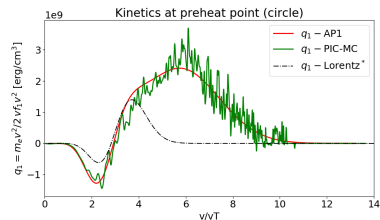
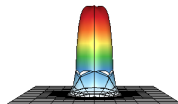
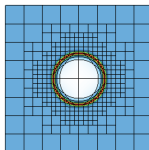


Multi-dimensional High-Order FEM Methods for Electron Kinetics in ALE Hydrodynamics

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1 Motivation - Nonlocal Magneto-Hydrodynamic model (Nonlocal-MHD)

2 S_N discontinuous Galerkin finite element approach to kinetics

3 P_N mixed finite element approach to kinetics

4 Conclusions

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Classical MHD

HYDRODYNAMICS

$$local \rightarrow \mathbf{q}_e = -\kappa_{SH} T_e^{2.5} \nabla T_e$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u},$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla(p_i + p_e) + \frac{c}{4\pi} \nabla \times \mathbf{B} \times \mathbf{B},$$

$$\rho \left(\frac{\partial \varepsilon_i}{\partial T_i} \frac{dT_i}{dt} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} \right) = -p_i \nabla \cdot \mathbf{u} - G(T_i - T_e),$$

$$\rho \left(\frac{\partial \varepsilon_e}{\partial T_e} \frac{dT_e}{dt} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} \right) = -p_e \nabla \cdot \mathbf{u} + \nabla \cdot (\kappa_{SH} T_e^{2.5} \nabla T_e) + \sigma \mathbf{E} \cdot \mathbf{E} + G(T_i - T_e) + Q_{IB}(\mathbf{E}_L),$$

MAXWELL EQUATIONS

$$resistive \rightarrow \mathbf{j} = \sigma \mathbf{E}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{d\mathbf{B}}{dt},$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sigma \mathbf{E},$$

KINETICS OF ELECTRONS

Landau – Fokker – Planck

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \int \nabla_v \nabla_v (\mathbf{v} - \tilde{\mathbf{v}}) \cdot (f \nabla_{\tilde{\mathbf{v}}} f - f \nabla_v f) d\tilde{\mathbf{v}} + \frac{\nu_{ei}}{2} \frac{\partial^2 f}{\partial \Omega^2}.$$

Nonlocal-MHD

HYDRODYNAMICS

4D

$$\begin{aligned}
 \frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u}, \\
 \rho \frac{d\mathbf{u}}{dt} &= -\nabla(p_i + p_e) + \mathbf{j}(f, \mathbf{E}) \times \mathbf{B}, \\
 \rho \left(\frac{\partial \varepsilon_i}{\partial T_i} \frac{dT_i}{dt} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_i \nabla \cdot \mathbf{u} - G(T_i - T_e), \\
 \rho \left(\frac{\partial \varepsilon_e}{\partial T_e} \frac{dT_e}{dt} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_e \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q}_e(f) + \mathbf{j}(f, \mathbf{E}) \cdot \mathbf{E} + G(T_i - T_e) + Q_{IB}(\mathbf{E}_L),
 \end{aligned}$$

MAXWELL EQUATIONS

4D

$$\begin{aligned}
 \nabla \times \mathbf{E} &= -\frac{1}{c} \frac{d\mathbf{B}}{dt}, \\
 \nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{j}(f, \mathbf{E}),
 \end{aligned}$$

KINETICS OF ELECTRONS

6D

$$\mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \nu \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f).$$

3D velocity space FEM discretizations

AWBS electron kinetic model 7D

$$C_V \frac{dT_e}{dt} = -\nabla \cdot \mathbf{q}_e(f) + \mathbf{j}(f) \cdot \mathbf{E} + S_H,$$

$$\mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \nu \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f).$$

- C7 (S_N - discontinuous Galerkin FEM) 3D velocity space in spherical coordinates (v, ϕ, θ)

$$\mathbf{n} \cdot \nabla f + \frac{\mathbf{E} \cdot \mathbf{n}}{v} \frac{\partial f}{\partial v} + \frac{E_\phi - v B_\theta}{v^2} \frac{\partial f}{\partial \phi} + \frac{E_\theta + v B_\phi}{v^2 \sin(\phi)} \frac{\partial f}{\partial \theta} = \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_M) + \frac{\nu_{scat}}{v} (f_0 - f).$$

- AP1 (P_N (VEF $\xi = 1/3$) - continuous Galerkin mixed FEM) *Physicists like it!*

