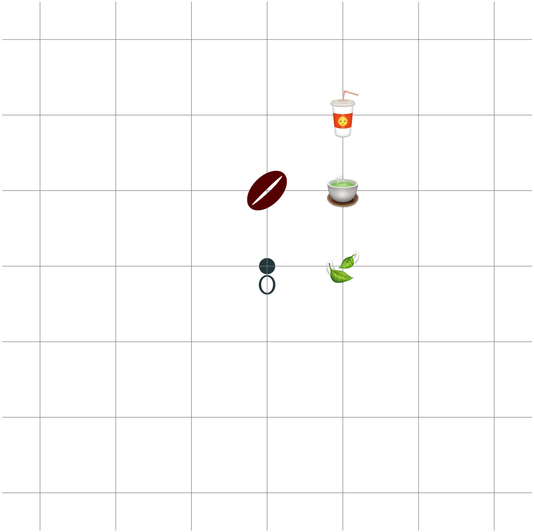
Now Shinchan would start of making drinks, again he has the following recipe

	🥤	🍵
🍃	1	1
🫘	2	1

🥤 uses $\boxed{🍃\ 🫘\ 🫘}$

🍵 uses $\boxed{🍃\ 🫘}$

This time he would like to use pictures to organize those data. He **put** each product to the corresponding point in the linear combination space of materials.



	🥤	🍵
🍃	1	1
🫘	2	1

Onece again, he wants to combine two tables.

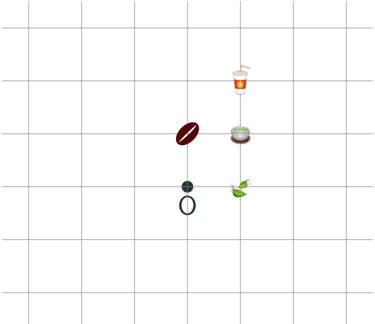
	🥤	🍵
🍃	1	1
🫘	2	1

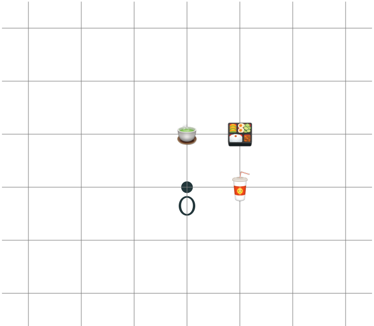
	🍱
🥤	1
🍵	1

=

	🍱
🍃	
🫘	

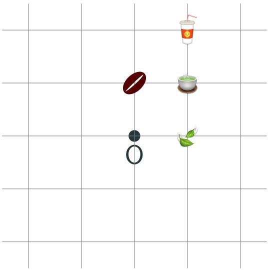
You know how to do it. But how can he do it by **purly with pictures**?





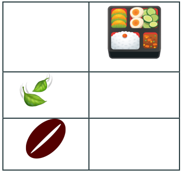
Linear Transformations

He is confusing...... How to do it..... How to do it.... **Without computation**... How can he found the position of 🍱in the left picture.... How he can do matrix multiplication **purely geometrically**.....



Left Picture

Right Picture

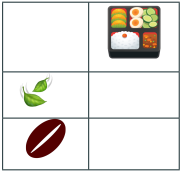


Linear Transformations

His chef come to his room and accidentally bumped on his picture....

Chef: Oh! I am sorry, Shinchan what are you doing here?

Left Picture

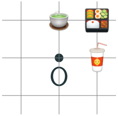


Right Picture

Shinchan: Oh!!! No!!! My pictures, you destroyed my picture...

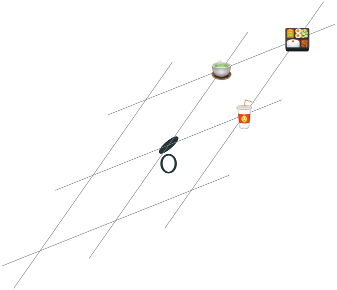
Linear Transformations

Chef: I am sorry, but... Are there any difference with those two pictures?



