Multi-dimensional High-Order FEM Methods for Electron Kinetics in ALE Hydrodynamics

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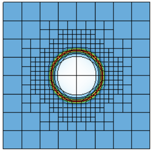

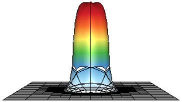

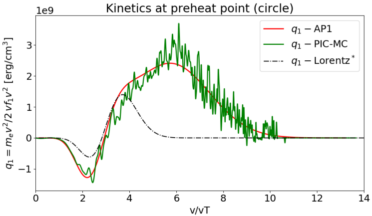

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Classical MHD

HYDRODYNAMICS *local* $\rightarrow \boldsymbol{q}_e = -\kappa_{SH} T_e^{2.5} \nabla T_e$

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho \nabla \cdot \boldsymbol{u}, \\
\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &= -\nabla(p_i + p_e) + \frac{c}{4\pi} \nabla \times \boldsymbol{B} \times \boldsymbol{B}, \\
\rho \left(\frac{\partial \varepsilon_i}{\partial T_i} \frac{\mathrm{d}T_i}{\mathrm{d}t} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) &= -p_i \nabla \cdot \boldsymbol{u} - G(T_i - T_e), \\
\rho \left(\frac{\partial \varepsilon_e}{\partial T_e} \frac{\mathrm{d}T_e}{\mathrm{d}t} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) &= -p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \left(\kappa_{SH} T_e^{2.5} \nabla T_e \right) + \sigma \boldsymbol{E} \cdot \boldsymbol{E} + G(T_i - T_e) + Q_{\mathrm{IB}}(\boldsymbol{E}_L),
\end{aligned}
$$

MAXWELL EQUATIONS *resistive* $\rightarrow \boldsymbol{j} = \sigma \boldsymbol{E}$

$$
\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{1}{c} \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}, \\
\nabla \times \boldsymbol{B} &= \frac{4\pi}{c} \sigma \boldsymbol{E},
\end{aligned}
$$

KINETICS OF ELECTRONS *Landau – Fokker – Planck*

$$
\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f = \Gamma \int \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot (f \nabla_{\tilde{\boldsymbol{v}}} f - f \nabla_{\boldsymbol{v}} f) \, \mathrm{d}\tilde{\boldsymbol{v}} + \frac{\nu_{ei}}{2} \frac{\partial^2 f}{\partial \Omega^2}.
$$

Nonlocal-MHD

HYDRODYNAMICS 4D

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\nabla\cdot\boldsymbol{u},\\
\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &= -\nabla(p_i+p_e)+\boldsymbol{j}(f,\boldsymbol{E})\times\boldsymbol{B},\\
\rho\left(\frac{\partial\varepsilon_i}{\partial T_i}\frac{\mathrm{d}T_i}{\mathrm{d}t}+\frac{\partial\varepsilon_i}{\partial\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) &= -p_i\nabla\cdot\boldsymbol{u}-G(T_i-T_e),\\
\rho\left(\frac{\partial\varepsilon_e}{\partial T_e}\frac{\mathrm{d}T_e}{\mathrm{d}t}+\frac{\partial\varepsilon_e}{\partial\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) &= -p_e\nabla\cdot\boldsymbol{u}-\nabla\cdot\boldsymbol{q}_e(f)+\boldsymbol{j}(f,\boldsymbol{E})\cdot\boldsymbol{E}+G(T_i-T_e)+Q_{\mathrm{IB}}(\boldsymbol{E}_L),
\end{aligned}
$$

MAXWELL EQUATIONS 4D

$$
\begin{aligned}
\nabla\times\boldsymbol{E} &= -\frac{1}{c}\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t},\\
\nabla\times\boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{j}(f,\boldsymbol{E}),
\end{aligned}
$$

KINETICS OF ELECTRONS 6D

$$
\boldsymbol{v}\cdot\nabla f+(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}f = v\tilde{\nu}_e\frac{\partial}{\partial v}\left(f-f_{MB}(T_e)\right)+\nu_{scat}\left(f_0-f\right).
$$

