Introduction to

Differential Equations

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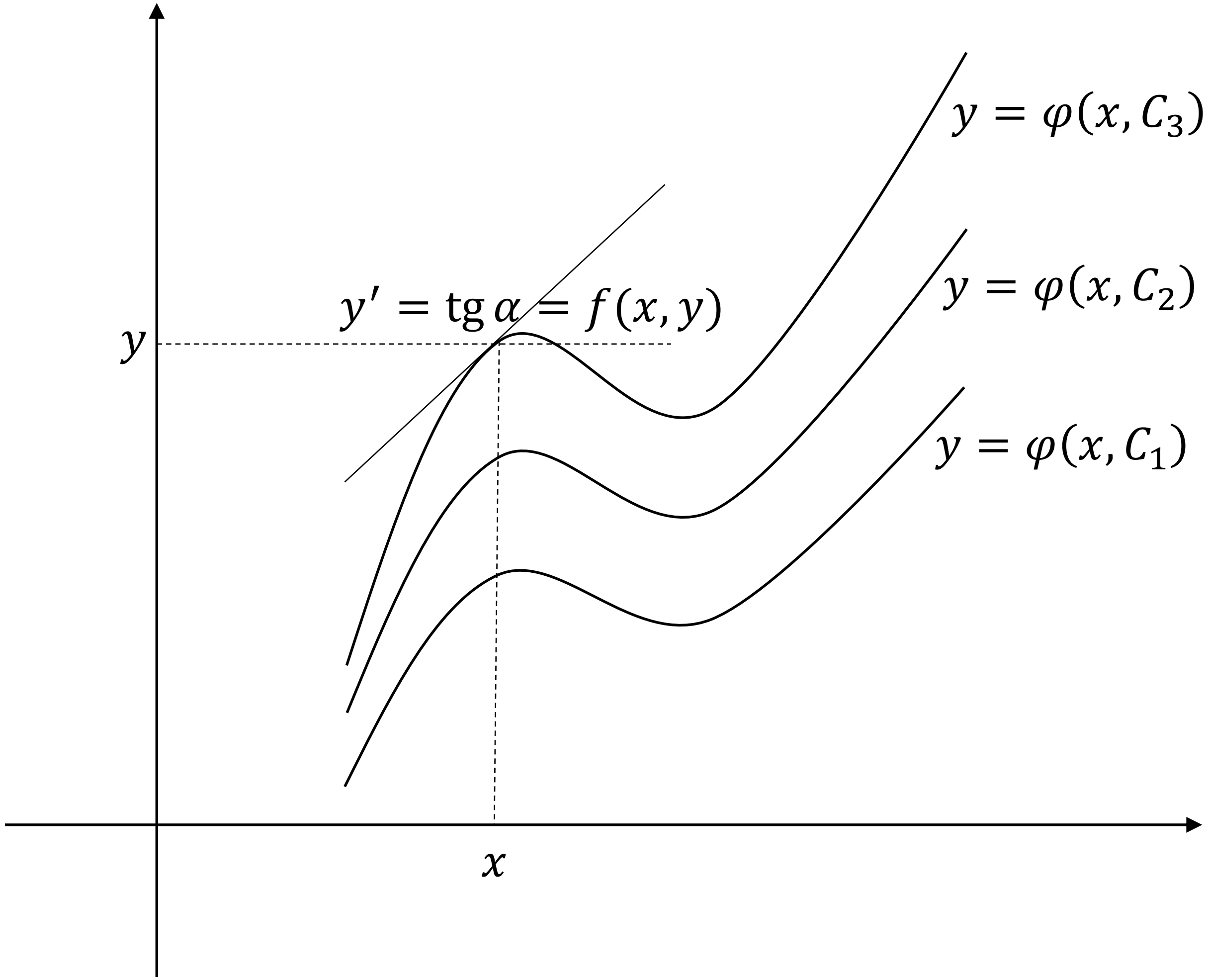
First-order ordinary differential equations

