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Second mid-term examination on Ordinary Differential Equations

Task 1

1.1. **Problem**: Solve IVP: $y''y' = \frac{1}{2}$, $y(1) = 12$, $y'(1) = 12$ and prove uniqueness of the solution

1.2. To guarantee uniqueness of the solution it is enough to find the most general solution, and then solve IVP. Let's do it.

Remark: It is enough because the most general solution represents all possible solutions. So, by solving IVP we consider all possible solutions. If after taking into account the initial conditions we have only one equation, then it is the unique one.

- 2.1. $y''y' = \frac{1}{2}$ - **Second-order nonlinear ordinary differential equation**
  - 3.1. Let's use one of the methods for order reduction. Without loss of generality, let $y' = p$: the equation transforms into $p'p = \frac{1}{2}$
  - 3.2. $(p)p' = \frac{1}{2}$ - **Separable equation**
    - 4.1. Let's integrate both parts (this action can be carried out without loss of generality): $\int p dp = \int \frac{1}{2} dx$
    - 4.2. $\frac{p^2}{2} = \frac{1}{2}x + C_1$, $C_1 \in \mathbb{R}$
  - 3.3. $p = \pm\sqrt{x + C_1} = y'$
- 2.2. $y' = \pm\sqrt{x + C_1}$
- 2.3. Let's integrate both parts (this action can be carried out without loss of generality): $\int dy = \pm \int \sqrt{x + C_1} dx$
- 2.4. $y = \pm\frac{2}{3}(x + C_1)^{1.5} + C_2$, $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, $x \in [-C_1; +\infty)$
- 2.5. To define the range for x properly, we need to substitute the answer into initial equation: $y' = \pm\sqrt{x + C_1}$, $y'' = \pm\frac{1}{2\sqrt{x+C_1}}$, $y''y' = \frac{\sqrt{x+C_1}}{2\sqrt{x+C_1}} = \frac{1}{2}$. So, $x \neq -C_1$
- 2.6. **Answer**: $y = \pm\frac{2}{3}(x+C_1)^{1.5} + C_2$, $C_1 \in \mathbb{R}, C_2 \in \mathbb{R}$, is the most general solution on $x \in (-C_1; +\infty)$

  Remark: The solution is the most general because all transitions and all actions have been done without loss of generality. There is no reason to think that any solution is lost. My solution describes all possible solutions. So, this solution is the most general one.

1.3. $y'(1) > 0 \quad \Rightarrow \quad y' = \sqrt{x + C_1} \quad \Rightarrow \quad y = \frac{2}{3}(x + C_1)^{1.5} + C_2$

1.4. $y'(1) = 12 \quad \Rightarrow \quad \sqrt{1 + C_1} = 12 \quad \Rightarrow \quad C_1 = 143$

1.5. $y(1) = 12 \quad \Rightarrow \quad \frac{2}{3}(1 + 143)^{1.5} + C_2 = 12 \quad \Rightarrow \quad C_2 = 1140$

1.6. **Answer**: $y = \frac{2}{3}(x + 143)^{1.5} + 1140$, is the unique solution on $x \in (-143; +\infty)$

Task 2

I dare to say there is a big difference between explaining the solution and explaining how I came up with the solution. The task says: "The solution process must be explained." So, the explanation of how I came up with the solution is not the part of this work. To provide a task solution is the main point, I consider. Therefore, here you can find a detailed description of the process of problem solving. If you are really interested in the course of my thoughts, or it is really necessary, I am ready to explain the solution in person.

Probably, this solution looks like magic. But again, to solve this task I do not need to explain how I came up with the solution. Mathemagic!

Problem: Find the most general solution of $y'' + 12y' - 12y = 0$ and prove that it is indeed the most general one.

Remark: if we solve this equation using a classical method, then we need to believe that solutions are in form $y = e^{rx}$. Professor Shilov does not like it. So, let's use an exotic method to solve this equation.

