Linear Transformations

Shinchan is making drinks with the following recipe

	🥤	🍵
🍃	1	1
🫘	2	1

🥤 uses $\boxed{\text{🍃 🫘 🫘}}$

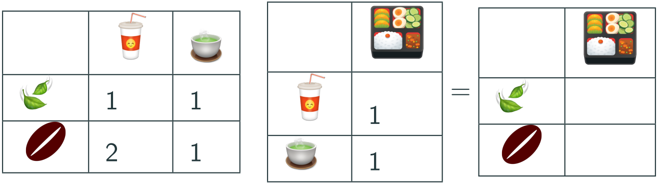
🍵 uses $\boxed{\text{🍃 🫘}}$

Linear Transformations

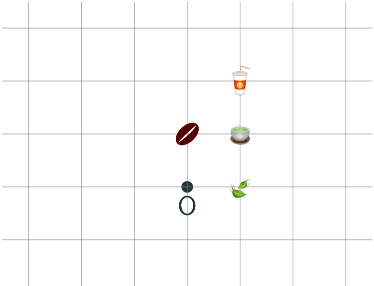
This time he would like to use pictures to organize the data. He plots each drink to the corresponding point in $\mathbb{R}^2$ to the linear combination space of materials.

	🥤	🍵
🍃	1	1
	2	1

Once again, he wants to combine the two tables



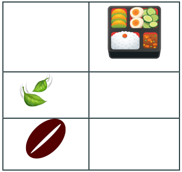
You know how to do it (matrix multiplication). But how can he do it by **purely with pictures**?



He is unsure how to do this **without computation**. How can he find the position of in the left picture? How can he do matrix multiplication **purely geometrically**?

Left Picture

Right Picture

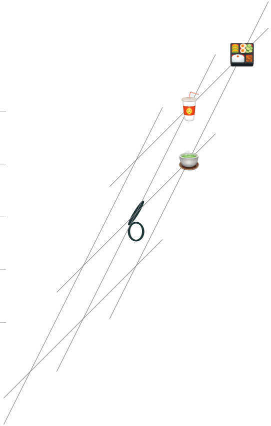


Linear Transformations

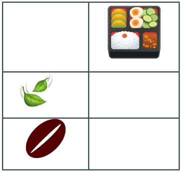
His chef comes into his room and accidentally bumps his picture....

Chef: Oh! I am sorry, Shinchan what are you doing here?

Left Picture

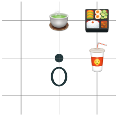


Right Picture



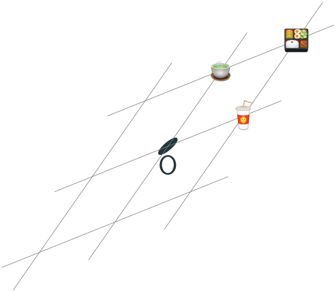
Shinchan: Oh!!! No!!! My pictures, you destroyed my picture...

Chef: I am sorry, but... are there any differences between these two pictures?



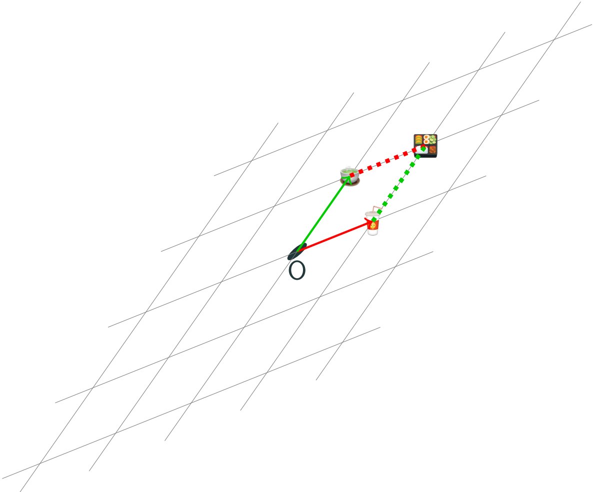
Right Picture
BEFORE

← Any difference? →



Right Picture AFTER

After the Chef′s explanation, Shinchan knows this crooked picture **keeps all information of the recipe table** because it **keeps** the **parallelograms**, the vector for tea 🍵and for cola 🥤, still adds to the vector for 🍱.