Part I

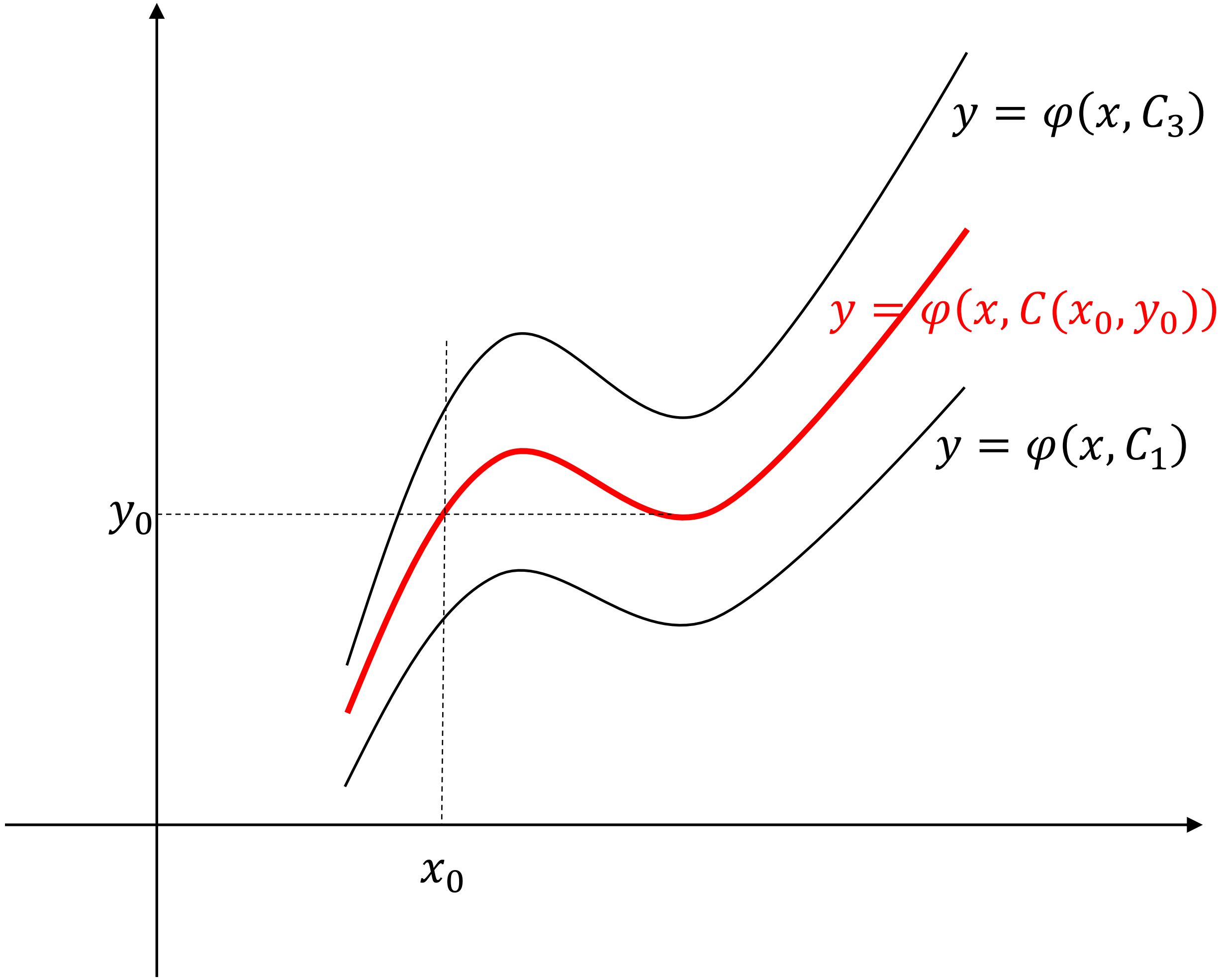
FO ODE's: implicit and explicit forms

- Equation in
  - implicit form: $F(x, y, y') = 0$;
  - explicit form: $y' = f(x, y)$.
- A general solution in
  - implicit form: $\Phi(x, y, C) = 0$;
  - explicit form: $y = \varphi(x, C)$.
- Each general solution (in explicit form) specifies a family of solutions.



Initial (Cauchy) problem for FO ODE's

- Initial (Cauchy) problem: integrate a given equation and find a solution with curve containing a given point (x_0, y_0).
- FYI: Cauchy–Kovalevskaya theorem is some general *sufficient* criterion that guaranties local existence and uniqueness of a solution for the initial value problems.



