Arbitrary Order Energy and Enstrophy Conserving Finite Element Methods for 2D Incompressible Fluid Dynamics and Drift-Reduced Magnetohydrodynamics

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Abstract

Maintaining conservation laws in the fully discrete setting is critical for accurate long-time behavior of numerical simulations and requires accounting for discrete conservation properties in both space and time. This paper derives arbitrary order finite element exterior calculus spatial discretizations for the two-dimensional (2D) Navier-Stokes and drift-reduced magnetohydrodynamic equations that conserve both energy and enstrophy to machine precision when coupled with generally symplectic time-integration methods. Both continuous and discontinuous-Galerkin (DG) weak formulations can ensure conservation, but only generally symplectic time integration methods, such as the implicit midpoint method, permit exact conservation in time. Moreover, the symplectic implicit midpoint method yields an order of magnitude speedup over explicit schemes. The methods are implemented using the MFEM library and the solutions are verified for an extensive suite of 2D neutral fluid turbulence test problems. Numerical solutions are verified via comparison to a semi-analytic linear eigensolver as well as to the finite difference Global Drift Ballooning (GDB) code. However, it is found that turbulent simulations that conserve both energy and enstrophy tend to have too much power at high wavenumber and that this part of the spectrum should be controlled by reintroducing artificial dissipation. The DG formulation allows upwinding of the advection operator which dissipates enstrophy while still maintaining conservation of energy. Coupling upwinded DG with implicit symplectic integration appears to offer the best compromise of allowing mid-range wavenumbers to reach the appropriate amplitude while still controlling the high-wavenumber part of the spectrum.

Keywords: Drift-Reduced-MHD, Finite element method, Conservative scheme, High-order methods

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⋆Energy and Enstrophy Conserving FEM scheme
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