II 1. 1) $y' + \frac{1}{x} y = \frac{7}{x^2} + 3$ $\quad p(x) = \frac{1}{x}$; $f(x) = \frac{7}{x^2} + 3$

$y_1' + p(x) y_1 = 0$; $\quad y_1' + \frac{y_1}{x} = 0$; $\quad \frac{dy_1}{dx} = -\frac{y_1}{x}$

$\int \frac{dy_1}{y_1} = -\int \frac{dx}{x}$; $\quad \ln|y_1| = -\ln|x| + C$

$e^{\ln|y_1|} = e^{-\ln|x| + C} = Ce^{-\ln|x|}$

$|y_1| = C \cdot \frac{1}{|x|}$; $\quad y_1 = \frac{1}{x}$; $\quad y_1' = -\frac{1}{x^2}$;

$u' = \frac{f(x)}{y_1(x)} = \frac{7 + 3x^2}{x^2} \cdot (x) = \frac{3x^2 + 7}{x}$

$u = \frac{3}{2} x^2 + 7\ln|x| + C$

$y = u y_1 = \left(\frac{3}{2} x^2 + 7\ln|x| + C\right) \frac{1}{x} = \cancel{-x^2 - 7 + \frac{C}{x}}$;

$= \frac{3}{2} x + \frac{7\ln|x|}{x} + \frac{C}{x}$

Answer: $y = \frac{3}{2} x + \frac{7\ln|x|}{x} + \frac{C}{x}$

2) $y' = 2x(x^2+y)$

$y' - 2xy = 2x^3 \qquad p(x) = -2x, \quad f(x) = 2x^3;$

$y_1' - 2xy_1 = 0 \qquad \dfrac{dy_1}{dx} = 2xy_1$

$\dfrac{dy_1}{y_1} = 2x\,dx \qquad \int \dfrac{dy_1}{y_1} = 2\int x\,dx$

$\ln|y_1| = x^2 + C \qquad |y_1| = e^{x^2+C} = Ce^{x^2}.$

$y_1 = \begin{cases} Ce^{x^2} \\ -Ce^{x^2} \end{cases} \qquad y_1 = e^{x^2}$

$y = uy_1 \qquad y' = u'y_1 + uy_1'$

$u' = \dfrac{f(x)}{y_1(x)} = \dfrac{2x^3}{e^{x^2}},$

$u = \int u'\,dx = 2\int x^3 e^{-x^2}\,dx = \left\{ t = x^2,\ dx = \dfrac{dt}{2x} \right\} =$

$= \dfrac{1}{2}\int te^{-t}\,dt = \left\{ \begin{matrix} p = t & dp = dt \\ q = -e^{-t} & dq = e^{-t}dt \end{matrix} \right\} =$

$= \dfrac{1}{2}\left(-te^{-t} + \int e^{-t}dt\right) = \dfrac{-te^{-t}}{2} - \dfrac{e^{-t}}{2} =$

$= -(x^2+1)e^{-x^2} + C = u.$

$y_1' = 2xe^{x^2}$

$y' = u'y_1 + uy_1' = \dfrac{2x^3}{e^{x^2}} \cdot e^{x^2} + [-(x^2+1)e^{-x^2} + C] \cdot 2xe^{x^2} =$

$= 2x^3 - 2(x^3 + x) + 2xe^{x^2}C =$

$= -2x + 2xe^{x^2}C$

$y = \int y'\,dx = -2\int x\,dx + 2C\int xe^{x^2}\,dx =$

$= -x^2 + Ce^{x^2} + C_2.$

~~Answer~~ $y = -x^2 + C_1e^{x^2} + C_2$

$-2x + 2xe^{x^2}C_1 = 2x(x^2 + (-x^2 + C_1e^{x^2} + C_2));$

$-2x + 2xe^{x^2}C_1 = 2xC_1e^{x^2} + 2xC_2$

$C_2 = -1$: $y = -x^2 + C_1e^{x^2} - 1.$

Answer: $y = -x^2 + C_1e^{x^2} - 1.$

4) $(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}; \quad y(0) = 1$ II

$p(x) = \frac{3}{x-1} \qquad f(x) = \frac{1}{(x-1)^4} + \frac{\sin x}{(x-1)^3}$

$y_1' + p(x) y_1 = 0 \qquad \frac{dy_1}{dx} = \frac{-y_1 \cdot 3}{x-1}$

$\int \frac{dy_1}{y_1} = -3 \int \frac{d(x-1)}{x-1} \qquad \ln|y_1| = -3 \ln|x-1| + C$