Differential form of FO ODE's

- Equation in
  - explicit form: $y' = f(x, y)$ or $\frac{dy}{dx} = f(x, y)$;
  - differential form: $P(x, y)dx + Q(x, y)dy = 0$.
- Since an equation in differential form is symmetric w.r.t. both variables than
  - each variable may be considered as a function of another independent variable (i.e. $y = y(x)$ and/or $x = x(y)$);
  - solution may be represented in a parameterized form (i.e. $y = y(t)$ and $x = x(t)$ where t is an independent variable).

Life is a hard experience

- There is no general method how to solve differential equations (and Cauchy initial problems) in analytical way.
- So we have
  - either consider special classes of ODE’s;
  - or solve them numerically.

Variable separation

Part II

Separable equations

- Separable equation:

  $h(y)y' = g(x)$ or $h(y)dy = g(x)dx$ (in differential form) .

- If h and g are continuous functions then (according to Cauchy theorem – a special case of Cauchy–Kovalevskaya theorem for equations in the form $z' = F(x)$) they both have antiderivatives (primitive function, primitive integral, etc.) H and G.
- Hence $H(y) = G(x) + C$ is an implicit solution of the equation $h(y)y' = g(x)$: if a solution $y = y(x)$ exists on some region then $H(y(x)) = G(x) + C$ must be a valid equality (because of integration by substitution rule).

Example 1

- Problem: Solve equation $y' = {}^{y}/_{x}$.
- Solution:
  1. Remark that $x \neq 0$.
  2. Assume that $y \neq 0$;
     - convert equation into a separable form $\frac{dy}{y} = \frac{dx}{x}$;
     - hence $\ln|y| = \ln|x| + \ln C$ (where $C > 0$) is the most general solution (in the implicit form) on $\boldsymbol{R}\backslash\{0\}$ of the last equation;
     - consequently $y = Cx$ (where $C \neq 0$) is the (more) general solution of the same equation on $\boldsymbol{R}\backslash\{0\}$.

Example 1 – cont.

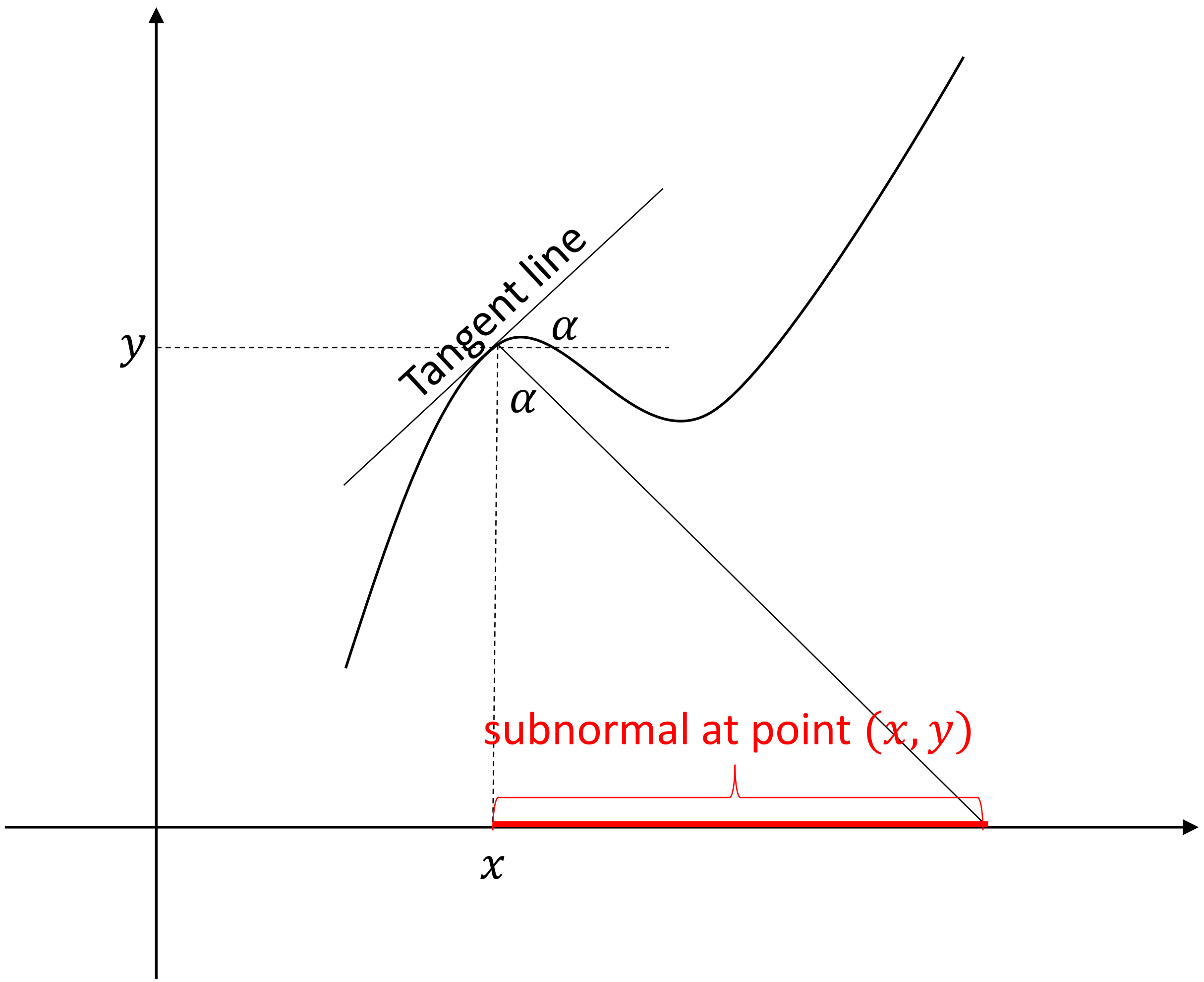
- Answer: $y = Cx$ is the most general solution of the original equation $y' = {}^{y}/_{x}$ on $\boldsymbol{R}\backslash\{0\}$.

