Aladin: P. Loiseau
Impact: J. Brodrick, Ch. Ridgers, R. Kingham

An efficient kinetic modeling in plasmas by using the AWBS transport equation

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Text of abstract.

I. INTRODUCTION

The first attempts of ~~modern~~ kinetic modeling of plasma can be tracked back to the fifties, when Cohen, Spitzer, and Routly (CSR) [1] ~~in detail~~ demonstrated ~~the fact,~~ that in the ionized gas the effect of Coulomb collisions between ~~electrons~~ and ions predominantly results from frequently occurring events of cumulative small deflections rather than occasional close encounters. This effect was originally described in [2] and Chandrasekhar [3] proposed to use the diffusion equation model of the Fokker-Planck type (FP) [4].

in a plasma with a temperature gradient accounting for e-e and e-i collisions

~~As a result,~~ a classical paper by Spitzer and Harm (SH) [5] provides the computed electron distribution function ~~spanning from low to high Z plasmas, and more importantly,~~ the expressions for current and heat flux ~~formulas, which~~ are widely used in ~~almost~~ every plasma hydrodynamic ~~code nowadays~~ models. The distribution function is of the form $f^0 + \mu f^1$, where f^0 and f^1 are isotropic and μ, is the direction cosine between the particle trajectory and some preferred direction in space. It should be emphasized that the SH solution expresses a small perturbation of equilibrium, i.e. that f^0 is the Maxwell-Boltzmann distribution and μf^1 represents a very small deviation.

This approximation is valid for $L_T \gg \lambda_{ei}$, the condition which is often invalid in fusion plasmas

The actual cornerstone of the modern FP simulations was set in place by Rosenbluth [6], when he derived a simplified form of the FP equation for a finite expansion of the distribution function, where all the terms are computed according to plasma conditions, including f^0, which of course needs to tend to the Maxwell-Boltzmann distribution.

In is the purpose of this paper to present an efficient alternative to FP model based on the Albritton-Williams-Bernstein-Swartz collision operator (AWBS) [7]. In Section II we propose a modified form of the AWBS collision operator, where its important properties are further presented in Section III with the emphasis on its comparison to the full FP solution in local diffusive regime. Section IV focuses on the performance of the AWBS transport equation model compared to modern kinetic codes including FP codes Aladin and Impact, and PIC code Calder, where the cases related to real laser generated plasma conditions are studied. Finally, the most important outcomes of our research are concluded in Section V.

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Kinetic simulations [Bell et al 1981 PRL, Luciani et al 1983 PRL] showed that in non-local conditions ($L_T \approx \lambda_{ei}$) the anisotropic correction cannot be considered to be small, $f^1 \sim f^0$, and there are significant deviations of f^0 from Maxwellian distribution.

II. THE AWBS KINETIC MODEL

Use physical non-normalized units!

The electrons in plasma can be modeled by the deterministic model of charged particles

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \frac{q_e}{m} \tilde{\boldsymbol{E}} \cdot \nabla_{\boldsymbol{v}} f = C_{ee}(f) + C_{ei}(f), \quad (1)$$

where $f(t, \boldsymbol{x}, \boldsymbol{v})$ represents the density function of electrons at time t, spatial point $\boldsymbol{x}$, and velocity $\boldsymbol{v}$, and ~~$\tilde{\boldsymbol{E}} = \frac{q_e}{m_e}\boldsymbol{E}$~~ $\boldsymbol{E}$ is the ~~existing~~ electric field in plasma.

The generally ~~accepted~~ form of the electron-electron collision operator C_{ee} is the Fokker-Planck form published by Landau [8]

$$C_{FP}(f) = \Gamma \int \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot (f\, \nabla_{\tilde{\boldsymbol{v}}} f - f\, \nabla_{\boldsymbol{v}} f)\, \mathrm{d}\tilde{\boldsymbol{v}}, \quad (2)$$

where $\Gamma = \frac{q_e^4 \ln\Lambda}{4\pi\epsilon^2 m_e^2}$ and $\ln\Lambda$ is the Coulomb logarithm. ~~In principal~~, The electron-ion collision operator C_{ei} could be expressed in ~~the~~ a simpler form ~~similar to (2), but~~ since ions are considered to be motionless compared to electrons thanks to their large mass. The scattering operator accounts for the change of electron velocity without ~~i.e. no~~ change in the velocity magnitude, expressed in spherical coordinates ~~is widely accepted~~. It is

$$C_{ei}(f) = \frac{\nu_{ei}}{2}\left(\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial f}{\partial \mu}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2}\right), \quad (3)$$

where $\mu = \cos(\phi)$, ϕ and θ are the polar and azimuthal angles, and $\nu_{ei} = \frac{Z n_e \Gamma}{v^3}$ is the electron-ion collision frequency.

