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WHAT IS OPERATIONS RESEARCH ?

1.1 INTRODUCTION : THE HISTORICAL DEVELOPMENT

In order to understand 'what Operations Research (OR)* is today,' we must know something of its history and evolution. The main origin of Operations Research was during the Second World–War. At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air and land defence of the country. Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, *e.g.* the efficient ocean transport, effective bombing, etc.

During World-War II, the Military Commands of U.K. and U.S.A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals and plans for aiding the Military Commands to arrive at the decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively. The OR teams were not actually engaged in military operations and in fighting the war. But, they were only advisors and significantly instrumental in winning the war to the extent that the scientific and systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence OR can be associated with ***"an art of winning the war without actually fighting it"***.

As the name implies, 'Operations Research' (sometimes abbreviated OR) was apparently invented because the team was dealing with *research* on (military) *operations*. The work of this team of scientists was named as *Operational Research* in England.

The encouraging results obtained by the British OR teams quickly motivated the United States military management to start with similar activities. Successful applications of the U.S. teams included the invention of new fight patterns, planning sea mining and effective utilization of electronic equipment. The work of OR team was given various names in the United States : ***Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation, Systems Research and Management Science.*** The name Operations Research was and is the most widely used so we shall also use it here.

Following the end of war, the success of military teams attracted the attention of *Industrial* managers who were seeking solutions to their complex executive-type problems. The most common problem was : what methods should be adopted so that the total cost is minimum or total profits maximum ? The first mathematical technique in this field (called the *Simplex Method* of linear programming) was developed in 1947 by American mathematician, **George B. Dantzig**. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in academic institutions and industry both.

Today, the impact of OR can be felt in many areas. A large number of management consultimg firms are currently engaged in OR activities. Apart from military and business applications, the OR activities include transportation system, libraries, hospitals, city planning, financial institutions, etc. Many of the Indian industries making use of OR activity are : ***Delhi Cloth Mills, Indian Railways, Indian Airlines, Defence Organizations, Hindustan Lever, Tata Iron & Steel Co., Fertilizer Corporation of India***, etc.

In business and other organizations, OR scientists and specialists always remain enagaged in the background. But, they help the top management officials and other line managers in doing their 'fighting' job better.

* The short word **'OR'** for **'Operations Research'** should not be confused with the word **'or'** throughout the book.

While making use of the techniques of OR, a mathematical model of the problem is formulated. This model is actually a simplified representation of the problems in which only the most important features are considered for reasons of simplicity. Then, an *optimal* or *most favourable* solution is found. Since the model is an idealized form of exact representation of real problem, the optimal solution thus obtained may not prove to be the best solution to the actual problem. Although, extremely accurate but highly complex mathematical models can be developed, but they may not be easily solvable. So from both the cost-minimising and mathematical simplicity point of view, it seems beneficial to develop a less accurate but simpler model, and to find a sequence of solutions consisting of a series of increasingly better approximations to the actual course of action. Thus, the apparent weaknesses in the initial solution are used to suggest improvements in the model, its input-data, and the solution procedure. A new solution is thus obtained and the process is repeated until the further improvements in the succeeding solutions become so small that it does not seem economical to make further improvements.

If the model is carefully formulated and tested, the resulting solution should reach to be good approximation to the ideal course of action for the real problem. Although, we may not get the best answers, but definitely we are able to find *the bad answers where worse exist.* Thus operations research techniques are always able to save us from worse situations of practical life.

Q. 1. Comment the following statements : **[Rewa (Maths.) 93]**
(i) O.R. is the art of winning war without actually fighting it.
(ii) O.R. is the art of finding bad answers where worse exist.

2. What is O.R. ? **[Garhwal 97, 96; Meerut (IPM) 90]**

3. Enumerate six applications of Operations Research (O.R.) and describe one briefly. **[IGNOU 2001 (June)]**

1.2 THE NATURE AND MEANING OF 'OR'

[IPM (PGDBA)* 82, 81; Meerut (Math.) 82]

'OR' has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. So it is important and interesting to give below a few opinions about the definition of OR which have been changed according to the development of the subject.

1. OR is a scientific method of providing executive departments with a *quantitative basis* for *decision* regarding the operations under their control. ***—Morse and Kimbal (1946)***
2. OR is a scientific method of providing executive with an *analytical* and *objective* basis for decisions. ***—P.M.S. Blackett (1948)***
3. The term 'OR' has hitherto-fore been used to connate various attempts to study operations of war by scientific methods. From a more general point of view, OR can be considered to be an attempt to study those *operations of modern society which involved organizations of men or of men and machines.* ***—P.M. Morse (1948)***
4. OR is the application of *scientific methods, techniques* and *tools* to problems involving the *operations of systems* so as to provide these in control of the operations with *optimum solutions* to the problem. ***—Churchman, Acoff, Arnoff (1957)***
5. OR is the art of giving *bad answers* to problems to which otherwise *worse answers* are given. ***—T. L. Saaty (1958)***
6. OR is a management *activity* pursued in two complementary ways—one half by the free and bold *exercise of commonsense* untrammelled by any routine, and other half by the application of a repertoire of *well established precreated methods* and techniques. ***—Jagjit Singh (1968)***
7. OR is the *attack* of modern methods on *complex problems* arising in the *direction and management* of large systems of men, machines, materials, and money in industry, business and defence. The distinctive approach is to developed a *scientific model* of the system, incorporating measurements of factors such as

* Wherever the name of the examination is not mentioned in the University Examination references, it should be understood M.A./M.Sc. throughout the book.
* IPM = Institute of Productivity Management. *PGDBA* = Post-Graduate Diploma in Business Administration.
* The symbol Q. will stand for '**EXAMINATION QUESTIONS**' throughout the book.

chance and risk with which to predict and *compare* the outcomes of alternative *decisions, strategies* or *controls*. The purpose is to help management to determine its policy and actions scientifically.

—***Operations Research Quarterly*** **(1971)**

8. Operations Research is the art of winning war without actually fighting it.
9. OR is an applied decision theory. It uses any *scientific mathematical or logical means* to attempt to cope with the problems that confront the executive when he tries to achieve a through going rationality in dealing with his decision problems. —***Miller and Starr.***
10. OR is a scientific approach to problem solving for executive management. —***H.M. Wagner***
11. OR is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. —***C. Kittel***
12. OR is the systematic method oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making. —***E.L. Arnoff & M.J. Netzorg***
13. OR is the application of *scientific methods* to problems arising from operations involving *integrated* systems of *men, machines and materials.* It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions. —***Fabrycky and Torgersen***
14. OR is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man–machine systems and OR workers are actively engaged in applying this knowledge to practical problems in business, government, and society. —***OR Society of America***
15. OR is the application of scientific method by inter-disciplinary teams to problems involving the controls of organized (man-machine) systems so as to provide solutions which *best serve the purpose of the organization as a whole.* —***Ackoff & Sasieni, (1968)***
16. OR utilizes the planned approach (*updated scientific method*) and an *inter-disciplinary* team in order to represent complex functional relationships as mathematical models for purpose of providing a *quantitative basis* for decision making and *uncovering new problems* for quantitative analysis. —***Thieanf and Klekamp*** **(1975)**

Comments on Definitions of OR :

From all above opinions, we arrive at the conclusion that whatever else 'OR' may be, it is certainly concerned with optimization problems. ***A decision, which taking into account all the present circumstances can be considered the best one, is called an optimal decision.*** **(Note)**

There are three main reasons for why most of the definitions of Operations Research are not satisfactory.

(i) First of all, Operations Research is not a science like any well-defined physical, biological, social phenomena. While chemists know about atoms and molecules and have theories about their interactions; and biologists know about living organisms and have theories about vital processes, *operations researchers* do not claim to know or have theories about operations. Operations Research is not a scientific research into the control of operations. It is essentially a collection of mathematical techniques and tools which in conjunction with a system approach are applied to solve practical decision problems of an *economic* or *engineering* nature. Thus it is very difficult to define Operations Research precisely.

(ii) Operations Research is inherently inter-disciplinary in nature with applications not only in military and business but also in medicine, engineering, physics and so on. Operations Research makes use of experience and expertise of people from different disciplines for developing new methods and procedures. Thus, inter-disciplinary approach is an important characteristic of Operations Research which is not included in most of its definitions. Hence most of the definitions are not satisfactory.

(iii) Most of the definitions of Operations Research have been offered at different times of development of 'OR' and hence are bound to emphasise its only one or the other aspect.

For example, 8th of the above definitions is only concerned with war alone. First definition confines 'OR' to be a scientific methodology applied for making operational decisions. It has no concern about the characteristics of different operational decisions and has not described how the scientific methods are applied in complicated situations. Many more definitions have been given by various authors but most of them fail to consider all basic characteristics of 'OR'. However, with further development of 'OR' perhaps more precise definitions should be forthcoming.

Q. 1. (a) Give any three definitions of Operations Research and explain them.
[J.N.T.U. (B. Tech.) 2003; Meerut (IPM) 91; Meerut (O.R.) 90]
(b) Give reasons for : why most of the definitions of Operations Research are not satisfactory.
2. Discuss the three approaches of MIS development. [CA (May) 2000]
3. What are the pre-requisites of a computer based MIS ? [MCI 2000]

1.3 MANAGEMENT APPLICATIONS OF OPERATIONS RESEARCH

Some of the areas of management decision making, where the 'tools' and 'techniques' of O R are applied, can be outlined as follows :

1. **Finance–Budgeting and Investments**
   (i) Cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.
   (ii) Credit policies, credit risks and delinquent account procedures.
   (iii) Claim and complaint procedures.
2. **Purchasing, Procurement and Exploration**
   (i) Rules for buying, supplies and stable or varying prices.
   (ii) Determination of quantities and timing of purchases.
   (iii) Bidding policies.
   (iv) Strategies for exploration and exploitation of raw material sources.
   (v) Replacement policies.
3. **Production Management**
   (i) ***Physical Distribution***
      (a) Location and size of warehouses, distribution centres and retail outlets.
      (b) Distribution policy.
   (ii) ***Facilities Planning***
      (a) Numbers and location of factories, warehouses, hospitals etc.
      (b) Loading and unloading facilities for railroads and trucks determining the transport schedule.
   (iii) ***Manufacturing***
      (a) Production scheduling and sequencing.
      (b) Stabilization of production and employment training, layoffs and optimum product mix.
   (iv) ***Maintenance and Project Scheduling***
      (a) Maintenance policies and preventive maintenance.
      (b) Maintenance crew sizes.
      (c) Project scheduling and allocation of resources.
4. **Marketing**
   (i) Product selection, timing, competitive actions.
   (ii) Number of salesman, frequency of calling on accounts per cent of time spent on prospects.
   (iii) Advertising media with respect to cost and time.
5. **Personnel Management**
   (i) Selection of suitable personnel on minimum salary.
   (ii) Mixes of age and skills.
   (iii) Recruitment policies and assignment of jobs.
6. **Research and Development**
   (i) Determination of the areas of concentration of research and development.
   (ii) Project selection.
   (iii) Determination of time cost trade-off and control of development projects.
   (iv) Reliability and alternative design.

From all above areas of applications, we may conclude that OR can be widely used in taking timely management decisions and also used as a corrective measure. The application of this tool involves certain data and not merely a personality of decision maker, and hence we can say : ***OR has replaced management by personality.***

Q. 1. "Operations Research replaces Management by personality." Discuss.
2. Explain applications of O.R. in Industry. **[Garhwal 97; Karnataka 95]**
3. Describe the various approaches used for development of MIS. **[MCI 2000]**

1.4 MODELLING IN OPERATIONS RESEARCH

Definition. *A model in the sense used in O R is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.*

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analysing the behaviour of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to 'examine the behaviour of a system without interfering with ongoing operations.

Models can be classified according to following characteristics :

1. Classification by Structure

(i) *Iconic models.* *Iconic models* represent the system as it is by scaling it up or down (*i.e.*, by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model of a real one. Other common examples of it are : photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct.

The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

(ii) *Analogue models.* The models, in which one set of properties is used to represent another set of properties, are called *analogue models.* After the problem is solved, the solution is reinterpreted in terms of the original system.

For example, graphs are very simple analogues because distance is used to represent the properties such as : *time, number, per cent, age, weight,* and many other properties. Contour-lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

(iii) *Symbolic (Mathematical) models.* The *symbolic or mathematical model* is one which employs a set of mathematical symbols (*i.e., letters, numbers,* etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. Classification by Purpose

Models can also be classified by purpose of its utility. *The purpose of a model may be* descriptive, predictive or prescriptive.

(i) *Descriptive models.* A descriptive model simply describe some aspects of a situation based on observations, survey, questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

(ii) *Predictive models.* Such models can answer 'what if' type of questions, *i.e.* they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

(iii) *Prescriptive models.* Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types :

(i) ***Deterministic models.*** Such models assume conditions of complete certainty and perfect knowledge. For example, linear programming, transportation and assignment models are deterministic type of models.

(ii) ***Probabilistic (or Stochastic) models.*** These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behaviour

(i) ***Static models.*** These models do not consider the impact of changes that takes place during the planning horizon, *i.e.* they are independent of time. Also, in a static model only one decision is needed for the duration of a given time period.

(ii) ***Dynamic models.*** In these models, *time* is considered as one of the important variables and admit the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent decisions is required during the planning horizon.

5. Classification by Method of Solution

(i) ***Analytical models.*** These models have a specific mathematical structure and thus can be solved by known analytical or mathematical techniques. For example, a *general linear programming* model as well as the specially structured *transportation* and *assignment models are analytical models.*

(ii) ***Simulation models.*** They also have a mathematical structure but *they cannot be solved* by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modelling has the advantage of being more flexible than mathematical modelling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. Classification by Use of Digital Computers

The development of the digital computer has led to the introduction of the following types of modelling in OR.

(i) ***Analogue and Mathematical models combined.*** Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in **classification 1** above.

For example, **simulation model** is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to '*simulate*' their decisions by summarizing the activities of industry in a scale-down period.

(ii) ***Function models.*** Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as–tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like in computer programming).

(iii) ***Quantitative models.*** Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are : (i) *transformation models* which are useful in converting a measurement of one scale to another (*e.g.*, Centigrade *vs* Fahrenheit conversion scale), and (ii) the *test models* that act as 'standards' against which measurements are compared (*e.g.*, business dealings, a specified standard production control, the quality of a medicine).

(iv) ***Heuristic models.*** These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well-defined strategies. Heuristic models do not claim to find the best solution to the problem.

Q. 1. Model building is the essence of the 'O.R. approach'. Discuss.

2. Discuss in detail the three types of models with special emphasis on the important logical properties and the relationship the three types bear to each other and to modelled entities. **[Meerut (OR) 90]**

3. What is meant by a mathematical model of real situation ? Discuss the importance of models in the solution of Operational Research probems ? **[Bhuwneshwar 2004]**

4. What is a model ? Discuss various classification schemes of models. **[Agra 95, 94; C.A. (May) 92; Meerut (IPM) 90]**

Trial 3, four hits	x	y	*Result*
1	− 2·015	− 0·594	Miss
2	− 0·623	− 1·047	Miss
3	− 0·699	− 1·347	Miss
4	0·481	0·996	Miss
5	0·586	− 1·023	Miss
6	0·579	0·551	**Hit***
7	0·120	0·418	**Hit***
8	0·191	0·074	**Hit***
9	0·071	0·524	**Hit***
10	− 3·001	0·479	Miss

These three trials give 3·33 as the average number of hits per mission. Many more trials should be conducted before we can have any real confidence in the result. One way of estimating : how many trials are necessary, is to list the cumulated mean at the end of each trial, and to stop the trials when the mean seems to have settled down to stable value. In this example, we have

after trial number	:	1	2	3
cummulated mean	:	4	3	3·33,

so that more trials are necessary.

The mean number of hits in a mission dropping 10 bombs is 3·69.

To compare the result with the exact value.

In this problem, unlike most Monte-Carlo problems, an exact calculation of the answer is much easier than the Monte Carlo calculation.

The probability of a hit with a single bomb is

$$\left[\int_{-1\cdot250}^{1\cdot250} f(x)\,dx\right]\times\left[\int_{-0\cdot625}^{0\cdot625} f(x)\,dx\right] = 2\cdot789\times 0\cdot468 \text{ (from the table of the normal integers)}$$

$$= 0\cdot369.$$

Thus the previous value 3·69 is ten times of this value.

Advantages :

1. These methods avoid unnecessary expenses and difficulties that arise during the trial and error experimentation.
2. By this technique, we find the solution of much complicated mathematical expression which is not possible by any other method.

Disadvantages :

1. This technique does not give optimal answers to the problems. The good results are obtained only when the sample size is quite large.
2. The computations are much complicated even in simple cases.
3. It is a costly procedure for obtaining a solution of any related problem.

Q. 1. Write a short note on Monte-Carlo Technique and their usefulness in real life situations. **[Meerut (Stat.) 98]**

2. Describe the use of Monte-Carlo methods in sampling experiments. Illustrate with possible examples.

1.8 MAIN CHARACTERISTICS (FEATURES) OF OPERATIONS RESEARCH

The main characteristics of OR are as follows :

1. Inter-disciplinary team approach. In **OR**, the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.

For example, while investigating the inventory management in a factory, perhaps we may require an engineer who knows the functions of various items of stores. We also require a cost accountant and a mathematician-cum-statistician. Each member of such OR team is benefitted by the view points of others, so that the workable solution obtained through such collaborative study has a greater chance of acceptance by management.

Furthermore, an OR team required for a big organization may include a statistician, an economist, a mathematician, one or more engineers, a psychologist, and some supporting staff like computer programmers, etc. A mathematician or a probabilist can apply his tools in a plant problem only if he gets to understand some of the physical implications of the plant from an engineer. Otherwise, he may give such a solution which may not be possible to apply.

2. Wholistic approach to the system. The most of the problems tackled by OR have the characteristic that OR tries to find the *best* (*optimum*) decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

For example, in attempting to solve a maintenance problem in a factory, OR tries to consider how this affects the production department as a whole. If possible, it also tries to consider how this effect on the production department in turn affects other department and the business as a whole. It may even try to go further and investigate how the effect on this particular business organization in turn affects the industry as a whole, etc. Thus OR attempts to consider inter-actions or chain of effects as far out as these effects are significant.

3. Imperfectness of solutions. By OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.

4. Use of scientific research. OR uses techniques of scientific research to reach the optimum solution.

5. To optimize the total output. OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

Q. 1. Give the main characteristics of Operations Research. [J.N.T.U. (B. Tech.) 2004, 03; C.A. (May) 92]
2. Define OR and discuss its characteristics and limitations.

1.9 MAIN PHASES OF OPERATIONS RESEARCH STUDY

About fourty years ago, it would have been difficult to get a single operations-researcher to describe a procedure for conducting OR project. The procedure for an OR study generally involves the following major phases :

Phase I : Formulating the problem. Before proceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model. To do so, the following information will be required.

(i) Who has to take the *decision* ?
(ii) What are the *objectives* ?
(iii) What are the ranges of *controlled variables* ?
(iv) What are the uncontrolled variables that may affect the possible solutions ?
(v) What are the restrictions or constraints on the variables ?

Since wrong formulation cannot yield a right decision (solution), one must be considerably careful while execution this phase.

Phase II : Constructing a mathematical model. The second phase of the investigations is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis. The most suitable form for this purpose is to construct a mathematical model representing the system under study. It requires the identification of both *static* and *dynamic* structural elements. A mathematical model should include the following three important basic factors :

(i) *Decision variables and parameters,* (ii) *Constraints or Restrictions,* (iii) *Objective function.*

Phase III : Deriving the solutions from the model. This phase is devoted to the computation of those values of decision variables that maximize (or minimize) the objective function. Such solution is called an *optimal solution* which is always in the best interest of the problem under consideration. The general techniques for deriving the solution of OR model are discussed in the following sections and further details are given in the text.

Phase IV : Testing the model and its solution (updating the model). After completing the model, it is once again tested as a whole for the errors if any. A model may be said to be valid if it can provide a reliable prediction of the system's performance. A good practitioner of Operations Research realises that his model be applicable for a longer time and thus he updates the model time to time by taking into account the past, present and future specifications of the problem.

1.11 SCOPE OF OPERATIONS RESEARCH

In its recent years of organized development, OR has entered successfully many different areas of research for military, government and industry. The basic problem in most of the developing countries in Asia and Africa is to remove *poverty* and *hunger* as quickly as possible. So there is a great scope for economists, statisticians, administrators, politicians and the technicians working in a team to solve this problem by an OR approach. Besides this, OR is useful in the following various important fields.

1. In Agriculture. With the explosion of population and consequent shortage of food, every country is facing the problem of—

(i) optimum allocation of land to various crops in accordance with the climatic conditions; and
(ii) optimum distribution of water from various resources like canal for irrigation purposes.

Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.

2. In Finance. In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of her country. OR-techniques can be fruitfully applied :

(i) to maximize the per capita income with minimum resources;
(ii) to find out the profit plan for the company;
(iii) to determine the best replacement policies, etc.

3. In Industry. If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using OR techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist of OR techniques in *business management*. Thus OR is useful to the *Industry Director* in deciding optimum allocation of various limited resources such as men, machines, material, money, time, etc., to arrive at the optimum decision.

4. In Marketing. With the help of OR techniques a *Marketing Administrator* (Manager) can decide :

(i) where to distribute the products for sale so that the total cost of transportation etc. is minimum,
(ii) the minimum per unit sale price,
(iii) the size of the stock to meet the future demand,
(iv) how to select the best advertizing media with respect to time, cost, etc.
(v) how, when, and what to purchase at the minimum possible cost ?

5. In Personnel Management. A personnel manager can use OR techniques :

(i) to appoint the most suitable persons on minimum salary,
(ii) to determine the best age of retirement for the employees,
(iii) to find out the number of persons to be appointed on full time basis when the workload is seasonal (not continuous).

6. In Production Management. A production manager can use OR techniques :

(i) to find out the number and size of the items to be produced;
(ii) in scheduling and sequencing the production run by proper allocation of machines;
(iii) in calculating the optimum product mix; and
(iv) to select, locate, and design the sites for the production plants.

7. In L.I.C. OR approach is also applicable to enable the L.I.C. offices to decide :

(i) what should be the premium rates for various modes of policies,
(ii) how best the profits could be distributed in the cases of with profit policies ? etc.

Finally, we can say : wherever there is a problem, there is OR. The applications of OR cover the whole extent of any thing. A recent publication of the OR society contains a summary of the applications of OR. The reader wishing more details on applications may consult the publication : ***'Progress in OR'* Vol. 2 by *Hertz., D.B.* and *R.T. Eddison*.**

Q. 1. Define O.R. and discuss its scope. **[Meerut (Stat.) 98; Garhwal 96; Kanpur 96; Rewa (Maths.) 93; Rohil. 93, 92]**
2. What are the areas of applications of O.R., **[Meerut (Maths) 91]**
3. (a) Explain the meaning, scope and methodology of O.R. **[VTU (BE Mech.) 2002]**
(b) Discuss the significance and scope of Operations Research in modern management. **[Delhi Univ. (MBA) HCA 2001]**
4. Write a critical essay on the definition and scope of Operations Research. **[JNTU (B. Tech) 2003, 02; Virbhadrah 2000]**

1.12 ROLE OF OPERATIONS RESEARCH IN DECISION-MAKING

The Operations Research may be regardeed as a tool which is utilized to increase the effectiveness of management decisions. In fact, OR is the objective suppliment to the subjective feeling of the administrator (decision-maker). Scientific method of OR is used to understand and describe the phenomena of operating system. OR models explain these phenomena as to what changes take place under altered conditions, and control these predictions against new observations. For example, OR may suggest the best locations for factories, warehouses as well as the most economical means of transportation. In marketing, OR may help in indicating the most profitable type, use and size of advertising compaigns subject to the financial limitations.

The advantages of OR study approach in business and management decision making may be classified as follows :

1. **Better Control.** The management of big concerns finds it much costly to provide continuous executive supervisions over routine decisions. An OR approach directs the executives to devote their attention to more pressing matters. For example, OR approach deals with production scheduling and inventory control.

2. **Better Co-ordination.** Sometimes OR has been very useful in maintaining the law and order situation out of chaos. For example, an OR based planning model becomes a vehicle for coordinating marketing decisions with the limitations imposed on manufacturing capabilities.

3. **Better System.** OR study is also initiated to analyse a particular problem of decision making such as establishing a new warehouse. Later, OR approach can be further developed into a system to be employed repeatedly. Consequently, the cost of undertaking the first application may improve the profits.

4. **Better Decisions.** OR models frequently yield actions that do improve an intuitive decision making. Sometimes, a situation may be so complicated that the human mind can never hope to assimilate all the important factors without the help of OR and computer analysis.

In the present text, we restrict ourselves to discuss the problems on : ***Inventory control, Replacement, Queues, Linear programming, Goal Programming, Transportation, Assignment, Games theory, Sequencing, Dynamic programming, Information theory, PERT/CPM, Simulation, and Decision theory*** etc.

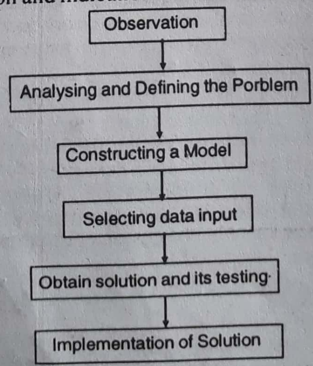
O.R. provides a logical and systematic approach for decision-making. The phases and process study must also be quite logical and systematic. There are six important steps in O.R. study, but in all each and every step does not necessarily follow logical order as below :

***Step I* : Observing the Problem Environment**

The activities in this step are visits, conferences, observations, research etc. With such activities analyst gets sufficient information and support to define the problem .

***Step II* : Analysing and Defining the Problem**

In this step the problem is defined, and objectives and limitations of the study are stated in its context. One thus gets clear grasp of need for a solution and indication of its nature.



***Step III* : Developing a Model**

Step III is to contruct a model. A model is representation of some real or abstract situations. O.R. models are basically mathematical models representing systems, processes or environment in the form of equations,

Management', ***'Materials Management Journal of India'***, ***'Defence Science Journal'***, ***'SCIMA'***, ***'Journal of Engineering Production'***, etc.

As far as the OR education in India is concerned University of Delhi was the first to introduce a complete M.Sc. course in OR in 1963. Simultaneously, Institute of Management at Calcutta and Ahemdabad started teaching OR in their MBA courses. Now-a-days, OR has become so popular subject that it has been introduced in almost all Institutes and Universities in various disciplines like, Mathematics, Statistics, Commerce, Economics, Management Science, Medical science, Engineering, etc. Also, realizing the importance of OR in Accounts and Administration, government has introduced this subject for the IAS, CA, ICWA examinations, etc.

Prof. Mahalanobis first applied OR in India by formulating second five-year plan with the help of OR techniques. Planning Commission made the use of OR techniques for planning the optimum size of the Caravelle fleet of Indian air lines. Some of the industries, namely, ***Hindustan Lever Ltd.; Union Carbide, TELCO, Hindustan Steel, Imperial Chemical Industries, Tata Iron & Steel Company, Sarabhai Group, FCI,*** etc. have engaged OR teams. ***Kirlosker Company*** is using the assignment technique of OR to maximize profit.

Textile firms like, DCM., Binni's and Calico, etc., are using linear programming techniques. Among other Indian organizations using OR are the **Indian Railways, CSIR, Tata Institute of Fundamental Research, Indian Institute of Science, State Trading Corporation,** etc.

****It is also worthnoting that the present text on*** **'OPERATIONS RESEARCH'** ***is the first book published in India to meet the requirements of various courses on this subject.***

1.15 ROLE OF COMPUTERS IN OPERATIONS RESEARCH

In fact, computers have played a vital role in the development of OR. But OR would not have achieved its present position for the use of computers. The reason is that—in most of the OR techniques computations are so complex and involved that these techniques would be of no practical use without computers. Many large scale applications of OR techniques which require only few minutes on the computer may take weeks, months and sometimes years even to yield the same results manually. So the computer has become as essential and integral part of OR. Now-a-days, OR methodology and computer methodology are growing up simultaneously. It seems that in the near future the line dividing the two methodologies will disappear and the two sciences will combine to form a more general and comprehensive science. It should also be noted that FORTRAN and C-programs are functionally equivalent.

The computor software packages are useful for rapid and effective calculations which is a necessary part of O.R. approach to solve the problems. These are :

(i) QSB+ (Quantitative System for Business Plus), Version 3.0, by Yih-long Chang and Robert S. Sullivan, is a software package that contains problem solving algorithms for OR/MS, as well as modules on basic statistics, non-linear programming and financial analysis.

(ii) QSOM (Quantitative Systems for Operations Management), by Yih-long, is an interactive user-friendly system. It contains problem-solving algorithms for operations management problems and associated information system.

(iii) Value STORM : MS quantitative Modelling for Decision Support, by Hamilton Emmons, A.D. Flowers, Chander Shekhar, M.Khot and Kamlesh Mathur, is a special version of Personal STORM version 3.0 developed for use in OR/MS.

(iv) Excel 97 by Gene Weiss Kopf and distributed by BPB publications, New Delhi, is an easy-to-use task-oriented guide to Excel Spread sheet applications.

(v) LINDO (Linear Interactive Discrete Optimization), developed by Linus Schrage Lindo in his book "An Optimization Modeling System, 4th ed. (Palo Alto, CA : Scientific Press 1991)

SELF-EXAMINATION QUESTIONS

1. (a) What is Operations Research ? A certain wine importer noticed that his sales of wine were not what they should be in comparison to other types of liquor. He hired you as a consultant to look into this problem, with the intention of improving the wine business. What would you do ?

   (b) How does one go about organising for effective Operations Research ? Explain.
2. Give a brief account of the methods used in model formulation.
3. Explain, how and why OR methods have been valuable in aiding executive decision. **[Meerut (Stat.) 90]**
4. Explain the concept, scope and tools of OR as applicable to business and industry.
5. Discuss the advantages and limitations of using results from a mathematical model to make decisions about operations.

