ECONOMETRICS
M.A. Program in Economics
YTU
Department of Economics

Unit Root Tests
Cointegration
and
Error Correction Models

Unit Root Tests

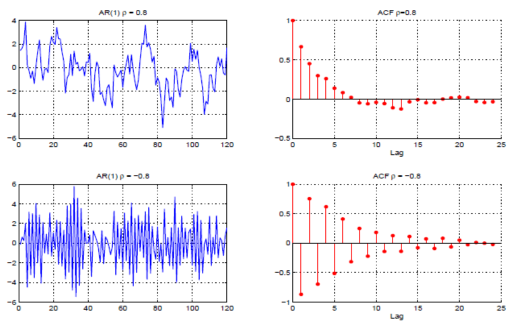
- How do we know if a time series contains a unit root?
- Recall that in the AR(1) process

$$y_t = \alpha + \rho y_{t-1} + e_t, |\rho| < 1$$

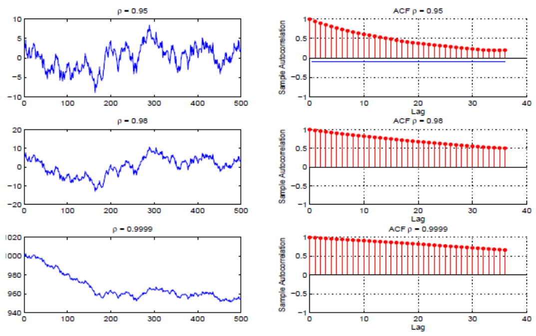
- The parameter Rho measures the degree of dependence to the past values:

$$Corr(y_t, y_{t-h}) = \rho^h$$

Two Simulated AR(1) Process and Correlograms



Highly Persistent AR(1) Processes



Random Walk

- In the AR(1) model if:

$$\alpha = 0, \rho = 1$$

Then we obtain random walk process.

$$y_t = y_{t-1} + e_t, e_t \text{ white noise}$$

- If the drift term is not zero we obtain random walk with drift:

$$y_t = \alpha + y_{t-1} + e_t$$

- Both of these processes are nonstationary.
- Random walk with drift has a stochastic trend.

Random Walk

$$y_t = y_{t-1} + e_t$$

By repeated substitution:

y(1) = ***y(0) + e(1)***

y(2) = y(1) + e(2) = ***y(0) + e(1)*** + e(2)

y(3) = y(2) + e(3) = ***y(0) + e(1) + e(2)***+ e(3)

…………….

y(t) = y(0) + e(1) + e(2) +….+e(t-1)+ e(t)

$$= y_0 + \sum_{t=1}^{t} e_t$$

$$\mathrm{E}(y_t) = \mathrm{E}(e_t) + \mathrm{E}(e_{t-1}) + \ldots + \mathrm{E}(e_1) + \mathrm{E}(y_0) = \mathrm{E}(y_0), \text{ for all } t \geq 1.$$

Random Walk

$$y_t = y_{t-1} + e_t$$

y(t) = y(0) + e(1) + e(2) +….+e(t-1)+ e(t)=

$$= y_0 + \sum_{t=1}^{t} e_t$$

Although expected value does not depend on time, Variance increases with time:

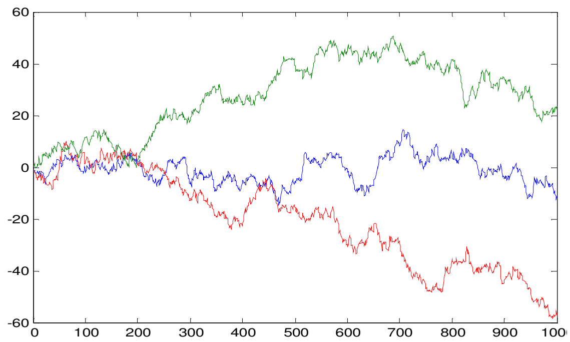
$$\mathrm{Var}(y_t) = \mathrm{Var}(e_t) + \mathrm{Var}(e_{t-1}) + \ldots + \mathrm{Var}(e_1) = \sigma_e^2 t.$$

