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Excercise. Compute the rank of the following matrix

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}$$

Left null space

Definition 1

We define the left null space of A to be $Null(A^T)$

Left null space

!!

Strictly speaking, the left null space for a $m \times n$ matrix A should be

$$\{x : xA = 0\}$$

However, x in the left null space is a $1 \times m$ row vector, so not an element in $\mathbb{R}^m$ (since $\mathbb{R}^m$ are set for column vectors). So **for adapting elementary level learners**, the textbook just transposes the whole expression

$$xA = 0 \iff A^T x^T = 0$$

to define left null space as $Null(A^T)$. You might find it unnatural.

Left null space

Definition 2

We define the row space of A to be $Col(A^T)$

Left null space

!!

Strictly speaking, the row space for a $m \times n$ matrix A should be

$$\{y : xA = y\}$$

However, y in the row space is a $1 \times n$ row vector, so not an element in $\mathbb{R}^n$ (since $\mathbb{R}^n$ are set for column vectors). So **for adapting elementary level learners**, the textbook just transposes the whole expression

$$xA = y \iff A^T x^T = y^T$$

to define row space as $Col(A^T)$. You might find it unnatural.

Left null space

We address the importance of the four subspaces here

$$Col(A^T), \qquad Null(A^T), \qquad Col(A), \qquad Null(A).$$

But in the future, you will see

Proposition 1

If $Col(B^T) = Null(A)$, then $Col(A^T) = Null(B)$.

In other words, the **row space** and **null space** are **determined each other**. This means they have the same amount of information.

!!

We will prove this proposition afterwards.

Left null space

Excercise. Suppose the null space of a matrix A is spanned by the following vectors

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Please determine the row space of A.

Solution: It is same of asking, if a row space of matrix B is given by the vector

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

then determine the null space of B, which is just

$$\mathrm{Col}(A^T) = \mathrm{Null}(B) = \mathrm{Null} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \mathrm{span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Row space

Let me copy the same proposition again, but replace A by A^T:

Proposition 2

If $Col(B^T) = Null(A^T)$, then $Col(A) = Null(B)$.

It says that the left null space of A if determined by the column space of A.

Conclusion for four fundamental subspaces

Therefore we have the following relation for the four fundamental subspaces.

Let A be $m \times n$ matrix of rank r.

$$\underbrace{\mathrm{Col}(A)}_{\substack{\text{Column space}\subset\mathbb{R}^m \\ \text{dim } = r}} \quad \underset{\text{determine each other}}{\leftrightarrow} \quad \underbrace{\mathrm{Null}(A^T)}_{\substack{\text{Left Null space}\subset\mathbb{R}^m \\ \text{dim } = m-r}}$$

$$\underbrace{\mathrm{Null}(A)}_{\substack{\text{Null space}\subset\mathbb{R}^n \\ \text{dim } = n-r}} \quad \underset{\text{determine each other}}{\leftrightarrow} \quad \underbrace{\mathrm{Col}(A^T)}_{\substack{\text{Row space}\subset\mathbb{R}^n \\ \text{dim } = r}}$$

Therefore, **Col(A) and Null(A) have included all informations** of four fundamental subspaces.

Column spaces

From this part, we require you to know the definition of one set contains in another:

$$W \subset V \iff (x \in W \implies x \in V)$$

The definition of two sets are equal is given by

$$W = V \iff (W \subset V \text{ and } V \subset W).$$

Column spaces

Recall the definition of Null space and column space.

Definition 3

Let A be an $m \times n$ matrix. So A has m rows and n columns.