The e-e collision operator needs to be linearized for efficient computations.

Fish introduced a linearized ~~alternative~~ form of C_{ee} ~~in~~ [9] ~~referred to as~~ in the high-velocity limit ($v \gg v_{th}$) ~~electron collision operator~~

$$C_H(f) = v\nu_e \frac{\partial}{\partial v}\left(f + \frac{v_{th}^2}{v}\frac{\partial f}{\partial v}\right)$$
$$+ \frac{\nu_e}{2}\left(1 - \frac{v_{th}^2}{2v^2}\right)\left(\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial f}{\partial \mu}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2}\right), \quad (4)$$

where $\nu_e = \frac{n_e \Gamma}{v^3}$ is the electron-electron collision frequency and $v_{th} = \sqrt{\frac{k_B T_e}{m_e}}$ is the electron thermal velocity. The linear form of C_H arises from an assumption that the fast electrons predominantly interact with the thermal (slow) electrons, which simplifies importantly the ~~nonlinear~~ form (2).

However numerical solution of the FP equations even in the Rosenbluth formulation remains very challenging computationally because the collision integral is nonlinear. More simple linearized forms of e-e collision operator are needed.

However the diffusion term in the ee operator still presents numerical difficulties.

A method of angular momenta for the solution of the electron kinetic equation with the

was proposed by

Here, the first term accounts for relaxation to equilibrium due to the ee collisions, and the second term accounts for the e-e contribution to the diffusion

~~The aim of this work is to use~~ a yet simpler form of the electron-electron collision operator, ~~i.e. the~~ AWBS ~~formulation~~ [7], ~~where we propose~~ ~~the following form~~ the

$$C_{AWBS}(f) = v\frac{\nu_e}{2}\frac{\partial}{\partial v}(f - f_M) + \frac{\nu_{ei} + \frac{\nu_e}{2}}{2}\left(\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial f}{\partial \mu}\right) + \frac{1}{\sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2}\right), \quad (5)$$

where f_M is the Maxwell-Boltzmann equilibrium distribution. ~~C_{AWBS} represents the complete $C_{ee} + C_{ei}$ collision operator in (1).~~

~~The complete form of~~ collision operator (5) was ~~previously~~ introduced in [10, 11], ~~nevertheless, we intentionally use~~ a half of the electron-electron collisional frequency ~~modification $\nu_e/2$~~, because this formulation provides very promising results compared to the full FP operator as emphasized in Section III.

~~The Maxwell-Boltzmann averaged e-e scattering in (4) can be approximated as $\nu_e \int \left(1 - \frac{v_{th}^2}{2v^2}\right) f_M 4\pi v^2 \mathrm{d}v = \frac{\nu_e}{2}$.~~

Following Fisch [9] we introduced

III. BGK, AWBS, AND FOKKER-PLANCK MODELS IN LOCAL DIFFUSIVE REGIME

where $L_T \gg \lambda_{ei}$

Modified forms of the collisional operators (4), (5) do not guarantee that the same expression for collisional fluxes were found by SH in the diffusive limit.

~~In a broad analysis of the electron transport, any qualitative information about its properties are highly welcome. Even better, if one can extract some qualitative information, which provides comparative and reliable results in a clear way, the confidence of using a transport model, e.g. (5), can lead to efficient yet relatively cheap computation cost predictions of real physics.~~

In this paper, we ~~can try to~~ find an approximate solution ~~to~~ in the ~~so-called~~ *local diffusive regime* of electron transport, ~~where~~ the *diffusive regime*, ~~in general~~, refers to a low anisotropy in angle given by μ, and *local* means that the mean free path of electrons λ_{ei} is rather restricted compared to the plasma spatial scale. In the words of mathematics this corresponds to the first order expansion in λ_{ei} and μ of the distribution function as

$$\tilde{f}(z, v, \mu) = f^0(z, v) + f^1(z, v)\lambda_{ei}\mu, \quad (6)$$

$f^1/f_0 \sim \lambda_{ei}/L_T$ where

where z is the spatial coordinate ~~along the axis z~~, v is the magnitude of ~~transport~~ electron velocity, and $\lambda_{ei} = \frac{v}{\nu_{ei}} = \frac{v^4}{Zn_e\Gamma}$. ~~In other words, one can say that by evaluating numerically $\tilde{f}$ in (6), we accept some error of the order $O(\lambda_{ei}^2) + O(\mu^2)$.~~ The ~~expansion in a~~ small parameter λ_{ei}/L_T is also coherent with a ~~time-steady approximation due to~~ the relation between the ~~mean free path and~~ inverse collision frequency, ~~where the higher the collision frequency the more steady the solution.~~

In order to start, we express the ~~time-steady~~ left hand side of (1) in 1D and insert the approximation (6), which

quasi-stationary

and the characteristic hydrodynamic time.

