Lecture 23

Kinetics of driven plasmas I: Fokker-Planck equation (FPE)

23.1   Notations used in the next two lectures

We use bold type for vectors, e.g. $\mathbf{A}$ (with components A_i) and sans-serif font with arrow on top for two dimensional tensors, e.g., $\overset{\leftrightarrow}{\mathsf{A}}$ (with components A_{ij}). The diadic product of two vectors constitutes a tensor, e.g., $\overset{\leftrightarrow}{\mathsf{C}} = \mathbf{a}\mathbf{b}^T$. Unit vectors are indicated with regular slanted type with a hat, e.g. $\hat{n}$. Averaging over some variable x is denoted with angular brackets as $\langle \cdot \rangle_x$. The following tensors are commonly used:

$$\overset{\leftrightarrow}{\mathsf{I}} \quad \text{— unit tensor;}$$
$$\overset{\leftrightarrow}{\mathsf{I}}_{\parallel} = \hat{n}\hat{n}^T \quad \text{— "parallel" projection tensor on a chosed direction } \hat{n};$$
$$\overset{\leftrightarrow}{\mathsf{I}}_{\perp} = \overset{\leftrightarrow}{\mathsf{I}} - \hat{n}\hat{n}^T \quad \text{— "perpendicular" projection tensor.}$$

The nabla operator ∇ is generally treated like a vector. The divergence of a tensor is denoted as $\mathbf{a} = \nabla\overset{\leftrightarrow}{\mathsf{A}}$, the result of which is a vector with components $a_j = \partial_i A_{ij}$. Sometimes we use a gradient of a vector $\overset{\leftrightarrow}{\mathsf{B}} = \nabla\mathbf{b}^T$, the result of which is a tensor with components $B_{ij} = \partial_i b_j$.

23.2 Fokker-Planck equation (FPE)

The Fokker-Planck (FP) equation approach is used for simplifying the collision integrals in the rhs of Boltzmann equation. In most of the Plasma Physics course, we assumed that plasma is collisionless and used Vlasov's equation (which is just a Boltzmann equation with zero rhs). However, often one has to deal with collisional plasmas. The electrons in plasma may collide with neutral molecules and atoms, ions or other electrons. Often the changes to electron velocities or energies are small, and in this case the collision integral can be simplified for computational purposes to a differential form (the FP equation).

The derivation of Fokker-Planck equation (FPE) is beautifully explained in D. R. Nicholson, "Introduction to Plasma theory", Appendix B, J. Wiley and Sons, New York (1983). Here we give just an outline of this derivation.

Consider a time-dependent distribution of particles $f(x,t)$ over a generalized coordinate x. The distribution at moment $t + \Delta t$ is dependent on value at t:

$$f(x, t + \Delta t) = \int f(x - \xi, t)\psi(x - \xi, \xi) \, d\xi \tag{23.1}$$

where $\psi(x', \xi)d\xi$ is the transition probability, i.e. the probability of the coordinate change in time Δt, from x' to a new coordinate $x' + \xi$ lying in an interval of size $d\xi$. Equation (23.1) implies that $f(t + \Delta t)$ is independent of values of f at times ealier than t, i.e., it describes a so-called *Markov process*.

Let us assume that the coordinate x does not change by large values in small time intervals. For example, if x is a particle velocity, the collisions change it only by a small amount. Then we can expand in Taylor series:

$$f(x - \xi, t)\psi(x - \xi, \xi) = f(x,t)\psi(x,\xi) + \sum_{n=1}^{\infty}(-1)^n \frac{\xi^n}{n!}\frac{d}{dx^n}\left(f(x,t)\psi(x,\xi)\right)$$

Substituting into equation for $f(x, t + \Delta t)$, we get

$$f(x, t + \Delta t) = f(x,t) + \sum_{n=1}^{\infty}(-1)^n \frac{d}{dx^n}\left\{f(x,t)\int \psi(x,\xi)\frac{\xi^n}{n!} \, d\xi\right\}$$

where we used the fact that ξ and x are independent variables and $\int \psi(x,\xi) \, d\xi = 1$.

Notice that $\int \psi(x,\xi)\xi^n \, d\xi = \langle \xi^n \rangle$, where the angular brackets $\langle \cdot \rangle$ denote the statistical (or ensemble) average. Now, to find the time derivative, let us take the limit $t \to 0$. This has to be done in such a way that Δt becomes small compared to any macroscopic time scale, while it still remains large compared with time intervals between individual coordinate change events (collisions), so that the changes ξ are independent at times differing more than by Δt. We will denote this as $t \to$ '0'. Let us introduce

$$D^{(n)} = \lim_{\Delta t \to '0'} \frac{\langle \xi^n \rangle}{n!\Delta t}$$

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The equation for time evolution of $f(x, t)$ with this notation is

$$\frac{\partial f(x, t)}{\partial t} = \sum_{n=1}^{\infty} (-1)^n \frac{d}{dx^n} \left(D^{(n)} f(x, t) \right)$$

The FPE is defined as this equation truncated after $n = 2$:

$$\frac{\partial f(x, t)}{\partial t} = -\frac{d}{dx} \left(D^{(1)} f(x, t) \right) + \frac{d}{dx^2} \left(D^{(2)} f(x, t) \right)$$

In problems considered in physics and later in this lecture, $D^{(3)}$ and higher terms are $\leq O(\Delta t)$, and therefore vanish as we take the limit $t \to {}'0'$.

This derivation can be generalized to an n-dimensional case, described by a distribution of particles $f(x_1, \ldots, x_n, t)$ in space with generalized coordinates x_k. We have

$$\partial_t f = \sum_{l=1}^{\infty} (-1)^l \sum_{k_1, \ldots, k_l} \partial_{x_{k_1}} \cdots \partial_{x_{k_l}} (D^{(l)}_{k_1, \ldots, k_l} f) \tag{23.2}$$

where ∂ denotes the partial derivative in respect to the variable in the subscript. The tensors $D^{(l)}_{k_1, \ldots, k_l}$ are

$$D^{(l)}_{k_1, \ldots, k_l} = \frac{1}{l!} \lim_{\Delta t \to {}'0'} \frac{\langle \Delta x_{k_1} \cdots \Delta x_{k_l} \rangle}{\Delta t}$$

where Δx_i are the changes in variables x_i in time Δt due to random collisions and other processes. Again, the FPE is understood as (23.2) truncated after $l = 2$.

23.3   FPE in momentum space and interpretation of coefficients D

We consider FPE in momentum space,

$$\frac{\partial f(\mathbf{p})}{\partial t} = -\nabla_{\mathbf{p}}[\mathbf{D}^{(1)} f(\mathbf{p})] + \nabla_{\mathbf{p}}[\nabla_{\mathbf{p}}(\overleftrightarrow{\mathsf{D}}^{(2)} f(\mathbf{p}))] \tag{23.3}$$

where

$$\mathbf{D}^{(1)} = \lim_{\Delta t \to {}'0'} \frac{\langle \Delta \mathbf{p} \rangle}{\Delta t}$$

and $\overleftrightarrow{\mathsf{D}}^{(2)}$ is a second rank tensor

$$\overleftrightarrow{\mathsf{D}}^{(2)} = \frac{1}{2} \lim_{\Delta t \to {}'0'} \frac{\langle \Delta \mathbf{p} \Delta \mathbf{p}^T \rangle}{\Delta t}$$

Here $\Delta \mathbf{p} \Delta \mathbf{p}^T$ is a dyadic product of vector $\Delta \mathbf{p}$ with itself. In the second term of (23.3), the inner $\nabla_{\mathbf{p}}$ is the divergence of a tensor (the result of which is a vector).

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In general, $\mathbf{D}^{(1)}$ is due to changes in particle momenta due to both external force and random collisions. Without external force, it is usually directed along $(-\mathbf{p})$, and slows the particle down. However, the decrease of average vector momentum can be due to spreading in particle directions (which is described by the term with $\overleftrightarrow{\mathsf{D}}^{(2)}$) and thus not equal to the decrease of the absolute value of momentum vector. Since $\Delta(p^2) = 2\mathbf{p} \cdot \Delta\mathbf{p} + (\Delta p_x)^2 + (\Delta p_x)^2 + (\Delta p_z)^2$, we have an effective "force" (the dynamic friction coefficient)

$$\mathbf{F}_1 = \hat{p}\frac{\langle \Delta |\mathbf{p}| \rangle}{\Delta t} = \frac{\hat{p}}{2|\mathbf{p}|}\frac{\langle \Delta(p^2) \rangle}{\Delta t} = \mathbf{D}^{(1)} + \frac{\hat{p}}{|\mathbf{p}|}\mathrm{Tr}\,\overleftrightarrow{\mathsf{D}}^{(2)}$$

where $\hat{p}$ is the unit vector along $\mathbf{p}$ and Tr denotes the trace operation.