- Exercise: Recall the system from the chemistry example (lecture for week 1, slide 33) and solve the first equation $\frac{dC_0}{dt} = -k_0C_0$ (i.e. find the most general solution) as a separable one. Is it possible to solve the equation without variable separation?

$$\begin{cases} \frac{dC_0}{dt} = -k_0C_0 \\ \frac{dC_1}{dt} = k_0C_0 - (k_1+k_2)C_1 \\ \frac{dC_2}{dt} = k_1C_1 \\ \frac{dC_3}{dt} = k_2C_1 - k_3C_3 \\ \frac{dC_4}{dt} = k_3C_3 \end{cases}$$

Example 2

- Problem: Find a smooth curve that passes through a point $(-1, 4)$ that has a *sub-normal* exactly 4 at every its point.
- Solution: Let $y = f(x)$ be an equation for a curve (if any exists).
    1. since $\operatorname{tg} \alpha = y'$ we have equation $yy' = 4$;



Example 2 – cont.

2. convert equation into a separable form $y\,dy = 4\,dx$;
3. it has the most general solution $y^2 = 8x + C$ (in an implicit form) or $x = \frac{y^2 - C}{8}$ (in an explicit form) on $\boldsymbol{R}$;
4. since we solve an initial value problem (a curve must pass through the point $(-1, 4)$) hence $C = 4^2 - 8(-1) = 24$.

- Answer: $x = \frac{y^2 - 24}{8}$ on $\boldsymbol{R}$ is the only smooth curve that passes through the point $(-1, 4)$ that has sub-normal exactly 4 in each its point.

Exercises about example 2

1. Explain why do we claim that we found the only curve that has the stated properties? (And what are these properties?)
2. Write down an equation for a smooth curve that passes through a point (a, b) that has a sub-normal exactly c at every its point. (Assume that $a, b, c \in \boldsymbol{R}$, $c \geq 0$.)

Example 3

- Newton's law of cooling reads that the rate of temperature change of an object (*B*) in a cooling medium (*E*) is proportional to temperature difference between the object and the environment:

$$\dot{T_B} = \frac{dT_B}{dt} = k(T_B - T_E)$$

where k is a temperature decay constant of the medium.

- Problem: A solid body was 20 minutes in a thermostatic medium with temperature 20°C. It is known that it was cooled down from 100°C to 60°C in. Find the process (as function of time).

