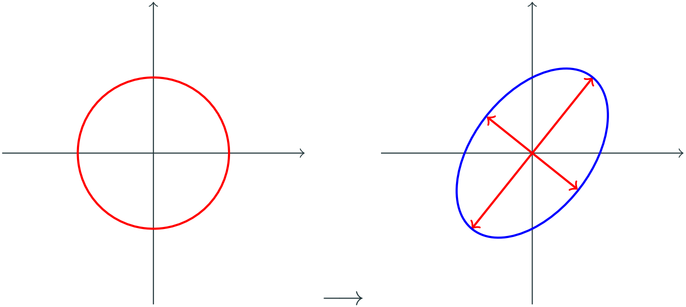
Singular Value Decomposition

Let $A = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$, the linear transformation of A maps the circle of radius 1 to an eclipse



The singular value of A is defined to be the length of the half-axes of eclipse after transformation.

Singular Value Decomposition

For this purpose, we consider a special case, if

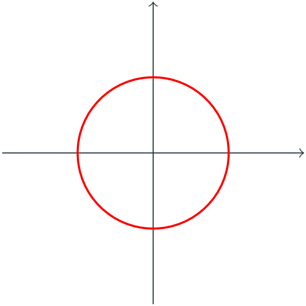
$$A = \Omega_1 \Lambda \Omega_2$$

where $\Omega_1^T \Omega_1 = I$; $\Omega_2^T \Omega_2 = I$; Λ a diagonal matrix.

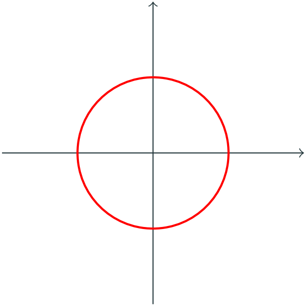
Applying the matrix A is the same as applying three linear transformations.

Singular Value Decomposition

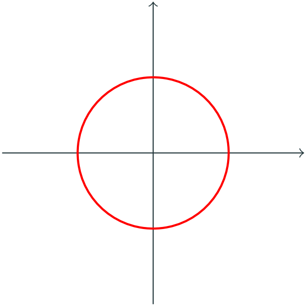
First, applying Ω_2, it does not change the length of any vector, the circle stays as a circle.



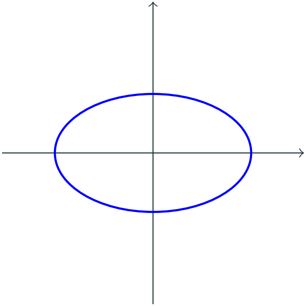
$\longrightarrow$



Second, applying the diagonal matrix, the circle would change only in the x and y direction

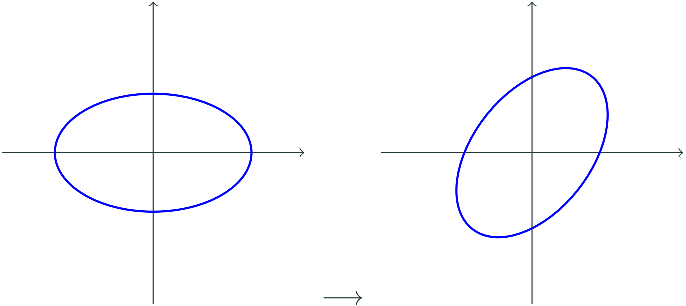


$\longrightarrow$



Singular Value Decomposition

Finally, another rotation rotate the ellipse to another angle



Singular Value Decomposition

Therefore, in the form

$$A = \Omega_1 \Lambda \Omega_2,$$

the diagonal matrix Λ exactly contains all the singular value of A.

Singular Value Decomposition

Next, we claim that any matrix A can be decomposed into the form

$$A = \Omega_1 \Lambda \Omega_2,$$

so that we may read singular values of A directly from Λ.

Singular Value Decomposition

Method for decomposition.

For $A = \Omega_1 \Lambda \Omega_2$, we realize that

$$A^T A = \Omega_2^T \Lambda \Omega_1^T \Omega_1 \Lambda \Omega_2 = \Omega_2^T \Lambda^2 \Omega_2$$
$$A A^T = \Omega_1 \Lambda \Omega_2 \Omega_2^T \Lambda \Omega_1^T = \Omega_1^T \Lambda^2 \Omega_1$$

!!