$$\mathrm{E}(y_{t+h}|y_t) = y_t, \text{ for all } h \geq 1.$$

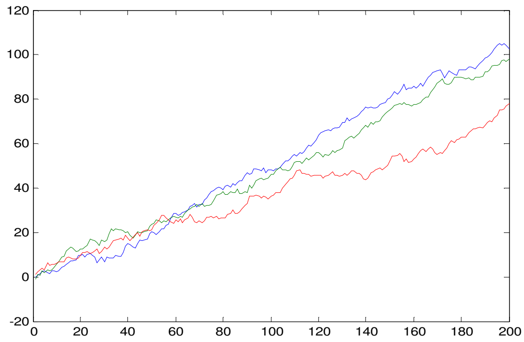
Random walk is nonstationary

$$\mathrm{Corr}(y_t, y_{t+h}) = \sqrt{t/(t+h)}.$$

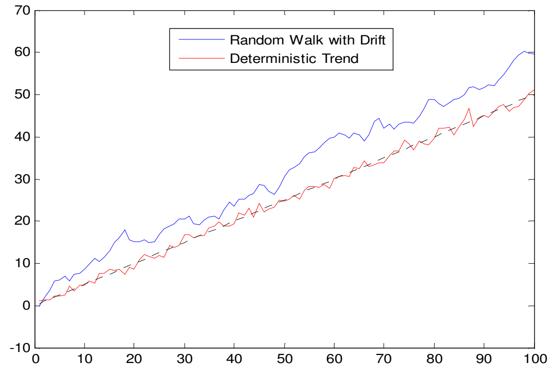
Three Realizations of a Random Walk Process
x(t) = x(t-1) + e(t)



Three Realizations of a Random Walk with Drift Process
x(t) = 0.5 + x(t-1) + e(t)



Random Walk with Drift vs Deterministic Trend



I(1) vs I(0)

- If the first difference of series is stationary, ie I(0), then the series is said to be integrated of order 1, denoted I(1)
- If we take the first difference of a random walk process we obtain a white noise process which is I(0) by definition:

$$\Delta y_t = y_t - y_{t-1} = e_t, e_t \text{ white noise}$$

- Similarly first difference of a random walk with drift process is I(0):

$$\Delta y_t = y_t - y_{t-1} = \alpha + e_t$$

- These series are called difference stationary series.

Trend-stationary series

- A trend-stationary process becomes stationary when it is detrended:

$$y_t = \beta_0 + \beta_1 t + e_t$$

- In general it is not easy to distinguish a trend-stationary process from a difference-stationary process (ie random walk with drift)
- Because, a random walk with drift process behaves very similarly to a trend-stationary process. It follows a clear time trend. But the trend is not deterministic, it is stochastic.
- Since we cannot use nonstationary variables in regression models, we need to appropriately transform them when necessary.
- Need to detrend when trend-stationary and first-difference when nonstationary
- Several unit root tests have been developed to help us decide.

Unit Root Tests

- Consider the following AR(1) model:

$$y_t = \alpha + \rho y_{t-1} + e_t$$

- where e(t) is a martingale-difference sequence:

$$\mathrm{E}(e_t | y_{t-1}, y_{t-2}, \ldots, y_0) = 0.$$

- e(t) has zero expectation, independent from the initial value y0 and it is iid
- We are interested in testing the following null and alternative hypotheses:

$$\mathrm{H}_0\colon \rho = 1. \quad \mathrm{H}_1\colon \rho < 1.$$

Unit Root Tests

- Alternatively we can write:

  H0 : I(1) vs H1: I(0)
- Subtract y(t-1) from both sides

$$\Delta y_t = \alpha + \theta y_{t-1} + e_t.$$

- Now the null and alternative are:

$$\mathrm{H}_0\colon \theta = 0 \qquad \mathrm{H}_1\colon \theta < 0.$$

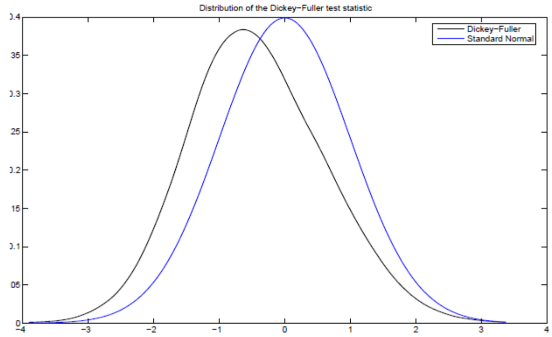
Unit Root Tests

$$\Delta y_t = \alpha + \theta y_{t-1} + e_t.$$

$$\mathrm{H}_0\colon \theta = 0 \qquad \mathrm{H}_1\colon \theta < 0.$$

- Can we use t-test for significance?
- NO. Because t-ratio does not follow the usual t-distribution under the null hypothesis.
- This distribution is called **Dickey-Fuller (DF)** distribution.
- This is more skewed to left as compared to t distribution.
- But we can use appropriate critical values from the DF distribution to make a decision.

