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Note: Preview of slides from (rowoperation.tex) by Qirui Li

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Linear Algebra Lectures

Column/Row operations

Learning Objectives

- What three kind of row/column operations are there?
- For the product AB = C,
 - Which two matrices is allowed to play simutaneous row operation without changing the equality?
 - Which two matrices is allowed to play simutaneous column operation without changing the equality?
- Will simutaneous (row/column) operation changes $A^{-1}B$ or AB^{-1} ?
- Let E be a matrix obtained by applying some row operations from I, what does EA mean?
- Let E be a matrix obtained by applying some column operations from I, what does AE mean?
- Why matrix equation Ax = b is a system of linear equations?
- How row operation help with solving system of linear equations?
 - What kind of matrix are we reducing to?
- How to use row/column operations to find inverse?

A custormer requests for a special **new drink** need the following ingradients

Old Drinks ingradients:

Old Di		0		
6	0	2	0	2
	0	1	0	1
	0	4	2	0
	1	1	0	0

New Drink requirement:

· cqac	ment.
	9
6	4
	2
	2
	4

Problem:









But the chef only have , at the hand, can he produce



by those materials?

The chef thought this problem is the same as a matrix product equation, indeed, replace those questionmarks by x, y, z, he need the following equation to be true

Old Drinks ingradients:

6	0	2	0	2		
	0	1	0	1		
	0	4	2	0		
	1	1	0	0		

Problem:



New Drink requirement:

6	4
	2
	2
	4

Mathematically, this equation is writting as

$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

4

By understanding by columns, solving it is the same as asking for

0	
0	
0	x +
1	

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$
y+

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$
 z +

This can be write as the following and we call it the **Linear equation**

$$\begin{cases} 0x + 2y + 0z + 2w = 4\\ 0x + 1y + 0z + 1w = 2\\ 0x + 4y + 2z + 0w = 2\\ 1x + 1y + 0z + 0w = 4 \end{cases}$$

Changing materials

Observation: The question is only asking for the meal's demand for semi-product meals. It does not asking anything related to the raw material.



No o, o, o, i appeared in this question. Therefore we can change materials to simplify the problem.

Row reduction

The clever chef changes the **material** so the ingradients table is easier

	Î	*	٥	8	ē
6	0	2	0	2	4
	0	1	0	1	2
	0	4	2	0	2
	1	1	0	0	4

	•	٥	8	9
1	1	0	0	4
0	2	1	0	1
0	1	0	1	2
0	0	0	0	0
	1 0 0	1 1 0 2 0 1	1 1 0 0 2 1 0 1 0	1 1 0 0 0 2 1 0 0 1 0 1

$$\begin{cases} 0x + 2y + 0z + 2w = 4\\ 0x + 1y + 0z + 1w = 2\\ 0x + 4y + 2z + 0w = 2\\ 1x + 1y + 0z + 0w = 4 \end{cases}$$

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 4y + 2z + 0w = 2 \\ 1x + 1y + 0z + 0w = 4 \end{cases} \implies \begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 0y + 0z + 0w = 0 \end{cases}$$

Row Multiplying

Doubling the material $O \mapsto O O$ will multiply 3rd row by $\frac{1}{2}$.

Î	*	٥	8	
0	2	0	2	4
0	1	0	1	2
0	4	2	0	2
1	1	0	0	4

	1	*	١	8	9
&	0	2	0	2	4
	0	1	0	1	2
00	0	2	1	0	1
	1	1	0	0	4

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 4y + 2z + 0w = 2 \\ 1x + 1y + 0z + 0w = 4 \end{cases} \implies \begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 1z + 0w = 1 \\ 1x + 1y + 0z + 0w = 4 \end{cases}$$

On equation, the third equation has been divided by 2. This is called **row multiplying**

Row Switching

Switching the order would not change problem,

		*	٥	8	9
6	0	2	0	2	4
	0	1	0	1	2
00	0	2	1	0	1
	1	1	0	0	4

	Î	*	٥	8	9
	1	1	0	0	4
00	0	2	1	0	1
	0	1	0	1	2
	0	2	0	2	4

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 1z + 0w = 1 \\ 1x + 1y + 0z + 0w = 4 \end{cases} \implies \begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 0z + 2w = 4 \end{cases}$$

This is called row switching

Row Adding

Replacing by S. Each time I use the package I save S. Then the row for sis reduced by 2 times the row for .

