Introduction to

Differential Equations

Innopolis University, BS-II

Fall semester 2020-21

Lecturer: Nikolay Shilov

About the course

Part 0

Lecturer

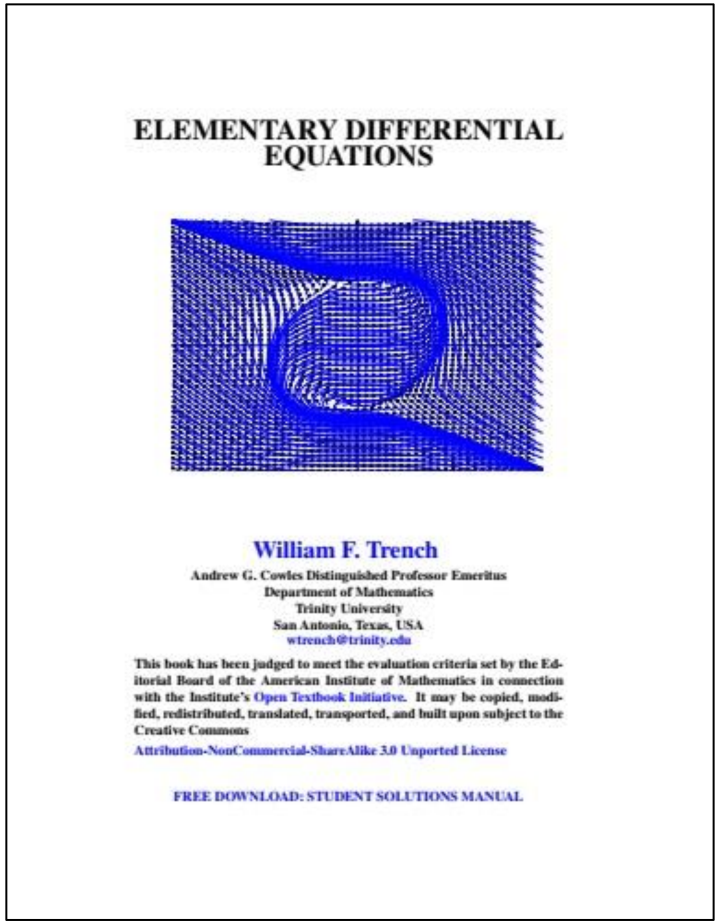
- Dr. Nikolay V. Shilov, Assistant Professor (office 507)
- Personal homepage: http://persons.iis.nsk.su/en/person/shilov
- Email (for IU-related staff): n.shilov@innopolis.ru
- Telegram (for urgent business or private non-business communication): @nick_shilov
- Skype (for online office hours or private non-business communication): nishilov

Official course links

- Moodle: https://moodle.innopolis.university/course/view.php?id=502;
    - Syllabus: https://moodle.innopolis.university/mod/resource/view.php?id=29351;
    - Advertisements: https://moodle.innopolis.university/mod/forum/view.php?id=28007;
    - Office hours mostly weekly: https://moodle.innopolis.university/mod/scheduler/view.php?id=40355;
    - Lectures mostly weekly from 9:00 to 10:30 (Moscow/Kazan) every Friday time from August 21 to November 27, 2020: https://moodle.innopolis.university/mod/forum/discuss.php?d=451.

Recommended textbook

- *Elementary Differential Equations* by William F. Trench. Brooks/Cole Thomson Learning, 2001.
- Available at http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_FREE_DIFFEQ_I.PDF.



ELEMENTARY DIFFERENTIAL EQUATIONS

William F. Trench
Andrew G. Cowles Distinguished Professor Emeritus
Department of Mathematics
Trinity University
San Antonio, Texas, USA
wtrench@trinity.edu

This book has been judged to meet the evaluation criteria set by the Editorial Board of the American Institute of Mathematics in connection with the Institute's Open Textbook Initiative. It may be copied, modified, redistributed, translated, transported, and built upon subject to the Creative Commons
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FREE DOWNLOAD: STUDENT SOLUTIONS MANUAL

Reference materials

- *Introduction to Differential Equations* (Lecture notes, The Hong Kong University of Science and Technology) by Jeffrey R. Chasnov. (Available at https://www.math.ust.hk/~machas/differential-equations.pdf.)
- *Differential Equations-I* (Lecture notes, University of Toronto) by Paul Selick. (Available at http://www.math.toronto.edu/selick/B44.pdf.)
- *Calculus I,II,III*, by J.E. Marsden and A. Weinstein (Springer-Verlag, 1985, is available at http://escholarship.org/uc/search?keyword=marsden+weinstein+calculus.)

Grading items

- In-class participation (for practice/lab classes): up to 1 point per week (i.e. 14 points in total).
- Each weekly practice quiz: up to 2 points (i.e. 24 points in total for 12 weeks with practice).
- Each mid-term exam and the final exam: up to 20 points (i.e. 60 points in total).
- Computational practicum assignment: up to 15 points for each task (i.e. up to 45 points for 3 tasks).
- Overall practice class contribution (to accumulate regular practice class activities beyond weekly in-class participation): up to 4 points.