M. Holec et al, *Phys. Plas.*, submitted (2019) / arXiv:1901.11378 .

3D velocity space FEM discretizations

AWBS electron kinetic model 7D

$$
\begin{aligned}
C_V \frac{\mathrm{d}T_e}{\mathrm{d}t} &= -\nabla \cdot \boldsymbol{q}_e(f) + \boldsymbol{j}(f) \cdot \boldsymbol{E} + S_H, \\
\boldsymbol{v} \cdot \nabla f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f &= v \tilde{\nu}_e \frac{\partial}{\partial v}\left(f - f_{MB}(T_e)\right) + \nu_{scat}\left(f_0 - f\right).
\end{aligned}
$$

- C7 (S_N - discontinuous Galerkin FEM) 3D velocity space in spherical coordinates (v, ϕ, θ)

$$
\boldsymbol{n} \cdot \nabla f + \frac{\boldsymbol{E} \cdot \boldsymbol{n}}{v} \frac{\partial f}{\partial v} + \frac{E_\phi - v\, B_\theta}{v^2} \frac{\partial f}{\partial \phi} + \frac{E_\theta + v\, B_\phi}{v^2 \sin(\phi)} \frac{\partial f}{\partial \theta} = \tilde{\nu}_e \frac{\partial}{\partial v}\left(f - f_M\right) + \frac{\nu_{scat}}{v}\left(f_0 - f\right).
$$

- AP1 (P_N (VEF $\xi = 1/3$) - continuous Galerkin mixed FEM) *Physicists like it!*

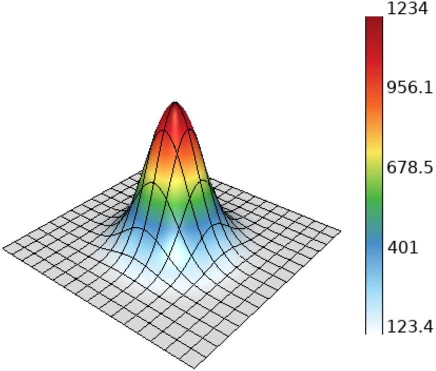
$$
\begin{aligned}
\xi \nabla \cdot \boldsymbol{f_1} + \xi \frac{q_e}{m_e v} \boldsymbol{E} \cdot \left(\frac{\partial \boldsymbol{f_1}}{\partial v} + \frac{2}{v} \boldsymbol{f_1}\right) &= \tilde{\nu}_e \frac{\partial}{\partial v}\left(f_0 - f_M\right), \\
\nabla f_0 + \frac{q_e}{m_e v} \boldsymbol{E} \frac{\partial f_0}{\partial v} + \frac{q_e \boldsymbol{B}}{m_e c v} \times \boldsymbol{f_1} &= \tilde{\nu}_e \frac{\partial \boldsymbol{f_1}}{\partial v} - \frac{\nu_{scat}}{v} \boldsymbol{f_1},
\end{aligned}
$$

S_N upwind DG , Multigrid Approximate-Ideal-Relaxation (AIR) solver

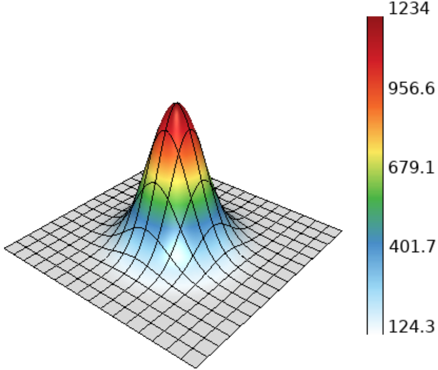
$$\mathbf{M}_{\left(\tilde{\nu_e}-\frac{\boldsymbol{E}\cdot\boldsymbol{n}_d}{v}\right)}\cdot\frac{\Delta\boldsymbol{f}_d}{\Delta v}=(\boldsymbol{n}_d\cdot\mathbf{G}+\mathbf{F}_d)\cdot\left(\tilde{\boldsymbol{f}}_d+\Delta\boldsymbol{f}_d\right)\;+\;\mathbf{M}_{\left(\frac{\nu_{scat}}{v}\right)}\cdot\left(\tilde{\boldsymbol{f}}_d+\Delta\boldsymbol{f}_d\right)\;+\;\mathbf{S}_{(\nu_{scat},\boldsymbol{E},\boldsymbol{B})}\cdot\tilde{\boldsymbol{f}}\;+\;\boldsymbol{S}_{\left(\tilde{\nu_e}\frac{\partial f_M}{\partial v}\right)}$$