6. "Mathematics of OR is mathematics of optimization". Discuss.
7. "OR is the application of scientific methods, techniques and tools to problem involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem." Discuss.
8. (a) Define Operations Research. Give the main characteristics of Operations Research.
   (b) Discuss the importance of Operations Research in decision-making process.
9. (a) Dicuss the significance and scope of Operations Research in modern management.
   (b) Describe in brief the uses of statistical techniques in Operations Research.
10. Write a detailed note on the use of models for decision-making. Your answers should specifically cover the following :
   (i) Need for model building.
   (ii) Type of model appropriate to the situation.
   (iii) Steps involved in the construction of the model.
   (iv) Setting up criteria for evaluating different alternatives.
   (v) Role of random numbers.
11. "Operations Research is an aid for the executive in making his decisions by providing him with the needed quantitative information, based on the scientific method analysis". Discuss this statement in detail, illustrating it with O.R methods that you know. **[Meerut (Stat.) 95]**
12. Give the essential characteristics of the following types of process :
   (a) Allocation (b) Competitive Games (c) Inventory (d) Waiting line.
13. What is the role of Operations Research in decision making ? Explain the scope and methodology of Operations Research, the main phases of Operations Research and techniques in solving an Operations Research problem.
14. Write short notes on the following :
   (i) Area of applications of Operations Research.
   (ii) Role of constraints and objectives in the construction of mathematical models.
   (iii) Statistician's role as member of O.R team.
15. Is Operations Research a discipline, or a profession, or set of techniques, or a philosophy, or a new name for an old thing ?
16. Outline broad features of the judgment phase and the research phase of scientific method in Operations Research. Discuss fully any one of these phases.
17. How can Operations Research models be classified ? Which is the best classification in terms of learning and understanding the fundamentals of Operations Research ?
18. What are the advantages and disadvantages of Operational Research models ? Why is it necessary to test models and how would you go about testing a model ?
19. What is Operations Research ? Describe four models used in Operations Research.
20. List any three Operations Research techniques and state in what conditions they can be used ?
21. Explain the role of quantitative techniques in the field of business and industry in modern times. Give a few examples in support of your answer.
22. What are the essential characteristics of Operations Research ? Mention different phases in an Operations Research study. Point out some limitations of Operations Research.
23. (a) Define OR as a decision making science.
   (b) Briefly explain the uses of OR-techniques in India. How are they found useful by the business executives ? Which of the three techniques are most commonly used in India ? Why
24. Explain the meaning and nature of OR.
25. State any four areas for the application of OR techniques in Financial Management, and how it improves the performance of the organisation.
26. (a) Comment on "Operations Research is a scientific and for enhancing creative and judicious capabilities of a decision maker".
   (b) Give any four processes of Operations Research and discuss their essential features.
27. Write a critical essay on the definition and scope of Operations Research.
28. Comment on the following statements :
   (a) OR is a bunch of mathematical techniques.
   (b) OR is no more than a quantitative analysis of the problem.
   (c) OR advocates a system's approach and is concerned with optimization. It provides a quantitative analysis for decision making.
   (d) OR has been defined semi-facetiously as the application of big minds to small problems.
29. What is Operations Research ? What areas of Operations Research have made a significant impact on decision making process ? Why is it important to keep an open mind in utilizing Operations Research techniques ?
30. Give a definition of Operations Research indicating the different types of models of the problem and the general methods of their solution.
31. (a) Write briefly about the following :
   (i) Iconic models, (ii) Analogue models, (iii) Mathematical models (or Symbolic models).

(b) Explain three types of models used in Operations Research, giving suitable example.

(c) What is the function of a model in decision making ? Name the types of models. What are the advantages of models? What are the pitfalls of models.

32. Distinguish the following models with suitable examples :

(i) Stochastic and deterministic models; (ii) Static and dynamic models.

33. Quantitative techniques complement the experience and judgement of an executive in decision making. They do not and cannot replace it. Discuss. **[Delhi Univ. (MBA) 1988]**

34. In construction any OR model, it is essential to realize that a most important purpose of the modelling process is "to help any manager better." Keeping this purpose in mind, state any four OR models that can be of help to Chartered Accountants in advising their clients. **[C.A. (May) 91]**

35. State three properties and three advantages of an OR model. **[C.A. (May) 92]**

36. Describe briefly the components of a proplem and mention the three major types of problems in decision making under different environment. **[C.A. (Nov.) 92]**

37. "Much of the success of OR applications in the last three decades is due to the computers." Discuss. **[C.A. (May) 93]**

38. Discuss the role and scope of quantitative methods for scientific decision making in a business environment. **[IPM (MBA) 2000]**

39. Explain briefly the various applications of O.R. **[VTU (BE Compu.) Aug. 2001]**

40. What are the advantages and limitations of OR studies ? **[VTU (BE VIth Sem.) Feb. 2002]**

41. What are the essential characteristics of operations research ? Mension different phases in the operations research study. Point out its limitations, if any. **[C.A. Nov 1992]**

42. (a) Why is the study of Operations Research important to the decision maker.

(b) Operation Research increases creative and judicious capabilities of a decision maker. Comment. **[Delhi Univ. (MBA) 1998]**

OBJECTIVE QUESTIONS

1. Operations research approach is
   (a) multi-disciplinary. (b) scientific. (c) intuitive. (d) all of the above.
2. Operations research analysts do not
   (a) predict future operations. (b) build more than one model.
   (c) collect relevant data. (d) recommend decision and accept.
3. For analysing a problem, decision-makers should normally study
   (a) its qualitative aspects. (b) its quantitative aspects. (c) both (a) and (b). (d) neither (a) nor (b).
4. Decision variables are
   (a) controllable. (b) uncontrollable. (c) parameters. (d) none of the above.
5. A model is
   (a) an essence of reality. (b) an approximation. (c) an idealization. (d) all of the above.
6. Managerial decisions are based on
   (a) an evaluation of quantitative data. (b) the use of qualitative factors.
   (c) numbers produced by formal models. (d) all of the above.
7. The use of decision models
   (a) is possible when the variable's value is known.
   (b) reduces the scope of judegement and intuition known with certainty in decision-making.
   (c) requires the knowledge of computer software use.
   (d) none of the above.
8. Every mathematical model
   (a) must be deterministic. (b) requires computer aid for its solution.
   (c) represents data in numerical form. (d) all of the above.
9. A physical model is example of
   (a) an iconic model. (b) an analogue model. (c) a verbal model. (d) a mathematical model.
10. An optimization model
    (a) mathematically provides the best decision. (b) provides decision within its limited context.
    (c) helps in evaluating various alternatives constantly. (d) all of the above.

Answers

1. (a) **2.** (a) **3.** (c) **4.** (a) **5.** (d) **6.** (d) **7.** (d) **8.** (c) **9.** (a) **10.** (d).

✱✱✱

LINEAR PROGRAMMING PROBLEM
(FORMULATION AND GRAPHICAL METHOD)

3.1. INTRODUCTION

In 1947, George Dantzig and his Associates, while working in the U.S. department of Air Force, observed that a large number of military programming and planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variables were restricted to values satisfying a system of linear constraints (a set of linear equations/or inequalities). A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + \ldots + a_nx_n$, where $a_1, a_2, \ldots, a_n$ are constants, and $x_1 . x_2, \ldots, x_n$ are variables. The term '*Programming*' refers to the process of determining a particular programme or plan of action. So Linear Programming (L.P.) is one of the most important optimization (maximization/minimization) techniques developed in the field of Operations Research (O.R.).

The methods applied for solving a linear programming problem are basically simple problems, a solution can be obtained by a set of simultaneous equations. However, a *unique* solution for a set of simultaneous equations in n-variables $(x_1, x_2, \ldots, x_n)$, at least one of them is non-zero, can be obtained if there are exactly n relations. When the number of relations is greater than or less than n, a unique solution does not exist, but a number of trial solutions can be found. In various practical situations, the problems are seen in which the number of relations is not equal to the number of variables and many of the relations are in the form of inequalities ($\leq$ or $\geq$) to maximize (or minimize) a linear function of the variables subject to such conditions. Such problems are known as *Linear Programming Problems* (LPP).

In this chapter, properties of LP problems are discussed and at present the graphical method of solving a LPP is applicable where two (or at most three) variables are involved. The most widely used method for solving LP problems of any number of variables is called the *simplex method* developed by G. Dantzig in 1947 and made generally available in 1951.

Definition. *The general LPP calls for optimizing (maximizing/minimizing) a linear function of variables called the '**OBJECTIVE FUNCTION**' subject to a set of linear equations and / or inequalities called the '**CONSTRAINTS**' or '**RESTRICTIONS**'.*

3.2. FORMULATION OF LP PROBLEMS

Now it becomes necessary to present a few interesting examples to explain the real-life situations where LP problems may arise. The outlines of formulation of the LP problems are explained with the help of these examples.

Model Examples on Formulation

Example 1. (Production Allocation Problem) *A firm manufactures two type of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H ; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day.*

Formulate the problem as a linear programming problem. **[Kanpur 96]**

Formulation. Let x_1 be the number of products of type A and x_2 the number of products of type B.

After carefully understanding the problem the given information can be systematically arranged in the form of the following table.

Table 3.1

Machine	Time of Products (minutes)		Available Time (minutes)
	Type A (x_1 units)	Type B (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 units of type A. Similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 units of B is given by

$$P = 2x_1 + 3x_2 \quad \textit{(objective function)}$$

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by : $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + x_2$.

But, machine G is not available for more than 6 hour 40 minutes (= 400 minutes). Therefore,

$$x_1 + x_2 \le 400 \quad \textit{(first constraint)}$$

Also, the machine H is available for 10 hours only, therefore,

$$2x_1 + x_2 \le 600 \quad \textit{(second constraint)*}$$

Since it is not possible to produce negative quantities,

$$x_1 \ge 0 \text{ and } x_2 \ge 0 \quad \textit{(non-negativity restrictions)}$$

Hence the allocation problem of the firm can be finally put in the form :

> **Find x_1 and x_2 such that the profit $P = 2x_1 + 3x_2$ is maximum,**
> **subject to the conditions :**
> $x_1 + x_2 \le 400, \quad 2x_1 + x_2 \le 600, \quad x_1 \ge 0, x_2 \ge 0.$

Example 2. *A company produces two types of Hats. Each hat of the first type requires twice as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type A and Rs. 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.*

Formulation. Let the company produce x_1 hats of type A and x_2 hats of type B each day. So the profit P after selling these two products is given by the linear function :

$$P = 8x_1 + 5x_2 \quad \textit{(objective function)}$$

Since the company can produce at the most 500 hats in a day and A type of hats require twice as much time as that of type B, production restriction is given by $2tx_1 + tx_2 \le 500\,t$, where t is the labour time per unit of second type, *i.e.*

$$2x_1 + x_2 \le 500.$$

But, there are limitations on the sale of hats, therefore further restrictions are :

$$x_1 \le 150, \quad x_2 \le 250.$$

Also, since the company cannot produce negative quantities,

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0.$$

Hence the problem can be finally put in the form :

> **Find x_1 and x_2 such that the profit $P = 8x_1 + 5x_2$ is maximum,**
> **subject to the restrictions :**
> $2x_1 + x_2 \le 500, x_1 \le 150, x_2 \le 250, x_1 \ge 0, x_2 \ge 0.$

*Here the constraint $2x_1 + x_2 = 600$ is not justified because using machine H for less than 10 hrs (if possible) will be more profitable.

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Example 3. *The manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredient available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B.*

(i) *Formulate this problem as a L.P.P.*

(ii) *How the manufacturer schedule his production in order to maximize profit.*

Formulation. (i) Suppose the manufacturer produces x_1 and x_2 thousand of bottles of medicines A and B, respectively. Since it takes three hours to prepare 1000 bottles of medicine A, the time required to fill x_1 thousand bottles of medicine A will be $3x_1$ hours. Similarly, the time required to prepare x_2 thousand bottles of medicine B will be x_2 hours. Therefore, total time required to prepare x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will be $3x_1 + x_2$ hours.

Now since the total time available for this operation is 66 hours, $3x_1 + x_2 \leq 66$.

Since there are only 45 thousand bottles available for filling medicines A and B, $x_1 + x_2 \leq 45$.

There are sufficient ingredients available to make 20 thousand bottles of medicine A and 40 thousand bottles of medicine B, hence $x_1 \leq 20$ and $x_2 \leq 40$.

Number of bottles being non-negative, $x_1 \geq 0$, $x_2 \geq 0$.

At the rate of Rs. 8 per bottle for type A medicine and Rs. 7 per bottle for type B medicine, the total profit on x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will become

$$P = 8 \times 1000\, x_1 + 7 \times 1000\, x_2 \quad \text{or} \quad P = 8000\, x_1 + 7000\, x_2.$$

Thus, the linear programming problem is :

Max. $P = 8000\, x_1 + 7000\, x_2$, **subject to the constraints :**

$$3x_1 + x_2 \leq 66,\ x_1 + x_2 \leq 45,\ x_1 \leq 20,\ x_2 \leq 40$$

and $x_1 \geq 0, x_2 \geq 0$.

(ii) See ***Example 28*** (page 79) for its solution by graphical method.

Example 4. *A toy company manufactures two types of doll, a basic version—doll A and a delux version—doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each doll should be produced per day in order to maximize the total profit. Formulate this problem.* **[Kanpur B.Sc. 90; Meerut 90]**

Formulation. Let x_1 and x_2 be the number of dolls produced per day of type A and B, respectively. Let the doll A require t hrs so that the doll B require $2t$ hrs. So the total time to manufacture x_1 and x_2 dolls should not exceed 2,000t hrs. Therefore, $tx_1 + 2tx_2 \leq 2000t$. Other constraints are simple. Then the linear programming problem becomes :

Maximize $P = 3x_1 + 5x_2$

subject to the restrictions

$$x_1 + 2x_2 \leq 2000 \quad \text{(time constraint)}$$
$$x_1 + x_2 \leq 1500 \quad \text{(plastic constraint)}$$
$$x_2 \leq 600 \quad \text{(dress constraint)}$$

and non–negativity restrictions

$$x_1 \geq 0, x_2 \geq 0.$$

Note : See ***Example 26*** (page 76) for its solution by graphical method.

6x60
360

Example 5. *In a chemical industry, two products A and B are made involving two operations. The production of B also results in a by-product C. The product A can be sold at Rs. 3 profit per unit and B at Rs. 8 profit per unit. The by-product C has a profit of Rs. 2 per unit, but it cannot be sold as the destruction cost is Re. 1 per unit. Forecasts show that up to 5 units of C can be sold. The company gets 3 units of C for each unit of A and B produced. Forecasts show that they can sell all the units of A and B produced. The manufacturing times are 3 hours per unit for A on operation one and two respectively and 4 hours and 5 hours per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18 and 21 hours of operation one and two respectively. The company question : how much A and B should be produced keeping C in mind to make the highest profit. Formulate LP model for this problem.*

Formulation. Let x_1 , x_2 , x_3 be the number of units produced of product A , B , C respectively. Then the profit gained by the industry is given by $\mathbf{P} = 3x_1 + 8x_2 + 2x_3$.

Here it is assumed that all the units of product A and B are sold.

In first operation, A takes 3 hours of manufacturer's time and B takes 4 hours of manufacturer's time, therefore total number of hours required in first operation becomes $3x_1 + 4x_2$.

In second operation, A takes 3 hours of manufacturer's time and B takes 5 hours of manufacturer's time, therefore the total number of hours used in second operation becomes $3x_1 + 5x_2$.

Since there are 18 hours available in first operation and 21 hours in second operation, the restrictions become : $3x_1 + 4x_2 \le 18$, $3x_1 + 5x_2 \le 21$.

Also, the company gets 3 units of by-product C for each unit of B produced, therefore the total number of units of product B and C produced becomes : $x_2 + 3x_3$.

But, the maximum number of units of C can be sold is 5, therefore $x_2 + 3x_3 \le 5$.

Thus the allocation problem of the industry can be finally put in the form :

Find the value of x_1 , x_2 , x_3 so as to maximize

$$\mathbf{P} = 3x_1 + 8x_2 + 2x_3 \textbf{ subject to the restrictions :}$$

$$\begin{aligned} 3x_1 + 4x_2 &\le 18 \\ 3x_1 + 5x_2 &\le 21 \\ x_2 + 3x_3 &\le 5, \end{aligned}$$

with non-negativity conditions : $x_1 \ge 0 , x_2 \ge 0 , x_3 \ge 0$ **.**

Example 6. *A firm can produce three types of cloth, say : A, B, and C. Three kinds of wool are required for it, say : red, green and blue wool. One unit length of type A cloth needs 2 meters of red wool and 3 meters of blue wool ; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool ; and one unit of type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3.00, of type B cloth is Rs. 5.00, and of type C cloth is Rs. 4.00.*

Determine, how the firm should use the available material so as to maximize the income from the finished cloth.

Formulation. It is often convenient to construct the *Table 3.2* after understanding the problem carefully.

Table 3.2

Quality of wool	*Type of Cloth*			*Total quantity of wool available (in meters)*
	A (x_1)	B (x_2)	C (x_3)	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	Rs. 3.00	Rs. 5.00	Rs. 4.00	

> **Find the value of x_1, x_2, x_3 so as to maximize**
>
> $$P = 3x_1 + 2x_2 + 4x_3$$
>
> **subject to the constraints :**
>
> $$4x_1 + 3x_2 + 5x_3 \le 2,000$$
>
> $$2x_1 + 2x_2 + 4x_3 \le 2,500$$
>
> $$100 \le x_1 \le 150,\ 200 \le x_2 \ge 0,\ 50 \le x_3 \ge 0.$$

Example 8. *A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Re. 1.00 per kg for tomatoes, Rs. 0·75 a head for lettuce and Rs. 2·00 per kg for radishes. The average yield per-acre is 2,000 kg of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs. 0·50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20·00 per man-day.*

Formulate this problem as a linear programming model to maximize the farmer's total profit.

[Kanpur (B.Sc.) 93, 92]

Formulation. Farmer's problem is to decide how much area should be alloted to each type of crop he wants to grow to maximize his total profit. Let the farmer decide to allot x_1, x_2 and x_3 acre of his land to grow tomatoes, lettuce and radishes respectively. So the farmer will produce $2000x_1$ kgs of tomatoes, $3000x_2$ heads of lettuce, and $1000x_3$ kgs of radishes.

Therefore, total sale will be $= $ Rs. $[2000\,x_1 + 0{\cdot}75 \times 3000\,x_2 + 2 \times 1000\,x_3]$

Fertilizer expenditure will be $=$ Rs. $[0{\cdot}50\,\{100\,(x_1 + x_2) + 50x_3\}]$

Labour expenditure will be $=$ Rs. $[20 \times (5x_1 + 6x_2 + 5x_3)]$

Therefore, farmer's net profit will be

$$\mathbf{P} = \text{Total sale (in Rs.)} - \text{Total expenditure (in Rs.)}$$

or $\mathbf{P} = [2000\,x_1 + 0{\cdot}75 \times 3000\,x_2 + 2 \times 1000\,x_3] - 0{\cdot}50 \times [100\,(x_1 + x_2) + 50\,x_3] - 20 \times [5x_1 + 6x_2 + 5x_3]$

or $\mathbf{P} = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$.

Since total area of the farm is restricted to 100 acre, $x_1 + x_2 + x_3 \le 100$.

Also, the total man-days labour is restricted to 400 man-days, therefore, $5x_1 + 6x_2 + 5x_3 \le 400$.

Hence the farmer's allocation problem can be finally put in the form :

> **Find the value of x_1, x_2, x_3 so as to maximize :**
>
> $$\mathbf{P = 1850\,x_1 + 2080\,x_2 + 1875\,x_3},$$
>
> **subject to the conditions :**
>
> $$x_1 + x_2 + x_3 \le 100,$$
>
> $$5x_1 + 6x_2 + 5x_3 \le 400,$$
>
> **and** $$x_1, x_2, x_3 \ge 0.$$

Example 9. *A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material (A and B) of which 4000 and 6000 units repectively are available. The raw material requirements per unit of the three models are given below :*

Raw Material	*Requirement per unit of given model*		
	I	*II*	*III*
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2500 units of model I. A market survey indicates that the minimum demand of the three models are 500, 500 and 375 units respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II

or
$$\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Therefore, $\mathbf{x} = (x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5)^T$, $\mathbf{c} = (2 \;\; 3 \;\; 4 \;\; 0 \;\; 0)$, $\mathbf{b} = (5 \;\; 7 \;\; 9)^T$, and
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix}.$$

3.8 SOME IMPORTANT DEFINITIONS

Following are defined a few important terms for standard LPP (3.1, 3.2, 3.3) which are necessary to understand further discussion.

1. **Solution to LPP.** Any set $\mathbf{x} = \{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a ***solution*** to LP problem, if it satisfies the set of constraints (3.12) only.
2. **Feasible Solution (FS).** Any set $\mathbf{x} = \{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a ***feasible solution*** (or programme) of L.P. problem, if it satisfies the set of constraints (3.12) and non-negativity restrictions (3.13) also.
3. **Basic Solution (BS).** A ***basic solution*** to the set of constraints (3.12) is a *solution* obtained by setting any n variables (among $m+n$ variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero. Such m variables (of course, some of them may be zero) are called ***basic variables*** and remaining n zero-valued variables are called ***non-basic variables.*** **[Bhubnaeshwar (IT) 2004; JNTU (B. Tech) 98]**

   The number of basic solutions thus obtained will be at the most ${}^{m+n}C_m = \dfrac{(m+n)\,!}{n\,!\,m\,!}$, which is the number of combinations of $n+m$ things taken m at a time.
4. **Basic Feasible Solution (BFS).** A *basic feasible solution* is a *basic solution* which also satisfies the non-negativity restrictions (3.13), that is, all basic variables are non-negative. **[JNTU. MCA (III) 2004]**

   Basic feasible solutions are of two types :

   (a) ***Non-degenerate BFS.*** A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_i $(i = 1, 2, \ldots, m)$. In other words, all m basic variables are positive, and the remaining n variables will be all zero.

   (b) ***Degenerate BFS.*** A basic feasible solution is called *degenerate*, if one or more basic variables are *zero-valued.* **[JNTU. MCA (III) 2004]**
5. **Optimum Basic Feasible Solution.** A basic feasible solution is said to be ***optimum,*** if it also optimizes (maximizes or minimizes) the objective function (3.11). **[JNTU. MCA(III) 2004; Meerut (L.P) 90]**
6. **Unbounded Solution.** If the value of the objective function z can be increased or decreased indefinitely, such solutions are called ***unbounded solutions.*** **[Meerut 90]**

Note. Unless otherwise stated, solution means a feasible solution. However, an optimum solution to a linear programming problem imply that z has a finite maximum or finite minimum.

Q. 1. For the system $AX = \mathbf{b}$ of m linear equations in n unknowns $(m < n)$ with rank$(A) = m$, define a basic solution. **[Meerut (IPM) 91]**

2. Explain the term optimal solution to a LPP. **[AIMS (Bangalore) MBA 2002]**

3. Define :
- (i) Feasible Solution
- (ii) Basic Solution
- (iii) Basic Feasible Solution **[JNTU MCA (III) 2004; (B. Tech.) 98]**
- (iv) Non-dgenerate BFS **[VTU (BE Mech.) 2002; Kanpur 96]**
- (v) Degenerate BFS **[JNTU. MCA (III) 2004]**
- (vi) Optimum Basic Feasible Solution **[JNTU. MCA (III) 2004; Kanpur 96; Meerut (Stat.) 95 ; (Math.) 90]**
- (vii) Unbounded Solution. **[JNTU (B. Tech) 98]**

Q. 1. State clearly the basic assumptions that are made in LPP.

2. What are the major assumptions in Linear Programming? **[JNTU (Mech. & Prod.) 2004]**

3.10. LIMITATIONS OF LINEAR PROGRAMMING

Inspite of wide area of applications, some limitations are associated with linear programming techniques. These are stated below :

1. In some problems objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.

2. There is no guarantee of getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.

3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large-scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

6. Linear programming deals with only single objective, whereas in real life situations problems come across with multiobjectives. *Goal programming* and *multi-objective programming* deal with such problems.

Q. What are the limitations of linear programming technique?

3.11. APPLICATIONS OF LINEAR PROGRAMMING

In this section, we discuss some important applications of linear programming in our life.

1. Personnel Assignment Problem. Suppose we are given m persons, n-jobs, and the expected productivity c_{ij} of ith person on the jth job. We want to find an assignment of persons $x_{ij} \geq 0$ for all i and j, to n jobs so that the average productivity of person assigned is maximum, subject to the conditions :

$$\sum_{j=1}^{n} x_{ij} \leq a_i \text{ and } \sum_{i=1}^{m} x_{ij} \leq b_j,$$

where a_i is the number of persons in personnel category i and b_j is the number of jobs in personnel category j. For details, refer the chaper of *Assignment Problems.*

2. Transportation Problem. We suppose that m factories (called sources) supply n warehouses (called destinations) with a certain product. Factory F_i ($i = 1\ 2, \ldots, m$) produces a_i units (total or per unit time), and warehouse W_j ($j = 1, 2, 3 \ldots, n$) requires b_j units. Suppose that the cost of shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped; and that the unit cost is c_{ij}. Let the decision variables, x_{ij}, be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ is minimized. In the mean time, the supply and demand must be satisfied exactly.

Mathematically, this problem is to find x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) in order to minimize the total transportation cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} (c_{ij})$, subject to the restrictions of the form

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m \text{ (factory)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n \text{ (warehouse)}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \text{ and } x_{ij} \geq 0, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n).$$

For detailed discussion, refer **chapter 10** on ***Transportation Problem.***

3. Efficiencing on Operation of System of Dams. In this problem, we determine variations in water storage of dams which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

4. Optimum Estimation of Executive Compensation. The objective here is to determine a consistent plan of executive compensation in an industrial concern. Salary, job ranking and the amounts of each factor required on the ranked job level are taken into consideration by the constraints of linear programming.

5. Agricultural Applications. Linear programming can be applied in agricultural planning for allocating the limited resources such as acreage, labour, water, supply and working capital, etc. so as to maximize the net revenue.

6. Military Applications. These applications involve the problem of selecting an air weapon system against gurillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets, and the problem of community defence against disaster to find the number of defence units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.

7. Production Management. Linear programming can be applied in production management for determining product mix, product smoothing, and assembly time-balancing.

8. Marketing Management. Linear programming helps in analysing the effectiveness of advertising compaign and time based on the available advertising media. It also helps travelling sales-man in finding the shortest route for his tour.

9. Manpower Management. Linear programming allows the personnel manager to analyse personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

10. Physical Distribution. Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centres for physical distribution.

Besids above, linear programming involves the applications in the area of administration, education, inventory control, fleet utilization, awarding contract, and capital budgeting etc.

Q. 1. Give a brief account of applications of linear programming problem.
2. Explain the meaning of a Linear Programming Problem stating its uses and give its limitations. **[C.A. (May) 95]**
3. State in brief uses of linear programming Technique.

3.12. ADVANTAGES OF LINEAR PROGARMMING TECHNIQUES

The advantages of linear programming techniques may be out-lined as follows :

1. Linear programming technique helps us in making the optimum utilization of productive resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.

2. The quality of decisions may also be improved by linear programming techniques. The user of this technique becomes more objective and less subjective.

3. Linear programming technique provides practically applicable solutions since there might be other constraints operating outside the problem which must also be taken into consideration just because, so many units must be produced does not mean that all those can be sold. So the necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.

4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet the demand while others remain idle for some time.

Q. 1. What are the advantages of Linear Programming Technique ? **[JNTU (B. Tech.) 2003]**
2. Give the properties of a Linear Programming Problem.

5

LINEAR PROGRAMMING PROBLEM : SIMPLEX METHOD

5·1. INTRODUCTION

It has not been possible to obtain the graphical solution to the LP problem of more than two variables. The analytic solution is also not possible because the tools of analysis are not well suited to handle inequalities. In such cases, a simple and most widely used simplex method is adopted which was developed by ***G. Dantzig*** in 1947.

The *simplex method*† provides an ***algorithm*** (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the ***fundamental theorem of linear programming.***

It is clear from Fig. 3.4 (page 78) that feasible solutions may be *infinite* in number (because there are infinite number of points in the feasible region, *OABCD*). So, it is rather impossible to search for the optimum solution amongst all the feasible solutions. But fortunately, the number of basic feasible solutions are finite in number (which are corresponding to extreme points O, A, B, C, D, respectively). Even then, a great labour is required in finding all the basic feasible solutions and to select that one which optimizes the objective function.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever repeat and there is no need to go back to vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. The procedure is explained in detail through a numerical example (*see Example 2,* ch. 5, page 70 (Unit 2)).

The simplex algorithm is an iterative (step-by-step) procedure for solving LP problems. It consists of—

(i) having a trial basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution,

(iii) improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

The computational procedure requires at most m [equal to the number of equations in (3·12)] non-zero variables in the solution at any step. In case of less than m non-zero variables at any stage of computations the degeneracy arises in LP problem. The case of degeneracy has also been discussed in detail in *this chapter*.

Further, it is very interesting to note that a feasible solution at any iteration is related to the feasible solution at the successive iteration in the following way. One of the non-basic variables (which are zero now) at one iteration becomes *basic* (non-zero) at the following iteration, and is called an ***entering variable.*** To compensate, one of the basic variables (which are non-zero now) at one iteration becomes non-basic (zero) at the following iteration, and is called a ***departing variable.*** The other non-basic variables remain zero, and the other basic variables, in general, remain non-zero (though their values may change).