Equation (23.3) is usually rewritten as

$$\frac{\partial f(\mathbf{p})}{\partial t} = -\nabla_{\mathbf{p}}[(\mathbf{D}^{(1)} - \nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)})f(\mathbf{p})] + \nabla_{\mathbf{p}}[\overleftrightarrow{\mathsf{D}}^{(2)}\nabla_{\mathbf{p}}f(\mathbf{p})] \qquad (23.4)$$

Thus, $\overleftrightarrow{\mathsf{D}}^{(2)}\nabla_{\mathbf{p}}f(\mathbf{p})$ can be interpreted as a flux of particles due to diffusion (in momentum space) with a diffusion coefficient $\overleftrightarrow{\mathsf{D}}^{(2)}$, and

$$\mathbf{F}_2 = \mathbf{D}^{(1)} - \nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)}$$

as the force acting on particles. If the diffusion does not change the absolute value $p = |\mathbf{p}|$, then both interpretations of the force are the same, $\mathbf{F}_1 = \mathbf{F}_2$, or

$$-\nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)} = \frac{\hat{p}}{|\mathbf{p}|}\mathrm{Tr}\,\overleftrightarrow{\mathsf{D}}^{(2)} \qquad (23.5)$$

23.4 Calculation of $D^{(1,2)}$ due to small-angle collisions with elastic fixed centers

Let the momentum before collision be $\mathbf{p} = p\hat{n}$ and after collision $\mathbf{p}' = p\hat{n}'$. The absolute value stays the same in elastic collisions. The change of momentum is

$$\Delta\mathbf{p} = p(\hat{n}' - \hat{n})$$

Then the needed coefficients are

$$\mathbf{D}^{(1)} = p\frac{\langle (\hat{n}' - \hat{n}) \rangle}{\Delta t}$$
$$\overleftrightarrow{\mathsf{D}}^{(2)} = \frac{p^2}{2}\frac{\langle (\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T \rangle}{\Delta t}$$

If we choose coordinate system so that $\mathbf{p} = p\hat{z}$, then

$$\hat{n}' - \hat{n} = -\hat{z}(1 - \cos\theta) + (\hat{x}\cos\phi + \hat{y}\sin\phi)\sin\theta$$

Here θ, ϕ give the direction of the scattering Ω. To find the average over Ω, we take into account axial symmetry to average over ϕ, so that $\langle \sin^2 \phi \rangle = \langle \cos^2 \phi \rangle = 1/2$ and $\langle \sin \phi \cos \phi \rangle = 0$. We get

$$\langle \hat{n}' - \hat{n} \rangle_\Omega = -\langle 1 - \cos \theta \rangle \hat{n} \tag{23.6}$$

$$\langle (\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T \rangle_\Omega = \frac{1}{2} \langle \sin^2 \theta \rangle \overset{\leftrightarrow}{\mathsf{I}}_\perp + \langle (1 - \cos \theta)^2 \rangle \overset{\leftrightarrow}{\mathsf{I}}_\parallel \tag{23.7}$$

where we introduced the projection operators $\overset{\leftrightarrow}{\mathsf{I}}_\parallel = \hat{n}\hat{n}^T$ and $\overset{\leftrightarrow}{\mathsf{I}}_\perp = \overset{\leftrightarrow}{\mathsf{I}} - \hat{n}\hat{n}^T$, $\overset{\leftrightarrow}{\mathsf{I}}$ being the unit operator. $\overset{\leftrightarrow}{\mathsf{I}}_\perp$ has the following properties:

$$\overset{\leftrightarrow}{\mathsf{I}}_\perp \hat{n} = 0 \tag{23.8}$$

$$\mathrm{Tr}\, \overset{\leftrightarrow}{\mathsf{I}}_\perp = 2 \tag{23.9}$$

$$\overset{\leftrightarrow}{\mathsf{I}}_\perp = p \nabla_{\mathbf{p}} \hat{n}^T = p \nabla_{\mathbf{p}} \nabla_{\mathbf{p}}^T p \tag{23.10}$$

where in the last equation a diadic product is taken. **Proof of (23.10)**: $\partial_k p_l = \delta_{kl} = \partial_k (p \hat{n}_l) = p \partial_k \hat{n}_l + \hat{n}_k \hat{n}_l$, where we used $\partial_k p = \hat{n}_k$. Equation (23.10) can be used to find the divergence $\nabla_{\mathbf{p}} \overset{\leftrightarrow}{\mathsf{I}}_\perp = -\nabla_{\mathbf{p}}(\hat{n}\hat{n}^T)$, $\sum_k \partial_k(\hat{n}_k \hat{n}_l) = (\overset{\leftrightarrow}{\mathsf{I}}_\perp \hat{n})_l/p + \hat{n}_l \mathrm{Tr}\, \overset{\leftrightarrow}{\mathsf{I}}_\perp/p = 2\hat{n}_l/p$:

$$\nabla_{\mathbf{p}} \overset{\leftrightarrow}{\mathsf{I}}_\perp = -\frac{2\hat{n}}{p}$$

and

$$\nabla_{\mathbf{p}}(f(p) \overset{\leftrightarrow}{\mathsf{I}}_\perp) = -f(p) \frac{2\hat{n}}{p} \tag{23.11}$$

for arbitrary function $f(p)$.

We use these properties to find

$$\langle |\hat{n}' - \hat{n}|^2 \rangle_\Omega = \mathrm{Tr}\, \langle (\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T \rangle_\Omega = 2\langle 1 - \cos \theta \rangle \tag{23.12}$$

Under the assumption of small-angle collisions equations (23.6–23.7) become

$$\langle \hat{n}' - \hat{n} \rangle_\Omega \approx -\frac{\langle \theta^2 \rangle}{2} \hat{n} \tag{23.13}$$

$$\langle (\hat{n}' - \hat{n})(\hat{n}' - \hat{n})^T \rangle_\Omega \approx \frac{\langle \theta^2 \rangle}{2} \overset{\leftrightarrow}{\mathsf{I}}_\perp \tag{23.14}$$

where we neglected 4-th power of θ and higher.

The small-angle collisions are best described in terms of *momentum transfer rate* ν_m, since the total cross-section, and therefore the collision frequency $\nu_\mathrm{c} = 1/\Delta t$ can be infinite (e.g., for Coulomb potential). It is defined as

$$\nu_\mathrm{m} = \frac{\langle 1 - \cos \theta \rangle}{\Delta t} \approx \frac{1}{2} \frac{\langle \theta^2 \rangle}{\Delta t}$$

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From (23.13–23.14) we get

$$
\begin{aligned}
\mathbf{D}^{(1)} &= -\nu_{\mathrm{m}}\mathbf{p} \\
\overleftrightarrow{\mathsf{D}}^{(2)} &= D(p)p^2\overleftrightarrow{\mathsf{I}}_\perp
\end{aligned}
$$

where $D(p) = \nu_{\mathrm{m}}/2$ can be considered as the angular diffusion coefficient. Note that any $\overleftrightarrow{\mathsf{D}}^{(2)} \sim \overleftrightarrow{\mathsf{I}}_\perp$, such as this one, conserves the absolute value of the momentum. Therefore, (23.5) is valid:

$$
\nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)} = -2\mathbf{p}D(p) = -\frac{\mathbf{p}}{p^2}\operatorname{Tr}\overleftrightarrow{\mathsf{D}}^{(2)} = -\nu_{\mathrm{m}}\mathbf{p}
$$

and the dynamic friction is $\mathbf{F} = \mathbf{D}^{(1)} - \nabla_{\mathbf{p}}\overleftrightarrow{\mathsf{D}}^{(2)} = 0$, as expected for elastic collisions.

23.5   Nonrelativistic applications of FPE

Consider collisions with centers of mass M, assuming that f describes particles of mass m. Example of this is electron distribution in an ambient gas of molecules, and we will call the particles by those names hereafter. We also will assume that molecules move with speed $\mathbf{V}$, which has properties $\langle\mathbf{V}\rangle = 0$, and $\langle V^2\rangle = 3T/M$, where T is an effective molecule temperature. We assume nonrelativistic motion here, so that to get velocities in the a different reference frame we use vector addition of velocities.

It is convenient, beside the velocities $\mathbf{v}$ and $\mathbf{V}$, to use the relative velocity and the center of mass velocity

$$
\begin{aligned}
\mathbf{v}_{\mathrm{r}} &= \mathbf{v} - \mathbf{V} & (23.15) \\
\mathbf{v}_{\mathrm{CM}} &= \frac{m\mathbf{v} + M\mathbf{V}}{M + m} & (23.16)
\end{aligned}
$$

These variables are convenient because $\mathbf{v}_{\mathrm{CM}}$ does not change in collisions and $\mathbf{v}_{\mathrm{r}}$ can be treated as a velocity of a particle having reduced mass

$$
\mu = \frac{mM}{M + m}
$$

moving in the field of a static scatterer (instead of particle of mass m moving with velocity $\mathbf{v}$ in the field of non-static scatterer of mass M). The inverse transformation from $\mathbf{v}_{\mathrm{r}}$, $\mathbf{v}_{\mathrm{CM}}$ to $\mathbf{v}$, $\mathbf{V}$ is

$$
\begin{aligned}
\mathbf{v} &= \frac{M}{M + m}\mathbf{v}_{\mathrm{r}} + \mathbf{v}_{\mathrm{CM}} & (23.17) \\
\mathbf{V} &= -\frac{m}{M + m}\mathbf{v}_{\mathrm{r}} + \mathbf{v}_{\mathrm{CM}} & (23.18)
\end{aligned}
$$

The collision process is most easily described in terms of relative velocity. From (23.6–23.7), (23.12) we obtain for averages over the scattering direction:

$$\nu_c(v_r)\langle\Delta\mathbf{v}_r\rangle_\Omega = -\nu_m(v_r)\mathbf{v}_r \tag{23.19}$$
$$\nu_c(v_r)\langle|\Delta\mathbf{v}_r|^2\rangle_\Omega = 2\nu_m(v_r)v_r^2 \tag{23.20}$$

(in these equation we did not use the requirement for $\Delta\mathbf{v}_r$ to be small).

The collision rates ν_c, ν_m now depend on v_r, not on v. This has to be taken into account when averaging over $\mathbf{v}$ or $\mathbf{V}$.

