The exponential function

Recall what we called exponential

$$e^x := \lim \left(1 + \frac{x}{n}\right)^n .$$

The exponential function

Recall the Binomial Expansion

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

The exponential function

We may think $e^x = (1 + \frac{x}{\infty})^\infty$, the binomial expansion is then

$$\left(1 + \frac{x}{\infty}\right)^\infty = 1 + \frac{\infty}{1!}\frac{x}{\infty} + \frac{\infty(\infty - 1)}{2!}\frac{x^2}{\infty^2} + \cdots + \frac{\infty(\infty - 1)(\infty - 2)}{3!}\frac{x^3}{\infty^3} + \cdots$$

Note that

$$\frac{\infty}{\infty} = \frac{\infty - 1}{\infty} = \frac{\infty - 2}{\infty} = \cdots = 1.$$

So finally we have the **Binomial Expansion**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Motivation of Exponential

Consider the differential equation

$$y' = 2y.$$

How to understand it?

The differential equation $y' = 2y$ represents a scenario where the rate of change of y (speed) is proportional to its current value (distance).

Motivation of Exponential

Consider $y(t)$ as a function of t. When we examine the value of y at a very small time, say $\frac{1}{n}$, we notice that $y\left(\frac{1}{n}\right)$ is almost the same as $y(0)$, since the time interval is extremely small.

Given our differential equation $y' = 2y$, the speed of change at $t = 0$ is $2y(0)$.

$$y\left(\frac{1}{n}\right) \approx y(0) + \frac{1}{n} \cdot 2y(0)$$

This leads to the expression:

$$y\left(\frac{1}{n}\right) = y(0)\left(1 + \frac{2}{n}\right)$$

Note that we have factored out $y(0)$ to highlight the proportional change in y over the interval $\frac{1}{n}$.

Motivation of Exponential

Since the rate of change $y' = 2y$ remains consistent over each interval, we can approximate the value of y at each step.

For the next small interval, starting from $\frac{1}{n}$, we use the updated value of y and apply the same formula:

$$y\left(\frac{2}{n}\right) = y\left(\frac{1}{n}\right)\left(1 + \frac{2}{n}\right)$$

Repeating this process n times to reach time $t = 1$, we have:

$$y(1) = y(0)\left(1 + \frac{2}{n}\right)^n$$

Motivation of Exponential

As n becomes very large, this expression approaches the form of an exponential function:

$$y(1) = \lim_{n \to \infty} y(0) \left(1 + \frac{2}{n}\right)^n = y(0)e^2$$

Motivation of Exponential

For finidng $y(s)$, consider extending our approach to a general time s using small time steps $\frac{1}{n}$.

To reach a general time s, we perform ns such steps. After ns steps, the approximation is:

$$y(s) \approx y(0)\left(1 + \frac{2}{n}\right)^{ns}$$

As n becomes very large, this expression approaches the form of an exponential function:

$$\lim_{n\to\infty} y(0)\left(1 + \frac{2}{n}\right)^{ns} = y(0)e^{2s}$$

Motivation of Exponential

This result

$$y(s) = y(0)e^{2s}$$

the solution to the differential equation $y' = 2y$.

Matrix Differential Equation

Consider the vector differential equation $\mathbf{y}' = A\mathbf{y}$, where $\mathbf{y}$ is a vector and A is a matrix. This equation describes a system where the rate of change of each component of the vector $\mathbf{y}$ is determined by a linear combination of all components of $\mathbf{y}$, with the coefficients provided by the matrix A.

To solve this equation, we introduce the concept of a matrix exponential, denoted as e^{At}. The matrix exponential is defined as:

$$e^{At} = \lim_{n \longrightarrow \infty} (1 + \frac{A}{n})^n$$

It has binomial expansion

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

where I is the identity matrix.

Now, let's solve the differential equation $\mathbf{y}' = A\mathbf{y}$ using a method analogous to the scalar case:

Matrix Differential Equation

Consider a small time step $\frac{1}{n}$. For each step, we approximate the change in $\mathbf{y}$ using the matrix A and the small time step, leading to:

$$\mathbf{y}\left(\frac{1}{n}\right) \approx \mathbf{y}(0) + \frac{1}{n}A\mathbf{y}(0)$$

To reach a general time s, perform ns such steps. The approximation after ns steps is:

$$\mathbf{y}(s) \approx \left(I + \frac{1}{n}A\right)^{ns} \mathbf{y}(0)$$

As n becomes very large, this expression approaches the form of the matrix exponential function:

$$\lim_{n\to\infty} \left(I + \frac{1}{n}A\right)^{ns} \mathbf{y}(0) = e^{As}\mathbf{y}(0)$$

This shows that the solution to the vector differential equation $\mathbf{y}' = A\mathbf{y}$ is given by the matrix exponential $e^{As}\mathbf{y}(0)$, illustrating how the state of the system $\mathbf{y}(s)$ evolves over time.