Let's start from something simple.
$y'' + 12y' - 12y = 0$ - **Second-order linear homogenous ordinary differential equation**
We know that $y'' + 12y' - 12y = 0$. It means that $y'' + 12y' - 12y = 0$ and $y'' + 12y' - 12y = 0$. (0)
We know that $y = 0$ is a trivial solution (1). Now, let's assume that $y \not\equiv 0$. (2)

Let's do some boring maths (just school maths, nothing interesting):

$$(0) \quad \begin{cases} y'' + 12y' - 12y = 0 \\ y'' + 12y' - 12y = 0 \end{cases} \Rightarrow \begin{cases} -y'' = -12y + 12y' \\ y'' = 12y - 12y' \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} -y'' = -12y + (4\sqrt{3}+6)y' - (4\sqrt{3}-6)y' \\ y'' = 12y + (4\sqrt{3}-6)y' - (4\sqrt{3}+6)y' \end{cases} \Rightarrow \begin{cases} (4\sqrt{3}-6)y' - y'' = -12y + (4\sqrt{3}+6)y' \\ (4\sqrt{3}+6)y' + y'' = 12y + (4\sqrt{3}-6)y' \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} ((4\sqrt{3}-6)y - y')' = -(4\sqrt{3}+6) \cdot ((4\sqrt{3}-6)y - y') \\ ((4\sqrt{3}+6)y + y')' = (4\sqrt{3}-6) \cdot ((4\sqrt{3}+6)y + y') \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a' = -(4\sqrt{3}+6) \cdot a \\ b' = (4\sqrt{3}-6) \cdot b \end{cases}, \text{ where } \begin{cases} a(x) = (4\sqrt{3}-6)y - y' \\ b(x) = (4\sqrt{3}+6)y + y' \end{cases} \quad (3)$$

Notice that:

$$a + b = (4\sqrt{3} - 6)y - y' + (4\sqrt{3} + 6)y + y' = 8\sqrt{3}y \quad \Rightarrow \quad y = \frac{a+b}{8\sqrt{3}} \quad (4)$$

Let's consider all possible cases for a and b:

1. $a \equiv 0$:

$$(3) \quad \Rightarrow \quad (4\sqrt{3} - 6)y - y' = 0 \quad \overset{(2)}{\Longrightarrow} \quad \frac{y'}{y} = (4\sqrt{3} - 6) \quad \Rightarrow \quad \int \frac{dy}{y} = \int (4\sqrt{3} - 6)dx \quad \Rightarrow$$

$$\Rightarrow \quad y = Ce^{(4\sqrt{3}-6)x}, \text{ where } C \in \mathbb{R}\backslash\{0\} \quad (5)$$

2. $b \equiv 0$:

$$(3) \quad \Rightarrow \quad (4\sqrt{3} + 6)y + y' = 0 \quad \overset{(2)}{\Longrightarrow} \quad \frac{y'}{y} = -(4\sqrt{3} + 6) \quad \Rightarrow \quad \int \frac{dy}{y} = -\int (4\sqrt{3} + 6)dx \quad \Rightarrow$$

$$\Rightarrow \quad y = Ce^{-(4\sqrt{3}+6)x}, \text{ where } C \in \mathbb{R}\backslash\{0\} \quad (6)$$

3. $a \not\equiv 0$ and $b \not\equiv 0$:

$$(3) \quad \Rightarrow \quad \begin{cases} \dfrac{a'}{a} = -(4\sqrt{3} + 6) \\ \dfrac{b'}{b} = (4\sqrt{3} - 6) \end{cases} \quad \Rightarrow \quad \begin{cases} \displaystyle\int \frac{da}{a} = -\int (4\sqrt{3} + 6)dx \\ \displaystyle\int \frac{db}{b} = \int (4\sqrt{3} - 6)dx \end{cases} \quad \Rightarrow$$

$$\Rightarrow \quad \begin{cases} a = C_1 e^{-(4\sqrt{3}+6)x} \\ b = C_2 e^{(4\sqrt{3}-6)x} \end{cases}, \text{ where } C_1, C_2 \in \mathbb{R}\backslash\{0\} \quad \overset{(4)}{\Longrightarrow} \quad y = \frac{C_1 e^{-(4\sqrt{3}+6)x} + C_2 e^{(4\sqrt{3}-6)x}}{8\sqrt{3}} \quad (7)$$

So, $(1), (5), (6), (7) \quad \Rightarrow \quad y = C_1 e^{-(4\sqrt{3}+6)x} + C_2 e^{(4\sqrt{3}-6)x}, C_1, C_2 \in \mathbb{R}$

Answer: $y = C_1 e^{-(4\sqrt{3}+6)x} + C_2 e^{(4\sqrt{3}-6)x}, C_1, C_2 \in \mathbb{R}$ is **the most general** solution of given equation on $x \in \mathbb{R}$

Remark: the solution is the most general because I do not introduce any assumptions and do not do anything that can cause loss of solutions. So, there are no reasons to lose some solutions.