1. Suppose $x \in B$ is integral over A, so $x^m + a_1x^{m-1} + ... + a_{m-1}x + a_m = 0$ for some $m \geq 1$ and $a_1, ..., a_m \in A$. Thus we can write x^n as an A-linear combination of $1, x, ..., x^{m-2}, x^{m-1}$, so $A[x] = \langle 1, x, ..., x^{m-2}, x^{m-1} \rangle$.

2. Suppose that $B = \langle b_1, ..., b_n \rangle$ as an A-module. We first use the determinant trick to prove (special case of) the Cayley-Hamilton theorem, following the lecture notes and Theorem 4.3 in Eisenbud's book. The statement of the theorem with our notation is:

If $\varphi \in \mathrm{End}_A(B)$ then there is a monic polynomial

$$p(x) = x^n + p_1x^{n-1} + ... + p_{n-1}x + p_n$$

such that $p \circ \varphi = 0$.

For the proof, we let $\varphi \in \mathrm{End}_A(B)$ and note (as shown in the notes) that we can write $\varphi(b_i) = \sum a_{ij}b_j$, a sum of the generators with coefficients $a_{ij} \in A$. We consider B as an $A[x]$-module by $\mu_x = \varphi$ (multiplication by x is φ). Denoting the $n \times n$ identity matrix by I, the above means that

$$(xI - (a_{ij})_{ij}) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = 0.$$

Multiplying on the left by the cofactor matrix of $(xI - (a_{ij})_{ij})$ we get

$$(\underbrace{\det(xI - (a_{ij})_{ij})}_{\Delta})I \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = 0.$$

In other words, $\Delta b_i = 0$ for all $i = 1, ..., n$, so with Δ as our choice of $p(x)$, we see that $p \circ \varphi = 0$.

From this, if $b \in B$, then choosing $\varphi = \mu_b$, multiplication by b, we have $p \circ \mu_b = 0$, ie. b is integral over A.

3. Suppose $x, y \in B$ are integral over A, so $A[x]$ is finitely generated over A, and since y is integral over A it is integral over $A[x]$, so $A[x, y]$ is finitely generated over $A[x]$. As $A[x]$ is finitely generated over A, $A[x, y]$ is finitely generated over A. Thus by problem 2 every element of $A[x, y]$ is integral over A, so in particular xy and $x - y$ are. As this is true for any two elements of B integral over A, the integral closure of A in B is a subring of B.

4. (a) Suppose $x \in B$ is integral over A, so

$$x^m + a_1 x^{m-1} + ... + a_{m-1}x + a_m = 0$$

for some $m \geq 1$ and $a_i \in \mathfrak{a}$, where $i = 1, ..., m$. So, moving the degree less than m terms over we see that

$$x^m = \sum_{k=1}^{m-1} (-a_k)x^k \in \mathfrak{a}B.$$

Thus $x \in \sqrt{\mathfrak{a}B}$.

(b) We view $\sqrt{\mathfrak{a}B}$ as an A-module, and we denote $B = \langle b_1, ..., b_n \rangle$ over A, so in particular $\sqrt{\mathfrak{a}B}$ is an A-submodule of $\langle b_1, ..., b_n \rangle$. We then follow the proof of problem 2, except we take the coefficients a_{ij} to be in $\mathfrak{a}$ instead of A. This shows that each element of $\mathfrak{a}B$ is integral over $\mathfrak{a}$, and so if $x \in \sqrt{\mathfrak{a}B}$ then x^n is integral over $\mathfrak{a}$, thus so is x.

5. We drop the function composition notation to save space, so fg means $f \circ g$. We will show that $d^{n+1}d^n = 0$. Let $f = (f^j)_{j\in\mathbb{Z}} \in \mathrm{Hom}_A^n(C^\bullet, D^\bullet)$, then

$$\begin{aligned} d^{n+1}(d^n(f)) &= d^{n+1}(\underbrace{d_{D^\bullet}^{n+j} f^j + (-1)^{n+1} f^{j+1} d_{C^\bullet}^j}_{g^j})_{j\in\mathbb{Z}} \\ &= (d_{D^\bullet}^{n+1+j} g^j + (-1)^{n+2} g^{j+1} d_{C^\bullet}^j)_{j\in\mathbb{Z}}. \end{aligned}$$

