Hypothesis Testing

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Testing Linear Hypotheses

Examples of linear hypotheses in the classical linear regression framework:

1. Testing significance of a regressor
$$H_0 : \beta_2 = 0$$
$$H_1 : \beta_2 \neq 0$$
2. Testing parameter value
$$H_0 : \beta_2 = \beta_2^*$$
$$H_1 : \beta_2 \neq \beta_2^*$$
3. Testing equality of two parameters
$$H_0 : \beta_2 = \beta_3$$
$$H_1 : \beta_2 \neq \beta_3$$
4. Testing significance of a subset of regressors
$$H_0 : \beta_2 = 0,\ \beta_3 = 0,\ \beta_4 = 0$$
$$H_1 : \beta_2 \neq 0,\ \beta_3 \neq 0,\ \beta_4 \neq 0$$
5. Testing overall significance of a regression
$$H_0 : \beta_2 = 0,\ \beta_3 = 0, \ldots, \beta_k = 0$$
$$H_1 : \beta_2 \neq 0,\ \beta_3 \neq 0, \ldots, \beta_k \neq 0$$

Testing Linear Hypotheses

- General linear hypothesis can be written as
$$H_0 : \boldsymbol{R\beta} = \boldsymbol{r}$$
$$H_1 : \boldsymbol{R\beta} \neq \boldsymbol{r}$$
where $\boldsymbol{R}$ is $q \times k$ matrix and $\boldsymbol{r}$ is $q \times 1$ vector. q is the number of restrictions.
- Linear restrictions may also be rewritten as
$$H_0 : \boldsymbol{R\beta} - \boldsymbol{r} = 0$$
$$H_1 : \boldsymbol{R\beta} - \boldsymbol{r} \neq 0$$
- As an example suppose that the model is:
$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + u$$
- Rewrite hypotheses in the previous slide using $\boldsymbol{R\beta} = \boldsymbol{r}$ notation.
- Also rewrite the following null hypothesis:
$$H_0 :\ \beta_1 = 0,\ \beta_2 = \beta_3,\ \beta_4 = 1$$

Testing Linear Hypotheses

- In the classical (Neyman-Pearson) testing framework we assume that the null hypothesis is true and try to find evidence against it using data.
- The truth about H_0 is unknown in practice. This leads to two types of error and associated probabilities:
- **Type I Error**: Rejecting **TRUE** H_0
- **Size of test**: Probability of rejecting **TRUE** H_0
- **Type II Error**: Accepting **FALSE** H_0
- **Power of test**: Probability of **NOT** making Type II Error. In other words, probability of rejecting the false H_0.
- **Consistent test**: Power approaches 1 as $n \to \infty$
- **Classical testing framework**: fix the significance level or α (probability of Type I Error) at a small value and conduct the test with the available data. The power cannot be preset by the analyst.

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Testing Linear Hypotheses

What are the properties of the model under the null hypothesis?

- Expected value:

$$\mathsf{E}(\boldsymbol{R}\hat{\boldsymbol{\beta}}) = \boldsymbol{R}\boldsymbol{\beta}$$

- Covariance matrix:

$$\begin{aligned}\mathsf{Var}(\boldsymbol{R}\hat{\boldsymbol{\beta}}) &= \mathsf{E}\left[(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{R}\boldsymbol{\beta})(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{R}\boldsymbol{\beta})^\top\right] \\ &= \mathsf{E}\left[\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^\top \boldsymbol{R}^\top\right] \\ &= \boldsymbol{R}\mathsf{Var}(\hat{\boldsymbol{\beta}})\boldsymbol{R}^\top \\ &= \sigma^2 \boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top\end{aligned}$$

- The last classical assumption states that the error term is multivariate normal:

$$\boldsymbol{u} \sim N(\boldsymbol{0},\ \sigma^2\boldsymbol{I})$$

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Testing Linear Hypotheses

- The exact sampling distribution of the OLS estimator under this assumption is also normal:

$$\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta},\ \sigma^2(\boldsymbol{X}^\top\boldsymbol{X})^{-1})$$

- Using the properties of normally distributed random variables we can write:

$$\boldsymbol{R}\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{R}\boldsymbol{\beta},\ \sigma^2\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top)$$

- or

$$\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \sim N(\boldsymbol{0}_k,\ \sigma^2\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top)$$

- If $\boldsymbol{R}\boldsymbol{\beta} = \boldsymbol{r}$ is true, then

$$(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r}) \sim N(\boldsymbol{0},\ \sigma^2\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top)$$

which is simply the sampling distribution of $\boldsymbol{R}\hat{\boldsymbol{\beta}}$.

