Spectural Decomposition

Recall our setting

$$F(x) = (x-\lambda_1)^{n_1} \dots (x-\lambda_i)^{n_i} \dots (x-\lambda_k)^{n_k}$$

We define $K_i(x)$ to be the **complementary factor** such that

$$F(x) = K_i(x)(x-\lambda_i)^{n_i}$$

Note that $K_i(\lambda_i) \neq 0$.

The symbol ϵ represents infinitesimal (very very small) and

$$\infty := \frac{1}{\epsilon}$$

represents infinity. We are allowing arithmetic calculation with symbols ∞ and ϵ so one can talk about $a_{-N}\infty + \cdots + a_0 + a_1\epsilon + \cdots$.

Spectural Decomposition

Recall our original formula of partial fraction decomposition

$$\frac{g(x)}{F(x)} = Q(x) + \sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot \frac{1}{\epsilon^{n_i - 1}} \cdot \frac{1}{K_i(\lambda_i + \epsilon)} \cdot \frac{1}{x - \lambda_i - \epsilon} \right)$$

We may multiply $F(x)$ on both sides and obtain

$$g(x) =$$

$$Q(x)F(x) + \sum_{i=1}^{k} \underbrace{F(x) \cdot \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot \infty^{n_i - 1} \cdot \frac{1}{K_i(\lambda_i + \epsilon)} \cdot \frac{1}{x - \lambda_i - \epsilon} \right)}_{\text{interpolation summand}}$$

Why each interpolation summand a polynomial?

Spectural Decomposition

Since

$$F(x) = \ldots + 0\infty^2 + 0\infty + F(x) + 0\epsilon + 0\epsilon^2 + \ldots$$

, we may put $F(x)$ inside, and note $F(x) = (x-\lambda_i)^{n_i} K_i(x)$ write

$$\text{Interpolation summand} = \mathrm{Const}_\epsilon \left(g(\lambda_i+\epsilon)\cdot\infty^{n_i-1}\cdot\frac{K_i(x)}{K_i(\lambda_i+\epsilon)}\cdot\frac{(x-\lambda_i)^{n_i}}{x-\lambda_i-\epsilon}\right)$$

At this far, since x appears in denomenator, we have no idea why this is a polynomial.

However, consider

$$\mathrm{Const}_\epsilon \left(\overbrace{\underbrace{g(\lambda_i+\epsilon)}_{=a_0+a_1\epsilon+\cdots}\cdot\infty^{n_i-1}\cdot K_i(x)\cdot\frac{1}{\underbrace{K_i(\lambda_i+\epsilon)}_{=b_0+b_1\epsilon+\ldots;}}\cdot\frac{\epsilon^{n_i}}{\underbrace{x-\lambda_i-\epsilon}_{=\epsilon^{n_i}+\frac{\epsilon^{n_i+1}}{x-\lambda_i}+\ldots;}}}^{=a_0 b_0 K_i(x)\epsilon+*\epsilon^2+*\epsilon^3+\cdots}\right) = 0$$

Spectural Decomposition

Subtract one equation from another, you see that the whole term is indeed a polynomial.

Interpolation summand =

$$\mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot \infty^{n_i - 1} \cdot \frac{K_i(x)}{K_i(\lambda_i + \epsilon)} \cdot \underbrace{\frac{(x - \lambda_i)^{n_i} - \epsilon^{n_i}}{x - \lambda_i - \epsilon}}_{\text{polynomial}} \right)$$

$$\text{Hint :} \qquad \frac{A^n - B^n}{A - B} = A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \cdots + B^{n-1}$$

Spectural Decomposition

Definition 1

Define the polynomial (involving ϵ and ∞ as coefficients)

$$f_{\lambda_i}(x) = \infty^{n_i - 1} \cdot \frac{K_i(x)}{K_i(\lambda_i + \epsilon)} \cdot \frac{(x-\lambda_i)^{n_i} - \epsilon^{n_i}}{x - \lambda_i - \epsilon}$$

and call it the **infinite interpolation polynomial at λ_i for $F(x) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k}$**

Proposition 1

We have infinite Lagurange interpolation theroem

$$g(x) = Q(x)F(x) + \sum_{i=1}^{k} \mathrm{Const}_{\epsilon}(g(\lambda_i + \epsilon) f_{\lambda_i}(x)).$$

In the previous argument, we are using the idea that small enough scalars does not change the computation of constant part

$$\mathrm{Const}((a_n\infty^n+\ldots+a_0+\underbrace{a_{-1}\epsilon+\ldots}_{\text{does not affect}})(b_0+b_1\epsilon+\ldots+b_n\epsilon^n+\underbrace{b_{n+1}\epsilon^{n+1}+\ldots}_{\text{does not affect}}))$$

$$= a_n b_n + a_{n-1} b_{n-1} + \cdots + a_0 b_0.$$

In other words, we do not need take care of all coefficient.

Big O notation

To make the notation lighter, we write any terms like

$$a_n\epsilon^n + a_{n+1}\epsilon^{n+1} + a_{n+2}\epsilon^{n+2}\cdots \sim O(\epsilon^n)$$

In other words, we may write both $\epsilon + 3\epsilon^2$ and $\epsilon + 5\epsilon^2 + 6\epsilon^3$ as $\epsilon + O(\epsilon^2)$. The notation $O(\epsilon^n)$ introduces some ambiguity.