For convenience, re-state the LP problem in standard form :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \dots + 0x_{n+m} \qquad \dots(5.1)$$

subject to the constraints :

$$\left.\begin{array}{llllll} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & + x_{n+1} & & & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & & + x_{n+2} & & & = b_2 \\ \dots \quad \dots \quad \dots & \dots & \dots & & & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & & & & + x_{n+m} & = b_m \end{array}\right\} \qquad \dots(5.2)$$

† For complete development of '**Simplex Method**' please see Appendix-A (**Theory of Simplex Method**) on page 1119.

and $$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0 \qquad \ldots(5.3)$$

For easiness, an obvious starting basic feasible solution of m equations (5·2) is usually taken as : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; $x_{n+1} = b_1, x_{n+2} = b_2, \ldots, x_{n+m} = b_m$. For this solution, the value of the objective function (5·1) is zero. Here $x_1, x_2, x_3, \ldots, x_n$ (each equal to zero) are ***non-basic variables*** and remaining variables $(x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m})$ are ***basic variables*** (some of them may also have the value zero).

5.2. SOME MORE DEFINITIONS AND NOTATIONS

The first basic feasible solution is : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; and $x_{n+1} = b_1$, $x_{n+2} = b_2$, $x_{n+3} = b_3 \ldots$, $x_{n+m} = b_m$ for the reformulated LP problem : Max $\mathbf{z} = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$ and $\mathbf{X} \geq \mathbf{0}$.

First denote the jth column of $m \times (n+m)$ matrix $\mathbf{A}$ by $\mathbf{a_j}$ $(j = 1, 2, 3, \ldots, n+m)$, so that

$$\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_{m+n}}]. \qquad \ldots(5.4)$$

Now form an $m \times m$ non-singular matrix $\mathbf{B}$, called ***basis matrix***, whose column vectors are m linearly independent columns selected from matrix $\mathbf{A}$ and renamed as $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$. Therefore,

$$\mathbf{B} = [\beta_1, \beta_2, \ldots, \beta_m] = [\mathbf{a_{n+1}}, \mathbf{a_{n+2}}, \ldots, \mathbf{a_{n+m}}] \qquad \ldots(5.5)$$

For initial basic feasible solution,

$$\mathbf{B} = [(1, 0, 0, \ldots, 0), (0, 1, 0, 0 \ldots, 0), \ldots, (0, 0, \ldots, 1)] = I_m \text{ (identity matrix)}.$$

The matrix $\mathbf{B}$ is evidently a basis matrix because column vectors in $\mathbf{B}$ form a basis set of m-dimensional Euclidean space (E^m).

Second, denote the basic variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ by $x_{B1}, x_{B2}, \ldots, x_{Bm}$ respectively, to give the basic feasible solution in the form :

$$\mathbf{X_B} = (x_{B1}, x_{B2}, x_{B3}, \ldots, x_{Bm}) = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m}) \qquad \ldots(5.6)$$

For initial basic feasible solution,

$$\mathbf{X}_B = (b_1, b_2, b_3, \ldots, b_m) = \text{ right side constants of (5.2).}$$

Next, the coefficients of basic variables $x_{B1}, x_{B2}, \ldots, x_{Bm}$ in the objective function z will be denoted by $c_{B1}, c_{B2}, \ldots, c_{Bm}$ respectively, so that

$$\mathbf{C}_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}).$$

For initial basic feasible solution,

$$\mathbf{C}_B = (0, 0, \ldots, 0) = \mathbf{0} \text{ (null vector)}$$

Consequently, the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots + 0x_{n+m} \text{ becomes}$$

$$z = c_1.0 + c_2.0 + \ldots + c_n.0 + c_{B1}x_{B1} + \ldots + c_{Bm}x_{Bm} \qquad [\text{since } x_1 = x_2 = x_3 = \ldots = x_n = 0]$$

or $$z = \mathbf{C}_B\, \mathbf{X}_B. \qquad \ldots(5.7)$$

Because $\mathbf{C}_B = \mathbf{0}$ (null vector) for initial solution, therefore

$$z = 0, \mathbf{X_B} = \mathbf{b}.$$

Since $\mathbf{B}$ is an $m \times m$ non-singular basis matrix, any vector in E^m can be expressed as a linear combination of vectors in $\mathbf{B}$ (by definition of basis for vector space). In particular, each vector $\mathbf{a_j}$ $(j = 1, 2, \ldots, n+m)$ of matrix $\mathbf{A}$ can be expressed as a linear combination of vectors β_i $(i = 1, 2, \ldots, m)$ in $\mathbf{B}$. The notation for such linear combination is given by

$$\mathbf{a}_j = x_{1j}\beta_1 + x_{2j}\beta_2 + \ldots + x_{mj}\beta_m = (\beta_1, \beta_2, \ldots, \beta_m)\begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix} = \mathbf{BX_j} \qquad \ldots(5.8)$$

where x_{ij} $(i = 1, 2, 3, \ldots, m)$ are scalars required to express each $\mathbf{a_j}$ $(j = 1, 2, 3, \ldots, n+m)$ as linear combination of basis vectors $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$.

Therefore, $\mathbf{X_j} = \mathbf{B}^{-1}\mathbf{a_j}$ and hence matrix $(\mathbf{X_j})$ will change if the columns of $(\mathbf{A})$ forming $(\mathbf{B})$ change.

For initial solution, $\mathbf{a}_j = \mathbf{I}_m\mathbf{X}_j = \mathbf{X}_j$.

Next define a new variable, say z_j, as

$$z_j = x_{1j}\, c_{B1} + x_{2j}\, c_{B2} + \ldots + x_{mj}c_{Bm} = \sum_{i=1}^{m} c_{Bi}\, x_{ij} = \mathbf{C}_B\, \mathbf{X}_j. \qquad \ldots(5.9)$$

Δ_j denotes the *net evaluation* which is computed by the formula :

$$\Delta_j = z_j - c_j = \mathbf{C_B X}_j - c_j \qquad \text{...(5.10)}$$

Lastly, these notations can be summerized in the following Starting Simplex Table 5.1.

Table 5.1 : Starting Simplex Table

		$c_j \rightarrow$	c_1	c_2	...	c_n	0	0	...	0	
BASIC VARIABLES	$\mathbf{C}_B$	$\mathbf{X_B}$	$\mathbf{X}_1 (= \mathbf{a}_1)$	$\mathbf{X}_2 (= \mathbf{a}_2)$	...	$\mathbf{X}_n (= \mathbf{a}_n)$	$\mathbf{X}_{n+1}$ (β_1)	$\mathbf{X}_{n+2}$ (β_2)	...	$\mathbf{X}_{n+m}$ (β_m)	MIN RATIO
$x_{n+1} (= s_1)$	$c_{B1} (= 0)$	$x_{B1} (= b_1)$	$x_{11} (= a_{11})$	$x_{12} (= a_{12})$	...	$x_{1n} (= a_{1n})$	1	0	...	0	
$x_{n+2} (= s_2)$	$c_{B2} (= 0)$	$x_{B2} (= b_2)$	$x_{21} (= a_{21})$	$x_{22} (= a_{22})$	...	$x_{2n} (= a_{2n})$	0	1	...	0	
:	:	:	:	:	:	:	:	:	:	:	
$x_{n+m} (= s_m)$	$c_{Bm} (= 0)$	$x_{Bm} (= b_m)$	$x_{m1} (= a_{m1})$	$x_{m2} (= a_{m2})$	...	$x_{mn} (= a_{mn})$	0	0	...	1	
	$z = \mathbf{C}_B\mathbf{X}_B$		Δ_1	Δ_2	...	Δ_n	0	0	...	0	$\Delta_j = \mathbf{C_B X}_j - c_j$

Note. Basic variables in the first column are always sequenced in the order of columns forming the unit matrix.

Above definitions and notations can be clearly understood by the following numerical example.

5.2-1. An Example to Explain Above Definitions and Notations

Example 1. *Illustrate definitions and notations by the linear programming problem :*

Maximize $z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5$, *subject to* $4x_1 + 2x_2 + x_3 + x_4 = 4$, $x_1 + 2x_2 + 3x_3 - x_5 = 8$.

Solution. First of all, constraint equations in matrix form may be written as

$$\overset{\mathbf{A}}{\begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \end{array}} \quad \quad \mathbf{X} \quad \quad \mathbf{B}$$

$$\begin{bmatrix} 4 & 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

or

$$\mathbf{AX} = \mathbf{b}.$$

A basis matrix $\mathbf{B} = (\beta_1, \beta_2)$ is formed using columns $\mathbf{a}_3$ and $\mathbf{a}_1$, so that

$$\beta_1 = \mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ \beta_2 = \mathbf{a}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The rank of matrix $\mathbf{A}$ is 2, and hence $\mathbf{a}_3$, $\mathbf{a}_1$ column vectors are linearly independent, and thus forms a basis for R^2.

Thus, basis matrix is

$$\mathbf{B} = (\beta_1, \beta_2) = \begin{array}{c} \begin{array}{cc} \mathbf{a}_3 & \mathbf{a}_1 \end{array} \\ \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \end{array}$$

Using (5·4) and (5.8), the basic feasible solution is

$$\mathbf{X_B} = \mathbf{B}^{-1}\,\mathbf{b} = \left[\frac{1}{|\mathbf{B}|} \text{adj } (\mathbf{B})\right]\mathbf{b} = \frac{-1}{11}\begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 28 \\ 4 \end{bmatrix}$$

or

$$\mathbf{X_B} = \begin{bmatrix} 28/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix}.$$

Therefore, basic variables are $x_{B1} = 28/11 = x_3$, $x_{B2} = 4/11 = x_1$, and remaining variables are non-basic (which are always zero) *i.e.*, $x_2 = x_4 = x_5 = 0$. Also,

$$c_{B1} = \text{coefficient of } x_{B1} = \text{coeff. of } x_3 = c_3 = 3$$

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$$c_{B2} = \text{coefficient of } x_{B2} = \text{coeff. of } x_1 = c_1 = 1$$

Hence $\qquad \mathbf{C_B} = (3, 1)$.

Now, using (5.7), the value of the objective function is

$$z = \mathbf{C_B X_B} = (3, 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a_j} = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\beta_1 + x_{22}\beta_2 = x_{12}\mathbf{a}_3 + x_{22}\mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (5.3) to get

$$\mathbf{X}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $\qquad x_{12} = 6/11, x_{22} = 4/11.$

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (5.6b), the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{C_B X_2} = (3, 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3 \times \frac{6}{11} + 1 \times \frac{4}{11}\right) = \frac{22}{11}.$$

Similarly z_1, z_3, z_4, z_5 can also be computed.

5·3. COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

The computational aspect of the simplex procedure is first explained by the following simple example.

Example 2. *Consider the linear programming problem :*

Maximize $z = 3x_1 + 2x_2$, subject to the constraints :

$x_1 + x_2 \le 4, x_1 - x_2 \le 2,$ *and* $x_1, x_2 \ge 0$. **[Kanpur 2000, 96; IAS (Maths.) 92]**

Solution. ***Step 1.*** First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by -1. In this example, all the b_i*'s* (right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative ***slack*** or ***surplus*** variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '≤', only slack variables s_1 and s_2 are needed. Therefore, given problem now becomes :

Maximize $\quad z = 3x_1 + 2x_2 + 0s_1 + 0s_2$, subject to the constraints :

$$x_1 + x_2 + s_1 \quad = 4$$
$$x_1 - x_2 \quad + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Step 3. Now, present the constraint equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4. Construct the starting simplex table using the notations already explained in *Sec 5·2*.

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column $\mathbf{X_B}$ gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from Table 5·2 as : $s_1 = 4, s_2 = 2, x_1 = 0, x_2 = 0$, and the value of the objective function is *zero*.

Note. In this step, the variables s_1 and s_2 are corresponding to the columns of basis matrix (identity matrix), so will be called ***basic variables***. Other variables, x_1 and x_2, are *non-basic variables* which always have the value zero.

Table 5·2 : Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3 (S_1) (β_1)	X_4 (S_2) (β_2)	MIN. RATIO X_B/X_k for $X_K > 0$
		$c_j \rightarrow$	3	2	0	0	
					BASIS MATRIX		
s_1	0	4	1	1	1	0	TO BE COMPUTED IN NEXT STEP.
s_2	0	2	1	− 1	0	1	
	$z = C_B X_B$		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = z_j - c_j = C_B X_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the *'net evaluation'* Δ_j for each variable x_j (column vector $\mathbf{X}_j$) by the formula

$$\Delta_j = z_j - c_j = \mathbf{C_B X}_j - c_j \qquad \text{[from (5·10)]}$$

Thus, we get

$\Delta_1 = \mathbf{C_B X}_1 - c_1 = (0, 0)(1, 1) - 3 = (0 \times 1 + 0 \times 1) - 3 = -3$

$\Delta_2 = \mathbf{C_B X}_2 - c_2 = (0, 0)(1, -1) - 2 = (0 \times 1 - 0 \times 1) - 2 = -2$

$\Delta_3 = \mathbf{C_B X}_3 - c_3 = (0, 0)(1, 0) - 0 = (0 \times 1 + 0 \times 0) - 0 = 0$

$\Delta_4 = 0$

Remark. Note that in the starting simplex table Δ_j's are same as $(-c_j)$'s. Also, Δ_j's corresponding to the columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test :

(i) If all $\Delta_j\ (= z_j - c_j) \geq 0$, the solution under test will be ***optimal.*** *Alternative optimal solutions* will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column $\mathbf{X}_j$ are negative or zero (≤ 0), then the solution under test will be ***unbounded.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in ***Step 6.***

Step 6. In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector'*** and ***'outgoing vector'*** respectively.

'Incoming vector'. The incoming vector $\mathbf{X}_k$ is always selected corresponding to the most negative value of Δ_j (say, Δ_k). Here $\Delta_k = \min\ [\Delta_1, \Delta_2] = \min\ [-3, -2] = -3 = \Delta_1$. Therefore, $k = 1$ and hence column vector $\mathbf{X}_1$ must enter the basis matrix. The column $\mathbf{X}_1$ is marked by an upward arrow (↑).

'Outgoing vector'. The outgoing vector β_r is selected corresponding to the minimum ratio of elements of $\mathbf{X_B}$ by the corresponding *positive* elements of predetermined incoming vector $\mathbf{X}_k$. This rule is called the ***Minimum Ratio Rule.*** In mathematical form, this rule can be written as

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Br}}{x_{ik}},\ x_{ik} > 0\right]$$

For $k = 1$,

$$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{4}{1}, \frac{2}{1}\right]$$

or

$$\frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}.$$

Comparing both sides of this equation, we get $r = 2$. So the vector β_2, *i.e.*, $\mathbf{X}_4$ marked with downward arrow (↓) should be removed from the basis matrix. The ***Starting Table 5·2*** is now modified to ***Table 5·3*** given below.

Table 5·3

key element

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	Ca	Xa	X_1	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$ (β_2)	MIN. RATIO (X_B/X_1)
s_1	0	4	1	1	1	0	4/1
s_2	0	2	$\boxed{1}$	1	0	1	2/1 ← MIN. RATIO
	$z = C_B X_B = 0$		-3 (min. Δ_j)	-2	0	0	$\leftarrow \Delta_j = z_j - c_j = C_B B_j - c_j$
			↑entering vector			↓leaving vector	

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked '□' position and zero at all other places of X_1. If the number in the marked '□' position is other than *unity, divide all elements of that row by the 'key element'. (The element at the intersection of minimum ratio arrow* (←) *and incoming vector arrow* (↑) *is called the **key element** or **pivot element**).*

Then, subtract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column X_1. Thus, the process can be fortified by simple matrix transformation as follows :

The intermediate coefficient matrix is :

	X_B	X_1	X_2	X_3	X_4	
R_1	4	1	1	1	0	
R_2	2	1	-1	0	1	
R_3	$z = 0$	-3	-2	0	0	$\leftarrow \Delta_j$

Apply $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow R_3 + 3R_2$ to obtain

X_B	X_1	X_2	X_3	X_4	
2	0	2	1	-1	
2	1	-1	0	1	
$z = 6$	0	-5	0	3	$\leftarrow \Delta_j$

Now, construct the improved simplex table as follows :

Table 5·4

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_2)	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$	MIN-RATIO $(X_B/X_2, X_2 > 0)$
s_1	0	2	0	$\boxed{2}$	1	-1	$\frac{2}{2}$ ← key row
x_1	3	2	1	-1	0	1	$\cancel{\frac{2}{-1}}$ (negative ratio is not counted)
	$z = C_B X_B = 6$		0	-5 ↑	0 ↓	3	$\leftarrow \Delta_j$
				key column			

From this table, the improved basic feasible solution is read as : $x_1 = 2$, $x_2 = 0$, $s_1 = 2$, $s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = C_B X_j - c_j$.

Step 8. Now repeat *Steps* 5 *through* 7 as and when needed until an optimum solution is obtained in Table 5·5.

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_2 .$$

Therefore, $k = 2$ and hence $\mathbf{X}_2$ should be the entering vector (key column). By minimum ratio rule :

$$\text{Minimum Ratio}\left(\frac{\mathbf{X_B}}{\mathbf{X}_2}, \mathbf{X}_2 > 0\right) = \text{Min}\left[\frac{2}{2}, -\right]$$ (since negative ratio is not counted, so the second ratio is not considered)

Since *first ratio* is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as :

	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	
R_1	1	0	$\boxed{1}$	1/2	− 1/2	
R_2	2	1	− 1	0	1	
R_3	$z = 6$	0	− 5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 5R_1$

1	0	1	1/2	− 1/2	
3	1	0	1/2	1/2	
$z = 11$	0	0	5/2	1/2	$\leftarrow \Delta_j$

Now construct the next improved simplex table as follows :

Final Simplex Table 5·5

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1$ (β_2)	$\mathbf{X}_2$ (β_1)	$\mathbf{S}_1$	$\mathbf{S}_2$	
$\rightarrow x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
	$z = \mathbf{C_B X_B} = 11$		0	0	5/2	1/2	$\leftarrow \Delta_j$

The solution as read from this table is : $x_1 = 3$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, and max. $z = 11$. Also, using the formula $\Delta_j = \mathbf{C_B X_j} - c_j$ verify that all Δ_j's are non-negative. Hence the optimum solution is

$$x_1 = 3\,,\; x_1 = 1\;;\; \max z = 11\,.$$

Note. If at the optimal stage, it is desired to bring s_1 in the solution, the total profit will be reduced from 11 (the optimal value) to 5/2 times of 2 units of s_1 in Table 3·4, i.e., $z = 11 - 5/2 \times 2 = 6$. This explains the *economic interpretation* of net-evaluations Δ_j .

5·4. SIMPLE WAY FOR SIMPLEX METHOD COMPUTATIONS

Complete solution with its different computational steps can be more conveniently represented by the following single table (see Table 5.6).

Table 5·6

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{S}_1$	$\mathbf{S}_2$	MIN RATIO ($\mathbf{X_B/X_K}$)
s_1	0	4	↓	1	1	0	4/1
$\leftarrow s_2$	0	2	$\boxed{1}$	1	0	1	2/1 ← Min
$x_1 = x_2 = 0$	$z = \mathbf{C_B X_B} = 0$		− 3* ↑	− 2	0	0 ↓	$\leftarrow \Delta_j = z_j - c_j$
$\leftarrow s_1$	0	2	0	$\boxed{2}$	1	−1	2/2 Min ←
$\rightarrow x_1$	3	2	1	−1	0	1	—
$x_2 = s_2 = 0$	$z = \mathbf{C_B X_B} = 6$		0	−5* ↑	0 ↓	3	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
$s_1 = s_2 = 0$	$z = \mathbf{C_B X_B} = \mathbf{11}$		0	0	5/2	1/2	← All $\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 11$.

Q. 1. What is a simplex ? Describe simplex method of solving linear programming problems. **[Kanpur (B. Sc.) 90]**

2. Write the steps used in the simplex method.

3. Describe a computational procedure of the simplex method for the solution of a maximization l.p.p.

Tips for Quick Solution :

1. In the first iteration only, since Δ_j's are the same as $-c_j$'s, so there is no need of calculating them separately by using the formula $\Delta_j = \mathbf{C_B X_j} - c_j$.
2. Mark *min* (Δ_j) by '↑' which at once indicates the column $\mathbf{X}_k$ needed for computing the minimum ratio $(\mathbf{X_B}/\mathbf{X}_k)$.
3. '*Key element*' is found at the place where the upward directed arrow '↑' of min Δ_j and the left directed arrow (←) of minimum ratio $(\mathbf{X_B}/\mathbf{X}_k)$ intersect each other in the simplex table.
4. '*Key element*' indicates that the current table must be transformed in such a way that the key element becomes 1 and all other elements in that column become 0.
5. Since Δ_j's corresponding to unit column vectors are always zero, there is no need of calculating them.
6. While transforming the table by row operations, the value of z and corresponding Δ_j's are also computed at the same time. Thus a lot of time and labour can be saved in adopting this technique.

Example 3. *Min* $z = x_1 - 3x_2 + 2x_3$, *subject to* :

$$3x_1 - x_2 + 3x_3 \le 7, -2x_1 + 4x_2 \le 12, -4x_1 + 3x_2 + 8x_3 \le 10, \text{ and } x_1, x_2, x_3 \ge 0.$$

[Kanpur (B.Sc.) 95, 93, (B.A.) 90]

Solution. This is the problem of minimization. Converting the objective function from minimization to maximization, we have

$$\text{Max.} -z = -x_1 + 3x_2 - 2x_3 = \text{Max. } z' \text{ where } -z = z',$$

Here we give only tables of solution. The students are advised to verify them.

Table 5·7. Simplex Table

BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$ $c_j \rightarrow$	$\mathbf{X_1}$ (−1)	$\mathbf{X_2}$ (3)	$\mathbf{X_3}$ (−2)	$\mathbf{X_4}$ (0)	$\mathbf{X_5}$ (0)	$\mathbf{X_6}$ (0)	MIN. RATIO $(\mathbf{X_B}/\mathbf{X}_k)$
x_4	0	7	3	−1	3	1	0	0	—
$\leftarrow x_5$	0	12	−2	← $\boxed{4}$	0	0	1	0	12/4 ← min.
x_6	0	10	−4	3	8	0	0	1	10/3
$x_1 = x_2 = x_3 = 0$	$z' = 0, z = 0$		1	−3* ↑	2	0 ↓	0	0	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	10	$\boxed{5/2}$	0	3	1	1/4	0	$\frac{10}{5/2}$ ←
$\rightarrow x_2$	3	3	−1/2	1	0	0	1/4	0	—
x_6	0	1	−5/2	0	8	0	−3/4	1	—
$x_1 = x_3 = x_5 = 0$	$z' = 9$ ∴ $z = -9$		−1/2* ↑	0	2	0 ↓	3/4	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	−1	4	1	0	6/5	2/5	1/10	0	
x_2	3	5	0	1	3/5	1/5	3/10	0	
x_6	0	11	0	0	11	1	−1/2	1	
$x_3 = x_4 = x_2 = 0$	$z' = 11$ ∴ $z = -11$		0	0	13/5	1/5	8/10	0	$\leftarrow \Delta_j \ge 0$

The optimal solution is : $x_1 = 4, x_2 = 5, x_3 = 0$, *Min* $z = -11$.

Example 4. *Max.* $z = 3x_1 + 2x_2 + 5x_3$, *subject to the constraints* :

$$x_1 + 2x_2 + x_3 \le 430, 3x_1 + 2x_3 \le 460, x_1 + 4x_2 \le 420, \text{ and } x_1, x_2, x_3 \ge 0.$$

[IAS (Main 94)]

Solution.

Table 5·8. Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO (X_B/X_k)
	$c_j \rightarrow$		3	2	5	0	0	0	
x_4	0	430	1	2	1	1	0	0	430/1
$\leftarrow x_5$	0	460	3	0	$\boxed{2}$	0	1	0	460/2 ←
x_6	0	420	1	4	0	0	0	1	—
$x_1 = x_2 = x_3 = 0$	$z = 0$		–3	–2	–5* ↑	0	0 ↓	0	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	200	– 1/2	$\boxed{2}$	0	1	–1/2	0	200/2 ←
$\rightarrow x_3$	5	230	3/2	0	1	0	1/2	0	—
x_6	0	420	1	4	0	0	0	1	420/4
$x_1 = x_2 = x_5 = 0$	$z = 1150$		9/2	–2* ↑	0	0 ↓	5/2	0	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	100	– 1/4	1	0	1/2	– 1/4	0	
x_3	5	230	3/2	0	1	0	1/2	0	
x_6	0	20	2	0	0	–2	1	1	
$x_1 = s_4 = x_5 = 0$	$z = 1350$		4	0	0	1	2	0	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, the solution is : $x_1 = 0$, $x_2 = 100$, $x_3 = 230$, max $z = 1350$.

Example 5. *Solve the LP problem :Max.* $z = 3x_1 + 5x_2 + 4x_3$, *subject to the constraints :*
$2x_1 + 3x_2 \leq 8$, $2x_2 + 5x_3 \leq 10$, $3x_1 + 2x_2 + 4x_3 \leq 15$, *and* $x_1, x_2, x_3 \geq 0$.

[Tamilnadu (ERODE) 97; Rewa 93; Kanpur (B.Sc.) 92; (B.A.) 90, Meerut (M.Sc. Stat. & B.Sc. Math.) 90]

Solution. After introducing slack variables, the constraint equations become :

$$\begin{aligned} 2x_1 + 3x_2 \quad + x_4 \quad\quad\quad &= 8 \\ 2x_2 + 5x_3 \quad\quad + x_5 \quad &= 10 \\ 3x_1 + 2x_2 + 4x_3 \quad\quad\quad + x_6 &= 15. \end{aligned}$$

Table 5·9. Starting Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO. (X_B/X_2)
	$c_j \rightarrow$		3	5	4	0	0	0	
$\leftarrow x_4$	0	8	2	$\boxed{3}$	0	1	0	0	8/3 ←
x_5	0	10	0	2	5	0	1	0	10/2
x_6	0	15	3	2	4	0	0	1	15/2
$x_1 = x_2 = x_3 = 0$	$z = C_B X_B = 0$		–3	–5* ↑ Incoming vector	–4	0 ↓ outgoing vector	0	0	$\leftarrow \Delta_j$

Now apply short-cut method for minimum ratio rule (min X_B/X_2), and find the key element 3. This key element indicates that unity should be at first place of X_2 , so the vector to be removed from the basis matrix is X_4.

Now, in order to get the second simplex table, calculate the intermediate coefficient matrices as follows :

First, divide the first row by 3 to get

R_1	8/3	2/3	1	0	1/3	0	0	
R_2	10	0	2	$\boxed{5}$	0	1	0	
R_3	15	3	2	4	0	0	1	
R_4	0	– 3	– 5	– 4	0	0	0	$\leftarrow \Delta_j$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 2R_1$, $R_4 \rightarrow R_4 + 5R_1$,

R_1	8/3	2/3	$\boxed{1}$	0	1/3	0	0	
R_2	14/3	−4/3	0	5	−2/3	1	0	
R_3	29/3	5/3	0	4	−2/3	0	1	
R_4	40/3	1/3	0	−4	5/3	0	0	$\leftarrow \Delta_j$

Now the second simplex table (Table 5.10) is constructed as below :

Table 5·10

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	3 X_1	5 X_2	4 X_3	0 X_4	0 X_5	0 X_6	MIN RATIO (X_B/X_3)
$\rightarrow x_2$	5	8/3	2/3	1	0	1/3	0	0	—
$\leftarrow x_5$	0	14/3	−4/3	0	$\leftarrow \boxed{5}$	−2/3	1	0	$\frac{14}{3}/5 \leftarrow$
x_6	0	29/3	5/3	0	4	−2/3	0	1	$\frac{29}{3}/4$
$x_4 = x_1 = x_3 = 0$		$z = \frac{40}{3}$	1/3	0	−4* ↑	5/3	0 ↓	0	$\leftarrow \Delta_j$

Incoming Outgoing

Now verify that

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = -3 + (5, 0, 0)(2/3, -4/3, 5/3) = 1/3$$

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = -4 + (5, 0, 0)(0, 5, 4) = -4$$

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = 0 + (5, 0, 0)(1/3, -2/3, -2/3) = 5/3.$$

The key-element is found to be 5. Hence the vector to be removed from the basis matrix is $\mathbf{X_5}$. Thus proceeding exactly in the same manner, the remaining simplex tables are obtained (Tables 5.11 and 5.12).

Table 5·11

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{2}{3}/\frac{2}{3}$
$\rightarrow x_3$	4	14/15	−4/15	0	1	−2/15	1/15	0	—
$\leftarrow x_6$	0	89/15	$\boxed{41/15}$	0	0	−2/15	−4/5	1	$\frac{89}{15}/\frac{41}{15} \leftarrow$
$x_1 = x_5 = x_4 = 0$		$z = 256/15$	−11/15* ↑	0	0	17/15	4/5	0 ↓	$\leftarrow \Delta_j$

Incoming Outgoing

Table 5·12. Final Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN RATIO
x_2	5	50/41	0	1	0	15/41	8/41	−10/41	
x_3	4	62/41	0	0	1	−6/41	5/41	4/41	
x_1	3	89/41	1	0	0	−2/41	−12/41	15/41	
$x_4 = x_5 = x_6 = 0$		$z = \mathbf{C_B X_B} = 765/41$	0	0	0	45/41	24/41	11/41	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, the solution given by $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$, max $z = 765/41$, is optimal.

