10.1 $f(t) = (1+t)^2$

$$L(f)(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} (1+t)^2 e^{-st}\, dt$$

$$\int (1+t)^2 e^{-st} dt = \left\{ \begin{matrix} u = (1+t)^2 & du = 2(1+t)dt \\ v = -\frac{1}{s}e^{-st} & dv = e^{-st}dt \end{matrix} \right\} =$$

$$= -\frac{1}{s} e^{-st} (1+t)^2 + 2\frac{1}{s} \int (1+t) e^{-st} dt$$

$$\int (1+t) e^{-st} dt = \left\{ \begin{matrix} u = 1+t & du = dt \\ v = -\frac{1}{s}e^{-st} & dv = e^{-st}dt \end{matrix} \right\} =$$

$$= -\frac{1}{s} e^{-st} (1+t) + \frac{1}{s} \int e^{-st} dt = -\frac{t+1}{s} e^{-st} - \frac{e^{-st}}{s^2} + C$$

$$\int (1+t)^2 e^{-st} dt = -\frac{(t+1)^2}{s} e^{-st} - \frac{2(t+1)}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} + C$$

$$L(f)(s) = \left(-\frac{(t+1)^2}{s} - \frac{2(t+1)}{s^2} - \frac{2}{s^3} \right) e^{-st} \Big|_{t=0}^{+\infty} =$$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}$$

Answer: $L(f)(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}, s > 0$

10.2 $f(t) = \sin^2 t$

$$L(f)(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt = \int_0^{+\infty} \sin^2 t\, e^{-st}\, dt$$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$\int \sin^2 t\, e^{-st}\, dt = \frac{1}{2}\int e^{-st}\, dt - \frac{1}{2}\int \cos 2t\, e^{-st}\, dt$$

$$\underbrace{\int \cos 2t\, e^{-st}\, dt}_{T} = \left\{\begin{matrix} u = \cos 2t & du = -2\sin 2t\, dt \\ v = -\frac{1}{s} e^{-st} & dv = e^{-st}\, dt \end{matrix}\right\} =$$

$$= -\frac{1}{s}\cos 2t\, e^{-st} - 2\cdot\frac{1}{s}\int \sin 2t\, e^{-st}\, dt$$

$$\int \sin 2t\, e^{-st}\, dt = \left\{\begin{matrix} u = \sin 2t & du = 2\cos 2t\, dt \\ v = -\frac{1}{s} e^{-st} & dv = e^{-st}\, dt \end{matrix}\right\} =$$

$$= -\frac{1}{s}\sin 2t\, e^{-st} + \frac{2}{s}\underbrace{\int \cos 2t\, e^{-st}\, dt}_{T}$$

$$T = -\frac{1}{s}\cos 2t\, e^{-st} - \frac{2}{s}\left(-\frac{1}{s}\sin 2t\, e^{-st} + \frac{2}{s}T\right)$$

$$T = -\frac{1}{s}\cos 2t\, e^{-st} + \frac{2}{s^2}\sin 2t\, e^{-st} - \frac{4}{s^2}T$$

$$T = \frac{s^2}{s^2+4}\left(-\frac{1}{s}\cos 2t + \frac{2}{s^2}\sin 2t\right)e^{-st} + C$$

$$\int \sin^2 t\, e^{-st}\, dt = -\frac{1}{2s}e^{-st} - \frac{1}{2}\cdot\frac{2\sin 2t - s\cos 2t}{s^2+4}e^{-st} + C$$

$$L(f)(s) = \left(-\frac{1}{2s} - \frac{\sin 2t}{s^2+4} + \frac{s\cos 2t}{2(s^2+4)}\right)e^{-st}\Bigg|_{t=0}^{+\infty} =$$

$$= \frac{2}{s^3+4s}$$

Answer: $L(f)(s) = \frac{2}{s^3+4s}$, $s>0$

103. $f(t) = \sin 2t + \cos 4t$

$$L(f)(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} (\sin 2t + \cos 4t)e^{-st}dt$$

$$\int (\sin 2t + \cos 4t)\, e^{-st}dt = \int \sin 2t\, e^{-st}dt + \int \cos 4t\, e^{-st}dt$$

$$\underbrace{\int \sin 2t\, e^{-st}dt}_{T_1} = \left\{\begin{matrix} u = \sin 2t & du = 2\cos 2t\, dt \\ v = -\frac{1}{s}e^{-st} & dv = e^{-st}dt \end{matrix}\right\} =$$

$$= -\frac{1}{s}\sin 2t\, e^{-st} + \frac{2}{s}\int \cos 2t\, e^{-st}dt$$

$$\int \cos 2t\, e^{-st}dt = \left\{\begin{matrix} u = \cos 2t & du = -2\sin 2t\, dt \\ v = -\frac{1}{s}e^{-st}\cancel{dt} & dv = e^{-st}dt \end{matrix}\right\} =$$

$$= -\frac{1}{s}\cos 2t\, e^{-st} - \frac{2}{s}\underbrace{\int \sin 2t\, e^{-st}dt}_{T_1}$$

$$T_1 = -\frac{1}{s}\sin 2t\, e^{-st} - \frac{2}{s^2}\cos 2t\, e^{-st} - \frac{4}{s^2}T_1$$

$$T_1 = \frac{-s\cdot\sin 2t - 2\cos 2t}{s^2+4}e^{-st} + C$$

$$\underbrace{\int \cos 4t \, e^{-st} dt}_{T_2} = \left\{ \begin{matrix} u = \cos 4t & du = -4\sin 4t \, dt \\ v = -\frac{1}{s} e^{-st} & dv = e^{-st} dt \end{matrix} \right\} =$$

