Multi-dimensional High-Order FEM Methods for Electron Kinetics in ALE Hydrodynamics

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Classical MHD

HYDRODYNAMICS $\quad$ *local* $\rightarrow \boldsymbol{q}_e = -\kappa_{SH} T_e^{2.5} \nabla T_e$

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho \nabla \cdot \boldsymbol{u}, \\
\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &= -\nabla p + \frac{c}{4\pi} \nabla \times \boldsymbol{B} \times \boldsymbol{B}, \\
\rho \frac{\partial \varepsilon}{\partial T} \frac{\mathrm{d}T}{\mathrm{d}t} &= -p \nabla \cdot \boldsymbol{u} + \nabla \cdot \left(\kappa_{SH} T_e^{2.5} \nabla T_e \right) + \sigma \boldsymbol{E} \cdot \boldsymbol{E},
\end{aligned}$$

MAXWELL EQUATIONS $\quad$ *resistive* $\rightarrow \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{B})$

$$\begin{aligned}
\nabla \times \boldsymbol{E} &= -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t}, \\
\nabla \times \boldsymbol{B} &= \frac{4\pi}{c} \sigma (\boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{B}),
\end{aligned}$$

KINETICS OF ELECTRONS $\quad$ *Landau − Fokker − Planck*

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f = \Gamma_e \int \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \cdot (f \nabla_{\tilde{\boldsymbol{v}}} f - f \nabla_{\boldsymbol{v}} f) \, \mathrm{d}\tilde{\boldsymbol{v}} + \frac{\nu_{ei}}{2} \frac{\partial^2 f}{\partial \Omega^2}.$$

Nonlocal-MHD

HYDRODYNAMICS 4D

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\nabla\cdot\boldsymbol{u}, \\
\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &= -\nabla p + \boldsymbol{j}(f)\times\boldsymbol{B}, \\
\rho\frac{\partial\varepsilon}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} &= -p\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{q}_e(f) + \boldsymbol{j}(f)\cdot\boldsymbol{E},
\end{aligned}$$

MAXWELL EQUATIONS 4D

$$\begin{aligned}
\nabla\times\boldsymbol{E} &= -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}, \\
\nabla\times\boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{j}(f),
\end{aligned}$$

KINETICS OF ELECTRONS 6D

$$\boldsymbol{v}\cdot\nabla f + (\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}} f = v\tilde{\nu}_e\frac{\partial}{\partial v}\left(f - f_{MB}(T_e)\right) + \nu_{scat}\left(f_0 - f\right).$$

M. Holec et al, *Phys. Plas.*, submitted (2019) / arXiv:1901.11378 .

3D velocity space discretizations

AWBS electron kinetic model 7D

$$
\begin{aligned}
C_V \frac{\mathrm{d} T_e}{\mathrm{d}t} &= -\nabla \cdot \boldsymbol{q}_e(f) + \boldsymbol{j}(f) \cdot \boldsymbol{E} + S_H, \\
&= \int_{4\pi} \int_v v \tilde{\nu}_e \frac{\partial f}{\partial v} v^4 \mathrm{d}v \mathrm{d}\boldsymbol{n} - \sigma T_e^{-0.5} + S_H, \\
\boldsymbol{v} \cdot \nabla f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f &= v \tilde{\nu}_e \frac{\partial}{\partial v} \left(f - f_{MB}(T_e)\right) + \nu_{scat} \left(f_0 - f\right).
\end{aligned}
$$

- C7 (S_N - discontinuous Galerkin FEM) 3D velocity space in spherical coordinates (v, ϕ, θ)

$$
\boldsymbol{n} \cdot \nabla f + \frac{\boldsymbol{E} \cdot \boldsymbol{n}}{v} \frac{\partial f}{\partial v} + \frac{E_\phi - v\ B_\theta}{v^2} \frac{\partial f}{\partial \phi} + \frac{E_\theta + v\ B_\phi}{v^2 \sin(\phi)} \frac{\partial f}{\partial \theta} = \tilde{\nu}_e \frac{\partial}{\partial v} \left(f - f_M\right) + \frac{\nu_{scat}}{v} \left(f_0 - f\right).
$$

