Before reviewing proof

Make sure you are able to do QR decomposition, calculating determinant, calculating A^n, calculating e^A, able to solve differential equations $y' = Ay$, able to solve applicational problems (like the one in homework).

Able to verify if a matrix is diagonalizable (by verifying if it satisfies a polynomial). Able to diagonalize a matrix, able to find Jordan canonical form....

You must be able to do all computations. In an efficient way.

Eigenvalue and eigenvectors for 2 by 2 matrices

Given 2×2 matrix. Its characteristic polynomial is given by

$$\det\left(tI_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = t^2 - (a+d)t + (ad - bc)$$

Suppose the polynomial is given by $(t - \lambda_1)(t - \lambda_2)$ with $\lambda_1 \neq \lambda_2$. In this case, it is diagonalizable.

Eigenvalue and eigenvectors for 2 by 2 matrices

Since $(A-\lambda_1)(A-\lambda_2)=0$, we have $A(A-\lambda_1)=\lambda_2(A-\lambda_1)$ and $A(A-\lambda_2)=\lambda_1(A-\lambda_2)$.

Therefore, the matrix

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix} \qquad \text{eigenmatrix of eigenvalue } \lambda_2$$

$$\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix} \qquad \text{eigenmatrix of eigenvalue } \lambda_1$$

Excercise. Suppose $2a + b \neq 0$. Given the following matrix

$$\begin{pmatrix} b & a \\ 2b & 2a \end{pmatrix}$$

What is its eigenvalue and what is its eigenvector?

Eigenvalue and eigenvectors for 2 by 2 matrices

No matter what, this matrix is always **not invertible** since columns are colinear. Therefore, 0 must be an eigenvalue of it. Therefore, by looking at the trace, the other eigenvalue must be $2a + b$.

Therefore $A - 0I = \begin{pmatrix} b & a \\ 2b & 2a \end{pmatrix}$ is the eigenmatrix of eigenvalue $2a + b$

$A - (2a + b)I = \begin{pmatrix} -2a & a \\ 2b & -b \end{pmatrix}$ is the eigenmatrix of eigenvalue 0.

So

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is eigenvector of eigenvalue $2a + b$. And

$$\begin{pmatrix} a \\ -b \end{pmatrix}$$

is an eigenvector of eigenvalue 0.

Eigenvalue and eigenvectors for 2 by 2 matrices

Therefore, we can even write down its diagonalization

$$\begin{pmatrix} 1 & a \\ 2 & -b \end{pmatrix}^{-1} \begin{pmatrix} b & a \\ 2b & 2a \end{pmatrix} \begin{pmatrix} 1 & a \\ 2 & -b \end{pmatrix} = \begin{pmatrix} 2a+b & 0 \\ 0 & 0 \end{pmatrix}$$

!!

When eigenvalues are given , finding eigenvectors is extremely easy, you should able to be compute within 10 seconds in mind.

Eigenvalue and eigenvectors for 2 by 2 matrices

Excercise. Write down an eigenvector of the following matrix

$$\begin{pmatrix} 0 & -2 \\ 3 & 5 \end{pmatrix}$$

of eigen value 2.

Excercise. Write down an eigenvector of the following matrix

$$\begin{pmatrix} 3 & 4 \\ 3 & 7 \end{pmatrix}$$

of eigen value 9.

The most important fact

Theorem 1

If $P_1, \ldots, P_n$ are projection matrices $P_i^2 = P_i$ and

$$P_1 + \ldots + P_n = I$$

Then $P_i P_j = 0$

Proof We consider a matrix

$$Q = \begin{pmatrix} P_1 & P_2 & \cdots & P_n \end{pmatrix}$$

$$R = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} \qquad S = \begin{pmatrix} P_1 & & \cdots & \\ & P_2 & \cdots & \\ & & \ddots & \\ & & & P_n \end{pmatrix}$$

This definition implies that $QR = I$.