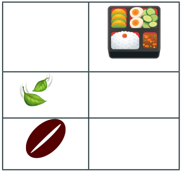
	🍱
🥤	1
🍵	1

Linear Transformations

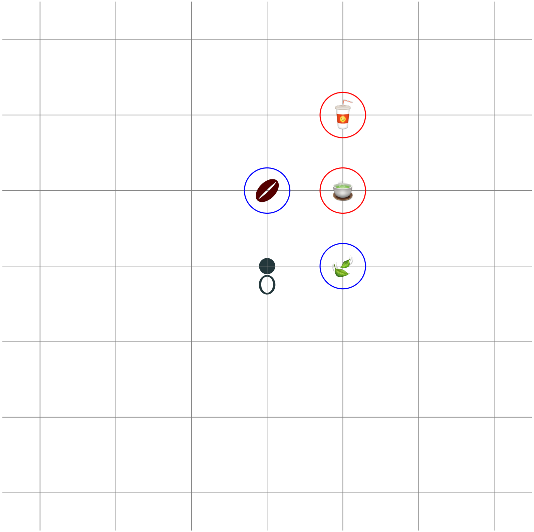
Shinchan: Great! I am trying to compute matrix multiplication geometrically. I have a recipe to make **intermediates** from **materials**, and to make **final meals** from **intermediates**. I wish to figure out how to make **final meals** from **materials**. What should I do?

Left Picture

Right Picture

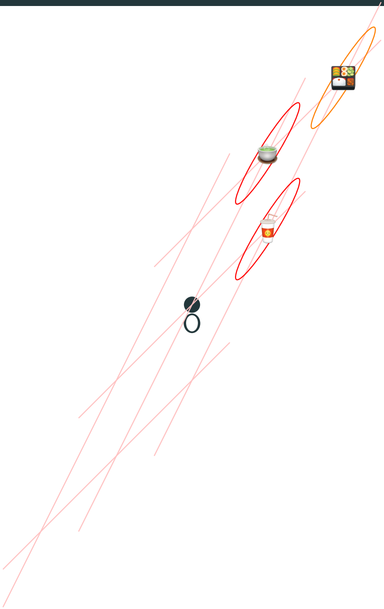


Linear Transformations



Chef: Now I see you have the Left Picture. That represents how you make drinks out of materials

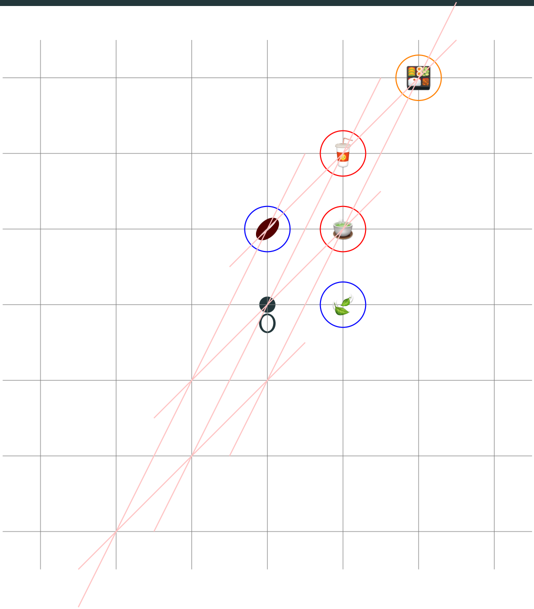
	🥤	🍵
🍃	1	1
🫘	2	1



Chef: And I see you have the Right Picture, which represents how to make meals out of drinks. Oh I am sorry that I bumped it... hopefully we did not lose any information.

	🍱
🥤	1
🍵	1

Linear Transformations



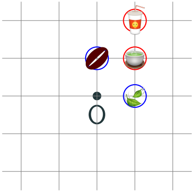
Oh! Let us simply put these two pictures together!! Then it shows us all the information. We now know how to make meals 🍱 by materials 🍃, 🫘.

	🍱
🍃	2
🫘	3

Linear Transformations

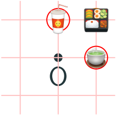
Let′s summarize the **Geometric method** of computing matrix multiplication.