Example 6. *Minimize* $z = x_2 - 3x_3 + 2x_5$ *,subject to the constraints :*

$$3x_2 - x_3 + 2x_5 \leq 7, -2x_2 + 4x_3 \leq 12, -4x_2 + 3x_3 + 8x_5 \leq 10, \textit{ and } x_2, x_3, x_5 \geq 0.$$

[JNTU (Mech.) 99; Kanpur 96; Madurai B.Sc. (Comp. Sc.) 92, (Appl. Math) 85; Kerala B.Sc. (Math.) 90]

Solution. Equivalently, max $z' = -x_2 + 3x_3 - 2x_5$ where $z' = -z$. Introducing x_1, x_4 and x_6 as slack variables, the constraint equations become :

$$x_1 + 3x_2 - x_3 + 0x_4 + 2x_5 + 0x_6 = 7$$

$$0x_1 - 2x_2 + 4x_3 + x_4 + 0x_5 + 0x_6 = 12$$

$$0x_1 - 4x_2 + 3x_3 + 0x_4 + 8x_5 + x_6 = 10.$$

Now proceeding as in above example the simplex computations are performed as follows :

Table 5.13

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	0 X_1	−1 X_2	3 X_3	0 X_4	−2 X_5	0 X_6	MIN RATIO (X_B/X_k)
x_1	0	7	1	3	−1	0	2	0	—
$\leftarrow x_4$	0	12	0	−2	$\boxed{4}$	1	0	0	12/4 ←
x_6	0	10	0	−4	3	0	8	1	10/3
		$z' = 0$	0	1	−3* ↑	0 ↓	2	0	$\leftarrow \Delta_j$
$\leftarrow x_1$	0	10	1	$\boxed{5/2}$	0	1/4	2	0	4 ←
$\rightarrow x_3$	3	3	0	−1/2	1	1/4	0	0	—
x_6	0	1	0	−5/2	0	−3/4	8	1	—
		$z' = 9$	0 ↓	−1/2* ↑	0	3/4	2	0	$\leftarrow \Delta_j$
x_2	−1	4	2/5	1	0	1/10	4/5	0	
x_3	3	5	1/5	0	1	3/10	2/5	0	
x_6	0	11	1	0	0	−1/2	10	1	
		$z' = 11$ or $z = -11$	1/5	0	0	4/5	12/5	0	$\leftarrow \Delta_j \geq 0$

Thus, optimal solution is : $x_2 = 4$, $x_3 = 5$, $x_5 = 0$, min. $z = -11$.

Alternative forms of Example 6 :

(i) *Min.* $z = x_1 - 3x_2 + 2x_3$, *subject to* $3x_1 - x_2 + 2x_3 \leq 7$, $-2x_1 + 4x_2 \leq 12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$ *and* $x_1 \,.\, x_2 \,,\, x_3 \geq 0$.

(ii) *Min.* $z = x_2 - 3x_3 + 2x_5$, *subject to the constraints :*

$$x_1 + 3x_2 - x_3 + 2x_5 = 7, -2x_2 + 4x_3 + x_4 = 12, -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \text{ and } x_1, x_2, \ldots, x_6 \geq 0 .$$

Example 7 (Bounded Variables). *A manufacturer of three products tries to follow a policy of producing those which continue most to fixed cost and profit. However, there is also a policy of recognising certain minimum sales requirements currently, these are :*

Product :	X_1	X_2	X_3
Units per week :	20	30	60

There are three producing departments. The product times in hour per unit in each department and the total times available for each week in each department are :

Departments	*Time required per product in hours* X_1	X_2	X_3	*Total hours available*
1	0·25	0·20	0·15	420
2	0·30	0·40	0·50	1048
3	0·25	0·30	0·25	529

The contribution per unit of product X_1 , X_2 , X_3 *is Rs.* 10·50, *Rs.* 9·00 *and Rs.* 8·00 *respectively. The company has scheduled 20 units of* X_1 , *30 units of* X_2 *and 60 units of* X_3 *for production in the following week, you are required to state :*

(i) *Whether the present schedule is an optimum one from a profit point of view and if it is not, what it should be;*

(ii) *The recommendations that should be made to the firm about their production facilities (following the answer to (i) above).*

Solution. The formulation of the problem is as follows :

Maximize $z = 10{\cdot}5X_1 + 9X_2 + 8X_3$, *subject to the constraints :*

$$0{\cdot}25X_1 + 0{\cdot}20X_2 + 0{\cdot}15X_3 \leq 420$$
$$0{\cdot}30X_1 + 0{\cdot}40X_2 + 0{\cdot}50X_3 \leq 1048$$

$$0{\cdot}25X_1 + 0{\cdot}30X_2 + 0{\cdot}25X_3 \le 529$$
$$0 \le X_1 \ge 20\ ,\ 0 \le X_2 \ge 30\ ,\ 0 \le X_3 \ge 60\ .$$

Since the company is already producting minimum of X_2 and X_3 it should, at least, produce maximum of X_1 limited by the first constraint. Lower bounds are specified in this problem, i.e., $X_1 \ge 20$, $X_2 \ge 30$, $X_3 \ge 60$. This can be handled quite easily by introducing the new variables x_1 , x_2 and x_3 such that

$$X_1 = 20 + x_1, X_2 = 30 + x_2, X_3 = 60 + x_3\ .$$

Substituting for X_1 , X_2 and X_3 in terms of x_1 , x_2 , x_3, the problem now becomes :

Maximize $z = 10{\cdot}5x_1 + 9x_2 + 8x_3 +$ constant, *subject to the constraints :* $0{\cdot}25x_1 + 0{\cdot}20x_2 + 0{\cdot}15x_3 \le 400$, $0{\cdot}30x_1 + 0{\cdot}40x_2 + 0{\cdot}50x_3 \le 1000$, $0{\cdot}25x_1 + 0{\cdot}30x_2 + 0{\cdot}25x_3 \le 500$, *and* $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

The students may now proceed to find the optimal solution by simplex method in the usual manner.

Example 8. *For a company engaged in the manufacture of three products, viz. X , Y and Z , the available data are given below :*

Minimum Sales Requirement

Product :	X	Y	Z
Min. sales requirement per month :	10	20	30

Operations, Required Processing Times and Capacity

Operations	*Time (hrs.) required per item of*			*Total available hours per month*
	X	*Y*	*Z*	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180

Profit (Rs.) per unit

Product :	*X*	*Y*	*Z*
Profit (Rs.)/unit :	10	15	8

Find out the product-mix to maximize profit. **[C.A. (Nov.) 89]**

Solution. Let x , y and z denote the number of units produced per month for the products X , Y and Z, respectively.

Minimum sales requirements give the constraints : $x \ge 10$, $y \ge 20$, $z \ge 30$, where x , y , $z \ge 0$.

Operations, processing times and capacity lead to the following constraints :

$$x + 2y + 2z \le 200 \ ...(i) \qquad 2x + y + z \le 220 \ ..(ii) \qquad 3x + y + 2z \le 180 \ ...(iii)$$

The objective function is : Max. $\mathbf{P} = 10x + 15y + 8z$. Thus we have to solve the following problem :

Max. $\mathbf{P} = 10x + 15y + 8z$, *subject to* $x + 2y + 2z \le 200$, $2x + y + z \le 220$, $3x + y + 2z \le 180$, and $0 \le x \ge 10$, $0 \le y \ge 20$, $0 \le z \ge 30$.

Let us make the substitutions : $x = a + 10, y = b + 20, z = c + 30$, where a , b , $c \ge 0$.

Substituting these values in the objective function and constraints (i), (ii) and (iii), the problem becomes :

Max. $\mathbf{P} = 10a + 15b + 8c + 640$, *subject to,*

$$(a + 10) + 2\,(b + 20) + 2\,(c + 30) \le 200$$
$$2(a + 10) + (b + 20) + (c + 30) \le 220$$
$$3(a + 10) + (b + 20) + 2\,(c + 30) \le 180$$

where $a \ge 0, b \ge 0, c \ge 0$.

Solving this problem by simplex method we get the solution : $a = 10$, $b = 40$ and $c = 0$. Substituting these values, we find the original values :

$x = 10 + 10 = 20$, $y = 40 + 20 = 60$, $z = 0 + 30 = 30$, and the maximum value of objective function is given by $P =$ Rs. 1340 .

The optimal product mix is to produce 20 units of X , 60 units of Y , and 30 units of Z to get a maximum profit of Rs. 1340.

Example 9. *Nooh's Boats makes three different kinds of boats. All can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table :*

Input	Row Boat	Canoe	Keyak	Monthly Available
Labour (Hours)	12	7	9	1.260 hrs.
Wood (Board feet)	22	18	16	19,008 board feet
Screws (Kg.)	2	4	3	396 Kg.
Selling price (in Rs.)	4,000	2,000	5,000	

(a) Formulate the above as a linear programming problem.

(b) Solve it by simplex method. From the optimal table of the solved linear programming problem, answer the following questions :

(c) How many boats of each type will be produced and what will be the resulting revenue ?

(d) Which, if any, of the resources are not fully utilized ? If so, how much of spare capacity is left ?

(e) How much wood will be used to make all of the boats given in the optimal solution ? **[C.A. (Nov.) 93]**

Solution. (a) Let x_1, x_2 and x_3 be the number of *Row Boats, Canoe* and *Keyak* made every month. The linear programming model can be formulated as follows :

Max. Revenue $z = 4{,}000x_1 + 2{,}000x_2 + 5{,}000x_3$, *subject to*

$12x_1 + 7x_2 + 9x_3 \leq 1260$, $22x_1 + 18x_2 + 16x_3 \leq 19008$, $2x_1 + 4x_2 + 3x_3 \leq 396$ and $x_1, x_2, x_3, \geq 0$.

(b) Adding slack variables s_1, s_2, s_3, the above formulated problem becomes

Max. $z = 4000x_1 + 2000x_2 + 5000x_3 + 0s_1 + 0s_2 + 0s_3$, *subject to :*

$12x_1 + 7x_2 + 9x_3 + s_1 = 1260$, $22x_1 + 18x_2 + 16x_3 + s_2 = 19008$, $2x_1 + 4x_2 + 3x_3 + s_3 = 396$, and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

The starting solution and subsequent simplex tables are given below :

		$c_j \rightarrow$	4000	2000	5000	0	0	0	
Basic Variables	Prog. C_B	Qty X_B	X_1	X_2	X_3	S_1	S_2	S_3	Replacement Ratio Min (X_B/X_K)
s_1	0	1,260	12	7	9	1	0	0	1260/9
s_2	0	19,008	22	18	16	0	1	0	19008/16
s_3	0	396	2	4	$\leftarrow \boxed{3}$	0	0	1	396/3 $\leftarrow$
	$z = 0$		−4000	−2000	−5000↑	0	0	0↓	$\leftarrow \Delta_j$ (NER)
s_1	0	72	$\boxed{6}$	−5	0	1	0	−3	12 $\leftarrow$
s_2	0	16,896	34/3	−10/.3	0	0	1	−16/3	1491
x_3	5000	132	2/3	4/3	1	0	0	1/3	198
	$z = 660000$		−2000/3	14,000/3↑	0	0↓	0	5000/3	$\leftarrow \Delta_j$
x_1	4000	12	1	−5/6	0	1/6	0	−1/2	
s_2	0	16,760	0	55/9	0	−17/9	1	1/3	
x_3	5000	124	0	17/9	1	−1/9	0	2/3	
	$z = 6{,}68{,}000$		0	37,000/9	0	1000/9	0	4000/3	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, the optimal solution is given by $x_1 = 12$, $x_2 = 0$ and $x_3 = 124$.

(c) The company should produce 12 *Row boats* and *124 Kayak boats* only. The maximum revenue will be Rs. 6,68,000.

(d) Wood is not fully utilized. Its share capacity is 16,760 board feet.

(e) The total wood used to make all of the boats given by the optimum solution is

$$= 22 \times 12 + 16 \times 124 = 2{,}248 \text{ board feet.}$$

EXAMINATION PROBLEMS

Solve the following problems by simplex method :

1. Max. $z = 5x_1 + 3x_2$, subject to
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
$x_1, x_2 \geq 0$. **[JNTU (B. Tech. III) 2003; Kanpur B.Sc. (ii) 2003]**
[**Ans.** $x_1 = 20/19$, $x_2 = 45/19$, Max. $z = 235/19$]

2. Max. $z = 7x_1 + 5x_2$, subject to
$-x_1 - 2x_2 \geq -6$
$4x_1 + 3x_2 \leq 12$,
$x_1, x_2 \geq 0$ **[Meerut (IPM) 91]**
[**Ans.** $x_1 = 3$, $x_2 = 0$, Max. $z = 21$]

3. Max. $z = 5x_1 + 7x_2$, subject to
$x_1 + x_2 \le 4$
$3x_1 - 8x_2 \le 24$
$10x_1 + 7x_2 \le 35$
and $x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 0$, $x_2 = 4$, $max.\ z = 28$]

4. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 40$
$x_1 + x_2 \le 24$
$2x_1 + 3x_2 \le 60$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 16$, $x_2 = 8$, $z^* = 64$]

5. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 5$
$x_1 + x_2 \le 3$
$x_1, x_2 \ge 0$
and $x_1, x_2 \ge 0$. [I.A.S (Main) 91]
[**Ans.** $x_1 = 6$, $x_2 = 12$, $z^* = 60$]

6. Max. $z = 2x_1 + 4x_2$, subject to
$2x_1 + 3x_2 \le 48$
$x_1 + 3x_2 \le 42$
$x_1 + x_2 \le 21$
and $x_1, x_2 \ge 0$
[**Ans.** Solution is unbounded]

7. Max. $z = 3x_1 + 4x_2$, subject to
$x_1 - x_2 \le 1$
$-x_1 + x_2 \le 2$
$x_1, x_2 \ge 0$.
[**Ans.** Sol. is unbounded]

8. Max. $z = 3x_1 + 2x_2$, subject to
$2x_1 + x_2 \le 10$
$x_1 + 3x_2 \le 6$
$x_1, x_2 > 0$
[**Ans.** $x_1 = 24/5$, $x_2 = 2/5$, $z^* = 76/5$]

9. Max. $z = 2x_1 + 5x_2$, subject to
$x_1 + 3x_2 \le 3$
$3x_1 + 2x_2 \le 6$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2$, $x_2 = 0$, $z^* = 4$]

10. Max. $z = 3x_1 + 5x_2$, subject to
$3x_1 + 2x_2 \le 18$
$x_1 \le 4$
$x_2 \le 6$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2$, $x_2 = 6$, $z^* = 36$]

11. Max. $z = 2x_1 + x_2$, subject to
$x_1 + 2x_2 \le 10$
$x_1 + x_2 \le 6$
$x_1 - x_2 \le 2$
$x_1 - 2x_2 \le 1$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 4$, $x_2 = 2$, $z^* = 10$]

12. Max. $z = 2x + 5y$, subject to
$x + y \le 600$
$0 \le x \le 400$
$0 \le y \le 300$
[**Ans.** Two Iterations.
$x = 300$, $y = 300$, max $z = 2100$]

13. Max. $z = x_1 - x_2 + 3x_3$, subject to
$x_1 + x_2 + x_3 \le 10$
$2x_1 - x_3 \le 2$
$2x_1 - 2x_2 + 3x_3 \le 0$
and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0$, $x_2 = 6$, $x_3 = 4$, $z^* = 6$]

14. Max. $z = x_1 + x_2 + x_3$, subject to
$4x_1 + 5x_2 + 3x_3 \le 15$
$10x_1 + 7x_2 + x_3 \le 12$
and $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0$, $x_2 = 0$, $x_3 = 5$, $z^* = 5$]

15. Max. $z = 8x_1 + 19x_2 + 7x_3$,
subject to
$3x_1 + 4x_2 + x_3 \le 25$
$x_1 + 3x_2 + 3x_3 \le 50$
$x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 7/3$, $x_2 = 9$, $x_3 = 0$]

16. Max. $z = x_1 + x_2 + 3x_3$,
subject to
$3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0$.
[VTU (BE common) 2002]
[**Ans.** $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $z^* = 3$]

17. Max. $z = 4x_1 + 3x_2 + 4x_3 + 6x_4$,
subject to
$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$
$2x_1 + 2x_3 + x_4 \le 60$
$3x_1 + 3x_2 + x_3 + x_4 \le 80$
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 280/13$, $x_2 = 0$,
$x_3 = 20/13$, $x_4 = 180/13$
$z^* = 2280/13$].

18. Max. $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$.
subject to
$x_1 + x_2 + x_3 + x_4 \le 15$
$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$
$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$
$x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 50/7$, $x_2 = 0$, $x_3 = 55/7$,
$x_4 = 0$, $z^* = 695/7$]

19. Max. $z = 2x_1 + 4x_2 + x_3 + x_4$,
subject to
$2x_1 + x_2 + 2x_3 + 3x_4 \le 12$,
$3x_1 + 2x_3 + 2x_4 \le 20$,
$2x_1 + x_2 + 4x_3 \le 16$,
$x_1, x_2, x_3, x_4 \ge 0$. **[JNTU (MCA) 2004]**
[**Ans.** $x_1 = x_3 = x_4 = 0$, $x_2 = 12$,
max. $z = 48$. On iteration only]

20. Max. $z = 5x_1 + 3x_2$,
subject to the constraints :
$x_1 + x_2 \le 2$
$5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12$
$x_1, x_2 \ge 0$.
[**Ans.** $x_1 = 2$, $x_2 = 0$, max. $z = 10$,
one iteration only]

21. Max. $z = 8x_1 + 11x_2$
subject to the constraints :
$3x_1 + x_2 \le 7$, $x_1 + 3x_2 \le 8$, $x_1, x_2 \ge 0$.
[**Ans.** Two iterations.
$x_1 = 13/8$, $x_2 = 17/8$, max $z = 291/8$]

22. Max. $z = 10x_1 + x_2 + 2x_3$, subject to the constraints
$x_1 + x_2 - 3x_3 \le 10$, $4x_1 + x_2 + x_3 \le 20$, $x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 5$, $x_2 = 0$, $x_3 = 0$, max. $z = 50$]

23. Max. $z = 2x_1 + 4x_2 + x_3 + x_4$, subject to the constraints : $x_1 + 3x_2 + x_4 \le 4$, $2x_1 + x_2 \le 3$, $x_2 + 4x_3 + x_4 \le 3$; $x_1, x_2, x_3, x_4 \ge 0$.
[**Ans.** $x_1 = 1$, $x_2 = 1$, $x_3 = 1/2$, $x_4 = 0$, max. $z = 13/2$]

24. Max. $z = 10x_1 + 6x_2$, subject to the constraints :
$x_1 + x_2 \le 2$, $2x_1 + x_2 \le 4$, $3x_1 + 8x_2 \le 12$, and
$x_1, x_2 \ge 0$.
[**Ans.** One iteration only.
$x_1 = 2$, $x_2 = 0$, max. $z = 20$]

25. Max. $z = 107x_1 + x_2 + 2x_3$, subject to the constraints :
$14x_1 + x_2 - 6x_3 + 3x_4 = 7$, $16x_1 + 1/2x_2 - 6x_3 \le 5$, $3x_1 - x_2 - x_3 \le 0$.
$x_1, x_2, x_3, x_4 \ge 0$.
[**Hint.** Divide the first equation by 3 (coefficient of x_4) and then treat x_4 as the slack variable]. [**Ans.** Unbounded solution].

...thod by carrying out one iteration in the following problem :
...subject to the constraints :

$x_1 + 2x_2 + 2x_3 + x_4 = 8$, $3x_1 + 4x_2 + x_3 + x_5 = 7$ and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$.

[**Ans.** One iteration only, $x_1 = x_2 = x_4 = 0$, $x_3 = 4$, $x_5 = 3$, max. $z = 15$]

27. Max. $z = 3x_1 + 2x_2 - 2x_3$
subject to the constraints :
$x_1 + 2x_2 + 2x_3 \leq 10$
$2x_1 + 4x_2 + 3x_3 \leq 15$
$x_1, x_2 < x_3 \geq 0$
[**Ans.** One iteration only.
$x_1 = 15/2$, $x_2 = x_3 = 0$, max. $z = 45/2$]

28 Max. $z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints :
$6x_1 + 5x_2 + 3x_3 \leq 52$
$6x_1 + 2x_2 + 5x_3 \leq 14$.
$x_1, x_2, x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = 0$, $x_2 = 7$, $x_3 = 0$, max. $z = 161$]

29. Max. $z = 7x_1 + x_2 + 2x_3$,
subject to the constraints :
$x_1 + x_2 - 2x_3 \leq 10$
$4x_1 + x_2 + x_3 \leq 20$
$x_1, x_2, x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = x_2 = 0$, $x_3 = 20$
max. $z = 40$]

30. Max. $R = 2x - 3y + z$
subject to the constraints :
$3x + 6y + z \leq 6$
$4x + 2y + z \leq 4$
$x - y + z \leq 3$
$x, y, z \geq 0$
[**Ans.** Two iterations. $x = 1/3$,
$y = 0$, $z = 8/3$, max. $R = 10/3$]

31. Max. $R = 2x + 4y + 3z$
subject to the constraints :
$3x + 4y + 2z \leq 60$
$2x + y + 2z \leq 40$
$x + 3y + 2z \leq 80$
$x, y, z \geq 0$.
[**Ans.** Two iterations. $x = 0$, $y = 20/3$,
$z = 50/7$, max. $R = 250/3$.]

32. Max. $z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$
subject to the constraints :
$x_1 + x_4 + 6x_6 = 9$
$3x_1 + x_2 - 4x_3 + 2x_6 = 2$
$x_1 + 2x_3 + x_5 + 2x_6 = 6$
$x_i \geq 0$, $i = 1, 2, 3, 4, 5, 6$.
[**Ans.** One iteration. $x_1 = 2/3$, $x_2 = x_3 = 0$,
$x_4 = 25/3$, $x_5 = 16/3$, $x_6 = 0$, max $z = 43/3$]

33. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yield a profit of Rs. 30. An acre of wheat cost Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans cost Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8,000 man-days of work, how masny acres should be allocated to each crop to maximize profit ? **[Jammu Univ. (MBA) Feb. 96]**

[**Hint.** Formulation of the problem is :
Max. $z = 30x_1 + 40x_2 + 20x_3$, s.t.
$10x_1 + 12x_2 + 7x_3 \geq 10{,}000$; $7x_1 + 10x_2 + 8x_3 \leq 8{,}000$
$x_1 + x_2 + x_3 \leq 1{,}000$; $x_1, x_2, x_3 \geq 0$.]

[**Ans.** Acreage for carn, wheat and soyabeans are 250, 625 and respectively with max. profit of Rs. 32,500]

5·5. ARTIFICIAL VARIABLE TECHNIQUES

5·5– 1. Two Phase Method
[Garhwal 97; Kanpur (B.Sc.) 90; Rohil. 90]

Linear programming problems, in which constraints may also have '≥' and '=' signs after ensuring that all b_i are ≥ 0, are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, the ***artificial variable.*** These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in ***Phase I*** of the solution, and ***Phase II*** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases, it is called ***'Two Phase Simplex Method'*** due to **Dantzig, Orden** and **Wolfe.**

Remarks :

1. **The objective of Phase I is to search for a B.F.S. to the given problem It ends up either giving a B.F.S. or indicating that the given L.P.P. has no feasible solution at all.**
2. **The B.F.S. obtained at the end of Phase 1 provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.**
3. **In Phase II, care must be taken to ensure that an artificial variable is never allowed to become positive, if were present in the basis. Moreover, whenever some artifical variable happens to leave the basis, its column must be deleted from the simplex table altogether.**

Q. 1. Explain the term 'Artificial variable' and its use in linear programming.

2. What do you mean by two phase-method in linear programming problems, why it is used ?

This technique is well explained by the following example.

Example 10. *Solve the problem : Minimize* $z = x_1 + x_2$, *subject to* $2x_1 + x_2 \geq 4$, $x_1 + 7x_2 \geq 7$, *and* $x_1 , x_2 \geq 0$. **[Kanpur (B.Sc.) 03; Delhi B.Sc. (Math.) 91, 88; Bharthidasan B.Sc. (Math.) 90; VTU (BE, Common) Aug. 02]**

Solution. First convert the problem of minimization to maximization by writing the objective function as :

$$\text{Max } (-z) = -x_1 - x_2 \quad \text{or} \quad \text{Max. } z' = -x_1 - x_2 \text{, where } z' = -z.$$

Since all b_i's (4 and 7) are positive, the 'surplus variables' $x_3 \geq 0$ and $x_4 \geq 0$ are introduced, then constraints become :

$$\begin{aligned} 2x_1 + x_2 - x_3 \qquad &= 4 \\ x_1 + 7x_2 \qquad - x_4 &= 7 . \end{aligned}$$

But the basis matrix **B** would not be an identity matrix due to negative coefficients of x_3 and x_4 . Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraint equations can be written as

$$\begin{aligned} 2x_1 + x_2 - x_3 \qquad + a_1 \qquad &= 4 \\ x_1 + 7x_2 \qquad - x_4 \qquad + a_2 &= 7 . \end{aligned}$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I. Construct the first table (*Table 5·14*) where $\mathbf{A}_1$ and $\mathbf{A}_2$ denote the artificial column-vectors corresponding to a_1 and a_2, respectively.

Table 5·14

BASIC VARIABLES	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$	$\mathbf{A}_2$
a_1	4	2	1	−1	0	1	0
a_2	7	1	$\boxed{7}$	0	−1	0	1
			↑	×	×	×	↓

Now remove each artificial column vector $\mathbf{A}_1$ and $\mathbf{A}_2$ from the basis matrix. To remove vector $\mathbf{A}_2$ first, select the entering vector either $\mathbf{X}_1$ or $\mathbf{X}_2$, being careful to choose any one that will yield a non-negative (feasible) revised solution. Take the vector $\mathbf{X}_2$ to enter the basis matrix. It can be easily verified that if the vector $\mathbf{A}_2$ is entered in place of $\mathbf{X}_1$, the resulting solution will not be feasible. Thus transformed table (Table 5·15) is obtained.

Table 5·15

BASIC VARIABLES	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$	$\mathbf{A}_2$
a_1	3	13/7	0	−1	$\boxed{1/7}$	1	−1/7
x_2	1	1/7	1	0	−1/7	0	1/7
					↑	↓	

(Delete column $\mathbf{A}_2$ for ever at this stage)

This table gives the solution : $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $a_1 = 3$, $a_2 = 0$. When the artificial variable a_2 becomes zero (non-basic), we forget about it and never consider the corresponding vector $\mathbf{A}_2$ again for re-entry into the basis matrix.

Similarly, remove $\mathbf{A}_1$ from the basis matrix by introducing it in place of $\mathbf{X}_4$ by the same method. Thus Table 5·16 is obtained.

Table 5·16

BASIC VARAIBLES	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$
x_4	21	13	0	−7	1	7
x_2	4	2	1	−1	0	1

(Delete column $\mathbf{A}_1$ for ever at this stage)

This table gives the solution : $x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21, a_1 = 0$. Since the artificial variable a_1 becomes zero (non-basic), so drop the corresponding column $\mathbf{A}_1$ from this table. Thus, the solution $(x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21)$ is the basic feasible solution and now usual simplex routine can be started to obtain the required optimal solution.

Phase II. Now in order to test the starting above solution for optimality, construct the starting simplex Table 5.17

Table 5·17

BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$ ($c_j \rightarrow$)	$\mathbf{X}_1$ (−1)	$\mathbf{X}_2$ (−1)	$\mathbf{X}_3$ (0)	$\mathbf{X}_4$ (0)	Min. Ratio $(\mathbf{X_B}/\mathbf{X}_1)$
$\leftarrow x_4$	0	21	$\leftarrow \boxed{13}$ ↑	0	−7	1	21/13 ←
x_2	−1	4	2	1	−1	0	4/2
	$z' = \mathbf{C_B X_B}$ = −4		−1 ↑	0	1	0 ↓	$\leftarrow \Delta_j$

Compute $\Delta_1 = -1, \Delta_3 = 1$

Key element 13 indicates that $\mathbf{X}_4$ should be removed from the basis matrix. Thus, by usual transformation method Table 5·18 is formed.

Table 5·18

BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$ ($c_j \rightarrow$)	$\mathbf{X}_1$ (−1)	$\mathbf{X}_2$ (−1)	$\mathbf{X}_3$ (0)	$\mathbf{X}_4$ (0)	MIN. RATIO COLUMN
$\rightarrow x_1$	−1	21/13	1	0	−7/13	1/13	
x_2	−1	10/13	0	1	1/13	−2/13	
	$z' = -31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Also, verify that

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-1, -1)(-7/13, 1/13) = 6/13$$
$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (-1, -1)(1/13, -2/13) = 1/13 .$$

Since all $\Delta_j \geq 0$, the required optimal solution is :

$$x_1 = 21/13, x_2 = 10/13 \text{ and min. } z = 31/13 \text{ (because } z = -z') .$$

5.5·2. Simple Way for Two-Phase Simplex Method

Phase I : Table 5·19

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$	$\mathbf{A}_2$
a_1	4	2	1	−1	0	1	0
$\leftarrow a_2$	7	1	$\boxed{7}$ ↑	0	−1	0	1 ↓
$\leftarrow a_1$	3	13/7	0	−1	$\boxed{1/7}$	1	−1/7
$\rightarrow x_2$	1	1/7	1	0	−1/7 ↑	0 ↓	1/7
$\rightarrow x_4$	21	13	0	−7	1	7	×
x_2	4	2	1	−1	0	1	×

Thus, initial basic feasible solution is : $x_1 = 0, x_2 = 4, x_3 = 0, x_4 = 21$. Now start to improve this solution in Phase II by usual simplex method.

Note.

1. Remove the artificial vector $\mathbf{A_2}$ and insert it anywhere such that $\mathbf{X_B}$ remains feasible (≥ 0) .
2. As soon as $\mathbf{A_2}$ is removed from the basis by matrix transformation or otherwise, delete $\mathbf{A_2}$ for ever.
3. Similar process is adopted to remove other artificial vectors one by one from the basis.
4. Purpose of introducing artificial vectors is only to provide an initial basic feasible solution to start with simplex method in Phase II. So, as soon as the artificial variables become non-basic (*i.e.* zero), delete artificial vectors to enter Phase II.
5. Then, start Phase II, which is exactly the same as original simplex method.

Phase II. Table 5·20

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	-1 X_1	-1 X_2	0 X_3	0 X_4	MIN. RATIO (X_B/X_k)
$\leftarrow x_4$	0	21	$\boxed{13}$	0	7	1	$21/13 \leftarrow$
x_2	-1	4	2	1	-1	0	4/2
	$z' = -4$		$-1^* \uparrow$	0	1	$0 \downarrow$	$\leftarrow \Delta_j$
$\rightarrow x_1$	-1	21/13	1	0	$-7/13$	1/13	
x_2	-1	10/13	0	1	1/10	2/13	
	$z' = -31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Thus, the desired solution is obtained as : $x_1 = 21/13$, $x_2 = 10/13$, max. $z = 31/13$.

