5.1. $2xy^2+4-2(3-x^2y)y'=0$

$(2xy^2+4)dx+(-6+2x^2y)dy=0$, $P\,dx+Q\,dy=0$

$\frac{dP}{dy}=\frac{d(2xy^2+4)}{dy}=4xy$; $\frac{dQ}{dx}=\frac{d(-6+2x^2y)}{dx}=4xy$

$\frac{dP}{dy}=\frac{dQ}{dx}$ $\Rightarrow$ exact diferential

$$\begin{cases} \frac{dF}{dx}=P \\ \frac{dF}{dy}=Q \end{cases}$$

$\int P\,dx=\int(2xy^2+4)\,dx=x^2y^2+4x+f(y)$

$\int Q\,dy=\int(-6+2x^2y)\,dy=-6y+x^2y^2+\varphi(x)$

$\int P\,dx=\int Q\,dy$

$F=x^2y^2+4x-6y=C$

Answer: $x^2y^2+4x-6y=C$

5.2. $y' - \frac{3y}{x+1} = (x+1)^4$

$(-3y - (x+1)^5)dx + (x+1)\,dy = 0 \qquad P dx + Q dy = 0$

$\frac{dP}{dy} = -3 \qquad \frac{dQ}{dx} = 1 \qquad \frac{dP}{dy} \neq \frac{dQ}{dx}$

$\Rightarrow$ Not exact differential

Try to find $\mu(x)$, such that:

$\frac{d(\mu P)}{dy} = \frac{d(\mu Q)}{dx} \qquad \mu(-3) = \mu'(x+1) + \mu,\ \mu' = \frac{d\mu}{dx}$

$\frac{dx}{(x+1)} = \frac{d\mu}{-4\mu} \qquad \ln|x+1| = -\frac{1}{4}\ln|\mu| + C$

$\mu = (x+1)^{-4}$.

Multiply by $\mu = (x+1)^{-4}$

$\left(\frac{-3y}{(x+1)^4} - (x+1)\right)dx + \frac{1}{(x+1)^3}\,dy = 0$

$\frac{dP}{dy} = -\frac{3}{(x+1)^4} \qquad \frac{dQ}{dx} = -\frac{3}{(x+1)^4} \qquad \frac{dP}{dy} = \frac{dQ}{dx}$

$\Rightarrow$ this is exact differential.

$F = \int P dx = -3y \int \frac{d(x+1)}{(x+1)^4} - \int d(x+1)(x+1) = \frac{y}{(x+1)^3} - \frac{(x+1)^2}{2} + f(y)$

$F = \int Q dy = \int \frac{dy}{(x+1)^3} = \frac{y}{(x+1)^3} + \varphi(x)$

$F = \frac{y}{(x+1)^3} - \frac{(x+1)^2}{2}$.

Answer: $\frac{y}{(x+1)^3} - \frac{(x+1)^2}{2} = C$

5.3. $ydx - (4x^2y + x)dy = 0$

$Pdx + Qdy = 0;\quad P = y \quad Q = -4x^2y - x$

$\frac{dP}{dy} = 1;\ \frac{dQ}{dx} = -8xy - 1;\ \frac{dP}{dy} \neq \frac{dQ}{dx} \Rightarrow$ not exact differential

Try to find $\mu(x)$, such that: $\frac{d(\mu P)}{dy} = \frac{d(\mu Q)}{dx}$

$\mu \cdot 1 = \frac{d\mu}{dx} \cdot (-4x^2y - x) + \mu \cdot (-8xy - 1)$

$\frac{\frac{d\mu}{dx}}{\mu} = -\frac{2}{x};\quad \int \frac{d\mu}{\mu} = -2\int \frac{dx}{x};\quad \ln|\mu| = -2\ln|x| + C$

$\mu = \frac{1}{x^2}$

Multiply by μ:

$\frac{y}{x^2}dx + (-4y - \frac{1}{x})dy = 0$

5.3. $\frac{dP}{dy} = \frac{1}{x^2}$, $\frac{dQ}{dx} = \frac{1}{x^2}$; $\frac{dP}{dy} = \frac{dQ}{dx} \Rightarrow$ this is exact differential

$F = \int P dx = -\frac{y}{x} + f(y)$

$F = \int Q dy = -2y^2 - \frac{y}{x} + \varphi(x)$

$F = -\frac{y}{x} - 2y^2$ Answer: $-\frac{y}{x} - 2y^2 = C$

5.4. $(x^2 + y^2 + x)dx + y\,dy = 0$

$P dx + Q dy = 0$

$\frac{dP}{dy} = 2y$ $\frac{dQ}{dx} = 0$ $\frac{dP}{dy} \neq \frac{dQ}{dx} \Rightarrow$ not exact differential

Try to find $\mu(x)$, such that $\frac{d(\mu P)}{dy} = \frac{d(\mu Q)}{dx}$

$\mu \cdot 2y = y \frac{d\mu}{dx}$ $\int \frac{d\mu}{\mu} = 2\int dx$

$\ln|\mu| = 2x + C$ $\mu = e^{2x}$

Multiply by μ:

$(x^2 + y^2 + x)e^{2x} dx + y e^{2x} dy = 0$

$\frac{dP}{dy} = 2e^{2x} y$; $\frac{dQ}{dx} = 2e^{2x} y$

$\frac{dP}{dy} = \frac{dQ}{dx} \Rightarrow$ this is exact differential

$F=\int P\,dx = \frac{1}{2}(x^2+y^2)e^{2x} + f(y)$

$F=\int Q\,dy = \frac{1}{2}y^2e^{2x} + \varphi(x)$

$F= \frac{1}{2}(x^2+y^2)e^{2x}$

Answer: $\frac{1}{2}(x^2+y^2)e^{2x}=C$

5.5. $(x^2+y^2+y)dx - x\,dy = 0$

$P\,dx + Q\,dy=0$; $P= x^2+y^2+y$; $Q=-x$

$\frac{dP}{dy}=2y+1$; $\frac{dQ}{dx}=-1$; $\frac{dP}{dy} \neq \frac{dQ}{dx}$ $\Rightarrow$ not exact differential

Try to find $\mu(x)$, such that $\frac{d(\mu P)}{dy}=\frac{d(\mu Q)}{dx}$

$\mu(2y+1)=\mu(-1)+\frac{d\mu}{dx}(-x)$

This function depends on both x and y

$\Rightarrow$ it can't be rewritten as exact differential

Rewrite: $x^2-xy'+y^2+y=0$

Substitution: $y=vx$; $y'=\frac{dy}{dx}=\frac{d(vx)}{dx}=xv'+v$

$x^2-x(xv'+v)+x^2v^2+xv=0$

$x^2-x^2v'+x^2v^2=0$; $x^2(1-v'+v^2)=0$

$x=0$ – partial solution. $1+v^2=\frac{dv}{dx}$; $\int\frac{dv}{1+v^2}=\int dx$

$\arctan v = x+C$; $y=\frac{x\tan(x+C)}{\cancel{x}}$

Answer: $y=x\cdot\tan(x+C)$, $x=0$