- AP1 (P_N (VEF $\xi = 1/3$) - continuous Galerkin mixed FEM) *Physicists like it!*

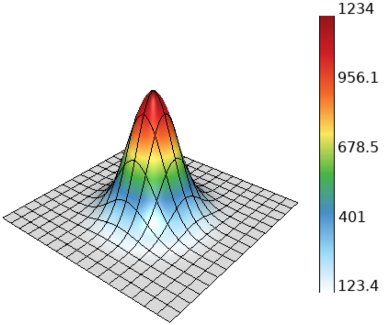
$$
\begin{aligned}
\xi \nabla \cdot \boldsymbol{f}_1 + \xi \frac{q_e}{m_e v} \boldsymbol{E} \cdot \left(\frac{\partial \boldsymbol{f}_1}{\partial v} + \frac{2}{v} \boldsymbol{f}_1 \right) &= \tilde{\nu}_e \frac{\partial}{\partial v} \left(f_0 - f_M\right), \\
\nabla f_0 + \frac{q_e}{m_e v} \boldsymbol{E} \frac{\partial f_0}{\partial v} + \frac{q_e \boldsymbol{B}}{m_e c v} \times \boldsymbol{f}_1 &= \tilde{\nu}_e \frac{\partial \boldsymbol{f}_1}{\partial v} - \frac{\nu_{scat}}{v} \boldsymbol{f}_1,
\end{aligned}
$$

S_N upwind High-Order DG, AMG Approximate-Ideal-Relaxation (AIR) solver

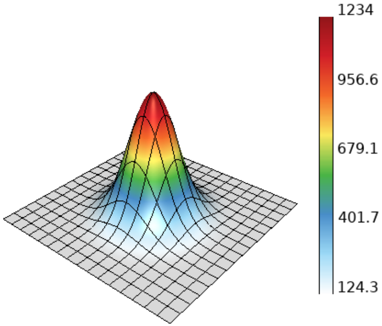
$$\mathbf{M}_{\left(\tilde{\nu_e} - \frac{\boldsymbol{E}\cdot\boldsymbol{n}_d}{v}\right)} \cdot \frac{\Delta \boldsymbol{f}_d}{\Delta v} - (\boldsymbol{n}_d \cdot \mathbf{G} + \mathbf{F}_d) \cdot \left(\tilde{\boldsymbol{f}}_d + \Delta \boldsymbol{f}_d\right) = \mathbf{M}_{\left(\frac{\nu_{scat}}{v}\right)} \cdot \left(\tilde{\boldsymbol{f}}_d + \Delta \boldsymbol{f}_d\right) + \boldsymbol{S}_{\left(\tilde{f}, \nu_{scat}, \boldsymbol{E}, \boldsymbol{B}, \tilde{\nu_e} \frac{\partial f_M}{\partial v}\right)}$$

Velocity groups	32	64	128	256	order
Backward Euler	1.595e-1	8.618e-2	4.456e-2	2.264e-2	0.98
SDIRK2	3.217e-2	8.888e-3	2.322e-3	5.924e-3	1.97
SDIRK3	2.455e-2	3.639e-3	4.887e-4	6.317e-05	2.96

Hydro temperature

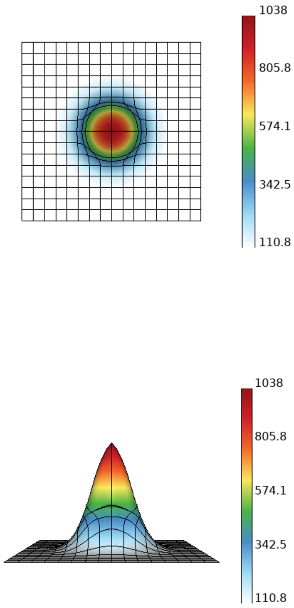


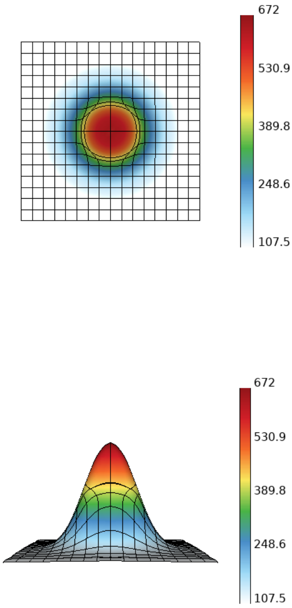
Kinetic temperature (50 groups)