5.5–3. *Alternative Approach* of Two-phase Simplex Method

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I. In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise :

(i) Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

(ii) Max $z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to *Phase–II.*

(iii) Max $z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to *Phase–II.*

Phase II. Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of *Phase–I*, until an optimum basic feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of *Phase–I* are removed.

Q. 1. What are artificial variables ? Why do we need them ? Describe briefly the two-phase method of solving a L.P. problem with artificial variables. **[Meerut M.Sc. (Math.) 93]**

2. What do you mean by two phase method for solving a given L.P.P. ? Why is it used ?

3. Explain steps in solving a linear programming problem by two-phase method.

The following examples will make the *alternative* two-phase method clear.

Example 11. *Use two-phase simplex method to solve the problem : Minimize* $z = x_1 - 2x_2 - 3x_3$, *subject to the constraints* :$- 2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, *and* $x_1 , x_2 , x_3 \geq 0$, **[Meerut (Maths) 91]**

Solution. First convert the objective function into maximization form :

$$\text{Max } z' = - x_1 + 2x_2 + 3x_3 , \text{ where } z' = - z .$$

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

$$-2x_1 + x_2 + 3x_3 + a_1 \quad = 2$$
$$2x_1 + 3x_2 + 4x_3 \quad + a_2 = 1$$
$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

Phase I. Auxiliary L.P. problem is : *Max.* $z'^* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$ *subject to above given constraints.*

The following solution table is obtained for auxiliary problem.

Table 5·21

		$c_j \rightarrow$	0	0	0	− 1	− 1	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	A_1	A_2	MIN. RATIO (X_B/X_k)
a_1	− 1	2	− 2	1	3	1	0	2/3
$\leftarrow a_2$	− 1	1	2	3	$\boxed{4}$	0	1	1/4 ←
		$z'^* = -3$	0	− 4	− 7* ↑	0	0 ↓	$\leftarrow \Delta_j$
a_1	− 1	5/4	− 7/2	− 5/4	0	1	− 3/4	
$\rightarrow x_3$	0	1/4	1/2	3/4	1	0	1/4	
		$z'^* = -5/4$	7/4	5/4	0	0	3/4	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution to the auxiliary L.P.P. has been attained. But at the same time max. z'^* is negative and the artificial variable a_1 appears in the basic solution at a positive level. Hence the original problem does not possess any feasible solution. Here there is no need to enter *Phase II.*

Example 12. *Use two-phase simplex method to solve the problem :*

Minimize $z = 15/2 \; x_1 - 3x_2$, *subject to the constraints :*

$$3x_1 - x_2 - x_3 \geq 3, \quad x_1 - x_2 + x_3 \geq 2, \textit{ and } x_1, x_2, x_3 \geq 0.$$

Solution. Convert the objective function into the maximization form : Maximize $z' = -15/2 \; x_1 + 3x_2$.

Introducing the surplus variables $x_4 \geq 0$ and $x_5 \geq 0$, and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints of the given problem become

$$3x_1 - x_2 - x_3 - x_4 \quad + a_1 \quad = 3$$
$$x_1 - x_2 + x_3 \quad - x_5 \quad + a_2 = 2$$
$$x_1, \; x_2, \; x_3, \; x_4, a_1, a_2 \geq 0.$$

Phase I. Assigning a cost − 1 to artificial variables a_1 and a_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes : Max. $z'^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 - 1a_1 - 1a_2$, subject to the above given constraints.

Now apply simplex method in usual manner, (see *Table 5·22*).

Phase I : Table 5·22

		$c_j \rightarrow$	0	0	0	0	0	− 1	− 1	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	A_2	MIN RATIO (X_B/X_k)
$\leftarrow a_1$	− 1	3	$\boxed{3}$	− 1	− 1	− 1	0	1	0	3/3 ←
a_2	− 1	2	1	− 1	1	0	− 1	0	1	2/1
		$z'^* = -5$	− 4* ↑	2	0	1	1	0 ↓	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	1	1	− 1/3	− 1/3	− 1/3	0	1/3	0	—
$\leftarrow a_2$	− 1	1	0	− 2/3	$\boxed{4/3}$	1/3	− 1	1/3	1	3/4 ←
		$z'^* = -1$	0	2/3	− 4/3* ↑	− 1/3	1	2/3	0 ↓	$\leftarrow \Delta_j$
x_1	0	5/4	1	− 1/2	0	− 1/4	− 1/4	1/4	1/4	
x_3	0	3/4	0	− 1/2	1	1/4	− 3/4	1/4	3/4	
		$z'^* = 0$	0	0	0	0	0	1	1	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been attained.

Phase 2. In this phase, now consider the actual costs associated with the original variables, the objective function thus becomes : Max. $z' = -15/2\,x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

Now apply simplex method in the usual manner.

Phase 2 : Table 5·23

		$c_j \rightarrow$	$-15/2$	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_k)
x_1	$-15/2$	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	
x_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	
	$z' = -75/8$		0	$3/4$	0	$15/8$	$15/8$	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been attained.

Hence optimum solution is : $x_1 = 5/4$, $x_2 = 0$, $x_3 = 3/4$, min $z = 75/8$.

EXAMINATION PROBLEMS

Solve the following LP problems by two-phase method :

1. Max. $z = 3x_1 - x_2$
subject to the constraints :
$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 2$
$x_2 \leq 4$
and $x_1, x_2 \geq 0$.
[Ans. $x_1 = 2$, $x_2 = 0$ Max $z = 6$]

2. Max. $z = 5x_1 + 8x_2$
subject to the constraints :
$3x_1 + 2x_2 \geq 3$
$x_1 + 4x_2 \geq 4$
$x_1 + x_2 \leq 5$
and $x_1, x_2 \geq 0$.
[JNTU (Mech. & Prod.) 2004]
[Ans. $x_1 = 0$, $x_2 = 5$, max. $z = 40$]

3. Max $z = x_1 + 1.5x_2 + 2x_3 + 5x_4$
with the conditions :
$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$
$2x_1 + x_2 + x_3 + 5x_4 \leq 4$
$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$
$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$
$x_i\ (i = 1, 2, 3, 4) \geq 0$
[Ans. $x_1 = 1.2$, $x_2 = 0$, $x_3 = 0.9$
$x_4 = 0$, max. $z = 19.8$]

4. Minimize $z = x_1 - 2x_2 - 3x_3$, subject to
$-2x_1 + x_2 + 3x_3 = 2$
$2x_1 + 3x_2 + 4x_3 = 1$,
$x_j \geq 0$, $j = 1, 2, 3$.
[Ans. Here all $\Delta_j \geq 0$, but at the same time artificial variable a_1 appears in the basis. Hence the given LPP has no feasible solution]

5. Max. $z = 3x_1 + 2x_2 + x_3 + 4x_4$
subject to
$4x_1 + 5x_2 + x_3 - 3x_4 = 5$
$2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$
$x_1 + 4x_2 + 2.5x_3 - 4x_4 = 6$
$x_1, x_2, x_3 \geq 0$
[Ans. No solution]

6. Max $z = 5x_1 - 2x_2 + 3x_3$
subject to
$2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \leq 3$
$x_2 + 3x_3 \leq 5$
$x_1, x_2, x_3, x_4 \geq 0$.
[AIMS (BE Ind.) Bang. 2002]
[Ans. $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$,
max. $z = 85/3$]

7. Max. $z = 2x_1 + 3x_2 + 5x_3$,
subject to the constraints :
$3x_1 + 10x_2 + 5x_3 \leq 15$,
$x_1 + 2x_2 + x_3 \geq 4$,
$33x_1 - 10x_2 + 9x_3 \leq 33$,
$x_1, x_2, x_3 \geq 0$.

8. Max. $500\,x_1 + 1400\,x_2 + 900x_3$,
subject to,
$x_1 + x_2 + x_3 = 100$
$12x_1 + 35x_2 + 15x_3 \geq 25$
$8x_1 + 3x_2 + 4x_3 \geq 6$;
$x_1, x_2, x_3 \geq 0$. **[Meerut (MA) 93]**

[Ans. There does not exist any feasible solution, because artificial variable is not removed in the problem]

9. A firm has an advertising budget of Rs. 7,20,000. It wishes to allocate this budget to two media : magazines and televisions, so that total exposure is maximized. Each page of magazine advertising is estimated to result in 60,000 exposures, whereas each spot on television is estimated to result in 1,20,000 exposures. Each page of magazine advertising costs Rs. 9,000 and each spot on television costs Rs. 12,000. An additional condition that the firm has specified is that at least two pages of magazine advertising be used and at least 3 spots on television. Determine the optimum media-mix for this firm.
[Hint. The problem is : **[Delhi (MBA) 97]**
Max. $z = 60{,}000x_1 + 12{,}000x_2$ s.t.
$9{,}000x_1 + 12{,}000x_2 \leq 7{,}20{,}000$, $x_1 \geq 2$, $x_2 \geq 3$, $x_1, x_2 \geq 0$,
where x_1 = no. of pages of magazine
x_2 = no. of spots on television]
[Ans. $x_1 = 2$, $x_2 = 58{\cdot}5$ and max. $z = 7{,}14{,}000$]

5.5-4 Big-*M*-Method (Charne's Penalty Method)

[Kanpur (B.Sc.) 92, 91]

Computational steps of big-*M*-method are as stated below :

Step 1. Express the problem in the standard form.

Step 2. Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type ($\geq$) and '='. When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists). On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by $-M$ for maximization problems ($+M$ for minimization problems), where $M > 0$.

Step 3. In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Q. 1. Explain the use of Big-*M*-method in solving L.P.P. What are its characteristics ?

Example 13. *Solve by using big-M method the following linear programming problem :*

$$\text{Max. } z = -2x_1 - x_2, \text{ subject to } 3x_1 + x_2 = 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 4, \text{ and } x_1, x_2 \geq 0.$$

Solution. [JNTU (B. Tech.) 2003]

Step 1. Introducing slack, surplus and artificial variables, the system of constraint equations become :

$$\begin{aligned} 3x_1 + x_2 \quad + a_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + a_2 &= 6 \\ x_1 + 2x_2 + x_4 \quad &= 4 \end{aligned}$$

which can be written in the matrix form as :

$$\begin{array}{cccccc} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \mathbf{X}_4 & \mathbf{A}_1 & \mathbf{A}_2 \end{array}$$
$$\begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

- 69 98

Step 2. Assigning the large negative price $-M$ to the artificial variables a_1 and a_2, the objective function becomes : Max. $z = -2x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$.

Step 3. Construct starting simplex table (Table 5·24)

Starting Simplex Table 5·24

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN. RATIO $(\mathbf{X_B}/\mathbf{X_1})$
$\leftarrow a_1$	$-M$	3	$\boxed{3}$	1	0	0	1	0	$3/3 \leftarrow$
a_2	$-M$	6	4	3	-1	0	0	1	$6/4$
x_4	0	4	1	2	0	1	0	0	$4/1$
	$z = -9M$		$(2-7M)$ $\uparrow$	$(1-4M)$	M	0	0 $\downarrow$	0	$\leftarrow \Delta_j$

To apply optimality test, compute

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (-M, -M, 0)\,(3, 4, 1) - (-2) = 2 + (-3M - 4M + 0) = 2 - 7M$$
$$\Delta_2 = \mathbf{C_B X_2} - c_2 = (-M, -M, 0)\,(1, 3, 2) - (-1) = 1 + (-M - 3M + 0) = 1 - 4M$$
$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-M, -M, 0)\,(0, -1, 0) + 0 = M$$

$\therefore\ \Delta_k = \min\,[\Delta_1, \Delta_2, \Delta_3] = \min\,[2 - 7M, 1 - 4M, M] = \Delta_1$. Therefore, $\mathbf{X_1}$ will be entered.

Using minimum ratio rule, find the key element 3 which indicates that $\mathbf{A_1}$ should be removed. Now the transformed table (*Table 5·25*) is obtained in usual manner.

First Improved Table 5·25

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN RATIO $(\mathbf{X_B/X_2})$
$\rightarrow x_1$	-2	1	1	1/3	0	0	1/3	0	$1/\frac{1}{3}$
$\leftarrow a_2$	$-M$	2	0	$\leftarrow \boxed{5/3}$ ↑	1	0	$-4/3$	1	$2/\frac{5}{3} \leftarrow$
x_4	0	3	0	5/3	0	1	$-1/3$	0	$3/\frac{5}{3}$
	$z = -2 - 2M$		0	$(1-5M)/3$ ↑	M	0	$(-2+7M)/3$	0 ↓	$\leftarrow \Delta_j$

Again compute, $\Delta_2 = \mathbf{C_B X_2} - c_2 = (-2, -M, 0)\,(1/3, 5/3, 5/3) + 1 = (1-5M)/3$, and similarly, $\Delta_3 = M$, $\Delta_5 = (-2+7M)/3$.

Since *minimum* Δ_j *rule and minimum ratio rule* decide the key element $5/3$, so enter $\mathbf{X_2}$ and remove $\mathbf{A_2}$. Therefore, the second improved table (Table 5.26) is formed.

Table 5·26

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN. RATIO
x_1	-2	3/5	1	0	1/5	0	3/5	$-1/5$	
x_2	-1	6/5	0	1	$-3/5$	0	$-4/5$	3/5	
x_4	0	1	0	0	1	1	1	-1	
	$z = \mathbf{C_B X_B} = -12/5$		0	0	1/5	0	$M - 2/5$	$M - 1/5$	$\leftarrow \Delta_j \geq 0$

To test the solution for optimality, compute

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-2, -1, 0)\,(1/5, -3/5, 1) - 0 = 1/5$$

$$\Delta_5 = \mathbf{C_B A_2} - c_5 = (-2, -1, 0)\,(3/5, -4/5, 1) + M = M - 2/5$$

$$\Delta_6 = \mathbf{C_B A_2} - c_6 = (-2, -1, 0)\,(-1/5, -3/5, -1) + M = M - 1/5.$$

Since M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all positive. Consequently, the optimal solution is: $x_1 = 3/5$, $x_2 = 6/5$, max $\mathbf{z} = -12/5$.

10 mark

Example 14. *Solve the following problem by Big-M-method : Max.* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to :*

$$x_1 + 2x_2 + 3x_3 = 15,\; 2x_1 + x_2 + 5x_3 = 20,\; x_1 + 2x_2 + x_3 + x_4 = 10, \textit{ and } x_1, x_2, x_3, x_4 \geq 0.$$

[IAS (Maths.) 95; Kanpur (B.Sc.) 92; Karala (B.Sc.) 91; Meerut (B.Sc.) 90]

Solution. Since the constraints of the given problem are equations, introduce the artificial variables $a_1 \geq 0$, $a_2 \geq 0$. The problem thus becomes :

Max. $z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$, subject to the constraints :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad\quad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad\quad &= 10 \\ \textit{and } x_1, x_2, x_3, x_4, a_1, a_2 \quad &\geq 0. \end{aligned}$$

Now applying the usual simplex method, the solution is obtained as given in the *Table 5·27.*

Table 5·27 (Example 14)

		$c_j \rightarrow$	1	2	3	−1	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_k)
a_1	$-M$	15	1	2	3	0	1	0	15/3
$\leftarrow a_2$	$-M$	20	2	1	$\boxed{5}$	0	0	1	20/5 $\leftarrow$
x_4	−1	10	1	2	1	1	0	0	10/1
	$z = (-35M - 10)$		$(-3M - 2)$	$(-3M - 2)$	$(-8M - 4)$ $\uparrow$	0	0	0 $\downarrow$	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	3	−1/5	$\boxed{7/5}$	0	0	1	×	$\frac{3}{7/5} \leftarrow$
$\rightarrow x_3$	3	4	2/5	1/5	1	0	0	×	$\frac{4}{1/5}$
x_4	−1	6	3/5	9/5	0	1	0	×	$\frac{6}{9/5}$
	$z = (-3M + 6)$		$(M - 2)/5$	$-(7M - 16)/5$ $\uparrow$	0	0	0 $\downarrow$	×	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	15/7	−1/7	1	0	0	×	×	—
x_3	3	25/7	3/7	0	1	0	×	×	25/3
$\leftarrow x_4$	−1	15/7	$\boxed{6/7}$	0	0	1	×	×	15/6 $\leftarrow$
	$z = 90/7$		$-6/7^*$ $\uparrow$	0	0	0 $\downarrow$	×	×	$\leftarrow \Delta_j$
x_2	2	15/6	0	1	0	1/6	×	×	
x_3	3	15/6	0	0	1	3/6	×	×	
$\rightarrow x_1$	1	15/6	1	0	0	7/6	×	×	
	$z = 15$		0	0	0	75/36	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been obtained as :

$$x_1 = x_2 = x_3 = \frac{15}{6} = \frac{5}{2}, \text{ max } z = 15 .$$

Example 15. *Use penalty (Big-M) method to maximize* : $z = 3x_1 - x_2$ *subject to the constraints* :

$$2x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 3,\ x_2 \leq 4, \text{ and } x_1, x_2 \geq 0 .$$

Solution. By introducing the surplus variable $x_3 \geq 0$, artificial variable $a_1 \geq 0$, and slack variables $x_4 \geq 0$, $x_5 \geq 0$, the problem becomes : Max. $z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + x_2 - x_3 \quad + a_1 &= 2 \\ x_1 + 3x_2 \quad + x_4 &= 3 \\ x_2 \quad + x_5 &= 4 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0 . \end{aligned}$$

$-M \cdot 1 + (-M \times 2) + (-1 \times 1)$

In matrix form,
$$\begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Now the solution is obtained as given in Table 5.28

Table 5.28 [Example 15]

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
$c_j \rightarrow$			3	−1	0	0	0	−M	
$\leftarrow a_1$	$-M$	2	$\boxed{2}$	1	1	0	0	1	$2/2 \leftarrow$
x_4	0	3	1	3	0	1	0	0	3/1
x_5	0	4	0	1	0	0	1	0	—
	$z = -2M$		$(-2M-3)\uparrow$	$-M+1$	M	0	0	$0\downarrow$	
$\rightarrow x_1$	3	1	1	1/2	−1/2	0	0	×	—
$\leftarrow x_4$	0	2	0	5/2	$\boxed{1/2}$	1	0	×	$2/\frac{1}{2} \leftarrow$
x_5	0	4	0	1	0	0	1	×	—
	$z = 3$		0	5/2	$(-3/2)\uparrow$	$0\downarrow$	0	×	$\leftarrow \Delta_j$
x_1	3	3	1	3	0	1	0	×	
$\rightarrow x_3$	0	4	0	5	1	2	0	×	
x_5	0	4	0	1	0	0	1	×	
	$z = 9$		0	10	0	3	0	×	$\leftarrow \Delta_j \geq 0$

Thus the optimum solution is obtained as : $x_1 = 3$, $x_2 = 0$, max. $z = 9$.

Example 16. ***(Unrestricted Variables)***

(a) Maximize $z = 8x_2$, *subject to the constraints :* $x_1 - x_2 \geq 0, 2x_1 + 3x_2 \leq -6$ *and* x_1 , x_2 *are unrestricted.*

[Meerut (Maths) 93]

(b) Solve the LPP : Max $z = 4x_1 + 6x_2$, *subject to :* $x_1 - 2x_2 \geq -4$, $2x_1 + 44x_2 \geq 12$, $x_1 + 3x_2 \geq 9$ *and* x_1, x_2 *are unrestricted.*

[JNTU (MCA) 2004; Meerut 96, 93]

Solution. (a) In this problem, the variables x_1 and x_2 are unrestricted in sign, *i.e.* x_1 and x_2 may be + ive, –ive or zero. But, the simplex method can be used only when the variables are non-negative (≥ 0) . This difficulty can be immediately removed by using the transformation :

$$x_1 = x_1' - x_1'' \text{ and } x_2 = x_2' - x_2'' \text{ such that } x_1' \geq 0 , x_1'' \geq 0 , x_2' \geq 0 , x_2'' \geq 0 .$$

Therefore, the given problem becomes : *maximize* $z = 8x_2' - 8x_2''$, *subject to the constraints :*

$$(x_1' - x_1'') - (x_2' - x_2'') \geq 0$$
$$-2(x_1' - x_1'') - 3(x_2' - x_2'') \geq 6$$
$$x_1' , x_1'' , x_2' , x_2'' \geq 0 .$$

Now introducing the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the given problem becomes : Max. $z = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0x_3 + 0x_4 - Ma_1 - Ma_2$, subject to :

$$x_1' - x_1'' - x_2' + x_2'' - x_3 + a_1 = 0$$
$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - x_4 + a_2 = 6$$
$$x_1' , x_1'' , x_2' , x_2'' , x_3 , x_4 , a_1 , a_2 \geq 0 .$$

Table 5.29

BASIC VAR.	C_B	X_B	X_1'	X_1''	X_2'	X_2''	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_k)
$c_j \rightarrow$			0	0	8	−8	0	0	$-M$	$-M$	
$\leftarrow a_1$	$-M$	0	1	−1	−1	$\boxed{1}$	−1	0	1	0	$0 \leftarrow$
a_2	$-M$	6	−2	2	−3	3	0	−1	0	1	6/3
	$z = -6M$		M	$-M$	$(4M-8)$	$(-4M+8)\uparrow$	M	M	$0\downarrow$	1	$\leftarrow \Delta_j$
$\rightarrow x_2'$	−8	0	1	−1	−1	1	−1	0	×	0	—
$\leftarrow a_2$	$-M$	6	−5	$\boxed{5}$	0	0	3	−1	×	1	$6/5 \leftarrow$
	$z = -6M$		$(5M-8)$	$(-5M+8)\uparrow$	0	0	$(-3M+8)$	M	×	$0\downarrow$	$\leftarrow \Delta_j$
x_2''	−8	6/5	0	0	−1	1	−2/5	1/5	×	×	
$\rightarrow x_1''$	0	6/5	−1	1	0	0	3/5	−1/5	×	×	
	$z = -48/5$		0	0	0	0	16/5	8/5	×	×	$\leftarrow \Delta_j \geq 0$

Remember that the coefficients of slack or surplus variables in the objective function are always zero and the coefficient of artificial variables is taken a *largest negative quantity* $-M$ *where* $M > 0$.

Applying the simplex method in the usual manner, the solution is obtained as given in *Table 5·29*.

Since all $\Delta_j \geq 0$, an optimum solution is obtained as : $x_1' = 0$, $x_1'' = 6/5$, $x_2' = 0$, $x_2'' = 6/5$.

Since $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$, transforming the solution to original variables, we get

$$x_1 = 0 - 6/5 = -6/5, \ x_2 = 0 - 6/5 = -6/5, \text{ max. } z = -48/5.$$

(b) Solve as (a).

Note. Whenever the range of a variable is not given in the problem, it should be understood that such variable is unrestricted in sign.

Example 17. (Imp.) *Maximize* $z = 4x_1 + 5x_2 - 3x_3 + 50$, *subject to the constraints :*

$$x_1 + x_2 + x_3 = 10 \quad \ldots(i)$$
$$x_1 - x_2 \geq 1 \quad \ldots(ii)$$
$$2x_1 + 3x_2 + x_3 \leq 40 \quad \ldots(iii)$$
$$x_1, x_2, x_3 \geq 0.$$

[Meerut (Maths.) 97 P]

Solution. *If any constant is included in the objective function (like 50 here) it should be deleted in the beginning and finally adjusted in optimum value of z and, if there is an equality in the constraints, then one variable can be eliminated from the inequalities with* $\leq$ *or* $\geq$ *sign.* ***(Note)***

Subtracting *(i)* from *(iii)* with a view to eliminate x_3 from (iii) and retaining x_3 in (i) to work as a slack variable, the restrictions are modified as follows :

$$x_1 + x_2 + x_3 = 10, x_1 - x_2 \geq 1, x_1 + 2x_2 \leq 30, \text{ and } x_1, x_2, x_3 \geq 0.$$

Now introducing the slack, surplus and artificial variables, the problem becomes :

Max. $z = 4x_1 + 5x_2 - 3x_3 + 0x_4 - Ma_1 + 0x_5$, subject to the constraints :

$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 - x_4 + a_1 = 1$$
$$x_1 + 2x_2 + x_5 = 30$$
$$x_1, x_2, x_3, x_4, x_5, a_1 \geq 0.$$

Applying the usual simplex method, the solution is obtained as given in Table 5·30.

Table 5·30 [Example 17]

	$c_j \rightarrow$		4	5	-3	0	$-M$	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	X_5	MIN. RATIO (X_B/X_k)
x_3	-3	10	1	1	1	0	0	0	10/1
$\leftarrow a_1$	$-M$	1	$\leftarrow \boxed{1}$	1	0	-1	1	0	$1/1 \leftarrow$
x_5	0	30	1	2	0	0	0	1	30/1
	$z = -30 - M$		$-7-M$ $\uparrow$	$-8+M$	0	M	0 $\downarrow$	0	$\leftarrow \Delta_j$
$\leftarrow x_3$	-3	9	0	$\boxed{2}$	1	1	$\times$	0	$9/2 \leftarrow$
$\rightarrow x_1$	4	1	1	-1	0	-1	$\times$	0	—
x_5	0	29	0	3	0	1	$\times$	1	29/3
	$z = 9/2$		0	-15^* $\uparrow$	0 $\downarrow$	-7	$\times$	0	$\leftarrow \Delta_j$
$\rightarrow x_2$	5	9/2	0	1	1/2	1/2	$\times$	0	
x_1	4	11/2	1	0	1/2	$-1/2$	$\times$	0	
x_5	0	31/2	0	0	$-3/2$	$-1/2$	$\times$	1	
	$z = 89/2$		0	0	15/2	1/2	$\times$	0	$\leftarrow \Delta_j \geq 0$

Hence the solution is : $x_1 = 11/2$, $x_2 = 9/12$, max. $z = 89/2 + 50 = 189/2$.

Example 18. *Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of product mix by simplex method.* **[Bharthidasan B.Sc. (Math.) 90]**

Solution. Let x_1 grams of food X and x_2 grams of food Y be purchased. Then the problem can be formulated as : Minimize $z = 12x_1 + 20x_2$, subject to the constraints : $6x_1 + 8x_2 \geq 100$, $7x_1 + 12x_2 \geq 120$, and $x_1, x_2 \geq 0$.

Introducing the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints become :

$$6x_1 + 8x_2 - x_3 + a_1 = 100$$
$$7x_1 + 12x_2 - x_4 + a_2 = 120.$$

Objective function becomes :

$$\text{Max. } z' = -12x_1 - 20x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2, \text{ where } z' = -z.$$

Now proceeding by usual simplex method, the solution is obtained as given in Table 5.31.

Table 5.31 [Example 18]

	$c_j \rightarrow$		-12	-20	0	0	$-M$	$-M$	
BASIC VAR.	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	MIN RATIO (X_B/X_k)
a_1	$-M$	100	6	8 ↑	-1	0	1	0	100/8
$\leftarrow a_2$	$-M$	120	7	← $\boxed{12}$	0	1	0	1	$-120/12 \leftarrow$
	$z' = -220M$		$(-13M+12)$	$(-20M+20)$ ↑	M	M	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	20	$\boxed{4/3}$	0	-1	2/3	1	×	$60/4 \leftarrow$
$\rightarrow x_2$	-20	10	7/12	1	0	$-11/2$	0	×	120/7
	$z' = -20M - 200$		$-(4M-1)/3$ ↑	0	M	$(-2M+5)/3$	0 ↓	×	$\leftarrow \Delta_j$
$\rightarrow x_1$	-12	15	1	0	$-3/4$	1/2	×	×	
x_2	-20	5/4	0	1	7/16	$-3/4$	×	×	
	$z' = -205$		0	0	1/4	9	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal solution is attained. Hence the optimal solution is :

$$x_1 = 15,\ x_2 = 5/4,\ \max z = -(-205) = 205.$$

Hence 15 grams of food X and 5/4 grams of food Y should be the required product-mix with minimum cost of Rs. 205.

5·6. DISADVANTAGES OF BIG-M-METHOD OVER TWO-PHASE METHOD

1. Although big-M method can always be used to check the existence of a feasbile solution, it may be computationally inconvenient because of the manipulation of the constant M. On the other hand, two-phase method eliminates the constant M from calculations.
2. Another difficulty arises specially when the problem is to be solved on a digital computer. To use a digital computer, M must be assigned some numerical value which is much larger than the values $c_1, c_2, \ldots$, in the objective function. But, a computer has only a fixed number of digits.

Q. 1. What is an artificial variable and why it is necessary to introduce it ? Describe the two phase process of solving an L.P.P. by simplex method. **[Delhi B.Sc. (Maths.) 90]**

2. Why is an artificial vector which leaves the basis once never considered again for re-entry into the basis ? **[Delhi B.Sc. (Math.) 91]**

3. In the two-phase method explain when phase 1 terminates.

4. Optimality criteria being satisfied; state what is indicated by each of the following :
   (i) One or more artificial vectors are in the basis at zero level.
   (ii) One or more artificial vectors are in the basis at positive level. **[Delhi B.Sc. (Math.) 91]**

EXAMINATION PROBLEMS

Solve the following LP problems using Charne's Big-M (Penalty) method:

1. Min. $z = 2x_1 + 9x_2 + x_3$, subject to the constraints : $x_1 + 4x_2 + 2x_3 \geq 5$, $3x_1 + x_2 + 2x_3 \geq 4$, $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 0$, $x_2 = 0$, $x_3 = 5/2$, min. $z = 5/2$].