Velocity groups	32	64	128	256	order
Backward Euler	1.595e-1	8.618e-2	4.456e-2	2.264e-2	0.98
SDIRK2	3.217e-2	8.888e-3	2.322e-3	5.924e-3	1.97
SDIRK3	2.455e-2	3.639e-3	4.887e-4	6.317e-05	2.96

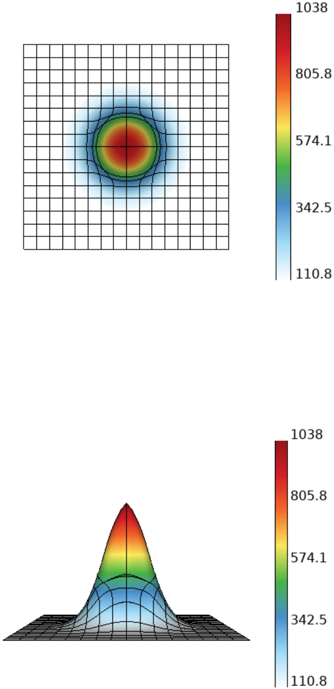
Hydro temperature



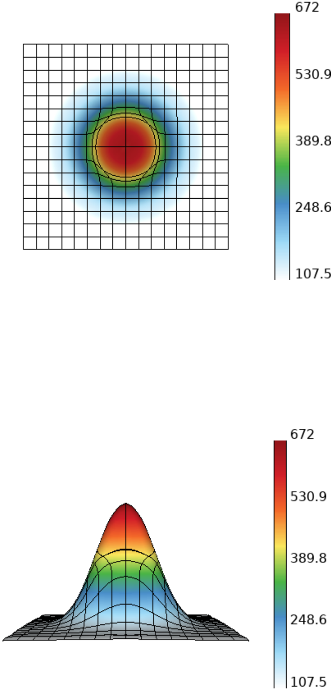
Kinetic temperature (50 groups)



Local hydro temperature



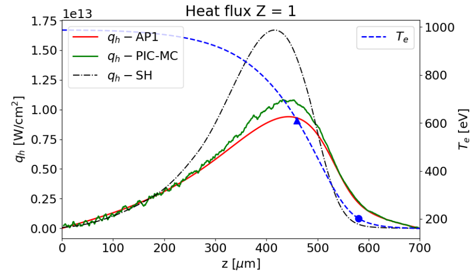
Nonlocal kinetic temperature

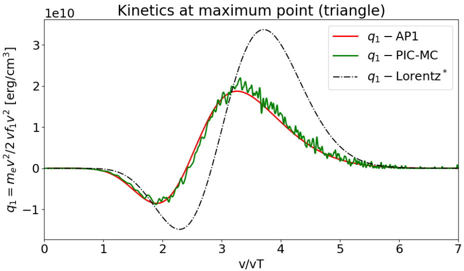


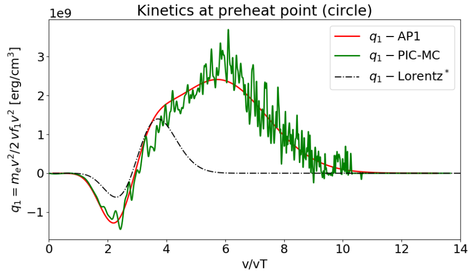
AP1 formulation, fixed P1 angular discretization PCG(AMG)

$$\mathbf{M}^{L_2}_{(\tilde{\nu_e})} \cdot \frac{\mathrm{d}\boldsymbol{f_0}}{\mathrm{d}v} - \mathbf{V}^{L_2}_{(\frac{\xi q_e \boldsymbol{E}}{m_e v})} \cdot \frac{\mathrm{d}\boldsymbol{f_1}}{\mathrm{d}v} = \mathbf{D}^{L_2}_{(\xi)} \cdot \boldsymbol{f_1} + \mathbf{M}^{L_2}_{(\frac{\xi 2 q_e \boldsymbol{E}}{m_e v^2})} \cdot \boldsymbol{f_1} + \boldsymbol{S}^{L_2}_{(\tilde{\nu_e}\frac{\partial f_M}{\partial v})},$$

$$\mathbf{M}^{H_1}_{(\tilde{\nu_e})} \cdot \frac{\mathrm{d}\boldsymbol{f_1}}{\mathrm{d}v} - \mathbf{V}^{H_1}_{(\frac{q_e \boldsymbol{E}}{m_e v})} \cdot \frac{\mathrm{d}\boldsymbol{f_0}}{\mathrm{d}v} = \mathbf{G}^{H_1} \cdot \boldsymbol{f_0} + \mathbf{M}^{H_1}_{(\frac{\nu_{scat}}{v})} \cdot \boldsymbol{f_1} + \mathbf{C}^{H_1}_{(\frac{q_e \boldsymbol{B}}{m_e c v}\times)} \cdot \boldsymbol{f_1},$$