23.5.1 Heavy scatterers of finite mass

We will now relax the requirement of small-angle collisions. Then it would be impossible to write down FP equation for the vector velocities $\mathbf{v}$. However, if $m \ll M$ (which is the case of electron colliding with molecules), the energy changes during the collisions are small, and therefore we still can write FP equation for electron energies $\mathcal{E} = mv^2/2$. We will assume that electrons have an isotropic distribution and that their state is completely defined by energy distribution $n(\mathcal{E})$ such that $n(\mathcal{E})\,d\mathcal{E}$ equals to the number density of electrons in energy interval $d\mathcal{E}$ around $\mathcal{E}$.

We will sometimes make use of the smallness of parameter $\alpha = \sqrt{m/M}$. From the energy equipartition theorem, $mv^2 \sim MV^2$, and therefore $V/v \sim \alpha$, $(mv)/(MV) \sim \alpha$.

Using transformations (23.17–23.18), we can exress the electron energy in terms of $\mathbf{v}_r$ and $\mathbf{v}_{CM}$:
$$\mathcal{E} = \frac{mv^2}{2} = \frac{mv_{CM}^2}{2} + \frac{M}{M+m}\frac{\mu v_r^2}{2} + \mu(\mathbf{v}_r \cdot \mathbf{v}_{CM})$$

The total energy in the collision is separated into "relative" energy and the energy of the system as a whole:

$$\mathcal{E}_{tot} = \frac{mv^2}{2} + \frac{MV^2}{2} = \frac{\mu v_r^2}{2} + \frac{(M+m)v_{CM}^2}{2}$$

For elastic collisions $\mathcal{E}_{tot}$ does not change and the change of electron energy is

$$\Delta\mathcal{E} = \mu(\Delta\mathbf{v}_r \cdot \mathbf{v}_{CM}) \tag{23.21}$$

Using (23.19), the average of this change over scattering direction is obtained:

$$\nu_c\langle\Delta\mathcal{E}\rangle_\Omega = -\mu\nu_m(v_r)(\mathbf{v}_r \cdot \mathbf{v}_{CM})$$

Using (23.15–23.16), we have

$$\mathbf{v}_r \cdot \mathbf{v}_{CM} = \frac{2(\mathcal{E} - \mathcal{E}_M)}{M+m} + \frac{M-m}{M+m}(\mathbf{v} \cdot \mathbf{V})$$

where $\mathcal{E}_M = MV^2/2$. Now, let us average over the electron and molecule velocities $\mathbf{v}$. The term $\sim (\mathbf{v} \cdot \mathbf{V})$ would go away if it were not for dependence $\nu_m(|\mathbf{v} - \mathbf{V}|)$. Let us use $V \ll v$ to expand

$$\nu_{\mathrm{m}}(v_{\mathrm{r}}) = \nu_{\mathrm{m}}(|\mathbf{v} - \mathbf{V}|) \approx \nu_{\mathrm{m}}(v) - m(\mathbf{v} \cdot \mathbf{V})\frac{d\nu_{\mathrm{m}}}{d\mathcal{E}}$$

(we used $d\nu_{\mathrm{m}}/d\mathcal{E} = (d\nu_{\mathrm{m}}/dv)/(mv)$). Neglecting terms $\sim m/M$, we get

$$\langle \nu_{\mathrm{c}}\Delta\mathcal{E} \rangle_{\mathbf{v}} = -\frac{2m}{M}\nu_{\mathrm{m}}(\mathcal{E} - \mathcal{E}_M) + m^2\frac{d\nu_{\mathrm{m}}}{d\mathcal{E}}\langle (\mathbf{v} \cdot \mathbf{V})^2 \rangle_{\mathbf{v}}$$

Using

$$\langle (\mathbf{v} \cdot \mathbf{A})(\mathbf{v} \cdot \mathbf{B}) \rangle_{\mathbf{v}} = \frac{v^2}{3}\mathbf{A} \cdot \mathbf{B} \tag{23.22}$$

we obtain the (almost) final answer

$$\langle \nu_{\mathrm{c}}\Delta\mathcal{E} \rangle_{\mathbf{v}} = -\frac{2m}{M}\nu_{\mathrm{m}}(\mathcal{E} - \mathcal{E}_M) + \frac{4m}{3M}\mathcal{E}\mathcal{E}_M\frac{d\nu_{\mathrm{m}}}{d\mathcal{E}}$$

Now, we can average over $\mathbf{V}$. Since the molecules have a thermal distribution, we use $\langle \mathcal{E}_M \rangle_{\mathbf{V}} = 3T/2$:

$$\langle \nu_{\mathrm{c}}\Delta\mathcal{E} \rangle = -\frac{2m}{M}\nu_{\mathrm{m}}\left(\mathcal{E} - \frac{3}{2}T\right) + \frac{2m}{M}T\mathcal{E}\frac{d\nu_{\mathrm{m}}}{d\mathcal{E}} \tag{23.23}$$

To write FP equation for the energy $\mathcal{E} = \frac{mv^2}{2}$, beside $\langle \nu_{\mathrm{c}}\Delta\mathcal{E} \rangle$ we also need $\langle \nu_{\mathrm{c}}(\Delta\mathcal{E})^2 \rangle$. Using (23.21), and $\mathbf{v}_{\mathrm{CM}} \approx \mathbf{V}$, we get:

$$(\Delta\mathcal{E})^2 \approx m^2(\Delta\mathbf{v}_{\mathrm{r}} \cdot \mathbf{V})^2$$

It is more convenient to average first over $\mathbf{V}$, keeping $\Delta\mathbf{v}_{\mathrm{r}}$ constant. We make a small error in doing this (due to the fact that $\mathbf{V}$ is not completely isotropic relative to $\mathbf{v}_{\mathrm{r}}$), but we neglect it. Using (23.22) again, we get

$$\langle (\Delta\mathcal{E})^2 \rangle_{\mathbf{V}} = \frac{m^2}{3}(\Delta\mathbf{v}_{\mathrm{r}})^2\langle V^2 \rangle = \frac{m^2}{M}T(\Delta\mathbf{v}_{\mathrm{r}})^2$$

Finally, use (23.20) to average over the scattering direction and the final answer is

$$\langle \nu_{\mathrm{c}}(\Delta\mathcal{E})^2 \rangle = \frac{4m}{M}\nu_{\mathrm{m}}\mathcal{E}T \tag{23.24}$$

The FP equation in the form (23.4) is

$$\frac{\partial n(\mathcal{E})}{\partial t} = \frac{\partial}{\partial \mathcal{E}}\left(D_{\mathcal{E}}\frac{\partial n}{\partial \mathcal{E}} - F_{\mathcal{E}}n\right) \tag{23.25}$$

where $n(\mathcal{E})$ is the energy distribution function and

$$
\begin{aligned}
F_{\mathcal{E}}(\mathcal{E}) &= \langle \nu_c \Delta \mathcal{E} \rangle - \frac{1}{2} \frac{\partial \langle \nu_c (\Delta \mathcal{E})^2 \rangle}{\partial \mathcal{E}} = -\frac{2m}{M} \nu_m \left(\mathcal{E} - \frac{T}{2} \right) && (23.26) \\
D_{\mathcal{E}}(\mathcal{E}) &= \frac{1}{2} \langle \nu_c (\Delta \mathcal{E})^2 \rangle = \frac{2m}{M} \nu_m T \mathcal{E} && (23.27)
\end{aligned}
$$

Note that in an K-dimensional gas there is a requirement that

$$
\frac{F_{\mathcal{E}}(\mathcal{E})}{D_{\mathcal{E}}(\mathcal{E})} = -\frac{1}{T} + \frac{K-2}{2\mathcal{E}} \tag{23.28}
$$

in order to get a stationary Maxwellian distribution $n(\mathcal{E}) = n_0 C \mathcal{E}^{(K-2)/2} e^{-\mathcal{E}/T}$ (where $C = \left[T^{K/2} \Gamma(K/2) \right]^{-1}$ for correct normalization, e.g., $C = 2/\sqrt{\pi T^3}$ for $K = 3$). Sometimes to shorten the equation one uses slightly different definitions $D = D_{\mathcal{E}}/\mathcal{E}$ and $F = -F_{\mathcal{E}} + D(K-2)/2$ and writes Boltzmann equation as

$$
\frac{\partial n(\mathcal{E})}{\partial t} = \frac{\partial}{\partial \mathcal{E}} \left[D\mathcal{E}^{K/2} \frac{\partial}{\partial \mathcal{E}} \left(\frac{n}{\mathcal{E}^{(K-2)/2}} \right) + Fn \right] \tag{23.29}
$$

The requirement (23.28) becomes simply $F/D = \mathcal{E}/T$.

We can introduce the elastic "energy loss rate"

$$
\nu_{\mathcal{E}} = \frac{2m}{M} \nu_m \tag{23.30}
$$

which is $\ll \nu_m$ since $m \ll M$. With this definition, $F = \nu_{\mathcal{E}} \mathcal{E}$ and $D = \nu_{\mathcal{E}} T$. The kinetic equation for $K = 3$ in form (23.29) is

$$
\frac{\partial n(\mathcal{E})}{\partial t} = \frac{\partial}{\partial \mathcal{E}} \left[\nu_{\mathcal{E}} T \mathcal{E}^{3/2} \frac{\partial}{\partial \mathcal{E}} \left(\frac{n}{\mathcal{E}^{1/2}} \right) + \nu_{\mathcal{E}} \mathcal{E} n \right] \tag{23.31}
$$

One can verify by integrating (23.31) multiplied by $\mathcal{E}$ that for $T = 0$ indeed

$$
\frac{\partial \langle \mathcal{E} \rangle}{\partial t} = -\langle \nu_{\mathcal{E}} \mathcal{E} \rangle
$$

For a Maxwellian distribution with electron temperature T_e,

$$
\frac{\partial \langle \mathcal{E} \rangle}{\partial t} = -\frac{T_e - T}{T_e} \langle \nu_{\mathcal{E}} \mathcal{E} \rangle
$$

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23.5.2 Rosenbluth potentials

Now, let us consider Coulomb collisions. They have the property that the change $\Delta\mathbf{v}$ is small in each collision, so we can again use the approach of Section 23.3 and write the FP equation for the vector velocity $\mathbf{v}$. The requirement $m \ll M$ can now be relaxed. In fact, we can even take $M = m$, so that this equation can now be used not just for collisions of electrons with heavy ions, but also for electron-electron collisions. It turns out that for Coulomb collisions the expressions for $\mathbf{D}^{(1)}$ and $\overset{\leftrightarrow}{\mathsf{D}}^{(2)}$ can be written in terms of so called *Rosenbluth potentials*, which satisfy certain relations that lead to some simplification in Boltzmann equation. Here, we will just show the basics. For more information, see I. P. Shkarofsky, T. W. Johnston and M. P. Bachinski "The Particle Kinetics of Plasmas", Addison-Wesley, Palo Alto (1966).