		*	٥		9
	1	1	0	0	4
00	0	2	1	0	1
	0	1	0	1	2
6	0	2	0	2	4

	9	*	٥	8	
	1	1	0	0	4
00	0	2	1	0	1
666	0	1	0	1	2
6	0	0	0	0	0

$$\begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 0z + 2w = 4 \end{cases} \implies \begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 0y + 0z + 0w = 0 \end{cases}$$

On equation, 4'th equation has been subtracted by 3rd equation. This is called **row adding**

In one words, row operation is updating the raw ingradient list.

We should stop if the matrix is **simple enough** for us to solve equations. So what is simple? There are many discussions.

Theory 1: We should stop if

• Each non-zero row has an entry 1, such that this 1 is the only non-zero entry on its columns.

				8	
ŢĄ.	1	1	0	0	4
00	0	2	1	0	1
0 + 6	0	1	0	1	2
6	0	0	0	0	0

Explaination to Theory 1: If the matrix has been reduced that way, we obtain an equation

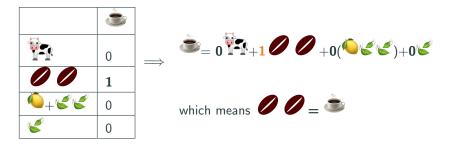
$$\begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \end{cases}$$

such that each equation has a variable, such that this variable does not appear in other equations. When this happens, we may move these variable to one side of equation. Assigning other variables arbitrary values would end up with a solution.

$$\begin{cases} x = 4 - y \\ z = 1 - 2y \\ w = 2 - y \end{cases}$$

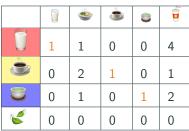
Put y = 0, then (x, y, z, w) = (4, 0, 1, 2) is a solution.

Explaination to Theory 1 without equation: A column with a single 1 and 0 elsewhere can help us to replace a material with some compunds. For example,



With this observation, we can replace **certain materials** by **certain meals**

	Î	•	٥	8	9
*	1	1	0	0	4
00	0	2	1	0	1
0 +66	0	1	0	1	2
8	0	0	0	0	0



Note that the leaves \checkmark is no longer needed for those packaged materials,

we can	aeiete	IT.			
				9	
Ū	1	1	0	0	4
	0	2	1	0	1
	0	1	0	1	2

which tell us directly the list we want, lets compare the original question

Output		
Ī	4	
	1	
	2	



So x = 4, y = 0, z = 1, w = 2. (This process is equivalent as setting y = 0 to obtain a solution.

$$\begin{cases} x = 4 \\ y = 0 \\ z = 1 \\ w = 2 \end{cases}$$

Theory 2: We should stop if

 After deleting zero rows, there are columns that can be rearranged into a triangular matrix with non-zero diagonal.

Example:

We beifly explain why , firstly, circle the entry that corresponding to the diagonal of the triangular matrix

$$\begin{pmatrix}
1 & 2 & 3 & 4 & 6 & 6 \\
2 & 0 & 3 & 2 & 1 & 1 \\
1 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We read equations from bottom to top, one by one

$$x + 0v + 0z + 2w + 1u = 0 \implies x = -2w - 1u$$
.

The circled variable would appear as a new variabl never appeared before

$$2x + 0y + 3z + 2w + 1u = 1 \implies z = \frac{-2x - 2w - u}{3}$$
.

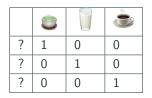
Then the next new variable is y.

$$x + 2y + 3z + 4u + 6w = 6 \implies y = \frac{6 - 6w - 4u - 3z - x}{2}.$$

Therefore, the equation can be solved. We call this process **backwards substituting**.

Identity matrix

Filling the following blanks.



?	?	?
1	0	0
0	1	0
0	0	1

Identity matrix

Suppose , any two can not blend to the third drink (called linealy independent). Filling the following blanks.

	9		
	?	?	?
	?	?	?
٥	?	?	?

Identity matrix

9		
1	0	0
0	1	0
0	0	1

This matrix is the ingradient list of making ingradient, i.e. do nothing.

Definition 1

The identity matrix is a $n \times n$ square matrix with 1 on the diagonal and 0 elsewhere.

Proposition 1

For any $n \times m$ matrix P, $I_n P = PI_m = P$.

Inverse Matrix

The chef is wondering if another guest coming with a special request, so he would like a list to produce the ingradient out of meals.

He has a list

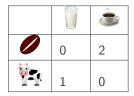
0 2

1

0

How c	ould he	make	another	list?
		2 2		
	?	?		
٥	?	?		

He realize this list should have a property, combinging them should be.



	0	
9	?	?
	?	?

