) \tag{23}$$

together with the analogous expansions for F_f, H_f, and G_f. The recurrence relations for the functions Y_{lm} (ref. 17, p. 181) would be utilized to express their derivatives and the products of their derivatives with the quantities μ, $(1 - \mu^2)^{1/2}$, and so forth, in terms of the functions Y_{lm} themselves. It would then be necessary to expand products of spherical harmonics as follows:

$$Y_{l_1 m_1}(\theta,\varphi) Y_{l_2 m_2}(\theta,\varphi) = \sum_{l=0}^{\infty} C_{l_1 m_1 l_2 m_2, l} Y_{l, m_1+m_2}(\theta,\varphi) \tag{B18}$$

where the constants $C_{l_1 m_1 l_2 m_2, l}$ are related to the Clebsch-Gordon coefficients (ref. 17, p. 514). Finally, the result would have to be rearranged so as to separate out the various terms in Y_{lm}. Although the procedure is straightforward, the amount of calculation would be rather large, and the usefulness of the general term in the expansion would probably not justify the labor in obtaining it. For this reason the present report is restricted to the first two terms ($l = 0,1$) in the spherical harmonics expansion of the Fokker-Planck equation. The nonlinearity of equation (B17) is responsible for the difficulty in working out the general term of the expansion. The expansion of a much simpler linear collision term appropriate for collisions with infinitely massive field particles (the so-called Lorentz model) has been carried out in reference 18. Reference 18 also contains the spherical harmonics expansion of the electric and magnetic field terms and the spatial gradient terms in the Boltzmann equation.

REFERENCES

1. Rosenbluth, Marshall N.; MacDonald, William M.; and Judd, David L.: Fokker-Planck Equation for an Inverse-Square Force. Phys. Rev., vol. 107, no. 1, July 1, 1957, pp. 1-6.

2. MacDonald, W. M.; Rosenbluth, M. N.; and Chuck, Wong: Relaxation of a System of Particles with Coulomb Interactions. Phys. Rev., vol. 107, no. 2, July 15, 1957, pp. 350-353.

3. Dreicer, H.: Electron Velocity Distributions in a Partially Ionized Gas. Phys. Rev., vol. 117, no. 2, Jan. 15, 1960, pp. 343-354.

4. Shkarofsky, I. P.: Values of the Transport Coefficients in a Plasma for Any Degree of Ionization Based on a Maxwellian Distribution. Can. J. Phys., vol. 39, no. 11, Nov. 1961, pp. 1619-1703.

5. Cahn, Julius H.: Electronic Interaction in Electrical Discharges in Gases. Phys. Rev., vol. 75, no. 2, Jan. 15, 1949, pp. 293-300.

6. Montgomery, D. C.; and Tidman, D. A.: Plasma Kinetic Theory. McGraw-Hill Book Co., Inc., 1964.

7. Longmire, Conrad L.: Elementary Plasma Physics. Intersci. Pub., 1963.

8. Jackson, John David: Classical Electrodynamics. John Wiley & Sons, Inc., 1962.

9. Johnston, T. Wyatt: Cartesian Tensor Scalar Product and Spherical Harmonic Expansions in Boltzmann's Equation. Phys. Rev., vol. 120, no. 4, Nov. 15, 1960, pp. 1103-1111.

10. Allis, W. P.: Motions of Ions and Electrons. Handbuch der Physik, S. Flügge, ed., vol. XXI, Springer-Verlag (Berlin), 1956, pp. 383-444.

11. Lorentz, H. A.: The Theory of Electrons. Second ed., Dover Pub., 1952, p. 272.

12. Cohen, Robert S.; Spitzer, Lyman, Jr.; and Routly, Paul McR.: The Electrical Conductivity of an Ionized Gas. Phys. Rev., vol. 80, no. 2, Oct. 15, 1950, pp. 230-238.

13. Hwa, Rudolph C.: Effects of Electron-Electron Interactions on Cyclotron Resonances in Gaseous Plasmas. Phys. Rev., vol. 110, no. 2, Apr. 15, 1958, pp. 307-313.

14. Balescu, R.: Statistical Mechanics of Charged Particles. Intersci. Pub., 1963, sec. 37-38, pp. 181-191.

15. Magnus, Wilhelm; and Oberhettinger, Fritz: Formulas and Theorems for the Special Functions of Mathematical Physics. Chelsea Pub. Co., 1954.

16. Spain, Barry: Tensor Calculus. Intersci. Pub., 1956.

17. Merzbacher, Eugen: Quantum Mechanics. John Wiley & Sons, Inc., 1961.

18. Carpenter, Chester P.; and Mezger, Fritz W.: General Spherical Harmonics Formulation of Plasma Boltzmann Equation. J. Mathematical Phys., vol. 2, no. 5, Sept.-Oct. 1961, pp. 694-701.

NASA-Langley, 1965 E-2814

2/22/85
98

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546