Throughout $\psi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha} z}$ denotes the Blaschke factor $\mathbb{D} \to \mathbb{D}$ swapping α and z. Also, I do not remember if we agreed on a convention in class, but I will denote $\mathbb{N} = \{1, 2, ...\}$, so that $0 \notin \mathbb{N}$.

1. (a) $\implies$ (b): Suppose that $f_n \to f$ uniformly on compact sets, so in particular $\lim_{n\to\infty} d_k(f_n, f) = 0$ for each $k \in \mathbb{N}$. Let $d_\infty(f_n, f)$ denote $\sup_{z\in\Omega} |f_n(z) - f(z)|$, which may be infinite. We remark that, by the definition of the sequence $(K_n)_{n\in\mathbb{N}}$ we have

$$d_1(f_n, f) \leq d_2(f_n, f) \leq ... \leq d_\infty(f_n, f).$$

We write $d(f_n, f)$ as a finite sum (denoted by A_n) plus its tail (denoted by B_n), with N some natural number:

$$d(f_n, f) = \sum_{k=1}^{N-1} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} 2^{-k} + \sum_{k=N}^{\infty} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} 2^{-k} = A_n + B_n.$$

Since $\lim_{n\to\infty} d_k(f_n, f) = 0$ we have $\lim_{n\to\infty} A_n = 0$. For B_n (we have omitted absolute value signs in these calculations since all terms are nonnegative),

$$B_n \leq \frac{d_N(f_n, f)}{1 + d_\infty(f_n, f)} \sum_{k=N}^{\infty} 2^{-k} \to 0.$$

Thus $\lim_{n\to\infty} d(f_n, f) = 0$.

(b) $\implies$ (a): Suppose that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} 2^{-k} = 0.$$

As a series of nonnegative terms, we have

$$\sum_{k=1}^{\infty} \frac{d_k(f_n, f)}{1 + d_k(f_n, f)} 2^{-k} \geq \frac{d_N(f_n, f)}{1 + d_N(f_n, f)}$$

for any $N \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} \frac{d_N(f_n, f)}{1 + d_N(f_n, f)} = 0. \tag{1}$$

The function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{x}{1+x}$ is only zero at $x = 0$, and $\lim_{x\to\infty} f(x) = 1$. Therefore, (1) implies that $\lim_{n\to\infty} d_N(f_n, f) = 0$, ie. $f_n \to f$ uniformly on K_N. Since this is true for any $N \in \mathbb{N}$, and for any compact $K \subset \Omega$ we have $K_N \supset K$ for some N, we conclude that $f_n \to f$

uniformly on compact subsets of Ω.

2. Since g is nonconstant and holomorphic, z_0 is an isolated zero of g, so we can write

$$g(z) = (z - z_0)^m h(z) \tag{2}$$

for all z in an open neighbourhood U_1 of z_0, and where $h : U_1 \to \mathbb{C}$ is holomorphic and nonvanishing. Take an open $U_2 \subset U_1$ such that $z_0 \in U_2$ and U_2 is simply connected, then we can define a holomorphic function $H : U_2 \to \mathbb{C}$ such that $e^{H(z)} = h(z)$ for all $z \in U_2$. We then define $\phi : U_2 \to \mathbb{C}$ by

$$\phi(z) = (z - z_0)e^{H(z)/m}$$

so that $\phi^m(z) = g(z)$ by (2). We then have

$$\phi'(z) = (1 - (z - z_0)\frac{H'(z)}{m})e^{H(z)/m}.$$

Then $\phi'(z_0) \neq 0$, so by problem 4 of homework 4 we find an open disk $U_3 \subset U_2$ centred at z_0 such that $\phi|_{U_3}$ is injective. By the open mapping theorem $\mathrm{im}(\phi|_{U_3})$ contains a disk $D_r(z_0)$ for some $r > 0$, so taking V to be $(\phi|_{U_3})^{-1}(D_r(z_0))$ we see that $\phi : V \to D_r(z_0)$ satisfies properties (a) and (b).

3. (a) Taking the hint, define $\Psi : \mathbb{D} \to \mathbb{D}$ by $\Psi = \psi_{f(w)} \circ f \circ \psi_w^{-1}$. As a composition of holomorphic functions Ψ is holomorphic, and

$$\Psi(0) = (\psi_{f(w)} \circ f \circ \psi_w^{-1})(0) = (\psi_{f(w)} \circ f)(w) = \psi_{f(w)}(f(w)) = 0.$$

By the Schwarz lemma, $|\Psi(z)| \leq |z|$ for all $z \in D$, ie.

$$\Big|\frac{f(\psi_w^{-1}(z)) - f(w)}{1 - \overline{f(w)}f(\psi_w^{-1}(z))}\Big| \leq |z|. \tag{3}$$

By substituting z with $\psi_w(z)$ in (3) we obtain $\rho(f(z), f(w)) \leq \rho(z, w)$. If $\varphi : \mathbb{D} \to \mathbb{D}$ is an automorphism, we apply the inequality twice:

$$\rho(z_1, z_2) = \rho((\varphi^{-1} \circ \varphi)(z_1), (\varphi^{-1} \circ \varphi)(z_2)) \leq \rho(\varphi(z_1), \varphi(z_2)) \leq \rho(z_1, z_2),$$

so $\rho(\varphi(z_1), \varphi(z_2)) = \rho(z_1, z_2)$.