Step 1: We have two pictures corresponding to the left factor and the right factor.



Left factor: Making drinks by materials

	🥤	🍵
🫘	2	1
🍃	1	1

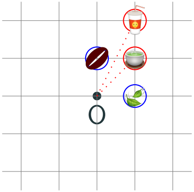


Right factor: Making meals by drinks

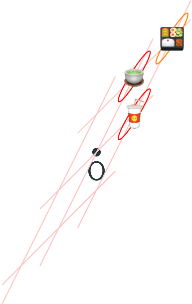
	🍱
🍵	1
🥤	1

Linear Transformations

Step 2: Skew the right picture so that the relative position of drinks matches its position in the left picture.



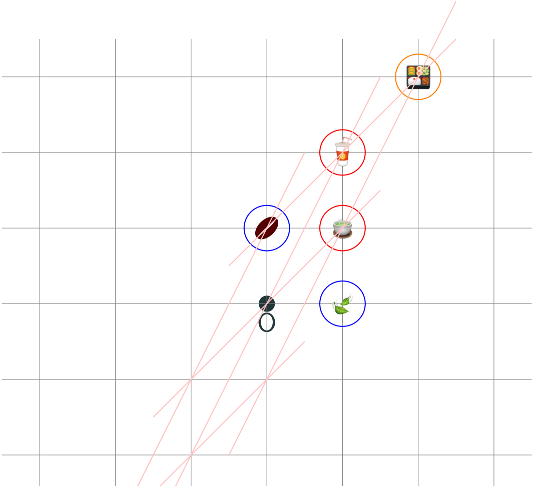
Left factor: Making drinks by materials



Right factor: Making meals by drinks

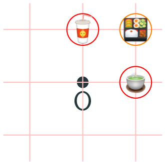
Linear Transformations

Step 3: Put them together, then you can see how to make a meal 🍱 by materials 🍃, 🫘 . You get the same effect as matrix multiplication.



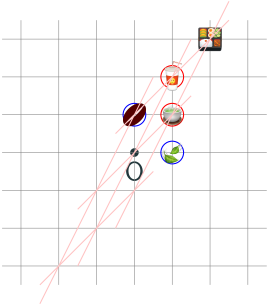
Linear Transformations

The whole process is a **map**. The domain of the map is the linear combination space of drinks. The codomain(target) of the map is the linear combination space of materials. The process skews the picture of the domain and put it to the codomain.



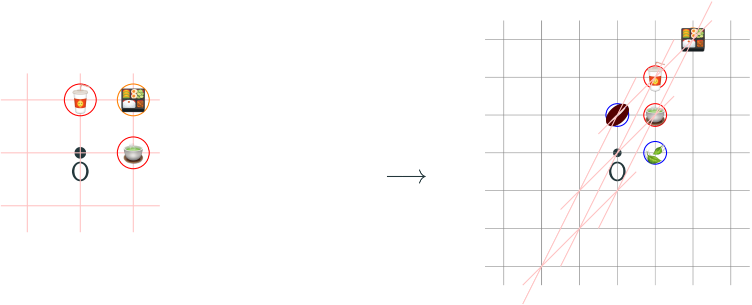
Domain: space of drinks.
Corresponds to **right factor**.

$\longrightarrow$



Codomain: space of materials.
Corresponds to **left factor**.

Linear Transformations



The only restriction for this map is that: The whole process maps parallelograms to parallelograms and it maps origin to origin. In Math, this is called a **Linear Transformation**.

Linear Transformations

Definition 1

A linear transformation is a map $T : V \longrightarrow W$ for linear spaces V, W over F such that

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \qquad T(\lambda \vec{v}) = \lambda T(\vec{v})$$

for any $\vec{v}, \vec{w} \in V$ and $\lambda \in F$.
We call V the **Domain** of T, W the **Codomain** of T.

Definition 2

The linear transformation $T : V \longrightarrow V$ in the case domain equals the codomain is called a **Linear Operator**.

Linear Transformations

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \qquad T(\lambda\vec{v}) = \lambda T(\vec{v})$$

is a condition of saying keeping parallelograms.

As we have shown here, linear transformation gives a **geometric understanding** of matrix multiplication.