- $\mathrm{Null}(A) = \{x \in \mathbb{R}^n | Ax = 0\} \subset \mathbb{R}^n$
- $\mathrm{Col}(A) = \{y \in \mathbb{R}^m | y = Ax\} \subset \mathbb{R}^m$

Proposition 3

Suppose A is $m \times n$ matrix

- $\mathrm{Null}(A) = \{\vec{0}\} \iff$ Columns of A **linealy independent**
- $\mathrm{Col}(A) = \mathbb{R}^m \iff$ Columns of A **span the whole space**

Column spaces

Proposition 4

Let A be $m \times n$ and B be $n \times q$ matrices, then both $\mathrm{Col}(A)$ and $\mathrm{Col}(AB)$ are subsets of $\mathbb{R}^m$. In particular,

$$\mathrm{Col}(AB) \subset \mathrm{Col}(A)$$

$$\begin{aligned} y \in \mathrm{Col}(AB) \quad &\Longrightarrow \quad y = ABx \text{ for some } x \\ &\Longrightarrow \quad y = At \text{ for } t = Bx \\ &\Longrightarrow \quad y \in \mathrm{Col}(A). \end{aligned}$$

Column spaces

One may understand $\text{Col}(AB) \subset \text{Col}(A)$ by follwing:

!!

Columns of AB are obtained from linear combination of columns of the left factor A. So $\text{Col}(AB) \subset \text{Col}(A)$

Excercise. Can you give some example that $\text{Col}(AB) \neq \text{Col}(A)$?

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{AB}$$

Conclusion: Multiplying a **right-factor**, column space **not getting bigger**.

Column spaces

Proposition 5

Let A be $m \times n$ and B be $n \times q$ matrices, then both $\text{Null}(B)$ and $\text{Null}(AB)$ are subsets of $\mathbb{R}^n$. In particular,

$$\text{Null}(AB) \supset \text{Null}(B)$$

$$x \in \text{Null}(B) \implies Bx = 0 \implies ABx = 0 \implies x \in \text{Null}(AB)$$

Column spaces

The null space are linear relations. In the following example

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The second column is double of the first, so

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \in \mathrm{Null}(B).$$

Note that doing whatever on rows, would not change the relation. For example, we double the second row and delete the last row

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}}_{AB}$$

Therefore $\mathrm{Null}(B) \subset \mathrm{Null}(AB)$

Column spaces

Conclusion: Multiplying a **left-factor**, null space **not getting smaller**.

Factors that not changing column space

Multiplying factors might makes the null space bigger or column space smaller. What kind of factor would **not** change them?

Factors that not changing column space

Warm up Question: What kind of matrix have **LEFT INVERSE**? (Choose two)

A. Columns **linealy independent**

B. Columns **span the whole space**

C. Rows **linealy independent**

D. Rows **span the whole space** .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Factors that not changing column space

Warm up Question: What kind of matrix have **RIGHT INVERSE**? (Choose two)

A. Columns **linealy independent**

B. Columns **span the whole space**

C. Rows **linealy independent**

D. Rows **span the whole space** .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Factors that not changing column space

Proposition 6

If the right factor B have **right inverse**, then

$$\text{Col}(AB) = \text{Col}(\text{A})$$

$$\text{Col}(AB) \subset \text{Col}(\text{A})$$

Let C be the right inverse of B, then $ABC = A$

$$\text{Col}(A) = \text{Col}(ABC) \subset \text{Col}(AB) \subset \text{Col}(A).$$

Factors that not changing null space

Proposition 7

If left factor A have **left inverse**, then

$$\text{Null}(AB) = \text{Null}(\text{B})$$

Factors that not changing null space

Excercise. Please find a matrix C such that

$$\mathrm{Col}(C) = \mathrm{span}\{\begin{pmatrix}1\\2\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\}$$

$$\mathrm{Null}(C) = \mathrm{span}\{\begin{pmatrix}1\\2\\3\end{pmatrix}\}$$

Our method is to write $C = AB$, where A has **left inverse** and B has **right inverse**.

