4.1 $xy' - 4y = x^2\sqrt{y}$

$y' - \frac{4y}{x} = x\sqrt{y}$ $\qquad y' + p(x)y = q(x)y^n$

This is Bernoulli equation

$z = \sqrt{y}$; $\qquad \frac{y'}{\sqrt{y}} - 4\sqrt{y} = x^2$

$z' = y' \cdot \frac{1}{2\sqrt{y}}$ $\qquad 2z' - 4z = x^2$

$2\frac{dz}{dx} = x^2 + 4z$; $\qquad e^{-2x}\frac{dz}{dx} - 2e^{-2x}z = \frac{1}{2}e^{-2x}x^2$

$e^{-2x}\frac{dz}{dx} + \frac{d(e^{-2x})}{dx}z = \frac{1}{2}e^{-2x}x^2$

$uv' + u'v = (uv)'$: $\frac{d(e^{-2x}z)}{dx} = \frac{1}{2}e^{-2x}x^2$

$\int d(e^{-2x}z) = \frac{1}{2}\int e^{-2x}x^2\,dx$

$e^{-2x}z = -\frac{e^{-2x}}{8}(2x^2+2x+1) + C$

$z = -\frac{1}{8}(2x^2+2x+1) + C = \sqrt{y}$

$y = \frac{1}{64}(2x^2+2x+1)^2 - \frac{C}{4}(2x^2+2x+1) + C^2$

Answer: $y = \frac{1}{64}(2x^2+2x+1)^2 - \frac{C}{4}(2x^2+2x+1) + C^2$

4.2. $7xy' - 2y = -\frac{x^2}{y^6}$

$y' - \frac{2}{7x}y = -\frac{x}{7y^6}$; $y' + p(x)y = q(x)y^n$

This is Bernoulli equation

$y'y^6 - \frac{2}{7x}y^7 = -\frac{x}{7}$, $z = y^7$, $z' = 7y^6y'$

$\frac{z'}{7} - \frac{2z}{7x} = -\frac{x}{7}$; $z' - \frac{2z}{x} = -x$

$\frac{z'}{x^2} - \frac{2z}{x^3} = -\frac{1}{x}$; $\frac{z'}{x^2} + \frac{d\left(\frac{1}{x^2}\right)}{dx} z = -\frac{1}{x}$

$uv' + u'v = (uv)'$, $\frac{d\left(\frac{z}{x^2}\right)}{dx} = -\frac{1}{x}$

$\int d\left(\frac{z}{x^2}\right) = -\int \frac{dx}{x}$, $\frac{z}{x^2} = -\ln|x| + C$

$\frac{z}{x^2} = -\ln|x| + C$ $\qquad \frac{y^7}{x^2} + \ln|x| = C$

Answer: $\frac{y^7}{x^2} + \ln|x| = C$

4.3. $y' = 1 + x - (1+2x)y + xy^2$, $y_1 = 1$

$y = y_1 + z = 1 + z$; $y' = z'$

$z' = 1 + x - (1+2x)(1+z) + x(z+1)^2$

$z' = \underline{1} + \underline{x} \underline{-1} - z \underline{-2x} \underline{-2xz} + xz^2 + \underline{x} + \underline{2xz}$

$z' = -z + xz^2$;

$z' + z = xz^2$; $z' + p(x)z = q(x)z^n$

This is Bernoulli equation

$\frac{z'}{z^2} + \frac{1}{z} = x$; $t = \frac{1}{z}$; $t' = -z' \cdot \frac{1}{z^2}$

$-t' + t = x$ $\qquad -\frac{dt}{dx} + t = x$

$e^{-x}\frac{dt}{dx} - e^{-x}t = -e^{-x}x$

$e^{-x}\frac{dt}{dx} + \frac{d(e^{-x})}{dx}t = -e^{-x}x$, $uv' + u'v = (uv)'$

$\frac{d(e^{-x}t)}{dx} = -e^{-x}x$ $\qquad \int d(e^{-x}t) = -\int e^{-x}x\,dx$

$e^{-x}t = e^{-x}(x+1) + C$; $e^{-x}(t - x - 1) = C$

$t = \underline{\frac{1}{y+1}}$; $y = -1$ - is not solution

Answer: $e^{-x}\left(\frac{1}{y+1} - x - 1\right) = C$