Proposition 1

If $T : V \longrightarrow W$ is a linear transformation, then for any $\vec{v}_1, \cdots \vec{v}_n \in V$, $a_1, \cdots, a_n \in \mathbb{R}$, we have

$$T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n).$$

In other words, linear transformation preserves the coefficient of linear combination.

Proof.

$$\begin{aligned} T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) &= T((a_1\vec{v}_1 + \cdots + a_{n-1}\vec{v}_{n-1}) + a_n\vec{v}_n) \\ &= T(a_1\vec{v}_1 + \cdots + a_{n-1}\vec{v}_{n-1}) + a_n T(\vec{v}_n) \end{aligned}$$

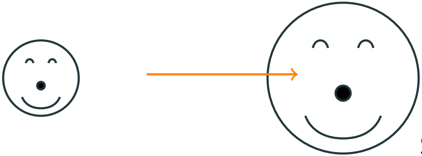
Then using induction □

Linear Transformations

There are other geometric examples of linear transformations.

Rotation by 90 degree.

Reflection horizontally



Scalling

Projecting

Linear Transformations

In the space of functions, linear transformations happen when we **change variables** or make **linear combinations of functions**. For example, let P_∞ be the space of all polynomials. The map defined by

$$T : P_\infty \longrightarrow P_\infty$$

$$f(x) \longmapsto f(x^2)$$

is **linear**, (**Excercise.**: Check it).

Remember to check that the expression $f(x^2)$ you defined is **actually** an element in the codomain. For example, if P_2 is the sapce of all polynomials of degree at most 2. The following argument

$$T : P_2 \longrightarrow P_2$$

$$f(x) \longmapsto f(x^2)$$

is not even a map $(x^2 \mapsto x^4 \notin P_2)$

Linear Transformations

You can also define linear transformation on set of functions by **linearly combine its function values**, **derivative**, or **integration** for example

$$T : f(x) \mapsto f(1+\sqrt{x}) + f(1-\sqrt{x}); \qquad T : f(x) \mapsto xf(x) + x^2 f(x)$$

$$T : f(x) \mapsto f'(x); \qquad T : f(x) \mapsto \int_0^x f(t)\mathrm{d}t.$$

Verify a map is a linear transformation

To verify a map $T : V \longrightarrow W$ is a linear transformation, we only need

- Write down expression of $\vec{v}_1, \vec{v}_2$ for arbitrary element $\vec{v} \in V$.
- Check the element is well-defined and it defined to be an element in codomain.
- Compute $T(\lambda\vec{v}_1 + \vec{v}_2)$ for arbitrary element $\lambda \in F$
- Compare with $\lambda T(\vec{v}_1) + T(\vec{v}_2)$.

We only verify $T(\lambda\vec{v}_1 + \vec{v}_2)$ is because the following

Proposition 2

For any map T, if $T(\lambda\vec{v}_1 + \vec{v}_2) = \lambda T(\vec{v}_1) + T(\vec{v}_2)$, this is a linear tranformation.

Proof.

$T(\lambda\vec{v}_1 + \mu\vec{v}_2) = \lambda T(\vec{v}_1) + T(\mu\vec{v}_2) = \lambda T(\vec{v}_1) + \mu T(\vec{v}_2)$ $\square$

Verify a map is a linear transformation

Excercise. Let P_2 be the space of polynomials of degree at most 2. Show that the following map

$$T : P_2 \longrightarrow P_2$$

$$f(x) \longmapsto f(1+\sqrt{x}) + f(1-\sqrt{x})$$

is a linear transformation.

Solution. Let f, g be arbitrary polynomials in P_2, so there exists unique $a, b, c, d, e, f \in F$ so f, g can be written as

$$f(x) = ax^2 + bx + c \qquad g(x) = dx^2 + ex + f. \tag{1}$$

We first show that $T[f] \in P_2$. Indeed,

$$T[f](x) = f(1+\sqrt{x}) + f(1-\sqrt{x})$$

Verify a map is a linear transformation

Plug (**??**) in, we have

$$T[f](x) = a(1+\sqrt{x})^2 + b(1+\sqrt{x}) + c + a(1-\sqrt{x})^2 + b(1-\sqrt{x}) + c$$

By computation, this expression equals to

$$a(2+x^2) + 2b + 2c \in P_2.$$

To check linearity, note that for any scalar $\lambda \in f$

$$T[\lambda f+g] = \lambda f(1+\sqrt{x})+\lambda f(1-\sqrt{x})+g(1+\sqrt{x})+g(1-\sqrt{x}) = \lambda T[f]+T[g]$$

So T is a linear transformation.