Right Picture
BEFORE

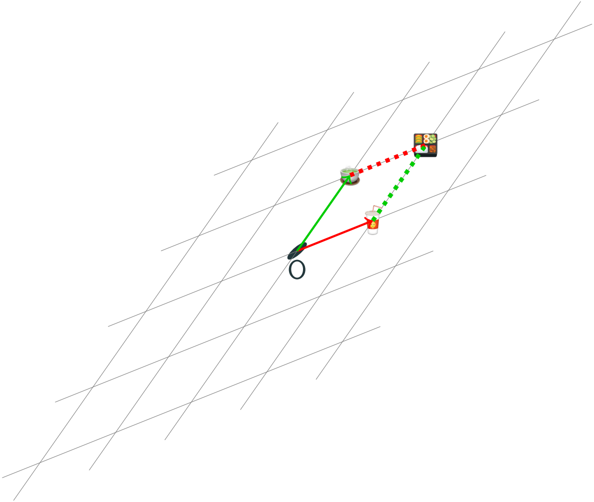
$\leftarrow$ Any difference? $\rightarrow$



Right Picture AFTER

Linear Transformations

After the Chef explained, Shinchan knows this crooked picture **keeps all information of the recipe table** because it **keeps** the **parallelgrams**, the vector for tea 🍵and for cola 🥤, still adds to the vector for 🍱.

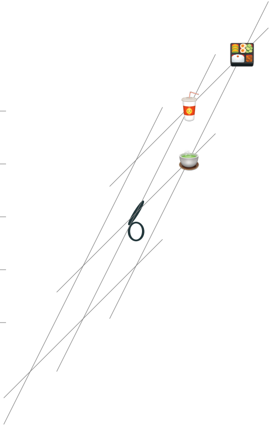


	🍱
🥤	1
🍵	1

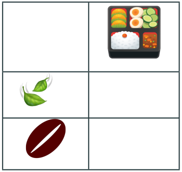
Linear Transformations

Shinchan: Great! I am trying to compute matrix multiplication geometrically. I have a recipe to make **intermidiates** from **materials**, and to make **final meals** from **intermidiates**. I wish to figure out how to make **final meals** from **materials**. What should I do?

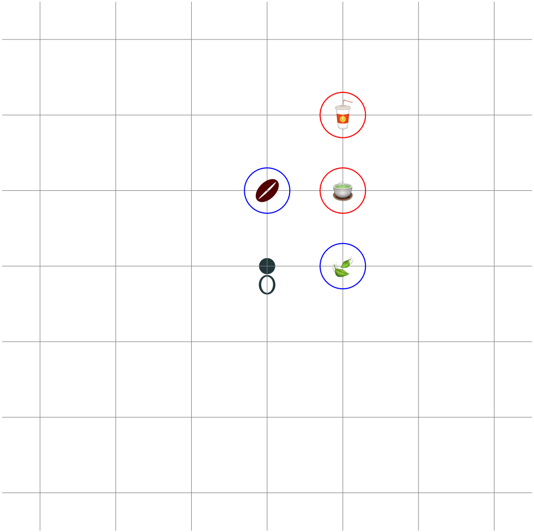
Left Picture



Right Picture

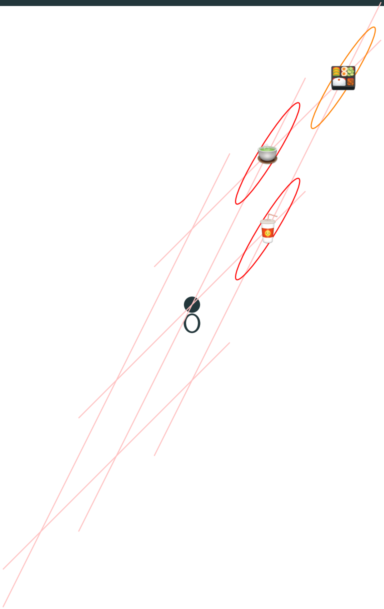


Linear Transformations



Chef: Now I see you have the Left Picture. That represents how you make drinks out of materials

