Using expressions (34) and (35) the Rosenbluth potentials can be written as:

$$\mathbb{H} = \frac{1+\mu}{\mu}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1/\upsilon}{2\ell+1}\left[I_\ell(F_\ell^m)+J_{-1-\ell}(F_\ell^m)\right]P_\ell^{|m|}(\cos\theta)e^{im\varphi}, \tag{36}$$

$$\mathbb{G} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\upsilon\left[\frac{I_{\ell+2}(F_\ell^m)+J_{-1-\ell}(F_\ell^m)}{(2\ell+1)(2\ell+3)} - \frac{I_\ell(F_\ell^m)+J_{1-\ell}(F_\ell^m)}{(2\ell-1)(2\ell+1)}\right]P_\ell^{|m|}(\cos\theta)e^{im\varphi}. \tag{37}$$

The isotropic part of the distributions F and f, i.e. F_0^0 and f_0^0, does not depend on angle and we can write $\nabla = \hat{\upsilon}\partial_\upsilon$ and $\nabla\nabla = [(\boldsymbol{I_2} - \hat{\boldsymbol{\upsilon}}\hat{\boldsymbol{\upsilon}})/\upsilon]\partial_\upsilon$ when these operators act on it. We also note that $\hat{\upsilon}\hat{\upsilon} : \nabla\nabla = \partial_\upsilon^2$ and $\boldsymbol{I_2}$: $\nabla\nabla = \nabla^2$. Exploiting these properties and substituting the Rosenbluth potentials into Eq. (32) we obtain:

$$\frac{1}{\Gamma_{zZ}}\left(\frac{\delta f_0^0}{\delta t}\right) = \frac{1}{3\upsilon^2}\frac{\partial}{\partial\upsilon}\left[\frac{3}{\mu}f_0^0 I_0\left(F_0^0\right) + \upsilon\left(I_2\left(F_0^0\right)+J_{-1}\left(F_0^0\right)\right)\frac{\partial f_0}{\partial\upsilon}\right]. \tag{38}$$

To derive an equation for the high-order harmonics we note that the only operator that acts on angle in Eq. (33) is ∇^2 and it can be substituted with $-\ell\times(\ell+1)/\upsilon^2$. Calculating the derivatives ∂_υ and ∂_υ^2 of the Rosenbluth potentials and substituting them into Eq. (33) yields:

$$\begin{aligned}\frac{1}{\Gamma_{zZ}}\left(\frac{\delta f_{\ell\geqslant 1}^m}{\delta t}\right) = {}& \frac{4\pi}{\mu}\left[F_0^0 f_\ell^m + f_0^0 F_\ell^m\right]\\ & -\frac{(\mu-1)}{\mu\upsilon^2}\left\{\frac{\partial f_0^0}{\partial\upsilon}\left[\frac{\ell+1}{2\ell+1}I_\ell(F_\ell^m) - \frac{\ell}{2\ell+1}J_{-1-\ell}(F_\ell^m)\right] + I_0\left(F_0^0\right)\frac{\partial f_\ell^m}{\partial\upsilon}\right\}\\ & + \frac{I_2\left(F_0^0\right)+J_{-1}\left(F_0^0\right)}{3\upsilon}\frac{\partial^2 f_\ell^m}{\partial\upsilon^2} + \frac{-I_2\left(F_0^0\right)+2J_{-1}\left(F_0^0\right)+3I_0\left(F_0^0\right)}{3\upsilon^2}\frac{\partial f_\ell^m}{\partial\upsilon}\\ & -\frac{\ell(\ell+1)}{2}\times\frac{-I_2\left(F_0^0\right)+2J_{-1}\left(F_0^0\right)+3I_0\left(F_0^0\right)}{3\upsilon^3}f_\ell^m\\ & + \frac{1}{2\upsilon}\frac{\partial^2 f_0^0}{\partial\upsilon^2}\left[C_1 I_{\ell+2}(F_\ell^m) + C_1 J_{-\ell-1}(F_\ell^m) + C_2 I_\ell(F_\ell^m) + C_2 J_{1-\ell}(F_\ell^m)\right]\\ & + \frac{1}{\upsilon^2}\frac{\partial f_0^0}{\partial\upsilon}\left[C_3 I_{\ell+2}(F_\ell^m) + C_4 J_{-\ell-1}(F_\ell^m) + C_5 I_\ell(F_\ell^m) + C_6 J_{1-\ell}(F_\ell^m)\right],\end{aligned} \tag{39}$$

where $C_1,\ldots,C_6$ are coefficients which depend on the order of the spherical harmonic ℓ:

$$\begin{aligned} C_1 &= \frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}\\ C_2 &= -\frac{(\ell-1)\ell}{(2\ell+1)(2\ell-1)}\\ C_3 &= \frac{-\ell(\ell+1)/2-(\ell+1)}{(2\ell+1)(2\ell+3)}\\ C_4 &= \frac{-\ell(\ell+1)/2+(\ell+2)}{(2\ell+1)(2\ell+3)}\\ C_5 &= \frac{\ell(\ell+1)/2+(\ell-1)}{(2\ell+1)(2\ell-1)}\\ C_6 &= \frac{\ell(\ell+1)/2-\ell}{(2\ell+1)(2\ell-1)}\end{aligned}$$