1	0
0	1

Inverse Matrix

Definition 2

For a $n \times n$ matrix A, an inverse is a matrix B, such that

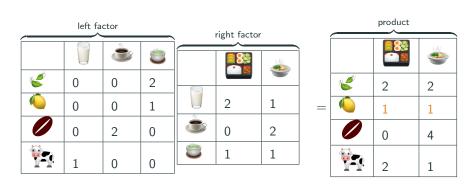
$$AB = BA = I_n$$
.

If such a B exists, A is called **invertible** and denote the inverse as A^{-1} .

For the product C = AB, changing raw materials only affect rows of A and C by certain simutaneous row operations. the matrix B would not change since it does not depend on raw materials. Therefore,

Proposition 2

The equality C=AB will still be true if we perform arbitrary simutaneous row operation on A and C



If A is invertible, we can write $B = A^{-1}C$, this actually tells us that

Corollary 1

If A is invertible, the product $A^{-1}C$ does not change if we perform simutaneous row operation on A and C.

Let me show you an application of this in calculation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 2 & 9 \end{pmatrix}$$

$$r_1 \mapsto r_1 - r_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & -6 \\ 1 & 2 & 1 \\ 0 & 2 & 9 \end{pmatrix}$$

$$r_2 \mapsto r_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 2 & 9 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -6 \\ 0 & 2 & 9 \\ 1 & 2 & 1 \end{pmatrix}$$

Simutaneous Column Operation

Similarly, Column operations corresponding to updating list when changing final meals by its order, amount, or packing them together. For the product C = AB, when final meals changes, the matrices affected is C and B. A is the list of making intermediates, it does not change.

Proposition 3

The equality C = AB will still be true if we perform arbitrary simutaneous column operation on B and C

If B is invertible, we can write $A = CB^{-1}$, this actually tells us that

Corollary 2

If B is invertible, the product CB^{-1} does not change if we perform simutaneous column operation on B and C.

We will introduce elementary matrices.

Let A be an $m \times n$ matrix, and let I_m be $m \times m$ identity matrix. We have a equation

$$A = I_m A$$

After some simutaneous row operation, A changes to A' and I_m changes to E, by what we discussed before, the following equation is true

$$A' = EA$$

We may view this fact by another perspective: The product EA is the same as applying row operations recorded in E to A.

Experiment:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 3 \end{pmatrix}$$

Explaination from another perspective:

left factor	row operation recorded	product: apply row operation
$ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} $	add 2nd row to 1st row.	$ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 3 \end{pmatrix} $

Proposition 4

Suppose E is a matrix obtained by applying some row operations from I_n , then EA is exactly the matrix by applying the same row operations on A.

Definition 3

An elementary matrix E is a matrix after one-step row operation from identity matrix.

Name	Elementary matrices example	Inverse
Switching	$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} $	$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} $
Multiplying	$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} $	$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} $
Adding	$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} $	$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} $

Exercise: What is the following product? calculate in mind.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 6 & 8 \end{pmatrix}$$

Look at the following matrix (maybe not elementary). By thinking how it changed from identity matrix, what kind of row operation does it record?

left-factor	row operation recorded.
$ \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] $	Switch two rows
$\begin{pmatrix} 0 & 1 \end{pmatrix}$	Delete the first row
$\begin{pmatrix} 1 & 1 \end{pmatrix}$	
1 1	
$\begin{pmatrix} 2 & 0 \end{pmatrix}$	
$\left(\begin{array}{cc} 0 & 1 \end{array}\right)$	
0 1	
$\begin{pmatrix} 0 & 1 \end{pmatrix}$	

The same logic is also for columns. Recall that AB = C stays as an equation when applying column oprtations on B and C. **Experiment:**

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \implies \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$$

Explaination from another perspective:

right factor	row operation recorded	product: apply row operation		
$ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} $	add 1st column to 2nd column.	$ \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix} $		

Proposition 5

Suppose E is a matrix obtained by applying some column operations from I_n , then AE is exactly the matrix by applying the same column operations on A.

right-factor	column operation recorded.
$ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} $	Switch two columns
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	
$ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} $	
$ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} $	

What is the meaning of the following product? From the perspective of column action?

$$\underbrace{A}_{3\times 4\text{Matrix}} \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}$$

What is the meaning of the following product? From the perspective of column action?

What is the meaning of the following matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\frac{A}{2 \times 2 Matrix}}_{2 \times 2 Matrix}$$

How about the following

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{matrix} \mathcal{A} \\ 2 \times 2 Matrix \end{matrix}}_{2 \times 2 Matrix}$$

Is this product depends on the second row of A?

Let A be a $m \times n$ matrix, and let I be the 2nd row of A, can you write I in terms of a matrix multiplication?

Let \vec{v} be the 3rd column of A, can you write \vec{v} into a matrix multiplication?