The infinite scalar can be thought of some expression as

$$a_n\infty^n + \cdots + a_0 + O(\epsilon).$$

The multiplication of formal and infinite scalar has already determined up to ϵ^n and nothing to do with ϵ^{n+1}

$$\mathrm{Const}\left((a_n\infty^n + \cdots + a_0 + O(\epsilon))(b_0 + b_1\epsilon + \cdots + b_n\epsilon^n + O(\epsilon^{n+1}))\right)$$

the result has determined and has nothing to do with $O(\epsilon)$ of first factor and $O(\epsilon^{n+1})$ of second factor.

Algorithms of big O

The big O notation follows the rule

$$O(\epsilon^n) \pm O(\epsilon^m) \sim O(\epsilon^{\min\{n,m\}})$$

$$O(\epsilon^n) \cdot O(\epsilon^m) \sim O(\epsilon^{n+m})$$

Algorithms of big O

Intuitively, we don't care $O(\epsilon)$ since it vanishes as $\epsilon \longrightarrow 0$, the infinitesimal part will have no influence of the limit at all.

Algorithms of big O

Definition 2

We call

$$a_{-n}\infty^n + a_{-n+1}\infty^{n-1} + \cdots + a_0 + O(\epsilon)$$

an **infinite scalar**, and when the context is clear, we simply write it as

$$a_{-n}\infty^n + a_{-n+1}\infty^{n-1} + \cdots + a_0$$

Algorithms of big O

!!

Each Laurent scalar corresponds to a unique infinite scalar, we denote such correspondence by

$$
\begin{aligned}
& a_{-n}\infty^n + a_{-n+1}\infty^{n-1} + \cdots + a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots \\
\equiv \quad & a_{-n}\infty^n + a_{-n+1}\infty^{n-1} + \cdots + a_0 + O(\epsilon)
\end{aligned}
$$

Algorithms of big O

Definition 3

Two different Laurent scalars α, β might corresponds to the same infinite scalar. When this happens, we simply denote by $\alpha \equiv \beta$.

Definition 4

Further abuse the notation, no matter α or β is infinite scalars or Laurent scalars, by $\alpha \equiv \beta$ we always mean they are the same as infinite scalars.

$$3\infty + 2 + \epsilon \equiv 3\infty + 2 + 3\epsilon + 2\epsilon^2 \equiv 3\infty + 2 + O(\epsilon).$$

Algorithms of big O

Because infinite scalars has ambiguity of $O(\epsilon)$, we **only allow infinite scalar to add or subtract** since $O(\epsilon) \pm O(\epsilon) = O(\epsilon)$, **multiplication or division** is not allowed.

Nevertheless, we can still multiply a formal scalar to an infinite scalar, the result is another infinite scalar.

$$(2 + 3\epsilon)(3\infty + 2) \equiv 6\infty + 13.$$

Algorithms of big O

$$\text{Infinite scalar} + \text{Infinite scalar} \equiv \text{Infinite scalar}$$

$$\text{Infinite scalar} - \text{Infinite scalar} \equiv \text{Infinite scalar}$$

$$\text{Infinite scalar} \times \text{Infinite scalar} = \text{NOT ALLOWED}$$

$$\text{Infinite scalar} \div \text{Infinite scalar} = \text{NOT ALLOWED}$$

$$\text{Formal scalar} + \text{Formal scalar} = \text{Formal scalar}$$

$$\text{Formal scalar} - \text{Formal scalar} = \text{Formal scalar}$$

$$\text{Formal scalar} \times \text{Formal scalar} = \text{Formal scalar}$$

$$\text{Formal scalar} \div \underbrace{\text{Formal scalar}}_{\text{non-zero}} = \text{Laurent scalar}$$

$$\text{Formal scalar} \times \text{Infinite scalar} \equiv \text{Infinite scalar}$$

$$\underbrace{\text{Laurent scalar}}_{\text{that not a formal scalar}} \times \text{Infinite scalar} = \text{NOT ALLOWED}$$

Degree of infinite scalar

Definition 5

Call an infinite scalar

$$a_n\infty^n + a_{n-1}\infty^{n-1} + \cdots + a_0$$

of degree n if $a_n \neq 0$.

Degree of infinite scalar

Definition 6

An infinite matrix is a matrix of infinite scalars. Define the degree of such matrix to be the maximal degree of its entries.

Definition 7

An infinite vector is a $n \times 1$ infinite matrix.

Proposition 2

The set of $m \times n$ infinite matrix is closed under addition, subtraction, and formal scalar multiplication. But we can not define product of two infinite matrix!

