Left and right inverse for Linear Transformations

We add those pages for completness, read those proofs by yourself as a homework. And try to prove it directly without using induced transformation and cancellation rule.

Left Inverse for injective

Proposition 1

$\boxed{R_I}$ Let V, W be finite dimensional vector spaces. If $T : V \longrightarrow W$ is an **injective** linear transformation , then it has a **left inverse** $R : W \longrightarrow V$ with

$$R \circ T = \mathrm{id}_V$$

Proof: Let $\mathcal{E}$ be a basis of V, since T is **injective** , then $T\mathcal{E}$ is also **linealy independent** in W(By Prop. $\boxed{C_I}$). Therefore there exists a linear transformation $R : W \longrightarrow V$ with $R(T\mathcal{E}) = \mathcal{E}$(By Prop. $\boxed{I \mapsto S}$), therefore,

$$R \circ T \circ L_{\mathcal{E}} = L_{\mathcal{E}} = \mathrm{id}_V \circ L_{\mathcal{E}}$$

Since $\mathcal{E}$ **span the whole space** , $L_{\mathcal{E}}$ is a **surjective** , use the **right cancellation** rule we have

$$R \circ T = \mathrm{id}_V.$$

Right Inverse for surjective

Proposition 2

$\boxed{R_S}$ Let V, W be finite dimensional vector spaces. If $T : V \longrightarrow W$ is an **surjective** linear transformation , then it has a **right inverse** $R : W \longrightarrow V$ with

$$T \circ R = \mathrm{id}_W$$

Proof: Let $\mathcal{E}$ be a basis of V, since T is **surjective** , then $T\mathcal{E}$ **span the whole space** W(By Prop. $\boxed{C_S}$). Let $\mathcal{F}$ be a subtuple of $\mathcal{E}$ such that $T\mathcal{F}$ is a **basis** of W. Then there exists a linear transformation $R : W \longrightarrow V$ with $R(T\mathcal{F}) = \mathcal{F}$(By Prop $\boxed{I \mapsto S}$). So $T(R(T\mathcal{F})) = T\mathcal{F}$

Now we have

$$T \circ R \circ L_{T\mathcal{F}} = L_{T\mathcal{F}} = \mathrm{id}_W \circ L_{T\mathcal{F}}$$

Since $T\mathcal{F}$ **span the whole space** , $L_{T\mathcal{F}}$ is a **surjective** , by **right cancellation** rule we have $T \circ R = \mathrm{id}_W$.