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- Overall score: 147 points (100%).

Opportunities for Bonus Points

- For lecture/tutorial in-class participation: up to 1 point per week (to engage students) for each individual contribution in a (14 points in total).
- Up to 6 points for extra-curricular activities (like achievements in the course contest for volunteers, for approved short presentations in lecture and tutorial classes, etc.).

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- Up to 20 bonus points in total (to be added to individual student score)

Final grade scale

- A: 117..147 points (at least 80% of the overall score);
- B: 88..116 points (at least 60% of the overall score);
- C: 73..87 points (at least 50% of the overall score);
- D: less than 73 points (less than 50% of the overall score).

Examination/live grading policy

- All tests and examinations are open-books, students may bring and use their own hand-written conspectus and printed books.
- Live-grading for computational practicum assumes that students may use third-party code but must write their own short reports (2-3 pages) and be able to explain individually “what is what” in the code and how it relates to the implemented computational method.
- In case of suspected cheating all involved students should explain individually problematic solution/submissions to the instructor.
- Students who do not take an assignment without legal excuse (e.g., documented medical) may try later but not later than one week after the scheduled date with 30% deduction.

Project-32 on weeks 10-11 (?) same time as the computational Practicum

https://github.com/br4ch1st0chr0n3/Project-32

- Description:
  - This project's purpose is to estimate the number of people who would like to use Wolfram Mathematica during their studies at the Innopolis University.
  - You can participate iff you are an Innopolis student and want to get a Mathematica|Desktop Student license.
  - As the outcome, we expect to create a community of students who know how to apply Mathematica in real studies and a set of solutions on which one can study Mathematica capabilities.

Project-32 on weeks 10-11 (?) same time as the computational Practicum

https://github.com/br4ch1st0chr0n3/Project-32

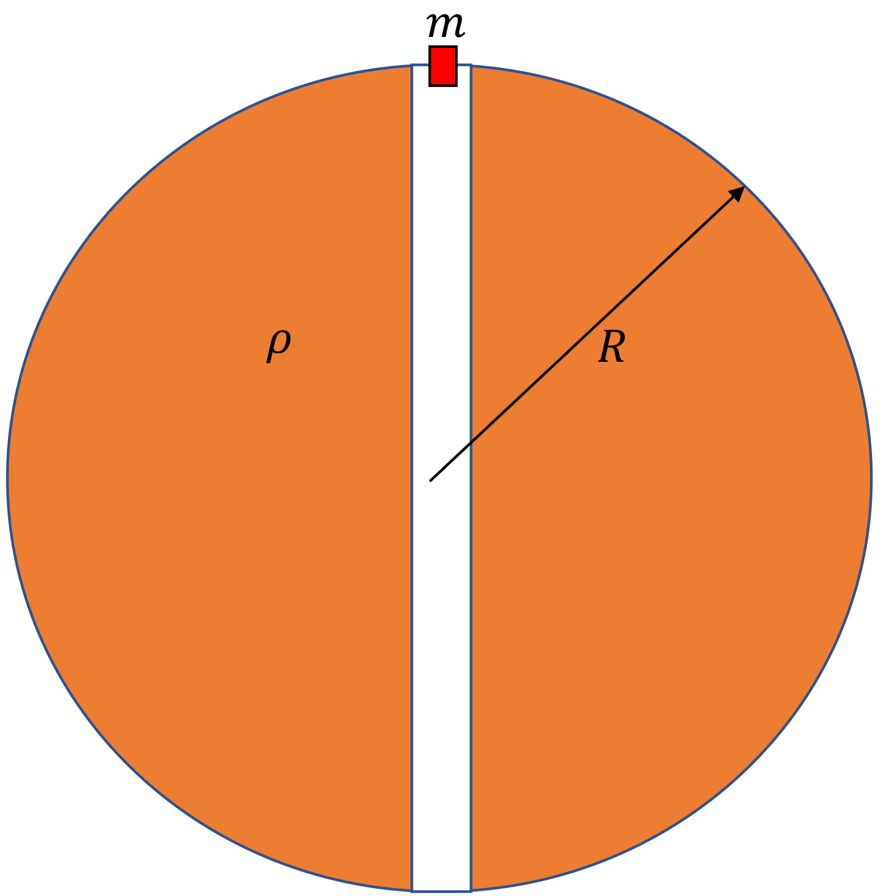
- Submission
  - For implementation of your project, you may use Mathematica's Free Trial version
  - You submit the projects written in Mathematica notebooks(.nb extension) to this repository via pull requests. All your projects will be stored in separate folders whose names should be relevant to your project.
  - You can submit one work only (?)