Factors that not changing null space

If we put

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \implies \mathrm{Col}(A) = \mathrm{span}\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}.$$

$$B = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix} \implies \mathrm{Null}(B) = \mathrm{span}\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\}$$

Clear A **has left inverse** and B **has right inverse**. If we put $C = AB$, we gonna get $\mathrm{Col}(C) = \mathrm{Col}(A)$ and $\mathrm{Null}(C) = \mathrm{Null}(B)$.

$$C = AB = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -1 \\ 7 & -2 & -1 \\ 3 & 0 & -1 \\ 5 & -1 & -1 \end{pmatrix}$$

Factors that not changing null space

Recall that **invertible row operations** including row adding, row multiplying and row switching, all of them are the same as left multiplying an **invertible left factor**

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\substack{\text{Row switching} \\ \text{invertible}}} A \qquad \underbrace{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}}_{\substack{\text{Row adding} \\ \text{invertible}}} A \qquad \underbrace{\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{Row multiplying} \\ \text{invertible}}} A$$

Call the elementary matrices as E. Since E is invertible, in particular it have **right inverse**. So

$$\operatorname{Null}(EA) = \operatorname{Null}(A)$$

Factors that not changing null space

Proposition 8

Invertible Row operations does not change the row space and null space of A

!!

However, deleting a row might change the row space or null space. Since deleting is not an invertible row action

Factors that not changing null space

Excercise. After several row operations, we reduce a matrix into the following

$$\begin{pmatrix} 1 & \square & 9 \\ 2 & \square & 0 \\ 3 & \square & 0 \end{pmatrix} \xrightarrow{\text{After invertible row operations}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Please fill in the missing number in $\square$!

Factors that not changing null space

A similar result holds for columns as well.

Proposition 9

Invertible Column operations does not change the column space and left null space of A

Left inverse and right inverse

Left Inverse

Thin full rank matrix $\iff$ rank = number of columns $\iff$ columns **linealy independent** $\iff$ having **left inverse**

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{fat matrix}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{thin matrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But left inverse not unique

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Right Inverse

Fat full rank matrxi $\iff$ rank = number of rows $\iff$ columns **span the whole space** $\iff$ having **right inverse**

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{fat matrix}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{thin matrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But right inverse may not unique

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Invertible

Square full rank matrix $\iff$ rank = number of cols = number of rows $\iff$ columns **linealy independent** and **span the whole space** (**basis**) $\iff$ have **both inverse**

Question: Is the inverse of square matrix unique?

Question: Is the left inverse and right inverse of a square matrix the same?

Inverse Pair

This slides study left and right inverse.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse Pair

A is $m \times n$ matrix, B is $n \times m$ matrix,

$$AB = I_m.$$

- B is right inverse of A
- A is left inverse of B

$$\mathrm{Col}(A) \supset \mathrm{Col}(AB) = \mathrm{Col}(I_m) = \mathbb{R}^m$$

$\Longrightarrow$ columns of A must **span the whole space**

$$\mathrm{Null}(B) \subset \mathrm{Null}(AB) = \mathrm{Null}(I_m) = \{\vec{0}\}$$

$\Longrightarrow$ columns of B must **linealy independent**

Inverse Pair

Proposition 10

Suppose A is $m \times n$ matrix and B is $n \times m$ matrix such that $Null(B) = \{\vec{0}\}$ and $Col(A) = \mathbb{R}^m$. Then

$$AB \text{ is } \textbf{invertible} \quad \Longleftrightarrow \quad \mathrm{Null}(A) \cap \mathrm{Col}(B) = \{\vec{0}\}.$$

(This is a homework)

Hint: $\Longrightarrow$ is easy. For $\Longleftarrow$ part, show that $\mathrm{Null}(AB) = 0$, and that AB is a square matrix.

Inverse Pair

!!

We have important observation! If AB is **invertible**

$(AB)^{-1}A$ is a **left inverse** of B.

$B(AB)^{-1}$ is a **right inverse** of A.