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Quadratic Form

- The quadratic form:

$$(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})^\top \left[\sigma^2\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top\right]^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r}) \sim\ \chi^2_q$$

has a chi-square distribution with q degrees of freedom.

- It can be shown that

$$\frac{\hat{\boldsymbol{u}}^\top\hat{\boldsymbol{u}}}{\sigma^2} \sim\ \chi^2_{n-k}$$

- Using these the WALD statistic can be written as

$$\frac{(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})^\top \left[\sigma^2\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top\right]^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})/q}{\hat{\boldsymbol{u}}^\top\hat{\boldsymbol{u}}/(n-k)} \sim\ F(q, n-k)$$

- Noting that $s^2 = \hat{\boldsymbol{u}}^\top\hat{\boldsymbol{u}}/(n-k)$ the Wald statistic is rewritten as

$$\frac{(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})^\top \left[s^2\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top\right]^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})}{q} \sim\ F(q, n-k)$$

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F Test

- Decision rule: Reject H_0 if the calculated F statistic is greater than the critical value at α. Or, compute the p-value.
- p-value: calculated probability of Type-I Error. Let F_c be the computed test statistic and let $F(q, n-k)$ is the distribution of this test statistic. Then the p-value, or simply p is

$$p = \mathbb{P}(F_c > F(q, n-k))$$

- Small values of p leads to the rejection of H_0
- As $n \to \infty$ the standard error of the regression decreases proportionally (at rate $\sqrt{n}$). Therefore, for large sample sizes we need to use smaller p or α, significance level, to reject the null

F Test

- Let the null hypothesis be $H_0 : \boldsymbol{\beta}_j = 0$. Let us derive the F statistic for this null.
- $\boldsymbol{R}\hat{\boldsymbol{\beta}}$ picks out $\hat{\beta}_j$, $\boldsymbol{R}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{R}^\top$ picks out $(\boldsymbol{X}^\top\boldsymbol{X})^{-1}_{jj}$, ie, jth diagonal element of $(\boldsymbol{X}^\top\boldsymbol{X})^{-1}$.
- The F statistic becomes

$$F = \frac{\hat{\beta}_j^2}{s^2(\boldsymbol{X}^\top\boldsymbol{X})^{-1}_{jj}} = \frac{\hat{\beta}_j^2}{\text{Var}(\hat{\beta}_j)} \sim \ F(1, n-k)$$

- Taking the square root we obtain the t statistic

$$t_j = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} \sim \ t_{n-k}$$

- Similarly, if the null hypothesis is $H_0 : \boldsymbol{\beta}_j = \beta_j^*$ then

$$t_j = \frac{\hat{\beta}_j - \beta_j^*}{se(\hat{\beta}_j)} \sim \ t_{n-k}$$

Monte Carlo Experiment: Size of t-test

- DGP Setup:

$$y = \beta_1 + \beta_2 x_1 + \beta_3 x_2 + u$$

  True values of parameters:

$$\beta_1 = 5, \quad \beta_2 = 1, \quad \beta_3 = 0$$

- We want to test $H_0 : \beta_3 = 0$ against $H_1 : \beta_3 \neq 0$
- Let $\boldsymbol{x} = [x_1 \ \ x_2]^\top$ be fixed in repeated samples and

$$\boldsymbol{x} \sim \ N(\boldsymbol{\mu}, \ \boldsymbol{\Sigma}), \ \boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  so that $Cor(x_1, x_2) = 0$
- Sample size $n = 30$, the number of Monte Carlo replications is 10000.
- $u_i \sim \ N(0, 1)$, $i = 1, 2, \ldots, n$ (Homoskedastic errors).

