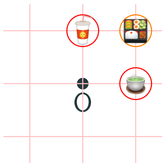
Matrix of linear transformations

To represent a linear transformation, we will use matrices.

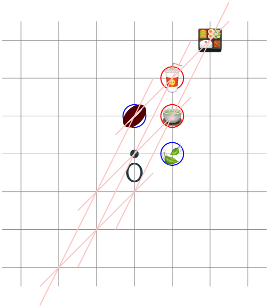
Matrix of linear transformations

In previous example, whenever we have a receipe table, it gives a linear transformation **from space of drink combinations** to **space of material combinations**.

	🥤	🍵
	2	1
🍃	1	1



$\longrightarrow$



Matrix of linear transformations

If we call this map T. Then we use 🥤, 🍵as symbols for those drinks in the domain. And

$$T\left(\text{🥤}\right) \qquad T\left(\text{🍵}\right)$$

as symbols for its position in the codomain. Since materials are all in the codomain, it makes more sense to write our table as

	$T\left(\text{🥤}\right)$	$T\left(\text{🍵}\right)$
🫘	2	1
🍃	1	1

Matrix of linear transformations

This table can be written as an expression

$$\begin{pmatrix} T\text{🥤} & T\text{🍵} \end{pmatrix} = \begin{pmatrix} \text{🫘} & \text{🍃} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Factor T out, we can write

$$T\begin{pmatrix} \text{🥤} & \text{🍵} \end{pmatrix} = \begin{pmatrix} \text{🫘} & \text{🍃} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Note that here (🥤 ,🍵) is a basis of the domain, and (🫘 ,🍃) is a basis of the codomain. We call the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The matrix representation of T in the basis (🥤 ,🍵) and (🫘 ,🍃). It determines the linear transformation completely.

Matrix Representation of Linear Transformation

Definition 1

For a linear transformation $T : V \longrightarrow W$, let

- $\mathcal{E} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ be a basis of domain V
- $\mathcal{F} = \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}$ be a basis of codomain W.

The **matrix representation** of T with respect to $\mathcal{E}$ and $\mathcal{F}$, is the matrix P such that

$$T \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} = \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix} P$$

In other words, the matrix representation is the recipe table to make $T \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ by materials $\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}$.

Matrix Representation of Linear Transformation

The matrix representation P of $T : V \longrightarrow W$ with basis $\mathcal{E}$ and $\mathcal{F}$, is the coordinate matrix of

$$T\mathcal{E} = \begin{pmatrix} T\vec{e}_1 & T\vec{e}_2 & \cdots & T\vec{e}_n \end{pmatrix}$$

in the following basis of codomain W

$$\mathcal{F} = \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}.$$

The matrix P fits into the following linear combination equation

$$\overbrace{\begin{pmatrix} T\vec{e}_1 & T\vec{e}_2 & \cdots & T\vec{e}_n \end{pmatrix}}^{T\mathcal{E}} = \overbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}^{\mathcal{F}} P$$

Each column of P is the coordinate of $T\vec{e}_i$ in the **basis** $\mathcal{F}$.

$$P = \begin{pmatrix} [T\vec{e}_1]^{\mathcal{F}} & [T\vec{e}_2]^{\mathcal{F}} & \cdots & [T\vec{e}_n]^{\mathcal{F}} \end{pmatrix}$$

Matrix Representation of Linear Transformation

Excercise. Let $V = P_{2,x} = \{ax^2 + bx + c, \text{ where } a, b, c \in F\}$,
$W = P_{2,t} = \{at^2 + bt + c, \text{ where } a, b, c \in F\}$

Consider a linear map

$$T : V \longrightarrow W, f(x) \longmapsto f(t+1)$$

Find matrix representation of T with bases

$$\mathcal{F} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \text{ in } V \qquad \mathcal{E} = \begin{pmatrix} 1 & 2x+1 & x^2+1 \end{pmatrix} \text{ in } W$$

Matrix Representation of Linear Transformation

Solution.: Apply the linear transformation T on each of the function on **basis** and write the coordinate in **basis** of the target. We find

$$T(1) = 1 = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{[T(1)]^{\mathcal{F}}}$$

$$T(2x+1) = 2(t+1)+1 = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \underbrace{\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}}_{[T(2x+1)]^{\mathcal{F}}}$$

$$T(x^2+1) = (t+1)^2+1 = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}_{[T(x^2+1)]^{\mathcal{F}}}$$

Matrix Representation of Linear Transformation

We write this into a matrix form

$$T \underbrace{\begin{pmatrix} 1 & 2x+1 & x^2+1 \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} 1 & t & t^2 \end{pmatrix}}_{\mathcal{F}} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

We know the matrix representation of T is $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$