

# Multi-dimensional High-Order FEM Methods for Electron Kinetics in ALE Hydrodynamics

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**1** Motivation - Nonlocal Magneto-Hydrodynamic model (Nonlocal-MHD)

**2**  $S_N$  discontinuous Galerkin finite element approach to kinetics

**3**  $P_N$  mixed finite element approach to kinetics

**4** Conclusions

# Classical MHD

HYDRODYNAMICS      *local*  $\rightarrow \mathbf{q}_e = -\kappa_{SH} T_e^{2.5} \nabla T_e$

$$\begin{aligned}\frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u}, \\ \rho \frac{d\mathbf{u}}{dt} &= -\nabla(p_i + p_e) + \frac{c}{4\pi} \nabla \times \mathbf{B} \times \mathbf{B}, \\ \rho \left( \frac{\partial \varepsilon_i}{\partial T_i} \frac{dT_i}{dt} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_i \nabla \cdot \mathbf{u} - G(T_i - T_e), \\ \rho \left( \frac{\partial \varepsilon_e}{\partial T_e} \frac{dT_e}{dt} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_e \nabla \cdot \mathbf{u} + \nabla \cdot (\kappa_{SH} T_e^{2.5} \nabla T_e) + \sigma \mathbf{E} \cdot \mathbf{E} + G(T_i - T_e) + Q_{IB}(\mathbf{E}_L),\end{aligned}$$

MAXWELL EQUATIONS      *resistive*  $\rightarrow \mathbf{j} = \sigma \mathbf{E}$

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{d\mathbf{B}}{dt}, \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c} \sigma \mathbf{E},\end{aligned}$$

KINETICS OF ELECTRONS      *Landau – Fokker – Planck*

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \Gamma \int \nabla_{\mathbf{v}} \nabla_{\mathbf{v}} (\mathbf{v} - \tilde{\mathbf{v}}) \cdot (f \nabla_{\tilde{\mathbf{v}}} f - f \nabla_{\mathbf{v}} f) d\tilde{\mathbf{v}} + \frac{\nu_{ei}}{2} \frac{\partial^2 f}{\partial \Omega^2}.$$

# Nonlocal-MHD

HYDRODYNAMICS

4D

$$\begin{aligned}\frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u}, \\ \rho \frac{d\mathbf{u}}{dt} &= -\nabla(p_i + p_e) + \mathbf{j}(f, \mathbf{E}) \times \mathbf{B}, \\ \rho \left( \frac{\partial \varepsilon_i}{\partial T_i} \frac{d T_i}{dt} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{d \rho}{dt} \right) &= -p_i \nabla \cdot \mathbf{u} - G(T_i - T_e), \\ \rho \left( \frac{\partial \varepsilon_e}{\partial T_e} \frac{d T_e}{dt} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{d \rho}{dt} \right) &= -p_e \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q}_e(f) + \mathbf{j}(f, \mathbf{E}) \cdot \mathbf{E} + G(T_i - T_e) + Q_{IB}(\mathbf{E}_L),\end{aligned}$$

MAXWELL EQUATIONS

4D

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{d\mathbf{B}}{dt}, \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{j}(f, \mathbf{E}),\end{aligned}$$

KINETICS OF ELECTRONS

6D

$$\mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = v \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f).$$

M. Holec et al, *Phys. Plas.*, submitted (2019) / arXiv:1901.11378 .

## 3D velocity space FEM discretizations

### AWBS electron kinetic model 7D

$$\begin{aligned} C_V \frac{dT_e}{dt} &= -\nabla \cdot \mathbf{q}_e(f) + \mathbf{j}(f) \cdot \mathbf{E} + S_H, \\ \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f &= v \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f). \end{aligned}$$

- C7 ( $S_N$  - discontinuous Galerkin FEM) 3D velocity space in spherical coordinates  $(v, \phi, \theta)$

$$\mathbf{n} \cdot \nabla f + \frac{\mathbf{E} \cdot \mathbf{n}}{v} \frac{\partial f}{\partial v} + \frac{E_\phi - v B_\theta}{v^2} \frac{\partial f}{\partial \phi} + \frac{E_\theta + v B_\phi}{v^2 \sin(\phi)} \frac{\partial f}{\partial \theta} = \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_M) + \frac{\nu_{scat}}{v} (f_0 - f).$$