see `OLStests1.m`

Monte Carlo Experiment: Size of t-test

```
n = 30; % sample size
MCrep = 10000; % number of Monte Carlo replications
beta1 = 5; beta2 = 1; beta3 = 0; % true parameter values
rng(33332,'twister'); % fix the RNG for replication
X = randn(n,2); % X is fixed in repeated samples
dof = n-3; % degrees of freedom
alpha = (0.01:0.01:0.5); % nominal significance levels
cv = tinv(1-alpha/2,dof); % critical value

for i=1:MCrep
    u = randn(n,1);
    y = beta1 + beta2*X(:,1) + beta3*X(:,2) + u; % DGP
    res = OLS(y,[ones(n,1) X],0);
    tratio(i,:) = res.tratio(3); % t-test for H0:beta3=0
    ind(i,:) = abs(tratio(i,:))>cv; % ind=1 rejects, ind=0 fail to reject
end
```

see `OLStests1.m`

Monte Carlo Experiment: Size of t-test

```
disp('Nominal alpha level and Size')
   [alpha' mean(ind)']

    0.0100    0.0092
    0.0200    0.0200
    0.0300    0.0298
    0.0400    0.0404
    0.0500    0.0509
    0.0600    0.0607
    0.0700    0.0702
    0.0800    0.0806
    0.0900    0.0895
    0.1000    0.0974
    0.1100    0.1082
    0.1200    0.1178
    0.1300    0.1304
    0.1400    0.1437
    0.1500    0.1528
     ....       ....
figure
plot(alpha,[alpha' mean(ind)'],'linewidth',2)
xlabel('Significance level (\alpha)')
ylabel('size')
legend('Nominal alpha','Size of t=test')
```

see `OLStests1.m`

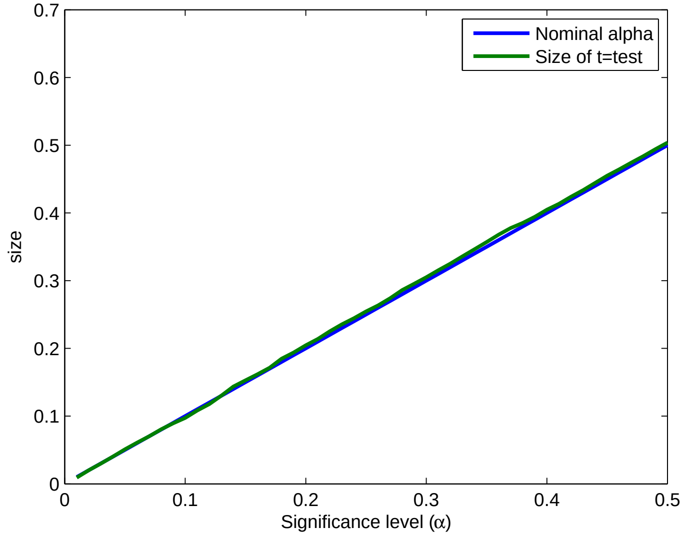

Figure 1 : Size of t-test (Homoskedastic Errors)

Monte Carlo Experiment: Size of t-test under heteroscedasticity

- DGP Setup:
$$y = \beta_1 + \beta_2 x_1 + \beta_3 x_2 + u$$
True values of parameters:
$$\beta_1 = 5, \quad \beta_2 = 1, \quad \beta_3 = 0$$
- We want to test $H_0 : \beta_3 = 0$ against $H_1 : \beta_3 \neq 0$
- Let $\boldsymbol{x} = [x_1 \quad x_2]^\top$ be fixed in repeated samples and
$$\boldsymbol{x} \sim \; N(\boldsymbol{\mu}, \; \boldsymbol{\Sigma}), \; \boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
so that $Cor(x_1, x_2) = 0$
- Sample size $n = 30$, the number of Monte Carlo replications is 10000.
- $u_i \sim \; N(0, \sigma_i^2)$, $\sigma_i = \exp(0.5x_{1i} + 0.5x_{2i})$ $i = 1, 2, \ldots, n$ (Heteroscedastic errors).

see `OLStests1.m`

Monte Carlo Experiment: Size of t-test under heteroscedasticity

```
disp('Nominal alpha level and Size')
   [alpha' mean(ind)']

    0.0100    0.0124
    0.0200    0.0243
    0.0300    0.0378
    0.0400    0.0514
    0.0500    0.0640
    0.0600    0.0768
    0.0700    0.0891
    0.0800    0.1019
    0.0900    0.1148
    0.1000    0.1257
    0.1100    0.1380
    0.1200    0.1492
    0.1300    0.1596
    0.1400    0.1707
    0.1500    0.1822
    ....       ....
figure
plot(alpha,[alpha' mean(ind)'],'linewidth',2)
xlabel('Significance level (\alpha)')
ylabel('size')
legend('Nominal alpha','Size of t=test')
```