Degree of infinite scalar

Recall the infinite interpolation formula

$$f_{\lambda_i}(x) = \infty^{n_i - 1} \cdot \frac{K_i(x)}{K_i(\lambda_i + \epsilon)} \cdot \frac{(x - \lambda_i)^{n_i} - \epsilon^{n_i}}{x - \lambda_i - \epsilon}$$

$$g(x) = Q(x)F(x) + \sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot f_{\lambda_i}(x)\right)$$

Let A be a matrix with $F(A) = 0$, we have specutral decomposition

Proposition 3

$$g(A) = \sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i}\right)$$

where

$$\mathscr{P}_{\lambda_i} := f_{\lambda_i}(A)$$

Degree of infinite scalar

Recall our settings

- $F(x) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k} = K_i(x)(x-\lambda_i)^{n_i}$ for all i.
- $F(A) = 0$
- $f_{\lambda_i}(x) = \infty^{n_i - 1} \cdot \frac{K_i(x)}{K_i(\lambda_i+\epsilon)} \cdot \frac{(x-\lambda_i)^{n_i} - \epsilon^{n_i}}{x-\lambda_i-\epsilon}$
- $\mathscr{P}_{\lambda_i} = f_{\lambda_i}(A)$

Degree of infinite scalar

Our goal in this lecture:

1. We have
$$A\mathscr{P}_{\lambda_i} \equiv \mathscr{P}_{\lambda_i} A \equiv (\lambda_i + \epsilon)\mathscr{P}_{\lambda_i}$$
Later on we call such thing an **infinite eigenmatrix** of A with eigenvalue $\lambda_i + \epsilon$.
2. We have
$$\mathscr{P}_{\lambda_i} \equiv P_{\lambda_i} + N_{\lambda_i}\infty + N_{\lambda_i}^2\infty^2 + \cdots + N_{\lambda_i}^{n_i-1}\infty^{n_i-1}$$
for some $P_{\lambda_i}^2 = P_{\lambda_i}$, $N_{\lambda_i}^{n_i} = 0$ and $N_{\lambda_i}P_{\lambda_i} = P_{\lambda_i}N_{\lambda_i}$. Later on, we will call such thing **infinite projection matrix**.
3. We have
$$P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_k} = I.$$

Degree of infinite scalar

Let us prove the first, note that our goal is equivalent to show

$$(A-(\lambda_i+\epsilon)I)\mathscr{P}_{\lambda_i} \equiv \mathscr{P}_{\lambda_i}(A-(\lambda_i+\epsilon)I) \equiv 0.$$

Note that

$$\cancel{(x-(\lambda_i+\epsilon))} \cdot \underbrace{\infty^{n_i-1} \cdot \frac{K_i(x)}{K_i(\lambda_i+\epsilon)} \cdot \frac{(x-\lambda_i)^{n_i}-\epsilon^{n_i}}{\cancel{x-\lambda_i-\epsilon}}}_{f_{\lambda_i}(x)}$$

$$= \infty^{n_i-1} \cdot \frac{K_i(x)\cdot((x-\lambda_i)^{n_i}-\epsilon^{n_i})}{K_i(\lambda_i+\epsilon)}$$

$$= \frac{\infty^{n_i-1}\cdot \overbrace{K_i(x)\cdot(x-\lambda_i)^{n_i}}^{=F(x)}}{K_i(\lambda_i+\epsilon)} - \frac{\overbrace{K_i(x)\cdot\infty^{n_i-1}\cdot\epsilon^{n_i}}^{\equiv 0}}{K_i(\lambda_i+\epsilon)}$$

Degree of infinite scalar

Conclusion: $(x-(\lambda_i+\epsilon))f_{\lambda_i}(x) \equiv \infty^{n_i-1}F(x)/K_i(\lambda_i+\epsilon)$

Since $F(A)=0$, $\mathscr{P}_{\lambda_i}=f_{\lambda_i}(A)$, plug in $x=A$,

$$(A-(\lambda_i+\epsilon))f_{\lambda_i}(A) \equiv \infty^{n_i-1}F(A)/K_i(\lambda_i+\epsilon) = \infty^{n_i-1}0/K_i(\lambda_i+\epsilon) = 0$$

we finish the proof.

Degree of infinite scalar

Definition 8

We call an infinite square matrix $\mathscr{P}$ an **infinite eigenmatrix** of eigenvalue $\lambda + \epsilon$ of A if

$$A\mathscr{P} \equiv \mathscr{P}A \equiv (\lambda + \epsilon)\mathscr{P}.$$

Definition 9

We call a **non-zero** infinite vector $\vec{v}$ the **infinite eigenvector** of eigenvalue $\lambda + \epsilon$ if

$$A\vec{v} \equiv (\lambda + \epsilon)\vec{v}.$$

Degree of infinite scalar

Example.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \vec{v} \equiv \begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

is an infinite eigenvector of eigenvalue $1+\epsilon$. Indeed

$$A\vec{v} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \infty \\ 1 \end{pmatrix} \equiv \begin{pmatrix} \infty + 1 \\ 1 \end{pmatrix} \equiv (1+\epsilon) \cdot \begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

Degree of infinite scalar

Infinite eigenvector are generalizations of eigenvector:

!!

- A (classical) eigenvector is an infinite eigenvector of degree 0.
- A constant matrix A has a **non-classical infinite eigenvector** of eigenvalue $\lambda + \epsilon$ ($\lambda \in \mathbb{C}$) if and only if it is **non-digaonalizable**.(Try to prove it yourself.)