The differential Coulomb cross-section for collision of particles of charges Q_1 and Q_2 is

$$\frac{d\sigma}{d\Omega_{\text{scat}}} = \frac{b_0^2}{4\sin^4(\theta/2)}, \quad \frac{d\sigma}{d\theta} = \frac{\pi b_0^2 \cos(\theta/2)}{\sin^3(\theta/2)}, \quad b_0 = \frac{|Q_1 Q_2|}{4\pi\varepsilon_0 \mu v_{\text{r}}^2} \qquad (23.32)$$

where v_{r} is the relative velocity, μ is the reduced mass, θ is the scattering angle for the relative velocity. We see that the total cross-section $\sigma = \int_0^\pi (d\sigma/d\theta)\,d\theta$ is infinite, because there is a large contribution from small angles θ. The angle θ is related to the target parameter b through the relation

$$b = \frac{b_0}{\tan(\theta/2)}$$

so that b_0 has the meaning of the target parameter needed for 90° scattering. In plasma, we can limit the maximum target parameter by the Debye length $b_m = \lambda_D$, which corresponds to a minimum scattering angle of θ_m given by

$$\sin(\theta_m/2) = \left[1 + \left(\frac{b_m}{b_0}\right)^2\right]^{-1/2} \approx \frac{b_0}{b_m}$$

since $b_m \gg b_0$. The total momentum transfer cross-section is

$$\sigma_{\text{m}} = \int_{\theta_m}^\pi \frac{d\sigma}{d\theta}(1 - \cos\theta)\,d\theta \approx 4\pi b_0^2 \log\Lambda = \frac{m^2 Y}{\mu^2 v_{\text{r}}^4}, \qquad Y = 4\pi\left(\frac{Q_1 Q_2}{4\pi\varepsilon_0 m}\right)^2 \log\Lambda$$

where $\log\Lambda = \log(b_m/b_0)$ is the Coulomb logarithm. For most problems, $\log\Lambda \gg 1$, which means that most collisions are small-angle collisions. The momentum transfer rate is

$$\nu_{\text{m}} = N_s v_{\text{r}} \sigma_{\text{m}}(v_{\text{r}}) = N_s \frac{m^2 Y}{\mu^2 v_{\text{r}}^3}$$

where N_s is the density of scatterers at velocity $\mathbf{V}$. Using (23.17), we have

$$\Delta\mathbf{v} = \frac{\mu}{m}\Delta\mathbf{v}_{\text{r}}$$

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From formulas for small-angle collisions (23.13–23.14), we have, for fixed $\mathbf{V}$ and $\mathbf{v}$:

$$\nu_c \langle \Delta\mathbf{v} \rangle_\Omega = -\frac{\mu}{m}\nu_m(v_r)\mathbf{v}_r = -N_s\frac{mY}{\mu v_r^3}\mathbf{v}_r = N_s\frac{mY}{\mu}\nabla_{\mathbf{v}_r}\left(\frac{1}{v_r}\right) \qquad (23.33)$$

$$\nu_c \langle \Delta\mathbf{v}\Delta\mathbf{v}^T \rangle_\Omega = \left(\frac{\mu}{m}\right)^2 \nu_m(v_r)v_r^2 \overset{\leftrightarrow}{\mathsf{I}}_\perp = N_s\frac{Y}{v_r}\overset{\leftrightarrow}{\mathsf{I}}_\perp = N_sY\nabla_{\mathbf{v}_r}\nabla_{\mathbf{v}_r}^T v_r \qquad (23.34)$$

Here we used (23.10) in the last equation. At the next step, we use $\nabla_{\mathbf{v}_r} \equiv \nabla_{\mathbf{v}}$ and integrate over scatterer velocities $\mathbf{V}$ using $N_s = F(\mathbf{V})\,d^3\mathbf{V}$ where $F(\mathbf{V})$ is their distribution. We get

$$\mathbf{D}^{(1)} = \langle \nu_c\Delta\mathbf{v} \rangle = Y\frac{(M+m)}{M}\nabla_{\mathbf{v}}\int\frac{F(\mathbf{V})}{|\mathbf{v}-\mathbf{V}|}\,d^3\mathbf{V}$$

$$\overset{\leftrightarrow}{\mathsf{D}}^{(2)} = \frac{1}{2}\langle \nu_c\Delta\mathbf{v}\Delta\mathbf{v}^T \rangle = \frac{Y}{2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}^T\int F(\mathbf{V})\,|\mathbf{v}-\mathbf{V}|\,d^3\mathbf{V}$$

We can introduce functions $H(\mathbf{v})$ and $G(\mathbf{v})$ called *Rosenbluth potentials*:

$$H(\mathbf{v}) = \frac{(M+m)}{M}\int\frac{F(\mathbf{V})}{|\mathbf{v}-\mathbf{V}|}\,d^3\mathbf{V}$$

$$G(\mathbf{v}) = \int F(\mathbf{V})\,|\mathbf{v}-\mathbf{V}|\,d^3\mathbf{V}$$

and get relatively simple expressions

$$\mathbf{D}^{(1)} = Y\nabla_{\mathbf{v}}H(\mathbf{v})$$

$$\overset{\leftrightarrow}{\mathsf{D}}^{(2)} = \frac{Y}{2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}^T G(\mathbf{v})$$

Some simplification in Boltzmann equation arises from these relations:

$$\nabla_{\mathbf{v}}^2 G = \frac{2MH}{M+m}$$

$$\nabla_{\mathbf{v}}^2 H = -\frac{4\pi(M+m)F(\mathbf{v})}{M}$$

$$\nabla_{\mathbf{v}}^4 G = -8\pi F(\mathbf{v})$$

etc.

For the case of electron-electron collisions, after a lot of math one can get the following collision integral for the energy distribution function $n(\mathcal{E})$, which is normalized in such a way that $\int n(\mathcal{E})\,d\mathcal{E} = N_e$. It is given here for reference [see, e.g., W. L. Morgan and B. M. Penetrante, *Computer Physics Communications*, **58**, 127 (1990)]:

$$\left(\frac{\partial n}{\partial t}\right)_{\text{e-e}} = \sqrt{\frac{m^3}{18}}Y\left[\frac{3}{\mathcal{E}^{1/2}}n^2 + 2\mathcal{E}^{3/2}\frac{\partial\psi}{\partial\mathcal{E}}\frac{\partial}{\partial\mathcal{E}}\left(\frac{\partial n}{\partial\mathcal{E}} - \frac{n}{2\mathcal{E}}\right) + \frac{\psi}{\mathcal{E}^{1/2}}\left(\frac{\partial n}{\partial\mathcal{E}} - \frac{n}{2\mathcal{E}}\right)\right]$$

where

$$\psi(\mathcal{E},t) = 3\int_0^\mathcal{E} n(\mathcal{E}')\,d\mathcal{E}' - \frac{1}{\mathcal{E}}\int_0^\mathcal{E}\mathcal{E}'n(\mathcal{E}')\,d\mathcal{E}' + 2\mathcal{E}^{1/2}\int_\mathcal{E}^\infty\frac{n(\mathcal{E}')}{\mathcal{E}'^{1/2}}\,d\mathcal{E}'$$

23.6 Small energy losses in inelastic collisions

Consider a process in which an electron excites an infinitely heavy molecule from discrete lower energy level to discrete upper (excited) energy level. For simplicity, assume levels are non-degenerate. The energy distance between levels is $\mathcal{E}_s$. In a thermal gas, if the population density of lower level is N_s, then the population density of the excited level is $N_s^* = N_s e^{-\mathcal{E}_s/T}$. The electrons can also collide with the excited molecules and gain energy from de-excitation. This process is called "superelastic collision". The inelastic cross-section $\sigma_s(\mathcal{E})$ and superelastic cross-section $\sigma_s^*(\mathcal{E})$ are related by the Klein-Rosseland relation [O. Klein and S. Rosseland (1921), *Über Zusammenstöße zwischen Atomen und freien Elektronen*, Zeitschrift für Physik, **4**(1), 46–51, doi:10.1007/BF01328041]:

$$\mathcal{E}\sigma_s(\mathcal{E}) = (\mathcal{E} - \mathcal{E}_s)\sigma_s^*(\mathcal{E} - \mathcal{E}_s) \tag{23.35}$$

