# Causal Inference Despite Limited Global Confounding via Mixture Models

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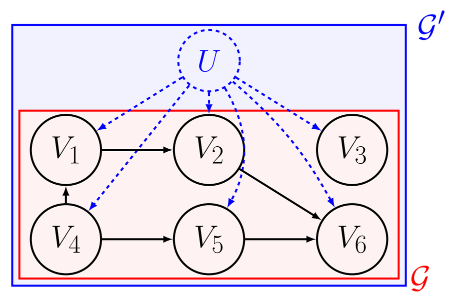
PAPER

## Problem

### Setup




- A *Bayesian Network* is a directed acyclic graph (DAG) $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.
- A *Bayesian Network Distribution* on $n$ random variables $\mathbf{V}$ is Markovian on Bayesian Network $\mathcal{G}$.
- A $k$-*mixture* of such distributions ($k$-MixBND) is represented using one additional vertex $U$ with $\mathsf{CH}(U) = \mathbf{V}$.

### Task

**Knowns:**

- The *marginal* probability distribution on $\mathbf{V}$:

$$\Pr(\mathbf{V}) = \sum_{u \in U} \Pr(u) \Pr(\mathbf{V} \mid u)$$

**Unknowns:**

- The probability distribution on $U$, i.e. $\Pr(u)$ for $u \in \{1, \dots, k\}$.
- The *within source* probability distribution $\Pr(\mathbf{V} \mid u) = \mathcal{P}_u(\mathbf{V})$ for $u \in \{1, \dots, k\}$.
  - $\mathcal{P}_u(\mathbf{V})$ is a Bayesian network distribution, so it suffices to find $\mathcal{P}_u(V \mid \mathsf{pa}(V))$ for all $V \in \mathbf{V}$, and assignments $\mathsf{pa}(V)$ to $\mathsf{PA}(V)$.

### Motivation

- Setting emerges when combining multiple...
  - -Populations
  - -Environments
  - -Datasets

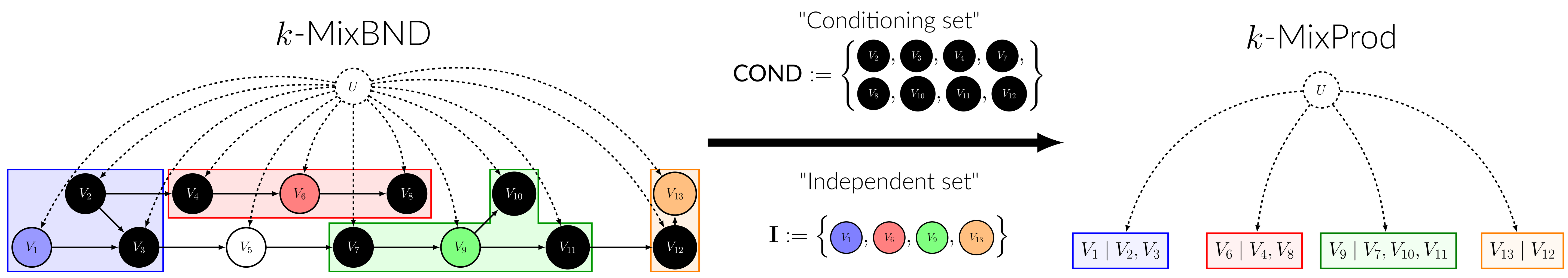
  } $k <$ observed support!
- Graphically, the causal relationships are considered unidentifiable.
- Interventional probabilities *can* be calculated using the *within source* distributions, $\Pr(\mathbf{V} \mid u)$.
- Solving the mixture model allows for a degree of **deconfounding**.

### Previous Work

Using 3-independent variabls and larger alphabets (linear in $k$):

- E. S. Allman, 2009 use algebraic methods to exploit within-source independence.
- Anandkumar, Hsu, and Kakade, 2012 follows a similar strategy using tensor decomposition.
- Wang and Blei, 2019 introduced deconfounders using multiple causes.
  - Criticized in Ogburn, Shpitser, and Tchetgen, 2019 and D'Amour, 2019
  - Criticism is linked to the difference between **learning** parameters that generate similar statistics and **identification** of the true parameters.