Local hydro temperature

Nonlocal kinetic temperature

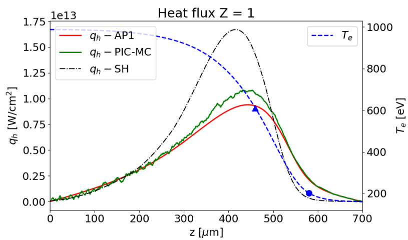


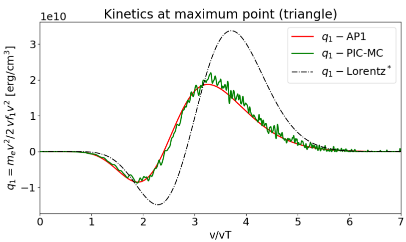


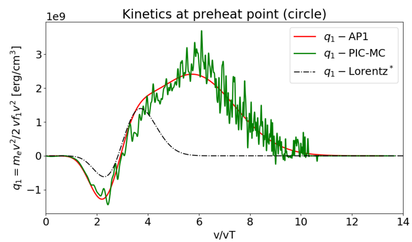
AP1 High-Order Mixed FEM formulation, fixed P1 angular discretization PCG(AMG)

$$\mathbf{M}^{L_2}_{(\tilde{\nu_e})} \cdot \frac{\mathrm{d}\boldsymbol{f_0}}{\mathrm{d}v} - \mathbf{V}^{L_2}_{(\frac{\xi q_e \boldsymbol{E}}{m_e v})} \cdot \frac{\mathrm{d}\boldsymbol{f_1}}{\mathrm{d}v} \quad = \quad \mathbf{D}^{L_2}_{(\xi)} \cdot \boldsymbol{f_1} + \mathbf{M}^{L_2}_{(\frac{\xi 2 q_e \boldsymbol{E}}{m_e v^2})} \cdot \boldsymbol{f_1} + \boldsymbol{S}^{L_2}_{(\tilde{\nu_e} \frac{\partial f_M}{\partial v})},$$

$$\mathbf{M}^{H_1}_{(\tilde{\nu_e})} \cdot \frac{\mathrm{d}\boldsymbol{f_1}}{\mathrm{d}v} - \mathbf{V}^{H_1}_{(\frac{q_e \boldsymbol{E}}{m_e v})} \cdot \frac{\mathrm{d}\boldsymbol{f_0}}{\mathrm{d}v} \quad = \quad \mathbf{G}^{H_1} \cdot \boldsymbol{f_0} + \mathbf{M}^{H_1}_{(\frac{\nu_{scat}}{v})} \cdot \boldsymbol{f_1} + \mathbf{C}^{H_1}_{(\frac{q_e \boldsymbol{B}}{m_e c v} \times)} \cdot \boldsymbol{f_1},$$

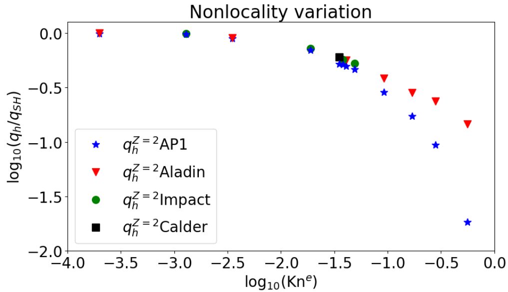






Electron velocity limit - friction vs. $\boldsymbol{E}$ stopping

$$\left(\tilde{\nu_e} - \frac{\boldsymbol{E}\cdot\boldsymbol{n}}{v}\right)\frac{\partial f}{\partial v} = \boldsymbol{n}\cdot\nabla f + \frac{\nu_{scat}}{v}(f - f_0) + \frac{E_\phi - v\ B_\theta}{v^2}\frac{\partial f}{\partial \phi} + \frac{E_\theta + v\ B_\phi}{v^2\sin(\phi)}\frac{\partial f}{\partial \theta} + \tilde{\nu_e}\frac{\partial f_M}{\partial v}.$$



$\boldsymbol{E}$ stopping overtakes collisions for Kn > 0.1 .