This equation can be derived in two ways. In the first way, we can consider detailed balance when the electrons also have thermal distribution with temperature T. The relation (23.35) has to be satisfied in order for the electron distribution $n(\mathcal{E})$ to be stationary for any T. Another way is to look at the dynamics of an electron. We can relax the infinite mass requirement and consider relative motion of the electron with reduced mass μ and relative velocity $\mathbf{v}_{\mathrm{r}}$. Let b be the target parameter. Denote with primes the variables after inelastic collision (or before superelastic collision), so that the energy conservation is

$$\mathcal{E}_{\mathrm{r}}' = \frac{\mu v_{\mathrm{r}}'^2}{2} = \mathcal{E}_{\mathrm{r}} - \mathcal{E}_s = \frac{\mu v_{\mathrm{r}}^2}{2} - \mathcal{E}_s$$

The "relative" angular momentum is conserved in the collision, so

$$\mu v_{\mathrm{r}} b = \mu v_{\mathrm{r}}' b'$$

By squaring the last equation and taking its differential we have $\mathcal{E}_{\mathrm{r}}(2\pi b\, db) = \mathcal{E}_{\mathrm{r}}'(2\pi b'\, db')$, or $\mathcal{E}_{\mathrm{r}}\, d\sigma_s = \mathcal{E}_{\mathrm{r}}'\, d\sigma_s^*$. The solid angle for direct and reverse process is the same, so we have

$$\mathcal{E}_{\mathrm{r}}\frac{d\sigma_s}{d\Omega}(\mathcal{E}_{\mathrm{r}}, \theta) = \mathcal{E}_{\mathrm{r}}'\frac{d\sigma_s^*}{d\Omega}(\mathcal{E}_{\mathrm{r}}', \theta) \tag{23.36}$$

by integrating which we get (23.35).

Using (23.35), we find that the collision rates $\nu = Nv\sigma$ for inelastic and superelastic process are related by

$$\nu_s^*(\mathcal{E}) = N_s^* v(\mathcal{E})\sigma_s^*(\mathcal{E}) = \nu_s(\mathcal{E} + \mathcal{E}_s)\frac{\sqrt{\mathcal{E} + \mathcal{E}_s}\,e^{-(\mathcal{E}+\mathcal{E}_s)/T}}{\sqrt{\mathcal{E}}\,e^{-\mathcal{E}/T}} \tag{23.37}$$

Now, assume that the energy losses are small, $\mathcal{E}_s \ll \mathcal{E}, T$. Then we can find the Fokker-Planck coefficients

$$\langle \nu_s \Delta \mathcal{E} \rangle = -\mathcal{E}_s \nu_s(\mathcal{E}) + \mathcal{E}_s \nu_s^*(\mathcal{E})$$

$$= \mathcal{E}_s \frac{1}{\sqrt{\mathcal{E}}e^{-\mathcal{E}/T}} \left[-\nu_s(\mathcal{E})\sqrt{\mathcal{E}}e^{-\mathcal{E}/T} + \nu_s(\mathcal{E}+\mathcal{E}_s)\sqrt{\mathcal{E}+\mathcal{E}_s}e^{-(\mathcal{E}+\mathcal{E}_s)/T} \right]$$

$$\approx \frac{\mathcal{E}_s}{\sqrt{\mathcal{E}}e^{-\mathcal{E}/T}} \mathcal{E}_s \frac{d}{d\mathcal{E}} \left[\nu_s(\mathcal{E})\sqrt{\mathcal{E}}e^{-\mathcal{E}/T} \right] = \mathcal{E}_s^2 \left[\frac{d\nu_s}{d\mathcal{E}} + \nu_s \left(-\frac{1}{T} + \frac{1}{2\mathcal{E}} \right) \right]$$

and

$$\frac{1}{2}\langle \nu_s (\Delta\mathcal{E})^2 \rangle = \frac{\mathcal{E}_s^2}{2} \left[\nu_s(\mathcal{E}) + \nu^*(\mathcal{E}) \right] \approx \mathcal{E}_s^2 \nu_s(\mathcal{E})$$

The Fokker-Planck equation in form (23.29) becomes

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial\mathcal{E}} \left[\mathcal{E}_s^2 \nu_s \sqrt{\mathcal{E}} \frac{\partial}{\partial\mathcal{E}} \left(\frac{n}{\sqrt{\mathcal{E}}} \right) + \frac{\mathcal{E}_s^2 \nu_s}{T} n \right]$$

We see that the requirement on the coefficients (23.28) holds. By considering also elastic losses, we see that we have to change in (23.31)

$$\nu_{\mathcal{E}} \to \nu_{\mathcal{E}} + \frac{\mathcal{E}_s^2}{T\mathcal{E}} \nu_s \tag{23.38}$$

Let us consider a common application of this result.

23.6.1   Energy correction for rotation inelastic collisions

In molecular gases such as N_2, O_2 and H_2, the rotational transitions ($\lesssim 10^{-3}$ eV) are easily excited by electrons with energies of a fracton of 1 eV. The rotational excitations may make a much greater contribution than the elastic collisions to the changes in the electron energies. This is due to the fact that the electron interacts with the quadrupole moment of the molecule at greater distances than elastic interaction. Here we just give the final result for the contribution of the rotation excitation mechanism to the kinetic equation (23.31). [For derivation, see E. Gerjuoy and S. Stein, *Phys. Rev.*, **97**, 1671 (1955)]. The continuous energy loss rate $\nu_{\mathcal{E}}$ should be changed in (23.31) and (24.15) in the next Lecture in the following way:

$$\nu_{\mathcal{E}} \to \nu_{\mathcal{E}} + Nv\sigma_0 \frac{4B}{\mathcal{E}}$$

where N is the density of molecules with quadrupole moment, B is the rotational constant (defined from the rotational energy levels $\mathcal{E}_J = BJ(J+1)$, $B \sim 10^{-4}$ eV) and

$$\sigma_0 = \frac{8\pi}{15} a_0^2 Q^2$$

where $Q \sim 1$ is the molecule quadrupole moment in units of ea_0^2 (cgs system of units) and $a_0 \approx 5.3 \times 10^{-11}$ m is the Bohr radius. Note that this result is not completely identical to (23.38) because we must sum over many different rotational transition processes.

23.7 *[Optional]* Effect of quasi-static electric field

In our derivation above for FP equation for the energy distribution function, we assumed large-angle scattering. The FP derivation was possible because we assumed that electron velocity distribution is isotropic. If an electric field is applied to the system, however, the distribution is not isotropic, and calculation of some moments cannot be done without *spherical harmonic expansion* of $f(\mathbf{v})$ (see, e.g., Dr. Victor Pasko's lecture for this course or the next Lecture). It is interesting, however, that we can still calculate some moments of $\Delta\mathcal{E}$. We have to be very careful to remember that the action of electric field on electron produces continuous change of electron energy $\Delta\mathcal{E}$. We have

$$\Delta_{\mathbf{E}}\mathcal{E} = m\mathbf{v} \cdot \dot{\mathbf{v}}\Delta t = -e\mathbf{E} \cdot \mathbf{v}\Delta t \tag{23.39}$$

One of the assumptions for FPE is that the change $\Delta\mathcal{E}$ in time interval Δt is independent from other time intervals. It means that Δt has to be large enough for the electron has to "forget" its velocity through collisions, so that $(\mathbf{E} \cdot \mathbf{v})$ at t becomes independent from its value at moment $t - \Delta t$. This requirement is equivalent to the requirement of the spherical harmonic expansion approach that the distribution has to be (almost) isotropic. Another note to be made here is that $\Delta_{\mathbf{E}}\mathcal{E}$ is independent in the statistical sense from $\Delta\mathcal{E}$ due to collisions so that the moments that we obtain now can be simply added to expressions (23.26–23.27).

Using (23.39), $\langle\Delta_{\mathbf{E}}\mathcal{E}\rangle/\Delta t = -e\langle\mathbf{E} \cdot \mathbf{v}\rangle$ and it would be zero for isotropic distribution. However, when electric field is applied to the system, the distribution is deviating from isotropic, and this is not zero. We cannot calculate it if we don't know the anisotropic part of the distribution function. However, in the calculation of the second moment $\langle(\Delta_{\mathbf{E}}\mathcal{E})^2\rangle$, the isotropic part gives the biggest nonzero contribution.

Assuming $\mathbf{E} \parallel \hat{z}$:

$$(\Delta D_{\mathcal{E}})_{\mathbf{E}} = \frac{\langle(\Delta_{\mathbf{E}}\mathcal{E})^2\rangle}{2\Delta t} = \frac{e^2 E^2}{2\Delta t}\left\langle\left[\int_0^{\Delta t} v_z(t)\,dt\right]^2\right\rangle = \frac{e^2 E^2}{2\Delta t}\int_0^{\Delta t}\int_0^{\Delta t}\langle v_z(t_1)v_z(t_2)\rangle\,dt_1\,dt_2$$

$$= \frac{e^2 E^2}{2}\int_{-\infty}^{+\infty}\langle v_z(0)v_z(\tau)\rangle\,d\tau = \frac{e^2 E^2}{6}\int_{-\infty}^{+\infty}\langle\mathbf{v}(0) \cdot \mathbf{v}(\tau)\rangle\,d\tau$$

Here in the second line we changed integration variables to $\tau = t_2 - t_1$ and $t = t_1$ and integrated over t (getting a factor Δt) assuming that the velocity correlation function $\langle v_z(t_1)v_z(t_2)\rangle$ depends only on time difference ($\mathbf{v}(t)$ is a stationary random process). We extended integration over τ to infinity, because of our assumptions of independence of $\mathbf{v}$ at time intervals $> \Delta t$. In the last step we used isotropicity of $\mathbf{v}$.