Dickey-Fuller (DF) Distribution



DF Unit Root Test

$$\Delta y_t = \alpha + \theta y_{t-1} + e_t.$$

$$\mathrm{H}_0\colon \theta = 0 \qquad \mathrm{H}_1\colon \theta < 0.$$

- Decision rule: if t-ratio is smaller than the critical value at a given significance level then we reject the null hypothesis that series contain a unit root, that is we reject that it is nonstationary.
- Note that we conduct a left-tail test.
- Most statistical packages calculate p-values or critical values automatically.
- If the p-value is sufficiently small then we reject the null hypothesis.
- Critical values depend on whether the test regression contains a constant, a time trend or both.

DF Unit Root Test

$$\Delta y_t = \alpha + \theta y_{t-1} + e_t.$$

$$\mathrm{H}_0\colon \theta = 0 \qquad \mathrm{H}_1\colon \theta < 0.$$

- Critical values when there is no time trend

Table 18.2

Asymptotic Critical Values for Unit Root *t* Test: No Time Trend

Significance Level	1%	2.5%	5%	10%
Critical Value	−3.43	−3.12	−2.86	−2.57

DF Unit Root Test: Example

- Is there a unit root in 3-month treasury bill rate? (intqrt.gdt)

$$\widehat{\Delta r3_t} = .625 - .091\ r3_{t-1}$$
$$(.261)\quad (.037)$$
$$n = 123,\ R^2 = .048,$$

$-.091/.037 = -2.46.$

Table 18.2

Asymptotic Critical Values for Unit Root *t* Test: No Time Trend

Significance Level	1%	2.5%	5%	10%
Critical Value	−3.43	−3.12	−2.86	−2.57

Augmented Dickey-Fuller (ADF) Test

- Residuals must be serially uncorrelated in the test regression
- Can add lagged values of dependent variable to the test regression to obtain a clean residuals. Thus the regression is augmented with lagged y's. For example:

$$\Delta y_t = \alpha + \theta y_{t-1} + \gamma_1 \Delta y_{t-1} + e_t,$$

where we added only one lag of y. In general

$$\Delta y_t \text{ on } y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-p}$$

- ADF test statistic is calculated as the t-ratio for the significance of y(t-1). DF critical values are still valid.

ADF Unit Root Test: Example

- Is infation nonstationary? (phillips.gdt)

$$\widehat{\Delta inf_t} = 1.36 - .310\ inf_{t-1} + .138\ \Delta inf_{t-1}$$
$$(.261)\quad (.103)\qquad\qquad (.126)$$
$$n = 47,\ R^2 = .172.$$

$-.310/.103 = -3.01.$

ADF Unit Root Test with Time Trend

- If the series follow a clear trend we may want to add a time trend in ADF test regression:

$$\Delta y_t = \alpha + \delta t + \theta y_{t-1} + e_t,$$

H_0: $\theta = 0$. H_1: $\theta < 0$.

- Notice that under the null hypothesis, y follows a RW with drift process.
- The alternative hypothesis says that the series is stationary around a time trend. (trend-stationary process).

ADF Unit Root Test with Time Trend

$$\Delta y_t = \alpha + \delta t + \theta y_{t-1} + e_t,$$

H_0: $\theta = 0$. H_1: $\theta < 0$.

- Critical values :

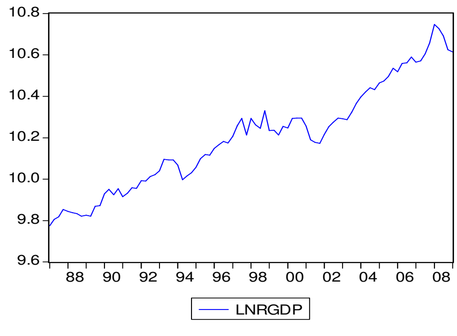
Table 18.3

Asymptotic Critical Values for Unit Root *t* Test: Linear Time Trend

Significance Level	1%	2.5%	5%	10%
Critical Value	−3.96	−3.66	−3.41	−3.12

ADF Unit Root Test with Time Trend: Example

- Is there a unit root in the Turkish Real GDP?