Inverse Pair

Corollary 1

If $AB = I_m$, then $\text{Null}(A) \cap \text{Col}(B) = \{\vec{0}\}$

Inverse Pair

Theorem 1

Let B be a full rank **thin** $n \times m$ matrix ($n \geq m$, $\mathrm{rank}(B) = m$). Then $\mathrm{Col}(B) \subset \mathbb{R}^n$. Let $W \subset \mathbb{R}^n$ be a subspace such that

$$W \cap \mathrm{Col}(B) = \{\vec{0}\} \qquad \dim(W) = n - m.$$

Then there exists a unique left inverse A, such that

$$AB = I_m, \qquad \mathrm{Null}(A) = W.$$

In other wrods, the left inverse is not unique, but it is uniquely determined by specifying a complement as a null space.

In this case, a basis in Null(A) together with columns of B gives a basis for the whole space $\mathbb{R}^n$. You may think of finding left inverse as extending **linealy independent** vectors into **basis** .

Inverse Pair

Theorem 2

Let A be a full rank **fat** $m \times n$ matrix ($n \geq m$, $\text{rank}(A) = m$). Then $\text{Null}(A) \subset \mathbb{R}^n$. Let $W \subset \mathbb{R}^n$ be a subspace such that

$$W \cap \text{Null}(A) = \{\vec{0}\} \qquad \dim(W) = n - m.$$

Then there exists a unique right inverse B, such that

$$AB = I_m, \qquad \text{Col}(B) = W.$$

In other wrods, the right inverse is not unique, but it is uniquely determined by specifying a complement as a column space.

In this case, a basis in Null(A) together with columns of B gives a basis for the whole space $\mathbb{R}^n$. You may think of finding right inverse as extending **linealy independent** row-vectors into **basis** .

Inverse Pair

Excercise. Let A be the following matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

Find the right inverse B of A such that

$$\text{Col}(B) \text{ is } \textbf{spanned by } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Try $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and calculate

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 14 \\ 3 & 6 \end{pmatrix}$$

Inverse Pair

The right inverse of A is then given by

$$B(AB)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 14 \\ 3 & 6 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 3 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 4 & -1 \\ 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{4}{3} \\ 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{pmatrix}$$

Inverse pair and space complement

Therefore, studying a pair A, B with $AB = I_m$ is the same as studying two subspaes W_A and W_B with

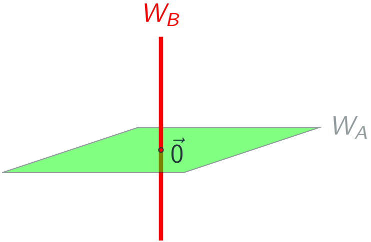
- $W_A \cap W_B = \{\vec{0}\}$
- $\dim(W_A) = n - m$
- $\dim(W_B) = m$
- $\mathrm{Null}(A) = W_A$
- $\mathrm{Col}(B) = W_B$

We call W_A and W_B complement to each other.

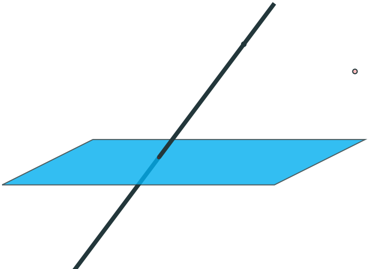
Studying a pair of subspaces $W_A, W_B \subset \mathbb{R}^n$ with $W_A \cap W_B = \{\vec{0}\}$ and $\dim(W_A) + \dim(W_B) = n$ is the same as studying **projection matrices**(wil defined later)

Inverse pair and space complement

Othorgonal Complement



Complement which not necessarily orthogonal



Inverse pair and space complement

Now, if $AB = I_m$, we are curious about BA.

Since B has **left inverse**

$$W_A = \mathrm{Null}(A) = \mathrm{Null}(\mathrm{BA})$$

Since A has **right inverse**

$$W_B = \mathrm{Col}(B) = \mathrm{Col}(\mathrm{BA})$$

Therefore, BA has all information of W_A and W_B!