~~leads to~~ The distribution function (6) verifies the following kinetic equation

$$\mu\left(\frac{\partial \tilde{f}}{\partial z} + \frac{\tilde{E}_z}{v}\frac{\partial \tilde{f}}{\partial v}\right) + \frac{\tilde{E}_z(1-\mu^2)}{v^2}\frac{\partial \tilde{f}}{\partial \mu} = C_{AWBS}$$

$$\mu\left(\frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v}\frac{\partial f^0}{\partial v}\right) + \frac{\tilde{E}_z \cancel{\lambda_{ei}}}{v^2}f^1 + \cancel{O(\mu^2)}, \quad (7)$$

and is truncated for low anisotropy, i.e. with error $O(\mu^2)$.

A. The BGK local diffusive electron transport

You need to introduce BGK operator and explain why you are using it.

Even though the BGK plasma collisional operator [12]

$$\cancel{\frac{1}{\nu_e}} C_{BGK}(\tilde{f}) = \nu_{ee}\frac{(\tilde{f} - f_M)}{\cancel{\lambda_e}} + \frac{\nu_{ei} + \nu_{ee}}{\cancel{2\lambda_{ei}}}\,2\,\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial \tilde{f}}{\partial \mu}, \quad (8)$$

where $\lambda_e = Z\lambda_{ei}$, is not actually used in our nonlocal transport simulations, we consider it useful to include this simplest form of the Boltzmann transport collision operator, because of two reasons: a) it can be treated analytically in the local diffusive regime; and b) it represents the so-called phenomenological collision operator by explicitly using the Maxwell-Boltzmann equilibrium distribution f_M, which proves to be very useful in coupling of the nonlocal electron transport to hydrodynamics.

If one applies the action of the right hand side, i.e. of (8), on the approximation (6) and sets the result to be equal to the left hand side (7), the corresponding terms in μ are governed by the following equations

use physical non-normalized units.

$$f^0 = f_M + \frac{\tilde{E}_z}{v^2}f^1 Z\lambda_{ei}^2, \quad (9)$$

$$f^1 = -\frac{Z}{Z+1}\left(\frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v}\frac{\partial f^0}{\partial v}\right), \quad (10)$$

In the steady state

i.e. $f^0 = f_M + O(\lambda_{ei}^2)$ and $f^1 = -\frac{Z}{Z+1}\left(\frac{\partial f_M}{\partial z} + \frac{\tilde{E}_z}{v}\frac{\partial f_M}{\partial v}\right)$. ~~Now~~ the electric current ~~expressing the contribution of every electron naturally tends to~~ must be zero, ~~i.e.~~ the *quasi-neutrality* constraint, ~~which~~ leads to an analytic formula of the self-consistent electric field

$$\boldsymbol{j} \equiv q_e \int \boldsymbol{v} f^1 \,\mathrm{d}v = \boldsymbol{0} \rightarrow \tilde{\boldsymbol{E}} = v_{th}^2\left(\frac{\nabla\rho}{\rho} + \frac{5}{2}\frac{\nabla T}{T}\right). \quad (11)$$

Consequently, based on (9), (10), and (11), the analytic formula (6) of the electron distribution function reads

$$\tilde{f} = f_M - \frac{Z}{Z+1}\left(\frac{v^2}{2v_{th}^2} - 4\right)\frac{1}{T}\frac{\partial T}{\partial z}f_M\lambda_{ei}\mu, \quad (12)$$

which is ~~nothing else than~~ the ~~famous~~ Lorentz electron-ion collision gas model [13] scaled by a constant depending on Z, naturally arising from the BGK model (8).

Say how these expressions compare with the original SH results, and why we need to improve it.

Unfortunately, the BGK collision operator does not conserve the number of particles, which may produce significant errors in the non-local limit.