ADF Test: Example

- Null Hypothesis: LNRGDP has a unit root
- Exogenous: Constant, Linear Trend
- Lag Length: 0 (Automatic based on SIC, MAXLAG=11)

	t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic	-2.707791	0.2362
Test critical values:		
1% level	-4.065702	
5% level	-3.461686	
10% level	-3.157121	

*MacKinnon (1996) one-sided p-values.

Spurious Regression

- In the cross-section analysis, spurious regression problem between y and x will arise when these variables are related through a third variable.
- Although y and x seem to be related, when we control for the effect of the third variable (by including it in the regression) the relationship disappears.
- A similar problem arises in the time series context.
- If y and x both have a clear upward or downward trend we may find a significant relationship between them.
- What we found may come from the time trend or common trend that both series follows. They may not have a significant relationship.

Spurious Regression

- If both series are weakly-dependent and stationary then the problem can be solved by adding time trend to the regression or by detrending them before analysis.
- But **if the series** are not trend-stationary but **difference stationary adding time trend will not solve the problem**.
- In the simple regression model, using two independent I(1) series may result in significant t-statistics an high R2.
- In fact the series are not related but regression results will not reflect this.

Spurious Regression

- Suppose that both x and y are independent I(1):

$$x_t = x_{t-1} + a_t \qquad y_t = y_{t-1} + e_t,\ t = 1,2,\ldots,$$

- Consider the following simple regression

$$\hat{y}_t = \hat{\beta}_0 + \hat{\beta}_1 x_t$$

- We should find insignificant t statistic for the slope parameter 95% of the time.
- But simulation studies show that rejection rates for the t-statistic is much larger than the nominal 5% level.
- This is called spurious regression problem.

Spurious Regression

- Although y and x are not related t-stats tend to be significant and R2 tends to be high.
- The source of the problem is that under the H0, y follows a random walk and the t-ratio does not follow the usual t-distribution.
- Similarly, R2 will not converge to the population value in this case.
- Instead R2 converges to a random number which has a large probability of having a high value.
- This is why we observe high R2 in the spurious regressions.

Spurious Regression

- The practical significance of this problem is rather obvious.
- We should be especially careful when running regressions with variables in levels.
- Because most series in economics and business tend to be I(1).
- Thus there is always the danger of finding a spurious regression.
- Under what circumstances we may be sure that we find a genuine relationship between two (or more) I(1) series?

Cointegration

- If I(1) variables are related in such a way that the regression reflects long-run relationship, in other words, if they are cointegrated, we can be sure that we do not have spurious regression.
- Consider again the simple regression model. Suppose that both y and x are I(1) variables.
- If a linear combination of them is I(0), ie, stationary then these two series are cointegrated:

$$y_t - \beta x_t$$

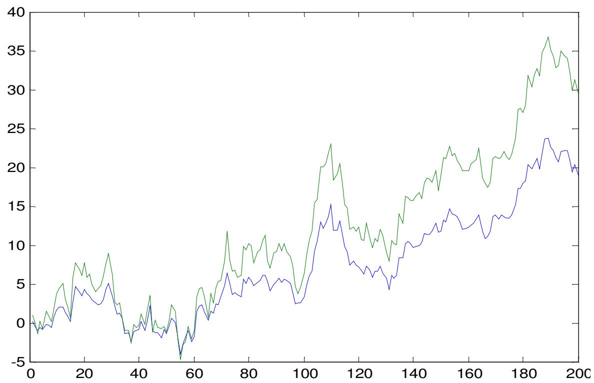
- If we can find a nonzero beta coefficient then it is called cointegrating parameter.

Cointegration

$$y_t - \beta x_t$$

- Reflects long run equilibrium relationship.
- It allows short-term deviations from this stable relationship.
- But these are assumed to be short-run in nature and the equilibrium will be attained at a certain speed.
- Example: Law of One Price
- P1 = a + b P2
- Example: Purchasing Power Parity

Two cointegrated series



Cointegration

- How do we know if two series are cointegrated?
- *Engle-Granger cointegration test:*
- Use OLS to estimate the LR relationship:

$$y_t = \hat{\alpha} + \hat{\beta} x_t$$

- Apply ADF unit root test to the residuals obtained from this regression.
- H0: residuals contain a unit root (series are not cointegrated)
- If we reject H0, in other words, if the residuals are stationary then x and y are said to be cointegrated.

