Throughout this homework, coefficients are in a field $\mathbf{k}$, ie. $H_n(X) = H_n(X, \mathbf{k})$ and $C_n(X) = C_n(X, \mathbf{k})$. I roughly follow the (terse!) proofs in *Infinite Cyclic Coverings* by John Milnor.

1. (i) We consider the sequence

$$0 \longrightarrow C_*(\widehat{X}) \xrightarrow{t-1} C_*(\widehat{X}) \xrightarrow{p_*} C_*(X) \longrightarrow 0 \tag{1}$$

where p_* is induced from the covering map $\widehat{X} \xrightarrow{p} X$. First we show exactness of each row. As mentioned in class, Π acts on $C_n(\widehat{X})$ by permuting generators, so the map $C_n(\widehat{X}) \xrightarrow{t-1} C_n(\widehat{X})$ induced by the action of $t-1$ must be injective. As $\widehat{X}/\Pi \cong X$, the covering map $\widehat{X} \xrightarrow{p} X$ is cellular, so the induced map $C_n(\widehat{X}) \to C_n(X)$ is surjective. Finally, if $e_i^n \in C_n(\widehat{X})$ then

$$(t-1)e_i^n = te_i^n - e_i^n = e_{i+1}^n - e_i^n \in \ker p_*$$

as e_i^n and e_{i+1}^n project to the same cell in X, and conversely if $p_*(z) = 0$ for some $z \in C_n(\widehat{X})$ then z must be of the form $\sum_j (t-1)e_j^n$ as $\widehat{X}/\Pi \cong X$, so $\ker(p_*) = \mathrm{im}\,(t-1)$.

To show (1) is exact it remains to show each square in the diagram below commutes (for each n). I don't know how to do this.

$$\begin{tikzcd}
0 \arrow[r] & C_n(\widehat{X}) \arrow[r, "t-1"] \arrow[d, "d_{\widehat{X}}"] & C_n(\widehat{X}) \arrow[r, "p_*"] \arrow[d, "d_{\widehat{X}}"] & C_n(X) \arrow[r] \arrow[d, "d_X"] & 0 \\
0 \arrow[r] & C_{n-1}(\widehat{X}) \arrow[r, "t-1"] & C_{n-1}(\widehat{X}) \arrow[r, "p_*"] & C_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

Here is a statement of the Snake Lemma.

Lemma (snek) *Let A be a commutative ring, and let*

$$\begin{tikzcd}
 & P \arrow[r, "\varphi"] \arrow[d, "\alpha"] & Q \arrow[r, "\psi"] \arrow[d, "\beta"] & R \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & P' \arrow[r, "\varphi'"] & Q' \arrow[r, "\psi'"] & R' &
\end{tikzcd} \tag{2}$$

be a commutative diagram of A-modules with exact rows. Then there is an A-linear map $\delta : \ker\gamma \to \mathrm{coker}\,\alpha$ such that the sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \mathrm{coker}\,\alpha \longrightarrow \mathrm{coker}\,\beta \longrightarrow \mathrm{coker}\,\gamma \tag{3}$$

is exact. Moreover, if φ is injective then the first map $\ker\alpha \to \ker\beta$ is injective, and if ψ' is surjective, the last map coker $\beta \to$ coker γ is surjective. The exact sequence (3) is functorial in the diagram (2).

Expanding (1) to make things more visually clear

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(\widehat{X}) & \xrightarrow{t-1} & C_0(\widehat{X}) & \longrightarrow & C_0(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1(\widehat{X}) & \xrightarrow{t-1} & C_1(\widehat{X}) & \longrightarrow & C_1(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_2(\widehat{X}) & \xrightarrow{t-1} & C_2(\widehat{X}) & \longrightarrow & C_2(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{4}$$

we apply the snake lemma to the 0^{th} row to get an exact sequence

$$0 \to Z_0(\widehat{X}) \xrightarrow{t-1} Z_0(\widehat{X}) \to Z_0(X) \xrightarrow{\partial} C_1(\widehat{X})/B_0(\widehat{X}) \xrightarrow{t-1} C_1(\widehat{X})/B_0(\widehat{X}) \to C_1(X)/B_0(X) \to 0$$

and in general apply the snake lemma to the n^{th} row to get an exact sequence

$$0 \to Z_n(\widehat{X}) \xrightarrow{t-1} Z_n(\widehat{X}) \to Z_n(X) \xrightarrow{\partial} C_{n+1}(\widehat{X})/B_n(\widehat{X}) \xrightarrow{t-1} C_{n+1}(\widehat{X})/B_n(\widehat{X}) \to C_{n+1}(X)/B_n(X) \to 0$$

So, for each n we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & C_{n+1}(\widehat{X})/B_n(\widehat{X}) & \longrightarrow & C_{n+1}(\widehat{X})/B_n(\widehat{X}) & \longrightarrow & C_{n+1}(X)/B_n(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n(\widehat{X}) & \longrightarrow & Z_n(\widehat{X}) & \longrightarrow & Z_n(X) & &
\end{array} \tag{5}$$

and applying the snake lemma to (5) for each n yields the long exact sequence

$$\dots \xrightarrow{\partial} H_n(\widehat{X}) \xrightarrow{t-1} H_n(\widehat{X}) \xrightarrow{p_*} H_n(X) \xrightarrow{\partial} H_{n-1}(\widehat{X}) \xrightarrow{t-1} \dots \tag{6}$$