Non-linear transformations

Non-linear transformations happens when we combine values of functions in a non-linear way, like $f(x) \mapsto f(x)^2$ or $f(x) \mapsto \sqrt{f(x)}$. To disprove linearity, we only need to choose some coefficient so that the definition of linear transformation fails to hold.

Sometimes, we write $T[f]$ to denote the output function when apply linear transformation of T.

Non-linear transformations

Excercise. Let V be the space of all polynomials over $\mathbb{R}$. Show that $T[f](x) = f(x)^2$ is not a linear transformation.

Solution.: Let

$$\begin{cases} f_1(x) = 1 \\ f_2(x) = x \end{cases}$$

, then

$$T[f_1 + f_2](x) = (1 + x)^2 = 1 + x^2 + 2x$$

But

$$T[f_1](x) + T[f_2](x) = 1 + x^2.$$

So

$$T[f_1 + f_2] \neq T[f_1] + T[f_2]$$

This is not a linear transformation.

Matrix of linear transformations

To represent a linear transformation, we will use matrices.

Matrix of linear transformations

Shinchan is making drinks with the following recipe

	🥤	🍵
🍃	1	1
🫘	2	1

🥤 uses 🍃 🫘 🫘

🍵 uses 🍃 🫘

Matrix of linear transformations

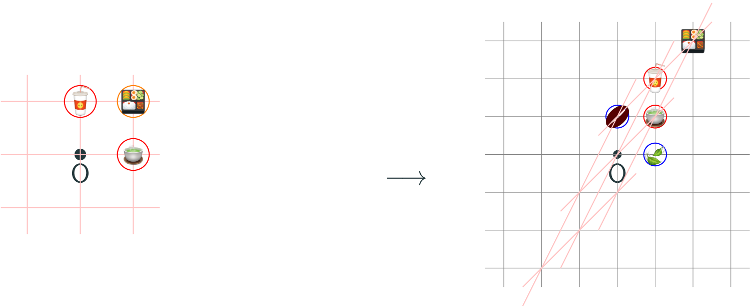
This time he would like to use pictures to organize the data. He plots each drink to the corresponding point in $\mathbb{R}^2$ to the linear combination space of materials.

	🥤	🍵
🍃	1	1
	2	1

Matrix of linear transformations

This process can be understood as a linear transformation **from the space of drink combinations** to **the space of material combinations**.

	🥤	🍵
🫘	2	1
🍃	1	1



Matrix of linear transformations

We call this map T. We use 🥤, 🍵 as symbols for the drinks in the domain and

$$T\left(🥤\right) \qquad T\left(🍵\right)$$

as symbols for its position in the codomain. Since materials are all in the codomain, it makes more sense to write our table as

	$T\left(🥤\right)$	$T\left(🍵\right)$
🫘	2	1
🍃	1	1

Matrix of linear transformations

This table can be written as an expression

$$\begin{pmatrix} T\text{🥤} & T\text{🍵} \end{pmatrix} = \begin{pmatrix} \text{🫘} & \text{🍃} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Factoring T out we can write

$$T\begin{pmatrix} \text{🥤} & \text{🍵} \end{pmatrix} = \begin{pmatrix} \text{🫘} & \text{🍃} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Note that (🥤,🍵) is a basis of the domain, and (🫘,🍃) is a basis of the codomain. We call the matrix

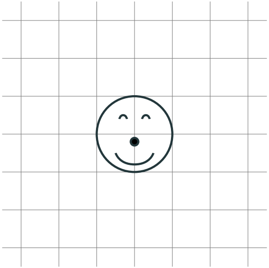
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

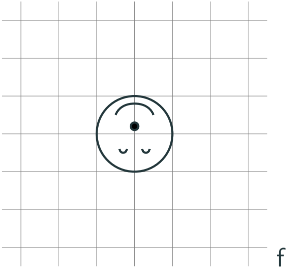
The matrix representation of T in the basis (🥤,🍵) and (🫘,🍃). It determines the linear transformation completely.