(b) By (a), for any $w, z \in \mathbb{D}$ we have

$$\Big|\frac{f(z) - f(w)}{1 - \overline{f(z)}f(w)}\Big| \leq \Big|\frac{z - w}{1 - \overline{z}w}\Big|.$$

Rearrange to get

$$\Big|\frac{f(z)-f(w)}{z-w}\frac{1}{1-\overline{f(z)}f(w)}\Big| \leq \Big|\frac{1}{1-\overline{z}w}\Big|, \tag{4}$$

so letting $w \to z$ in (4) we see that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.$$

4. (a) Let γ be a curve from z_1 to z_2. By the chain rule, we have

$$\int_0^1 \|(f\circ\gamma)'(t)\|_{(f\circ\gamma)(t)}dt \leq \int_0^1 \|\gamma'(t)\|_{\gamma(t)}dt.$$

Taking the infimum of both sides over the set of curves γ, and then noting that the infimum of the integral on the LHS over all curves from z_1 to z_2 can only decrease, as there may be some not of the form $f\circ\gamma$, we see that $d(f(z_1), f(z_2) \leq d(z_1, z_2)$.

(b) Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an automorphism. Using part (a),

$$d(z_1,z_2) = d((\varphi^{-1}\circ\varphi)(z_1), (\varphi^{-1}\circ\varphi)(z_2)) \leq d(\varphi(z_1), \varphi(z_2)) \leq d(z_1, z_2),$$

so $d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2)$.

Conversely, suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is a function such that $d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2)$.

(c) We construct a family of automorphisms $\mathbb{D} \to \mathbb{D}$ of the form

$$\varphi_{z_1,z_2} = e^{-i\theta}\psi_{z_1},$$

where $\theta = \arg(\psi_{z_1}(z_2)$. We see that $\varphi_{z_1,z_2}(z_1) = 0$ and $\varphi_{z_1,z_2}(z_2) = |\psi_{z_1}(z_2)|$. Since $|\psi_\alpha|$ is continuous in α (homework 1, composed with $z \mapsto |z|$), by the intermediate value theorem each $s \in [0, 1)$ is of the form $|\psi_{z_1}(z_2)|$ for some $z_1, z_2 \in \mathbb{D}$.

(d) We first show that $d(0, s) \leq \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$ by calculating the integral for

$\gamma(t) = st$, the straight line from 0 to s.

$$\begin{aligned}\int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt &= \int_0^1 \frac{s}{1-(st)^2} dt \\ &= \frac{s}{2}\int_0^1 \Big(\frac{1}{1+st} + \frac{1}{1-st}\Big) dt \\ &= \frac{1}{2}\log\Big(\frac{1+s}{1-s}\Big).\end{aligned}$$

Next we show $d(0,s) \geq \frac{1}{2}\log\big(\frac{1+s}{1-s}\big)$. We write $\gamma(t) = x(t) + iy(t)$ and note that the integral is minimized if $x'(t)$ is monotone and does not change sign. Thus

$$\int_0^1 \frac{|\gamma'(t)|}{1+|\gamma(t)|^2} dt = \int_0^1 \frac{x'(t)+iy'(t)}{1+x(t)^2+y(t)^2} dt \leq \int_0^1 \frac{|x'(t)|}{1+x(t)^2} dt. \tag{5}$$

We substitute $\eta = x(t)$ so that (5) equals

$$\int_0^s \frac{d\eta}{1-\eta^2} dt = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$$

by the same calculation as above. Thus $d(0,s) = \frac{1}{2}\log\big(\frac{1+s}{1-s}\big)$.

(e) By (b), d is preserved by automorphisms, so

$$d(z_1, z_2) = d(\varphi_{z_1,z_2}(z_1), \varphi_{z_1,z_2}(z_2)) = d(0, |\psi_{z_1}(z_2)|) = \frac{1}{2}\log\left(\frac{1+|\psi_{z_1}(z_2)|}{1-|\psi_{z_1}(z_2)|}\right).$$

5. By proposition 1.1 in chapter 8 of the textbook, $f'(z) \neq 0$ for all $z \in \mathbb{C}$. If f is a polynomial, then f must have degree 1, otherwise f' has a zero by the fundamental theorem of algebra. Therefore it suffices to show f is a polynomial of any degree.