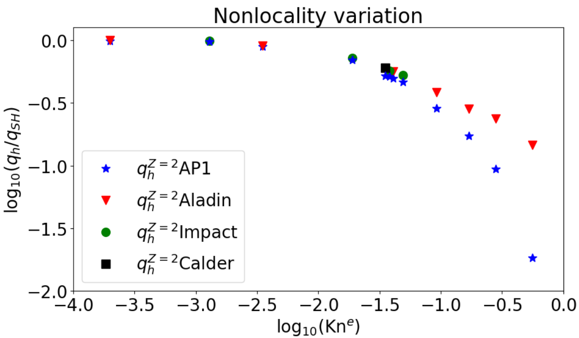




M. Holec et al, *Phys. Plas.*, submitted (2019) / arXiv:1901.11378 .

Electron velocity limit - friction vs. $\boldsymbol{E}$ stopping

$$\left(\tilde{\nu_e} - \frac{\boldsymbol{E}\cdot\boldsymbol{n}}{v}\right)\frac{\partial f}{\partial v} = \boldsymbol{n}\cdot\nabla f + \frac{\nu_{scat}}{v}(f - f_0) + \frac{E_\phi - v\,B_\theta}{v^2}\frac{\partial f}{\partial \phi} + \frac{E_\theta + v\,B_\phi}{v^2\sin(\phi)}\frac{\partial f}{\partial \theta} + \tilde{\nu_e}\frac{\partial f_M}{\partial v}.$$



$\boldsymbol{E}$ stopping overtakes collisions for Kn > 0.1 .

Kn = $\frac{\lambda}{L}$	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1
v_{lim}/v_{th}	70.8	22.4	7.3	3.1	1.8

M. Holec et al, *Phys. Plas.*, submitted (2019) / arXiv:1901.11378 .

Adaptive DSA preconditioner for Hydro-Kinetics coupling

AWBS electron kinetic model 7D

$$\begin{aligned} C_V \frac{\mathrm{d}T_e}{\mathrm{d}t} &= -\nabla \cdot \boldsymbol{q}_e(f) + \boldsymbol{j}(f) \cdot \boldsymbol{E} + S_H, \\ &= \int_{4\pi} \int_v v\tilde{\nu}_e \frac{\partial f}{\partial v} v^4 \mathrm{d}v \mathrm{d}\boldsymbol{n} - \sigma T_e^{-0.5} + S_H, \\ \boldsymbol{v} \cdot \nabla f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f &= v\tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f) . \end{aligned}$$

Continuum analysis of local (Kn$\ll$1) transport regime $\rightarrow$ DIFFUSION

$$C_V \frac{\mathrm{d}T_e}{\mathrm{d}t} = \nabla \cdot \lambda(T_e^{2.5}) \nabla T_e + O(\mathrm{Kn}^2)$$

Fixed-point iteration with preconditioner $\mathbf{E}(\Delta \boldsymbol{T}) = (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma(\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T})^\alpha + \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T})^\alpha$

$$\mathbf{M}_{C_V} \cdot \frac{\Delta \boldsymbol{T_e}^{k+1}}{\Delta t} + \mathbf{E}(\Delta \boldsymbol{T_e}^{k+1}) - \mathbf{E}(\Delta \boldsymbol{T_e}^{k}) = \mathbf{K}(f^k) - \mathbf{M}_\sigma(\tilde{\boldsymbol{T}}_{\boldsymbol{e}} + \Delta \boldsymbol{T_e}^k)^\alpha .$$

Finally, we get an unconditionally stable backward Euler (SDIRK) fast iterating scheme