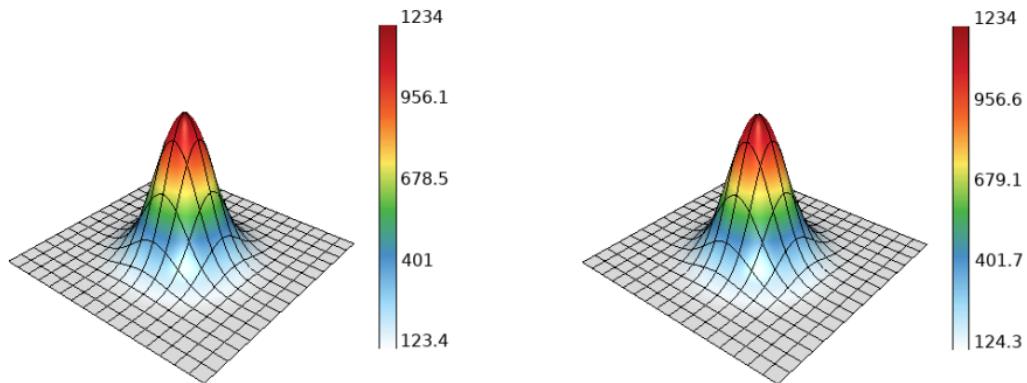
- AP1 ( $P_N$  (VEF  $\xi = 1/3$ ) - continuous Galerkin mixed FEM) *Physicists like it!*

$$\begin{aligned} \xi \nabla \cdot \mathbf{f}_1 + \xi \frac{q_e}{m_e v} \mathbf{E} \cdot \left( \frac{\partial \mathbf{f}_1}{\partial v} + \frac{2}{v} \mathbf{f}_1 \right) &= \tilde{\nu}_e \frac{\partial}{\partial v} (f_0 - f_M), \\ \nabla f_0 + \frac{q_e}{m_e v} \mathbf{E} \frac{\partial f_0}{\partial v} + \frac{q_e \mathbf{B}}{m_e c v} \times \mathbf{f}_1 &= \tilde{\nu}_e \frac{\partial \mathbf{f}_1}{\partial v} - \frac{\nu_{scat}}{v} \mathbf{f}_1, \end{aligned}$$

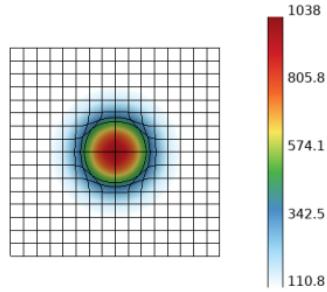
## $S_N$ upwind High-Order DG , AMG Approximate-Ideal-Relaxation (AIR) solver

$$\mathbf{M}_{(\tilde{\nu}_e - \frac{E \cdot \mathbf{n}_d}{v})} \cdot \frac{\Delta \mathbf{f}_d}{\Delta v} = (\mathbf{n}_d \cdot \mathbf{G} + \mathbf{F}_d) \cdot (\tilde{\mathbf{f}}_d + \Delta \mathbf{f}_d) + \mathbf{M}_{(\frac{\nu_{scat}}{v})} \cdot (\tilde{\mathbf{f}}_d + \Delta \mathbf{f}_d) + \mathbf{S}_{(\nu_{scat}, E, B)} \cdot \tilde{\mathbf{f}} + \mathbf{S}_{(\tilde{\nu}_e \frac{\partial f_M}{\partial v})}$$

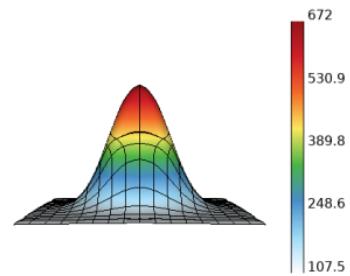
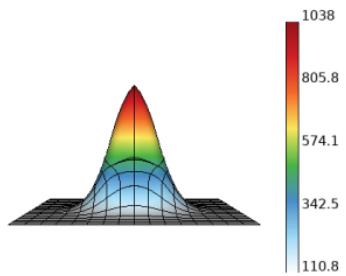
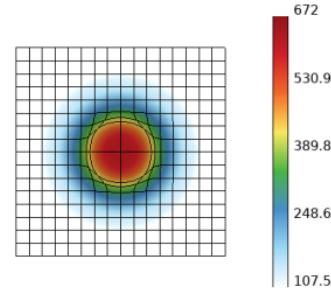
Velocity groups	32	64	128	256	order	
Backward Euler	1.595e-1	8.618e-2	4.456e-2	2.264e-2	0.98	
SDIRK2	3.217e-2	8.888e-3	2.322e-3	5.924e-3	1.97	
SDIRK3	2.455e-2	3.639e-3	4.887e-4	6.317e-05	2.96	
Hydro temperature		Kinetic temperature (50 groups)				