The second line in Eq. (39) can be identified as the additional terms that need to be included if $M \neq m$. This term comes from the potential $\mathbb{H}$, which is associated with the dynamical friction coefficient, and becomes particularly important for the scattering of massive particles off much lighter ones ($\mu \ll 1$). The last two lines in Eq. (39) can be seen as the effect of the perturbed Rosenbluth potentials $\mathbb{H}(F_\ell^m), \mathbb{G}(F_\ell^m)$ on the isotropic part of the distribution of the scattering particles (f_0^0). Substituting $\ell = 1$ ($\ell = 2$) in Eq. (39) we recover the equation for the first (second) order cartesian tensor expansion in Shkarofsky et al. [42].

Let us consider electrons ($m = m_e, z = 1$) scattering off immobile ions and substitute $\mu \gg 1$ and $F(\upsilon) = n_i\,\delta(\upsilon)/(4\pi\upsilon^2)$ in Eq. (39). For $\upsilon > 0$ all integrals vanish except $I_0\left(F_0^0\right) = n_i$. The terms that survive in the second and third line cancel and Eq. (39) reduces to:

$$\left(\frac{\delta f_\ell^m}{\delta t}\right)_{ei} = -\frac{\ell(\ell+1)}{2}\times\frac{n_i\Gamma_{ei}}{\upsilon^3}f_\ell^m, \tag{40}$$

where $\Gamma_{ei} = 4\pi(Ze^2)^2\ln\Lambda/m_e^2$. This is the expression for angular scattering of electrons and it amounts to a damping rate proportional to $\ell\,(\ell+1)$ as a result of which high-order harmonics ($\ell \gg 1$) decay rapidly. In Eq. (39), except for the general "angular scattering" term that appears in the forth line, i.e. $-[\ell(\ell+1)/2] \times \left[-I_2\left(F_0^0\right) + 2J_{-1}\left(F_0^0\right) + 3I_0\left(F_0^0\right)\right)]/(3\upsilon^3) f_\ell^m$ and scales as $O(\ell^2)$, all other terms scale as $O(\ell^0)$.

For collisions between electrons $\mu = 1,\ z = Z = 1,\ F_0^0 = f_0^0,\ F_\ell^m = f_\ell^m$ Eq. (39) reduces to:

$$\begin{aligned}\frac{1}{\Gamma_{ee}}\left(\frac{\delta f_\ell^m}{\delta t}\right) &= 8\pi f_0^0 f_\ell^m + \frac{I_2(f_0^0) + J_{-1}(f_0^0)}{3\upsilon}\frac{\partial^2 f_\ell^m}{\partial \upsilon^2} + \frac{-I_2(f_0^0) + 2J_{-1}(f_0^0) + 3I_0(f_0^0)}{3\upsilon^2}\frac{\partial f_\ell^m}{\partial \upsilon} - \frac{\ell(\ell+1)}{2} \\ &\times \frac{-I_2(f_0^0) + 2J_{-1}(f_0^0) + 3I_0(f_0^0)}{3\upsilon^3} f_\ell^m + \frac{1}{2\upsilon}\frac{\partial^2 f_0^0}{\partial \upsilon^2}\left[C_1 I_{\ell+2}(f_\ell^m) + C_1 J_{-\ell-1}(f_\ell^m) + C_2 I_\ell(f_\ell^m) + C_2 J_{1-\ell}(f_\ell^m)\right] + \frac{1}{\upsilon^2} \\ &\times \frac{\partial f_0^0}{\partial \upsilon}\left[C_3 I_{\ell+2}(f_\ell^m) + C_4 J_{-\ell-1}(f_\ell^m) + C_5 I_\ell(f_\ell^m) + C_6 J_{1-\ell}(f_\ell^m)\right],\end{aligned} \tag{41}$$

where $\Gamma_{ee} = 4\pi e^4 \ln\Lambda/m_e^2$. Eq. (41) may also be derived from Eq. (10) in Shkarofsky et al. [43] by carrying out the differentiation with respect to υ (and similarly for Eq. (4) in Alouani-Bibi et al. [1]).

3.2. Characteristic collisional quantities

The relaxation of an isotropic distribution due to collisions between electrons (and more generally for identical particles) is described by Eq. (38) with $\mu = 1$ which can be written as:

$$\frac{1}{4\pi\Gamma_{ee}/3}\frac{\delta f_0^0}{\delta t} = \frac{1}{\upsilon^2}\frac{\partial}{\partial \upsilon}\left\{\frac{1}{\upsilon}\frac{\partial}{\partial \upsilon}\left[f_0^0\int_0^\upsilon f_0^0 u^4 du + \upsilon^3 f_0^0 \int_\upsilon^\infty f_0^0 u du - 3\int_\upsilon^\infty f_0^0 u du \int_0^\upsilon f_0^0 u^2 du\right]\right\}. \tag{42}$$