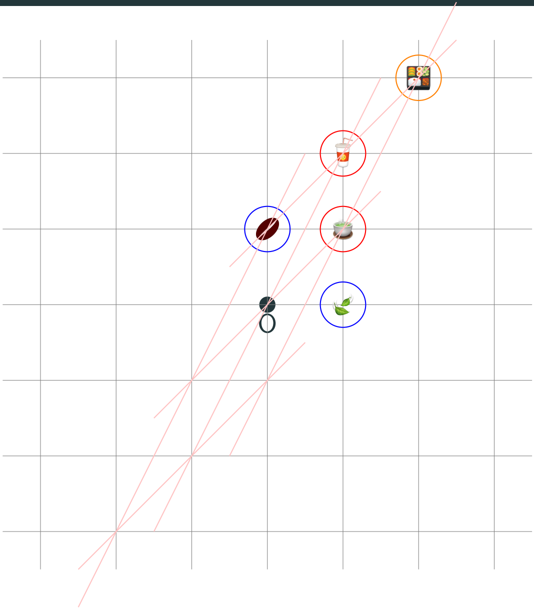
	🥤	🍵
🍃	1	1
	2	1



Chef: And I see you have the Right Picture, which represents how to make meals out of drinks. Oh I am sorry to bump it...Hopefully we do not lose any information.

	🍱
🥤	1
🍵	1

Linear Transformations

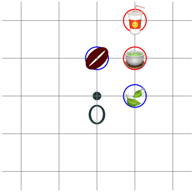


Oh! Let us simply put those two picture together!! Then it shows up all information. We now know how to make meals 🍱by materials 🍃, 🫘.

	🍱
🍃	2
🫘	3

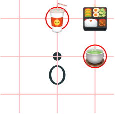
Let′s summarise the **Geometric method** of computing matrix multiplication.

Step 1: We have two pictures corresponding to the left factor and the right factor.



Left factor: Making drinks by materials

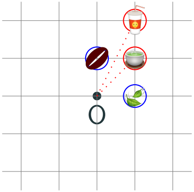
	🥤	🍵
	2	1
🍃	1	1



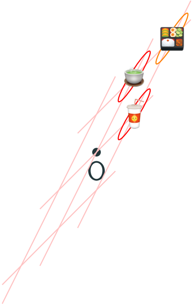
Right factor: Making meals by drinks

	🍱
🍵	1
🥤	1

Step 2: Skew the right picture so that the relative position of drinks matches its position in the left picture.

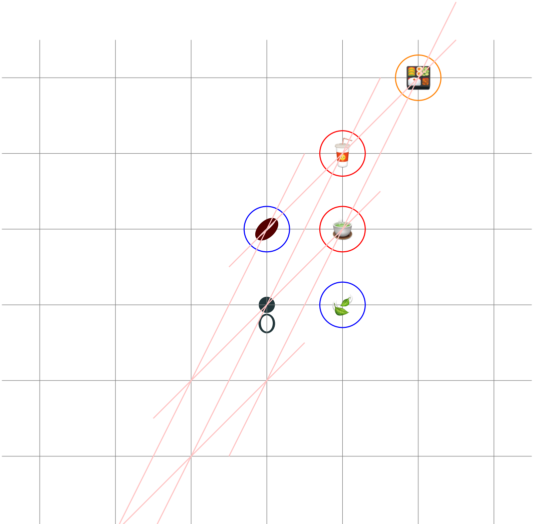


Left factor: Making drinks by materials



Right factor: Making meals by drinks

Step 3: Put them together, then you can see how to make a meal 🍱 by materials 🍃, 🫘. You got the matrix multiplication.

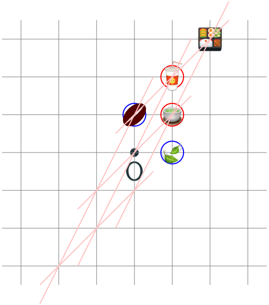


Linear Transformations

The whole process is a **map**. The domain of the map is the linear combination space of drinks. The codomain(target) of the map is the linear combination space of materials. The process skews the picture of the domain and put it to the codomain.

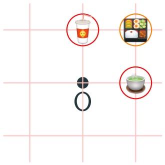
Domain: space of drinks.
Corresponds to **right factor**.

$\longrightarrow$

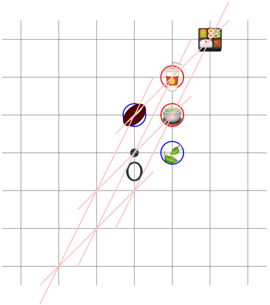


Codomain: space of materials.
Corresponds to **left factor**.

