The final examination on Ordinary Differential Equations

Task 1

Subproblem 1: Is function $f(t) = e^{t^{(12-12)}}$ on $[0, +\infty)$ a piece-with (probably should be piecewise) continuous?

Solution for subproblem 1:
Definition of piecewise continuity: a function is called **piecewise continuous** on an interval $J \subset \mathbb{R}$ if it is pointwise continuous apart from a set $M \subset J$ of isolated points $\xi \in M$ where only the one-sided limits $\lim_{x \to \xi^-} f(x)$ and $\lim_{x \to \xi^+} f(x)$ exist.

Let's denote set of points which we will ignore: $M = \{0\}$
Now we need to check if the function $f(t)$ is pointwise continuous on interval $[0; +\infty) \backslash M$.

$$f(t) = e^{t^{(12-12)}} = e,\ t \neq 0 \quad \Rightarrow \quad f(t) = e, t \in (0; +\infty) \quad \Rightarrow \quad f(t) \text{ - pointwise continuous on } (0; +\infty)$$

Check one-side limits (need to check only one limit):

$$\lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} e^{t^0} = e \text{ - limit exist}$$

Answer for subproblem 1: Function is piecewise continuous on $[0, +\infty)$

Subproblem 2: Does function $f(t) = e^{t^{(12-12)}}$ have exponential order?

Solution for subproblem 2:
Definition of exponential order from presentations: a function $f : [0; +\infty)$ have exponential order $\alpha \in \mathbb{R}$ if

$$\exists C \in \mathbb{R} : \ \forall t \in [0; +\infty) \ |f(t)| < C \cdot e^{\alpha t}$$

Given function is not defined at point $t = 0$. It meant that this definition is not applicable to given function. Let's use Google to find a more precise definition: A function $f(t)$ is said to be of exponential order α if

$$\exists \alpha \ \exists M > 0 \ \exists T > 0 : \ \forall t > T \ \ |f(t)| < Me^{\alpha t}$$

Let $\alpha = 0$, $M = 4$, $T = 1$: $\forall t > 1 \ \ |e^1| < 4$ - correct
Answer for subproblem 2: function $f(t)$ has exponential order 0

Subproblem 3: Determine range for s where Laplace transform $L[e^{t^{12-12}}](s)$ is defined.

Solution for subproblem 3:
By definition, Laplace transform for function $f(x)$ is define if and only if $\int_0^\infty e^{-st}f(t)dt$ converges. So, we need to find all values of s for which improper integral converges.
If $s = 0$:

$$\int_0^\infty e^{-st}f(t)dt = \int_0^\infty edt \text{ - diverges} \quad (1)$$

If $s \neq 0$ (2):

$$\int_0^\infty e^{-st}f(t)dt = \lim_{a\to 0^+}\lim_{b\to +\infty} -\frac{e}{s}\int_a^b d(e^{-st}) = \lim_{a\to 0^+}\lim_{b\to +\infty} \frac{e}{s}(e^{-sa} - e^{-sb}) = \frac{e}{s}(1 - \lim_{b\to +\infty} e^{-sb})$$

$\lim_{b\to +\infty} e^{-sb}$ has finite value only for $s >= 0$ (3)
(1), (2), (3) $\Rightarrow$ improper integral converges only for $s > 0$ and diverges overwise.
Answer for subproblem 3: Laplace transform defined only for $s > 0$

Task 2

Problem: Find the most general solution of $y_1' = y_1 - y_2$ and $y_2' = 12y_1 + 12y_2$ and prove that it is indeed the most general one. Use Elimination method

Solution starts here: We need to use Elimination method, but before using this method we have to prove that this method will give us **the most general** solution. The only problem this method has is that by now we need to assume that y_1 can be differentiated at least two times, which can lead to loss of some solutions. But the point is that we do not need to assume it.

Let's prove by induction that if y_1 and y_2 is a solution of the initial system then $y_1^{(n)}$ and $y_2^{(n)}$ exist for any $n \in \mathbb{N}$:

1. Base of induction: y_1 and y_2 exist. It follows from the fact that y_1 and y_2 is a solution.
2. Step of induction: Assume that $y_1^{(k)}$ and $y_2^{(k)}$ exist (1). Now let's use initial equation and differentiate k-times both equations:

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = 12y_1 + 12y_2 \end{cases} \Rightarrow \begin{cases} a = y_1^{(k+1)} & (2) \\ a = y_1^{(k)} - y_2^{(k)} & (3) \\ b = y_2^{(k+1)} & (4) \\ b = 12y_1^{(k)} + 12y_2^{(k)} & (5) \end{cases}$$

   (1) $\Rightarrow$ right part of (3), (5) exists $\Rightarrow$ left part of (3), (5) exists $\Rightarrow$ left part of (2), (4) exists $\Rightarrow$ right part of (2), (4) exists $\Rightarrow$ $y_1^{(k+1)}$ and $y_2^{(k+1)}$ exists
3. According to induction principal, $y_1^{(n)}$ and $y_2^{(n)}$ exist for any $n \in \mathbb{N}$

Let's use Elimination method to obtain the solution:

1. Express y_2 from the first equation. Differentiate the first equation and substitute y_2' and y_2 into it:

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = 12y_1 + 12y_2 \end{cases} \Leftrightarrow \begin{cases} y_1'' = y_1' - y_2' \\ y_2' = 12y_1 + 12y_2 \\ y_2 = y_1 - y_1' \end{cases} \Leftrightarrow \quad y_1'' - 13y_1' + 24y_1 = 0$$

2. Solve linear second-order homogeneous equation using method of characteristic equation (we know that this method give us the most general solution):
   (a) Calculate roots of characteristic equation $\lambda^2 - 13\lambda + 24 = 0$: $\lambda_{1,2} = \frac{13 \pm \sqrt{73}}{2}$
   (b) The most general solution of a second-order linear homogeneous equation with constant coefficients is $C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$, $C_1, C_2 \in \mathbb{R}$:

$$y_1 = C_1 e^{\frac{13+\sqrt{73}}{2}x} + C_2 e^{\frac{13-\sqrt{73}}{2}x},\ C_1, C_2 \in \mathbb{R}$$

   (c) I hope I do not need to prove once again that it is the most general solution of a second-order linear homogeneous equation with constant coefficients. If it is realy necessary, I am ready to prove it with using the same method I used in midterm 2 (second task) or in midterm 1 (bonus task) or even method presented in lecture (week 14, slide 10).

3. Use equation $y_2 = y_1 - y_1'$ (from initial system) to obtain the solution for y_2:

$$y_2 = C_1(1 - \frac{13+\sqrt{73}}{2})e^{\frac{13+\sqrt{73}}{2}x} + C_2(1 - \frac{13-\sqrt{73}}{2})e^{\frac{13-\sqrt{73}}{2}x},\ C_1, C_2 \in \mathbb{R}$$

4. **Answer**:

$$\begin{cases} y_1 = 2C_1e^{\frac{13+\sqrt{73}}{2}x} + 2C_2e^{\frac{13-\sqrt{73}}{2}x} \\ y_2 = C_1(-11-\sqrt{73})e^{\frac{13+\sqrt{73}}{2}x} + C_2(-11+\sqrt{73})e^{\frac{13-\sqrt{73}}{2}x} \end{cases},\ C_1, C_2 \in \mathbb{R}$$

is **the most general** solution of given system of equations on $x \in \mathbb{R}$

Remark: the solution is the most general because all transitions I make are equivalent and I do not introduce any assumption. So, there are no reasons to lose some solutions.