## Strategy




### Reduction to $k$-MixProd

- The *conditional* probability distribution $\mathbf{I} \mid \mathsf{COND}$ is an instance of $k$-MixProd on the **independent set** $\mathbf{I}$.
- We will create many such instances (called "runs") and stitch together the results. These runs will need to:
  1. **Cover** all of the variables and assignments to their parents .
  2. Be **alignable** with eachother so their results can be synthesized.
- **Runtime of** $k$**-MixBND**: $n2^{\Delta^2}$ executions of $k$-MixProd where $\Delta$ is a bound on the degree of $\mathcal{G}$.
- **Runtime of** $k$**-MixProd**: $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(k)}$ in Gordon et al., 2021 and $2^{\mathcal{O}(k \log(k))} n^{\mathcal{O}(k)}$ in upcoming work.

### Difficulties

- $k$-MixProd is symmetric to permutations in the labels of $U$.
  **Solution: Alignment variables**
- Conditioning (via post-selection) limits the variables whose information we have access to. How can we ensure we have obtained all of the necessary information?
  **Solution: "Good sets of runs" and alignment spanning trees**
- The parameters returned from $k$-MixProd instances are of the form $\mathcal{P}_u(V \mid \mathsf{mb}(V))$ -- we want them of the form $\mathcal{P}_u(V \mid \mathsf{pa}(V))$ (which is standard for Bayesian networks).
  **Solution: Bayesian unzipping**

### Assumptions

1. We have access to a $k$-MixProd oracle requiring $O(k)$ variables that are independent within each source.
2. The observable variables in our BND are binary and discrete.
   - Extensions exist for larger alphabets.
3. The mixture is supported on $\leq k$ sources.
4. The underlying Bayesian DAG is sufficiently sparse.
   - The algorithm works if $n \geq (\Delta + 1)^4 N_{\mathrm{mp}}$
   - Milder requirements exist for specific graphs.
5. The resulting product mixtures are non-degenerate.
6. The DAG structure $\mathcal{G}$ is known.
   - Upcoming work on how to do causal discovery to learn $\mathcal{G}$.

## Alignment of Source Labels

### Alignment Variables

Run $a$:

| $U^{(a)}$ | $\mathcal{P}_{u^{(a)}}(X_1)$ | $\mathcal{P}_{u^{(a)}}(X_2)$ |
|---|---|---|
| 0 | .7 | .4 |
| 1 | .3 | .6 |

Run $b$:

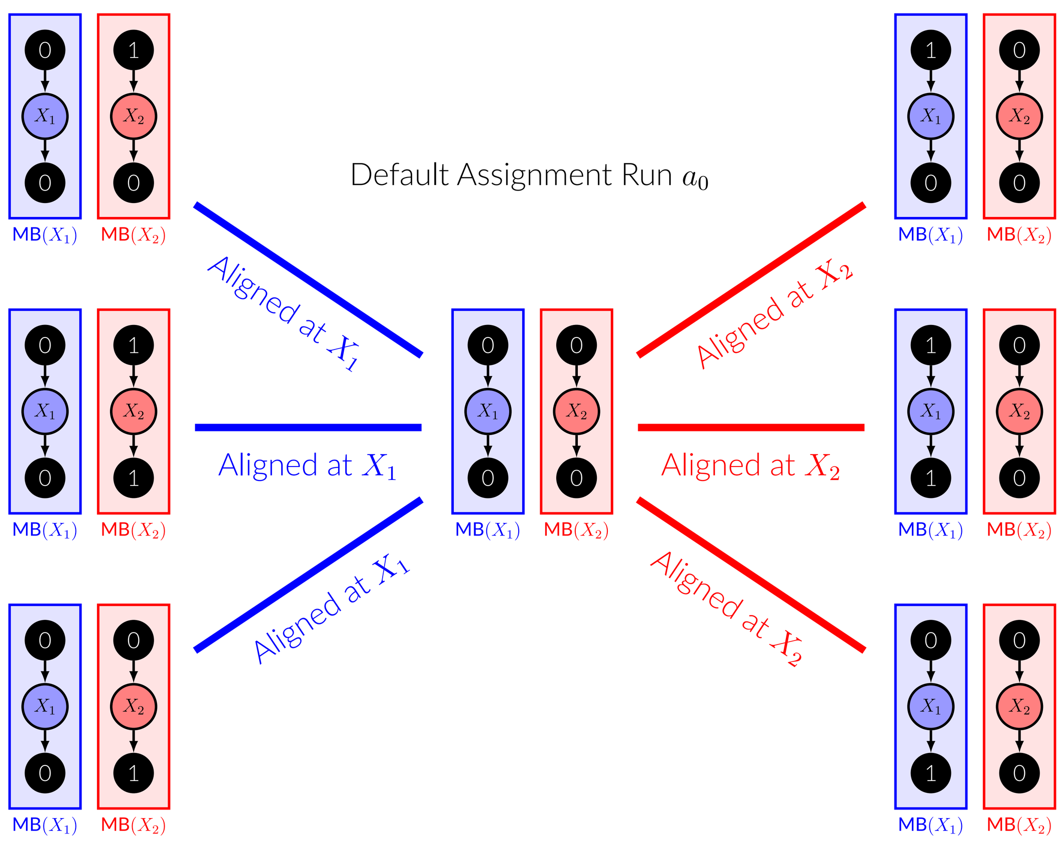
| $U^{(b)}$ | $\mathcal{P}_{u^{(b)}}(X_1)$ | $\mathcal{P}_{u^{(b)}}(X_3)$ |
|---|---|---|
| 0 | .3 | .2 |
| 1 | .7 | .8 |