Example 3 – cont.

- Solution:
  1. Newton’s law for this concrete problem can be written as $\frac{dT}{dt} = k(T - 20)$;
  2. Assume that $T \neq 20$;
     - variable separation leads to the equation $\frac{dT}{T-20} = k\ dt$;
     - a (“the” maybe?) general solution of the last equation is $\ln(T - 20) = kt + \ln C$ (assuming $T > 20$ and $C > 0$);
  3. So we come to a general solution for the original equation $T = 20 + Ce^{kt}$ on $\boldsymbol{R}$ where $C \in \boldsymbol{R}$.

Example 3 – cont.

4. The problem isn't an initial value problem, but the solution must meet some additional constraints that may be expressed as
$$\begin{cases} 100 = 20 + C \\ 60 = 20 + Ce^{20k} \end{cases};$$

- Answer: The cooling process is described by the following law
$$T = 20 + 80 \times (1/2)^{t/20} \text{ for } t \geq 0.$$

Exercises

1. What is the most general solution of $\frac{dT}{dt} = k(T - 20)$? What is the most general solution of $\frac{dT}{dt} = k(T - c)$ where $c \in \boldsymbol{R}$ is a constant.
2. A solid body was M minutes in a thermostatic medium with temperature $T_E = \mathrm{const}$ and was cooled down (warmed up) from T_{in} to T_{out} in a thermostat. Find the process (as a function of time).

Other techniques to solve first-order ordinary differential equations

Part III

Homogeneous functions

- A function F of 2 real arguments is called *homogeneous* with degree n if $F(kx, ky) = k^n F(x, y)$ for all $k \in \boldsymbol{R}$ and all x, y (if kx and ky both are in the domain of F).
- Exercises
  1. Which of two polynomials below are homogeneous functions? What is/are degree(s) in the case of homogeneousness?
     - $10^6 x^{1098} y^{-98} - \frac{1}{2^{100}} xy^{999}$;
     - $x^3 + y^2$.
  2. Formulate and prove (a) necessary and (b) sufficient conditions for a polynomial to be a homogeneous function.

FO ODE’s with homogeneous coefficients

- Hint: Solving a differential equation with homogeneous coefficients (with same degree) $P(x, y)dx + Q(x, y)dy = 0$, try one of two following substitution $u = {}^{y}/_{x}$ or $v = {}^{x}/_{y}$ to reduce the equation to a separable one.
- Sample problem: Solve equation $(x + y)dx + x\,dy = 0$.
- Solution:
  1. Observe that the equation isn’t in a separable form, but its coefficients are homogeneous with degree 1;

FO ODE's with homogeneous coefficients – cont.

2. assuming $x \neq 0$, try substitution $u = {}^{y}/_{x}$, i.e. $y = ux$:
   - $dy = x\,du + u\,dx$;
   - $(x+y)dx + x\,dy = (x+ux)dx + x(x\,du + u\,dx) =$
     $= x(1+2u)dx + x^2 du$;
   - since $x \neq 0$, we come to the following separable equation
     - $(1+2u)dx + x\,du = 0$
     - or (assuming $2u+1 \neq 0$) $\frac{dx}{x} = -\frac{du}{2u+1}$;
   - solving the separable equation we find an implicit general solution $2u+1 = {}^{C}/_{x^2}$ or $y = \frac{C-x^2}{2x}$;

FO ODE’s with homogeneous coefficients – cont.

3. Recall that in item 2 we assume that $x \neq 0$ and $2u + 1 \neq 0$…

- Answer: $x \equiv 0$ is the only trivial solution and $y = \frac{C-x^2}{2x}$ is the most general non-trivial solution of the equation $(x + y)dx + x\,dy = 0$ on $\boldsymbol{R}\backslash\{0\}$.
- Exercise: Solve the following equations
    - $(x + y)dx - x\,dy = 0$;
    - $(x^2 + y^2)dx + xy\,dy = 0$.

Bernoulli equation

- A Bernoulli equation a generic name for equations having the form $y' + g(x)y = f(x)y^k, k \in \boldsymbol{R}$.
- Observe that
  - if $k = 0$ then Bernoulli equation is a linear non-homogeneous equation $y' + g(x)y = f(x)$,
  - if $k = 1$ then Bernoulli equation is a linear homogeneous separable equation $y' + (g(x) - f(x))y = 0$.