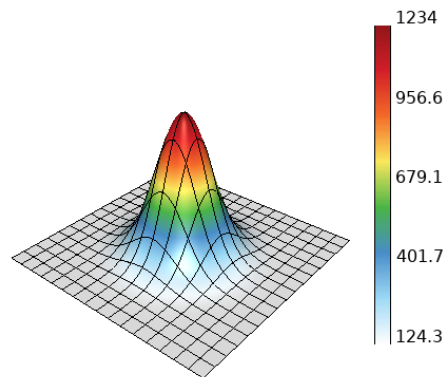
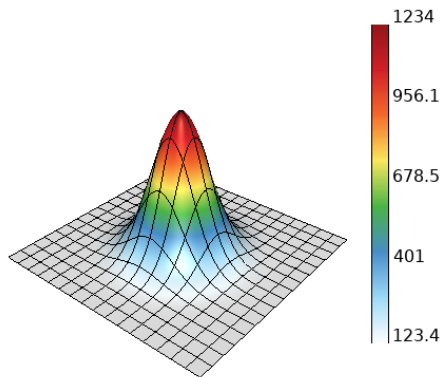
$$\xi \nabla \cdot \mathbf{f}_1 + \xi \frac{q_e}{m_e v} \mathbf{E} \cdot \left(\frac{\partial \mathbf{f}_1}{\partial v} + \frac{2}{v} \mathbf{f}_1 \right) = \tilde{\nu}_e \frac{\partial}{\partial v} (f_0 - f_M),$$

$$\nabla f_0 + \frac{q_e}{m_e v} \mathbf{E} \frac{\partial f_0}{\partial v} + \frac{q_e \mathbf{B}}{m_e c v} \times \mathbf{f}_1 = \tilde{\nu}_e \frac{\partial \mathbf{f}_1}{\partial v} - \frac{\nu_{scat}}{v} \mathbf{f}_1,$$

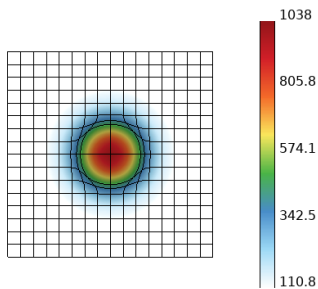
S_N upwind DG

$$\mathbf{M}_{(\tilde{\nu}_e - \frac{\mathbf{E} \cdot \mathbf{n}_d}{v})} \cdot \frac{\Delta \mathbf{f}_d}{\Delta v} = (\mathbf{n}_d \cdot \mathbf{G} + \mathbf{F}_d) \cdot (\tilde{\mathbf{f}}_d + \Delta \mathbf{f}_d) + \mathbf{M}_{(\frac{\nu_{scat}}{v})} \cdot (\tilde{\mathbf{f}}_d + \Delta \mathbf{f}_d) + \mathbf{S}_{(\nu_{scat}, \mathbf{E}, \mathbf{B})} \cdot \tilde{\mathbf{f}} + \mathbf{S}_{(\tilde{\nu}_e \frac{\partial \mathbf{f}_M}{\partial v})}$$

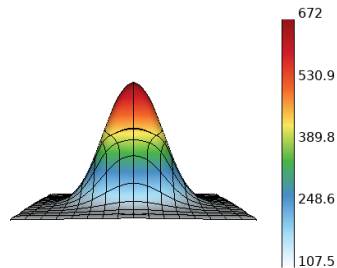
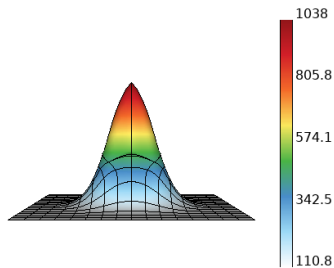
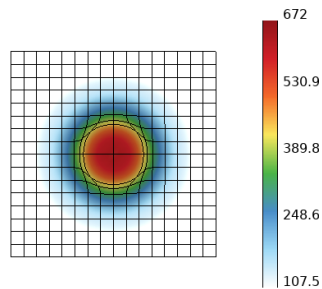
Velocity groups	32	64	128	256	order
Backward Euler	1.595e-1	8.618e-2	4.456e-2	2.264e-2	0.98
SDIRK2	3.217e-2	8.888e-3	2.322e-3	5.924e-3	1.97
SDIRK3	2.455e-2	3.639e-3	4.887e-4	6.317e-05	2.96
Hydro temperature	Kinetic temperature (50 groups)				