Matrix Differential Equation

Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$ and the vector differential equation $\mathbf{y}' = A\mathbf{y}$, where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is a vector. This equation represents a system of linear differential equations.

The system can be written explicitly as:

$$\begin{pmatrix} {y_1}' \\ {y_2}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

This breaks down to the following componentwise equations:

$$\begin{aligned} {y_1}' &= -y_2 \\ {y_2}' &= 2y_1 + 3y_2 \end{aligned}$$

Matrix Differential Equation

As we discussed, the solution of this system is given by

$$\mathbf{y}(s) = e^{As}\mathbf{y}(0)$$

Now let us directly calculate e^{As}.

Firstly, A has characteristic polynomial $\det(tI - A) = (t-1)(t-2)$.
Therefore, A satisfies

$$(A - I)(A - 2I) = 0.$$

Matrix Differential Equation

Using **interpolation polynomials**, we may decompose every polynomial $g(x)$ into the form

$$g(x) = Q(x)(x-1)(x-2) + g(1)\frac{x-2}{1-2} + g(2)\frac{x-1}{2-1}.$$

Use interpolations to construct **eigenspace projections**

$$P_1 = \frac{A-2I}{1-2} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \qquad P_2 = \frac{A-I}{2-1} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$$

Plug in A into the equation, we have a spectural decomposition, for any polynomial g that

$$g(A) = g(1)P_1 + g(2)P_2.$$

Matrix Differential Equation

How to calculate $e^A s$? We actually use polynomials first

$$\left(I + \frac{A}{n}\right)^{ns} = \left(1 + \frac{1}{n}\right)^{ns} P_1 + \left(1 + \frac{2}{n}\right)^{ns} P_2$$

Now letting $s \longrightarrow \infty$, we obtain

$$e^{As} = e^{s} \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + e^{2s} \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$$

Going back to the original equation, we have

$$\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = e^{As} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} (2e^{s} - e^{2s}) y_1(0) + (e^{s} - e^{2s}) y_2(0) \\ (-2e^{s} + 2e^{2s}) y_1(0) + (-e^{s} + 2e^{2s}) y_2(0) \end{pmatrix}$$

We solved it!

Geometric understanding via eigenvectors

We will now explore how eigenvectors and eigenvalues of a matrix can be used to solve the differential equation $\mathbf{y}' = A\mathbf{y}$. But first, let's review what eigenvectors and eigenvalues are.

Geometric understanding via eigenvectors

An eigenvector of a matrix A is a non-zero vector $\mathbf{v}$ that, when multiplied by A, results in a scalar multiple of itself. This scalar is known as the eigenvalue. Mathematically, it is expressed as:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where:

- $\mathbf{v}$ is the eigenvector,
- λ is the eigenvalue associated with $\mathbf{v}$,
- A is the matrix in question.

Eigenvectors point in directions that are unaffected by the transformation, except for being scaled by their corresponding eigenvalues.

Geometric understanding via eigenvectors

Recall that we have used interpolations to construct **eigenspace projections**

$$P_1 = \frac{A - 2I}{1 - 2} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \qquad P_2 = \frac{A - I}{2 - 1} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$$

We have $P_1 + P_2 = I$, this gives a decomposition to any vector, in particular

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

Our idea is to name this two components as

$$\mathbf{w}(s) = \begin{pmatrix} w_1(s) \\ w_2(s) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix}$$

and

$$\mathbf{u}(s) = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix}$$

Geometric understanding via eigenvectors

With this decomposition,

$$\mathbf{y}(s) = \mathbf{w}(s) + \mathbf{u}(s).$$

and

$$A\mathbf{y}(s) = A\mathbf{w}(s) + A\mathbf{u}(s) = \mathbf{w}(s) + 2\mathbf{u}(s).$$

Therefore, we are in fact solving two equations

$$\mathbf{w}'(s) = A\mathbf{w}(s) = \mathbf{w}(s) \implies \mathbf{w}(s) = e^{s}\mathbf{w}(0)$$

$$\mathbf{u}'(s) = A\mathbf{u}(s) = 2\mathbf{u}(s) \implies \mathbf{u}(s) = e^{2s}\mathbf{u}(0)$$