Matrix Representation of a Linear Transformation

The idea of matrix representation is to represent linear transformation as matrix.

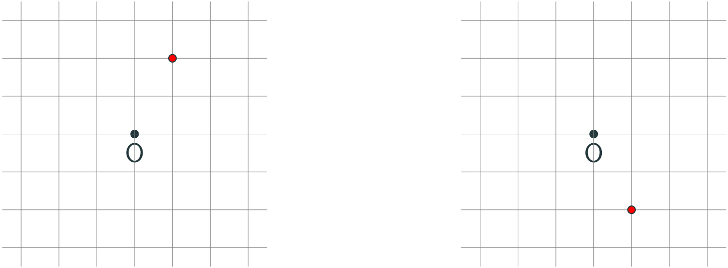
Look at the following example. Reflection vertically.





Matrix Representation of a Linear Transformation

The idea is to fina a matrix to realize the action on all vectors by matrix multiplication. Clear, we find out that the formula for reflection vertically is given by



The formula is mapping $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} x \\ -y \end{pmatrix}$.

Matrix Representation of a Linear Transformation

However, this map can be written as a matrix multiplication.

$$\underbrace{\begin{pmatrix} x \\ -y \end{pmatrix}}_{\text{new position}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{old position}}$$

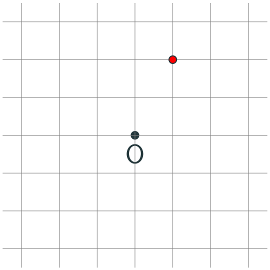
We call the matrix

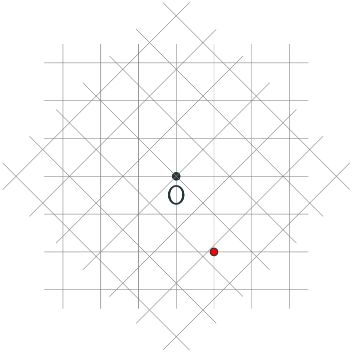
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrix representation of the linear transformation.

Matrix Representation of a Linear Transformation

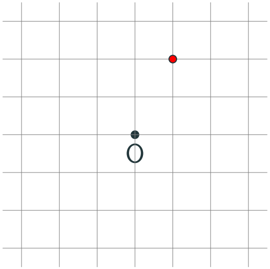
However, the notion of coordinate only make sense if we draw a grid, if we draw a different grid, the coordinate would change

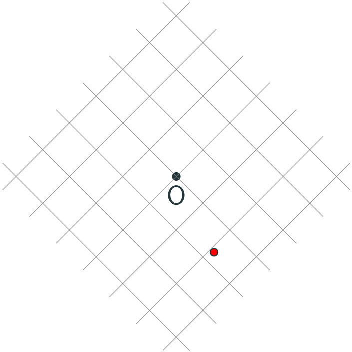




Matrix Representation of a Linear Transformation

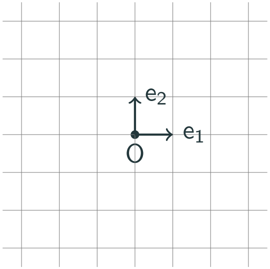
For the following example, the coordinate of the right picture changes because we choose a different basis.



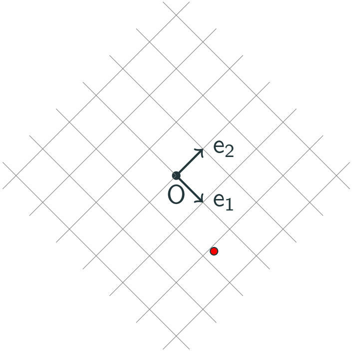


Matrix Representation of a Linear Transformation

In linear algebra, a basis is the information of how to draw such a grid,



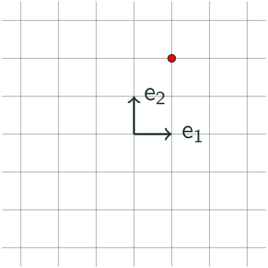
basis of domain



basis of codomain

Matrix Representation of a Linear Transformation

In general, the coordinate of a point can only be identified **after choosing a basis.** For example,