see `OLStests1.m`

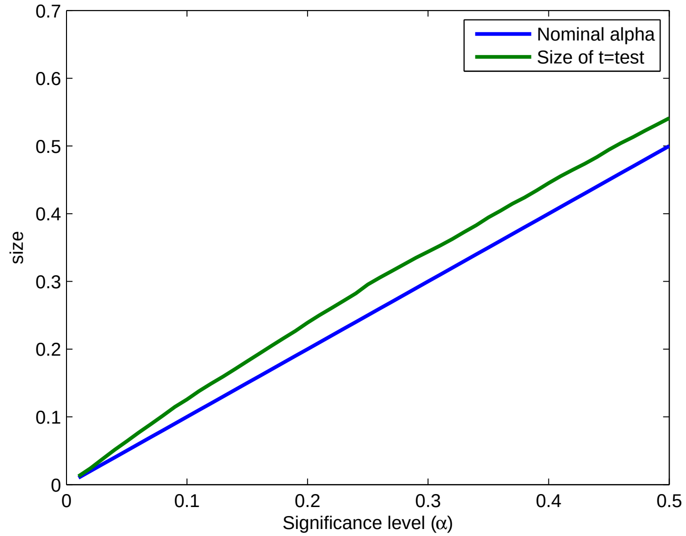

Figure 2 : Size of t-test (Heteroscedastic Errors)

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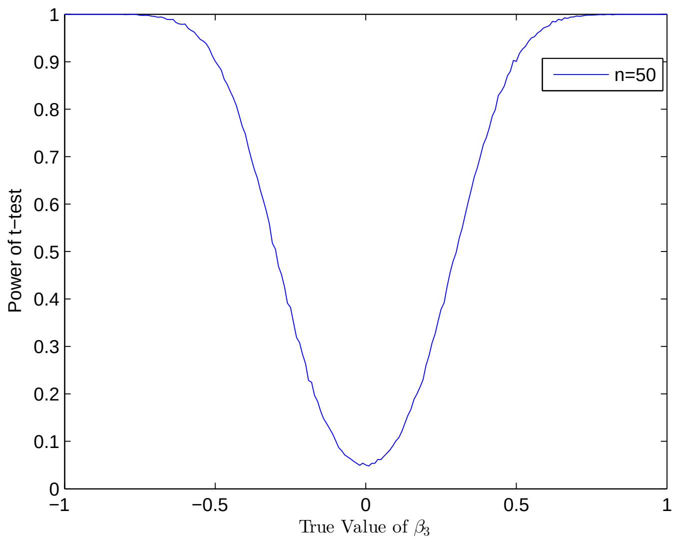

Figure 3 : Power of t-test

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Monte Carlo Experiment: Power of t-test

```
nn = [30 50 100 200 500]; % sample sizes
for h=1:length(nn)
n = nn(h);
beta1 = 5; % true parameter values
beta2 = 1;
beta3 = (-1:0.05:1);
dof = n-3;
alpha = 0.05;
cv = tinv(1-alpha/2,dof);
for i=1:MCrep
    for j=1:length(beta3)
        [i j]
        u = randn(n,1);
        y = beta1 + beta2*X(:,1) + beta3(j)*X(:,2) + u; % true DGP
        res = OLS(y,[ones(n,1) X],0);
        tratio(i,j) = res.tratio(3);
        ind(i,j,h) = abs(tratio(i,j))>cv;
    end
end
end
power = squeeze(mean(ind(:,:,:)));
```

see OLStests1.m

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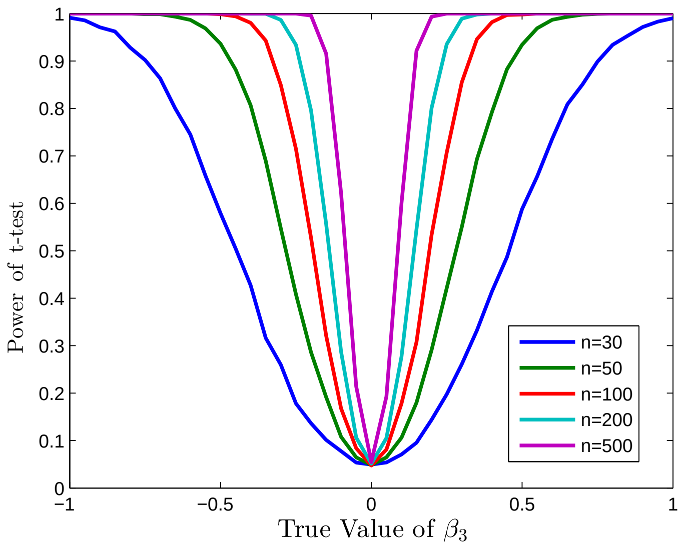

