integrals with powers of sec and tan:

I: Integrals like

$$\int \sec^{11} x \tan^5 x \, dx$$

(*odd* powers of sec and tan) can be solved the following way:

$$\begin{aligned}
\int \sec^{11} x \tan^5 x \, dx &= \int \sec x \tan x \sec^{10} x \tan^4 x dx && \text{(break off } \sec x \tan x\text{)} \\
&= \int \sec x \tan x \sec^{10} x (\tan^2 x)^2 dx \\
&= \int \sec x \tan x \sec^{10} x (\sec^2 x - 1)^2 dx && \text{(use identity } \tan^2 x + 1 = \sec^2 x\text{)} \\
&= \int u^{10}(u^2 - 1)^2 du && \text{(substitute } u = \sec x\text{)}
\end{aligned}$$

From here we can expand out and then break up the integral to finish solving it.

II: Integrals like

$$\int \sec^6 x \tan^{12} x \, dx$$

(*even* powers of sec and tan) can be solved the following way:

$$\begin{aligned}
\int \sec^6 x \tan^{12} x \, dx &= \int \sec^2 x \sec^4 x \tan^{12} x \, dx && \text{(break off} \sec^2 x\text{)} \\
&= \int \sec^2 x (\sec^2 x)^2 x \tan^{12} x \, dx \\
&= \int \sec^2 x (\tan^2 x + 1)^2 x \tan^{12} x \, dx && \text{(use identity } \tan^2 x + 1 = \sec^2 x\text{)} \\
&= \int (u^2 + 1)^2 u^{12} \, du && \text{(substitute } u = \tan x\text{)}
\end{aligned}$$

From here we can expand out and then break up the integral.

trig substitution: uses the identities

$$\cos^2 \theta + \sin^2 \theta = 1 \quad \text{and} \quad \tan^2 \theta + 1 = \sec^2 \theta$$

for integrals like these (where n is usually 1,2 or 3)

$$(1) \int \frac{dx}{(x^2 - a^2)^{n/2}} \quad \text{or} \quad (2) \int \frac{dx}{(a^2 - x^2)^{n/2}} \quad \text{or} \quad (3) \int \frac{dx}{(x^2 + a^2)^{n/2}}.$$

For (1) substitute $x = a \sec \theta$ so we can use the identity $\sec^2 \theta - 1 = \tan^2 \theta$.

For (2) substitute $x = a \sin \theta$ so we can use the identity $1 - \sin^2 \theta = \cos^2 \theta$.

For (3) substitute $x = a \tan \theta$ so we can use the identity $\tan^2 \theta + 1 = \sec^2 \theta$.

If there's an x term, like in

$$\int \frac{dx}{x^2 + x + \frac{17}{4}}$$

we'll have to complete the square:

$$x^2 + x + \frac{17}{4} = x^2 + x + (\frac{1}{2})^2 - (\frac{1}{2})^2 + \frac{17}{4} = (x + \frac{1}{2})^2 + 4.$$

Then for this one we can substitute $x + \frac{1}{2} = 2\tan\theta$.

integrals with powers of sin or cos:

I: even powers of cos: write the integral as

$$\int \cos^{2n} x dx = \int (\cos^2 x)^n dx$$

and use the formula

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

and then expand and it will break up into other integrals of powers of cos.

II: even powers of sin (really similar to even powers of cos): write the integral as

$$\int \sin^{2n} x dx = \int (\sin^2 x)^n dx$$

and use the formula

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

and then expand and it will break up into other integrals of powers of cos.

III: odd powers of sin or cos (these ones are usually less annoying than the even powers): for odd powers of sin write like this: (usually n will be like 1,2 or 3)

$$\int \sin^{2n+1} x \, dx = \int \sin^{2n} x \, \sin x \, dx = \int (\sin^2 x)^n \sin x \, dx$$

and plug the pythagorean identity $\sin^2 x = 1 - \cos^2 x$ to get

$$\int (1 - \cos^2 x)^n \sin x \, dx$$

and then substitute $u = \cos x$ to finish solving. Integrals with odd powers of cos are solved the same way, just use $\cos^2 x = 1 - \sin^2 x$ instead (and substitute $u = \sin x$ instead).