Degree of infinite scalar

Scaling property

Proposition 4

Any formal scalar multiple of infinite eigen vector $\mu\vec{v}$ is either zero vector or an infinite eigenvector of the same eigenvalue.

$$A\vec{v} = (\lambda + \epsilon)\vec{v} \implies A\mu\vec{v} = \mu A\vec{v} = \mu(\lambda + \epsilon)\vec{v} = (\lambda + \epsilon)\mu\vec{v}$$

Degree of infinite scalar

Example:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \infty \\ 1 \end{pmatrix} \equiv (1+\epsilon) \begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

Infinite eigenvector of degree 1

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv (1+\epsilon) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Classical eigenvector;
Infinite eigenvector of degree 0

Relation of these two infinite eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \underbrace{\epsilon}_{\substack{\text{formal} \\ \text{scalar} \\ \text{multiple}}} \begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

Degree of infinite scalar

The appearance of infinite eigenvectors indicates that there are multiple eigenspaces repeated to each other. And the infinite eigenvector have some cosins.

Philosophy Recall the theory of interpolation polynomials, the appearance of interpolation polynomial

$$f_{\lambda_i}(x) = *\infty^{n_i - 1} + *\infty^{n_i - 2} + \ldots + *\infty + *$$

indicates the existence of repeated root of multiplicity n_i, and f_{λ_i} has $n_i - 1$ other cosins, when summing them up, **the infinity cancels** and it gives a finite value $\mathrm{Const}(f_{\lambda_i})$.

Degree of infinite scalar

How to understand the infinite eigenvector ? Let′s introduce a variable a.

$$\begin{pmatrix} 1+a & 1 \\ 0 & 1 \end{pmatrix}$$

When $a = 0$, the matrix is NOT diagonalizable and results an infinite eigenvector

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \infty \\ 1 \end{pmatrix} = (1+\epsilon) \begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

However, **when $a \neq 0$**, the matrix is **always diagonalizable** and admits a spectural decomposition

$$g \begin{pmatrix} 1+a & 1 \\ 0 & 1 \end{pmatrix} = g(1+a) \begin{pmatrix} 1 & \frac{1}{a} \\ 0 & 0 \end{pmatrix} + g(1) \begin{pmatrix} 0 & -\frac{1}{a} \\ 0 & 1 \end{pmatrix}.$$

Degree of infinite scalar

To visualize the change of eigenvector along a, we fix a vector and look its eigenvector decomposition (which is unique)

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{a} \\ 0 \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =1+a}} + \underbrace{\begin{pmatrix} -\frac{1}{a} \\ 1 \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =1}}$$

Degree of infinite scalar

Let′s visualize what happens when $a \longrightarrow 0$ The blue vector is our fixed vector, and red vector is its decomposition as eigenvectors

This is what happend for $a = 0.1$

This is what happend for $a = 0.05$

This is what happend for $a = 0.03$

When $a = 0$, **eigenvectors goes to infinities** in opposite direction. **But the sum** of this two eigenvector **is a finite vector**, which is our fixed one.

Degree of infinite scalar

The supper big red vector is described by infinite eigenvector

$$\begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

The sum of these two supper big eigenvector, is described as what we known as the constant part

$$\text{Const} \begin{pmatrix} \infty \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Degree of infinite scalar

From the perspective of eigenspace, as $a \longrightarrow 0$, the eigenspaces are running close to each other.

This is what happend for $a = 0.1$

This is what happend for $a = 0.05$

This is what happend for $a = 0.03$

This is what happend for $a = 0.01$

Degree of infinite scalar

We now answer the second question, determine the structure of $\mathscr{P}_{\lambda_i}$.

Recall that

$$f_{\lambda_i}(x) = \infty^{n_i - 1} \cdot \frac{K_i(x)}{K_i(\lambda_i + \epsilon)} \cdot \frac{(x - \lambda_i)^{n_i} - \epsilon^{n_i}}{x - \lambda_i - \epsilon},$$

and

$$\mathscr{P}_{\lambda_i} := f_{\lambda_i}(A).$$

the coefficients of f_{λ_i} has ∞-degree at most $n_i - 1$.

Therefore, $\mathscr{P}_{\lambda_i}$ is a matrix with ∞-degree at most $n_i - 1$, we have

$$\mathscr{P}_{\lambda_i} \equiv P_0 + N_1 \infty + N_2 \infty^2 + \ldots + N_{n_i - 1} \infty^{n_i - 1}$$

for some constant matrices $P_0, N_1, \ldots, N_{n_i - 1}$. Now we determine the properties of these matrices.

Degree of infinite scalar

Using spectral decomposition, we obtain two expressions for $g(A)h(A)$

$$\underbrace{\sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot h(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i} \right)}_{g(A)h(A)} =$$

$$\underbrace{\left(\sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i} \right) \right)}_{g(A)} \cdot \underbrace{\left(\sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(h(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i} \right) \right)}_{h(A)}$$

Degree of infinite scalar

Let α,β be two arbitrary formal scalars, put

	$\lambda_1 + \epsilon$	$\lambda_2 + \epsilon$	$\cdots$	$\lambda_i + \epsilon$	$\cdots$	$\lambda_k + \epsilon$
$g(x)$	$0 + O(\epsilon^{n_1})$	$0 + O(\epsilon^{n_2})$	$\cdots$	$\alpha + O(\epsilon^{n_i})$	$\cdots$	$0 + O(\epsilon^{n_k})$
$h(x)$	$0 + O(\epsilon^{n_1})$	$0 + O(\epsilon^{n_2})$	$\cdots$	$\beta + O(\epsilon^{n_i})$	$\cdots$	$0 + O(\epsilon^{n_k})$
$g(x)h(x)$	$0 + O(\epsilon^{n_1})$	$0 + O(\epsilon^{n_2})$	$\cdots$	$\alpha\beta + O(\epsilon^{n_i})$	$\cdots$	$0 + O(\epsilon^{n_k})$