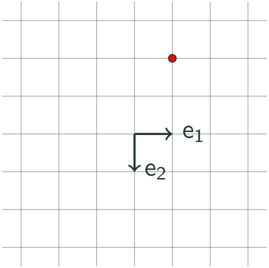


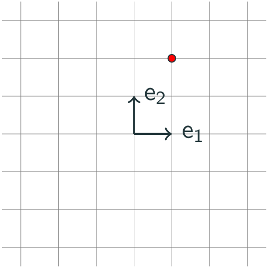
The point is in fact given by $\vec{v} = \vec{e}_1 + 2\vec{e}_2 = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\vec{v} = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix}}_{\text{Basis}} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{Coordinates}}$$

Matrix Representation of a Linear Transformation

In fact, the coordinate would change when choosing a different basis,





$$\vec{v} = \underbrace{\begin{pmatrix} \vec{e_1} & \vec{e_2} \end{pmatrix}}_{\text{Basis}} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{Coordinates}} = \underbrace{\begin{pmatrix} \vec{e_1} & -\vec{e_2} \end{pmatrix}}_{\text{Basis}} \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\text{Coordinates}}$$

Coordinates depends on the choice of basis!

Matrix Representation of a Linear Transformation

For a linear transformation $T : V \longrightarrow W$, to represent vectors in V and W, we need a basis $\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ for V, and also a basis $\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}$ for W.

Matrix Representation of a Linear Transformation

basis $\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ for V; basis $\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}$ for W.

So we may write a vector $\vec{v} \in V$ and $T\vec{v} \in W$as

$$\underbrace{\vec{v} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{Vector in } V} \longmapsto \underbrace{T\vec{v} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}_{\text{Vector in } W}$$

A matrix representation of T under the basis $\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ and $\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}$ means we want to find the matrix M so that the fomula holds for all $\vec{v}$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Do such a formula exists? Yes, in fact

$$T\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix} = \begin{pmatrix}T\vec{e}_1 & T\vec{e}_2 & \cdots & T\vec{e}_n\end{pmatrix}$$

is a vector list in W, so we may find the list M with each column collects the coordinate of $T\vec{e}_i$ under the basis $\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix}$, so that one can write

$$T\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix} = \begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix} M.$$

Matrix Representation of a Linear Transformation

This M is what we want, because

$$\underbrace{T\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}}_{\vec{v}} = \begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix} M \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

On the other hand, we want

$$T\vec{v} = \begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix}\begin{pmatrix}y_1\\y_2\\\vdots\\y_m\end{pmatrix},$$

Matrix Representation of a Linear Transformation

This implies that

$$\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix} M \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

Since linearly independent vectors have left-inverse, we cancel it from left and have

$$M \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

Matrix Representation of a Linear Transformation

Definition 3

For a linear transformation $T : V \longrightarrow W$, let

- $\mathcal{E} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ be a basis of domain V
- $\mathcal{F} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}$ be a basis of codomain W.

The **matrix representation** of T with respect to $\mathcal{E}$ and $\mathcal{F}$, is the matrix P such that

$$T \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix} P$$

In other words, the matrix representation is the recipe table to make $T \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ by materials $\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}$.

Note: Different basis will result different matrix representation for the same linear transforamtion $T : V \longrightarrow W$!

Excercise. Let $V = P_{2,x} = \{ax^2 + bx + c, \text{ where } a, b, c \in F\}$,
$W = P_{2,t} = \{at^2 + bt + c, \text{ where } a, b, c \in F\}$

Consider a linear map

$$T : V \longrightarrow W$$

$$f(x) \longmapsto f(t+1)$$

Find matrix representation of T with bases

$$\mathcal{F} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \text{ in } W \qquad \mathcal{E} = \begin{pmatrix} 1 & 2x+1 & x^2+1 \end{pmatrix} \text{ in } V$$

Matrix Representation of a Linear Transformation

Solution.: Apply the linear transformation T on each of the function on **basis** and write the coordinate in **basis** of the target. We find

$$T(1) = 1 = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$T(2x+1) = 2(t+1)+1 = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \underbrace{\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}}$$

$$T(x^2+1) = (t+1)^2+1 = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

Matrix Representation of a Linear Transformation

We write this into a matrix form

$$T \underbrace{\begin{pmatrix} 1 & 2x+1 & x^2+1 \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