The most important fact

$$RQ = \begin{pmatrix} P_1^2 & P_1 P_2 & \cdots & P_1 P_n & \\ P_2 P_1 & P_2^2 & \cdots & P_2 P_n & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_n P_1 & P_n P_2 & \cdots & P_n^2 & \end{pmatrix}$$

Furthermore, $QS = Q$, $SR = R$. So

$$(S - RQ)^2 = S^2 - SRQ - RQS + RQRQ = S - RQ$$

This implies $S - RQ$ is a projection matrix. However, $P_i^2 = P_i$ so $\text{tr}(S - RQ) = 0 \implies$ S-RQ=0 $\implies P_i P_j = 0$.

The most important fact

Theorem 2

Suppose $P_i\vec{v}_i = \vec{v}_i$ for $1 \leq i \leq k$ and $P_iP_j = 0$ for $1 \leq i, j \leq k$. Then

$$\vec{v}_1, \ldots, \vec{v}_k$$

is linearly independent.

For $i \neq j$,

$$P_i\vec{v}_j = P_iP_j\vec{v}_j = 0$$

So if

$$a_1\vec{v}_1 + \ldots + a_k\vec{v}_k = 0$$

Multiply P_i we have

$$0 + 0 + \ldots + 0 + a_iP_i\vec{v}_i + 0 + \cdots = P_i0 = 0 \implies a_i\vec{v}_i = 0 \implies a_i = 0.$$

Of different eigenvalue

Excercise. Show that eigenvectors from different eigenvalue must be linearly independent.

Suppose $A\vec{v}_i = \lambda_i \vec{v}_i$ with $\lambda_i \neq \lambda_j$. Assume

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k = 0.$$

Multiply by A, we have

$$a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 + \ldots + a_k\lambda_k\vec{v}_k = 0.$$

Subtract this equation by λ times the first equation, we have

$$a_2(\lambda_2 - \lambda_1)\vec{v}_2 + \ldots + a_k(\lambda_k - \lambda_1)\vec{v}_k = 0$$

By induction hypothesis, $a_2 = \ldots = a_k = 0$. So $a_1\vec{v}_1 = 0$.

Of different eigenvalue

Proof by spectural decomposition. Assume $\vec{v}$ is an eigenvector of eigenvalue μ

$$(\lambda + \epsilon)\mathscr{P}_\lambda \vec{v} = A\mathscr{P}_\lambda \vec{v} = \mathscr{P}_\lambda A\vec{v} = \mathscr{P}_\lambda \mu \vec{v}$$

This implies

$$(\lambda - \mu + \epsilon)\mathscr{P}_\lambda \vec{v} = 0$$

If $\lambda \neq \mu$, this implies $\mathscr{P}_\lambda \vec{v} = 0$, in particular, taking constant part, we have $P_\lambda \vec{v} = 0$. Therefore $P_{\mu_1} + \ldots + P_{\mu_k} + P_\lambda + \ldots + P_{\mu_m} = I$, multiply $\vec{v}$ we have $P_\mu \vec{v} = \vec{v}$.

Some simple proof

We have already proved eigenvectors of normal matrices are perpendicular each other. Now let us write some shorter proof for some cases.

Some simple proof

Excercise. Let $A = A^H$ be Hermitian matrices, Show that eigenvectors of different eigenvalues of A are automatically Hermitian orthogonal.

Solution. Let $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$ with $\lambda \neq \mu$. Since $A = A^H$, λ, μ are all real numbers.

$$\vec{v}^H \mu \vec{w} = \vec{v}^H A \vec{w} = \vec{v}^H A^H \vec{w} = \vec{v}^H \lambda \vec{w}.$$

This implies that

$$\vec{v}^H \vec{w} = 0.$$

Some simple proof

Excercise. Let $A = -A^H$ be skew Hermitian matrices, Show that eigenvectors of different eigenvalues of A are automatically Hermitian orthogonal.