Local hydro temperature



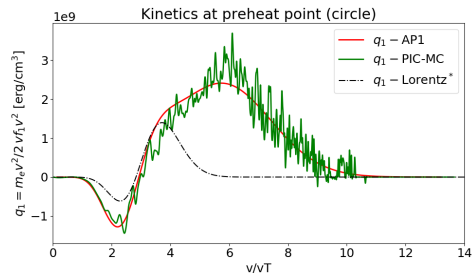
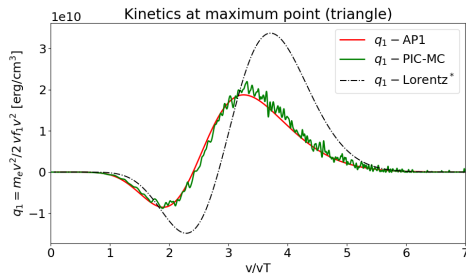
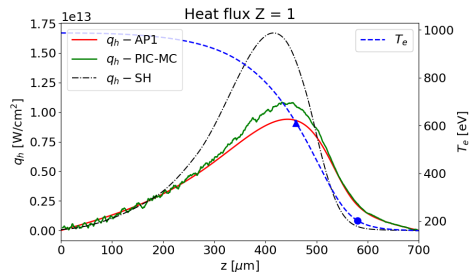
Nonlocal kinetic temperature



AP1 formulation PCG(AMG), fixed P1 angular discretization

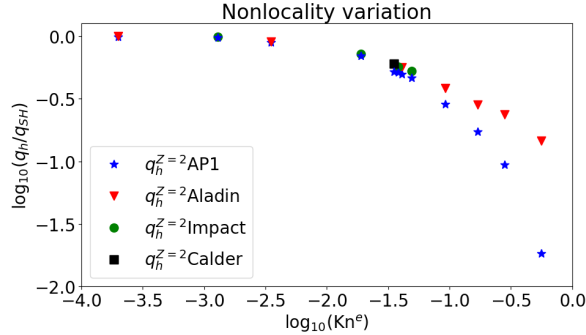
$$\mathbf{M}_{(\tilde{\nu}_e)}^{L_2} \cdot \frac{d\mathbf{f}_0}{dv} - \mathbf{V}_{(\frac{\xi q_e E}{m_e v})}^{L_2} \cdot \frac{d\mathbf{f}_1}{dv} = \mathbf{D}_{(\xi)}^{L_2} \cdot \mathbf{f}_1 + \mathbf{M}_{(\frac{\xi 2 q_e E}{m_e v^2})}^{L_2} \cdot \mathbf{f}_1 + \mathbf{S}_{(\tilde{\nu}_e \frac{\partial f_M}{\partial v})}^{L_2},$$

$$\mathbf{M}_{(\tilde{\nu}_e)}^{H_1} \cdot \frac{d\mathbf{f}_1}{dv} - \mathbf{V}_{(\frac{q_e E}{m_e v})}^{H_1} \cdot \frac{d\mathbf{f}_0}{dv} = \mathbf{G}^{H_1} \cdot \mathbf{f}_0 + \mathbf{M}_{(\frac{\nu_{scat}}{v})}^{H_1} \cdot \mathbf{f}_1 + \mathbf{C}_{(\frac{q_e B}{m_e c v} \times)}^{H_1} \cdot \mathbf{f}_1,$$



Electron velocity limit - friction vs. \mathbf{E} stopping

$$\left(\tilde{\nu}_e - \frac{\mathbf{E} \cdot \mathbf{n}}{\nu} \right) \frac{\partial f}{\partial \nu} = \mathbf{n} \cdot \nabla f + \frac{\nu_{scat}}{\nu} (f - f_0) + \frac{E_\phi - \nu B_\theta}{\nu^2} \frac{\partial f}{\partial \phi} + \frac{E_\theta + \nu B_\phi}{\nu^2 \sin(\phi)} \frac{\partial f}{\partial \theta} + \tilde{\nu}_e \frac{\partial f_M}{\partial \nu}.$$



\mathbf{E} stopping overtakes collisions for $\text{Kn} > 0.1$.

$\text{Kn} = \frac{\lambda}{L}$	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1
v_{lim}/v_{th}	70.8	22.4	7.3	3.1	1.8

Adaptive DSA preconditioner for Hydro-Kinetics coupling

AWBS electron kinetic model 7D

$$\begin{aligned}
 C_V \frac{dT_e}{dt} &= -\nabla \cdot \mathbf{q}_e(f) + \mathbf{j}(f) \cdot \mathbf{E} + S_H, \\
 &= \Gamma \int_{\mathbf{v}} \frac{\partial f}{\partial \mathbf{v}} v^2 d\mathbf{v} - \sigma T_e^{-0.5} + S_H, \\
 \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f &= \nu \tilde{\nu}_e \frac{\partial}{\partial \mathbf{v}} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f).
 \end{aligned}$$

Continuum analysis of local ($\text{Kn} \ll 1$) transport regime \rightarrow DIFFUSION

$$C_V \frac{dT_e}{dt} = \nabla \cdot \lambda(T_e^{2.5}) \nabla T_e + O(\text{Kn}^2)$$