Applications leading to differential equations

Part I

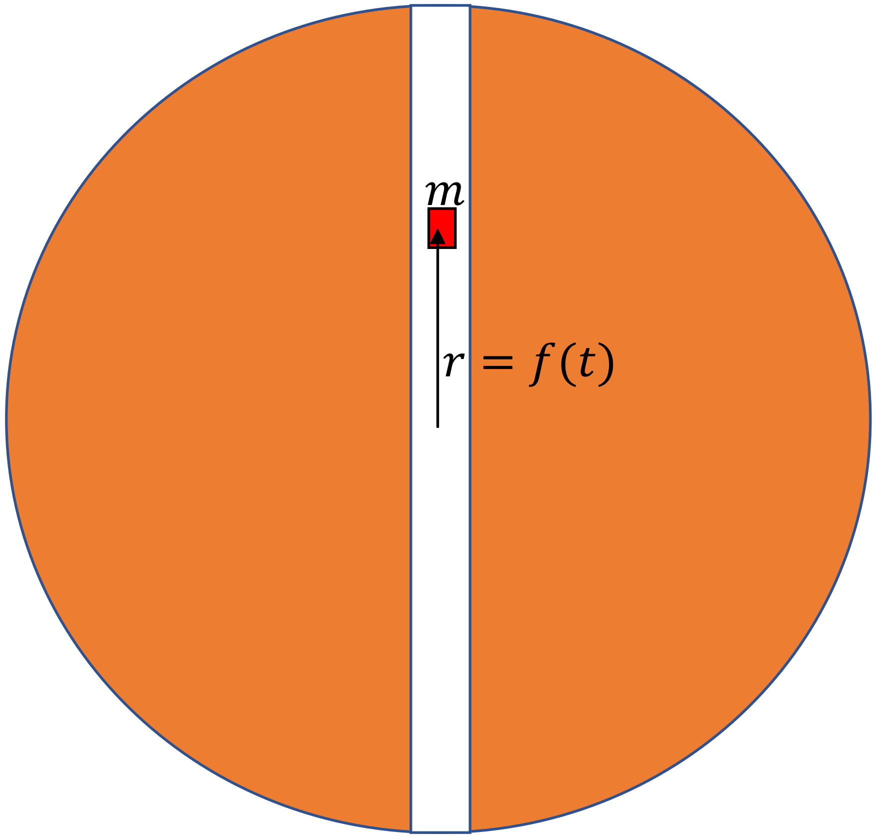
Problem: Journey to the Center of the Earth

- A homogenous ball-shaped planet (with density ρ and radius R) has a straight end-to-end tunnel goes through the planet center.
- A team of travelers plan an expedition through the tunnel on a shell (with total mass altogether with the crew m).



Problem (cont.)

- The initial position of the shell is on the level of the planet surface, the initial speed is zero.
- What is location of the shell at time t after expedition start? – It means that we need to write an explicit formula $r = f(t)$ where $f: [0, \infty) \to [-R, R]$ is a known function.



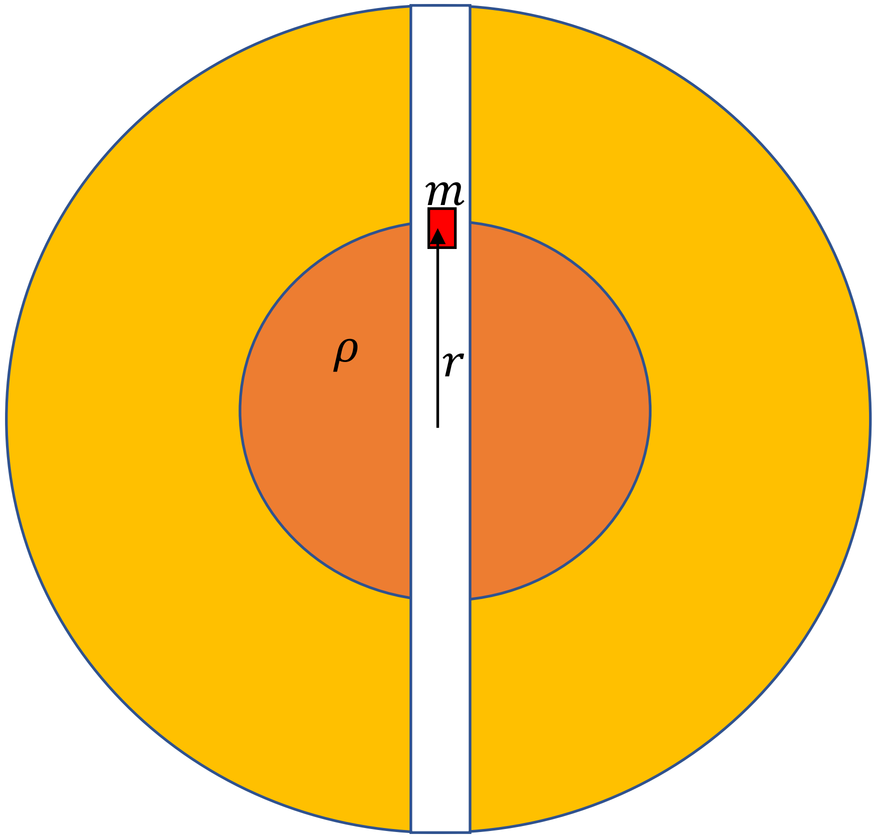
Force applied to the shell

- According to the Newton's law

$$F = ma = mv' = mr''$$

where

- a is the instant acceleration,
- v *is* the instant speed (velocity),
- r is the instant location,
- $'$ is the derivative operator.