Bernoulli equation – cont.

- Remark also that a (so-called) complementary equation
$$y' + g(x)y = 0$$
is separable, has a trivial solution $y \equiv 0$ and (if $y \neq 0$) can be solved by variable separation.
- Exercise: Assume that function $g(x)$ is continuous in some region; write (using indefinite integral with a variable boundary) the most general non-trivial solution for the equation $y' + g(x)y = 0$.

Bernoulli equation – cont.

- Let y_c be any non-trivial solution of the complementary equation.
- Try a substitution $y = uy_c$ for the original equation:
  $$y' + g(x)y = (u'y_c + uy_c') + g(x)uy_c =$$
  $$= u'\,y_c + u(y_c' + g(x)y_c) =$$
  $$= u'\,y_c = f(x)(uy_c)^k = f(x)u^k y_c^k.$$
- We come to a separable equation

$$u'y_c = f(x)u^k y_c^k \text{ or } \frac{du}{u^k} = f(x)y_c^{k-1}dx.$$

Bernoulli equation examples

- Let us solve several equations in the form $y' - y = xy^k, k \in \boldsymbol{N}$. All (almost all) these equations use a non-trivial solution of the complementary equation $y' - y = 0$, for example $y = e^x$.
- Sample problem 0: Solve linear non-homogeneous equation

$$y' - y = x.$$

- Solution:
    - Let $y = e^x u$ in $y' - y = (e^x u + e^x u') - e^x u = e^x u' = x$;
    - hence $du = \frac{x\,dx}{e^x}, u = -\frac{x+1}{e^x} + C, y = Ce^x - x - 1.$
- Answer: $y = Ce^x - x - 1$ is a general solution of the equation.

Exercises

1. What is missed in the answer for the sample problem 0 on slide 26?
2. Is $y = Ce^{x} - x - 1$ the most general solution of the equation $y' - y = x$?
3. What if instead of $y = e^{x}$ to use in the solution on slide 26 another non-trivial solution of $y' - y = 0$?

Bernoulli equation examples (cont.)

- Sample problem 1: Solve linear non-homogeneous equation

$$y' - y = xy.$$

- Solution: We have a separable equation $y' = y(x+1)$; assuming $y \neq 0$ we get $\frac{dy}{y} = (x+1)dx$ and hence $\ln|y| = \frac{x^2}{2} + x + \ln C$ ($C > 0$).
- Answer: $y = Ce^x e^{x^2/2}$, where $C \in \boldsymbol{R}$, is the most general solution of the equation on $\boldsymbol{R}$.

Exercises

- Consider Bernoulli equation $y' - y = xy^2$.
  - Exercise 1: Show that $y = -\frac{e^x}{xe^x - e^x + C}$, where $C \in \boldsymbol{R}$, is a general solution of the equation on $\boldsymbol{R}$.
  - Exercise 2: Get the above solution using the specified method with substitution $y = e^x u$.
- Solve equation $y' - y = xy^k$ where $k \in \boldsymbol{N}$, $k > 2$.

Exact equations

- Equation $P(x, y)dx + Q(x, y)dy = 0$ is said to be exact if there exists a function $F(x, y)$ such that $P = \frac{\partial F}{\partial x}$ and $Q = \frac{\partial F}{\partial y}$ (where $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are *partial derivatives*).
- For self-study: Section 2.5 EXACT EQUATIONS from the textbook.

Linear first-order ordinary differential equations

Part IV

FO linear equations as Bernoulli equations

- A general form a linear FO ODE is $a(x)y' + b(x)y = f(x)$.
- We already remarked that when a linear first-order equation has Bernoulli form $y' + g(x)y = f(x)$ then a general solution of the equation can be found in three steps:
  1. find any non-trivial solution y_c of the complementary equation $y' + g(x)y = 0$;
  2. find the most general solution u of a separable equation $u'y_c = f(x)$;
  3. the most general solution of the original equation is uy_c.