B. The AWBS local diffusive electron transport

The main object of this work presented in Sec II simplifies in 1D to a relatively simple form of the Boltzmann transport collision operator (compared to (2))

$$\frac{1}{v}C_{AWBS}(f) = \frac{v}{2\lambda_e}\frac{\partial}{\partial v}\left(f - f_M\right) + \frac{1}{2}\left(\frac{1}{\lambda_{ei}} + \frac{1}{2\lambda_e}\right)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}. \quad (13)$$

refer to eq 5.

but it conserves the number of particles which

Similarly to the BGK model, AWBS [13] is also ~~referred to as~~ a phenomenological model, since it explicitly uses the Maxwell-Boltzmann equilibrium distribution f_M, ~~and also~~, makes it a very attractive model of the nonlocal electron transport to be coupled to hydrodynamics via the plasma electron temperature and density.

A ~~qualitative information about~~ the AWBS model is obtained while using Eq ~~repeating the action on~~ (6) ~~by the left hand side (7) and by the right hand side (13) and setting the equality~~. The corresponding ~~terms in μ~~ Solution are then ~~governed by the following~~ equations for f_0 and f_1 given as

$$\frac{\partial}{\partial v}\left(f^0 - f_M\right) = \frac{\tilde{E}_z}{v^3}f^1 2Z\lambda_{ei}^2, \quad (14)$$

$$\frac{v}{2Z\lambda_{ei}}\frac{\partial (f^1\lambda_{ei})}{\partial v} - \frac{2Z+1}{2Z}f^1 = \frac{\partial f^0}{\partial z} + \frac{\tilde{E}_z}{v}\frac{\partial f^0}{\partial v}, \quad (15)$$

i.e. $f^0 \cancel{= f_M + O(\lambda_{ei}^2)}$ remains a Maxw. function in the first order on λ_{ei}/L, however, the f^1 does not have a straightforward analytic formula. In ~~reality, f^1~~ It arises from the ordinary differential equation (~~by inserting f_M into (15)~~)

$$\frac{\partial f^1}{\partial v} + \frac{1}{v}(3-2Z)f^1 = \frac{2Z}{v}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial z} + \left(\frac{v^2}{2v_{th}^2} - \frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial z} - \frac{\tilde{E}_z}{v_{th}^2}\right)f_M. \quad (16)$$

We will stick with a numerical solution of (16), where the details about the resulting distribution function can be found in Section III D.

C. The Fokker-Planck local diffusive electron transport

The Fokker-Plank (2) collision operator can be also written as [14]

$$\frac{1}{v}C_{FP}(f) = \frac{\Gamma}{v}\left(4\pi f^2 + \frac{\nabla_v\nabla_v f : \nabla_v\nabla_v g}{2}\right), \quad (17)$$

where $g(\boldsymbol{v}) = \int |\boldsymbol{v} - \tilde{\boldsymbol{v}}| f(\tilde{\boldsymbol{v}})\, \mathrm{d}\tilde{\boldsymbol{v}}$ is the Rosenbluth potential [6]. Since we are interested in the approximate solution in the local diffusive regime, it is convenient to use a low anisotropy approximation $\tilde{g} = g^0(f^0) + g^1(f^1)\lambda_{ei}\mu$, which arises based on Eq. 45 of [6].

compare it with SH solution and make conclusions.

For a better clarity we present the action of (17) in 1D

$$\begin{aligned} C_{FP}(\tilde{f}) = \Gamma\Bigg(&4\pi {f^0}^2 + \frac{1}{2}\frac{\partial^2 f^0}{\partial v^2}\frac{\partial^2 g^0}{\partial v^2} + \frac{1}{v^2}\frac{\partial f^0}{\partial v}\frac{\partial g^0}{\partial v}\Bigg) \\ &+ \frac{\mu}{Zn_e}\Bigg[8\pi f^0 f^1 v^4 - v\left(\frac{\partial f^0}{\partial v}g^1 + \frac{\partial g^0}{\partial v}f^1\right) \\ &+ \frac{1}{v^2}\left(\frac{\partial f^0}{\partial v}\frac{\partial (g^1 v^4)}{\partial v} + \frac{\partial g^0}{\partial v}\frac{\partial (f^1 v^4)}{\partial v}\right) \\ &+ \frac{1}{2}\left(\frac{\partial^2 f^0}{\partial v^2}\frac{\partial^2 (g^1 v^4)}{\partial v^2} + \frac{\partial^2 g^0}{\partial v^2}\frac{\partial^2 (f^1 v^4)}{\partial v^2}\right)\Bigg] + O(\lambda_{ei}^2, \mu^2), \end{aligned} \quad (18)$$

truncated by the quadratic terms in the angular anisotropy and the transport localization.