From the table,

$$g(A) = \sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i}\right) = \mathrm{Const}_\epsilon \left(\alpha \cdot \mathscr{P}_{\lambda_i}\right)$$

$$h(A) = \sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(h(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i}\right) = \mathrm{Const}_\epsilon \left(\beta \cdot \mathscr{P}_{\lambda_i}\right)$$

$$g(A)h(A) = \sum_{i=1}^{k} \mathrm{Const}_\epsilon \left(g(\lambda_i + \epsilon)h(\lambda_i + \epsilon) \cdot \mathscr{P}_{\lambda_i}\right) = \mathrm{Const}_\epsilon \left(\alpha\beta \cdot \mathscr{P}_{\lambda_i}\right)$$

Degree of infinite scalar

The above argument proves that

$$\mathrm{Const}_\epsilon\left(\alpha \cdot \mathscr{P}_{\lambda_i}\right)\mathrm{Const}_\epsilon\left(\beta \cdot \mathscr{P}_{\lambda_i}\right) = \mathrm{Const}_\epsilon\left(\alpha\beta \cdot \mathscr{P}_{\lambda_i}\right)$$

for any formal scalar α,β.

Definition 10

We call an infinite matrix $\mathscr{P}$ an **infinite projection** matrix, if

$$Const(x\mathscr{P})Const(y\mathscr{P}) = Const(xy\mathscr{P})$$

for any **formal scalar** x, y

Degree of infinite scalar

Definition 11

A constant matrix P is called **projection** if $P^2 = P$.

!!

To see analogue with infinite projection, $P^2 = P$ is equivalent to $(aP)(bP) = (abP)$.

Definition 12

A constant matrix N is called **nilpotent** if $N^k = 0$ for some integer k.

Degree of infinite scalar

Theorem 1

An infinite matrix $\mathscr{P}$ of degree at most $m-1$ is an infinite projection matrix if and only if

$$\mathscr{P} = P + N\infty + N^2\infty^2 + \cdots + N^{m-1}\infty^{m-1}$$

with $P^2 = P$, $PN = NP = N$, and $N^m = 0$.

Proof: If $\mathscr{P}$ is an infinite projection, assume

$$\mathscr{P} = P + N\infty + N_2\infty^2 + \cdots + N_{m-1}\infty^{m-1}$$

$$PP = \mathrm{Const}(1\cdot\mathscr{P})\mathrm{Const}(1\cdot\mathscr{P}) = \mathrm{Const}(1\cdot 1\cdot\mathscr{P}) = P.$$

$$PN = \mathrm{Const}(1\cdot\mathscr{P})\mathrm{Const}(\epsilon\cdot\mathscr{P}) = \mathrm{Const}(1\cdot\epsilon\cdot\mathscr{P}) = \mathrm{Const}(\epsilon\cdot\mathscr{P}) = N.$$

$$NP = \mathrm{Const}(\epsilon\cdot\mathscr{P})\mathrm{Const}(1\cdot\mathscr{P}) = \mathrm{Const}(\epsilon\cdot 1\cdot\mathscr{P}) = \mathrm{Const}(\epsilon\cdot\mathscr{P}) = N.$$

$$N_k = \mathrm{Const}(\epsilon^k\mathscr{P}) = \mathrm{Const}(\epsilon\mathscr{P})^k = N^k \implies N_k = N^k$$

$$N^m = \mathrm{Const}(\epsilon\cdot\mathscr{P})^m = \mathrm{Const}(\epsilon^m\cdot\mathscr{P}) = 0.$$

Degree of infinite scalar

Proof of the other direction: Suppose $P^2 = P$, $PN = NP = N$, and $N^m = 0$, and

$$\mathscr{P} = P + N\infty + N^2\infty^2 + \cdots + N^{m-1}\infty^{m-1}$$

To prove

$$\begin{aligned}
&\mathrm{Const}(\underbrace{(a_0 + a_1\epsilon + \ldots + O(\epsilon^m))}_{\alpha}\mathscr{P})\mathrm{Const}(\underbrace{(b_0 + b_1\epsilon + \ldots + O(\epsilon^m))}_{\beta}\mathscr{P}) \\
&= \mathrm{Const}(\underbrace{(a_0 + a_1\epsilon + \ldots + O(\epsilon^m))(b_0 + b_1\epsilon + \ldots + O(\epsilon^m))}_{\alpha\beta}\mathscr{P})
\end{aligned}$$

It suffices to prove

$$\mathrm{Const}(\epsilon^k\mathscr{P})\mathrm{Const}(\epsilon^n\mathscr{P}) = \mathrm{Const}(\epsilon^{k+n}\mathscr{P})$$

for $m, n \geq 0$. This is the same as proving

$$P^2 = P, PN^n = N^n, N^kP = N^k, N^kN^n = N^{k+n}, N^m = 0,$$

which is obviously true.

Degree of infinite scalar

!!