2. Max. $R = 5x - 2y - z$, subject to the constraints : $2x + 2y - z \geq 2$, $3x - 4y \geq 3$, $y + 3z \geq 5$, $x, y, z \geq 0$.
[Ans. $x = 13/9$, $y = 1/3$, $z = 14/9$, max. $R = 5$]

3. Min. $z = x_1 + x_2 + 3x_3$, subject to $3x_1 + 2x_2 + x_3 \leq 3$, $2x_1 + x_2 + 2x_3 \geq 3$, $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 3/4$, $x_2 = 0$, $x_3 = 3/4$, min. $z = 3$]

4. Min. $z = 4x_1 + 8x_2 + 3x_3$, subject to $x_1 + x_2 \geq 2$, $2x_1 + x_3 \geq 5$ and $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$, min. $z = 10$]

5. Maximize $z = 2x_1 + x_2 + 3x_3$, subject to $x_1 + x_2 + 2x_3 \leq 5$, $2x_1 + 3x_2 + 4x_3 = 12$, and $x_1, x_2, x_3 \geq 0$.
What is the maximum number of basic solutions to the L.P. problem ?
[Ans. $x_1 = 3$, $x_2 = 2$, $x_3 = 0$, max. $z = 8$]

6. Min. $P = 5x + 4y + 4z$, subject to $x + y + z = 100$, $x \geq 20$, $y \geq 30$, $z \leq 40$ and $x, y, z \geq 0$.
[JNTU (B. Tech.) 2003]

7. Maximize $z = 3x_1 + 2{\cdot}5x_2$, subject to the constraints : $2x_1 + 4x_2 \geq 40$, $3x_1 + 2x_2 \geq 50$ $x_1, x_2 \geq 0$
[Hint. First constraint can be divided by the common factor 2]
[Ans. Unbounded solution]

8. Min. $z = 0{\cdot}6x_1 + x_2$, subject to the constraints : $10x_1 + 4x_2 \geq 20$, $5x_1 + 5x_2 \geq 20$, and $x_1, x_2 \geq 0$.
[Hint. First divide the constraints by the common factors 2,5,2 respectively]
[Ans. $x_1 = 4$, $x_2 = 0$, min. $z = 12/5$]

9. Min. (cost) $z = 2y_1 + 3y_2$, subject to the constraints : $y_1 + y_2 \geq 5$, $y_1 + 2y_2 \geq 6$, $y_1 \geq 0$, $y_2 \geq 0$
[Ans. $y_1 = 4$, $y_2 = 1$ min. $z = 11$]

10. Max. $z = 3x_1 + 2x_2$, subject to the constraints : $2x_1 + x_2 \leq 2$, $3x_1 + 4x_2 \geq 12$, and $x_1, x_2 \geq 0$.
[Ans. Pseudo-optimum basic feasible solution exist.]

11. Min. $z = 4x_1 + 2x_2$, subject to the constraints : $3x_1 + x_2 \geq 27$, $x_1 + x_2 \geq 21$, and $x_1, x_2 \geq 0$.
[Ans. $x_1 = 3$, $x_2 = 18$, min $z = 48$]

12. Max. $z = 2x_1 + 3x_2 - 5x_3$, subject to the constraints : $x_1 + x_2 + x_3 = 7$, $2x_1 + 5x_2 + x_3 \geq 10$, and $x_1, x_2, x_3 \geq 0$.
[Hint. Since first constraint is an equation, we can subtract this equation from second constraint n order to reduce the number of artificial variables]
[Ans. $x_1 = 45/7$, $x_2 = 4/7$, $x_3 = 0$, max. $z = 102/7$]

13. Min. $z = 3x_1 + 2x_2 + x_3$, subject to : $2x_1 + 5x_2 + x_3 = 12$, $3x_1 + 4x_2 = 11$, x_1 is unrestricted, $x_2 \geq 0$, $x_3 \geq 0$.
[Ans. $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$, max $z = 47/3$.]

14. Min. $z = 5x_1 + 6x_2$, subject to $2x_1 + 5x_2 \geq 1500$, $3x_1 + x_2 \geq 1200$, and $x_1, x_2 \geq 0$. Verify the result graphically.
[Kanpur (B.Sc.) 95, 91]

15. A cabinet manufacturer produces wood cabinets for T.V., sets, stereo systems and radios, each of which must be assembled and crated. Each T.V. cabinet requires 3 hrs. to assemble, 5 hrs. to decorate and 1/10 hr. to crate and returns a profit of Rs. 10. Each stereo cabinet requires 10 hrs. to assemble 8 hrs. to decorate and 3/5 hr. to crate and returns a profit Rs. 25. Each radio cabinet requires 1 hr. to assemble, 1 hr. to decorate and 1/10 hr. to, crate and returns a profit of Rs. 3. The manufacturer has the maximum of 30,000, 40,000 and 120 hrs. available for assembling, decorating and crating respectively.
(i) Formulate the above problem as a LPP.
(ii) Use simplex method to find how many units of each product should be manufactured to maximize profit.
(iii) Does the problem have unique solution.
[Delhi (MBA) Nov. 98]
[Ans. (i) Max. $z = 10x_1 + 25x_2 + 3x_3$, s.t.

$$3x_1 + 10x_2 + x_3 \leq 30{,}000,\ 5x_1 + 8x_2 + x_3 \leq 40{,}000,$$
$$\frac{1}{10}x_1 + \frac{3}{5}x_2 + \frac{1}{10}x_3 \leq 120,\ x_1, x_2, x_3 \geq 0$$

(ii) $x_1 = 1{,}200$, $x_2 = x_3 = 0$ with max. profit $z = 12{,}000$.
(iii) No.]

16. Product *A* offers a profit of Rs. 25 per unit and product *B* yields a profit of Rs. 40 per unit. To manufacture the products-leather, wood and glue are required in the amount shown below :

Resources Required for one unit

Product	Leather	Woods (in sq. units)	Glue (in litres)
A	0·50	4	0·2
B	0·25	7	0·2

Available resources include 2,200 kgs of leather, 28,000 square metres of wood and 1,400 litres of glue :
(i) State the objective function and constraints in mathematical form.
(ii) Find the optimum situation.
(iii) Which resources are fully consumed ? How much of each resource remains unused ?
(iv) What are the shadow prices of resources ?
[C.S. (Final) June 97]
[Hint.(i) Max. $z = 25x_1 + 40x_2$, s.t.

$$0{\cdot}50x_1 + 0{\cdot}25x_2 \geq 2{,}200,\ 4x_1 + 7x_2 \leq 28{,}000,$$
$$0{\cdot}20x_1 + 0{\cdot}20x_2 \leq 1{,}400,\ x_1, x_2 \geq 0.$$

(ii) $x_1 = 3{,}360$, $x_2 = 2{,}080$ with max. $z =$ Rs. 1,67,200

(iii) used and unused reosurces.

Resources	Used	Unused	Total
Leather	(0·50) (3360) + (0·25) (2080) = 2200 kg.	0 kg.	2,200 kg.
Wood	28,000 sq. mt.	0 sq. mt.	28,000 sq. mt.
Glue	1080 litres	312 litres	1,400 litres

16. Two products *A* and *B* are processed on 3 machine, M_1, M_2 and M_3. The processing times per unit, machine availability and profit per unit are

Machine	Processing	Time (hrs.)	Availability (hrs.)
M_1	2	3	1,500
M_2	3	2	1,500
M_3	1	1	100
Profit (Rs.)	10	12	

Any unutilized time on machine M_3 can be given on rental basis to others at an hourly rated of Rs. 1·50. Solve the problem by simplex method to determine the maximum profit. **[IGONU (MBA) Dec. 98]**

[**Hint.** Since any amount of unused time on M_3 can be rented out at a rate of Rs. 1·50 per hour, the total rent will be 1·5 [1000 – ($x_1 + x_2$)]. Thus the total profit is equal to $10x_1 + 12x_2 + 1{,}500 - 1{\cdot}5x_1 - 1{\cdot}5x_2$. Thus formulation of LPP is :

Max. $z = 8{\cdot}5x_1 + 10{\cdot}5x_2 + 1{,}500$, s.t. $2x_1 + 3x_2 \le 1{,}500$, $3x_1 + 2x_2 \le 1{,}500$, $x_1 + x_2 \le 1{,}000$; $x_1, x_2 \ge 0$.

[**Ans.** 300 units of *A* and *B* both with max. profit of Rs. 7,200]

17. A factory manufactures three products which are processed through three different production stages. The time required to manufacture, one unit of each of the three products and their daily capacity of the stages are given in the following table :

Stage	Time per unit in minutes			Stage capacity (in minutes)
	Product 1	Product 2	Product 3	
1	1	1	1	430
2	3	—	2	460
3	1	4	—	420
Profit per unit				—

(i) Set the data in simplex table.
(ii) Find the table for optimum solution.
(iii) State from the table-min. profit, production pattern and surplus capacity at any stage.
(iv) What is the meaning of shadow price ? Where is it shown in the table ? Explain it in respect of resource of stages having shadow price.
(v) How many units of other resources will be required so as to completely utilise the surplus resource ?

[Osmania (MBA) Feb. 97]

18. Ashok Chemicals Co. manufactures two chemicals *A* and *B* which are sold to the manufacturers of soaps and detergents. On the basis of the next month's demand, the management has decided that the total production for chemicals *A* and *B* should be at least 350 kilograms. Moreover, a major customer's order for 125 kgs. of product *A* must also be supplied. Product *A* requires 2 hours of processing time per kg. and product *B* requires one hour of processing time per kg. For the coming month, 600 hours of processing time are available. The company wants to meet the above requirements at a minimum total production cost. The production costs are Rs. 2/– per kg. for product *A* and Rs. 3/– per kg for product *B*.

Ashok Chemicals Co. wants to determine its optimum productwise and the total minimum cost relevant thereto.
(i) Formulate the above as a linear programming problem.
(ii) Solve the problem with the simplex method.
(iii) Does the problem have multiple optimum solutions ? **[Delhi (M. Com.) 98]**

19. A firm manufacturing office furniture provides you the following information regarding resource consumption and availability and profit contribution :

Resources	Usage per unit			Daily availability
	Tables	Chairs	Bookcases	
Timber (cu. ft)	8	4	3	640
Assembly department (man-hours)	4	6	2	540
Finishing department (man-hours)	1	1	1	100
Profit contribution per unit (Rs.)	30	20	12	
Minimum production requirement	0	50	0	

The firm wants to determine its optimal product mix.

(i) Formulate the linear programming problem with the help of the above data.
(ii) Solve the problem with the Simplex Method and find the optimal product mix and the total maximum profit contribution.
(iii) Identify the shadow prices of the resources.
(iv) What other information can be obtained from the optimal solution of the problem ? **[Delhi (M. Com.) 97]**

20. Use penalty (Big M) method to solve the following LP problem :
Min. $z = 5x_1 + 3x_2$, s.t. $2x_1 + 4x_2 \leq 12$, $2x_1 + 2x_2 = 10$, $5x_1 - 2x_2 \geq 10$, and x_1 , $x_2 \geq 0$ **[IPM (PGDBM) 2000]**

Problem of Degeneracy (Tie for Minimum Ratio)

5.7. WHAT IS DEGENERACY PROBLEMS ?

At the stage of improving the solution during simplex procedure, minimum ratio $\mathbf{X_B/X_k}$ $(\mathbf{X_k > 0})$ is determined in the last column of simplex table to find the key row (*i.e.*, a row containing the *key element*). But, sometimes this ratio may not be unique, *i.e.*, the key element (hence the variable to leave the basis) is not uniquely determined or at the very first iteration, the value of one or more basic variables in the $\mathbf{X_B}$ column become equal to zero, this causes the problem of degeneracy.

However, if the minimum ratio is zero for two or more basic variables, degeneracy may result the simplex routine to cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. Fortunately, such phenomenon very rarely occurs in practical problems.

5.7–1. Method to Resolve Degeneracy (Tie)

The following systematic procedure can be utilised to avoid cycling due to degeneracy in L.P. problems.

Step 1. First pick up the rows for which the min. non-negative ratio is same (tied). To be definite, suppose such rows are first, third, etc., for example.

Step 2. Now rearrange the columns of the usual simplex table so that the columns forming the original unit matrix come first in proper order.

Step 3. Then find the minimum of the ratio :

$$\left[\frac{\textbf{elements of first column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique. That is, for the rows *first, third,* etc. as picked up in step 1. (*key column* is that one for which Δ_j is minimum).

(i) If this minimum is attained for third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is also not unique, then go to next step.

Step 4. Now compute the minimum of the ratio :

$$\left[\frac{\textbf{elements of second column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 3.

(i) If this min. ratio is unique for the first row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique then go to next step.

Step 5. Next compute the *minimum* of the ratio :

$$\left[\frac{\textbf{elements of third column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 4.

(i) If this min. ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this min. is still not unique, then go on repeating the above outlined procedure till the unique min. ratio is obtained to resolve the degeneracy. After the resolution of this tie, simplex method is applied to obtain the optimum solution. Following example will make the procedure clear.

Q. 1. What do you understand by degeneracy ? Discuss a method to resolve degeneracy in a LPP.

2. Explain the concept of degeneracy in simplex method. [VTU 2003]

Example 19. *Maximize* $\mathbf{z} = 3x_1 + 9x_2$, *subject to the constraints :*

$$x_1 + 4x_2 \le 8,\ x_1 + 2x_2 \le 4, \text{ and } x_1, x_2 \ge 0.$$

Solution. Introducing the slack variables $s_1 \ge 0$ and $s_2 \ge 0$, the problem becomes :

$$\text{Max. } z = 3x_1 + 9x_2 + 0s_1 + 0s_2$$

subject to the constraints :

$$x_1 + 4x_2 + s_1 \quad = 8$$
$$x_1 + 2x_2 \quad + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Table 5.32. Starting Simplex Table

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	3 X_1	9 X_2	0 S_1	0 S_2	MIN. RATIO (X_B/X_k)
s_1	0	8	1	4	1	0	$\left.\begin{matrix} 8/4 = 2 \\ 4/2 = 2 \end{matrix}\right\}$ Tie
s_2	0	4	1	2	0	1	
		$z = 0$	-3	-9	0	0	$\leftarrow \Delta_j$

Since min. ratio 2 in the last column of above table is not unique, both the slack variables s_1 and s_2 may leave the basis. This is an indication for the existence of degeneracy in the given LP problem. So we apply the above outlined procedure to resolve degeneracy (tie).

First arrange the columns $\mathbf{X}_1, \mathbf{X}_2, \mathbf{S}_1$ and $\mathbf{S}_2$ in such a way that the initial identity (basis) matrix appears first. Thus the initial simplex table becomes :

Table 5.33

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	0 S_1	0 S_2	3 X_1	9 X_2	MIN RATIO (S_1/X_2)
s_1	0	8	1	0	1	4	1/4
$\leftarrow s_2$	0	4	0	1	1	$\leftarrow \boxed{2}$	$0/2 \leftarrow$
		$z = 0$	0	0 $\downarrow$	-3	-9 $\uparrow$	$\leftarrow \Delta_j \ge 0$

Now using the *step 3* of the procedure for resolving degeneracy, we find

$$\min\left[\frac{\textbf{elements of first column } (\mathbf{S}_1)}{\textbf{corres. elements of \textit{key column} } (\mathbf{X}_2)}\right] = \min\left[\frac{1}{4}, \frac{0}{2}\right] = 0$$

which occurs for the second row. Hence $\mathbf{S}_2$ must leave the basis, and the key element is 2 as shown above.

First Iteration. By usual matrix transformation introduce $\mathbf{X}_2$ and leave $\mathbf{S}_2$.

Table 5.34. First Improvement Table

BASIC VARIABLES	C_B	$C_j \rightarrow$ X_B	0 S_1	0 S_2	3 $\mathbf{X}_1$	9 $\mathbf{X}_2$	MIN RATIO
s_1	0	0	1	-2	-1	0	
$\rightarrow x_2$	9	2	0	1/2	1/2	1	
		$z = 18$	0	9/2	3/2	0	$\leftarrow \Delta_j \ge 0$

Since all $\Delta_j \ge 0$, an optimal solution has been reached. Hence the optimum basic feasible solution is : $x_1 = 0$, $x_2 = 2$, max. $\mathbf{z} = 18$.

Example 20. *Max.* $z = 2x_1 + x_2$, *subject to* $4x_1 + 3x_2 \le 12$, $4x_1 + x_2 \le 8$, $4x_1 - x_2 \le 8$, *and* $x_1, x_2 \ge 0$.

Solution. Introducing the slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$, and proceeding in the usual manner, the starting simplex table is given below :

Table 5.35

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	2 X_1	1 X_2	0 S_1	0 S_2	0 S_3	MIN. RATIO (X_B/X_k)
s_1	0	12	4	3	1	0	0	12/4
s_2	0	8	4	1	0	1	0	8/4
s_3	0	8	4	−1	0	0	1	8/4
	$z = 0$		−2	−1	0	0	0	$\leftarrow \Delta_j$

Since min. ratio in the last column of above table is 2 which is same for *second* and *third* rows. This is an indication of *degeneracy*. So arrange the columns in such a way that the initial identity (basis) matrix comes first. Then starting simplex table becomes.

Table 5.36

BASIC VARIABLES	C_B	X_B	S_1	S_2	S_3	X_1	X_2	MIN (S_1/X_1)	MIN (S_2/X_1)
s_1	0	12	1	0	0	4	3	—	—
s_2	0	8	0	1	0	4	1	0/4	1/4
s_3	0	8	0	0	1	$\boxed{4}$	1	0/4	0/4 ←
	$z = 0$		0	0	0 ↓	−2 ↑	−1		

Using the procedure of degeneracy, compute

$$\left[\frac{\textbf{elements of first column } (S_1) \textbf{ of unit matrix}}{\textbf{corres. elements of key column } (X_1)}\right]$$

only for *second* and *third* rows. Therefore, min $[-, \frac{0}{4}, \frac{0}{4}]$ which is not unique.

So again compute

$$\min\left[\frac{\textbf{element of second column } (S_2) \textbf{ of unit matrix}}{\textbf{corres. element of key column } (X_1)}\right]$$

only for second and third rows. Therefore, min $[-, \frac{1}{4}, \frac{0}{4}] = 0$ which occurs corresponding to the third row. Hence the key element is 4.

Now improve the simplex **Table 5.36** in the usual manner to get Table 5.37.

Table 5.37

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	0 S_1	0 S_2	0 S_3	2 X_1	1 X_2	MIN. (X_B/X_k)
s_1	0	4	1	0	−1	0	4	4/4
s_2	0	0	0	1	−1	0	$\boxed{2}$	0/2 ←
x_1	2	2	0	0	1/4	1	−1/4	—
	$z = 4$		0	0 ↓	1/2	0	−3/2 ↑	$\leftarrow \Delta_j$
s_1	0	4	1	−2	$\boxed{1}$	0	0	4/1 ←
x_2	1	0	0	1/2	−1/2	0	1	—
x_1	2	2	0	1/8	1/8	1	0	$2/\frac{1}{8}$
	$z = 4$		0 ↓	3/4	−1/4 ↑	0	0	$\leftarrow \Delta_j$
s_3	0	4	1	−2	1	0	0	
x_2	1	2	1/2	−1/2	0	0	1	
x_1	2	3/2	−1/8	3/8	0	1	0	
	$z = 5$		1/4	1/4	0	0	0	$\leftarrow \Delta_j \ge 0$

Since all $\Delta_j \ge 0$, an optimum solution is obtained as : $x_1 = 3/2$, $x_2 = 2$, max $z = 5$.

Example 21. *Max.* $z = 5x_1 - 2x_2 + 3x_3$, *subject to* $2x_1 + 2x_2 - x_3 \geq 2$, $3x_1 - 4x_2 \leq 3$, $x_2 + 3x_3 \leq 5$, *and* $x_1, x_2, x_3 \geq 0$. **[Kanpur 96]**

Solution. Introducing the surplus variable $s_1 \geq 0$, slack variables $s_2 \geq 0$, $s_3 \geq 0$ and an artificial variable $a_1 \geq 0$, the constraints of the problem become :

$$\begin{aligned} 2x_1 + 2x_2 - x_3 - s_1 \qquad\quad + a_1 &= 2 \\ 3x_1 - 4x_2 \qquad + s_2 \qquad &= 3 \\ x_2 + 3x_3 \qquad\quad + s_3 \quad &= 5. \end{aligned}$$

and using big-M technique objective function becomes :

$$\text{Max. } z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - Ma_1.$$

In the usual manner, the starting simplex table is obtained as below :

Table 5.38

		$c_j \rightarrow$	5	−2	3	0	0	0	−M	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	A_1	MIN. RATIO (X_B/X_k)
$\leftarrow a_1$	−M	2	$\boxed{2}$ ↑	2	−1	−1	0	0	1	2/2 ←
s_2	0	3	3	−4	0	0	1	0	0	3/3
s_3	0	5	0	1	3	0	0	1	0	–
	$z = -2M$		−2M − 5 ↑	−2M + 2	M − 3	M	0	0	0 ↓	$\leftarrow \Delta_j$

Net evaluations Δ_j are computed by the formula $\Delta_j = \mathbf{C}_B \mathbf{X}_j - c_j$ in the usual manner. Since Δ_1 is the most negative, $\mathbf{X}_1$ enters the basis. Further, since the min. ratio in the last column of above table is 1 for both the first and second rows, therefore either $\mathbf{A}_1$ or $\mathbf{S}_2$ tends to leave the basis. This is an indication of the existence of degeneracy. But, $\mathbf{A}_1$ being an artificial vector will be prefered to leave the basis. *Note that there is no need to apply the procedure for resolving degeneracy under such circumstances.*

Continuing the simplex routine, the computations are presented in the following tabular form.

Table 5.39

		$c_j \rightarrow$	5	−2	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	MIN. RATIO (X_B/X_k)
$\rightarrow x_1$	5	1	1	1	−1/2	−1/2	0	0	–
$\leftarrow s_2$	0	0	0	−7	$\boxed{3/2}$	3/2	1	0	$0/\frac{3}{2}$ ←
s_3	0	5	0	1	3	0	0	1	5/3
	$z = 5$		0	7	−11/2 ↑	−5/2	0 ↓	0	$\leftarrow \Delta_j$
x_1	5	1	1	−4/3	0	0	1/3	0	–
$\rightarrow x_3$	3	0	0	−14/3	1	1	2/3	0	–
$\leftarrow s_3$	0	5	0	$\boxed{15}$	0	−3	−2	1	5/15 ←
	$z = 5$		0	−56/3 ↑	0	3	11/3	0 ↓	$\leftarrow \Delta_j$
x_1	5	13/9	1	0	0	−4/15	7/45	4/45	–
$\leftarrow x_3$	3	14/9	0	0	1	$\boxed{1/15}$	2/45	14/45	70/3 ←
$\rightarrow x_2$	−2	1/3	0	1	0	−1/5	−2/15	1/15	–
	$z = 101/9$		0	0	0 ↓	−11/15 ↑	53/45	56/45	$\leftarrow \Delta_j$
x_1	5	23/3	1	0	4	0	1/3	4/3	
$\rightarrow s_1$	0	70/3	0	0	15	1	2/3	14/3	
x_2	−2	5	0	1	3	0	0	1	
	$z = 85/3$		0	0	11	0	5/3	14/3	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is : $x_1 = 23/3, x_2 = 5, x_3 = 0$, max. $z = 85/3$.

Q. 1. What is degeneracy ? Discuss a method to resolve degeneracy in L.P. problems.

2. Explain what is meant by degeneracy and cycling in linear programming. How their effects overcome ?

[Meerut (L.P.) 90]

EXAMINATION PROBLEMS

Solve the following LP problems :

1. Max. $z = 5x_1 + 3x_2$
subject to
$x_1 + x_2 \leq 2$
$5x_1 + 2x_2 \leq 10$
$3x_1 + 8x_2 \leq 12$
$x_1, x_2, \geq 0.$
[Ans. $x_1 = 2, x_2 = 0, z = 10$]

2. Max. $R = 22x + 30y + 25z$
subject to
$2x + 2y \leq 100$
$2x + y + z \leq 100$
$x + 2y + 2z \leq 100$
$x, y, z, \geq 0$.
[Ans. $x = 100/3, y = 50/3, z = 50/3,$ $R = 1650$]

3. Max. $z = 2x_1 + 3x_2 + 10x_3$
subject to
$x_1 + 2x_3 = 0$
$x_2 + x_3 = 1$
$x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 0, x_2 = 1, x_3 = 0$ and max. $z = 3$]

4. Max. $z = 3x_1 + 5x_2$
subject to the constraints
$x_1 + x_3 = 4, x_2 + x_4 = 6,$
$3x_1 + 2x_2 + x_5 = 12,$ and
$x_1, x_2, x_3, x_4, x_5 \geq 0$
Does the degeneracy occur in this problem ?
[Ans. $x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 0,$
$z* = 30$. Yes, degeneracy occurs.]

5. Max. $z = 2x_1 + x_2$
subject to the constraints
$x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6,$
$x_1 - x_2 \leq 2, x_1 + 2x_2 \leq 1,$
$2x_1 - 3x_2 \leq 1$, and $x_1, x_2 \geq 0$.
[Ans. $x_1 = 5/7, x_2 = 1/7$
max. $z = 11/7$]

6. Max. $z = 3/4\, x_1 - 150\, x_2 + 1/50\, x_3 - 6x_4,$
subject to the constraints
$1/4\, x_1 - 60\, x_2 - 1/26\, x_3 + 9\, x_4 \leq 0,$
$1/2\, x_1 - 90\, x_2 - 1/50\, x_3 + 3x_4 \leq 0,$
$x_3 \leq 1$ and $x_1, x_2, x_3, x_4 \geq 0$.
[Ans. $x_1 = 1/25, x_2 = 0, x_3 = 1$
and $x_4 = 0$, max. $z = 1/20$]

7. Max. $z = 2x_1 + 3x_2 + 10x_3$, subject to
$x_1 + 2x_3 = 1, x_2 + x_3 = 1,$
and $x_1, x_2, x_3 \geq 0.$
[Ans. $x_1 = 0, x_2 = 1/2, x_3 = 1/2$, max. $z = 13/2$]

8. Min. $z = -\, 3/4\, x_1 + 20x_2 - 1/2\, x_3 + 6x_4$, subject to
$1/4\, x_1 - 8x_2 - x_3 + 9x_4 \leq 0, 1/4\, x_1 - 12x_2 - 1/2\, x_3 + 3x_4 \leq 0$
and $x_1, x_2, x_3, x_4, \geq 0.$
[Ans. Unbounded solution.]

5·8. SPECIAL CASES : ALTERNATIVE SOLUTIONS, UNBOUNDED SOLUTIONS AND NON-EXISTING SOLUTIONS

In this section, some important cases (except degeneracy) are discussed which are very often encountered during simplex procedure. The properties of these cases have already been visualised in the graphical solution of two variable LP problems.

5·8-1 Alternative Optimum Solutions

Example 22. *Use penalty (or Big-M) method to solve the problem :*

Max. $z = 6x_1 + 4x_2$, *subject to* $2x_1 + 3x_2 \leq 30$, $3x_1 + 2x_2 \leq 24$, $x_1 + x_2 \geq 3$, and $x_1, x_2 \geq 0$.

Is the solution unique ? If not, give two different solutions.

Solution. Introducing the slack variables $x_3 \geq 0$, $x_4 \geq 0$, surplus variable $x_5 \geq 0$, and artificial variable $a_1 \geq 0$, the problem becomes :

Max. $z = 6x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + 3x_2 + x_3 \qquad\qquad &= 30 \\ 3x_1 + 2x_2 \qquad + x_4 \qquad &= 24 \\ x_1 + x_2 \qquad\quad - x_5 + a_1 &= 3 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0. \end{aligned}$$

Now the solution is obtained as follows :

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Table 5.40

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN RATIO (X_B/X_k)
	$c_j \rightarrow$		6	4	0	0	0	$-M$	
x_3	0	30	2	3	1	0	0	0	30/2
x_4	0	24	3	2	0	1	0	0	24/3
$\leftarrow a_1$	$-M$	3	$\leftarrow \boxed{1}$	1	0	0	-1	1	$3/1 \leftarrow$
	$z = -3M$		$(-M-6)$ ↑	$(-M-4)$	0	0	M	0 ↓	$\leftarrow \Delta_j$
x_3	0	24	0	1	1	0	2	×	24/2
$\leftarrow x_4$	0	15	0	-1	0	1	$\boxed{3}$	×	$15/3 \leftarrow$
$\rightarrow x_1$	6	3	1	1	0	0	-1	×	—
	$z = 18$		0	2	0	0 ↓	-6 ↑	×	$\leftarrow \Delta_j$
$\leftarrow x_3$	0	14	0	$\boxed{5/3}$	1	$-2/3$	0	×	$\frac{14}{5/3} = 42/5 \leftarrow$
$\rightarrow x_5$	0	5	0	$-1/3$	0	1/3	1	×	—
x_1	6	8	1	2/3	0	1/3	0	×	$\frac{8}{2/3} = 12$
	$z = 48$		0	0* ↑	0 ↓	2	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is obtained as : $x_1 = 8$, $x_2 = 0$, max $z = 48$.

Alternative Solutions. Since Δ_2 corresponding to non-basic variable x_2 is obtained zero, this indicates that the *alternative solutions* also exist. Therefore, the solution as obtained above is not unique.

Thus we can bring X_2 into the basis in place of X_3. Therefore, introducing X_2 into the basis in place of X_3, the new optimum simplex table is obtained as follows :

Table 5.41

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_2	4	42/5	0	1	3/5	$-2/5$	0	×	
x_5	0	39/5	0	0	1/5	1/5	1	×	
x_1	6	12/5	1	0	$-2/5$	3/5	0	×	
	$z = 48$		0	0	0	2	0	×	$\leftarrow \Delta_j \geq 0$

From this table we get a different optimum solution : $x_1 = 12/5$, $x_2 = 42/5$, max. $z = 48$.