Local hydro temperature

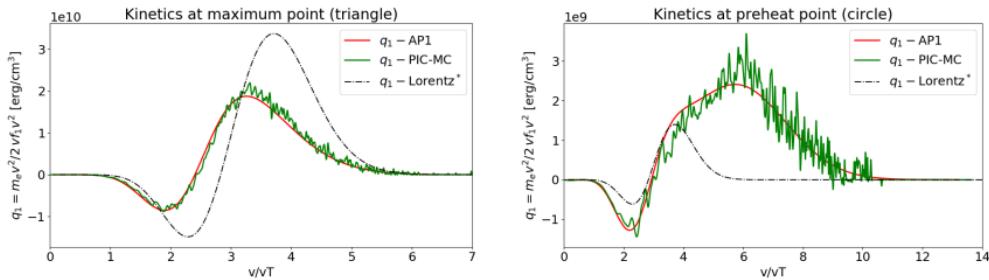
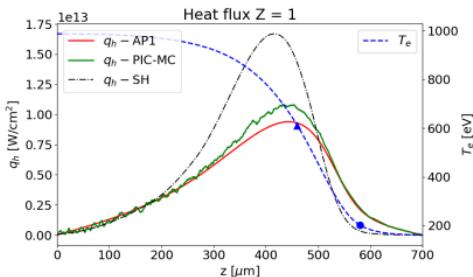


Nonlocal kinetic temperature



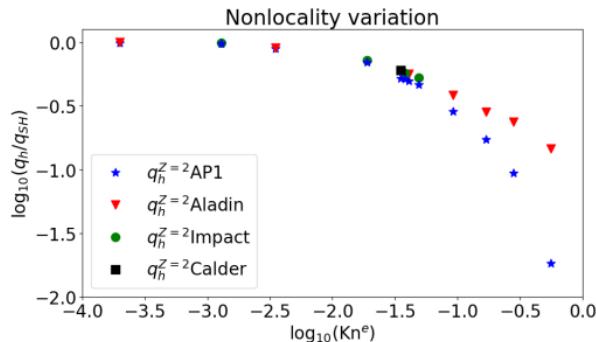
# AP1 High-Order Mixed FEM formulation, fixed P1 angular discretization PCG(AMG)

$$\begin{aligned} \mathbf{M}_{(\nu_e)}^{L_2} \cdot \frac{d\mathbf{f}_0}{dv} - \mathbf{V}_{(\frac{\xi q_e E}{m_e v})}^{L_2} \cdot \frac{d\mathbf{f}_1}{dv} &= \mathbf{D}_{(\xi)}^{L_2} \cdot \mathbf{f}_1 + \mathbf{M}_{(\frac{\xi^2 q_e E}{m_e v^2})}^{L_2} \cdot \mathbf{f}_1 + \mathbf{S}_{(\nu_e \frac{\partial f_M}{\partial v})}^{L_2}, \\ \mathbf{M}_{(\nu_e)}^{H_1} \cdot \frac{d\mathbf{f}_1}{dv} - \mathbf{V}_{(\frac{q_e E}{m_e v})}^{H_1} \cdot \frac{d\mathbf{f}_0}{dv} &= \mathbf{G}^{H_1} \cdot \mathbf{f}_0 + \mathbf{M}_{(\frac{\nu_{scat}}{v})}^{H_1} \cdot \mathbf{f}_1 + \mathbf{C}_{(\frac{q_e B}{m_e c v})}^{H_1} \cdot \mathbf{f}_1, \end{aligned}$$



## Electron velocity limit - friction vs. $E$ stopping

$$\left(\tilde{\nu}_e - \frac{\mathbf{E} \cdot \mathbf{n}}{v}\right) \frac{\partial f}{\partial v} = \mathbf{n} \cdot \nabla f + \frac{\nu_{scat}}{v} (f - f_0) + \frac{E_\phi - v B_\theta}{v^2} \frac{\partial f}{\partial \phi} + \frac{E_\theta + v B_\phi}{v^2 \sin(\phi)} \frac{\partial f}{\partial \theta} + \tilde{\nu}_e \frac{\partial f_M}{\partial v}.$$



$E$  stopping overtakes collisions for  $Kn > 0.1$ .