Force on distance r from the planet center

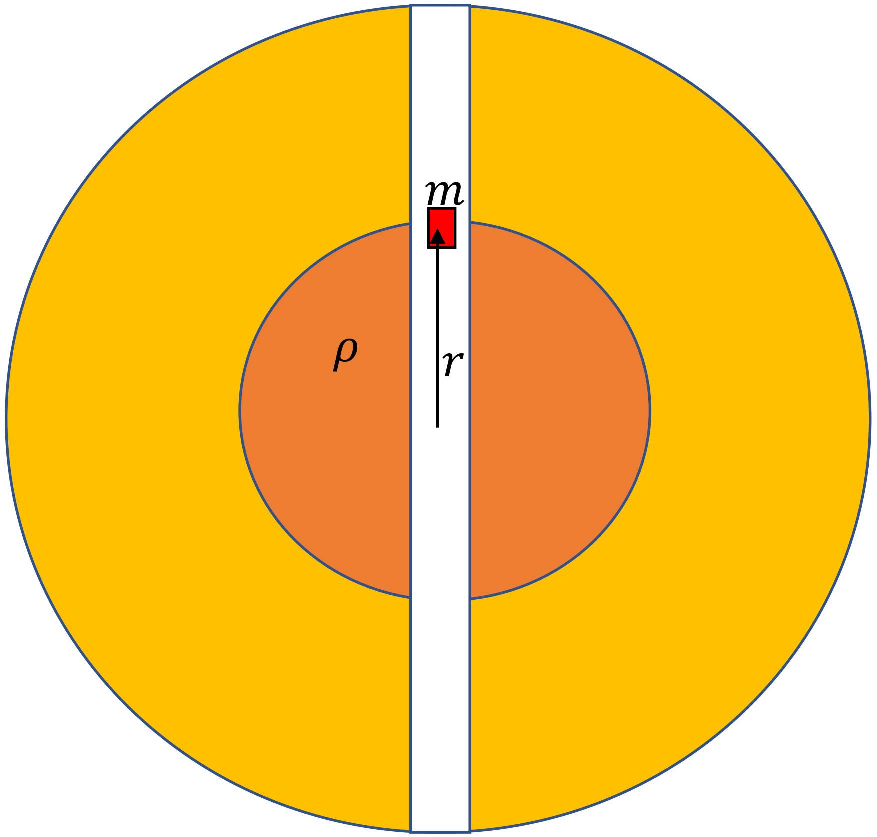
- $F = -G\dfrac{m(\rho V_r)}{r^2} = -G\dfrac{m\left(\rho\frac{4\pi r^3}{3}\right)}{r^2} =$

$$= -G\frac{4\pi\rho m}{3}r,$$

  where G is the gravity constant.

- Question: Why $F = -G\dfrac{m(\rho V_r)}{r^2}$?

  (In other words: why external part of the ball w.r.t. to inside ball of radius r doesn't contribute to F?)