Example

- Remark, that it isn't necessary to transform an equation into Bernoulli form but just apply the method directly as in the following example.
- Problem: Solve equation $xy' + 2y = x^2$.
- Solution:
    - find any non-trivial solution of the complementary equation $xy' + 2y = 0$, e.g. $y_c = {}^1/_{x^2}$;
    - find the most general solution u of the equation $xu'y_c = x^2$; e.g. $u = {}^{x^4}/_4 + C$. (Exercise: Explain details of this step!)
- Answer: $y = \frac{x^2}{4} + \frac{C}{x^2}$, $C \in \boldsymbol{R}$, is the most general solution on $\boldsymbol{R}\backslash\{0\}$.

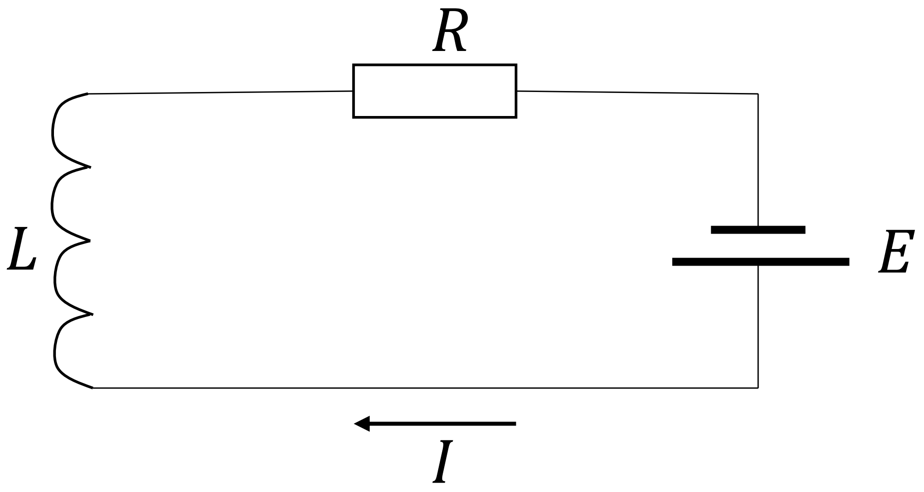
Exercises

1. Solve Cauchi problem for equation $(x+y)y' = 1$ and initial values $x_0 = -1$ and $y_0 = 0$. (Hint: Consider $x = x(y)$ and transform (reduce) the equation into the linear first-order.)
2. back to Chemistry example: Solve the system from slide 29 of lecture notes for week 1:

$$\begin{cases} \dfrac{dC_0}{dt} = -k_0 C_0 \\ \dfrac{dC_1}{dt} = k_0 C_0 - (k_1 + k_2) C_1 \\ \dfrac{dC_2}{dt} = k_1 C_1 \\ \dfrac{dC_3}{dt} = k_2 C_1 - k_3 C_3 \\ \dfrac{dC_4}{dt} = k_3 C_3 \end{cases}$$

Case study: electric circuit with resistance and inductance

- Find the current (as a function of time) $I = I(t)$ through the circuit (depicted right) consisting of resistance R, inductance L and electromotive force $E = E_0 \cos \omega t$.



Recall from electro-physics

- Ohm’s law: The resistance (R) of an object equals to the ratio of voltage across it (V) to current (I).
- Lenz's law: A changing electric current (I) through a circuit that contains inductance induces a voltage proportional (with a coefficient L) to rate of a change (i.e. $\dot{I} = {}^{dI}/_{dt}$).
- Kirchhoff's voltage law (KVL): The sum of all the voltages around a loop is equal to zero (assuming electromotive power with negative sign).

Case study: solution

- Equation: $V_L + V_R = L\frac{dI}{dt} + RI = E = E_0 \cos \omega t$.
- Validate:
  - $y_c = e^{-Rt/_L}$ is a non-trivial solution of the complementary equation;
  - $u = \frac{E_0}{L}(C + \int_0^t e^{Rx/_L} \cos \omega x \; dx)$ is a general solution of $uy_c = \frac{E_0}{L} \cos \omega t$. (Exercise: Find $\int_0^t e^{Rx/_L} \cos \omega x \; dx$!)
  - $I = \frac{E_0}{L\left(\omega^2 + R^2/_{L^2}\right)}(\omega \sin \omega t + \frac{R}{L} \cos \omega t - \frac{R}{L} e^{-Rt/_L})$ is the current (as a function of time) through the circuit .