Thus, if two alternative optimum solutions can be obtained, then any number of optimum solutions can be obtained, as given below :

Variables	First Sol.	Second. Sol.	General Solution
x_1	8	12/5	$x_1 = 8\lambda + (12/5)(1-\lambda)$
x_2	0	42/5	$x_2 = 0\lambda + (42/5)(1-\lambda)$
x_3	14	0	$x_3 = 14\lambda + 0(1-\lambda)$
x_4	0	0	$x_4 = 0\lambda + 0(1-\lambda)$
x_5	5	39/5	$x_5 = 5\lambda + (39/5)(1-\lambda)$
a_1	0	0	$a_1 = 0\lambda + 0(1-\lambda)$

For any arbitrary value of λ, same optimal value of z will be obtained.

Note. If two optimum solutions of an LP problem are obtained. thus the mean of these two solutions will give us the third optimum solution. This process can be continued indefinitely to get as many alternative solutions as we want.

Example 23. *Maximize* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to the constraints :*

$x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, *and* $x_1, x_2, x_3, x_4 \geq 0$.

Solution. Introducing artificial variables a_1 and a_2 in the *first* and *second* constraint equations, respectively, and the *original variable* x_4 can be treated to work as an artificial variable for the *third* constraint equation to obtain :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad &= 10 . \end{aligned}$$

Phase 1 : Table 5·42

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
a_1	15	1	2	3	0	1	0
a_2	20	2	1	5	0	0	1
$\leftarrow x_4$	10	$\boxed{1}$	2	1	1	0	0
		↑			↓		

By the same arguments as given in the previous examples of two-phase method insert $\mathbf{X_4}$ in place of $\mathbf{X_1}$. The transformed table (*Table* 5·43) is obtained by applying row transformations $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$.

Table 5·43

BASIC VARIABES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
a_1	5	0	0	2	−1	1	0
$\leftarrow a_2$	0	0	−3	$\boxed{3}$	−2	0	1
$\to x_1$	10	1	2	1	1	0	0
				↑			↓

In spite of the fact that the artificial variable x_4 has served its purpose, the column $\mathbf{X_4}$ cannot be deleted from *Table 5·43,* because x_4 is the original variable also. Although the value of the artificial variable a_2 also becomes zero at this stage, the column $\mathbf{A_2}$ cannot be deleted unless it is inserted at one of the places $\mathbf{X_2}$ or $\mathbf{X_3}$ or $\mathbf{X_4}$ (wherever it is possible). Now, it is observed that $\mathbf{A_2}$ can be inserted in place of $\mathbf{X_3}$. Hence transformation *Table* 5·44 is obtained by applying the row transformations : $R_2 \to \frac{1}{3} R_2$, $R_1 \to R_1 - \frac{2}{3} R_2$, $R_3 \to R_3 - \frac{1}{3} R_2$.

Table 5·44

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
$\leftarrow a_1$	5	0	2	0	1/3	1	−2/3
$\to x_3$	0	0	−1	1	−2/3	0	1/3
x_1	10	1	3	0	5/3	0	−4/3
			↑			↓	

Now removing $\mathbf{A_1}$ and inserting it in the suitable position of $\mathbf{X_2}$, the next transformed *Table 5·45* is obtained by row transformations : $R_1 \to \frac{1}{2} R_1$, $R_2 \to R_2 + \frac{1}{2} R_1$, $R_3 \to R_3 - \frac{3}{2} R_1$.

Table 5·45

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$
x_2	5/2	0	1	0	1/6	1/2
x_3	5/2	0	0	1	−1/2	1/2
x_1	5/2	1	0	0	7/6	−3/2

Delete column $\mathbf{A_1}$ $(a_1 = 0)$. The starting basic feasible solution is obtained : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$. Further, proceed to test this solution for optimality in *Phase II.* For this, compute

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (2 , 3 , 1)\,(1/6 , -1/2 , 7/6) - 0 = 0 .$$

Phase II. Table 5·46

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	Min. Ratio
x_2	2	5/2	0	1	0	1/6	
x_3	3	5/2	0	0	1	− 1/2	
x_1	1	5/2	1	0	0	7/6	
	$z = C_B X_B = 15$		0	0	0	0*	$\leftarrow \Delta_j$

Since all Δ_j 's are zero, the solution : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$, is optimal to give us $z^* = 15$. Further, Δ_4 being zero indicates that *alternative* optimal solutions are also possible.

Note. Here Δ_j corresponding to nonbasic vector $\mathbf{X}_4$ also becomes zero. This indicates that alternative optimum solutions are possible. However, the other optimal solutions can be obtained as : $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$, max. $z = 15$.

Now, given the *two* alternative basic solutions ;

(i) $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$ (ii) $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$

an infinite number of *non-basic* solutions can be obtained and by realizing them any weighted average of these two basic solutions is also an alternative optimum solution.

To verify this, *third* solution will be obtained as :

$$x_1 = \frac{5/2 + 0}{2}, \ x_2 = \frac{5/2 + 15/7}{2}, \ x_3 = \frac{5/2 + 25/7}{2}, \ x_4 = \frac{0 + 0}{2}$$

i.e., $$x_1 = 5/4 , \ x_2 = 65/28 , \ x_3 = 85/28 , \ x_4 = 0 ,$$

yielding the maximum value of $z = 15$.

Note. Also see example 14 (ch. 5, unit 2, page 88).

Example 24. *Following is the LP problem : Maximize* $z = x_1 + x_2 + x_4$, *subject to the constraints :*

$$x_1 + x_2 + x_3 + x_4 = 4, x_1 + 2x_2 + x_3 + x_5 = 4, \ x_1 + 2x_2 + x_3 = 4, x_1 , x_2 , x_3 , x_4 , x_5 \geq 0 .$$

(i) *Find out all the optimal basic feasible solutions by using penalty (or Big-M) method.*

(ii) *Write-down the general form of an optimal solution.*

Solution. Since the constraints of the given problem are already equations, only artificial variables are required to form the basis matrix. In order to bring the basis matrix as unit matrix, only artificial variable $a_1 \geq 0$ is needed in the third constraint. So the problem may be re-written in the form :

Max. $z = x_1 + x_2 + 0x_3 + x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 \quad\quad\quad &= 4 \\ x_1 + 2x_2 + x_3 \quad + x_5 \quad &= 4 \\ x_1 + 2x_2 + x_3 \quad\quad + a_1 &= 4 \\ x_1 , x_2 , \ldots, x_5 , a_1 &\geq 0 \end{aligned}$$

These constraints may be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} .$$

Applying the usual simplex method, the solution is obtained as follows :

Table 5·47

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_4	1	4	1	1	1	1	0	0	4/1
x_5	0	4	1	2	1	0	1	0	4/2
$\leftarrow a_1$	$-M$	4	1	$\boxed{2}$	1	0	0	1	4/2 ←(Note)
	$z = -4M + 4$		$-M$	$-2M$ ↑	$-M + 1$	0	0	0 ↓	$\leftarrow \Delta_j$

Note. Here it is observed that the minimum 4/2 occurs at two places (2nd and 3rd) in the last column. Although one of these two may be chosen by degeneracy rule (see 5.7-1, page 95), but minimum at 3rd place has been chosen to remove artificial basis vector A_1 from the basis matrix.

Table 5·48

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_4	1	2	1/2	0	1/2	1	0	×	
x_5	0	0	0	0	0	0	1	×	
$\leftarrow x_2$	1	2	1/2	1	1/2	0	0	×	
	$z = 4$		0*	0	1	0	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution has been attained. Thus the optimum solution is given by

$$x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 2, x_5 = 0, \text{ max. } z = 4.$$

Since $\Delta_1 = 0$, alternative optimum solutions also exist.

5·8–2. Unbounded Solutions

The case of unbounded solutions occurs when the feasible region is unbounded such that the value of the objective function can be increased indefinitely. It is not necessary, however, that an unbounded feasible region should yield an unbounded value for the objective function. The following examples will illustrate these points.

Example 25. (Unbounded Optimal Solution)

Max. $z = 2x_1 + x_2$, *subject to* : $x_1 - x_2 \leq 10$, $2x_1 - x_2 \leq 40$, *and* $x_1 \geq 0, x_2 \geq 0$.

Solution. The starting simplex table is as follows :

Table 5.49

BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1 (β_1)	S_2 (β_2)
s_1	0	10	1	−1	1	0
s_2	0	40	2	−1	0	1
	$z = C_B X_B = 0$		−2	−1	0	0

It can be seen from the starting simplex table that the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are candidates for the entering vector. Since Δ_1 has the minimum value, $\mathbf{X}_1$ should be selected as the entering vector. It is noticed, however, that if $\mathbf{X}_2$ is selected as the entering vector, the value of x_2 (and hence the value of z) can be increased indefinitely without affecting the feasibility of the solution (since it has all x_{i2} negative). It is thus concluded that the problem has no bounded solution. This can also be seen from the graphical solution of the problem in Fig. 5.1.

In general, *an unbounded solution can be detected if, at any iteration, any of the candidates for the entering vector* $\mathbf{X}_k$ *(for which* $\Delta_k < 0$, *i.e.* $z_k - c_k < 0$*) has all* $x_{ik} \leq 0$, $i = 1, 2, ..., m$, *i.e., all elements of the entering column are* ≤ 0.

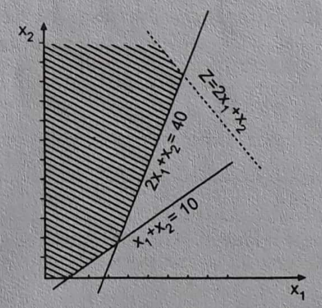

Fig. 5.1

Example 26. (Unbounded Solution)

Maximize $z = 107x_1 + x_2 + 2x_3$, *subject to* :

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7,\ 16x_1 + \tfrac{1}{2}x_2 - 6x_3 \leq 5,\ 3x_1 - x_2 - x_3 \leq 0, \text{ and } x_1, x_2, x_3 \geq 0.$$

Solution. By introducing slack variables, $x_5 \geq 0$, $x_6 \geq 0$, the set of constraints is converted into the system of equations :

$$\left\{\begin{array}{lr} 14x_1 + x_2 - 6x_3 + 3x_4 & = 7 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 \quad + x_5 & = 5 \\ 3x_1 - x_2 - x_3 \quad + x_6 & = 0 \end{array}\right\} \text{ or } \left\{\begin{array}{lr} \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 & = 7/3 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 \quad + x_5 & = 5 \\ 3x_1 - x_2 - x_3 \quad + x_6 & = 0 \end{array}\right\}$$

or
$$\begin{bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

Here original variable x_4 has been treated as slack variable as its coefficient in the objective function is zero, *i.e.*, *Maximize* $z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Now start simplex method as follows :

Table 5·50

		$c_j \rightarrow$	107	1	2	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO
x_4	0	7/3	14/3	1/3	–2	1	0	0	7/14
x_5	0	5	16	1/2	–6	0	1	0	5/16
x_6	0	0	$\boxed{3}$	–1	–1	0	0	1	0/3 ←
	$z = 0$		–107 ↑	–1	–2	0	0	0 ↓	← Δ_j
x_4	0	7/3	0	17/9	– 4/9	1	0	– 14/9	
x_5	0	5	0	35/6	– 2/3	0	1	– 16/3	
x_1	107	0	1	– 1/3	– 1/3	0	0	1/3	
	$z = 0$		0	$-\frac{110}{3}$	$-\frac{113}{3}$	0	0	$\frac{107}{3}$	← Δ_j

Since corresponding to negative Δ_3, all elements of $\mathbf{X}_3$ column are negative, so $\mathbf{X}_3$ cannot enter into the basis matrix. Consequently, this is an indication that there exists an *unbounded solution* to the given problem.

Example 27. (***Unbounded feasible region but bounded optimal solution***)

Max. $z = 6x_1 - 2x_2$, *subject to* $2x_1 - x_2 \le 2$, $x_1 \le 4$, *and* $x_1, x_2 \ge 0$.

Solution. We only give the successive tables here. Students are advised to fill up the details.

Table 5·51. Starting Simplex Table

		$c_j \rightarrow$	6	–2	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3 (β_1)	X_4 (β_2)	MIN. RATIO (X_B/X_1)
x_3	0	2	↑ ← $\boxed{2}$	–1	1	0	2/2 ←
x_4	0	4	1	0	0	1	4/1
	$z = C_B X_B = 0$		–6 ↑	2	0 ↓	0	← Δ_j

First Improvement. We enter X_1 and remove β_1.

Table 5·52

		$c_j \rightarrow$	6	–2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_1)	X_2	X_3	X_4 (β_2)	MIN. RATIO (X_B/X_2)
x_1	6	1	1	– 1/2	1/2	0	—
x_4	0	3	0	← $\boxed{1/2}$	–1/2	1	$3/\frac{1}{2}$ ←
	$z = C_B X_B = 6$		0	–1 ↑	3	0 ↓	← Δ_j

Second Improvement. Enter $\mathbf{X}_2$ and remove β_2.

Table 5·53

BASIC VARABLES	C_B	X_B	X_1 (β_1)	X_2 (β_2)	X_3	X_4	Min. Ratio
x_1	6	4	1	0	0	1	
x_2	–2	6	0	1	–1	2	
	$z = C_B X_B = 12$		0	0	2	2	$\leftarrow \Delta_j$

The optimal solution is : $x_1 = 4$, $x_2 = 6$, and $z = 12$.

It is now interesting to note from starting table that the elements of $\mathbf{X}_2$ are negative or zero (–1 and 0). This is an immediate indication that the feasible region is not bounded (see Fig. 5·2). From this, we conclude that a problem may have unbounded feasible region but still the optimal solution is bounded.

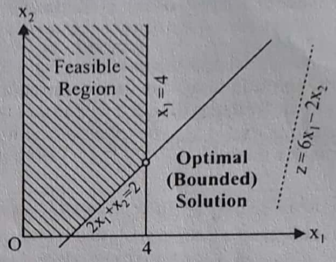

Fig. 5.2

5·8–3 . Non-existing feasible solutions

In this case, the feasible region is found to be empty which indicates that the problem has no feasible solution. The following example shows how such a situation can be detected by simplex method.

Example 28. (*Problem with no feasible solution*).

Max. $z = 3x_1 + 2x_2$, *subject to* $2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, *and* $x_1, x_2 \ge 0$.

[Garhwal 97; Meerut (O.R.) 90]

Solution. Introducing slack variable x_3, surplus variable x_4 together with the artificial variable a_1, the constraints become :

$$2x_1 + x_2 + x_3 \qquad = 2$$
$$3x_1 + 4x_2 \qquad - x_4 + a_1 = 12 .$$

Here we use M-technique for dealing with artificial variable a_1. For this, we write the objective function as

$$\text{Max. } z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Ma_1 .$$

The starting simplex table will be as follows.

Table 5·54

		$c_j \rightarrow$	3	2	0	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3 (β_1)	X_4	A_1 (β_2)	MIN. RATIO (X_B/X_K)
$\leftarrow x_3$	0	2	1	$\boxed{1}$	1	0	0	$2/1 \leftarrow$
a_1	$-M$	12	3	4	0	–1	1	12/4
	$z = C_B X_B = -12M$		$(-3M-3)$	$(-4M-2)$ $\uparrow$	0 $\downarrow$	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_B X_1 - c_1 = (0, -M)(2, 3) - 3 = (0 - 3M) - 3 = -3 - 3M$$
$$\Delta_2 = C_B X_2 - c_2 = (0, -M)(1, 4) - 2 = (0 - 4M) - 2 = -2 - 4M$$
$$\Delta_4 = C_B X_4 - c_4 = (0, -M)(0, -1) - 0 = M .$$

First improvement. Inserting $\mathbf{X}_2$ and removing β_1, *i.e.* $\mathbf{X}_3$

Table 5·55

		$c_j \rightarrow$	3	2	0	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	MIN. RATIO
x_2	2	2	2	1	1	0	0	
a_1	$-M$	4	–5	0	–4	–1	1	
	$z = C_B X_B = 4 - 4M$		$(1 + 5M)$	0	$(2 + 4M)$	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_B Y_1 - c_1 = (2, -M)(2, -5) - 3 = (4 + 5M) - 3 = (1 + 5M)$$

$$\Delta_3 = \mathbf{C_B Y_3} - c_3 = (2, -M)(1, -4) - 0 = (2 + 4M) - 0 = (2 + 4M)$$
$$\Delta_4 = \mathbf{C_B Y_4} - c_4 = (2, -M)(0, -1) - 0 = (0 + M) = M.$$

Here all Δ_j are positive since $M > 0$. So according to the optimality condition, this solution is optimal.

Note. Here we should, however, note that the optimal (basic) solution :

$$x_1 = 0,\ x_2 = 2,\ x_3 = 0,\ x_4 = 0,\ a_1 = 4,$$

includes the artificial variable a_1 with positive value 4. This immediately indicates that the problem has no feasible solution, because the positive value of a_1 violates the second constraint of given problem. This situation can be observed by the graphical representation of this example in Fig. 5·3.

Such solution may be called "*pseudo-optimal*", since (as clear from the Figure 5·3) it does not satisfy all the constraints, but it satisfies the optimality condition of the simplex method. **[JNTU (B.Tech) 98]**

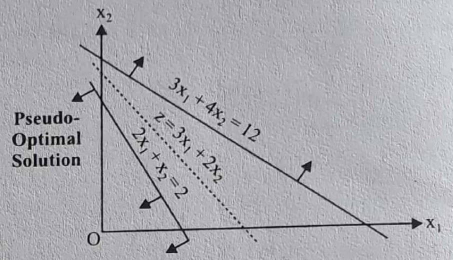

Fig. 5.3.

Q. 1. Show on graphy it (i) Unbounded solution space (ii) No feasible solution space. **[JNTU (B. Tech.) 2003]**

2. How are the following detected in simplex method ?

(i) Alternative solution (ii) Unbounded solution (iii) Infeasible solution. **[JNTU (B. Tech. III) 2003]**

5.9. BOUNDED VARIABLES PROBLEMS IN LP

In case the value of some or all variables is restricted with lower or upper bounds, the standard form of LP problem may be expressed as : Max (or Min) $z = \mathbf{cx}$, subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$ where $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ stand for *lower* and *upper* bounds for $\mathbf{x}$, respectively. Other symbols have their usual meaning.

Then the constraint $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$ can be converted into equality by introducing *slack* or *surplus* variables s' and s'' as follows :

$$\mathbf{x} \le \mathbf{u} \quad \Rightarrow \quad \mathbf{x} + s' = \mathbf{u},\ s' \ge 0$$

and

$$\mathbf{x} \ge \mathbf{l} \quad \Rightarrow \quad \mathbf{x} - s'' = \mathbf{l},\ s'' \ge 0.$$

Therefore given *LP* has $m + n$ constraints as equations with $3n$ variables. However, we may also reduce this size in the form of $\mathbf{Ax} = \mathbf{b}$.

The lower bound constraint $\mathbf{l} \le \mathbf{x}$ can also be written as : $\mathbf{x} = \mathbf{l} + s'', s'' \ge 0$ and therefore $\mathbf{x}$ can be eliminated from all the constraints with the help of this substitution. Similarly, the upper bound $\mathbf{x} \le \mathbf{u}$ can also be written as : $\mathbf{x} = \mathbf{u} - s', s' \ge 0$. However, such substitution may not ensure non-negative value of $\mathbf{x}$. Thus in order to overcome this difficulty, usual simplex method can also be modified to deal with bounded variables.

In bounded variable simplex method, the optimality condition remains the same. But the substitution of constraint $\mathbf{x} + \mathbf{s}' = \mathbf{u}$ in the simplex table require modification in the feasibility condition due to following reasons :

(i) A *basic variable* should become *non-basic* at its upper bound (in usual simplex method all *non-basic variables* are always *zero.*).

(ii) Whenever a non-basic variable becomes basic, its value should not be more than its upper bound and also should preserve the non-negativity and upper bound conditions of all basic variables that exist.

5.10. MODIFIED SIMPLEX ALGORITHM

The modified simplex algorithm for bounded variables can best be understood by following interative steps :

Step 1. If the objective function of the given problem is of minimization form, then change it to maximization form by using the relationship : Min $z = -$ Max z', $z' = -z$.

Step 2. Check whether all b_i, $i = 1, 2, \ldots, m$) are positive. If any of them is (are) negative. Then multiply the corresponding constraint by -1 in order to make it positive.

Step 3. Express the given *LP* problem in standard form by introducing slack/surplus variables.

Step 4. Obtain an Initial Basic Feasible Solution. If any of the basic variables is at a positive lower bound, then substitute it out at its lower bound.

Step 5. Compute $\Delta_j = C_B X_j - c_j, j = 1, 2, \ldots, n$ and examine the values of all $\Delta_j\ (= z_j - c_j)$.

Step 6. (i) If all $\Delta_j \geq 0$, then the basic feasible solution is optimum.

(ii) If at least one of them is negative and the corresponding column has at least one entry positive (*i.e.*, $x_{ij} > 0$) for some row i, then it indicates that further improvement in z is possible.

Step 7. If case **(ii)** of ***step 6*** is possible, select a non-basic variable x_k to enter into new solution according to the most negative $\Delta_j\ (= \Delta_k \uparrow)$.

Step 8. After identifying the non-basic variable (column vector) to enter the basis matrix **B**, the vector to be removed is determined. For this we first compute the quantities :

$$\theta_1 = \min_i \left\{ \frac{x_{Bi}}{x_{ir}}, x_{ir} > 0 \right\}, \theta_2 = \min_i \left\{ \frac{u_r - x_{Bi}}{-x_{ir}}, x_{ir} < 0 \right\}$$

and $$\theta = \min\,[\theta_1, \theta_2, u_r]$$

where u_r is the upper bound for the variable x_r in the current basic feasible solution. Clearly, if all $x_{ir} > 0$, $\theta_2 = \infty$. Then following *three* possibilities may occur :

(i) If $\theta_1 = \theta$, then basis column vector $\mathbf{a}_k$ (basic varaible x_k) will be removed from the basis and thus it is replaced by column vector, say $\mathbf{a}_r$ (non-basic variable x_r) in the usual manner.

(ii) If $\theta_2 = \theta$, then basis column vector $\mathbf{a}_k$ (basic variable x_k) will be removed from the basis and is replaced by a column vector $\mathbf{a}_r$ (non-basic variable x_r). But the value of basic variable $x_r = x_{Br}$ will not be at upper bound at this stage. Now this must be substituted by using the relationship :

$$(x_{Bk})_r = (x_{Bk})'_r - x_{kr} u_r,\ 0 \leq (x_{Bk})'_r \leq u_r$$

where $(x_{Bk})'_r$ represents the value of x_r.

The value of non-basic variable x_r is given at its upper bound while the remaining non-basic variables are put at zero level by using the relationship : $x_r = u_r - x'_r, 0 \leq x'_r \leq u_r$.

Step 9. Go to ***step 7*** and repeat the procedure until all θ entries in $\Delta_j\ (= z_j - c_j)$ row become either positive or zero.

5.11 ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following LP problem :*

Max $z = 3x_1 + 2x_2$, *subject to the constraints* :

$x_1 - 3x_2 \leq 3,\ x_1 - 2x_2 \leq 4,\ 2x_1 + x_2 \leq 20,\ x_1 + 3x_2 \leq 30,\ -x_1 + x_2 \leq 6$

and $0 \leq x_1 \leq 8, 0 \leq x_2 \leq 6$.

Solution. The given problem can be easily written in the following standard form :

Max $z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$

subject to the constraints :

$$\begin{aligned} x_1 - 3x_2 + s_1 \quad\quad\quad\quad &= 3 \\ x_1 - 2x_2 \quad + s_2 \quad\quad\quad &= 4 \\ 2x_1 + x_2 \quad\quad + s_3 \quad\quad &= 20 \\ x_1 + 3x_2 \quad\quad\quad + s_4 \quad &= 30 \\ -x_1 + x_2 \quad\quad\quad\quad + s_5 &= 6 \end{aligned}$$

and $$x_1, x_2, s_1, s_2, s_3, s_4, s_5 \geq 0.$$

Then the initial basic feasible solution is : $s_1 = 3$, $s_2 = 4$, $s_3 = 20$, $s_4 = 30$ and $s_5 = 6$. Since no upper bounds are given to these basic variables, we arbitrarily assume that all of them have upper bounds tending to ∞ (*i.e.*, $s_1 = s_2 = s_3 = s_4 = s_5 = \infty$). This solution may also be read from the following initial simplex *Table 5.56*.

Table 5.56 Initial Solution

Basic Var.	U_B	C_B	X_B	X_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
$u_j \rightarrow$				8	6	∞	∞	∞	∞	∞	
$c_j \rightarrow$				3	2	0	0	0	0	0	
s_1	∞	0	3	$\boxed{1}$	– 3	1	0	0	0	0	$(\infty - 3) \rightarrow \infty$
s_2	∞	0	4	1	– 2	0	1	0	0	0	$(\infty - 4) \rightarrow \infty$
s_3	∞	0	20	2	1	0	0	1	0	0	$(\infty - 20) \rightarrow \infty$
s_4	∞	0	30	1	3	0	0	0	1	0	$(\infty - 30) \rightarrow \infty$
s_5	∞	0	6	– 1	1	0	0	0	0	1	$(\infty - 6 \rightarrow \infty)$
		$z = 0$		$\boxed{-3}$ ↑	– 2	0	0	0	0	0	$\leftarrow \Delta_j$

Since $\Delta_1 = z_1 - c_1 = -3$ is largest negative, x_1 will *enter into the basis.*

As none of the basic variables s_1 to s_5 are at their upper bound, thus to decide about the variable to *leave the basic solution* we calculate.

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right) = \text{Min}\left(\frac{3}{1}, \frac{4}{1}, \frac{20}{2}, \frac{30}{1}\right) = 3 \qquad \text{(corresponding to } x_1\text{)}$$

$$\theta_2 = \underset{i}{\text{Min}}\left(\frac{u_i - x_{Bi}}{-x_{i1}}, x_{i1} < 0\right) = \text{Min}\left(\frac{\infty - 6}{-(-1)}\right) = \infty \qquad \text{(corresponding to } s_5\text{)}$$

and $u_1 = 8$.

$$\therefore \qquad \theta = \text{Min}\,(\theta_1, \theta_2, u_1) = \text{Min}\,(3, \infty, 8) = 3 \qquad \text{(corresponding to } \theta_1\text{)}$$

Hence the non-basic variable s_1 is eligible to leave the basic solution and therefore $\boxed{x_{11} = 1}$ will become the *key element.*

Thus introducing x_1 into the basis and *removing* s_1 from the basis using row operations in an usual manner we get the improved solution *Table 5.57*

Table 5.57

Basic Var.	U_B	C_B	X_B	X_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
$u_j \rightarrow$				8	6	∞	∞	∞	∞	∞	
$c_j \rightarrow$				3	2	0	0	0	0	0	
x_1	8	3	3	1	– 3	1	0	0	0	0	$(8 - 3) \rightarrow 5$
s_2	∞	0	1	0	$\boxed{1}$	– 1	1	0	0	0	$(\infty - 1) \rightarrow \infty$
s_3	∞	0	14	0	7	– 2	0	1	0	0	$(\infty - 14) \rightarrow \infty$
s_4	∞	0	27	0	6	– 1	0	0	1	0	$(\infty - 27) \rightarrow \infty$
s_5	∞	0	9	0	– 2	1	0	0	0	1	$(\infty - 9) \rightarrow \infty$
∴		$z = 9$		0	– 11↑	3	0	0	0	0	$\leftarrow \Delta_j$

Since $\Delta_2 = -11$ is the largest negative quantity, variable x_2 must enter the basis. So, to find the variable to leave the basis, we compute

$$\theta_1 = \underset{i}{\text{Min}}\left[\frac{x_{Bi}}{x_{i2}}, x_{i2} > 0\right] = \text{Min}\left[\frac{1}{1}, \frac{14}{7}, \frac{27}{6}\right] = 1 \qquad \text{(corresponding to } x_2\text{)}$$

$$\theta_2 = \underset{i}{\text{Min}}\left[\frac{u_i - x_{Bi}}{-x_{i2}}, x_{i2} < 0\right] = \text{Min}\left[\frac{8-3}{-(-3)}, \frac{\infty}{-(-2)}\right] = \frac{5}{3} \qquad \text{(corresponding to } x_1\text{)}$$

and $\qquad u_2 = 6.$

Therefore, $\theta = \text{Min}\,[\theta_1, \theta_2, u_2] = \text{Min}\,[1, 5/3, 6] = 1$ (corresponding to s_2)

Hence s_2 must leave the basis and $x_{22} = 1$ will be the key element. So introduce x_2 into the basis and remove s_2 from the basis in the usual manner. The improved solution is given in *Table 5.58.*

Table 5.58

			$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	3	2	0	0	0	0	0	
Basic Var.	$\mathbf{U_B}$	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$	$\mathbf{S_4}$	$\mathbf{S_5}$	$\mathbf{u_i - x_{Bi}}$
x_1	8	3	6	1	0	$\boxed{-2}$	3	0	0	0	$8-6=2$
x_2	6	2	1	0	1	-1	1	0	0	0	$6-1=5$
s_3	∞	0	7	0	0	5	-7	1	0	0	$\infty - 7 \rightarrow \infty$
s_4	∞	0	21	0	0	5	-6	0	1	0	$\infty - 21 \rightarrow \infty$
s_5	∞	0	11	0	0	-1	2	0	0	1	$\infty - 11 \rightarrow \infty$
		$z=20$		0	0	-8 ↑	11	0	0	0	$\leftarrow \Delta_j$

In this table, $\Delta_3 = z_3 - c_3 = -8$ is the largest negative quantity. Therefore s_1 will *enter into the basis.*

Now to find the variable to be removed from the basis we calculate :

$$\theta_1 = \underset{i}{\text{Min}} \left(\frac{x_{Bi}}{x_{i3}}, x_{i3} > 0 \right) = \text{Min} \left(\frac{7}{5}, \frac{21}{5} \right) = \frac{7}{5} \qquad \text{(corresponding to } s_3\text{)}$$

$$\theta_2 = \underset{i}{\text{Min}} \left(\frac{u_i - x_{Bi}}{-x_{i1}}, x_{i1} < 0 \right) = \text{Min} \left(\frac{8-6}{-(-2)}, \frac{6-1}{-(-1)}, \frac{\infty}{-(-1)} \right) = 1 \qquad \text{(corresponding to } x_1\text{)}$$

and $u_3 = \infty.$

$$\therefore \quad \theta = \text{Min } (\theta_1, \theta_2, u_3) = \text{Min } (7/5, 1, \infty) = 1 \qquad \text{(corresponding to } x_1\text{)}$$

Therefore, the variable x_1 will leave the basis and $x_{13} = -2$ will become the key element.