If once more repeated the action on (6) by the left hand side (7) and by the right hand side (17) and setting the equality, the equation governing f^0 corresponding to μ^0 takes the form

$$\begin{aligned} 4\pi {f^0}^2 + \frac{1}{2}\frac{\partial^2 f^0}{\partial v^2}\frac{\partial^2 g^0}{\partial v^2} + \frac{1}{v^2}\frac{\partial f^0}{\partial v}\frac{\partial g^0}{\partial v} &= \frac{\tilde{E}_z}{v^5}f^1 Z n_e \lambda_{ei}^2 \\ - \frac{2}{v^2}\left(\frac{\partial f^1\lambda_{ei}}{\partial v} - \frac{f^1\lambda_{ei}}{v}\right)&\left(\frac{\partial g^1\lambda_{ei}}{\partial v} - \frac{g^1\lambda_{ei}}{v}\right), \end{aligned} \quad (19)$$

where the fundamental property of the Fokker-Planck collision operator tending to the Maxwell-Boltzmann distribution f_M [15], leads to $f^0 = f_M + O(\lambda_{ei}^2)$, where we write an explicit form of the quadratic term $O(\lambda_{ei}^2)$ obtained from the truncation (18). The equality corresponding to μ takes the form

$$\begin{aligned} \frac{1}{Zn_e}\Bigg[&\frac{1}{2}\left(\frac{\partial^2 f_M}{\partial v^2}\frac{\partial^2 (g^1 v^4)}{\partial v^2} + \frac{\partial^2 g_M}{\partial v^2}\frac{\partial^2 (f^1 v^4)}{\partial v^2}\right) \\ &+ \frac{1}{v^2}\left(\frac{\partial f_M}{\partial v}\frac{\partial (g^1 v^4)}{\partial v} + \frac{\partial g_M}{\partial v}\frac{\partial (f^1 v^4)}{\partial v}\right) \\ &- v\left(\frac{\partial f_M}{\partial v}g^1 + \frac{\partial g_M}{\partial v}f^1\right) + 8\pi f_M f^1 v^4\Bigg] - vf^1 \\ &= v\frac{\partial f_M}{\partial z} + \tilde{E}_z\frac{\partial f_M}{\partial v}, \end{aligned} \quad (20)$$

which is the equation governing the unknown f^1.

In principle, the solution to the equation (20) is very ambitious, as demonstrated in [1, 3, 6], fortunately, one can use the explicit evaluation of the electron distribution function published in [5], which takes the following form

$$f^1(z,v) = \frac{1}{\lambda_{ei}}\frac{m_e^2}{4\pi q_e^4 \ln\Lambda}\frac{v_{2th}^4}{Z}\left(2d_T(v/v_{2th}) + \frac{3}{2}\frac{\gamma_T}{\gamma_E}d_E(v/v_{2th})\right)\frac{f_M}{n_e}\frac{1}{T}\frac{\partial T_e}{\partial z}, \quad (21)$$

where $d_T(x) = ZD_T(x)/B$, $d_E(x) = ZD_E(x)/A$, γ_T, and γ_E are represented by numerical values in TABLE I, TABLE II, and TABLE III in [5], and $v_{2th} = \sqrt{\frac{k_B T_e}{2m_e}}$.

There is no need to reproduce the SH solution → just give the result.

Here we need a serious discussion of the BGK and AWBS models in the diff. limit in function of Z.

introduce P1 before or introduction

	$Z=1$	$Z=2$	$Z=4$	$Z=16$	$Z=116$
$\bar{\Delta} q_{AWBS}$	0.057	0.004	0.038	0.049	0.004

TABLE I. Relative error $\bar{\Delta} q_{AWBS} = |q_{AWBS} - q_{SH}|/q_{SH}$ of the AWBS kinetic model equation (5) showing the discrepancy (maximum around 5%) with respect to the original solution of the heat flux given by numerical solution in Spitzer and Harm [5].

D. Summary of the BGK, AWBS, and Fokker-Planck local diffusive transport

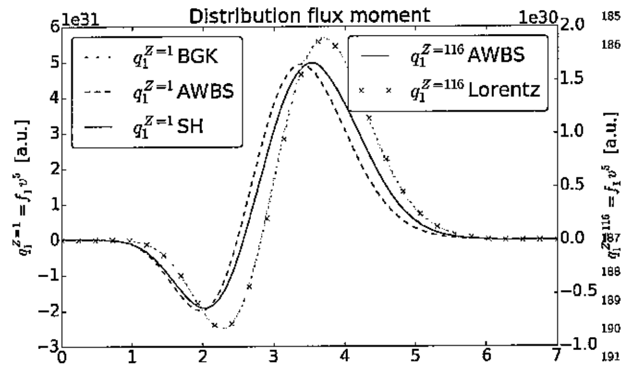

FIG. 1. The flux velocity moment of the anisotropic part of the electron distribution function in low $Z = 1$ and high $Z = 116$ plasmas in diffusive regime.