Figure 4 : Power of t-test as a function of sample size

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Confidence Interval

- %100$(1-\alpha)$ confidence interval for β_j:

$$\hat{\beta}_j \pm t_{\alpha/2,n-k} se(\hat{\beta}_j)$$

- Let null hypothesis be $H_0 : \boldsymbol{\beta}_j = \beta_j^*$.
- We reject the null hypothesis at the significance level α if β_j^* is outside the confidence interval.
- Interpretation of the confidence interval

Testing linear hypotheses

- Testing the overall significance:
$$F = \frac{ESS/(k-1)}{SSR/(n-k)} \sim F(k-1, n-k)$$
or equivalently
$$F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k)$$
- Testing general linear restrictions:
$$F = \frac{(\tilde{\boldsymbol{u}}^\top \tilde{\boldsymbol{u}} - \hat{\boldsymbol{u}}^\top \hat{\boldsymbol{u}})/q}{\hat{\boldsymbol{u}}^\top \hat{\boldsymbol{u}}/(n-k)} \sim F(q, n-k)$$
where $\tilde{\boldsymbol{u}}^\top \tilde{\boldsymbol{u}}$ is the Restricted SSR, $\hat{\boldsymbol{u}}^\top \hat{\boldsymbol{u}}$ is the Unrestricted SSR and q is the number of linear restrictions.
- Alternative form of the F statistic:
$$F = \frac{(R^2_{ur} - R^2_r)/q}{(1-R^2_{ur})/(n-k)} \sim F(q, n-k)$$

Testing Nonlinear Hypotheses

- Suppose that there is a single nonlinear restriction that we want to test:
$$H_0 : R(\boldsymbol{\beta}) = r$$
where $R(\boldsymbol{\beta})$ is a nonlinear function of the parameter vector.
- To test this null we can use the t test:
$$z = \frac{R(\hat{\boldsymbol{\beta}}) - r}{est.se\left(R(\hat{\boldsymbol{\beta}})\right)} \sim t(n-k)$$
or its square which would have $F(1, n-k)$ distribution.
- How to estimate the standard error of the nonlinear restriction $est.se\left(R(\hat{\boldsymbol{\beta}})\right)$?
- Note that by the Slutsky's theorem $R(\hat{\boldsymbol{\beta}})$ is a consistent estimator of $R(\boldsymbol{\beta})$ (we cannot use unbiasedness here, why?)

Testing Nonlinear Hypotheses

- Taking first order Taylor series expansion around the true value we obtain:
$$R(\hat{\boldsymbol{\beta}}) \approx R(\boldsymbol{\beta}) + \left(\frac{\partial R(\boldsymbol{\beta})}{\boldsymbol{\beta}}\right)^\top (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$$
- Then the variance is
$$\text{Var}\left(R(\hat{\boldsymbol{\beta}})\right) \approx \left(\frac{\partial R(\boldsymbol{\beta})}{\boldsymbol{\beta}}\right)^\top \text{Var}(\hat{\boldsymbol{\beta}}) \left(\frac{\partial R(\boldsymbol{\beta})}{\boldsymbol{\beta}}\right)$$
- Note that derivatives are unknown. We can use parameter estimates instead of unknown parameters. These are valid asymptotically.

Testing Nonlinear Hypotheses

- As an example see Greene, p.132 example 5.6.
- Consider the following model for the long run marginal propensity to consume:
$$\log C_t = \alpha + \beta \log Y_t + \gamma \log C_{t-1} + u_t$$
where C is real consumption and Y is real disposable income.
- The quarterly data comes from the US for the 1950-2000 period.
- In this distributed lag model the short run MPC is β.
- The long run MPC is
$$\delta = \frac{\beta}{1-\gamma}$$
- We want to test $H_0 : \delta = 1$.