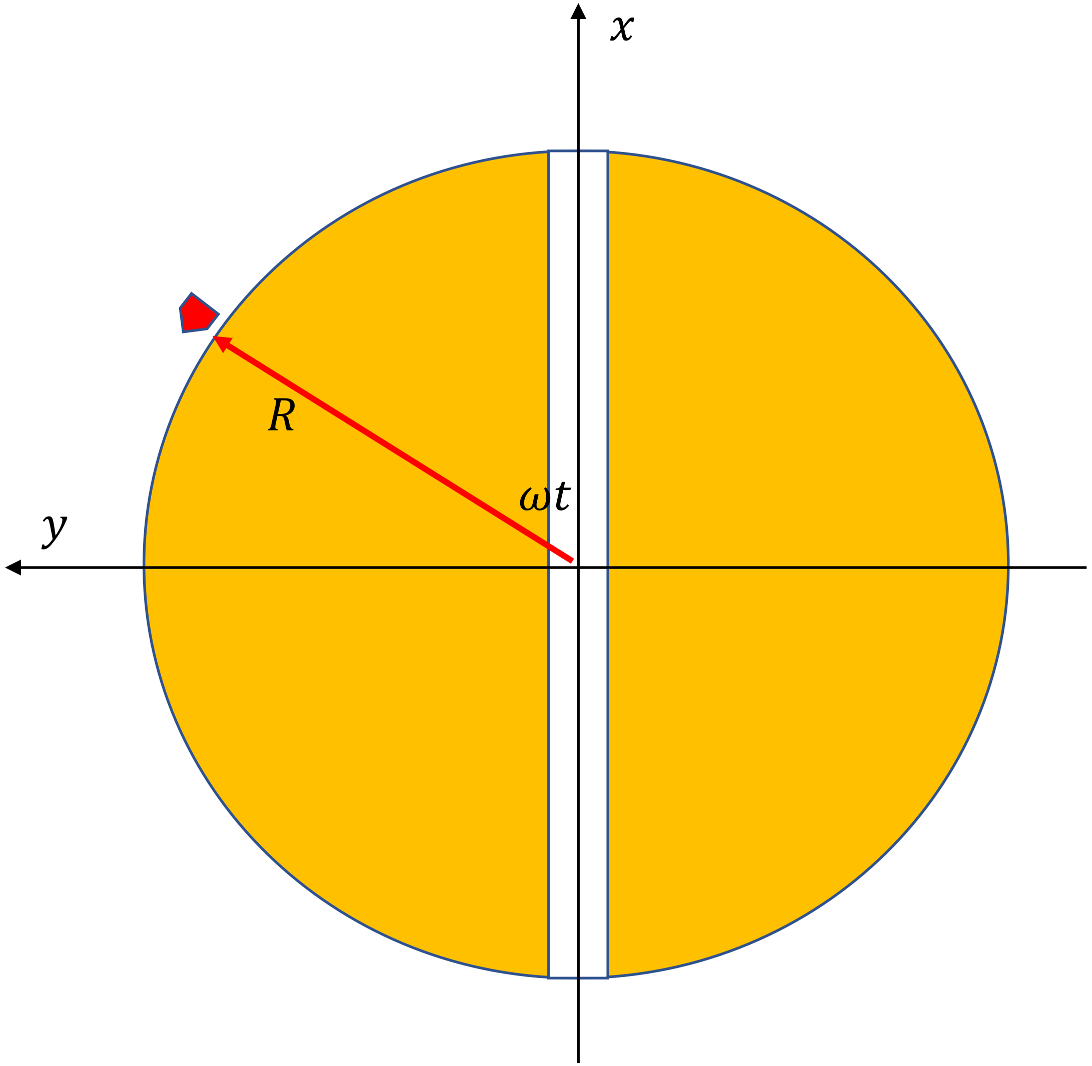


Differential equation for the shell motion

- Hence $mr'' = -G\,\frac{4\pi\rho m}{3}r,$
- or $3r'' = -4G\pi\rho\, r,$
- or $3r'' + 4G\pi\rho\, r = 0.$

How to solve the equation

- Let us send another shell on the circular orbit just above the planet surface. Let ω be the angular speed of this circular motion.
- The polar angle of the shell at time t is ωt, the centripetal acceleration of the shell is $\omega^2 R$.

