Vladislav Lamzenkov, 11.11.2001

Task 1

We need to solve the following initival value problem:

$$y''y' = \frac{1}{2} \, with \, y(1) = 11, y'(1) = 11 \tag{1}$$

Before solving, we analyze the equation above: second-order nonlinear ordinary differential equation. So, to solve it we can use the method for order reduction(Please, refer to lecture 8&9, slide 3).

Step 1. Since our equation doesn't depend y variable, do the following substitution $p = y'$ where $p = p(x)$, $p' = y''$(*Assuming that chain rule can be applied:* $\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx}$):

$$pp' = \frac{1}{2} \tag{2}$$

Now we have a separable equation, transform the equation and multiply both parts of the equation by 2:

$$(2p)dp = 1(dx) \tag{3}$$

Integrate both parts of the equation:

$$\int (2p)dp = \int dx \tag{4}$$

$$p^2 = x + c, \, where \, c \in R \tag{5}$$

$$p = \pm\sqrt{x + c}, \, where \, c \in R \, and \, x > -c \tag{6}$$

Please, note, $x + c \geq 0$ *by def. of square root, however, if* $x = c$*, then* $p = 0$ *which implies* $y' = 0$*, but* $y''y' = \frac{1}{2}$

Remember that $p = y'$, so substitute it:

$$y' = \pm\sqrt{x + c} \tag{7}$$

Step 2. Now we need to find y, transform the equation above:

$$dy = \pm(\sqrt{x + c})dx \tag{8}$$

Integrate both sides of the expression:

$$\int dy = \pm \int (\sqrt{x + c})dx \tag{9}$$

As a result, we obtain:

$$y = \pm\frac{2}{3}(x + c)^{\frac{3}{2}} + C_1, where \, C_1 \in R, \, c \in R \, and \, x > -c \tag{10}$$

Please, note, we didn't do any assumptions and didn't lose any solution, thus, $y = \pm\frac{2}{3}(x+c)^{\frac{3}{2}} + C_1, \textit{where } C_1 \in R,\ c \in R \textit{ and } x > -c$ *is the most general solution.*

Step 3. Find c and C_1.
Firslty, find c substituting $y'(1) = 11$ into (7):
$y' = \sqrt{x+c}$, (*As* $y' > 0$ *(since* $y'(1) = 11$*), so there is no meaning to consder* $y' = -\sqrt{x+c}$)

$11 = \sqrt{1+c}$
$c = 120$
So, we obtained the following equation:

$$y = \pm\frac{2}{3}(x+120)^{\frac{3}{2}} + C_1, \textit{where } C_1 \in R \textit{ and } x > -120 \tag{11}$$

Finally, find C_1 substituting $y(1) = 11$ into (11):
$y = \frac{2}{3}(x+120)^{\frac{3}{2}} + C_1$, (*As* $y > 0$ *(since* $y(1) = 11$*), so there is no meaning to consder* $y = -\frac{2}{3}(x+120)^{\frac{3}{2}} + C_1$)

$11 = \frac{2}{3} *$ [illegible] $* 11 + C_1$
$C_1 = -\frac{2629}{3}$

After all, we get the following unique solution(*Since we get the most general solution before(10), thus, our initial value problem has a unique solution*) for the given initial value problem.

$$y = \frac{2}{3}(x+120)^{\frac{3}{2}} - \frac{2629}{3}, \textit{ where } x > -120 \tag{12}$$

Answer: $y = \frac{2}{3}(x+120)^{\frac{3}{2}} - \frac{2629}{3}$, *where* $x > -120$ is a unique solution for the given initial value problem.

Task 2

We need to solve the following problem:

$$y'' + 11y' - 11y = 0 \tag{13}$$

Before solving, we analyze the equation above: homogeneous linear second-order ordinary differential equation with constant coefficients.

Step 1. We can try the following substitution $y = e^{ax}$ as a solution of the given equation(*Please, refer to lecture 8&9, slide 26*) and $y' = ae^{ax}$, $y'' = a^2e^{ax}$. So, we get:

$$a^2e^{ax} + 11ae^{ax} - 11e^{ax} = 0 \tag{14}$$

$$e^{ax}(a^2 + 11a - 11) = 0 \tag{15}$$

Since $e^{ax} \neq 0$ for any x and a, we can divide both parts of the equation by e^{ax} and get the characteristic equation:

$$a^2 + 11a - 11 = 0 \tag{16}$$

Step 2. Find roots of the characteristic(quadratic) equation:
$a_1 = \frac{-11+\sqrt{165}}{2} = -5.5 + \sqrt{165}$
$a_2 = \frac{-11-\sqrt{165}}{2} = -5.5 - \sqrt{165}$
So, we get 2 real roots, and $y = c_1e^{(-5.5+\sqrt{165})x} + c_2e^{(-5.5-\sqrt{165})x}$ where $c_1 \in R$, $c_2 \in R$, $x \in R$ is the most general solution of the equation(*Please, refer to lecture 8&9, slide 27*).

Answer: $y = c_1e^{(-5.5+\sqrt{165})x} + c_2e^{(-5.5-\sqrt{165})x}$ where $c_1 \in R$, $c_2 \in R$, $x \in R$ is the most general solution.