To find the correlation $\langle\mathbf{v}(0) \cdot \mathbf{v}(\tau)\rangle$ we have to sum over the number of collisions that electron experiences between $t = 0$ and $t = \tau$. The probability of n collisions is given by Poisson distribution:

$$P(n, |\tau|) = \frac{(\nu_{\mathrm{c}}\,|\tau|)^n}{n!}e^{-\nu_{\mathrm{c}}|\tau|}$$

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So,

$$\langle \mathbf{v}(0) \cdot \mathbf{v}(\tau) \rangle = v^2 e^{-\nu_c |\tau|} \sum_{n=0}^{\infty} \frac{(\nu_c |\tau|)^n}{n!} \langle \cos \gamma_n \rangle$$

where γ_n is the angle between original velocity and velocity after n collisions. Since after 1 collision $\langle \mathbf{v}_1' \rangle = \mathbf{v} \langle \cos \theta \rangle$ after averaging over angles, after n collisions we have $\langle \mathbf{v}_n' \rangle = \mathbf{v} \langle \cos \theta \rangle^n$. Therefore,

$$\langle \cos \gamma_n \rangle = \langle \cos \theta \rangle^n$$

where θ is the angle change in a single collisions. Thus, we get

$$\langle \mathbf{v}(0) \cdot \mathbf{v}(\tau) \rangle = v^2 e^{-\nu_c |\tau|} \sum_{n=0}^{\infty} \frac{(\nu_c |\tau| \langle \cos \theta \rangle)^n}{n!} = v^2 e^{-\nu_c (1 - \langle \cos \theta \rangle)|\tau|} = v^2 e^{-\nu_m |\tau|}$$

Substituting this into expression for $D_{\mathbf{E}}$ and integrating over τ, we get

$$(\Delta D_{\mathcal{E}})_{\mathbf{E}} = \frac{e^2 E^2}{6} v^2 \frac{2}{\nu_m} = \frac{2 e^2 E^2}{3 m \nu_m} \mathcal{E}$$

A derivation for the complete equation for $n(\mathcal{E})$ is given in the next Lecture by utilizing spherical harmonic expansion which also takes care of the anisotropic part of the distribution function.

Lecture 24

Kinetics of driven plasmas II: Spherical harmonic expansion of Boltzmann equation

24.1 Spherical harmonic expansion

Let $f(\mathbf{v}, t)$ be the electron distribution function in velocity space, normalized so that $\int f(\mathbf{v}, t)d^3\mathbf{v} = N_e$. The kinetic equation for $f(\mathbf{v}, t)$ is

$$\frac{\partial f}{\partial t} - \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}_E) \cdot \nabla_\mathbf{v} f = \left(\frac{\partial f}{\partial t}\right)_c \tag{24.1}$$

Here $e = -q_e > 0$, $\mathbf{E} = \mathbf{E}(t)$ is the electric field of the wave and $\mathbf{B}_E = const$ is the local geomagnetic field. We neglect the action of the magnetic field of the wave on the non-relativistic electrons, since this is small compared to the action of electric field.

We use the fact that $f(\mathbf{v}, t)$ is not excessively anisotropic to expand it in a spherical harmonic series:

$$f(\mathbf{v}, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\Omega) f_l^m(v, t)$$

where $v = |\mathbf{v}|$ and $\Omega = (\theta, \phi)$ determines the direction of $\mathbf{v}$, θ and ϕ being the polar and azimuthal angle. The spherical harmonics are

$$Y_l^m(\Omega) = Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\phi}$$

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where the associated Legendre polynomials P_l^m are given by

$$P_l^m(\cos\theta) = \frac{(-1)^m}{2^l l!}(\sin\theta)^m \frac{d^{l+m}}{(d\cos\theta)^{l+m}}(\sin\theta)^{2l}$$

$$P_l^{-m}(\cos\theta) = (-1)^m \frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)$$

where $m \geq 0$. The spherical harmonics satisfy the orthonormality relation:

$$\int_{4\pi} Y_l^m(\Omega)(Y_{l'}^{m'}(\Omega))^* d\Omega = \delta_{mm'}\delta_{ll'}$$

We neglect all terms except $l = 0$ and $l = 1$ and use the explicit formulas

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

and therefore look for the solution in the form

$$f(\mathbf{v},t) = \frac{1}{\sqrt{4\pi}}f_0^0(v,t) + \sqrt{\frac{3}{4\pi}}\left[f_1^0(v,t)\cos\theta + \left(\frac{e^{-i\phi}f_1^{-1}(v,t) + e^{i\phi}f_1^1(v,t)}{\sqrt{2}}\right)\sin\theta\right]$$

This can be written as

$$f(\mathbf{v},t) = f_0(v,t) + \hat{v}\cdot\mathbf{f}_1(v,t) \tag{24.2}$$

where $\hat{v} = \mathbf{v}/v = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$ is the unit vector in the direction of $\mathbf{v}$ and

$$f_0(v,t) = \frac{1}{\sqrt{4\pi}}f_0^0$$

$$\mathbf{f}_1(v,t) = \sqrt{\frac{3}{4\pi}}\left(\frac{f_1^{-1}+f_1^1}{\sqrt{2}}\hat{x} + \frac{-if_1^{-1}+if_1^1}{\sqrt{2}}\hat{y} + f_1^0\hat{z}\right)$$

We substitute the expansion (24.2) into (24.1) and use the orthogonality of spherical harmonics to obtain equations for f_0 and $\mathbf{f}_1$. We utilize the following useful relations

which involve $\overleftrightarrow{\mathsf{I}}_\perp = \overleftrightarrow{\mathsf{I}} - \hat{v}\hat{v}^T$ and $\overleftrightarrow{\mathsf{I}}_\| = \hat{v}\hat{v}^T$, operators projecting onto the plane perpendicular to $\hat{v}$ and on the direction of $\hat{v}$. Their sum is $\overleftrightarrow{\mathsf{I}}_\perp + \overleftrightarrow{\mathsf{I}}_\| = \overleftrightarrow{\mathsf{I}}$, the unit operator. We have

$$\nabla_{\mathbf{v}}(\hat{v}\mathbf{f}_1) = \overleftrightarrow{\mathsf{I}}_\perp \frac{\mathbf{f}_1}{v} + \overleftrightarrow{\mathsf{I}}_\| \frac{\partial \mathbf{f}_1}{\partial v} \tag{24.3}$$

$$\left\langle \overleftrightarrow{\mathsf{I}}_\| \right\rangle_\Omega = \frac{1}{3}\overleftrightarrow{\mathsf{I}} \tag{24.4}$$

$$\left\langle \overleftrightarrow{\mathsf{I}}_\perp \right\rangle_\Omega = \frac{2}{3}\overleftrightarrow{\mathsf{I}} \tag{24.5}$$

The last two are equivalent to $\left\langle \overleftrightarrow{\mathsf{I}}_\|\mathbf{A} \right\rangle_\Omega = \langle \hat{v}(\hat{v}\cdot\mathbf{A})\rangle_\Omega = \frac{1}{3}\mathbf{A}$ and $\left\langle \overleftrightarrow{\mathsf{I}}_\perp\mathbf{A} \right\rangle_\Omega = \mathbf{A} - \frac{1}{3}\mathbf{A} = \frac{2}{3}\mathbf{A}$ where $\mathbf{A}$ is an arbitrary vector independent of Ω. The relation (23.22) can be also obtained from (24.4): $\langle (\mathbf{A}\cdot\hat{v})(\mathbf{B}\cdot\hat{v})\rangle_\Omega = \left\langle \mathbf{A}^T(\hat{v}\hat{v}^T)\mathbf{B} \right\rangle_\Omega = \mathbf{A}^T \left\langle \overleftrightarrow{\mathsf{I}}_\| \right\rangle_\Omega \mathbf{B} = \frac{1}{3}\mathbf{A}^T\mathbf{B}$.

Averaging (24.1) over Ω (equivalent to taking a projection onto Y_0^0), we get

$$\frac{\partial f_0}{\partial t} - \frac{e\mathbf{E}}{m}\cdot\left(\frac{2}{3}\frac{\mathbf{f}_1}{v} + \frac{1}{3}\frac{\partial \mathbf{f}_1}{\partial v}\right) = \left\langle \left(\frac{\partial f}{\partial t}\right)_c \right\rangle_\Omega \tag{24.6}$$

Multiplying (24.1) by $\hat{v}$ and averaging over Ω is equivalent to taking a projection onto $Y_1^{-1,0,1}$. After this operation, we get

$$\frac{1}{3}\frac{\partial \mathbf{f}_1}{\partial t} - \frac{1}{3}\frac{e}{m}\left(\mathbf{E}\frac{\partial f_0}{\partial v} + \mathbf{B}_E \times \mathbf{f}_1\right) = \left\langle \hat{v}\left(\frac{\partial f}{\partial t}\right)_c \right\rangle_\Omega$$

24.1.1 Collision integral

We treat the collision integral as described in Chapter 5 of Yu. P. Raizer, "Gas Discharge Physics", Springer, New York (1997). It consists of elastic and inelastic collisions:

$$\left(\frac{\partial f}{\partial t}\right)_c = \left(\frac{\partial f}{\partial t}\right)_{ec} + \left(\frac{\partial f}{\partial t}\right)_{ic}$$

The elastic collisions only change the particle direction (there also a small elastic loss term which was considered earlier using FP approach):

$$\left(\frac{\partial f}{\partial t}\right)_{ec} = \nu_c(v)\int_{\Omega'} d\Omega'\, q(\theta)\left[f(\Omega') - f(\Omega)\right]$$

where $\nu_c(v)$ is the collision rate and

$$q(\theta) = \frac{1}{2\pi\sigma_{el}}\frac{d\sigma_{el}}{d(\cos\theta)} = \frac{1}{\sigma_{el}}\frac{d\sigma_{el}}{d\Omega_{scat}}$$