A classical projection is an infinite projection $\mathscr{P} = P$

$$\mathscr{P} = P + N\infty + N^2\infty^2 + \cdots + N^{m-1}\infty^{m-1}$$

with the case $N = 0$.

Therefore, the appearence of non-zero nilpotent matrix indicates the eigenspace projection could go to infinity, making non-classical infinite eigenvectors possible, and therefore implies the matrix non-diagonalizable.

Degree of infinite scalar

Recall that when A satisfies $(A-\lambda_1)^{n_1}\dots(A-\lambda_k)^{n_k}=0$, our method to check diagonalizability of A is by checking whether

$$(A-\lambda_1)\dots(A-\lambda_k)=0.$$

The reason behind it is because that

$$(A-\lambda_1)^{n_1}\dots(A-\lambda_k)^{n_k}=0 \implies (A-\lambda_1)\dots(A-\lambda_k) \text{ nilpotent.}$$

Intuitively, if the nilpotent part $=0$, we were in the situation that no eigenspace projection goes to infinity, and therefore the matrix is diagonalizable.

Degree of infinite scalar

The philosophy that nilpotent matrix is the infinite part of infinite projection seems deep. Let′s consider this.

It is true that

$$\frac{wv^T}{v^T w}$$

is always a rank 1 projection matrix. What happens if $v^T w = 0$? Then the denomenator is 0, the whole matrix goes to infinity. However, when $v^T w = 0$, the matrix wv^T must be nilpotent since

$$(wv^T)^2 = wv^T wv^T = w \underbrace{(v^T w)}_{=0} v^T = 0.$$

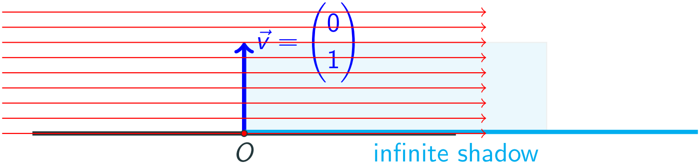
Therefore, when $v^T w = 0$, you may view this as a

$$\underbrace{\frac{1}{v^T w}}_{\substack{\text{scalar} \\ \text{roughly } \infty}} \cdot \underbrace{wv^T}_{\text{nilpotent matrix}}$$

This gives you an idea why nilpotency is an inifnite part of infinite projection.

Degree of infinite scalar

Sun set: Please visualize infinite projection $\mathscr{P}$ in the following picture.



The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is on the groud, so its shadow is itself. We have

$$\mathscr{P} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stands up, and its shadow goes to infinity, so

$$\mathscr{P} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \infty \\ 1 \end{pmatrix}$$

Degree of infinite scalar

Therefore, this infinite projection takes the form

$$\mathscr{P} = \mathscr{P}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \infty \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{projection matrix}} + \infty \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{nilpotent matrix}} .$$

Degree of infinite scalar

Excercise. For $\mathscr{P}_{\lambda_i} = P_{\lambda_i} + N_{\lambda_i}\infty + \ldots + N_{\lambda_i}^{n_i-1}\infty^{n_i-1}$. What is the relation between A, P_{λ_i} and N_{λ_i}?

Solution.$A\mathscr{P}_{\lambda_i} = (\lambda_i + \epsilon)\mathscr{P}$

So $(A - \lambda_i)\mathscr{P}_{\lambda_i} = \epsilon\mathscr{P}_{\lambda_i}$. Taking the constant part on both sides,

$$(A - \lambda_i I)\mathrm{Const}(\mathscr{P}_{\lambda_i}) = \mathrm{Const}(\epsilon\mathscr{P}_{\lambda_i})$$

so

$$(A - \lambda_i I)P_{\lambda_i} = N_{\lambda_i}$$

Infinite eigenvector decomposition

We have left the last property

$$P_{\lambda_1} + P_{\lambda_2} + \ldots + P_{\lambda_k} = I.$$

This is easist property by taking $g(x) = 1$, then

$$I = g(A) = \sum_{i=1}^{k} (g(\lambda_i + \epsilon)\mathscr{P}_{\lambda_i}) = \sum_{i=1}^{k} (\mathscr{P}_{\lambda_i}) = \sum_{i=1}^{k} P_{\lambda_i}.$$

Infinite eigenvector decomposition

Theorem 2

For $(A-\lambda_1)^{n_1}\dots(A-\lambda_k)^{n_k}=0$, there are infinite-matrix $\mathscr{P}_{\lambda_i}$ with

1. $\mathscr{P}_{\lambda_i}$ is infinite eigenmatrix of eigenvalue $\lambda_i+\epsilon$
$$A\mathscr{P}_{\lambda_i}\equiv\mathscr{P}_{\lambda_i}A\equiv(\lambda_i+\epsilon)\mathscr{P}_{\lambda_i}$$
2. $\mathscr{P}_{\lambda_i}$ is infinite projection
$$\mathscr{P}_{\lambda_i}\equiv P_{\lambda_i}+N_{\lambda_i}\infty+N_{\lambda_i}^2\infty^2+\cdots+N_{\lambda_i}^{n_i-1}\infty^{n_i-1}$$
3. We have
$$P_{\lambda_1}+P_{\lambda_2}+\cdots+P_{\lambda_k}=I.$$
4. We have for any polynomial g,
$$g(A)=\sum^{k}\mathrm{Const}(g(\lambda_i)\mathscr{P}_{\lambda_i}).$$

Infinite eigenvector decomposition

In this lecture, we introduce two big picture:

- **Infinite eigenvector decomposition**: Decomposing every vector into infinite eigenvectors.
- **Infinite eigenbasis**: Fix some infinite eigenvectors so that every vector can be represents as their linear combination.