- Rescale BGK according to Lorentz to match SH heat flux.
- Emphasize the outstanding match between our modified $\frac{\nu_e}{2}$ AWBS model and the SH profiles of q_1 (define) in TABLE I.
- Stress the visual results of AWBS compared to FP presented in FIG. 1.
- Raise a discussion about the structure of f^1 equations (10), (16), and (20).
- Point out the behavior of the maximum velocity of q_1.

Please do all that and also compare the heat flux and electric field.

IV. BENCHMARKING THE AWBS NONLOCAL TRANSPORT MODEL

A. ~~AP1~~ implementation of the model

Here, we ~~AP1 represents the abbreviation AWBS-P1,~~ i.e. the use of collision operator (5) and the P1 angular discretization, i.e. the lowest order anisotropy approximation. AP1 in general belongs to the so-called angular moments method and the electron distribution function takes the form

$$\tilde{f} = \frac{f_0}{4\pi} + \frac{3}{4\pi} \boldsymbol{n} \cdot \boldsymbol{f_1},$$

We use the same notations as in the previous section.

which consists of the isotropic part $f_0 = \int_{4\pi} \tilde{f} \mathrm{d}\boldsymbol{n}$ and the directional part $\boldsymbol{f_1} = \int_{4\pi} \boldsymbol{n} \tilde{f} \mathrm{d}\boldsymbol{n}$, where $\boldsymbol{n}$ is the transport direction (the solid angle).

The first two angular moments applied to the steady form of (1) with collision operator (5) lead to the AP1 model equations ?

$$v \frac{\nu_e}{2} \frac{\partial}{\partial v} \left(f_0 - 4\pi f_M \right) = v \nabla \cdot \boldsymbol{f_1} + \tilde{\boldsymbol{E}} \cdot \frac{\partial \boldsymbol{f_1}}{\partial v} + \frac{2}{v} \tilde{\boldsymbol{E}} \cdot \boldsymbol{f_1}, \quad (22)$$

$$v \frac{\nu_e}{2} \frac{\partial \boldsymbol{f_1}}{\partial v} - \nu_{scat} \boldsymbol{f_1} = \frac{v}{3} \nabla f_0 + \frac{\tilde{\boldsymbol{E}}}{3} \frac{\partial f_0}{\partial v}, \quad (23)$$

where $\nu_{scat} = \nu_{ei} + \frac{\nu_e}{2}$. The strategy of solving (22) and (23) resides in integrating $\frac{\partial f_0}{\partial v}$ and $\frac{\partial \boldsymbol{f_1}}{\partial v}$ in velocity magnitude while starting the integration from infinite velocity to zero velocity, which corresponds to decelerating electrons. It should be noted, that in practice we start the integration from $v = 7 v_{th}$, which represents a sufficiently high velocity.

1. Nonlocal electric field treatment

These parts need to be significantly developed!

Similarly to (11), one can obtain the model equation of the electric field $\tilde{\boldsymbol{E}}$ by evaluating the zero current condition (a velocity integration of (23))

$$\int_v \left(\frac{\frac{\nu_e}{2} v^2}{\nu_{scat}} \frac{\partial \boldsymbol{f_1}}{\partial v} - \frac{v^2}{3 \nu_{scat}} \nabla f_0 - \frac{v}{3 \nu_{scat}} \frac{\partial f_0}{\partial v} \tilde{\boldsymbol{E}} \right) v^2 \, \mathrm{d}v = 0, \quad (24)$$

from which it is easy to express $\tilde{\boldsymbol{E}}$ once f_0 and $\boldsymbol{f_1}$ are known, or in other words, the integral-differential model equations need to be solved simultaneously, which is achieved by k-iteration of $f_0^k(\tilde{\boldsymbol{E}}^k), \boldsymbol{f_1}^k(\tilde{\boldsymbol{E}}^k)$, i.e. (22), (23), and $\tilde{\boldsymbol{E}}^{k+1}(f_0^k, \boldsymbol{f_1}^k)$, i.e. (24), until the current evaluation (24) converges to zero. In particular, the first iteration starts with $\tilde{\boldsymbol{E}} = 0$ in (22) and (23).

Interestingly, we have encountered a very specific property of the AP1 model with respect to the electric field magnitude. The easiest way how to demonstrate this is to write the model equations (22) and (23) in 1D and eliminate one of the partial derivatives with respect to v.

this is very important. Should be developed and commented!