For any matrix A, $A^T A$ and AA^T are all positive semidefinite, therefore, all its eigenvalues are real and non-negative, and there exists orthogonal matirix Ω_2, but not unique, such that

$$\Omega_2 A^T A \Omega_2^T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Singular Value Decomposition

After finding Ω_2, our goal is to find Ω_1 and Λ such that

$$A\Omega_2^T = \Omega_1 \Lambda$$

Now write

$$\Omega_2^T = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix} \qquad \implies \Omega_1 \Lambda = \begin{pmatrix} Av_1 & Av_2 & \dots & Av_n \end{pmatrix}$$

Note that Av_i already orthogonal with Av_j because

$$(Av_i)^T(Av_j) = v_i^T A^T A v_j = \lambda_j v_i^T v_j = 0.$$

Recall that $v_1, \dots, v_n$ are all eigenvectors. We may rearrange them so that $v_1, \dots, v_m$ are eigenvectors of **non-zero eigenvalues**, and $v_{m+1}, \dots, v_n$ are eigenvectors of **zero eigenvalues**.

So

$$\begin{pmatrix} Av_1 & Av_2 & \dots & Av_m & Av_{m+1} & \dots & Av_n \end{pmatrix}$$
$$= \begin{pmatrix} Av_1 & Av_2 & \dots & Av_m & 0 & \dots & 0 \end{pmatrix}$$

Note that

$$(Av_i)^T(Av_i) = {v_i}^T A^T A v_i = \lambda_i {v_i}^T v_i = \lambda_i$$

Therefore, to obtain unit vector we do

$$\begin{pmatrix} \frac{Av_1}{\sqrt{\lambda_1}} & \frac{Av_2}{\sqrt{\lambda_2}} & \dots & \frac{Av_m}{\sqrt{\lambda_m}} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & & & & & \\ & \sqrt{\lambda_2} & & & & & \\ & & \ddots & & & & \\ & & & \sqrt{\lambda_m} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

Singular Value Decomposition

Now we may complete this into a orthonormal basis

$$\begin{pmatrix} \frac{Av_1}{\sqrt{\lambda_1}} & \frac{Av_2}{\sqrt{\lambda_2}} & \dots & \frac{Av_m}{\sqrt{\lambda_m}} & u_{m+1} & \dots & u_n \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & & & & \\ & \sqrt{\lambda_2} & & & & \\ & & \ddots & & & \\ & & & \sqrt{\lambda_m} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & \end{pmatrix}$$

Therefore, we find $\Omega_2\Lambda$.

Therefore $A\Omega_2^T = \Omega_1\Lambda$ so $\Omega_1^T A\Omega_2^T = \Lambda$ and

$$A = \Omega_1\Lambda\Omega_2$$

.

Excercise. Find a singluar value decomposition of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Solution. We put

$$AA^T = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\det(tI - AA^T) = t^2 - 4t + 3 = (t-1)(t-3).$$

Note that $\det(tI_m - AB) = t^{m-n}\det(tI_n - BA)$, we have of course

$$\det(tI - A^TA) = t(t^2 - 4t + 3) = t(t-1)(t-3)$$

Singular Value Decomposition

Find an eigenvector v_3 of eigenvalue 3, calculate

$$AA^T - I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

We have $v_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Find an eigenvector v_1 of eigenvalue 1, calculate

$$AA^T - 3I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

We have $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Singular Value Decomposition

To get eigenvector of $A^T A$, we automatically have eigenvectors of eigenvalue 1 and 3.

$$u_1 = A^T v_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$u_3 = A^T v_3 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Singular Value Decomposition

$$u_1 u_1^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[\text{to trace 1}]{\text{normalize}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$u_3 u_3^T = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow[\text{to trace 1}]{\text{normalize}} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Therefore, the third vector, an eigenvector of eigenvalue 0 is given by

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \qquad \text{We can take} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Singular Value Decomposition

Here is the collection of orthogonal vectors

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Singular Value Decomposition

Therefore

$$A^T \begin{pmatrix} \frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} \end{pmatrix} = \begin{pmatrix} \frac{u_1}{||u_1||} & \frac{u_2}{||u_2||} & \frac{u_3}{||u_3||} \end{pmatrix} \Lambda$$

We know that Λ are diagonal matrix of squareroot of eigenvalues. We have

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

Singular Value Decomposition

We have

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

Singular value decomposition

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Singular Value Decomposition

Take transpose, we have singular value decomposition

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$