We know the matrix representation of T is $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Matrix Representation of a Linear Transformation

Excercise. For the map

$$T : V \longrightarrow W$$

$$f(x) \longmapsto f(t+1)$$

the matrix representation has been found

$$T\underbrace{\begin{pmatrix}1 & 2x+1 & x^2+1\end{pmatrix}}_{\text{basis in } V} = \underbrace{\begin{pmatrix}1 & t & t^2\end{pmatrix}}_{\text{basis in } W}\begin{pmatrix}1 & 3 & 2\\ 0 & 2 & 2\\ 0 & 0 & 1\end{pmatrix}$$

Suppose $f(x) = 3 + 5(2x+1) + 2(x^2+1)$, use your matrix, find Tf

Matrix Representation of a Linear Transformation

Solution. We find the coordinate of f is just $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ Therefore, the coordinate of Tf in codomain is just obtained by multiplying the matrix to the coordinate of f in domain, we get

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 22 \\ 14 \\ 2 \end{pmatrix}$$

Therefore $Tf = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \begin{pmatrix} 22 \\ 14 \\ 2 \end{pmatrix} = 22 + 14t + 2t^2.$

Matrix Representation of a Linear Transformation

Solution. You may also proceed by the standard notation

$$Tf = T\begin{pmatrix} 1 & 2x+1 & x^2+1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t & t^2 \end{pmatrix}\begin{pmatrix} 22 \\ 14 \\ 2 \end{pmatrix} = 22 + 14t + 2t^2.$$

Polynomials on Linear Operators

From now we only consider the linear operators which is transformations with the same domain and codomain. $T : V \longrightarrow V$

When represnting the matrix of this linear transformation, we keep the basis in domain and codomain the same. So only one choice of basis is required.

Excercise. Represent the matrix of taking derivative $T : P_3 \longrightarrow P_3$ with respect to the basis $\begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix}$

$$T \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} = \begin{pmatrix} T(1) & T(t) & T(t^2) & T(t^3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 2t & 3t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{matrix represntation of } T \text{ under basis}}$$

Polynomials on Linear Operators

Since domain and codomain are the same, we must choose the same basis both in domain and codomain.

$$T \underbrace{\begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix}}_{\text{choice of basis of domain}} = \underbrace{\begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix}}_{\text{must be the same}} \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{matrix represntation of } T \text{ under basis}}$$

Polynomials on Linear Operators

Linear Operator $T : V \longrightarrow V$ can be raise into powers. By

$$T^n := \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ many } T}$$

This is the *n-th power* of the operator T. Using this notion of (integer) powers, we can define many new operators on V given by *polynomials*. For example we can define things like

$$T^2 + 2T + \mathrm{id}_V$$

You will find it is exactly equal to the operator

$$(T + \mathrm{id}_V)^2.$$

For our convenience, we abbreviate id_V as I.

Polynomials on Linear Operators

The algebra of linear operators can be concretly evaluated on its matrix representations. Because if $\mathscr{E}$ is a basis and A is the matrix of T under $\mathscr{E}$, in other words, $T\mathscr{E} = \mathscr{E}A$, then

$$T^n\mathscr{E} = T^{n-1}\mathscr{E}A = T^{n-2}\mathscr{E}A^2 = ... = \mathscr{E}A^n$$

In other words, under the same basis, if matrix representation of T is given by A, then the matrix represenrtation of T^n is given by A^n.

Polynomials on Linear Operators

In general we can apply a polynomial of degree $m \in \mathbb{N}$

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$$

to the operator T by evaluating P at $x = T$, obtaining

$$p(T) := a_m T^m + a_{m-1} T^{m-1} + \cdots + a_0 \cdot I$$

Excercise. Suppose T is a linear operator on V such that

$$(T-I)^2 = 0.$$

Show that T is invertible.

Proof. We have $T^2 - 2T + I = 0$, therefore $-T^2 + 2T = I$, so $T(2-T) = I$. This implies T is invertible and

$$T^{-1} = 2 - T.$$