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Task 1

We need to solve the following equation:

$$11x^2 + 11y^2 + 2001xyy' = 0 \tag{1}$$

Before solving, we analyze the equation above: it has only first order derivative, thus, it is the differential equation with the first order. All coefficients have the same degree, so, the equation is homogeneous. Such equation can not be written in the following form: $a(x)y' + b(x)y = f(x)$*, and it means that the equation is non-linear. Finally, we have first-order non-linear homogeneous equation.*

Step 1. We assume that $x \neq 0$ and do the following substitution $u = \frac{y}{x}$, so that $y = xu$. Take the derivative of y, so $y' = u + xu'$, and substitute it into (1), then we get:

$$11x^2 + 11u^2x^2 + 2001x^2u(u + xu') = 0 \tag{2}$$

Since we assumed that $x \neq 0$ divide both parts of the equation by $x^2 \neq 0$:

$$11 + 11u^2 + 2001u^2 + 2001xuu' = 0 \tag{3}$$

Then, after simplification we have:

$$2001xuu' = -11 - 2012u^2 \tag{4}$$

Now divide both parts of the equation by $\frac{1}{(11+2012u^2)x} \neq 0$, because $x \neq 0$(we assumed it before) and $2012u^2 - 11 \neq 0$ $(2012u^2 \neq -11)$. We do it to transorm the equation to separable form:

$$(\frac{2001u}{11 + 2012u^2})du = (\frac{-1}{x})dx \tag{5}$$

Then integrate both parts to solve the separable equation:

$$\int (\frac{2001u}{11 + 2012u^2})du = \int (\frac{-1}{x})dx \tag{6}$$

$$\frac{2001}{4024} \int (\frac{1}{11 + 2012u^2})d(11 + 2012u^2) = \int (\frac{-1}{x})dx \tag{7}$$

Thus, we get:

$$\frac{2001}{4024} \ln|11 + 2012u^2| = -\ln|x| + c, \textit{where } c \in R \tag{8}$$

After simplifying we obatin:

$$\ln|11 + 2012u^2|^{\frac{2001}{4024}} = \ln|\frac{1}{x}| + \ln|c_1|, \textit{where } c = \ln|c_1|,\ c_1 \in R \setminus \{0\} \textit{ by def. of logarithm} \tag{9}$$

After exponantiating ln we get:

$$(11 + 2012u^2)^{\frac{2001}{4024}} = \frac{c_1}{x}, \textit{where } c_1 \in R\backslash\{0\} \tag{10}$$

To get the most general solution we need to express y:

$$11 + 2012u^2 = \frac{c_2}{x^{\frac{4024}{2001}}}, \textit{where } c_2 = c_1^{\frac{4024}{2001}},\ c_2 \in R\backslash\{0\} \tag{11}$$

Move 11 to the right part and lead it to the common denominator, then divide both parts of the equation by 2012. The result is the following:

$$u^2 = \frac{c_2 - 11x^{\frac{4024}{2001}}}{2012x^{\frac{4024}{2001}}}, \textit{where } c_2 \in R\backslash\{0\} \tag{12}$$

Remember that $u = \frac{y}{x}$, so substitute:

$$(\frac{y}{x})^2 = \frac{c_2 - 11x^{\frac{4024}{2001}}}{2012x^{\frac{4024}{2001}}}, \textit{where } c_2 \in R\backslash\{0\} \tag{13}$$

Multiply both parts of the equation by x^2:

$$y^2 = \frac{c_2 - 11x^{\frac{4024}{2001}}}{2012x^{\frac{22}{2001}}}, \textit{where } c_2 \in R\backslash\{0\} \tag{14}$$

So, we have:

$$y = \pm\sqrt[2]{\frac{c_2 - 11x^{\frac{4024}{2001}}}{2012x^{\frac{22}{2001}}}}, \textit{where } c_2 \in R\backslash\{0\} \textit{ and } c_2 \geq 11x^{\frac{4024}{2001}} \textit{ (by def. of sqrt)} \tag{15}$$

Step 2. Initially we assumed that $x \neq 0$, thus, we must consider a case when $x = 0$ to check a possible trivial solution. Check it substituting $x = 0$ to (1):

$$11 * 0 + 11y^2 + 2001 * 0 * y' = 0 \tag{16}$$

$$11y^2 = 0 \tag{17}$$

So (17) holds only if $y = 0$, thus, $x = 0$ is not a solution.

Answer: the most general solution of the original equation on R $\backslash\{0\}$ is $y = \pm\sqrt[2]{\frac{c_2 - 11x^{\frac{4024}{2001}}}{2012x^{\frac{22}{2001}}}}$.