Thus introducing s_1 into the basis and removing x_1 from the basis by row transformation, we set the following improved solution *Table 5.59.*

Table 5.59

			$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	3	2	0	0	0	0	0	
Basic Var.	$\mathbf{U_B}$	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$	$\mathbf{S_4}$	$\mathbf{S_5}$	$\mathbf{u_i - x_{Bi}}$
s_1	∞	0	-3	$-1/2$	0	1	$-3/2$	0	0	0	$\infty - (-3) \rightarrow \infty$
x_2	6	2	-2	$-1/2$	1	0	$-1/2$	0	0	0	$6-(-2)=8$
s_3	∞	0	22	$5/2$	0	0	$1/2$	1	0	0	$\infty - 22 \rightarrow \infty$
s_4	∞	0	36	$5/2$	0	0	$3/2$	0	1	0	$\infty - 36 \rightarrow \infty$
s_5	∞	0	8	$-1/2$	0	0	$1/2$	0	0	1	$\infty - 8 \rightarrow \infty$
		$z=-4$		-4 ↑	0	0	-1	0	0	0	$\leftarrow \Delta_j$

Since $\Delta_1 = z_1 - c_1 = -4$ is the largest negative quantity, x_1 will enter into the basis. But upper bound for the variable x_1 is 8, therefore we can update the value of basic variables by using the relations and data of *Table 5.59.*

$$\begin{aligned}
s_1 &= x_{B1} = x'_{Bi} - x_{11}\, u_1 = -3 - (-1/2)\, 8 = 1 \\
x_2 &= x_{B2} = x'_{B2} - x_{21}\, u_1 = -2 - (-1/2)\, 8 = 2 \\
s_3 &= x_{B3} = x'_{B3} - x_{31}\, u_1 = 22 - (5/2)\, 8 \quad = 2 \\
s_4 &= x_{B4} = x'_{B4} - x_{41}\, u_1 = 36 - (5/2)\, 8 \quad = 16 \\
s_5 &= x_{B5} = x'_{B5} - x_{51}\, u_1 = 8 - (-1/2)\, 8 \quad = 12.
\end{aligned}$$

Also, the non-basic variable x_1 having its upper bound can be made non-basic (*i.e.* zero) by using the substitution

$$x_1 = u_1 - x'_1 = 8 - x'_1, \; 0 \le x'_1 \le 8.$$

Now we can update the data of *Table* 5.59 by substituting new values of basic variables as well as non-basic variables as given below in *Table* 5.60.

Table 5.60

			$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	−3	2	0	0	0	0	0	
Basic Var.	u_B	c_B	x_B	X'_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	1	1/2	0	1	−3/2	0	0	0	$\infty - 1 \rightarrow \infty$
x_2	6	2	2	1/2	1	0	−1/2	0	0	0	$6 - 2 = 4$
s_3	∞	0	2	−5/2	0	0	$\boxed{1/2}$	1	0	0	$\infty - 2 \rightarrow \infty$
s_4	∞	0	16	−5/2	0	0	3/2	0	1	0	$\infty - 16 \rightarrow \infty$
s_5	∞	0	12	1/2	0	0	1/2	0	0	1	$\infty - 12 \rightarrow \infty$
	$z = 24 + 4 = 28$			4	0	0	−1 ↑	0	0	0	$\leftarrow \Delta_j$

Since Δ_4 only is negative (*i.e.*, − 1), s_2 will *enter into the basis*. To find the variable to be *removed from the basis*, we calculate

$$\theta_1 = \min_i \left(\frac{x_{Bi}}{x_{i4}}, x_{i4} > 0 \right) = \text{Min}\left(\frac{2}{1/2}, \frac{16}{3/2}, \frac{12}{1/2} \right) = \text{Min } (4, 32/3, 24) = 4 \text{ (corresponding to } s_3)$$

$$\theta_2 = \min_i \left(\frac{u_i - x_{Bi}}{-x_{i4}}, x_{i4} < 0 \right) = \text{Min}\left(\frac{\infty}{-(-3/2)}, \frac{6-2}{-(-1/2)} \right) = 8 \qquad \text{(corresponding to } x_2)$$

and $u_4 = \infty$.

$\therefore$ $\theta = \text{Min }(\theta_1, \theta_2, u_4) = \text{Min }(4, 8, \infty) = 4$ (corresponding to s_3).

Therefore, the basic variable s_3 will be removed from the basis and thus $x_{34} = 1/2$ will become the key element. Thus *introducing* s_2 into the basis and *removing* s_3 from the basis, we get the following improved solution *Table* 5.61.

Table 5.61

			$u_i \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	−3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	7	−7	0	1	0	3	0	0	$\infty - 1 \rightarrow \infty$
x_2	6	2	4	$\boxed{-2}$	1	0	0	1	0	0	$6 - 4 = 2$
s_2	∞	0	4	−5	0	0	1	2	0	0	$\infty - 4 \rightarrow \infty$
s_4	∞	0	10	5	0	0	0	−3	1	0	$\infty - 10 \rightarrow \infty$
s_5	∞	0	10	3	0	0	0	−1	0	1	$\infty - 10 \rightarrow \infty$
	$z = 28 + 4 = 32$			−1	0	0	0	2	0	0	$\leftarrow \Delta_j$

Since Δ_1 is the only negative value, variable x_1 will enter into the basis. To find the variable to be removed from the basis, we calculate

$$\theta_1 = \min_i \left(\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0 \right) = \text{Min}\left(\frac{10}{5}, \frac{10}{3} \right) = \frac{10}{5} \qquad \text{(corresponding to } s_4)$$

$$\theta_2 = \min_i \left(\frac{u_i - x_{Bi}}{-x_{i1}}, x_{i1} < 0 \right) = \text{Min}\left(\frac{\infty}{-(-7)}, \frac{6-4}{-(-2)}, \frac{\infty}{-(-5)} \right) = 1 \qquad \text{(corresponding to } x_2)$$

and $u_1 = 8$.

$\therefore$ $\theta = \text{Min }(\theta_1, \theta_2, u_1) = \left(\frac{12}{5}, 1, 8 \right) = 1$ (corresponding to x_2)

Thus, the variable x_2 will leave the basis and $x_{21} = -2$ will become the key element. Hence introducing x'_1 into the basis and removing x_2 from the basis, we get the following improved solution *Table* 5.62.

Table 5.62

				$u_j \rightarrow$ 8	6	∞	∞	∞	∞	∞	
				$c_j \rightarrow$ 3	2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X_2	S_1	S_2	S_3	S_4	S_5	$u_i - x_{Bi}$
s_1	∞	0	−7	0	−7/2	1	0	−1/2	0	0	$\infty + 7 \rightarrow \infty$
x'_1	8	−3	−2	1	−1/2	0	0	−1/2	0	0	$-3+2=-1$
s_2	∞	0	−6	0	−5/2	0	1	−1/2	0	0	$\infty + 6 \rightarrow \infty$
s_4	∞	0	22	0	5/2	0	0	−1/2	1	0	$\infty - 22 \rightarrow \infty$
s_5	∞	0	16	0	3/2	0	0	1/2	0	1	$\infty - 16 \rightarrow \infty$
	$z = 32 - 2 = 30$			0	−1/2	0	0	3/2	0	0	$\leftarrow \Delta_j$

Since the upper bound for the variable x_2 is 6, we can update the basic variables by the following relations and the data of *Table* 10.7.

$$s_1 = x_{B1} = x'_{B1} - x_{12}u_2 = -7 - (-7/2)\,6 = 14$$
$$x'_1 = x_{B2} = x'_{B2} - x_{22}u_2 = -2 - (1/2)\,6 = 1$$
$$s_2 = x_{B3} = x'_{B3} - x_{32}u_2 = -6 - (-5/2)\,6 = 9$$
$$s_4 = x_{B4} = x'_{B4} - x_{42}u_2 = -22 - (5/2)\,6 = 7$$
$$s_5 = x_{B5} = x'_{B5} - x_{52}u_2 = 16 - (3/2)\,6 = 7.$$

Therefore, the non-basic variable x_2 at its upper bound can be made zero by using the substitution

$$x_2 = u_2 - x'_2 = 6 - x'_2,\; 0 \le x'_2 \le 6.$$

Now the data of *Table* 5.62 can be updated by substituting new values of basic variables and non-basic variables as given below in *Table* 5.63.

Table 5.63

			$u_j \rightarrow$	8	6	∞	∞	∞	∞	∞	
			$c_j \rightarrow$	−3	−2	0	0	0	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X'_2	S_1	S_2	S_3	S_4	S_5	
s_1	∞	0	14	0	7/2	1	0	−1/2	0	0	
x'_1	8	−3	1	1	1/2	0	0	−1/2	0	0	
s_2	∞	0	9	0	5/2	0	1	−1/2	0	0	
s_4	∞	0	7	0	−5/2	0	0	−1/2	1	0	
s_5	∞	0	7	0	−3/2	0	0	1/2	0	1	
	$z = 12 + 21 = 33$			0	1/2	0	0	3/2	0	0	$\leftarrow \Delta_j$

Since all $\Delta_j \ge 0$, an optimum solution is obtained with the values of variables as :

$$x'_1 = 1 \text{ or } x_1 = u_1 - x'_1 = 8 - 1 = 7;\; x_2 = u_2 - x'_2 = 6 - 0 = 6 \text{ and max } z = 33.$$

Example 2. *Solve the following LP problem :*

Maximize $z = 3x_1 + 5x_2 + 2x_3$, *subject to*

$$x_1 + 2x_2 + 2x_3 \le 14$$
$$2x_1 + 4x_2 + 3x_3 \le 23$$
$$0 \le x_1 \le 4$$
$$2 \le x_2 \le 5$$
$$0 \le x_3 \le 3.$$

Solution. The variable x_2 has a positive lower bound, therefore we can take the substitution $x'_2 = x_2 - 2$ or $x_2 = x'_2 + 2$. Thus the fourth constraint of the given problem can be written as $0 \le x'_2 \le 3$ and thus new LP problem becomes :

$$\text{Maximize} \quad z = 3x_1 + 5(x'_2 + 2) + 2x_3 = 3x_1 + 5x'_2 + 2x_3 + 10, \text{ subject to}$$

$$x_1 + 2(x'_2 + 2) + 2x_3 \le 14 \text{ or } x_1 + 2x'_2 + 2x_3 \le 10$$

$$2x_1 + 4(x'_2 + 2) + 3x_3 \le 23 \text{ or } 2x_1 + 4x'_2 + 3x_3 \le 15$$

$$0 \le x_1 \le 4$$

$$0 \le x'_2 \le 3$$

$$0 \le x_3 \le 3$$

with the help of non-negative slack variables s_1 and s_2, inequality constraints are converted to equations and thus standard form of LP problem becomes :

$$\text{Max. } z = 3x_1 + 5x'_2 + 2x_3 + 0s_1 + 0s_2 + 10, \text{ subject to the constraints}$$

$$x_1 + 2x'_2 + 2x_3 + s_1 = 10$$

$$2x_1 + 4x'_2 + 3x_3 + s_2 = 15$$

and $\qquad x_1, x'_2, x_3, s_1, s_2 \ge 0.$

The initial basic feasible solution becomes

$$s_1 = x_{B1} = 10,\ s_2 = x_{B2} = 15.$$

But, there are no upper bounds for the basic variables s_1 and s_2. Therefore, we may assume that both of these varaibles have an upper bound at ∞. Thus the initial basic feasible solution can be read from the initial simplex *Table* 5.64.

Table 5.64

			$u_j \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X'_2	X_3	S_1	S_2	$u_i - x_{Bi}$
s_1	∞	0	10	1	2	2	1	0	$\infty - 10 \rightarrow \infty$
s_2	∞	0	15	2	$\boxed{4}$	3	0	1	$\infty - 15 \rightarrow \infty$
	$z = 10$			-3	-5 $\uparrow$	-2	0	0	$\leftarrow \Delta_j$

Since $\Delta_2 = -5$ is the most negative, quantity x'_2 will enter into the basis. Also none of the basic variables (s_1 and s_2) are at their upper bounds. Therefore, to decide about the leaving variable, we calculate

$$\theta_1 = \underset{i}{\text{Min}}\left(\frac{x_{Bi}}{x_{i2}},\ x_{i2} > 0\right) = \text{Min}\left(\frac{10}{2}, \frac{15}{4}\right) = \frac{15}{4} \qquad \text{(corresponding to } s_2\text{)}$$

$$\theta_2 = \infty. \qquad (\because \text{ all entries in 2nd column are + ive, } x_{i2} > 0 \text{ for } i)$$

and $\qquad u_2 = 3.$

$$\therefore \qquad \theta = \text{Min}(\theta_1, \theta_2, u_2) = \text{Min}\left(\frac{15}{4}, \infty, 3\right) = 3 \qquad \text{(corresponding to } u_2\text{)}$$

Thus, the non-basic variable x'_2 can be substituted at its upper bound and will remain non-basic. Now the non-basic variable x'_2 at its upper bound will have the value zero by substitution :

$$x'_2 = u_2 - x''_2 = 3 - x''_2,\ 0 \le x''_2 \le 3.$$

The value of basic variables can be updated by using

$$s_1 = x_{B1} = x'_{B1} - x_{12}u_2 = 10 - 2 \times 3 = 4$$

$$s_2 = x_{B2} = x'_{B2} - x_{22}\,u_2 = 15 - 4 \times 3 = 3.$$

Now the *Table* 5.64 can be updated by substituting these new values of basic variables and non-basic variable x'_2 as given below in *Table* 5.65.

Table 5.65

			$u_j \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X''_2	X_3	s_1	s_2	$u_i - x_{Bi}$
s_1	∞	0	4	1	−2	2	1	0	$\infty - 4 \rightarrow \infty$
s_2	∞	0	3	$\boxed{2}$	−4	3	0	1	$\infty - 3 \rightarrow \infty$
	$z = 10 + 15 = 25$			−3	5	−2	0	0	$\leftarrow \Delta_j$

In this table, $\Delta_1 = -3$ is the most negative value, therefore x_1 will be the *entering variable* into the basis. Then, to decide about the variable *leaving* from the basis, we calculate

$$\theta_1 = \text{Min}\left(\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right) = \text{Min}\left(\frac{4}{1}, \frac{3}{2}\right) = 3/2 \qquad \text{(corresponding to } s_2)$$

$$\theta_2 = \infty \qquad (\because \text{ all entries in 1st column are + ive, } i.e.\ x_{i1} > 0\ \forall\ i)$$

and $\quad u_1 = 4.$

$\therefore \quad \theta = \text{Min}\,(\theta_1, \theta_2, u_1) = \text{Min}\,(3/2, \infty, 4) = 3/2 \qquad$ (corresponding to θ_1)

Hence basic variable s_2 will leave the basis and then $x_{21} = 2$ will become the *key element.* Therefore, introducing x_1 into the basis and removing s_2 from the basis in usual manner, we get the following improved solution *Table* 5.66.

Table 5.66

			$u_j \rightarrow$	4	3	3	∞	∞	
	$c_j \rightarrow$			3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X''_2	X_3	S_1	S_2	$u_i - x_{Bi}$
s_1	∞	0	5/2	0	0	1/2	1	−3/2	$\infty - 5/2 \rightarrow \infty$
x_1	3	3	3/2	1	$\boxed{-2}$	3/2	0	1/2	$3 - 3/2 = 3/2$
	$z = 25 + 9/2 = 59/2$			0	−1 ↑	5/2	0	3/2	$\leftarrow \Delta_j$

In this table, $\Delta_2 = -1$ is the only negative value, therefore x''_2 will be the variable *entering* into the basis. Then to decide about the variable *leaving* from the basis, we calculate.

$$\theta_1 = \infty \qquad (\because \text{ All entries in 2nd column are either 0 or negative, } i.e.\ x_{i2} \le 0\ \forall\ i)$$

$$\theta_2 = \text{Min}\left(\frac{u_i - x_{Bi}}{-x_{i2}}, x_{i2} < 0\right) = \text{Min}\left(\infty, \frac{3/2}{-(-2)}\right) = \frac{3}{4} \qquad \text{(corresponding to } x_1)$$

and $\quad u_2 = 3.$

$\therefore \quad \theta = \text{Min}\,(\theta_1, \theta_2, u_2) = \text{Min}\,(\infty, 3/4, 3) = 3/4 \qquad$ (corresponding to θ_2)

Thus, variable x_1 will leave the basis. Substituting $x_1 = 4 - x'_1$ in *Table* 5.65 for putting x_1 at its upper bound, we get the following improved solution *Table* 5.67.

Table 5.67

			$u_i \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	3	−5	2	0	0	
Basic Var.	U_B	C_B	X_B	X_1	X''_2	X_3	S_1	S_2	
s_1	∞	0	5/2	0	0	1/2	1	−1/2	
x''_2	3	−5	−3/4	−1/2	1	−3/4	0	−1/4	
	$z = 25 + 15/4$			−1/2 ↑	0	7/4	0	5/4	$\leftarrow \Delta_j$

The value of non-basic variable x_1 at its upper bound 4 can be put by substituting

$$x_1 = 4 - x'_1,\ 0 \le x'_1 \le 4.$$

The values of other basic variables can be updated by using

$$s_1 = x_{B1} = x'_{B1} - x_{11} u_1 = 5/2 - 0 \times 4 = 5/2$$

$$x''_2 = x_{B2} = x'_{B2} - x_{21}\, u_1 = -3/4 - (-1/2) \times 4 = 5/4$$

Now the *Table* 5.67 can be updated by taking new values of basic variables and non-basic x'_1 as shown below in *Table* 5.68.

Table 5.68

			$u_j \rightarrow$	4	3	3	∞	∞	
			$c_j \rightarrow$	-3	-5	2	0	0	
Basic Var.	U_B	C_B	X_B	X'_1	X''_2	X_3	S_1	S_2	
s_1	∞	0	5/2	0	0	1/2	1	$-1/2$	
x''_2	3	-5	5/4	1/2	1	$-3/4$	0	$-1/4$	
	$z = 123/4$			1/2	0	7/4	0	5/4	$\leftarrow \Delta_j$

Since all $\Delta_j \ge 0$, the optimal solution becomes :

$$x'_1 = 0 \quad \text{or} \quad 4 - x_1 = 0 \quad \text{or} \quad x_1 = 4$$

$$x''_2 = 5/4 \quad \text{or} \quad 3 - x'_2 = 5/4 \quad \text{or} \quad 3 - (x_2 - 2) = 5/4 \quad \text{or} \quad x_2 = 15/4$$

and $\max z = 123/4.$

EXAMINATION PROBLEMS

Solve the following LP *problems :*

1. Max. $z = 2x_1 + x_2$, subject to the constraints :
   $x_1 + 2x_2 \le 10,\ x_1 + x_2 \le 6,\ x_1 - x_2 \le 2,\ x_1 - 2x_2 \le 1,\ 0 \le x_1 \le 3,\ 0 \le x_2 \le 2.$
   [**Ans.** $x_1 = 3,\ x_2 = 2$ and max $z = 8$]
2. Max $z = x_2 + 3x_3$, subject to
   $x_1 + x_2 + x_3 \le 10,\ x_1 - 2x_3 \le 0,\ 0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ x_3 \le 0.$
   [**Ans.** $x_1 = 20/3,\ x_2 = 0,\ x_3 = 10/3$ and max. $z = 10$]
3. Max $z = x_1 + x_2 + 3x_3$, subject to $x_1 + x_2 + x_3 \le 12,\ -x_1 + x_2 \le 5,\ x_2 + 2x_3 \le 8,$
   $0 \le x_1 \le 3,\ 0 \le x_2 \le 6,\ 0 \le x_3 \le 4.$
4. Max. $z = 4x_1 + 4x_2 + 3x_3$, subject to
   $-x_1 + 2x_2 + 3x_3 \le 15,\ -x_2 + x_3 \le 4,\ 2x_1 + x_2 - x_3 \le 6,\ x_1 - x_2 + 2x_3 \le 10,$
   $0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ 0 \le x_3 \le 4.$
   [**Ans.** $x_1 = 17/5,\ x_2 = 16/5,\ x_3 = 4$ and Max. $z = 192/5$]
5. Max. $z = 2x_1 + 3x_2 - 2x_3$, subject to
   $x_1 + x_2 + x_3 \le 8,\ 2x_1 + x_2 - x_3 \ge 3,\ 0 \le x\, x_1 \le 4,\ -2 \le x_2 \le 6,\ x_3 \ge 2$
6. Min. $z = x_1 + 2x_2 + 3x_3 - x_4$, subject to
   $x_1 - x_2 + x_3 - 2x_4 \le 6,\ -x_1 + x_2 - x_3 + x_4 \le 8,\ 2x_1 + x_2 - x_3 \le 2,$
   $0 \le x_1 \le 3,\ 1 \le x_2 \le 4,\ 0 \le x_3 \le 10,\ 2 \le x_4 \le 5.$
7. Max. $z = 4x_1 + 10x_2 + 9x_3 + 11x_4$, subject to
   $2x_1 + 2x_2 + 2x_3 + 2x_4 \le 5$
   $48x_1 + 80x_2 + 160x_3 + 240x_4 \le 257$
   and $0 \le x_j \le 1,\ j = 1, 2, 3, 4.$
   [**Ans.** $x_1 = 9/16,\ x_2 = 1,\ x_3 = 15/16,\ x_4 = 0$ and Max. $z = 331/16$]
8. Max $z = 3x_1 + x_2 + x_3 + 7x_4$ subject to
   $2x_1 + 3x_2 - x_3 + 4x_4 \le 40,\ -2x_1 + 2x_2 + 5x_3 - x_4 \le 35,$
   $x_1 + x_2 - 2x_3 + 3x_4 \le 100$ and $x_1 \ge 2,\ x_2 \ge 1,\ x_3 \ge 3,\ x_4 \ge 4.$
   [**Ans.** $x_1 = 71/4,\ x_2 = 1,\ x_3 = 29/2,\ x_4 = 4$, Max. $z = 287/4$].
9. Min. $z = -2x_1 - 4x_2 - x_3$, subject to
   $2x_1 + x_2 + x_3 \le 10,\ x_1 + x_2 - x_3 \le 4$ and
   $0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 < 4.$
   [**Ans.** $x_1 = 2/3,\ x_2 = 6,\ x_3 = 8/3$ and Min. $z = -28$.]

5·12. SOLUTION OF SIMULTANEOUS EQUATIONS BY SIMPLEX METHOD

For the solution of n simultaneous linear equations in n variables a *dummy* objective function is introduced as

$$\text{Max.} \quad z = 0\,\mathbf{X} - 1\,\mathbf{X}_a$$

where $\mathbf{X}_a$ are artificial variables, and $x_r = x_r' - x_r''$, such that $x_r' \geq 0$, $x_r'' \geq 0$.

The reformulated linear programming problem is then solved by simplex method. The optimal solution of this problem gives the values of the variables (**X**).

The following example will illustrate the procedure.

Example 29. *Use simplex method to solve the following system of linear equations :*

$$x_1 - x_3 + 4x_4 = 3\,,\ 2x_1 - x_2 = 3\,,\ 3x_1 - 2x_2 - x_4 = 1\,,\ \text{and } x_1\,,\ x_2\,,\ x_3\,,\ x_4 \geq 0\,.$$

[Meerut M. Com. Jan. 98 (BP)]

Solution. Since the objective function for the given constraint equation is not prescribed, so a dummy objective function is introduced as :

Max. $z = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1a_1 - 1a_2 - 1a_3$, where $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$ are artificial variables.

Introducing artificial variables, the given equations can be written as :

$$\begin{aligned} x_1 \quad - x_3 + 4x_4 + a_1 \qquad\qquad &= 3 \\ 2x_1 - x_2 \qquad\qquad + a_2 \qquad &= 3 \\ 3x_1 - 2x_2 \quad - x_4 \qquad\qquad + a_3 &= 1\,. \end{aligned}$$

Now apply simplex method to solve the reformulated problem as shown in *Table 5·69*.

Table 5·69

BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	A_3	MIN RATIO (X_B/X_k)
		$c_j \to$	0	0	0	0	−1	−1	−1	
a_1	−1	3	1	0	−1	4	1	0	0	3/1
a_2	−1	3	2	−1	0	0	0	1	0	3/2
$\leftarrow a_3$	−1	1	$\boxed{3}$	−2	0	−1	0	0	1	1/3 ←
	$z = -10$		−6 ↑	3	1	−3	0	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	−1	8/3	0	2/3	−1	$\boxed{13/3}$	1	0	×	8/13 ←
a_2	−1	7/3	0	1/3	0	2/3	0	1	×	7/2
$\to x_1$	0	1/3	1	−2/3	0	−1/3	0	0	×	—
	$z = -5$		0	−1	1	−5 ↑	0 ↓	0	×	$\leftarrow \Delta_j$
$\leftarrow x_4$	0	8/13	0	$\boxed{2/13}$	−3/13	1	×	0	×	8/2 ←
a_2	−1	25/13	0	3/13	2/13	0	×	1	×	25/3
x_1	0	7/13	1	−8/13	1/13	0	×	0	×	—
	$z = 25/13$		0	−3/13 ↑	−2/13	0 ↓	×	0	×	$\leftarrow \Delta_j$
$\to x_2$	0	4	0	1	−3/2	13/2	×	0	×	—
$\leftarrow a_2$	−1	1	0	0	$\boxed{1/2}$	3/2	×	1	×	$1/\frac{1}{2}$ ←
x_1	0	3	1	0	−1	4	×	0	×	—
	$z = -1$		0	0	−1/2 ↑	3/2	×	0 ↓	×	$\leftarrow \Delta_j$
x_2	0	7	0	1	0	2	×	×	×	
$\to x_3$	0	2	0	0	1	−3	×	×	×	
x_1	0	5	1	0	0	1	×	×	×	
	$z = 0$		0	0	0	0	×	×	×	$\leftarrow \Delta_j = 0$

Since all $\Delta_j = 0$, an optimum solution has been attained. Thus the solution of simultaneous equations is given by, $x_1 = 5$, $x_2 = 7$, $x_3 = 2$, and $x_4 = 0$.

24. A small paint factory produces three types of paints as follows :

Paint	Production (Kg./day)	Profit (Units/Kg)
1	x	10
2	y	4
3	z	1

25. A firm manufactures three products which are processed on three different machines. The relevant data are as follows :

Time per unit (hrs.)

Machine	Product I	Product II	Product III
M_1	2	3	2
M_2	4	—	3
M_3	2	5	—

The machine capacities for M_1, M_2 and M_3 are respectively 440, 470 and 430 hrs while the unit profits for PI, PII and PIII are Rs. 400, 300 and 600 respectively. Assume that all the products are sold.

(i) Formulate the problem as an LPP.

(ii) Express the LPP in standard form.

(iii) Find an initial basic feasible solution.

(iv) Carry out up to three iterations towards optimal solution using simplex algorithm. Clearly explain our key steps.

[AIMS Bang. (MBA) 2002]

26. Solve the following LPP using Big-M method. Minimize $z = 2x_1 + 5x_2$, subject to $x_1 + x_2 = 100$, $x_1 \le 40$, $x_2 \le 30$, $x_1, x_2 \ge 0$. **[VTU 2003]**

SELF–EXAMINATION QUESTIONS

1. Establish the difference between (i) feasible solution, (ii) Basic feasible solution and (iii) degenerate basic feasible solution.
2. (a) Define a basic solution to a given system of m simultaneous linear equations in n unknowns.
   (b) How many basic feasible solutions are there to a given system of 3 simultaneous linear equations in 4 unknown.
3. Define the following terms :
   (i) basic variable (ii) basic solution (iii) basic feasible solution (iv) degenerate solution.
4. Give outlines of simplex method in linear programming. Why is it so called.
5. What do you mean by two phase method for solving a given L.P.P. Why is it used.
6. What are the various methods known to you for solving a linear programming problem ?
7. What is the pivoting process ?
8. Name the three basic parts of the simplex technique.
9. Give the geometric interpretation of the simplex procedure.
10. Write the role of pivot element in a simplex table. **[Madurai B.Sc. (Com. Sc.) 92]**
11. In the course of simplex table calculations, describe how you will detect a degenerate, an unbounded and a non existing feasible solution.
12. What is degeneracy in simplex ? Solve the following LP problem using simplex :
    Max. $z = 3x_1 + 9x_2$, s.t. $4x_1 + 4x_2 \le 8$, $x_1 + 2x_2 \le 4$ and $x_1, x_2 \ge 0$. **[IPM (PGDBM) 2000]**
13. With reference to the solution of LPP by simplex method/table when do you conclude as follows :
    (i) LPP has multiple solutions, (ii) LPP has no limit for the improvement of the objective function, (iii) LPP has no feasible solution. **[VTU 2002]**

OBJECTIVE QUESTIONS

1. The role of artificial variables in the simplex method is
   (a) to aid in finding an initial solution.
   (b) to find optimal dual prices in the final simplex table.
   (c) to start phases of simplex method.
   (d) all of the above.
2. For a maximization problem, the objective function coefficient for an artificial variable is
   (a) + M. (b) – M. (c) zero. (d) none of the above.
3. If a negative value appears in the solution values (x_B) column of the simplex table, then
   (a) the solution is optimal.
   (b) the solution is infeasible.
   (c) the solution is unbounded.
   (d) all of the above.
4. At every iteration of simplex method, for minimization problem, a variable in the current basis is replaced with another variable that has
   (a) a negative $z_j - c_j$ value.
   (b) a positive $z_j - c_j$ value.
   (c) $z_j - c_j = 0$.
   (d) none of the above.