Combined:

| $U$ | $\mathcal{P}_u(X_1)$ | $\mathcal{P}_u(X_2)$ | $\mathcal{P}_u(X_3)$ |
|---|---|---|---|
| 0 | .7 | .4 | .8 |
| 1 | .3 | .6 | .2 |

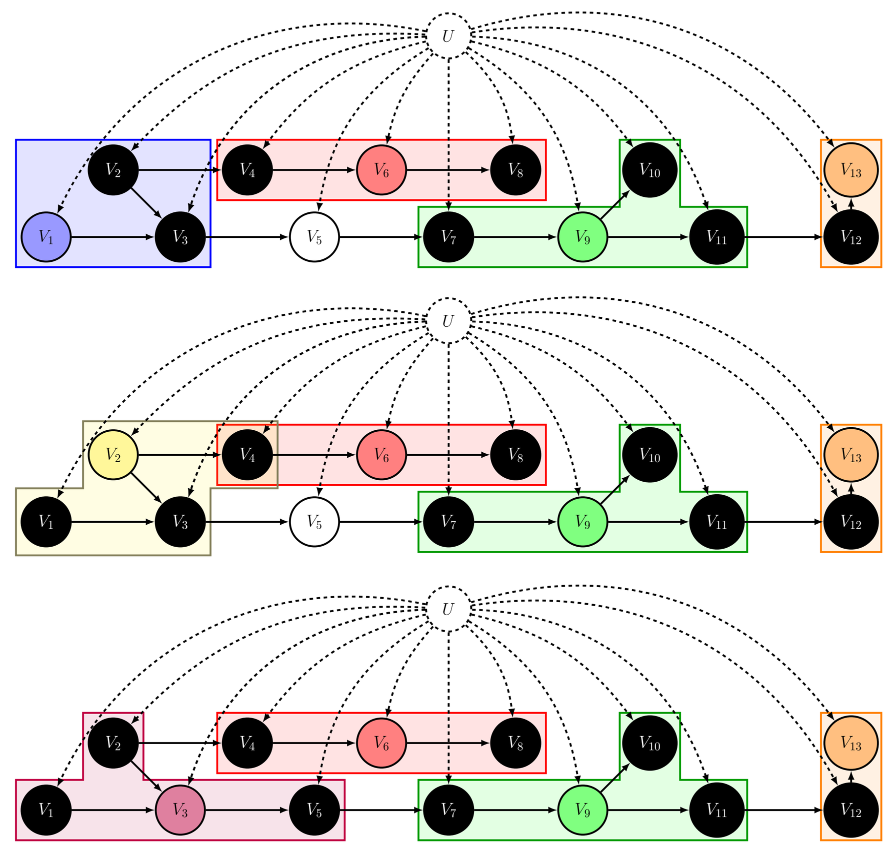
- Runs $a$ and $b$ are aligned at $X_1$, allowing $U^{(a)}$ and $U^{(b)}$ to be aligned.
- To ensure we have a "good collection of runs," any two runs must be alignable via a chain of alignment variables - i.e. we must have an **alignment spanning tree**.