Solution. Let $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$ with $\lambda \neq \mu$. Since $A = -A^H$, $\overline{\lambda} = -\lambda, \overline{\mu} = -\mu$ are all purely imaginary numbers.

$$\vec{v}^H\mu\vec{w} = \vec{v}^H A\vec{w} = -\vec{v}^H A^H\vec{w} = -\vec{v}^H\overline{\lambda}\vec{w} = \vec{v}^H\lambda\vec{w}.$$

Some simple proof

Excercise. Let $A^{-1} = A^H$ be Unitary matrices, Show that eigenvectors of different eigenvalues of A are automatically Hermitian orthogonal.

Solution. Let $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$ with $\lambda \neq \mu$. Since $A^{-1} = A^H$, $\overline{\lambda} = \frac{1}{\lambda}, \overline{\mu} = \frac{1}{\mu}$.

$$\vec{v}^H \mu \vec{w} = \vec{v}^H A \vec{w} = \vec{v}^H (A^H)^{-1} \vec{w} = \vec{v}^H (\overline{\lambda})^{-1} \vec{w} = \vec{v}^H \lambda \vec{w}.$$

Some simple proof

Excercise. Let $AA^H = A^H A$ be Normal matrices, Show that eigenvectors of different eigenvalues of A are automatically Hermitian orthogonal.

Solution. We first prove a lemma. That if $AA^H = A^H A$, then $Ax = 0$ implies $A^H x = 0$. Indeed,

$$Ax = 0 \implies x^H A^H A x = 0 \implies x^H A A^H x = 0 \implies A^H x = 0$$

Therefore $Ax = \lambda x \implies A^H x = \overline{\lambda} x \implies x^H A = \lambda x^H$.

$$\vec{v}^H \mu \vec{w} = \vec{v}^H A \vec{w} = \vec{v}^H \lambda \vec{w}.$$

Some simple proof

Excercise. Let $A = A^T$ be real, positive definite matrix. Then all eigenvalues of A are positive

Solution. If $Av = \lambda v$, then $0 < v^T A v = \lambda \underbrace{v^T v}_{>0}$ therefore $\lambda > 0$

Exponential Functions

$$e^X = \lim_{n \longrightarrow \infty} \left(1 + \frac{X}{n}\right)^n$$

Corollary 1

For any $X_1, X_2, \ldots, X_k$

$$\lim_{n \longrightarrow \infty} \left(1 + \frac{X_1}{n} + \frac{X_2}{n^2} + \ldots + \frac{X_k}{n^k}\right)^n = e^{X_1}$$

In other words, all $X_2, \ldots, X_k$ is ignorable.

Excercise. Show that if A is real matrix, $A = A^T$ and all eigenvalues of A are positive, then A is positive definite.

Exponential Functions

Excercise. If $AB = BA$, then $e^A e^B = e^{A+B}$

$$e^A e^B = \lim_{n \longrightarrow \infty} (I + A/n)^n (I + B/n)^n = \lim_{n \longrightarrow \infty} ((I + A/n)(I + B/n))^n$$

$$= \lim_{n \longrightarrow \infty} (I + (A + B)/n + AB/n^2)^n = e^{A+B}$$

Excercise. If $A = -A^H$, show that e^A is a unitary matrix

$$e^A (e^A)^H = e^A e^{A^H} = e^{A+A^H} = e^0 = I$$

Exponential Functions

Excercise. If $A = A^H$, show that e^A is a positive definite matrix.

Since $A = A^H$, all eigenvalues are real numbers. We have $(e^A)^H = e^{A^H} = e^A$. Also e^A is a real matrix. Furthermore, the eigenvalue of e^A is $e^\lambda > 0$. So e^A is a real symmetric matrix with all positive eigenvalues, it is positive definite

Excercise. Show that any real positive definite matrix A can be written as e^X for a unique matrix X. We call this X as $X = \ln A$

Excercise. Show that if A B are two positive definite matrix with $AB = BA$, then AB is also a positive definite matrix.

$$AB = e^{\ln A} e^{\ln B} = e^{\ln A + \ln B}$$

so AB is a positive definite matrix.

Excercise. If A is a normal matrix. Show that e^A admits a decomposition $e^A = PU$ where P is a positive definite matrix, U is a unitary matrix commute with P.

Eigenvalue and normal matrices

Show that if A is skew-Hermitian, then e^A is unitary