Kn $= \frac{\lambda}{L}$	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1
v_{lim}/v_{th}	70.8	22.4	7.3	3.1	1.8

Adaptive DSA preconditioner for Hydro-Kinetics coupling

AWBS electron kinetic model 7D

$$
\begin{aligned}
C_V \frac{\mathrm{d} T_e}{\mathrm{d}t} &= \int_{\boldsymbol{v}} \sigma K(f) \mathrm{d}\boldsymbol{v} - \sigma T_e^{\alpha}, \\
\boldsymbol{v} \cdot \nabla f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f &= v \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f) .
\end{aligned}
$$

Continuum analysis of local (Kn$\ll$1) transport regime $\rightarrow$ DIFFUSION

$$
C_V \frac{\mathrm{d} T_e}{\mathrm{d}t} = \nabla \cdot \lambda(T_e^{2.5}) \nabla T_e + O(\mathrm{Kn}^2)
$$

Precodnitioned fixed-point iteration $\mathbf{E}(\Delta \boldsymbol{T}) \equiv (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma (\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T})^\alpha - \mathbf{C} \cdot \mathbf{D}_\lambda (\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T})^\alpha$

$$
\mathbf{M}_{C_V} \cdot \frac{\Delta \boldsymbol{T}^{k+1}}{\Delta t} + \mathbf{E}(\Delta \boldsymbol{T}^{k+1}) - \mathbf{E}(\Delta \boldsymbol{T}^{k}) = \mathbf{K}_\sigma(f^k) - \mathbf{M}_\sigma(\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T}^k)^\alpha .
$$

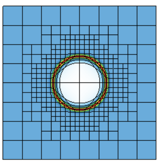
Finally, we get an unconditionally stable backward Euler (SDIRK) fast iterating scheme

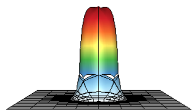
$$
\mathbf{M}_{C_V} \cdot \frac{\Delta \boldsymbol{T}^{k+1}}{\Delta t} + (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma (\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T}^{k+1})^\alpha - \mathbf{C} \cdot \mathbf{D}_\lambda (\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T}^{k+1})^\alpha = \mathbf{K}(f^k) - \mathbf{C} \cdot \mathbf{M}_\sigma (\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T}^{k})^\alpha - \mathbf{C} \cdot \mathbf{D}_\lambda (\tilde{\boldsymbol{T}} + \Delta \boldsymbol{T}^{k})^\alpha ,
$$

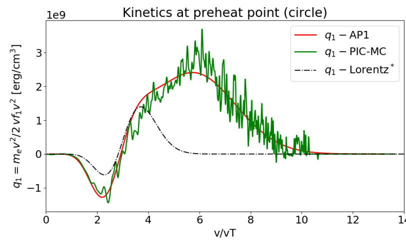
where adaptive coefficient diffusion $\mathbf{C} \xrightarrow{\mathrm{Kn} \ll 1} 1$ and nonlocal transport $\mathbf{C} \xrightarrow{\mathrm{Kn} > 1} 0$ and $\mathbf{C} \in (1, 0)$ in between.
T. Haut et al, *SIAM*, submitted (2018) / arXiv:1810.11082 .

Conclusions

- 7D microscopic world of electrons in hydro simulations.
- S_N high-order DG finite element approach.
- P_1 high-order mixed finite element approach.
- $\boldsymbol{E}$ field dominated stopping (P_1 fails).
- Adaptive DSA preconditioner for Hydro-Kinetics coupling (ML on **C**).
- Algebraic Multigrid solver pAIR scales $\log(P)^{1.22}$.







MS85 - Developments in Algebraic Multigrid for Nonsymmetric and Hyperbolic Problems - Ben S. Southworth
https://github.com/CEED/Laghos/tree/master/amr
https://mfem.org
holec1@llnl.gov

Thank you for your attention. Any questions?

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