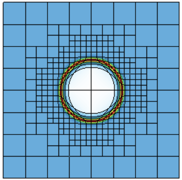
$$\mathbf{M}_{C_V} \cdot \frac{\Delta \boldsymbol{T_e}^{k+1}}{\Delta t} + (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma(\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T_e}^{k+1})^\alpha + \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\boldsymbol{T}}_{\boldsymbol{e}} + \Delta \boldsymbol{T_e}^{k+1})^\alpha = \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\boldsymbol{T}}_{\boldsymbol{e}} + \Delta \boldsymbol{T_e}^{k})^\alpha + \mathbf{K}(f^k) - \mathbf{C} \cdot \mathbf{M}_\sigma(\tilde{\boldsymbol{T}}_{\boldsymbol{e}} + \Delta \boldsymbol{T_e}^{k})^\alpha ,$$

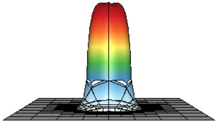
where the adaptive coefficient handles diffusion $\mathbf{C} \xrightarrow{\mathrm{Kn}\ll 1} 1$ and nonlocal transport $\mathbf{C} \xrightarrow{\mathrm{Kn}>1} 0$ and $\mathbf{C} \in (1, 0)$ in between.

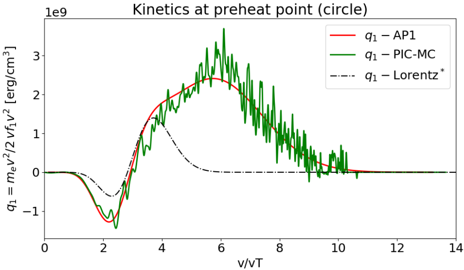
T. Haut et al, *SIAM*, submitted (2018) / arXiv:1810.11082 .

Conclusions

- 7D microscopic world of electrons in hydro simulations.
- S_N high-order DG finite element approach.
- P_1 high-order mixed finite element approach.
- $\boldsymbol{E}$ field dominated stopping (P_1 fails).
- Adaptive DSA preconditioner for Hydro-Kinetics coupling (deep learning on **C**).
- Algebraic Multigrid solver pAIR scales $\log(P)^{1.22}$.







MS85 - Developments in Algebraic Multigrid for Nonsymmetric and Hyperbolic Problems - Ben S. Southworth
https://github.com/CEED/Laghos/tree/master/amr
https://mfem.org

Nonlocal Ohm's law

We found that the quality of the electric field evaluation is essential for a correct plasma modeling

Nonlocal current in plasma

$$\boldsymbol{j}(f, \boldsymbol{E}) = e \int v \mathbf{f_1} v^2 \, \mathrm{d}v = e \int v \frac{\nu_{ei}^2 \boldsymbol{E}^* + \boldsymbol{\omega}_B \, \boldsymbol{\omega}_B \cdot \boldsymbol{E}^* + \nu_{ei} \, \boldsymbol{\omega}_B \times \boldsymbol{E}^*}{\nu_{ei}(\boldsymbol{\omega}_B^2 + \nu_{ei}^2)} v^2 \, \mathrm{d}v, \tag{1}$$

where $\boldsymbol{E}^* = v \frac{\nu_e}{2} \frac{\partial \mathbf{f_1}}{\partial v} - v \nabla \cdot \mathbf{f}_2 - \frac{q_e}{m_e} \boldsymbol{E} \frac{\partial f_0}{\partial v}$ is an effective electric field in plasma. A comparison to the *Generalized Ohm's law* shows a correct local behavior of (1), especially that $\nabla \times \nabla \cdot \mathbf{f}_2 \sim \nabla \times \frac{\nabla p_e}{n_e} \sim \nabla n_e \times \nabla T_e$

Generalized Ohm's law vs. nonlocal Ohm's law

$$\begin{aligned} \boldsymbol{E} &= (\boldsymbol{R}_T - \nabla p_e) &+& \quad \frac{\boldsymbol{j}}{\sigma} &+& \quad \boldsymbol{j} \times \boldsymbol{B}, \\ \boldsymbol{E} v \frac{q_e}{m_e} \frac{\partial f_0}{\partial v} &= v^2 \nabla \cdot \mathbf{f}_2 &+& \quad v^2 \frac{\nu_e}{2} \frac{\partial \mathbf{f_1}}{\partial v} - v \nu_{ei} \mathbf{f_1} &+& \quad v \mathbf{f_1} \times \frac{q_e}{m_e c} \boldsymbol{B}. \end{aligned}$$