Denote the image of f by U. Since f is holomorphic and injective, U is simply connected, and by the open mapping theorem U is open, so if U is a proper subset of $\mathbb{C}$ then by the Riemann mapping theorem there exists a conformal map $\phi : U \to \mathbb{D}$. Then we obtain a holomorphic injective map $\phi \circ f : \mathbb{C} \to \mathbb{D}$, a contradiction to Liouville's theorem. Thus $U = \mathbb{C}$. In particular, there exists a unique zero of f, and so we assume without loss of generality that $f(0) = 0$, denote the multiplicity of this zero by $n \in \mathbb{N}$. Denote the positively oriented unit circle by C and the negatively oriented unit circle by $C^{\leftarrow}$. By the argument principle we have

$$n = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz, \tag{6}$$

and so a short calculation shows, where $g = f(\frac{1}{z})$, that

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}dz &= -\frac{1}{2\pi i}\int_C \frac{1}{z^2}\frac{f'(\frac{1}{z})}{f(\frac{1}{z})}dz \\
&= \frac{1}{2\pi i}\int_{C^{\leftarrow}} \frac{f'(\zeta)}{f(\zeta)}d\zeta && \text{where } \zeta = \frac{1}{z} \\
&= -n && \text{by (2).}
\end{aligned}$$

Since g has no zeros inside or on C, by the argument principle g has a pole of order n at 0, so by problem 5 (b) of homework 3, f is a polynomial of degree n, completing the proof.

6. If $\Omega = \mathbb{C}$ then by problem 5 we have $f_1(z) = \alpha_1 z + \beta_1$ and $f_2 = \alpha_2 z + \beta_2$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}\backslash\{0\}$. Then $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$ imply $a = \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} = b$, a contradiction to a and b being distinct. Thus Ω is a proper subset of $\mathbb{C}$.

As Ω is proper and simply connected, by the Riemann mapping theorem there exists a conformal map $F : \Omega \to \mathbb{D}$ such that $F(f_1(a)) = 0$. We define two maps $\varphi_1, \varphi_2 : \mathbb{D} \to \mathbb{D}$ by

$$\varphi_j = F \circ f_j \circ F^{-1} \circ \psi_{F(a)},$$

where $j = 1, 2$. As a composition of conformal maps, φ_j is an automorphism, so

$$\varphi_j = e^{i\theta_j}\psi_{\zeta_j}$$

for some $\theta_j \in [0, 2\pi)$ and $\zeta_j \in \mathbb{D}$. However, we calculate $\varphi_j(0) = 0$, so φ_j is a rotation. Then, because φ_1 and φ_2 agree at $(\psi_{F(a)} \circ F)(b)$, and are nonzero at that point, we must have $\varphi_1 = \varphi_2$. Composing both sides of this inequality with $\psi_{F(a)} \circ F$ on the right and F^{-1} on the left, we obtain $f_1 = f_2$.

7. By Montel's theorem it suffices to show $\mathcal{G}$ is uniformly bounded on compact subsets of U. Let $K \subset U$ be compact, and let γ be a rectifiable loop in $U\backslash K$ such that any homotopy from γ to a point contains all of K in its image (ie. K is in the region bounded by γ), and such that there exists an $a > 0$ such that $\text{dist}(z, K) \geq a$ for all $z \in \gamma$. Denote the length of γ by L. Since $f \in \mathcal{F}$ and $\mathcal{F}$ is a normal family, by Montel's theorem there exists an $M > 0$, independent of f, such that $\sup_{z\in U}|f(z)| \leq M$. Let $z_0 \in K$, then by the Cauchy integral

formula and ML-estimate we have

$$\begin{aligned}
|f'(z_0)| &= \Big|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}dz\Big| \\
&\le \frac{L}{2\pi}\max_{z\in\gamma}\Big|\frac{f(z)}{(z-z_0)^2}\Big| \\
&\le \frac{L}{2\pi}\frac{M}{a^2}.
\end{aligned}$$

Since this bound is independent of f or z_0, f' is uniformly bounded on K, completing the proof.

8. Let $f_n : \mathbb{D} \to \mathbb{C}$ be defined by $f_n = z + nz^2$ for each $n \in \mathbb{N}$. Evidently we have f_n holomorphic, $f_n(0) = 0$, and $f'(0) = 1$ for each $n \in \mathbb{N}$, so $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$. Applying the results of problem 7 twice, we see that if $\mathcal{H} = \{f'' \mid f \in \mathcal{F}\}$ is not a normal family then neither is $\mathcal{F}$. Indeed, the sequence $\{f_n'' = 2n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ has no convergent subsequence, since every subsequence is unbounded. Thus $\mathcal{F}$ is not a normal family.

9. We imitate the proof in the textbook of the Hurwitz theorem. Suppose that there is a $z \in \Omega$ such that $f(z) = 0$. If f is not identically zero then z is an isolated zero, so choose a circle γ (positively oriented) with z in its interior, small enough so that $f(\zeta) \neq 0$ for all $\zeta \neq z$ in an open set containing γ and its interior. Then $\frac{1}{f_n} \to \frac{1}{f}$ and $f_n' \to f'$ uniformly on γ so

$$\int_\gamma \frac{f_n'(\zeta)}{f_n(\zeta)}d\zeta \to \int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta.$$

However, this is a contradiction since the argument principle implies

$$\frac{1}{2\pi i}\int_\gamma \frac{f_n'(\zeta)}{f_n(\zeta)} = 0 \quad \text{for all } n \in \mathbb{N}, \text{ and} \qquad \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta \ge 1.$$

Therefore f is either nonvanishing or identically zero on Ω.