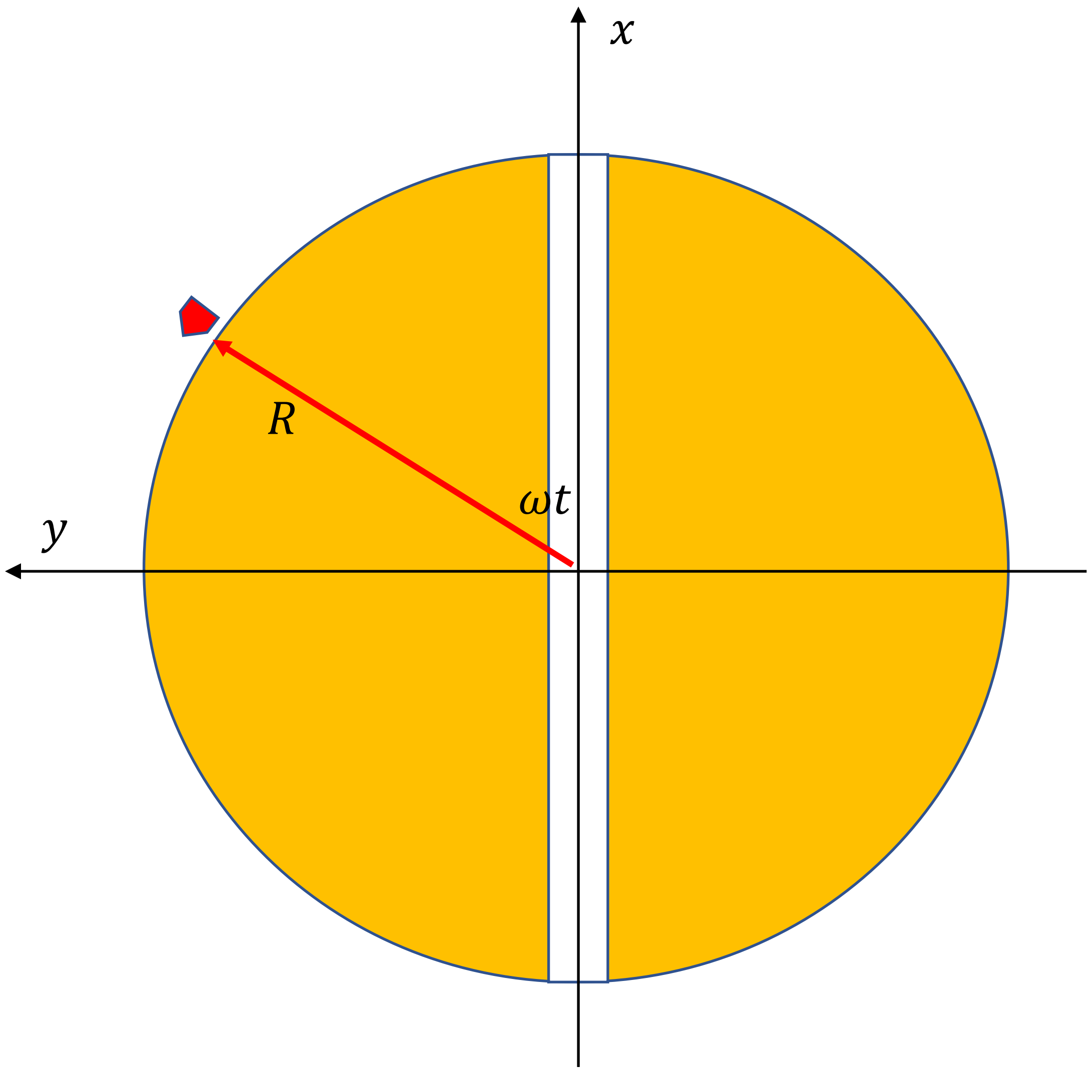


How to solve the equation (cont.)

- Since the orbiter shell moves around due to the gravity then its acceleration is

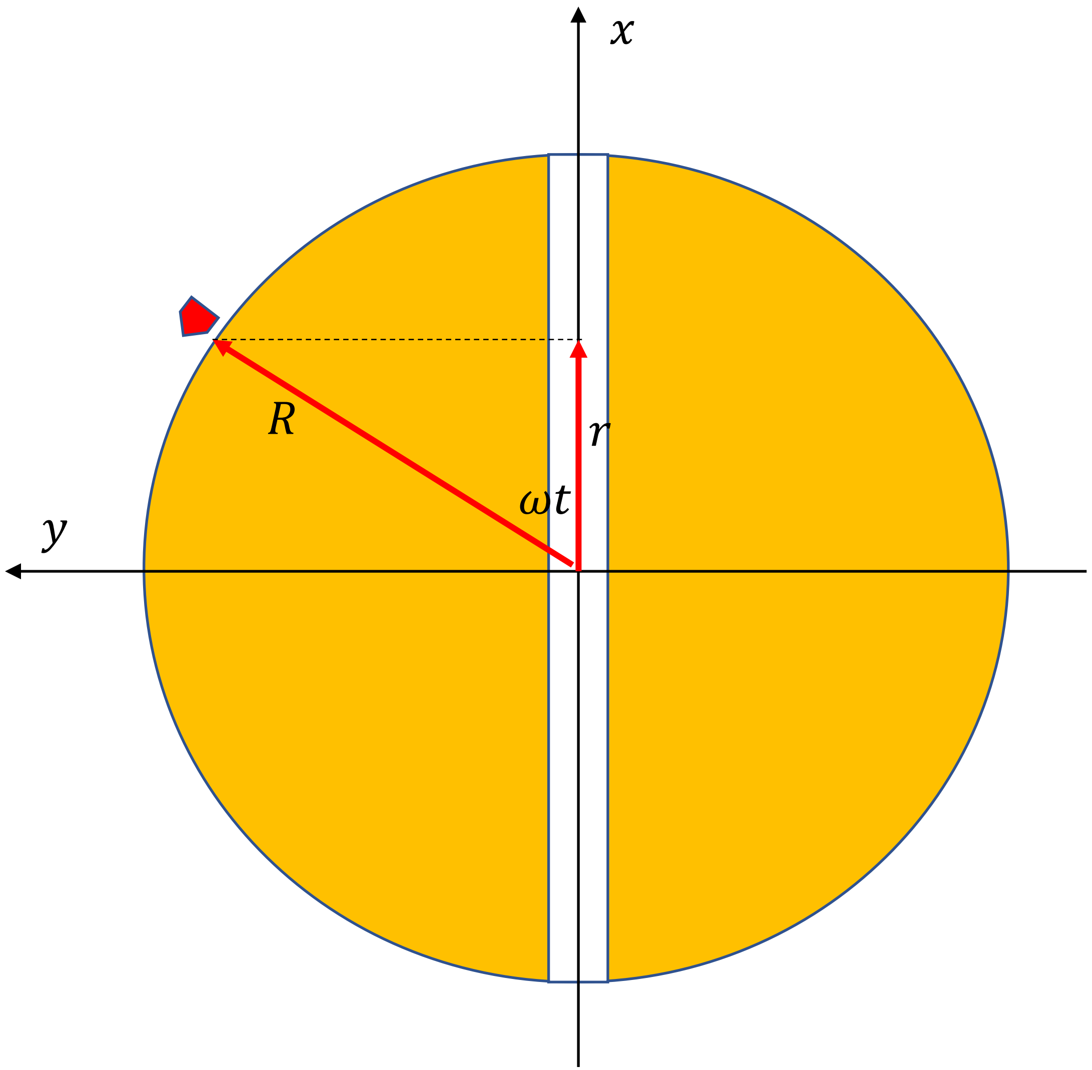
$$G\,\frac{(\rho V_R)}{R^2} = G\,\frac{\left(\rho\frac{4\pi R^3}{3}\right)}{R^2} = G\,\frac{4\pi\rho}{3}R.$$

- Hence $\omega = \sqrt{{}^{4G\pi\rho}\!/_{3}}$.