For a Maxwellian distribution $\frac{n_e}{(\sqrt{2\pi}\upsilon_t)^3} e^{-\upsilon^2/(2\upsilon_t^2)}$, a characteristic relaxation time can be defined from Eq. (42) as:

$$\tau_e = \left(\frac{4\pi\Gamma_{ee}}{3}\frac{n_e}{(\sqrt{2\pi}\upsilon_t)^3}\right)^{-1} = \frac{3\sqrt{m_e}(kT_e)^{3/2}}{4\sqrt{2\pi}n e^4 \ln\Lambda} \simeq 3.44\times 10^5 \times \frac{(T_e[eV])^{3/2}}{n[cm^{-3}]\times\ln\Lambda} sec. \tag{43}$$

This is identical to the electron collision time in Braginskii [10, p.205]. Alternatively, $\lambda_0 \equiv \frac{9m_e^2}{4\pi e^4 \ln\Lambda}$ can be used [3] instead of Γ_{ee}, where $12\pi/\lambda_0 = 4\pi\ \Gamma_{ee}/3$. In terms of the quantities used to normalize the equations in the code $\Gamma_{ee} = (\omega_p c^3/n_e)\times(k_p r_e)\ln\Lambda$, where $r_e = e^2/(m_e c^2) \simeq 2.82\times 10^{-13}$ is the classical electron radius. A characteristic electron–ion collision time can be defined using Γ_{ei} and assuming quasi-neutrality $n_e \simeq Z n_i$. This yields $\tau_{ei} = \tau_e/Z$ which is the characteristic momentum isotropization time. The mean free path for a thermal electron can be defined as:

$$\lambda_{mfp} = \upsilon_t \times \tau_{ei} = \frac{9\ N_D}{\sqrt{2/\pi}\ln\Lambda}\times\lambda_D. \tag{44}$$

The Coulomb logarithm $\ln\Lambda$ is a weak function of the number of particles in a Debye sphere and its classical value is $\ln\Lambda = \ln(9N_D/Z)$. For hot plasmas ($T_e > 4.2\times 10^5\ K \simeq 36.2$ eV) the classical value for Λ must be reduced due to quantum mechanical effects. Using the formulas from [20] for the Coulomb logarithm we have:

$$\ln\Lambda_{ee} = 23.5 - \ln(n_e^{1/2}T_e^{-5/4}) - [10^{-5} + (\ln T_e - 2)^2/16]^{1/2}, \tag{45}$$

$$\ln\Lambda_{ei} = 23 - \ln\left(n_e^{1/2} Z T_e^{-3/2}\right), \quad T_i m_e/m_i < T_e < 10Z^2\ eV, \tag{46}$$

$$\ln\Lambda_{ei} = 24 - \ln\left(n_e^{1/2} T_e^{-1}\right), \quad T_i m_e/m_i < 10Z^2\ eV < T_e. \tag{47}$$

3.3. Numerical scheme for electron–electron collisions

3.3.1. Energy-conserving numerical scheme for the relaxation of an isotropic system of electrons

The nonlinear Eq. (42) for the relaxation of the isotropic part of a distribution of electrons may written as:

$$\left(\frac{\delta f_0^0}{\delta t}\right)_{ee} = \frac{4\pi\Gamma_{ee}}{3}\frac{1}{\upsilon^2}\frac{\partial}{\partial \upsilon}\left[\frac{1}{\upsilon}\frac{\partial W(\upsilon)}{\partial \upsilon}\right], \tag{48}$$

$$W(\upsilon) = f_0^0\int_0^\upsilon f_0^0 u^4 du + \upsilon^3 f_0^0\int_\upsilon^\infty f_0^0 u du - 3\int_\upsilon^\infty f_0^0 u du\int_0^\upsilon f_0^0 u^2 du. \tag{49}$$

This formulation allows for a numerical scheme that conserves electron energy and number density [9]. In finite difference form Eq. (48) becomes:

$$\frac{\delta f_n}{\delta t} = \frac{4\pi\Gamma_{ee}}{3}\frac{1}{\upsilon_n^2}\frac{1}{\Delta_n}\left[\frac{1}{\upsilon_{n+\frac{1}{2}}}\frac{W_{n+1} - W_n}{\Delta_{n+\frac{1}{2}}} - \frac{1}{\upsilon_{n-\frac{1}{2}}}\frac{W_n - W_{n-1}}{\Delta_{n-\frac{1}{2}}}\right], \tag{50}$$

where f_n is used to denote $f_0^0(\upsilon_n),\ \upsilon_{n+\frac{1}{2}} = \frac{1}{2}(\upsilon_n + \upsilon_{n+1}),\ \Delta_n = \frac{1}{2}(\upsilon_{n+1} - \upsilon_{n-1}),\ \Delta_{n+\frac{1}{2}} = \upsilon_{n+1} - \upsilon_n$ and $W_n \equiv W(\upsilon_n)$ to be evaluated using Eq. (49). To calculate $\delta f_0/\delta t$ we set $\upsilon_{-1} = 0$, which yields $W_{-1} \equiv W(0) = 0$. We define the discrete nonrelativistic number