(ii) Suppose that X has the homology of a circle. First, we have $H_0(\widehat{X}) \cong \mathbf{k}$ since $\widehat{X}$ is connected, so $H_0(\widehat{X})$ has no $\mathbf{k}\Pi$-free summands. If $i \geq 2$ then $H_i(X) = 0$, so we get an exact sequence

$$H_i(\widehat{X}) \xrightarrow{t-1} H_i(\widehat{X}) \to 0$$

which means we can write each $x \in H_i(\widehat{X})$ as $(t-1)y$ for some $y \in H_i(\widehat{X})$, so $H_i(\widehat{X})$ does not have any $\mathbf{k}\Pi$-free summand.

For $i = 1$, as X is a $\mathbf{k}HS^1$ the long exact sequence (6) ends like

$$\dots \xrightarrow{0} H_1(X) \xrightarrow{\sim} H_0(\widehat{X}) \xrightarrow{0} H_0(\widehat{X}) \xrightarrow{\sim} H_0(X) \longrightarrow 0$$

so the situation is similar to the one described for $i \geq 2$ above. From this $H_i(\widehat{X})$ is finitely generated as a $\mathbf{k}$-vector space for all i.

(iii) As the maps induced on chains by $t-1$ and p are injective and surjective, respectively, we break up the long exact sequence (6) into short exact sequences

$$\begin{array}{c} \vdots \\ 0 \longrightarrow H_i(\widehat{X}) \xrightarrow{t-1} H_i(\widehat{X}) \longrightarrow H_i(X) \longrightarrow 0 \\ 0 \longrightarrow H_{i+1}(\widehat{X}) \xrightarrow{t-1} H_{i+1}(\widehat{X}) \longrightarrow H_{i+1}(X) \longrightarrow 0 \\ \vdots \end{array}$$

which split as the homology modules are finitely generated $\mathbf{k}$-vector spaces by (ii). So, for each i we have

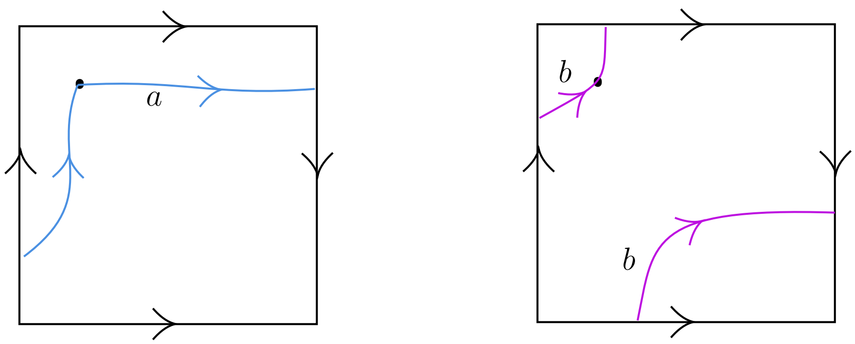
$$\operatorname{rank} H_i(\widehat{X}) = \operatorname{rank} H_i(\widehat{X}) + \operatorname{rank} H_i(X)$$

thus $\chi(\widehat{X}) = \chi(\widehat{X}) + \chi(X)$, so $\chi(X) = 0$.

2. (i) We write $\pi_1 X$ with the presentation $\pi_1 X = \langle a, b \mid a^2 b^{-2} \rangle$, with a and

b pictured below. We define a surjective homomorphism $\pi_1 X \xrightarrow{\varphi} \Pi$ by $a \mapsto t$ and $b \mapsto t$. From this we see that

$$\pi_1 \widehat{X} \cong \ker \varphi = \langle a^i b^{-i} \rangle_{i \in \mathbb{Z}}.$$



If we tile $\mathbb{R}^2$ with these squares to get the universal cover of X, we note that $a^i b^{-i}$ corresponds to moving the basepoint down by i squares. From covering space theory this means that $\widehat{X}$ corresponds to a 1-by-$\mathbb{Z}$ vertical strip in the tiling, which we can think of as a Mobius strip with infinite width, thus having homotopy type S^1. Therefore $H_i(\widehat{X}) = 0$ for $i > 1$. As a $\mathbf{k}\Pi$ module

$$H_0(\widehat{X}) \cong \mathbf{k}[t, t^{-1}]/(t-1)$$

and

$$H_1(\widehat{X}) \cong \mathbf{k}[t, t^{-1}]/(t+1).$$

(ii) We know $\pi_1 X = \langle a \rangle$ where a is a generator for $\pi_1 S^1 \cong \mathbb{Z}$. So, we get the identity map $\pi_1 X \to \Pi = \langle a \rangle$, thus $\pi_1 \widehat{X}$ is trivial so $\widehat{X}$ is the universal cover of X, thus $\widehat{X}$ has homotopy type S^2. As X has the homology of S^1 and Π acts trivially on X we must have

$$H_0(\widehat{X}) \cong \mathbf{k}[a, a^{-1}]/(a-1) \cong H_1(\widehat{X})$$

as $\mathbf{k}\Pi$-modules.