1.1

$$y' + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{\frac{7}{x^2} + 3}_{f(x)}$$

$$z' + \frac{1}{x} z = 0$$

$$z' = -\frac{z}{x}$$

$z \neq 0$

$$\int \frac{dz}{z} = -\int \frac{dx}{x}$$

$$\ln|z| = -\ln|x| + C$$

$$|z| = \frac{C_1}{|x|}$$

$$z = \frac{C_2}{x}, \quad C_2 \neq 0$$

$z = 0$

$$z' = 0$$

$$0 + \frac{0}{x} = 0$$

$$\Downarrow$$

$z = 0$ - trivial solution

$$y = uz = C_2 \frac{u}{x} \qquad y' = C_2 \frac{u'x - u}{x^2}$$

$$C_2 \frac{u'x \cancel{- u}}{x^2} \cancel{+ C_2 \frac{u}{x^2}} = \frac{7}{x^2} + 3$$

$$C_2 \frac{u'}{x} = \frac{7}{x^2} + 3 \quad \Big| \cdot x$$

$$u' = \frac{\frac{7}{x} + 3x}{C_2} \qquad u = \frac{1}{C_2}\left(7\int \frac{dx}{x} + 3\int x\,dx\right)$$

$$u = \frac{1}{C_2}\left(7\ln|x| + \frac{3}{2}x^2\right) + C_3$$

$$y = \frac{7\ln|x| + \frac{3}{2}x^2 + C_4}{x} = 7\,\frac{\ln|x| + C}{x} + \frac{3}{2}x$$

Answer

1.2

$$y' = 2x(x^2 + y)$$

$$y' \underbrace{- 2xy}_{p(x)} = \underbrace{2x^3}_{f(x)}$$

$$z' - 2xz = 0$$

$$z' = 2xz$$

$z = 0$

$0 = 2x \cdot 0 \Rightarrow z = 0$ – trivial solution

$z \neq 0$

$$\int \frac{dz}{z} = 2\int x\,dx$$

$$\ln|z| = x^2 + C$$

$$|z| = C_1 e^{x^2}$$

$$z = C_2 e^{x^2}, \quad C_2 \neq 0$$

$$y = uz = g e^{x^2} \qquad y' = g' e^{x^2} + 2gxe^{x^2}$$

$$g' e^{x^2} + \cancel{2gxe^{x^2}} = 2x(x^2 + \cancel{ge^{x^2}})$$

$$d(-x^2)$$

$$g' e^{x^2} = 2x^3$$

$$g' = 2\frac{x^3}{e^{x^2}} \qquad g = 2\int e^{-x^2} x^3\,dx = \int (-x^2)\,d e^{-x^2} =$$

$$= -x^2 e^{-x^2} - \int \underbrace{e^{-x^2} d(-x^2)}_{d(e^{-x^2})} = -x^2 e^{-x^2} - e^{-x^2} + C$$

$$y = (-x^2 e^{-x^2} - e^{-x^2} + C)e^{x^2}$$

$$y = -x^2 - 1 + Ce^{x^2}$$

Answer

1.3)

$$y' + \underbrace{\frac{2x}{1+x^2}}_{f(x)}\, y = \underbrace{\frac{e^{-x}}{1+x^2}}_{g(x)}$$

$$z' + \frac{2x}{1+x^2} z = 0$$

$z=0$

$$0 + \frac{2x}{1+x^2}\cdot 0 = 0$$

$\Downarrow$

$z=0$ - trivial solution

$z \neq 0$

$$\int \frac{dz}{z} = -\int \frac{2x\,dx}{1+x^2}$$

$$\ln|z| = -\ln|1+x^2| + C$$

$$z = \frac{C_1}{1+x^2}$$

$$y = uz = \frac{g}{1+x^2} \qquad y' = \frac{g'(1+x^2) - 2xg}{(1+x^2)^2}$$

$$\frac{g'(1+x^2) \cancel{-2xg}}{(1+x^2)^2} + \cancel{\frac{2xg}{(1+x^2)^2}} = \frac{e^{-x}}{1+x^2}$$

$$\frac{g'}{1+x^2} = \frac{e^{-x}}{1+x^2}$$

$$g' = e^{-x} \qquad g = \int e^{-x}dx = -e^{-x} + C$$

$$y = \frac{-e^{-x} + C}{1+x^2}$$

Answer

1.4

$$y' + \underbrace{\frac{3}{x-1}}_{f(x)} y = \underbrace{\frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}}_{g(x)} \qquad z' + \frac{3}{x-1} z = 0$$

