Composition of lienar transformations

Next part, we will see that the matrix multiplication interpretes the composition of linear transformations.

Composition of lienar transformations

Proposition 1

Let V, W, U be linear spaces with bases

$$\mathcal{E} \text{ basis for } V \quad \mathcal{F} \text{ basis for } W \quad \mathcal{H} \text{ basis for } U$$

Consider the following linear transformation

$$V \xrightarrow{T} W \xrightarrow{S} U$$

Then the matrix representation of $S \circ T$ is the product of matrix representations of S and T in corresponding bases.

Composition of lienar transformations

We suppose the linear transformation has the following matrix representations

$$T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} A$$

$$S\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} = \underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p\end{pmatrix}}_{\mathcal{H}} B$$

The proof is straightforward when looking at the commutative diagram

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[r, "S"] & U \\ F^n \arrow[u, "L_{\mathcal{E}}"] \arrow[r, "L_A"] & F^m \arrow[u, "L_{\mathcal{F}}"] \arrow[r, "L_B"] & F^p \arrow[u, "L_{\mathcal{H}}"] \end{tikzcd}$$

Composition of lienar transformations

Induced transformation proof:

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{S} & U \\ {\scriptstyle L_{\mathcal{E}}}\uparrow & & {\scriptstyle L_{\mathcal{F}}}\uparrow & & {\scriptstyle L_{\mathcal{H}}}\uparrow \\ F^n & \xrightarrow{L_A} & F^m & \xrightarrow{L_B} & F^p \end{array}$$

Let C be the matrix of $S \circ T$ on the basis $\mathcal{E}$ and $\mathcal{H}$. We want to show $C = BA$. To do so, we only need to prove

$$L_C = L_B \circ L_A$$

Since $L_C = L_{\mathcal{H}}^{-1} \circ S \circ T \circ L_{\mathcal{E}}$, $L_B = L_{\mathcal{H}}^{-1} \circ S \circ L_{\mathcal{F}}$, $L_A = L_{\mathcal{F}}^{-1} \circ T \circ L_{\mathcal{E}}$. Then

$$L_B \circ L_A = L_{\mathcal{H}}^{-1} \circ S \circ L_{\mathcal{F}} \circ L_{\mathcal{F}}^{-1} \circ T \circ L_{\mathcal{E}} = L_{\mathcal{H}}^{-1} \circ S \circ T \circ L_{\mathcal{E}} = L_C.$$

Composition of lienar transformations

Standard Proof: Since we have

$$T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} A$$

$$S\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} = \underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p\end{pmatrix}}_{\mathcal{H}} B$$

$$\begin{aligned} S \circ T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} &= S\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} A \\ &= \underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p\end{pmatrix}}_{\mathcal{H}} BA \end{aligned} \tag{1}$$

This implies

$$S \circ T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p\end{pmatrix}}_{\mathcal{H}} BA$$

Composition of lienar transformations

Corollary 1

Let V, W be linear spaces with bases

$$\mathcal{E} \text{ basis for } V \quad \mathcal{F} \text{ basis for } W$$

Suppose $T : V \longrightarrow W$ is **invertible** and represented by matrix A on those bases, then its inverse $T^{-1} : W \longrightarrow V$ is represented by A^{-1}

Composition of lienar transformations

Standard Proof: By what given, we have

$$T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} A$$

Apply T^{-1} on both hand, we have

$$\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = T^{-1}\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} A$$

Right multiplying A^{-1}, we have

$$\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} A^{-1} = T^{-1}\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}}$$

This proves the corollary. ■