Where you are using vectors and where scalars?
Explain the problem you are solving!

Kn^e	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1
v_{lim}/v_{th}	70.8	22.4	7.3	3.1	1.8

TABLE II. $\sqrt{3}v\frac{\nu_e}{2} > |\tilde{\boldsymbol{E}}|$.

I cannot follow your explanation
Please write them in an article form!

?

In the case of elimination of $\frac{\partial f_0}{\partial v}$ one obtains the following equation

$$\left(v\frac{\nu_e}{2} - \frac{2\tilde{E}_z^2}{3v\nu_e}\right)\frac{\partial f_{1_z}}{\partial v} = \frac{2\tilde{E}_z}{3\nu_e}\frac{\partial f_{1_z}}{\partial z} + \frac{4\pi\tilde{E}_z}{3}\frac{\partial f_M}{\partial v} + \frac{v}{3}\frac{\partial f_0}{\partial z} + \left(\frac{4\tilde{E}_z^2}{3v^2\nu_e} + \left(\nu_{ei} + \frac{\nu_e}{2}\right)\right)f_{1_z}. \quad (25)$$

It is convenient to write the left hand side of (25) as $\frac{2}{3v\nu_e}\left(\left(\sqrt{3}v\frac{\nu_e}{2}\right)^2 - \tilde{E}_z^2\right)$ from where it is clear that the bracket is negative if $\sqrt{3}v\frac{\nu_e}{2} = \sqrt{3}\frac{n_e\Gamma}{2v^2} < |\tilde{\boldsymbol{E}}|$, i.e. there is a velocity limit for a given magnitude $|\tilde{\boldsymbol{E}}|$, when the collisions are no more fully dominant and the electric field introduces a comparable effect to friction in the electron transport.

Since the last term on the right hand side of (25) is dominant, the solution behaves as $\boldsymbol{f_1} \sim \exp\left(-\left(\frac{4\tilde{E}_z^2}{3v^2\nu_e} + \left(\nu_{ei} + \frac{\nu_e}{2}\right)\right) / \left(v\frac{\nu_e}{2} - \frac{2\tilde{E}_z^2}{3v\nu_e}\right)v\right)$, which becomes ill-posed for velocities above the limit.

In order to provide a stable model, we introduce a reduced electric field

$$|\tilde{\boldsymbol{E}}_{red}| = \sqrt{3}v\frac{\nu_e}{2}, \quad (26)$$

ensuring that the bracket on the left hand side of (25) remains positive. Further more we define two quantities

$$\omega_{red} = \frac{|\tilde{\boldsymbol{E}}_{red}|}{|\tilde{\boldsymbol{E}}|}, \quad \nu_{scat}^E = \frac{|\tilde{\boldsymbol{E}}| - |\tilde{\boldsymbol{E}}_{red}|}{v}.$$

introducing the reduction factor of the electric field to be applied ω_{red} and the compensation of the electric field effect in terms of scattering ν_{scat}^E. Consequently, the AP1 model (22), (23), and (24) can be formulated as well posed with the help of ω_{red} and ν_{scat}^E. However, before doing so, we introduce a slightly different approximation to the electron distribution function as

$$\tilde{f} = \frac{4\pi f_M + \delta f_0}{4\pi} + \frac{3}{4\pi}\boldsymbol{n}\cdot\boldsymbol{f_1}. \quad (27)$$

where δf_0 represents the departure of isotropic part from the Maxwell-Boltzmann equilibrium distribution f_M, which we keep intentionally in the distribution function approximation.

Then, the stable AP1 model reads

$$v\frac{\nu_e}{2}\frac{\partial \delta f_0}{\partial v} = v\nabla\cdot\boldsymbol{f_1} + \tilde{\boldsymbol{E}}\cdot\left(\omega_{red}\frac{\partial \boldsymbol{f_1}}{\partial v} + \frac{2}{v}\boldsymbol{f_1}\right), \quad (28)$$

$$v\frac{\nu_e}{2}\frac{\partial \boldsymbol{f_1}}{\partial v} = \tilde{\nu}_{scat}\boldsymbol{f_1} + \frac{v}{3}\nabla\left(4\pi f_M + \delta f_0\right) + \frac{\tilde{\boldsymbol{E}}}{3}\left(4\pi\frac{\partial f_M}{\partial v} + \omega_{red}\frac{\partial \delta f_0}{\partial v}\right), \quad (29)$$

