13.1. A $\quad y'' + y = 0, \; X_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left((n+2)(n+1) a_{n+2} + a_n \right) x^n \right] = 0$$

$$(n+2)(n+1) a_{n+2} + a_n = 0$$

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

$$a_0 = a_0 \qquad a_1 = a_1 \qquad a_2 = \frac{-a_0}{2 \cdot 1} = -\frac{a_0}{2}$$

$$a_3 = \frac{-a_1}{3 \cdot 2} = -\frac{a_1}{6}$$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Answer: $y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

13.1.B. $(1-x^2)y'' - 8xy' - 12y = 0, \; x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} -$$

$$- 8\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - 12 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n -$$

$$- \sum_{n=0}^{\infty} 8 n a_n x^n - \sum_{n=0}^{\infty} 12 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [((n+2)(n+1) a_{n+2} - n(n-1) a_n - 8n a_n - 12 a_n) x_n] = 0$$

$$(n+2)(n+1) a_{n+2} - n(n-1) a_n - 8n a_n - 12 a_n = 0$$

$$a_{n+2} = \frac{n(n-1)a_n + 8n a_n + 12 a_n}{(n+2)(n+1)} = \frac{(n+3)(n+4)}{(n+1)(n+2)} a_n$$

$$a_{2n} = \frac{(2n+1)(2n+2)}{2} a_0 \qquad a_{2n+1} = \frac{(2n+2)(2n+3)}{6} a_1$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(2n+1)(2n+2)}{2} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(2n+2)(2n+3)}{6} x^{2n+1}$$

Answer: $y = a_0 \sum_{n=0}^{\infty} \frac{(2n+1)(2n+2)}{2} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(2n+2)(2n+3)}{6} x^{2n+1}$

13.1.C $(1+x^2)\, y'' - 8x\, y' + 20\, y = 0, \quad x_0 = 0$

$y = \sum_{n=0}^{\infty} a_n x_n \qquad y' = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^{n}$

$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n$

$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n+2} -$

$- \sum_{n=0}^{\infty} 8(n+1)\, a_{n+1}\, x^{n+1} + \sum_{n=0}^{\infty} 20\, a_n\, x^n = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^n -$

$- \sum_{n=0}^{\infty} 8n\, a_n\, x^n + \sum_{n=0}^{\infty} 20\, a_n\, x^n = 0$

$\sum_{n=0}^{\infty} [((n+2)(n+1)\, a_{n+2} + n(n-1)\, a_n - 8n\, a_n + 20\, a_n)\, x^n] = 0$

$(n+2)(n+1)\, a_{n+2} + n(n-1)\, a_n - 8n\, a_n + 20\, a_n = 0$

$a_0 = a_0 \qquad a_1 = a_1 \qquad a_2 = -10\, a_0 \qquad a_3 = -2\, a_1$

$a_4 = 5\, a_0 \qquad a_5 = \frac{1}{5}\, a_1 \qquad \forall n \geq 6 \quad a_n = 0$

$y = \sum_{n=0}^{\infty} a_n x^n = a_0 (1 - 10x^2 + 5x^4) + a_1 (x - 2x^3 + \frac{1}{5} x^5)$

Answer: $y = a_0 (1 - 10x^2 + 5x^4) + a_1 (x - 2x^3 + \frac{1}{5} x^5)$

13.2 A $y''+(x-3)y'+3y=0 \quad y(3)=-2 \quad y'(3)=3$

$x_0=3$

$$y=\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad y'=\sum_{n=0}^{\infty} (n+1)a_{n+1}(x-x_0)^n$$

$$y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n \qquad \begin{cases} a_0=-2 \\ a_1=3 \end{cases}$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-3)^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-3)^{n+1}+$$

$$+\sum_{n=0}^{\infty}3a_n(x-3)^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-3)^n+\sum_{n=0}^{\infty}n\,a_n(x-3)^n+$$

$$+\sum_{n=0}^{\infty}3a_n(x-3)^n=0$$

$$\sum_{n=0}^{\infty}\left[((n+2)(n+1)a_{n+2}+n\,a_n+3a_n)(x-3)^n\right]=0$$

$$(n+2)(n+1)a_{n+2}+n\,a_n+3a_n=0$$

$$a_{n+2}=\frac{-n-3}{(n+2)(n+1)}a_n$$

$a_0=-2 \quad a_1=3 \quad a_2=3 \quad a_3=-2$

$a_4=-\frac{5}{4} \quad a_5=\frac{3}{5} \quad a_6=\frac{7}{24}$

Answer: $a_0=-2 \quad a_1=3 \quad a_2=3 \quad a_3=-2$

$a_4=-\frac{5}{4} \quad a_5=\frac{3}{5} \quad a_6=\frac{7}{24}$

13.2.B. $(x^2-8x+14)y''-8(x-4)y'+20y=0$

$y(4)=3 \quad y'(4)=-4$

$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n$

$y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n \quad x_0=4 \qquad \begin{cases} a_0=3 \\ a_1=-4 \end{cases}$

$(x^2-8x+14)=(x-4)^2-2$

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-4)^{n+2}-\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}\cdot$

$\cdot(x-4)^n-\sum_{n=0}^{\infty}8(n+1)a_{n+1}(x-4)^{n+1}+\sum_{n=0}^{\infty}20a_n(x-4)^n=0$

$\sum_{n=0}^{\infty}n(n-1)a_n(x-4)^n-\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}\cdot(x-4)^n-$

$-\sum_{n=0}^{\infty}8na_n(x-4)^n+\sum_{n=0}^{\infty}20a_n(x-4)^n=0$

$\sum_{n=0}^{\infty}[(n(n-1)a_n-2(n+2)(n+1)a_{n+2}-8na_n+20a_n)(x-4)^n]=0$

$n(n-1)a_n-2(n+2)(n+1)a_{n+2}-8na_n+20a_n=0$

$a_{n+2}=\dfrac{(n-4)(n-5)}{2(n+2)(n+1)}a_n$

$a_0=3 \quad a_1=-4 \quad a_2=15 \quad a_3=-4 \quad a_4=\frac{15}{4} \quad a_5=-\frac{1}{5} \quad a_6=0$

Answer: $a_0=3 \quad a_1=-4 \quad a_2=15 \quad a_3=-4 \quad a_4=\frac{15}{2}$

$a_5=-\frac{1}{5} \quad a_6=0$