8.1. $y'' + \frac{y'}{x+1} = 9(x+1)$

Rewrite equation: $y'' = -\frac{y'}{x+1} + 9(x+1) = f(x, y')$

Let $y' = p$, then $y'' = p'$

$p' = -\frac{p}{x+1} + 9(x+1)$ $\quad p' + \frac{p}{x+1} = 9(x+1)$

This is linear non-homogenious equation

Apply substitution: $p = uv$, $p' = uv' + u'v$

$uv' + u'v + uv\frac{1}{x+1} = 9(x+1)$

$u'v + u\left(v' + \frac{v}{x+1}\right) = 9(x+1)$

$$\begin{cases} v' + \frac{v}{x+1} = 0 \\ u'v = 9(x+1) \end{cases}$$

$\int \frac{dv}{v} = \int -\frac{dx}{x+1}$ $\quad \ln|v| = -\ln|x+1| + C$

$v = \frac{C_1}{x+1}$ $\quad u' = \frac{x+1}{C_1} 9(x+1) = \frac{9}{C_1}(x+1)^2$

$u = \int u' dx = \frac{3}{C_1}(x+1) + C_2$; $uv = 3(x+1)^2 + \frac{C_3}{x+1} = p$

$p = 3(x+1)^2 + \frac{C_2}{x+1}$

$y = \int p\, dx = (x+1)^3 + C_2 \ln|x+1| + C_3$

Answer: $y = (x+1)^3 + C_2 \ln|x+1| + C_3$

8.2 $\begin{cases} y^3 y'' = 4(y^4 - 1) \\ y(0) = \sqrt{2}, \quad y'(0) = \sqrt{2} \end{cases}$

Rewrite equation: $y'' = 4y - \frac{4}{y^3} = f(y)$

Let $y' = p$. Then $y'' = pp'_y$

$pp' = 4y - \frac{4}{y^3}$. Separate variables:

$p\,dp = 4y\,dy - \frac{4}{y^3}dy$. Integrate both sides:

$\int p\,dp = 4\int y\,dy - 4\int \frac{dy}{y^3}$

$\frac{p^2}{2} = 2y^2 + 2y^{-2} + C$; $p = \sqrt{4y^2 + \frac{4}{y^2} + C} = \frac{dy}{dx}$

$\frac{dy}{\sqrt{4y^2 + \frac{4}{y^2} + C}} = dx$

Substitute $x = 0 \Rightarrow \begin{cases} y = \sqrt{2} \\ y' = \sqrt{2} \end{cases}$

$C = -8$

Integrate both sides:

$\int \frac{dy}{\sqrt{4y^2 + 4y^{-2} - 8}} = \int dx$. Rewrite: $\int \frac{y\,dy}{(y-1)(y+1)} = \int dx$

Apply substitution: $u = y^2$, $u' = 2y$; $\frac{1}{2}\int \frac{du}{u-1} = \int dx$

$\frac{1}{4}\ln(y^2 - 1) = x + C$. Substitute $x = 0$, $y = \sqrt{2}$

$C = 0$, $\ln(y^2 - 1) = 4x$

Answer: $\ln(y^2 - 1) = 4x$

8.3. $(x+1)y''' + y'' = x+1$

Rewrite equation: $y''' = -\frac{y''}{x+1} + 1 = f(x, y'')$

Let $y'' = p$, then: $p' = -\frac{p}{x+1} + 1$, $p' + \frac{p}{x+1} = 1$

This is linear non-homogenious FO DE

Apply substitution $p = uv$, $p' = uv' + u'v$

$uv' + u'v + uv \cdot \frac{1}{x+1} = 1$; $u'v + u(v' + \frac{v}{x+1}) = 1$

$$\begin{cases} v' + \frac{v}{x+1} = 0 \\ u'v = 1 \end{cases} \quad \int \frac{dv}{v} = -\int \frac{dx}{x+1} \quad v = C_1(x+1)^{-1}$$

$$u' = \frac{x+1}{C_1}; \; u = \int \frac{x+1}{C_1} dx = \frac{(x+1)^2}{2C_1} + C_2$$

$$p = uv = \frac{(x+1)}{2} + \frac{C_1C_2}{x+1} = \frac{x+1}{2} + \frac{C_3}{x+1}$$

$$y' = \int y'' dx = \frac{(x+1)^2}{4} + C_3 \ln|x+1| + C_4$$

$$y = \int y' dx = \frac{(x+1)^3}{12} + C_3(x+1)(\ln|x+1| - 1) + C_4x + C_5$$

Answer: $y = \frac{(x+1)^3}{12} + C_3(x+1)(\ln|x+1| - 1) + C_4x + C_5$