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is the probability of scattering into angle θ per unit solid angle. After substitution of expansion (24.2), we get

$$
\begin{aligned}
\left(\frac{\partial f}{\partial t}\right)_{ec} &= \nu_c(v) \int_{\Omega'} d\Omega' \, q(\theta) \left[(\hat{v}' - \hat{v}) \cdot \mathbf{f}_1(v) \right] \\
&= -(\hat{v} \cdot \mathbf{f}_1)\nu_c(v) \int_{\Omega'} d\Omega' \, q(\theta)(1 - \cos\theta) = -(\hat{v} \cdot \mathbf{f}_1)\nu_m(v)
\end{aligned}
$$

where $\nu_m = \nu_c \langle 1 - \cos\theta \rangle$ is the elastic momentum transfer rate. It is equal to ν_c for isotropic scattering and usually is a better measure of elastic collision frequency than ν_c for anisotropic scattering. We see that

$$
\begin{aligned}
\left\langle \left(\frac{\partial f}{\partial t}\right)_{ec} \right\rangle_\Omega &= \left\langle \left(\frac{\partial f_0}{\partial t}\right)_{ec} \right\rangle_\Omega = 0 \\
\left\langle \hat{v}\left(\frac{\partial f}{\partial t}\right)_{ec} \right\rangle_\Omega &= \left\langle \hat{v}\left(\hat{v}\cdot\left(\frac{\partial \mathbf{f}_1}{\partial t}\right)_{ec}\right) \right\rangle_\Omega = -\frac{1}{3}\nu_m \mathbf{f}_1
\end{aligned}
$$

The last equation can include the inelastic scattering, too. In this case, ν_m is to be understood as the total momentum transfer rate.

The equation for $\mathbf{f}_1$ can be rewritten so that it is expressed in terms of f_0:

$$
\frac{\partial \mathbf{f}_1}{\partial t} + \nu_m \mathbf{f}_1 - \omega_H \hat{z} \times \mathbf{f}_1 = \frac{e\mathbf{E}}{m}\frac{\partial f_0}{\partial v} \tag{24.7}
$$

where $\omega_H = eB_E/m$ is the electron gyrofrequency, and $\hat{z}$ is the direction of $\mathbf{B}_E$.

24.1.2   Harmonic E field

Let us assume $\mathbf{E} = \mathrm{Re}\,\tilde{\mathbf{E}}e^{-i\omega t}$ and $\mathbf{f}_1 = \mathrm{Re}\,\tilde{\mathbf{f}}_1 e^{-i\omega t}$. The isotropic part f_0 is assumed to change very little during the period of oscillations. Then (24.7) is the real part of the following equation:

$$
(\nu_m - i\omega)\tilde{\mathbf{f}}_1 - \omega_H \hat{z} \times \tilde{\mathbf{f}}_1 = \overset{\leftrightarrow}{\mathsf{A}}(\omega)\tilde{\mathbf{f}}_1 = \frac{e\tilde{\mathbf{E}}}{m}\frac{\partial f_0}{\partial v}
$$

so that

$$
\tilde{\mathbf{f}}_1 = \overset{\leftrightarrow}{\mathsf{A}}^{-1}(\omega)\tilde{\mathbf{E}}\frac{e}{m}\frac{\partial f_0}{\partial v}
$$

where

$$
\overset{\leftrightarrow}{\mathsf{A}}(\omega) = (\nu_m - i\omega)\overset{\leftrightarrow}{\mathsf{I}} - \omega_H \overset{\leftrightarrow}{\mathsf{T}}
$$

Here we introduced one more operator $\overset{\leftrightarrow}{\mathsf{T}}$ defined by $\overset{\leftrightarrow}{\mathsf{T}}\mathbf{A} = \hat{z} \times \mathbf{A}$. This operator has the properties $\overset{\leftrightarrow}{\mathsf{T}}\overset{\leftrightarrow}{\mathsf{I}}_\parallel = 0$, $\overset{\leftrightarrow}{\mathsf{T}}^2 = -\overset{\leftrightarrow}{\mathsf{I}}_\perp$, $\overset{\leftrightarrow}{\mathsf{T}}\overset{\leftrightarrow}{\mathsf{I}}_\perp = \overset{\leftrightarrow}{\mathsf{T}}$ (where now $\overset{\leftrightarrow}{\mathsf{I}}_\parallel$, $\overset{\leftrightarrow}{\mathsf{I}}_\perp$ are in respect to $\hat{z}$, the

direction of $\mathbf{B}_E$). In the coordinates it is written as

$$\overset{\leftrightarrow}{\mathsf{T}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find the inverse of $\overset{\leftrightarrow}{\mathsf{A}}$, we use the fact that for arbitrary complex numbers α, β ($\alpha \neq 0$)

$$(\alpha \overset{\leftrightarrow}{\mathsf{I}} + \beta \overset{\leftrightarrow}{\mathsf{T}})^{-1} = \left[(\alpha - i\beta) \overset{\leftrightarrow}{\mathsf{I}}_+ + (\alpha + i\beta) \overset{\leftrightarrow}{\mathsf{I}}_- + \alpha \overset{\leftrightarrow}{\mathsf{I}}_\parallel \right]^{-1} = \frac{1}{\alpha - i\beta} \overset{\leftrightarrow}{\mathsf{I}}_+ + \frac{1}{\alpha + i\beta} \overset{\leftrightarrow}{\mathsf{I}}_- + \frac{1}{\alpha} \overset{\leftrightarrow}{\mathsf{I}}_\parallel$$

Here we introduced projection operators $\overset{\leftrightarrow}{\mathsf{I}}_+ = (\overset{\leftrightarrow}{\mathsf{I}}_\perp + i\overset{\leftrightarrow}{\mathsf{T}})/2$ and $\overset{\leftrightarrow}{\mathsf{I}}_- = (\overset{\leftrightarrow}{\mathsf{I}}_\perp - i\overset{\leftrightarrow}{\mathsf{T}})/2$ on counterclockwise (positive, $\hat{x} + i\hat{y}$) and clockwise ($\hat{x} - i\hat{y}$) directions, correspondingly. As projection operators, they have a property of *idempotence* $(\overset{\leftrightarrow}{\mathsf{I}}_k)^2 = \overset{\leftrightarrow}{\mathsf{I}}_k$. The rank of $\overset{\leftrightarrow}{\mathsf{I}}_\pm$, $\overset{\leftrightarrow}{\mathsf{I}}_\parallel$ is 1, so these projections can be considered as a transformation to a different system of coordinates with a complete basis $\{\hat{n}_\pm, \hat{z}\}$. We have

$$\overset{\leftrightarrow}{\mathsf{A}}^{-1}(\omega) = \frac{1}{\nu_m - i(\omega - \omega_H)} \overset{\leftrightarrow}{\mathsf{I}}_+ + \frac{1}{\nu_m - i(\omega + \omega_H)} \overset{\leftrightarrow}{\mathsf{I}}_- + \frac{1}{\nu_m - i\omega} \overset{\leftrightarrow}{\mathsf{I}}_\parallel \tag{24.8}$$

For propagation along $\mathbf{B}_E$, the counter-clockwise $\tilde{\mathbf{E}}_+ = \overset{\leftrightarrow}{\mathsf{I}}_+\tilde{\mathbf{E}}$ and clockwise $\tilde{\mathbf{E}}_- = \overset{\leftrightarrow}{\mathsf{I}}_-\tilde{\mathbf{E}}$ electric field components are usually referred to as *extaordinary* and *ordinary* modes. Note that the extraordinary mode has a resonance at $\omega = \omega_H$.

Let us now turn to the solution of (24.6), which becomes

$$\frac{\partial f_0}{\partial t} - \frac{e}{3mv^2} \frac{\partial(v^2 \mathbf{E} \cdot \mathbf{f}_1)}{\partial v} = Q[f_0]$$

where $Q[f_0] = \left\langle \left(\frac{\partial f}{\partial t} \right)_{ic} \right\rangle_\Omega$ is the inelastic collision integral. It has a functional dependence on f_0 (which is indicated by square brackets) and is a function of v.

When substituting $\mathbf{f}_1$, we have to find the average of the product $\mathbf{E} \cdot \mathbf{f}_1$ over a period of oscillations. It is written in terms of complex amplitudes as $\frac{1}{2}\mathrm{Re}\,(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{f}}_1)$:

$$\frac{\partial f_0}{\partial t} - \frac{e^2}{6m^2 v^2} \frac{\partial}{\partial v} \left(\mathrm{Re}\, \left(\tilde{\mathbf{E}}^* \cdot \overset{\leftrightarrow}{\mathsf{A}}^{-1}(\omega) \tilde{\mathbf{E}} \right) v^2 \frac{\partial f_0}{\partial v} \right) = Q[f_0]$$

Let us switch from $f_0(v)$ to

$$n(\mathcal{E}, t) = 2\pi \left(\frac{2}{m} \right)^{3/2} f_0 \left(\sqrt{\frac{2\mathcal{E}}{m}}, t \right) \mathcal{E}^{1/2}$$

which is normalized so that $\int n(\mathcal{E})d\mathcal{E} = N_e$. Using $\partial/\partial v = \sqrt{2m\mathcal{E}}\,\partial/\partial\mathcal{E}$, we get the kinetic equation for n in form (23.29):

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial \mathcal{E}} \left(D_{\mathbf{E}} \mathcal{E}^{3/2} \frac{\partial}{\partial \mathcal{E}} \frac{n}{\mathcal{E}^{1/2}} \right) + Q[n] \tag{24.9}$$

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(**NOTE:** The "diffusion coefficient" used here is D from equation (23.29), not $D_\mathcal{E}$ of the previous Lecture.)