$Kn = \frac{\lambda}{L}$	$10^{-4}$	$10^{-3}$	$10^{-2}$	$10^{-1}$	1
$v_{lim}/v_{th}$	70.8	22.4	7.3	3.1	1.8

# Adaptive DSA preconditioner for Hydro-Kinetics coupling

## AWBS electron kinetic model 7D

$$\begin{aligned}
 C_V \frac{dT_e}{dt} &= -\nabla \cdot \mathbf{q}_e(f) + \mathbf{j}(f) \cdot \mathbf{E} + S_H, \\
 &= \int_{4\pi} \int_V v \tilde{\nu}_e \frac{\partial f}{\partial v} v^4 dv d\mathbf{n} - \sigma T_e^{-0.5} + S_H, \\
 \mathbf{v} \cdot \nabla f + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f &= v \tilde{\nu}_e \frac{\partial}{\partial v} (f - f_{MB}(T_e)) + \nu_{scat} (f_0 - f).
 \end{aligned}$$

Continuum analysis of local ( $\text{Kn} \ll 1$ ) transport regime → DIFFUSION

$$C_V \frac{dT_e}{dt} = \nabla \cdot \lambda(T_e^{2.5}) \nabla T_e + O(\text{Kn}^2)$$

Precodnitioned fixed-point iteration  $\mathbf{E}(\Delta \mathbf{T}) \equiv (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma(\tilde{\mathbf{T}} + \Delta \mathbf{T})^\alpha + \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\mathbf{T}} + \Delta \mathbf{T})^\alpha$

$$\mathbf{M}_{C_V} \cdot \frac{\Delta \mathbf{T}^{k+1}}{\Delta t} + \mathbf{E}(\Delta \mathbf{T}^{k+1}) - \mathbf{E}(\Delta \mathbf{T}^k) = \mathbf{K}(f^k) - \mathbf{M}_\sigma(\tilde{\mathbf{T}} + \Delta \mathbf{T}^k)^\alpha.$$

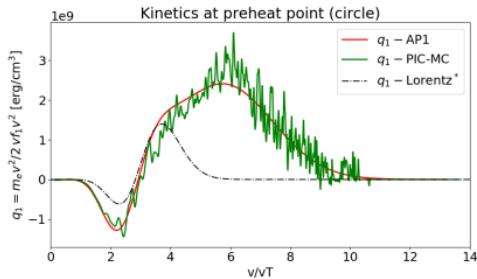
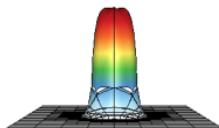
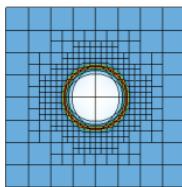
Finally, we get an unconditionally stable backward Euler (SDIRK) fast iterating scheme

$$\mathbf{M}_{C_V} \cdot \frac{\Delta \mathbf{T}^{k+1}}{\Delta t} + (\mathbf{I} - \mathbf{C}) \cdot \mathbf{M}_\sigma(\tilde{\mathbf{T}} + \Delta \mathbf{T}^{k+1})^\alpha + \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\mathbf{T}} + \Delta \mathbf{T}^{k+1})^\alpha = \mathbf{C} \cdot \mathbf{D}_\lambda(\tilde{\mathbf{T}} + \Delta \mathbf{T}^k)^\alpha + \mathbf{K}(f^k) - \mathbf{C} \cdot \mathbf{M}_\sigma(\tilde{\mathbf{T}} + \Delta \mathbf{T}^k)^\alpha,$$

where adaptive coefficient diffusion  $\mathbf{C} \xrightarrow{\text{Kn} \ll 1} 1$  and nonlocal transport  $\mathbf{C} \xrightarrow{\text{Kn} > 1} 0$  and  $\mathbf{C} \in (1, 0)$  in between.  
 T. Haut et al, SIAM, submitted (2018) / arXiv:1810.11082 .

# Conclusions

- 7D microscopic world of electrons in hydro simulations.
- $S_N$  high-order DG finite element approach.
- $P_1$  high-order mixed finite element approach.
- $E$  field dominated stopping ( $P_1$  fails).
- Adaptive DSA preconditioner for Hydro-Kinetics coupling (ML on **C**).
- Algebraic Multigrid solver pAIR scales  $\log(P)^{1.22}$ .



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MS85 - Developments in Algebraic Multigrid for Nonsymmetric and Hyperbolic Problems - Ben S. Southworth  
<https://github.com/CEED/Laghos/tree/master/amr>  
<https://mfem.org>  
[holec1@llnl.gov](mailto:holec1@llnl.gov)

# Thank you for your attention. Any questions?



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