How to solve the equation (cont.)

- Let us project the motion and acceleration of the orbiter on the x-axis:
  - $R\cos\omega t = r$,
  - $(G\frac{4\pi\rho}{3}R)\cos\omega t = G\frac{4\pi\rho}{3}r$.
- Hence $r'' = a_x = -G\frac{4\pi\rho}{3}r$, i.e. the projection of the motion satisfies the same differential equation as the motion in the tunnel: $3r'' + 4G\pi\rho\, r = 0$.



Motion equation for the shell in the tunnel

- Hence the solution of the differential equation $3r'' + 4G\pi\rho\, r = 0$ is $r = k\cos\left(\sqrt{{4G\pi\rho}/{3}}\, t\right)$ where k is a constant;
- since the initial conditions of the problem are
  - $r(0) = R$ and
  - $v(0) = r'(0) = 0$,

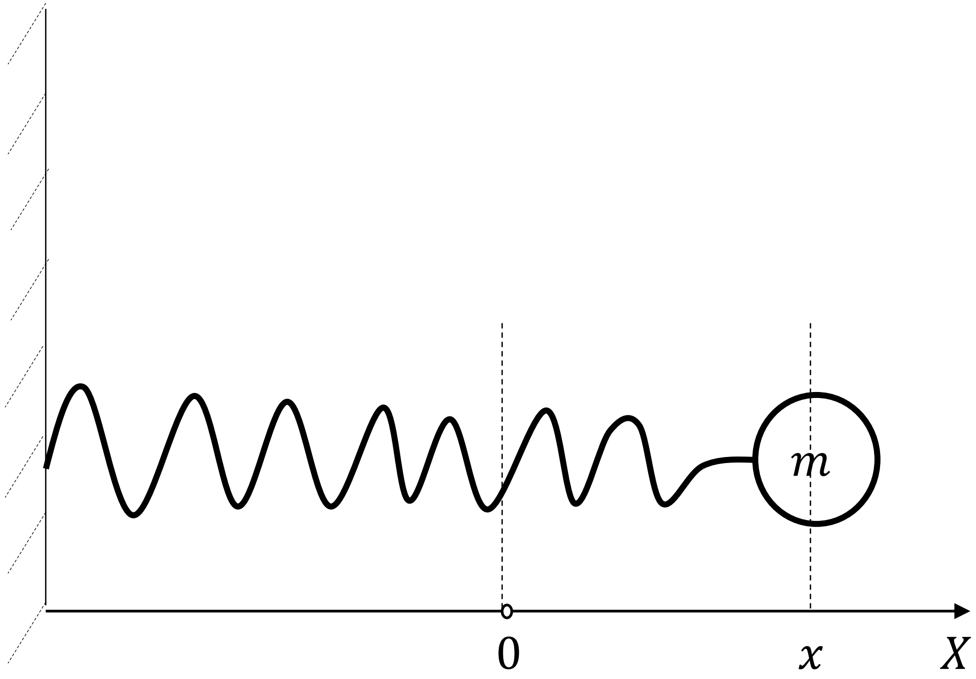
  then $r = R\cos\left(\sqrt{{4G\pi\rho}/{3}}\, t\right)$.

Thanks to my teachers…

I like the above physical solution very much… – Many thanks for introducing the solution to Prof. Pavel Ivanovich Zubkov (born in 1940) who taught me Physics at school in 1977-78.

Exercise

- Using 2-nd Newton's law and Hook's law for a spring with fixed stiffness k write the motion equation for a weight m; consider the equation as a differential one and solve it (use similarity with a shell in a tunnel through a homogenous planet) for initial spring deformation X_0 and weight's speed $v_0 = 0$. (Neglect gravity, friction, and the spring mass.)



Ordinary differential equations: basic definitions

Part II

General definitions

- An *ordinary differential equation* (o.d.e., ODE) of order $n \geq 0$ is equation
$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$
where
    - F is a known real function of (at most) $(n+2)$- real arguments,
    - x is a real variable (ranging over some “region” $D \subseteq \boldsymbol{R}$),
    - y is variable for a real n-times differentiable function on D.
- Question: What is “region”?

General definitions (cont.)

- A solution of the equation is any real n-times differentiable function $y: D \to \boldsymbol{R}$ that makes the equation a valid equality.
- To solve (or integrate) the equation means to find all solutions of the equation; if the range isn't specified explicitly with the equation, then each solution must specify its domain (the region) explicitly.
- Solution curve is a graph (chart) of a solution of the equation; integral curve is such a graph that any function that is part of this graph is a partial solution automatically.

General definitions (cont.)