The inelastic collision intergral $Q[n]$ is now a function of $\mathcal{E}$. For $T \neq 0$ it includes superelastic terms, describing the gain of energy by electrons from excited molecules:

$$Q[n] = Q^i[n] + Q^s[n] \tag{24.10}$$

where the inelastic and superelastic parts are

$$Q^i[n] = \sum_s [-\nu_s(\mathcal{E})n(\mathcal{E}) + \nu_s(\mathcal{E} + \mathcal{E}_s)n(\mathcal{E} + \mathcal{E}_s)] \tag{24.11}$$

$$Q^s[n] = \sum_s [-\nu_s^*(\mathcal{E})n(\mathcal{E}) + \nu_s^*(\mathcal{E} - \mathcal{E}_s)n(\mathcal{E} - \mathcal{E}_s)] \tag{24.12}$$

The sums are taken over different inelastic processes, with rates $\nu_s(\mathcal{E}) = N_s v(\mathcal{E})\sigma_s(\mathcal{E})$, where N_s is the density of the species that participate in process s, $v(\mathcal{E})$ is the velocity at energy $\mathcal{E}$ and σ_s is the inelastic cross-section. The superelastic collision rate ν_s^* is given by an analogous formula and is related to ν_s by equation (23.37).

The "diffusion coefficient" in energy space due to elastic collisions is given by

$$D_\mathbf{E}(\mathcal{E}) = \frac{e^2 \operatorname{Re}(\tilde{\mathbf{E}}^* \cdot \overleftrightarrow{\mathsf{A}}^{-1}(\omega)\tilde{\mathbf{E}})}{3m} \tag{24.13}$$

For constant electric field, equation (24.13) becomes

$$D_\mathbf{E}(\mathcal{E}) = \frac{2e^2\mathbf{E} \cdot \overleftrightarrow{\mathsf{A}}^{-1}(0)\mathbf{E}}{3m}$$

where the factor of 2 comes from differences in time averaging of oscillating and constant cases.

In electric field components,

$$
\begin{aligned}
D_\mathbf{E}(\mathcal{E}) &= D_+(\mathcal{E}) + D_-(\mathcal{E}) + D_z(\mathcal{E}) \tag{24.14}\\
&= \frac{e^2|\tilde{\mathbf{E}}_+|^2}{3m}\frac{\nu_m}{(\omega - \omega_H)^2 + \nu_m^2} + \frac{e^2|\tilde{\mathbf{E}}_-|^2}{3m}\frac{\nu_m}{(\omega + \omega_H)^2 + \nu_m^2} + \frac{e^2|\tilde{\mathbf{E}}_z|^2}{3m}\frac{\nu_m}{\omega^2 + \nu_m^2}
\end{aligned}
$$

where $\tilde{\mathbf{E}}_{\pm,z} = \overleftrightarrow{\mathsf{I}}_{\pm,\|}\tilde{\mathbf{E}}$. For each component

$$(D_\mathbf{E})_{\pm,z} = \frac{e^2|\tilde{\mathbf{E}}_{\pm,z}|^2}{3m}\frac{\nu_m}{\omega_{\text{eff}}^2 + \nu_m^2} = \frac{2e^2\langle E_{\pm,z}^2\rangle_t}{3m}\frac{\nu_m}{\omega_{\text{eff}}^2 + \nu_m^2}$$

where $\omega_{\text{eff}} = \omega \mp \omega_H$ for $\tilde{\mathbf{E}}_\pm$ field and $\omega_{\text{eff}} = \omega$ for $\tilde{\mathbf{E}}_z$ field, and there is an additional factor of two if we want to use RMS fields $\langle E^2\rangle_t$ instead of amplitudes. With elastic

energy losses from collisions with molecules of mass M and temperature T the kinetic equation becomes

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial \mathcal{E}}\left[(D_{\mathbf{E}} + \nu_{\mathcal{E}}T)\,\mathcal{E}^{3/2}\frac{\partial}{\partial \mathcal{E}}\left(\frac{n}{\mathcal{E}^{1/2}}\right) + \nu_{\mathcal{E}}\mathcal{E}n\right] + Q[n] \qquad (24.15)$$

where we used the result (23.31) from the previous Lecture obtained by Fokker-Planck approach. The same result can be obtained using the spherical harmonic expansion of the collision integral, but that way is more cumbersome and computationally intensive. The collision integral $Q(n)$ for $T \neq 0$ includes superelastic terms, describing the gain of energy by electrons from excited molecules. If $\omega = 0$, the kinetic equation is the same (24.15) except for the expression for $D(\mathcal{E})$:

$$D(\mathcal{E})|_{\omega_{\text{eff}}=0} = \frac{2e^2|\mathbf{E}|^2}{3m\nu_m}$$

Comparing to (24.14), we see that the oscillatory electric field has approximately the same effect as in the DC case, but with a reduced effective value:

$$E_{\text{eff}} = \frac{E}{\sqrt{1 + \left(\frac{\omega_{\text{eff}}}{\nu_{m,\text{eff}}}\right)^2}} \qquad (24.16)$$

with some effective (average) value of collision frequency $\nu_{m,\text{eff}}$. When $\nu_m \ll \omega_{\text{eff}}$, the reduction is by a factor of $\nu_{m,\text{eff}}/\omega_{\text{eff}}$.

24.2 Analytical solutions of kinetic equation

If there are only inelastic collisions in continuous approximation, so that they can be all included in $\nu_{\mathcal{E}}$ and $Q[n] \equiv 0$, then the steady-state solution of the kinetic equation (24.15) is

$$n(\mathcal{E}) = C\sqrt{\mathcal{E}}\exp\left\{-\int_0^{\mathcal{E}}\frac{\nu_{\mathcal{E}}\,d\mathcal{E}}{D_{\mathbf{E}} + \nu_{\mathcal{E}}T}\right\} = C\sqrt{\mathcal{E}}\exp\left\{-\int_0^{\mathcal{E}}\frac{d\mathcal{E}}{\frac{Me^2E^2}{3m^2(\nu_m^2+\omega_{\text{eff}}^2)} + T}\right\} \qquad (24.17)$$

where $E^2 = \langle|\mathbf{E}|^2\rangle_t = |\tilde{\mathbf{E}}_\pm|^2/2$ is the square of the RMS field and $\omega_{\text{eff}} = \omega \mp \omega_H$. Under assumption of a constant free path length $l = v/\nu_m = (N\sigma_m)^{-1} = const$ and absence of inelastic collisions, the electrons for $T = 0$ have a so-called *Margenau distribution* (H. Margenau (1946),*Phys. Rev.*, **69**, p. 508):

$$f_0(v) = C\exp\left[-\frac{3m^3}{4Me^2E^2l^2}(v^4 + 2v^2\omega_{\text{eff}}^2l^2)\right]$$

where $E^2 = \langle|\mathbf{E}|^2\rangle = |\tilde{\mathbf{E}}_\pm|^2/2$ is the square of the RMS field and $\omega_{\text{eff}} = \omega \mp \omega_H$. For $\omega_{\text{eff}} = 0$, it turns into a *Druyvesteyn distribution*, which is significantly different in shape from a Maxwellian distribution.

24.3   Conductivity

The conductivity tensor is expressed through the anisotropic part of the distribution function:

$$\tilde{\mathbf{J}} = \overleftrightarrow{\sigma}\tilde{\mathbf{E}} = -e\int \mathbf{v}(\hat{v}\cdot\tilde{\mathbf{f}}_1(v))d^3\mathbf{v}$$

from which we obtain the conductivity tensor

$$\overleftrightarrow{\sigma} = -\frac{4\pi e^2}{3m}\int \overleftrightarrow{\mathsf{A}}^{-1}(\omega)v^3\frac{\partial f_0}{\partial v}dv$$

where $\overleftrightarrow{\mathsf{A}}^{-1}(\omega)$ is given by (24.8). In terms of $n(\mathcal{E})$,

$$\overleftrightarrow{\sigma} = -\frac{2e^2}{3m}\int \overleftrightarrow{\mathsf{A}}^{-1}(\omega)\mathcal{E}^{3/2}\frac{\partial}{\partial\mathcal{E}}\frac{n}{\mathcal{E}^{1/2}}d\mathcal{E}$$

For extraordinary and ordinary waves and the parallel component

$$\sigma_{\pm} = -\frac{2e^2}{3m}\int \frac{\mathcal{E}^{3/2}}{\nu_m - i(\omega \mp \omega_H)}\frac{\partial}{\partial\mathcal{E}}\frac{n}{\mathcal{E}^{1/2}}d\mathcal{E} \tag{24.18}$$

$$\sigma_z = -\frac{2e^2}{3m}\int \frac{\mathcal{E}^{3/2}}{\nu_m - i\omega}\frac{\partial}{\partial\mathcal{E}}\frac{n}{\mathcal{E}^{1/2}}d\mathcal{E} \tag{24.19}$$

The usual conductivity tensor representation in Pedersen, Hall and parallel components is

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_p & -\sigma_h & 0 \\ \sigma_h & \sigma_p & 0 \\ 0 & 0 & \sigma_z \end{pmatrix} \tag{24.20}$$

which can be written in our notation as $\overleftrightarrow{\sigma} = \sigma_p\overleftrightarrow{\mathsf{I}}_\perp + \sigma_h\overleftrightarrow{\mathsf{T}} + \sigma_z\overleftrightarrow{\mathsf{I}}_\|$, with $\sigma_p = \frac{1}{2}(\sigma_+ + \sigma_-)$ and $\sigma_h = \frac{i}{2}(\sigma_+ - \sigma_-)$.