Linear Transformations



$\longrightarrow$



The only restriction for this map is that: The whole process maps parallelograms to parallelograms and it maps origin to origin. In Math, this is called a **Linear Transformation**.

Linear Transformations

Definition 1

A linear transformation is a map $T : V \longrightarrow W$ for linear spaces V, W over F such that

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \qquad T(\lambda\vec{v}) = \lambda T(\vec{v})$$

for any $\vec{v}, \vec{w} \in V$ and $\lambda \in F$.
We call V the **Domain** of T, W the **Codomain** of T.

Definition 2

The linear transformation $T : V \longrightarrow V$ in the case domain equals the codomain is called a **Linear Operator**.

Linear Transformations

$$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) \qquad T(\lambda\vec{v}) = \lambda T(\vec{v})$$

is a condition of saying keeping parallelograms.

As we have shown here, linear transformation gives a **geometric understanding** of matrix multiplication.

Proposition 1

If $T : V \longrightarrow W$ is a linear transformation, then for any $\vec{v}_1, \cdots \vec{v}_n \in V$, $a_1, \cdots, a_n \in \mathbb{R}$, we have

$$T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = a_1T(\vec{v}_1) + \cdots + a_nT(\vec{v}_n).$$

In other words, linear transformation preserves the coefficient of linear combination.

Proof.

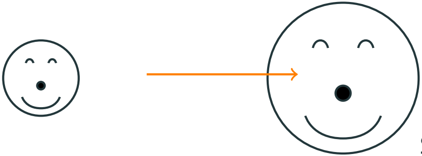
$$\begin{aligned} T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) &= T((a_1\vec{v}_1 + \cdots + a_{n-1}\vec{v}_{n-1}) + a_n\vec{v}_n) \\ &= T(a_1\vec{v}_1 + \cdots + a_{n-1}\vec{v}_{n-1}) + a_nT(\vec{v}_n) \end{aligned}$$

Then using induction $\square$

There are other geometric examples of linear transoformations.

Rotation by 90 degree.

Reflection horizontally



Scalling

Projecting

Linear Transformations

In space of functions, linear transformations happens when we **change varibles** or make **linear combination of functions**. For example, let P_∞ be the space of all polynomials. The map defined by

$$T : P_\infty \longrightarrow P_\infty, f(x) \longmapsto f(x^2)$$

is **linear**, (**Excercise.**: Check it).

Remember to check that the expression $f(x^2)$ you defined is **actually** an element in the codomain. For example, if P_2 is the sapce of all polynomials of degree at most 2. The following argument

$$T : P_2 \longrightarrow P_2, f(x) \longmapsto f(x^2)$$

is not even a map ($x^2 \mapsto x^4 \notin P_2$)

Linear Transformations

You can also define linear transformation on set of functions by **linearly combine its function values**, **derivative**, or **integration** for example

$$T : f(x) \mapsto f(1+\sqrt{x}) + f(1-\sqrt{x}); \qquad T : f(x) \mapsto xf(x) + x^2 f(x)$$

$$T : f(x) \mapsto f'(x); \qquad T : f(x) \mapsto \int_0^x f(t)\mathrm{d}t.$$

Linear space of linear transformation

Let S be a set, V a linear space, consider the set of all **maps**

$$\mathrm{Hom}_{\mathrm{set}}(S, V) := \{f | f : S \longrightarrow V\}$$

This set is **a linear space** under the following addition and scalar multiplication

For any $f, g \in \mathrm{Hom}_{\mathrm{set}}(S, V)$ and $\lambda, \mu \in F$, $s \in S$

$$(\lambda f + \mu g)(s) := \lambda f(s) + \mu g(s).$$

Excercise.: Verify $\mathrm{Hom}_{\mathrm{set}}(S, V)$ is a linear space.

Linear space of linear transformation

Excercise. Let V, W be linear spaces over F, use the following symbol to denote the set all linear transformations from W to V

$$\mathcal{L}(W, V) := \{f | f : W \longrightarrow V \text{ linear transformation } \}$$

We define addition and scalar multiplication for any $f, g \in \mathcal{L}(W, V)$ and $\lambda, \mu \in F$, $\vec{w} \in W$ by

$$(\lambda f + \mu g)(s) := \lambda f(s) + \mu g(s).$$

Verify $\mathcal{L}(W, V)$ is a linear space.