This illustrates the idea that

$$\mathbf{y}(s) = \mathbf{w}(s) + \mathbf{u}(s) = e^{s}\mathbf{w}(0) + e^{2s}\mathbf{u}(0).$$

Exponential of Differential Operator and Talor Expansion

To try to understand exponential function better, we give an intuitive understanding of

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)[f](x) = f(x+1)$$

Since $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. You can also write it as

$$f(x) + f'(x) + \frac{f''(x)}{2!} + \frac{f'''(x)}{3!} + \cdots = f(x+1)$$

This is the same as **Talor** expansion. You know mathematically, but **why**?

Exponential of Differential Operator and Talor Expansion

Remember exponential functions are defined by

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Therefore, in nature,

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \lim_{n\to\infty}\left(I+\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n$$

What does the linear operator $I + \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}$ do for functions?

It maps $f(x)$ to $f(x) + \frac{1}{n}f'(x)$. We claim this is approximately $f\left(x+\frac{1}{n}\right)$.
To understand why, see next slides.

Exponential of Differential Operator and Talor Expansion

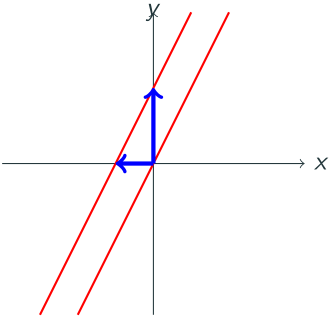
To understand why

$$f(x) + \frac{1}{n}f'(x) \approx f\left(x + \frac{1}{n}\right)$$

We firstly assume $f(x)$ is a linear function $f(x) = kx + b$, then

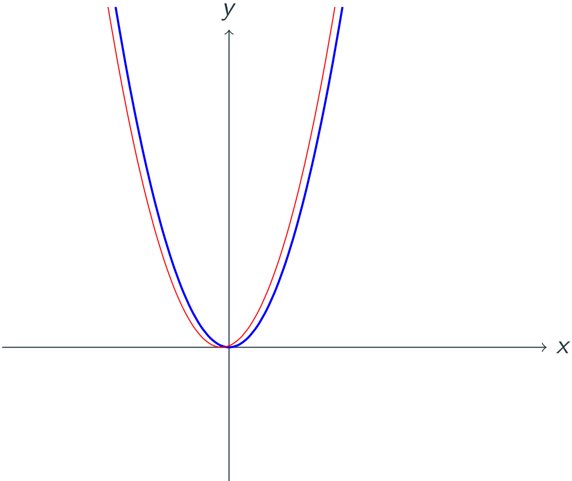
$$f(x) + \frac{1}{n}f'(x) = kx + b + \frac{k}{n} = k(x + \frac{1}{n}) + b$$

Geometrically, shifting a line $y = kx + b$ up by $\frac{k}{n}$ is shifting it left by $\frac{1}{n}$.



Exponential of Differential Operator and Talor Expansion

Since at the range $(x, x+\frac{1}{n})$, any function will become more straight. It can be approximate by line segments, therefore, the effect of $f(x)+\frac{1}{n}f'(x)$ when $n \to \infty$, is shifting the whole function to the left by $\frac{1}{n}$. It changes $f(x)$ to $f\left(x+\frac{1}{n}\right)$.



Exponential of Differential Operator and Talor Expansion

When n is large. Apply this change n times,

$$\underbrace{f(x) \to f(x+\frac{1}{n}) \to f(x+\frac{2}{n}) \to \cdots \to f(x+1)}_{n \text{ times}}$$

the total will change $f(x)$ to $f(x+1)$. Therefore

$$\exp\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)[f](x) = f(x+1)$$

This is a geometric understanding of Talor Expansion.

Exponential of Differential Operator and Talor Expansion

Now use the binomial expansion we have that

$$e^{\frac{\mathrm{d}}{\mathrm{d}x}} = \left(1 + \frac{1}{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\infty} = 1 + \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + \cdots$$

Apply this operator to f, we know that

$$f(x+1) = f(x) + f'(x) + \frac{f''(x)}{2!} + \frac{f'''(x)}{3!} + \cdots$$

This is Taylor expansion! Taylor expansion is nothing more than binomial expansion.

Excercise. Use the same method, deduce a formula for $f(x+s)$ for general s.