It suffices to show that the term above is zero for each j. This term is

$$d_{D^\bullet}^{n+1+j}(d_{D^\bullet}^{n+j} f^j + (-1)^{n+1} f^{j+1} d_{C^\bullet}^j) + (-1)^{n+2}(d_{D^\bullet}^{n+1+j} f^{j+1} + (-1)^{n+2} f^{j+2} d_{C^\bullet}^{j+1})(d_{C^\bullet}^j)$$

which is equal to

$$d_{D^\bullet}^{n+1+j} d_{D^\bullet}^{n+j} f^j + (-1)^{n+1} d_{D^\bullet}^{n+1+j} f^{j+1} d_{C^\bullet}^j + (-1)^{n+2} d_{D^\bullet}^{n+1+j} f^{j+1} d_{C^\bullet}^j + (-1)^{2n+4} f^{j+2} d_{C^\bullet}^{j+1} d_{C^\bullet}^j$$

and since the composition of differentials is zero, rearranging the signs we see the above is

$$(-1)^{n+1}(d_{D^\bullet}^{n+1+j} f^{j+1} d_{C^\bullet}^j - d_{D^\bullet}^{n+1+j} f^{j+1} d_{C^\bullet}^j) = 0.$$

6. Suppose that $f = (f^j)_{j\in\mathbb{Z}} \in Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet))$, so

$$0 = d^0(f) = (d_{D^\bullet}^j \circ f^j - f^{j+1} \circ d_{C^\bullet}^j)_{j\in\mathbb{Z}},$$

which is equivalent to saying the following diagram commutes for each j, ie. f is a cochain map.

$$\begin{array}{ccc} C^j & \xrightarrow{d^j_{C^\bullet}} & C^{j+1} \\ \Big\downarrow{\scriptstyle f^j} & & \Big\downarrow{\scriptstyle f^{j+1}} \\ D^j & \xrightarrow{d^j_{D^\bullet}} & D^{j+1} \end{array}$$

So, $Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)) \subset \mathrm{Hom}(C^\bullet, D^\bullet)$, and since all the statements above work in the converse direction we have $Z^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)) = \mathrm{Hom}(C^\bullet, D^\bullet)$.

7. Let $f = (f^j)_{j\in\mathbb{Z}} \in B^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet)$. So, f is a cochain map (problem 6) and $f = d^{-1}(g)$ for some $g \in \mathrm{Hom}^{-1}(C^\bullet, D^\bullet)$. Note the -1 here denotes the index, not some inverse. As

$$d^{-1}(g) = (d^{j-1}_{D^\bullet} \circ g^j + g^{j+1} \circ d_{C^\bullet})_{j\in\mathbb{Z}}$$

this is the same as saying as $f \sim 0$. So,

$$B^0(\mathrm{Hom}^\bullet(C^\bullet, D^\bullet) = \{f \in \mathrm{Hom}(C^\bullet, D^\bullet) \mid f \sim 0\}.$$

8. This proof is adapted from the proof of proposition A3.12 in Eisenbud's book. Let $f, g \in \mathrm{Hom}(C^\bullet, D^\bullet)$ with $f \sim g$, so $f - g \sim 0$. We first show that $H^n(f-g) = H^n(f) - H^n(g)$, so it suffices to show $H^n(f-g) = 0$ (for all n). This is immediate from the definition: let $x \in C^n$, then

$$\begin{aligned} H^n(f)(x + B^n(C^\bullet)) - H^n(g)(x + B^n) &= f(x) - g(x) + B^n(D^\bullet) \\ &= H^n(f-g)(x + B^n(C^\bullet)). \end{aligned}$$

So, as $f - g \sim 0$ there exists a $k = (k^n)_n \in \mathrm{Hom}^{-1}(C^\bullet, D^\bullet)$ such that $d^{n-1}_{D^\bullet} \circ k^n + k^{n+1} \circ d^n_{C^\bullet} = (f-g)^n$ for all n. Thus if $z \in Z^n(C^\bullet)$ we have

$$\begin{aligned} (f-g)^n(z) &= (d^{n-1}_{D^\bullet} \circ k^n)(z) + (k^{n+1} \circ d^n_{C^\bullet})(z) \\ &= (d^{n-1}_{D^\bullet} \circ k^n)(z) + k^{n+1}(0) \\ &= (d^{n-1}_{D^\bullet} \circ k^n)(z) \in B^n(D^\bullet) \end{aligned}$$

so $H^n(f-g)$ is the zero map on homology for all n.