Cointegration

- Notice that under the H0 we have a spurious regression.
- We need to use the following critical values (no time trend case):

Table 18.4

Asymptotic Critical Values for Cointegration Test: No Time Trend

Significance Level	1%	2.5%	5%	10%
Critical Value	−3.90	−3.59	−3.34	−3.04

Cointegration

- If the series follow a clear trend then we can add time trend to the regression:

$$\hat{y}_t = \hat{\alpha} + \hat{\eta} t + \hat{\beta} x_t$$

- In this case the critical values are:

Table 18.5

Asymptotic Critical Values for Cointegration Test: Linear Time Trend

Significance Level	1%	2.5%	5%	10%
Critical Value	−4.32	−4.03	−3.78	−3.50

Cointegration: Example

- Are fertility rate (gfr) and tax exemptions (pe) cointegrated?

$$\widehat{\text{gfr}} = \underset{(3.4753)}{109.930} - \underset{(0.10899)}{0.905188}\,\text{time} + \underset{(0.034626)}{0.186662}\,\text{pe}$$

$$\widehat{\text{d\_uhat}} = \underset{(0.67143)}{-0.183045} - \underset{(0.048939)}{0.118669}\,\text{uhat\_1} + \underset{(0.11696)}{0.244979}\,\text{d\_uhat\_1}$$

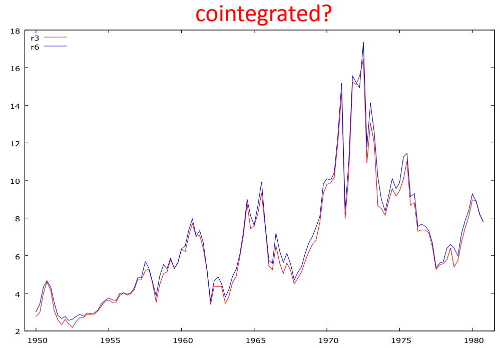
Model 6: OLS, using observations 1915-1984 (T = 70)
Dependent variable: d_uhat

	coefficient	std. error	t-ratio	p-value		
const	-0.183045	0.671431	-0.2726	0.7860		
uhat_1	-0.118669	0.0489389	-2.425	0.0180	**	%10 c=-3.5
d_uhat_1	0.244979	0.116958	2.095	0.0400	**	No cointegration

Example: Are 3-month and 6-month interest rates cointegrated?

- r3: interest rate on 3-month treasury bills
- r6: interest rate on 6-month treasury bills
- These two series will not wander too far away from each other because of arbitrage.
- LR relationship:
- $$r6_t = r3_t + \mu + e_t,$$
- Deviations from this relationship should be short lived becuase of arbitrage.

Are 3-month and 6-month interest rates cointegrated?



Are 3-month and 6-month interest rates cointegrated?

- First make sure that both series are I(1). Conduct unit root tests.

$$r6_t = r3_t + \mu + e_t,$$

- Then apply Engle-Granger cointegration test.
- Which is simply ADF test applied to the residuals from the regression above.
- Use appropriate critical values.

Are 3-month and 6-month interest rates cointegrated?

- OLS regression

$$\widehat{\text{r6}} = \underset{(0.054867)}{0.135374} + \underset{(0.0077088)}{1.02590}\;\text{r3}$$

- Test regression
- OLS, using observations 1951:2-1980:4 (T = 119)
- Dependent variable: d_uhat

	coefficient	std. error	t-ratio	p-value	
const	-3.45587e-05	0.0229010	-0.001509	0.9988	
uhat_1	-0.812633	0.160728	-5.056	1.67e-06	***
d_uhat_1	0.0513596	0.149446	0.3437	0.7317	
d_uhat_2	0.205066	0.132861	1.543	0.1255	
d_uhat_3	0.125392	0.119041	1.053	0.2944	
d_uhat_4	0.178414	0.0937341	1.903	0.0595	*

Error Correction Models (ECM)

- An ECM is a dynamic model that reflects the long term relationship between two or more variables.
- When x and y are not cointegrated but both are I(1) then we can estimate the following model

$$\Delta y_t = \alpha_0 + \alpha_1 \Delta y_{t-1} + \gamma_0 \Delta x_t + \gamma_1 \Delta x_{t-1} + u_t,$$

- Note that all variables are in first differences that is they are all I(0). We cannot use I(1) variables in the regression.
- This is in fact an FDL model which we saw before.
- Paramaters can be interpreted as in FDL models.