$e^{\ln|y_1|} = e^{-3\ln|x-1| + C} = C e^{-3\ln|x-1|} =$

$= C \left(e^{\ln|x-1|}\right)^{-3};$

$|y_1| = C (x-1)^{-3}; \quad y_1 = \frac{1}{(x-1)^3};$

$y_1' = \frac{-3(x-1)^2}{(x-1)^6} = \frac{-3}{(x-1)^4};$

$u' = \frac{f(x)}{y_1(x)} = \left(\frac{1}{(x-1)^4} + \frac{\sin x}{(x-1)^3}\right) \cdot (x-1)^3 =$

$= \frac{1}{x-1} + \sin x.$

$u = \int u' dx = \ln|x-1| - \cos x + C$

$y = u y_1 = (\ln|x-1| - \cos x + C)\left(\frac{1}{(x-1)^3}\right) =$

$= \frac{\ln|x-1| - \cos x + C}{(x-1)^3},$

$\frac{0 - 1 + C}{-1} = 1. \quad C = 0$

Answer: $\frac{\ln|x-1| - \cos x + C}{(x-1)^3}, \quad C = 0$

II 2. 1) $y' + x(y^2+y)=0$, $y(2)=1$

$$\frac{dy}{dx} = -x(y^2+y)$$

$$\frac{dy}{y^2+y} = -x\,dx; \quad \int\frac{dy}{y^2+y} = -\int x\,dx$$

$$\int\frac{dy}{y^2+y} = \int\frac{dy}{(\frac{1}{y}+1)y^2} = \left\{ t=\frac{1}{y}+1;\ dy=-y^2dt \right\} =$$

$$= -\int\frac{dt}{t} = -\ln|t| = -\ln\left|\frac{1}{y}+1\right| + C$$

$$-\ln\left|\frac{1}{y}+1\right| = -\frac{x^2}{2}+C; \quad \ln\left|\frac{1}{y}+1\right| = \frac{x^2}{2}+C$$

$y(2)=1$: $\ln\left|\frac{1}{1}+1\right| = \frac{2^2}{2}+C$; $\ln|2| = 2+C$

$C = \ln|2| - 2.$

Answer: $\ln\left|\frac{1}{y}+1\right| = \frac{x^2}{2} + \ln|2| - 2.$

2) $y' + \dfrac{(y+1)(y-1)(y-2)}{(x+1)} = 0, \quad y(1) = 0$

$$\frac{dy}{(y+1)(y-1)(y-2)} = -\frac{dx}{x+1}$$

$$\int\left(\frac{1}{6(y+1)} - \frac{1}{2(y-1)} + \frac{1}{3(y-2)}\right)dy = -\int\frac{dx}{x+1}$$

$$\frac{1}{6}\left(\ln|y+1| - 3\ln|y-1| + 2\ln|y-2|\right) = -\ln|x+1| + C$$

$$\ln|y+1| - 3\ln|y-1| + 2\ln|y-2| = -6\ln|x+1| + C$$

$y(1)=0:$ $\ln|1| - 3\ln|-1| + 2\ln|-2| = -6\ln|2| + C$

$C = 8\ln(2)$

Answer: $\ln|y+1| - 3\ln|y-1| + 2\ln|y-2| = -6\ln|x+1| + 8\ln 2$

3) $xy' - 2y = \frac{x^6}{y+x^2}$

$t = \frac{y}{x^2}$ $\qquad t' = y'\frac{1}{x^2} + y\left(\frac{1}{x^2}\right)' = \frac{y'}{x^2} - \frac{2y}{x^3}$

$y = tx^2$ $\qquad y' = \frac{x^3 z' + 2y}{x}$

$x \cdot \frac{x^3 z' + 2y}{x} - 2tx^2 = \frac{x^6}{tx^2 + x^2}$

$x^3 t' + 2tx^2 - 2tx^2 = \frac{x^4}{t+1}$

$t' = \frac{x}{t+1}$ $\qquad dt(t+1) = x\,dx$ $\qquad \int (t+1)\,dt = \int x\,dx$

$\frac{(t+1)^2}{2} = \frac{x^2}{2} + C$ $\qquad t = \pm\sqrt{x^2 + C} - 1$

$(t+1)^2 = x^2 + C$

Answer: $\pm\sqrt{x^2 + C} - 1$