Task 2

We need to solve the following equation:

$$dy - y(11 + xy^{11})dx = 0 \tag{18}$$

Rewrite (18) in the form of Bernoulli's equation $y' + g(x)y = f(x)y^n$:

$$y' - 11y = xy^{12} \tag{19}$$

Before solving, we analyze the equation above: it has only first order derivative, thus, it is the differential equation with the first order. All coefficients do not have the same degree, so, the equation is non-homogeneous. Such equation can not be written in the following form: $a(x)y' + b(x)y = f(x)$, and it means that the equation is non-linear. Finally, we have first-order non-linear non-homogeneous equation.

Step 1. To solve Bernoulli's equation we need to find a non-trivial solution y_c of (19), let us do it by solving the complementary equation of (19):

$$y_c{}' - 11y_c = 0 \tag{20}$$

Please, remark: we get the separable equation (20).
We assume that $y_c \neq 0$ and divide both parts of the equation by y_c to solve the separable equation:

$$\frac{dy_c}{y_c} = 11dx \tag{21}$$

Then we integrate both parts of the equation to solve the separable equation:

$$\int \frac{dy_c}{y_c} = \int 11dx \tag{22}$$

After the integration we get:

$$\ln|y_c| = 11x + c, \textit{ where } c \in R \tag{23}$$

After exponantiating ln we obtain:

$$y_c = e^{11x}c_1, \textit{where } c_1 = e^c \textit{ and } c_1 \in R\backslash\{0\} \textit{ (since } e^c \neq 0 \textit{ for a[illegible])} \tag{24}$$

Then according to the tecnique of solving Bernoulli's equation do the following substitution $y = uy_c$ where $u = u(x)$. So, $y = ue^{11x}$ and find its derivative $y' = e^{11x}u' + 11e^{11x}u$. Substitute it to (19):

$$e^{11x}u' + 11e^{11x}u - 11e^{11x}u = xe^{132x}u^{12} \tag{25}$$

Please, remark: we get the separable equation (25).
Assuming that $u \neq 0$ we divide both parts of the equat[illegible]by $e^{11x}u^{12}$ (also, we know that $e^{11x} \neq 0$) to solve the separable equation:

$$(\frac{1}{u^{12}})du = xe^{121x}dx \tag{26}$$

To solve the separable equation we integrate both parts:

$$\int (\frac{1}{u^{12}})du = \int (xe^{121x})dx \tag{27}$$

Please, remark: to solve the 2 integral we integrate by parts
Thus, we get:

$$\frac{-1}{11u^{11}} = \frac{xe^{121x}}{121} - \frac{e^{121x}}{121^2} + c_*, \textit{where } c_* \in R \tag{28}$$

Now we need to express u, so initially we multiply both parts of the expression by (-11) and lead to a common denominator the second part of the expression:

$$\frac{1}{u^{11}} = \frac{-121xe^{121x} + e^{121x} - 1331c_*}{1331}, \textit{where } c_* \in R \tag{29}$$

Knowing that $-121xe^{121x} + e^{121x} - 1331c_* \neq 0$ (since $u \neq 0$) we get:

$$u^{11} = \frac{1331}{-121xe^{121x} + e^{121x} - 1331c_*}, \textit{where } c_* \in R \tag{30}$$

Then we obtain:

$$u = \sqrt[11]{\frac{1331}{-121xe^{121x} + e^{121x} - 1331c_*}}, \textit{where } -121xe^{121x} + e^{121x} - 1331c_* \neq 0 \tag{31}$$

We remember that $y = uy_c$, then we substitute u back:

$$y = \sqrt[11]{\frac{1331}{-121xe^{121x} + e^{121x} - 1331c_*}} * e^{11x}, \textit{where } -121xe^{121x} + e^{121x} - 1331c_* \neq 0 \tag{32}$$

Step 2. Initially we assumed that $u \neq 0$, thus, we must consider a case when $u = 0$ to check a possible trivial solution. Check it substituting $x = 0$ to $y = uy_c$:

$$y = 0 * y_c = 0 \tag{33}$$

$$y' = 0 \tag{34}$$

Substitute $y = 0$ to (19):

$$0 - 11 * 0 = x * 0 \tag{35}$$

$$0 = 0 \tag{36}$$

So, $y = 0$ is a trivial solution.

Answer: trivial solution is $y = 0$ and the most general solution of the original equation on R is $y = (\sqrt[11]{\frac{1331}{-121xe^{121x}+e^{121x}-1331c_*}} * e^{11x})$, *where* $-121xe^{121x} + e^{121x} - 1331c_* \neq 0$