Review classical decomposition

We review eigenvector decomposition in diagonalizable cases. Let us just say a matrix

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}$$

with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Then every vector can be decomposed into eigenvectors of A

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \underbrace{\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =\lambda_1}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ c \\ 0 \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =\lambda_2}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =\lambda_3}}$$

This is eigenvector decomposition.

Review classical decomposition

However, eigenbasis means a fixed eigen vector, and we are writing any vector into linear combination of them. For example, the list

$$\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{\substack{\text{eigenvalue}\\=\lambda_1}}, \quad \underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\substack{\text{eigenvalue}\\=\lambda_1}}, \quad \underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{\substack{\text{eigenvalue}\\=\lambda_2}}, \quad \underbrace{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\substack{\text{eigenvalue}\\=\lambda_3}}$$

is an eigenbasis and we gonna write any vector by linear combination of them

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} a + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} b + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} c + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} d$$

Review classical decomposition

For non-diagonalizable matrices, its anihilating polynomial have repeated roots. We imagine its roots come together as clusters.

It is hard to seperate them, so we should not consider just one eigenvalue, consider them together.

Review classical decomposition

Let′s come back tO easieR diAgonalizaBle CAsEs

Review classical decomposition

In the above decomposition,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \underbrace{\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =\lambda_1}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ c \\ 0 \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =\lambda_2}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix}}_{\substack{\text{eigenvalue} \\ =\lambda_3}}$$

We can say

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} \quad \text{the } \lambda_1 - \text{ component} \qquad \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \end{pmatrix} \quad \text{the } \lambda_2 - \text{ component}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} \quad \text{the } \lambda_3 - \text{ component}$$

Review classical decomposition

$$\begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} \qquad \text{the}(\lambda_1, \lambda_2)\text{−component} \qquad \begin{pmatrix} a \\ b \\ 0 \\ d \end{pmatrix} \qquad \text{the}(\lambda_1, \lambda_3)\text{−component}$$

$$\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix} \qquad \text{the}(\lambda_2, \lambda_3)\text{−component} \qquad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \qquad \text{the}(\lambda_1, \lambda_2, \lambda_3)\text{−component}$$

Note that multivalue component are no longer eigenvectors!

Review classical decomposition

The idea of multi-eigenvalue component is that when eigenvalue are given

$$(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Any partitioN oF the above list would give a decomposition, where $(\lambda_i, \lambda_j, ..., \lambda_k)$-component means a vector $\vec{w}$ with

$$(A - \lambda_i I)(A - \lambda_j I) \ldots (A - \lambda_k I)\vec{w} = \vec{0}.$$

Review classical decomposition

λ_1-component means $(A - \lambda_1 I)\vec{v} = \vec{0}$

(λ_1, λ_2)-component means $(A - \lambda_1 I)(A - \lambda_2 I)\vec{v} = \vec{0}$

Review classical decomposition

For $\mathscr{P}_{\lambda_i} = P_{\lambda_i} + N_{\lambda_i}\infty + \ldots + N_{\lambda_i}^{n_i-1}\infty^{n_i-1}$.

Any classical vector $\vec{v}$ can be decomposed into

$$\vec{v} = I\vec{v} = P_{\lambda_1}\vec{v} + P_{\lambda_2}\vec{v} + \ldots + P_{\lambda_k}\vec{v}.$$

Note that

$$(A - \lambda_i I)^{n_i} P_{\lambda_i} = (A - \lambda_i I)^{n_i}\mathrm{Const}(\mathscr{P}_{\lambda_i}) = \mathrm{Const}((A - \lambda_i I)^{n_i}\mathscr{P}_{\lambda_i})$$

$$= \mathrm{Const}(\epsilon^{n_i}\mathscr{P}_{\lambda_i}) = 0.$$

Therefore,

$$P_{\lambda_i}\vec{v}$$

is an $(\underbrace{\lambda_i, \lambda_i, \ldots \lambda_i}_{n_i \text{many}})$-component of $\vec{v}$

Review classical decomposition

Similar to diagonalizable cases, in the decomposition

$$\vec{v} = I\vec{v} = P_{\lambda_1}\vec{v} + P_{\lambda_2}\vec{v} + \ldots + P_{\lambda_k}\vec{v}.$$

Each $P_{\lambda_i}\vec{v}$ is not an eigenvector if $n_i \geq 2$. Instead, it represents the sum of certain eigenvectors. In the non-diagonalizable cases, it can be viewed as the sum of infinite eigenvectors, and it is finite since the infinite part has been canceled.