where $\tilde{\nu}_{scat} = \nu_{ei} + \nu_{scat}^E + \frac{\nu_e}{2}$. The reason for keeping f_M in the distribution function approximation (27) can be seen in the last term on the right hand side of (29), which provides the effect of electric field on directional quantities as current or heat flux. In principle, the explicit use of f_M ensures the proper effect of $\tilde{\boldsymbol{E}}$ if $\delta f_0 \ll f_M$, i.e. no matter what the reduction ω_{red} is. Apart from its stability, it also exhibits much better convergence of the electric field, which is now given by the zero current condition of (29) as

$$\tilde{\boldsymbol{E}} = \frac{\int_v\left(\frac{\nu_e}{2\tilde{\nu}_{scat}}v^2\frac{\partial \boldsymbol{f_1}}{\partial v} - \frac{v^2}{3\tilde{\nu}_{scat}}\nabla\left(4\pi f_M + \delta f_0\right)\right)v^2\,\mathrm{d}v}{\int_v\frac{v}{3\tilde{\nu}_{scat}}\left(4\pi\frac{\partial f_M}{\partial v} + \omega_{red}\frac{\partial \delta f_0}{\partial v}\right)v^2\,\mathrm{d}v}. \quad (30)$$

For practical reasons we present in TABLE II some explicit values of velocity limit corresponding to varying transport conditions expressed in terms of Knudsen number $\mathrm{Kn}^e = \frac{\lambda_e|\nabla T_e|}{T_e}$, where $\frac{T_e}{|\nabla T_e|}$ stands for the length scale of plasma.

Justify that your approach is valid and in what conditions

B. Aladin, Impact, and Calder kinetic codes

- Brief description of the Aladin code FIG. 2, FIG. 3.
- Brief description of the Impact code FIG. 4.
- Brief description of the Calder code FIG. 5.

C. Simulation results

- Multiple runs analyzing the performance of AP1 with respect to Aladin/Impact/Calder along wide range of Kn^e shown in FIG. 6.
- Realistic hydro simulation setting provided by HYDRA, a comparison between AP1, Impact, and SNB shown in FIG. 7.
- Comment on and summarize the velocity limits for all figs.

how it can be:
E_{red} is a function of coordinate an $v\nu_e$ is function of v!

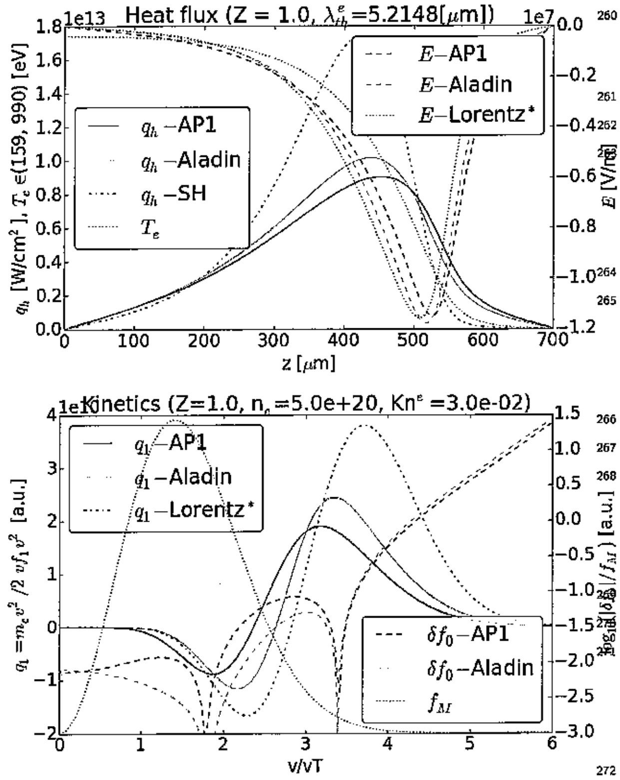

FIG. 2. Snapshot 20 ps. Left: correct steady solution of heat flux. Right: Aladins results are correct. Velocity limit 4.4 v_{th}..

V. CONCLUSIONS

- The most important point is that we introduce a collision operator, which is coherent with the full FP, i.e. no extra dependence on Z.
- Touch pros/contras of linearized FP in Aladin and Impact vs AWBS
- Raise discussion about what is the weakest point of AP1 for high Kns: the velocity limit or phenomenological Maxwellization?
- Summarize useful outcomes related to plasma physics as the tendency of the velocity maximum in q_1 with respect to Z and Kn^e.
- Emphasize the good results of Aladin (compared to Impact) and also outstanding results of Calder while being PIC.

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