$z = 0$

$$0 + \frac{3}{x-1} 0 = 0$$

$$\Downarrow$$

$z = 0$ - trivial solution

$z \neq 0$

$$\frac{z'}{z} = -\frac{3}{x-1}$$

$$\int \frac{dz}{z} = -3 \int \frac{d(x-1)}{x-1}$$

$$\ln|z| = -3\ln|x-1| + C$$

$$|z| = \frac{C_1}{|x-1|^3} \longrightarrow z = \frac{C_2}{(x-1)^3}$$

$$y = uz = \frac{g}{(x-1)^3} \qquad y' = \frac{g'(x-1)^3 - 3(x-1)^2 g}{(x-1)^6}$$

$$\frac{g'(x-1)^3 \cancel{- 3(x-1)^2 g}}{(x-1)^5} + \cancel{\frac{3g(x-1)^5}{(x-1)^5}} = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}$$

$$g' = \frac{1}{x-1} + \sin x \qquad g = \int \frac{d(x-1)}{x-1} - \int d(\cos x) =$$

$$= \ln|x-1| - \cos x + C$$

$$y = \frac{\ln|x-1| - \cos x + C}{(x-1)^3}$$

$$y(0) = \frac{0 - 1 + C}{-1} = 1 - C = 1 \Rightarrow C = 0$$

$$y = \frac{\ln|x-1| - \cos x}{(x-1)^3}$$

Answer

2.1

$$y' + x(y^2+y) = 0, \quad y(2) = 1$$

$y=0$:

$0 + x \cdot 0 = 0$

$\Downarrow$

$y=0$ - solution

$y(2) \neq 1$

$y=-1$:

$0 + x \cdot 0 = 0$

$\Downarrow$

$y=-1$ - solution

$y(2) \neq 1$

$y^2+y \neq 0$:

$$\frac{y'}{y^2+y} = -x$$

$$\int \frac{dy}{y(y+1)} = -\int x\,dx \qquad \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

$$\int \frac{dy}{y} - \int \frac{d(y+1)}{y+1} = -\int x\,dx$$

$$\ln|y| - \ln|y+1| = -\frac{x^2}{2} + C$$

$$\frac{|y|}{|y+1|} = e^{-\frac{x^2}{2}} \cdot C_1$$

$y(2)=1$: $\quad \frac{1}{2} = e^{-2} \cdot C_1$

$$C_1 = \frac{1}{2e^{-2}}$$

$$\frac{y}{y+1} = \frac{e^{-\frac{x^2}{2}}}{2e^{-2}}$$

$$y - \frac{e^{-\frac{x^2}{2}}}{2e^{-2}}\, y = \frac{e^{-\frac{x^2}{2}}}{2e^{-2}}$$

$$y \cdot \left(1 - \frac{e^{-\frac{x^2}{2}}}{2e^{-2}}\right) = \frac{e^{-\frac{x^2}{2}}}{2e^{-2}}$$

$$y = \frac{\frac{e^{-\frac{x^2}{2}}}{2e^{-2}}}{1 - \frac{e^{-\frac{x^2}{2}}}{2e^{-2}}} = \frac{2e^{-2}}{2e^{-2} - e^{-\frac{x^2}{2}}} - 1$$

Answer

2.2 maybe wrong

$$y' = -\frac{(y+1)\cdot(y-1)\cdot(y-2)}{x+1} \qquad \begin{cases} y \neq 2 \\ y \neq 1 \\ y \neq -1 \end{cases} \qquad y(1)=0$$

$y=-1$: $y(1) \neq 0$ ✗

$y=1$: $y(1) \neq 0$ ✗

$y=2$: $y(1) \neq 0$ ✗

$$\int \frac{dy}{(y+1)\cdot(y-1)\cdot(y-2)} = -\int \frac{d(x+1)}{x+1}$$

$$\frac{A}{y+1} + \frac{B}{y-1} + \frac{C}{y-2} = \frac{A(y-1)\cdot(y-2) + B(y+1)\cdot(y-2) + C(y+1)\cdot(y-1)}{(y+1)\cdot(y-1)\cdot(y-2)} \Rightarrow$$

$$\Rightarrow \begin{cases} A+B+C=0 \longrightarrow C=2A \\ 3A+B=0 \longrightarrow B=-3A \\ 2A-2B-C=1 \longrightarrow 2A+6A-2A=1 \end{cases} \Rightarrow \begin{cases} A=1/6 \\ B=-3/6 \\ C=2/6 \end{cases}$$

$$\frac{1}{6}\int \frac{d(y+1)}{y+1} - \frac{3}{6}\int \frac{d(y-1)}{y-1} + \frac{2}{6}\int \frac{d(y-2)}{y-2} = -\int \frac{d(x+1)}{x+1}$$

$$\ln|y+1| - 3\ln|y-1| + 2\ln|y-2| = -6\ln|x+1| + C$$

$$\frac{|y+1||y-2|^2}{|y-1|^3} = \frac{C_1}{|x+1|^6}$$

$y(1)=0$

$$\frac{C}{64} = \frac{1\cdot 2^2}{1^3} = 4 \Rightarrow C = 256$$

$$\frac{(y+1)(2-y)^2}{(1-y)^3} = \frac{256}{(x+1)^6}$$

Answer

2.3

$$xy' - 2y = \frac{x^6}{y + x^2}$$

Let $z = \frac{y}{x^2} \Rightarrow z' = \frac{y'x^2 - 2xy}{x^4}$

$$\Downarrow$$

$$y'x - 2y = x^3 z'$$

$$x^3 z' = \frac{x^4}{z+1}$$

$$(z+1)z' = x$$

$$\int (z+1)\,d(z+1) = \int x\,dx$$

$$(z+1)^2 = x^2 + C \Rightarrow z = \pm\sqrt{x^2+C} - 1$$

$$y = \pm x^2\sqrt{x^2+C} - x^2$$

Answer