Fixed-point iteration with preconditioner $\mathbf{E}(\Delta T) = (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma(\tilde{\mathbf{T}} + \Delta T)^\alpha + \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\mathbf{T}} + \Delta T)^\alpha$

$$\mathbf{M}_{C_V} \cdot \frac{\Delta T_e^{k+1}}{\Delta t} + \mathbf{E}(\Delta T_e^{k+1}) - \mathbf{E}(\Delta T_e^k) = \mathbf{K}(f^k) - \mathbf{M}_\sigma(\tilde{\mathbf{T}}_e + \Delta T_e^k)^\alpha.$$

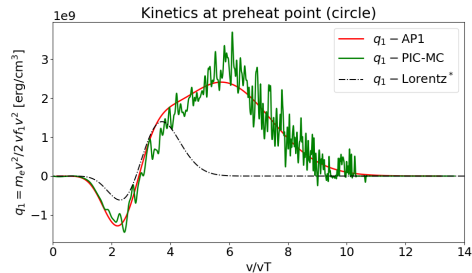
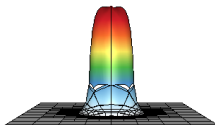
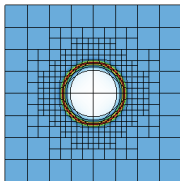
Finally, we get an unconditionally stable backward Euler (SDIRK) fast iterating scheme

$$\mathbf{M}_{C_V} \cdot \frac{\Delta T_e^{k+1}}{\Delta t} + (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma(\tilde{\mathbf{T}} + \Delta T_e^{k+1})^\alpha + \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\mathbf{T}}_e + \Delta T_e^{k+1})^\alpha = \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\mathbf{T}}_e + \Delta T_e^k)^\alpha + \mathbf{K}(f^k) - \mathbf{C} \cdot \mathbf{M}_\sigma(\tilde{\mathbf{T}}_e + \Delta T_e^k)^\alpha,$$

where the adaptive coefficient handles diffusion $\mathbf{C} \xrightarrow{\text{Kn} \ll 1} 1$ and nonlocal transport $\mathbf{C} \xrightarrow{\text{Kn} > 1} 0$ and $\mathbf{C} \in (1, 0)$ in between.

Conclusions

- 7D microscopic world of electrons in hydro simulations.
- S_N high-order DG finite element approach.
- P_1 high-order mixed finite element approach.
- \mathbf{E} field dominated stopping (P_1 fails).
- Adaptive DSA preconditioner for Hydro-Kinetics coupling (deep learning on \mathbf{C}).
- Algebraic Multigrid solver pAIR scales $\log(P)^{1.22}$.



MS85 - Developments in Algebraic Multigrid for Nonsymmetric and Hyperbolic Problems - Ben S. Southworth

<https://github.com/CEED/Laghos/tree/master/amr>

<https://mfem.org>

Nonlocal Ohm's law

We found that the quality of the electric field evaluation is essential for a correct plasma modeling

Nonlocal current in plasma

$$\mathbf{j}(\mathbf{f}, \mathbf{E}) = e \int v \mathbf{f}_1 v^2 dv = e \int v \frac{\nu_{ei}^2 \mathbf{E}^* + \boldsymbol{\omega}_B \boldsymbol{\omega}_B \cdot \mathbf{E}^* + \nu_{ei} \boldsymbol{\omega}_B \times \mathbf{E}^*}{\nu_{ei}(\omega_B^2 + \nu_{ei}^2)} v^2 dv, \quad (1)$$

where $\mathbf{E}^* = \nu_{ei} \frac{\partial \mathbf{f}_1}{\partial v} - v \nabla \cdot \mathbf{f}_2 - \frac{q_e}{m_e} \mathbf{E} \frac{\partial f_0}{\partial v}$ is an effective electric field in plasma. A comparison to the *Generalized Ohm's law* shows a correct local behavior of (1), especially that $\nabla \times \nabla \cdot \mathbf{f}_2 \sim \nabla \times \frac{\nabla p_e}{n_e} \sim \nabla n_e \times \nabla T_e$

Generalized Ohm's law vs. nonlocal Ohm's law

$$\begin{aligned} \mathbf{E} &= (\mathbf{R}_T - \nabla p_e) + \frac{\mathbf{j}}{\sigma} + \mathbf{j} \times \mathbf{B}, \\ E v \frac{q_e}{m_e} \frac{\partial f_0}{\partial v} &= v^2 \nabla \cdot \mathbf{f}_2 + v^2 \frac{\nu_{ei}}{2} \frac{\partial \mathbf{f}_1}{\partial v} - v \nu_{ei} \mathbf{f}_1 + v \mathbf{f}_1 \times \frac{q_e}{m_e c} \mathbf{B}. \end{aligned}$$