$$= -\frac{1}{s} \cos 4t \, e^{-st} - \frac{4}{s} \int \sin 4t \, e^{-st} dt$$

$$\int \sin 4t \, e^{-st} dt = \left\{ \begin{matrix} u = \sin 4t & du = 4\cos 4t \, dt \\ v = -\frac{1}{s} e^{-st} \cancel{dt} & dv = e^{-st} dt \end{matrix} \right\} =$$

$$= -\frac{1}{s} \sin 4t \, e^{-st} + \frac{4}{s} \underbrace{\int \cos 4t \, e^{-st} dt}_{T_2}$$

$$T_2 = -\frac{1}{s} \cos 4t \, e^{-st} + \frac{4}{s^2} \sin 4t \, e^{-st} - \frac{16}{s^2} T_2$$

$$T_2 = \frac{4\sin 4t - s\cos 4t}{s^2+16} e^{-st} + C$$

$$L(f)(s) = e^{-st} \left[\frac{-s \cdot \sin 2t - 2\cos 2t}{s^2+4} + \frac{4\sin 4t - s\cos 4t}{s^2+16} \right] \Bigg|_{t=0}^{+\infty}$$

$$= \frac{2}{s^2+4} + \frac{s}{s^2+16}$$

Answer: $L(f)(s) = \dfrac{2}{s^2+4} + \dfrac{s}{s^2+16}, \; s>0$

10.4. $f(t)=\begin{cases} te^{+t}, & 0\le t<1 \\ e^{t}, & t\ge 1 \end{cases}$

$$L(f)(t)=\int_0^1 te^{t}e^{-st}\,dt+\int_1^{+\infty}e^{t}e^{-st}\,dt=$$

$$=\int_0^1 te^{t(1-s)}\,dt+\int_1^{+\infty}e^{t(1-s)}\,dt$$

$$\int te^{t(1-s)}dt=\begin{Bmatrix} u=t & du=dt \\ v=\frac{1}{1-s}e^{t(1-s)} & dv=e^{t(1-s)}dt \end{Bmatrix}=$$

$$=\frac{t}{1-s}e^{t(1-s)}-\frac{1}{1-s}\int e^{t(1-s)}dt=$$

$$=\frac{t}{1-s}e^{t(1-s)}-\frac{1}{(1-s)^2}e^{t(1-s)}+C$$

$$\int_0^1 te^{t(1-s)}dt=e^{t(1-s)}\left(\frac{t}{1-s}-\frac{1}{(1-s)^2}\right)\Big|_0^1=\frac{1-se^{1-s}}{(1-s)^2}$$

$$\int_1^{+\infty}e^{t(1-s)}=+\frac{1}{1-s}e^{t(1-s)}\Big|_1^{+\infty}=\begin{cases}\frac{-e^{1-s}}{1-s}, & \text{if } s>1 \\ \text{doesn't define, otherwise}\end{cases}$$

$$L(f)(t)=\frac{1-se^{1-s}}{(1-s)^2}+\frac{-e^{1-s}}{1-s}$$

Answer: $L(f)(t)=\frac{1-se^{1-s}}{(1-s)^2}+\frac{-e^{1-s}}{1-s}$, $s>1$

10.5. if $f(t) \leftrightarrow F(s)$ then $t^n f(t) \leftrightarrow$
$(-1)^n F^{(n)}(s)$

$$F(s) = L(f(t))(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$\left[\int_a^b f(x,y)\,dx\right]'_y = \int_a^b f'_y(x,y)\,dx$$

$$\left(f(t)\, e^{-st}\right)^{(n)}_s = (-t)^n f(t)\, e^{-st} =$$
$$= (-1)^n t^n f(t)\, e^{-st}$$

$$F^{(n)}(s) = \int_0^{+\infty} (-1)^n t^n f(t)\, e^{-st}\, dt$$

$$(-1)^n F^{(n)}(s) = \int_0^{+\infty} [t^n f(t)]\, e^{-st}\, dt$$

$$(-1)^n F^{(n)}(s) \leftrightarrow t^n f(t)$$

Proved

10.6 $L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s} L(f(t))$

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \int_0^{+\infty} \left[\int_0^t f(\tau)\,d\tau\right] e^{-st}\, dt =$$

$$= \left\{ \begin{matrix} u = \int_0^t f(\tau)\,d\tau & du = f(t)\,dt \\ v = -\frac{1}{s} e^{-st} dt & dv = e^{-st}\,dt \end{matrix} \right\} =$$

$$\left[\int_{a(y)}^{b(y)} F(x,y)\,dx\right]'_y = \int_{a(y)}^{b(y)} \frac{\partial F(x,y)}{\partial y}\,dx + F(b(y),y)\cdot b'_y(y) -$$
$$- F(a(y),y)\cdot a'_y(y)$$

$$\textcircled{=} -\frac{e^{-st}}{s}\int_0^t f(\tau)d\tau\Big|_0^{+\infty} + \frac{1}{s}\int_0^{+\infty} f(t)e^{-st}dt =$$

$$= 0 + \frac{1}{s}\int_0^{+\infty} f(t)e^{-st}dt = \frac{1}{s}L(f(t))$$

Proved.

10.7. $L(t^n)$

Notice, that $t^n = \int_0^t n\tau^{n-1}d\tau$

Then $L(t^n) = L(\int_0^t n\tau^{n-1}d\tau) =$

$= \frac{1}{s}L(nt^{n-1})$ $\quad L(c) = \frac{c}{s}$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

Answer: $L(t^n) = \frac{n!}{s^{n+1}}$