Verify a map is a linear transformation

To verify a map $T : V \longrightarrow W$ is a linear tranformation, we only need

- Write down expression of $\vec{v}_1, \vec{v}_2$ for arbitrary element $\vec{v} \in V$.
- Check the element is well-defined and it defined to be an element in codomain.
- Compute $T(\lambda\vec{v}_1 + \vec{v}_2)$ for arbitrary element $\lambda \in F$
- Compare with $\lambda T(\vec{v}_1) + T(\vec{v}_2)$.

We only verify $T(\lambda\vec{v}_1 + \vec{v}_2)$ is because the following

Proposition 2

For any map T, if $T(\lambda\vec{v}_1 + \vec{v}_2) = \lambda T(\vec{v}_1) + T(\vec{v}_2)$, this is a linear tranformation.

Proof.

$T(\lambda\vec{v}_1 + \mu\vec{v}_2) = \lambda T(\vec{v}_1) + T(\mu\vec{v}_2) = \lambda T(\vec{v}_1) + \mu T(\vec{v}_2)$ $\square$

Verify a map is a linear transformation

Excercise. Let P_2 be the space of polynomials of degree at most 2. Show that the following map

$$T : P_2 \longrightarrow P_2, f(x) \longmapsto f(1+\sqrt{x}) + f(1-\sqrt{x})$$

is a linear transformation.

Solution. Let f, g be arbitrary polynomials in P_2, so there exists unique $a, b, c, d, e, f \in F$ so f, g can be written as[1]

$$f(x) = ax^2 + bx + c \qquad g(x) = dx^2 + ex + f. \tag{1}$$

We first show that $T[f] \in P_2$. Indeed,

$$T[f](x) = f(1+\sqrt{x}) + f(1-\sqrt{x})$$

[1] The use of **parameters** a, b, c, d, e, f is an example of **constructive language**

Verify a map is a linear transformation

Plug (**??**) in, we have

$$T[f](x) = a(1+\sqrt{x})^2 + b(1+\sqrt{x}) + c + a(1-\sqrt{x})^2 + b(1-\sqrt{x}) + c$$

By computation, this expression equals to

$$a(2+x^2) + 2b + 2c \in P_2.$$

To check linear, note that for any scalar $\lambda \in f$

$$T[\lambda f+g] = \lambda f(1+\sqrt{x}) + \lambda f(1-\sqrt{x}) + g(1+\sqrt{x}) + g(1-\sqrt{x}) = \lambda T[f] + T[g]$$

So T is a linear tranformation.

Verify a map is a linear transformation

Excercise. Choose an element $s \in S$. Let V be a linear space, $\mathrm{Hom}_{\mathrm{set}}(S, V)$ is the set of maps with domain S and codomain V. Previously we proved it is a linear space. Consider the **map of evaluation at s**

$$E_s : \mathrm{Hom}_{\mathrm{set}}(S, V) \longrightarrow V, f \longmapsto f(s)$$

Prove that this is a linear transformation.

Non-linear transformations

Non-linear transformations always happens when we combine values of functions in a non-linear way, like $f(x) \mapsto f(x)^2$ or $f(x) \mapsto \sqrt{f(x)}$. To disprove linearity, we only need to choose some coefficient such that the definition of linear transformation not work.

Sometimes, we write $T[f]$ to denote the output function when apply linear transfomation of T.

Non-linear transformations

Excercise. Let V be the space of all polynomials over $\mathbb{R}$. Show that $T[f](x) = f(x)^2$ is not a linear transformation.

Solution.: Let

$$\begin{cases} f_1(x) = 1 \\ f_2(x) = x \end{cases}$$

, then

$$T[f_1 + f_2](x) = (1 + x)^2 = 1 + x^2 + 2x$$

But

$$T[f_1](x) + T[f_2](x) = 1 + x^2.$$

So

$$T[f_1 + f_2] \neq T[f_1] + T[f_2]$$

This is not a linear transformation.