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Testing Nonlinear Hypotheses: Example

Estimated model in equation form is

$$\widehat{\text{l\_realcons}} = \underset{(0.010553)}{0.00314157} + \underset{(0.028727)}{0.0749579}\,\text{l\_realdpi} + \underset{(0.028594)}{0.924625}\,\text{l\_realcons\_1}$$

$$n = 203 \quad \bar{R}^2 = 0.9997 \quad F(2,200) = 3.4760\text{e+}005 \quad \hat{\sigma} = 0.0087425$$

(standard errors in parentheses)

and the coefficient covariance matrix is

Coefficient covariance matrix

const	l_realdpi	l_realcons_1	
0.00011136	−0.00013445	0.00012240	const
	0.00082522	−0.00082068	l_realdpi
		0.00081760	l_realcons_1

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Testing Nonlinear Hypotheses: Example

Based on these results estimated long run MPC is

$$\hat{\delta} = \frac{\hat{\beta}}{1-\hat{\gamma}} = \frac{0.0749579}{1-0.924625} = 0.99403$$

Derivatives are

$$\frac{\partial\hat{\delta}}{\partial\hat{\beta}} = \frac{1}{1-\hat{\gamma}} = 13.2626$$

$$\frac{\partial\hat{\delta}}{\partial\hat{\gamma}} = \frac{\hat{\beta}}{(1-\hat{\gamma})^2} = 13.1834$$

Estimated asymptotic variance of $\hat{\delta}$ is 0.0002585 and the standard error is 0.016078. The t statistic is

$$z = \frac{0.99403-1}{0.016078} = -0.37131.$$

which leads to nonrejection of the null hypothesis.

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Testing Nonnested Models

- So far we considered testing nested models. F-tests can only be applied if the restricted model is nested in the unrestricted model.
- Consider the following null and alternative hypotheses:

$$H_0 : \boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}_0$$
$$H_1 : \boldsymbol{y} = \boldsymbol{Z\gamma} + \boldsymbol{u}_1$$

- F-test cannot be used.
- Two approaches became popular
- Encompassing Principle (Mizon and Richard)
- J-test (Davidson and MacKinnon)

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Encompassing Principle

$$H_0 : \boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}_0$$
$$H_1 : \boldsymbol{y} = \boldsymbol{Z\gamma} + \boldsymbol{u}_1$$

- Model 0 encompasses Model 1 if the features of Model 1 can be explained by Model 0 but the reverse is not true.
- Construct a “supermodel”:

$$\boldsymbol{y} = \bar{\boldsymbol{X}}\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{Z}}\bar{\boldsymbol{\gamma}} + \boldsymbol{X\delta} + \boldsymbol{u}$$

$\bar{\boldsymbol{X}}$: set of variables in $\boldsymbol{X}$ that are not in $\boldsymbol{Z}$
$\bar{\boldsymbol{Z}}$: set of variables in $\boldsymbol{Z}$ that are not in $\boldsymbol{X}$
$\boldsymbol{W}$: common variables

- Estimate the supermodel. Use F test.
- If $\bar{\boldsymbol{\gamma}} = \boldsymbol{0}$, reject H_1
- If $\bar{\boldsymbol{\beta}} = \boldsymbol{0}$, reject H_0

Encompassing Principle

- Problem 1: this test does not really distinguish between H_0 and H_1, but between H_1 and a hybrid model.
- Problem 2: large number of regressors, multicollinearity
- Both models may be rejected or not rejected.

Nonnested Models: J-test

$$H_0 : \boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}_0$$
$$H_1 : \boldsymbol{y} = \boldsymbol{Z\gamma} + \boldsymbol{u}_1$$

- Consider
$$\boldsymbol{y} = (1-\lambda)\boldsymbol{X\beta} + \lambda(\boldsymbol{Z\gamma}) + \boldsymbol{u}$$
- Test $\lambda = 0$ supports H_0 against H_1
- But λ cannot separately estimated. Instead use a two-step procedure suggested by Davidson and MacKinnon
- Step 1: Regress $\boldsymbol{y}$ on $\boldsymbol{Z}$ obtain $\boldsymbol{Z}\hat{\boldsymbol{\gamma}}$ fitted values
- Step 2: Regress $\boldsymbol{y}$ on $\boldsymbol{X}$ and $\boldsymbol{Z}\hat{\boldsymbol{\gamma}}$. Carry out a t-test $H_0 : \lambda = 0$
- Significant t statistic implies rejection of H_0