Error Correction Models

- If the series are cointegrated, that is there exists a cointegration relation between them such that:

$$s_t = y_t - \beta x_t,$$

- We can add one-period lagged s as an additional regressors:

$$\begin{aligned}\Delta y_t &= \alpha_0 + \alpha_1 \Delta y_{t-1} + \gamma_0 \Delta x_t + \gamma_1 \Delta x_{t-1} + \delta s_{t-1} + u_t \\ &= \alpha_0 + \alpha_1 \Delta y_{t-1} + \gamma_0 \Delta x_t + \gamma_1 \Delta x_{t-1} + \delta(y_{t-1} - \beta x_{t-1}) + u_t,\end{aligned}$$

- This model is called Error Correction Model (ECM).

Error Correction Models

- For simplicity consider:

$$\Delta y_t = \alpha_0 + \gamma_0 \Delta x_t + \delta(y_{t-1} - \beta x_{t-1}) + u_t,$$

- Where $\delta < 0.$
- is called the error correction term.
- Since it is negative large deviations will fade away and LR equilibrium will be attained at a certain rate.
- ECM can be estimated using OLS.
- But we need to estimate the cointegration relationship first.

ECM

- General ECM with two variables:

$$\Delta y_t = \gamma_{10} + \alpha_1 e_{t-1} + \sum_{i=1}^{p} \phi_{1i} \Delta y_{t-i} + \sum_{i=1}^{q} \varphi_{1i} \Delta x_{t-i} + error$$

$$\Delta x_t = \gamma_{20} + \alpha_2 e_{t-1} + \sum_{i=1}^{p} \phi_{2i} \Delta y_{t-i} + \sum_{i=1}^{q} \varphi_{2i} \Delta x_{t-i} + error$$

If these coefficients are jointly significant then x Granger-causes y. Can use F-test.

Speed of adjustment parameter
One of these must be statistically significant.

ECM: Example

- A version of Wagner hypothesis says that real government expenditures (G) is positively related to real per capita income (Y/N) in a long run relationship:

$$\ln(G)_t = \beta_0 + \beta_1 \ln(Y/N)_t + \varepsilon_t$$

- In other words, it says that the variables must be cointegrated and the slope parameter should be positive.
- Unit root tests indicate that both variables are I(1).

ECM: Example

- Engle-Granger test regression is estimated as follows:

$$\ln(G)_t = 17.27 + 2.86 \ln(Y/N)_t + \hat{\varepsilon}_t$$

- ADF test statistic on residuals is -3.5817. It is significant at 5% level.
- Thus, we reject the null hypothesis that the series are not cointegrated.
- Real government expenditures and real per capita income are cointegrated.
- Slope parameter is positive and significant.
- These results support the Wagner hypothesis.

ECM Example: Wagner Law

Estimated cointegration relationship:

$$e_{t-1} = \ln G_{t-1} - 17.27 - 2.86 \ln(Y/N)_{t-1}$$

Error Correction Model. Lagged variables are excluded as they are insignificant.

$$\Delta \ln G_t = 0.0641 - 0.1879\, e_{t-1}$$

$$\Delta \ln(Y/N)_t = 0.0246 + 0.0538\, e_{t-1}$$

ECM Example: Expectations Hypothesis

- hy6(t): 3-month holding return on treasury bills: buy 6-month bill at period (t-1) and sell it as a 3-month bill three months later.
- hy3(t-1): return that we get from buying 3-month T-bill at time (t-1)
- hy3(t-1): this return is known at time (t-1)
- hy6(t): this return is not known at (t-1) because we do not know the price of 3-month bill at time t
- According to the expectations hypothesis these two investment strategies should have the same return because of arbitrage.

ECM Example: Expectations Hypothesis

- Expectations hypothesis says that, conditional on information set available at time t-1:

$$\mathrm{E}(hy6_t | I_{t-1}) = hy3_{t-1},$$

- Can test this using the following model:

$$hy6_t = \beta_0 + \beta_1 hy3_{t-1} + u_t,$$

- Are hy6(t) and hy3(t-1) cointegrated? Is slope parameter equal to 1?

ECM Example: Expectations Hypothesis

- Engle-Granger test results indicate that these two series are cointegrated. This implies that we can estimate an ECM:

$$\Delta hy6_t = \alpha_0 + \gamma_0 \Delta hy3_{t-1} + \delta(hy6_{t-1} - hy3_{t-2}) + u_t,$$

- ECM estimation results:

$$\Delta \widehat{hy6}_t = \underset{(.043)}{.090} + \underset{(.264)}{1.218}\, \Delta hy3_{t-1} - \underset{(.244)}{.840}\, (hy6_{t-1} - hy3_{t-2})$$

$$n = 122,\ R^2 = .790.$$