### Changing the Conditioned Values



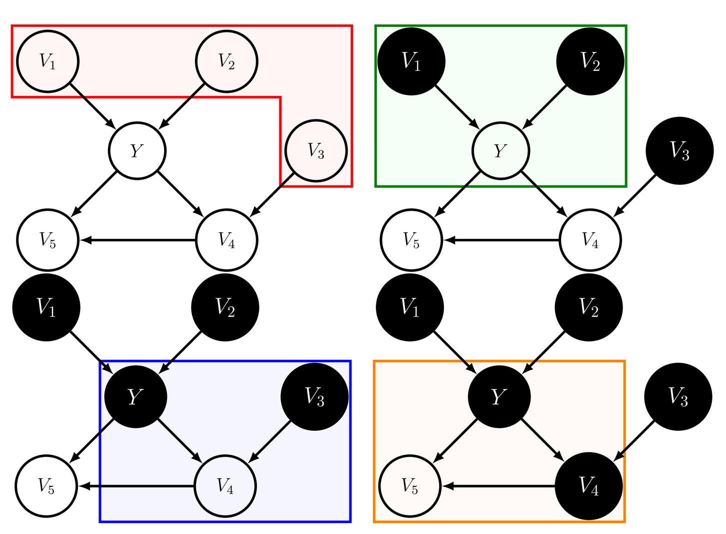

Change the assignments to Markov boundaries while leaving at least one assignment the same.

### Varying the Independent Set



Disjoint Markov boundaries ensure that a single variable can be swapped without requiring others in the independent set to be conditioned on.

## Bayesian Unzipping: From $\Pr(V \mid \mathsf{MB}(V)) \to \Pr(V \mid \mathsf{PA}(V))$



Let $y^0$ denote ``$y = 0$'' and $y^1$ denote ``$y = 1$''.

$$\mathcal{P}_u(y^1 \mid \mathsf{mb}(Y)) = \frac{\mathcal{P}_u(y^1, \mathsf{mb}(Y))}{\mathcal{P}_u(y^1, \mathsf{mb}(Y)) + \mathcal{P}_u(y^0, \mathsf{mb}(Y))}$$

We apply the standard factoring,

$$\mathcal{P}_u(y, \mathsf{mb}(Y)) = \mathcal{P}_u(v_1, v_2, v_3)\mathcal{P}_u(y \mid v_1, v_2)\mathcal{P}_u(v_4 \mid y, v_3)\mathcal{P}_u(v_5 \mid y, v_4),$$

to all three terms.

$$\mathcal{P}_u(y^1 \mid \mathsf{mb}(Y)) = \frac{\mathcal{P}_u(y^1 \mid v_1, v_2)\mathcal{P}_u(v_4 \mid y^1, v_3)\mathcal{P}_u(v_5 \mid y^1, v_4)}{\mathcal{P}_u(y^1 \mid v_1, v_2)\mathcal{P}_u(v_4 \mid y^1, v_3)\mathcal{P}_u(v_5 \mid y^1, v_4)) + \mathcal{P}_u(y^0 \mid v_1, v_2)\mathcal{P}_u(v_4 \mid y^0, v_3)\mathcal{P}_u(v_5 \mid y^0, v_4)}$$

- $\mathcal{P}_u(v_1, v_2, v_3)$ appears in both numerator and denominator, so it cancels out.
- If we traverse in **reverse topological order**, then $\mathcal{P}_u(v_4 \mid y, v_3)$ and $\mathcal{P}_u(v_5 \mid y, v_4)$ terms are previously calculated for both $y \in \{y^0, y^1\}$.
- $\mathcal{P}_u(y^0 \mid v_1, v_2) + \mathcal{P}_u(y^1 \mid v_1, v_2) = 1$, so we can solve for green terms.
- Iterating this process incurs stability costs proportional to the depth of the graph.
  - This can be avoided by not conditioning on the children of the deepest variables in the independent set.

## References

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