- A general solution of the equation is representation of a family of solutions in the form $y = \varphi(x, C_1, \dots C_n)$ where
  - $C_1, \dots C_n$ are real parameters with specified range (for example: $C_1 \geq 0, \dots C_n \in \{1,2,3,4\}$, etc.)
  - φ is n-times differentiable (w.r.t. x) known function (for all combinations of parameters values).
- A partial solution results from a general by instantiating concrete values for real parameter.
- The (most) general solution of the equation is a general solution such that any solution of the equation can be represented in this form (for appropriate parameter values).

Linear differential equations

- If the function F is linear (affine) on $y, y', y'', \dots, y^{(n)}$ then the equation is said to be linear differential equation:

$$a_0(x)y^{(n)} + \ \dots + a_n(x)y = f(x);$$

- Linear equation is said to be homogeneous if the right-hand-side function is identically 0.

Exercises

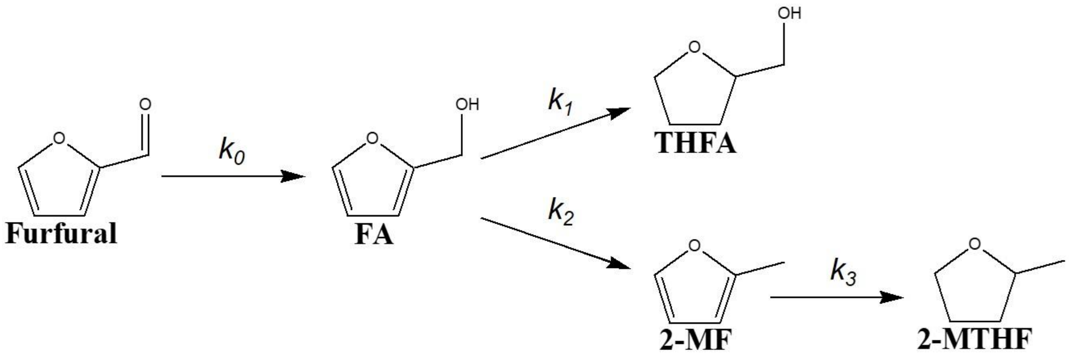
Let us consider a linear homogeneous equation $y'' + y = 0$ on real numbers.

- Check that
  - $y = C_1 \sin x + C_2 \cos x$ is a general solution of the equation (where C_1 and C_2 are real parameters);
  - give examples of partial solutions of the equation.
- Give (if possible) an example of a solution that isn't a partial solution instantiated from the above general one.

Example from Chemistry

- Shilov I.N., Smirnov A.A., Bulavchenko O.A. and Yakovlev V.A. Effect of Ni–Mo Carbide Catalyst Formation on Furfural Hydrogenation. Catalysts 2018, 8(11), 560; https://doi.org/10.3390/catal8110560

- *Based on the reaction products distribution, furfural hydrogenation scheme was proposed for catalysts $Ni_xMoC\text{-}SiO_2(400/600)$.*



Example from Chemistry – cont.

This scheme includes the route of furfural hydrogenation to FA, which can be converted to 2-MF or THFA. The aromatic ring of 2-MF can be hydrogenated to form 2-MTHF. All kinetic constants k_0-k_3 were defined according to the proposed scheme and the reaction products. Since the hydrogen pressure did not change during the experiment, the dependence of the reaction rates on the hydrogen pressure was taken into account in the rate constants. For calculation, a system of linear differential equations was compiled according to the above scheme.

Example from Chemistry – cont.

$$
\begin{cases}
\dfrac{dC_0}{dt} = -k_0 C_0 \\
\dfrac{dC_1}{dt} = k_0 C_0 - (k_1 + k_2) C_1 \\
\dfrac{dC_2}{dt} = k_1 C_1 \\
\dfrac{dC_3}{dt} = k_2 C_1 - k_3 C_3 \\
\dfrac{dC_4}{dt} = k_3 C_3
\end{cases}
$$

- C_0 – *furfural*
- C_1 – *FA*
- C_2 – *THFA*
- C_3 – *2-MF*
- C_4 – *2-MTHF*

Example from Chemistry – cont.

An analytical solution of the equations system was found:

$$\begin{cases} C_0 = e^{-k_0 t} \\ C_1 = k_0 \dfrac{e^{-k_0 t} - e^{-(k_1+k_2)t}}{k_1 + k_2 - k_0} \\ C_2 = \dfrac{k_0 k_1}{k_1 + k_2 - k_0}\left(\dfrac{1 - e^{-k_0 t}}{k_0} - \dfrac{1 - e^{-(k_1+k_2)t}}{k_1 + k_2}\right) \\ C_3 = \dfrac{k_0 k_2}{k_1 + k_2 - k_0}\left(\dfrac{e^{-k_0 t} - e^{-k_3 t}}{k_3 - k_0} - \dfrac{e^{-(k_1+k_2)t} - e^{-k_3 t}}{k_3 - (k_1+k_2)}\right) \\ C_4 = 1 - C_0 - C_1 - C_2 - C_3 \end{cases}$$

Exercises

- Validate (check) that functions listed on slide 34 are (partial) solutions of the system from slide 33.
- Explain how it is possible to find this partial solution and characterize differential equations that appear in the process in terms of order, homogeneity, linearity, etc.