Review classical decomposition

To decompose further, we may see

$$\underbrace{\underbrace{P_{\lambda_i}\vec{v}}_{(\lambda_i,\lambda_i,\ldots,\lambda_i)-\text{component}}}_{\text{Not an eigenvector if } n_i\geq 2} = \overbrace{\text{Const}(\underbrace{\mathscr{P}_{\lambda_i}\vec{v}}_{\substack{\lambda_i-\text{component}\\ \text{eigenvector}}})}^{\text{the sum of each } \lambda_i-\textit{component}}$$

$$\underbrace{(\lambda_i,\lambda_i,\ldots,\lambda_i)}_{n_i-\text{many}}$$

Taking constant part of an infinite vector is analogue of summing up infinite eigenvector of **the same, but infinitesimally different** eigenvalues. This euqaion represents the decomposition of $(\lambda_i,\lambda_i,\ldots,\lambda_i)$-component to λ_i-component, which is a true infinite eigenvector.

Review classical decomposition

Therefore, we regard

$$\vec{v} = \mathrm{Const}(\mathscr{P}_{\lambda_1}\vec{v}) + \mathrm{Const}(\mathscr{P}_{\lambda_2}\vec{v}) + \ldots + \mathrm{Const}(\mathscr{P}_{\lambda_k}\vec{v})$$

as the **infinite eigenvector decomposition** of any vector $\vec{v}$.

Review classical decomposition

Review In diagonalization theory, we also have a theorem

Theorem 3

Suppose A is diagonalizable matrix, eigenvectors with different eigenvalues must be linearly independent.

The proof is easy, since we have spectural decomposition

$$P_{\lambda_1} + P_{\lambda_2} + \ldots + P_{\lambda_k} = I$$

Suppose $A\vec{v} = \lambda_i \vec{v} \implies P_{\lambda_j}\vec{v} = \vec{0}$ for $i \neq j$ because the construction of P_{λ_j} has the factor of $(A - \lambda_i I)$. Therefore only $P_{\lambda_i}\vec{v} = I\vec{v} = \vec{v}$.

Suppose $\vec{v}_{\lambda_1}, \ldots, \vec{v}_{\lambda_k}$ are eigenvectors of eigenvalues $\lambda_1, \ldots \lambda_k$.

$$a_1\vec{v}_{\lambda_1} + a_2\vec{v}_{\lambda_2} + \ldots + a_k\vec{v}_{\lambda_k} = \vec{0}.$$

$$a_i\vec{v}_{\lambda_i} = P_{\lambda_i}\vec{0} = \vec{0}. \implies a_i = 0.$$

Review classical decomposition

To summarise, if $A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i}$, then

$$a_1 \vec{v}_{\lambda_1} + \cdots + a_k \vec{v}_{\lambda_k} = \vec{0} \implies a_i \vec{v}_{\lambda_i} = \vec{0} \text{ for all } i.$$

Review classical decomposition

We carry out the similar situation.

Lemma 1

Suppose $\vec{v}$ is an infinite eigenvector of A with eigenvalue $\lambda + \epsilon$, then

$$\text{Const}(\vec{v}) = 0 \iff \vec{v} \equiv 0.$$

Understanding(not proof):Const($\vec{v}$) may be understood as the sum of the inifinte eigenvector with its friends, different friend have slightly different eigenvalue, so the sum is zero implies each individual is zero.

Review classical decomposition

Proof: If $\vec{v} \equiv 0$, then clearly $\mathrm{Const}(\vec{v}) = 0$.

On the contary, suppose $\vec{v} = \vec{v}_0 + \vec{v}_1\infty + \vec{v}_2\infty^2 + \dots$

$$\vec{v}_0 = \mathrm{Const}(\vec{v}) = \vec{0}$$

$$\implies \quad \vec{v}_n = \mathrm{Const}(\epsilon^n \vec{v}) = \mathrm{Const}((A - \lambda I)^n \vec{v}) = (A - \lambda I)^n \mathrm{Const}(\vec{v})$$

$$= (A - \lambda I)^n \vec{0} = \vec{0}.$$

Therefore $\vec{v} = \vec{0} + \vec{0}\infty + \vec{0}\infty^2 + \dots = \vec{0}$.

Review classical decomposition

Now we proceed to **infinite eigenbasis**.

Definition 13

An **infinite eigenbasis** is a list of infinite eigenvectors

$$\vec{w}_{\lambda_1,1}, \qquad \vec{w}_{\lambda_1,2} \qquad \dots, \qquad \vec{w}_{\lambda_k,s_k},$$

such that for all classical vector $\vec{v}$, **there exists** formal scalars $a_{\lambda_1,1},\dots,a_{\lambda_k,s_k}$ and **unique** $a_{\lambda_1,1}\vec{w}_{\lambda_1,1},\dots,a_{\lambda_k,s_k}\vec{w}_{\lambda_k,s_k}$ such that we may represent $\vec{v}$ as the constant part of a linear combination of fixed those vectors

$$\vec{v} = \mathrm{Const}(a_{\lambda_1,1}\vec{w}_{\lambda_1,1}) + \mathrm{Const}(a_{\lambda_1,2}\vec{w}_{\lambda_1,2}) + \dots + \mathrm{Const}(a_{\lambda_k,s_k}\vec{w}_{\lambda_k,s_k})$$

In this part, we will give algorithm on finding those $\vec{w}_{\lambda_1,1}$. Again, when finding a basis, we use **cross-filling**.

Review classical decomposition

We say unique $a\vec{w}$ instead of unique a since $\vec{w}$ is an infinite vector, and a is a formal scalar, we would view a, b equivalent when $a\vec{w} \equiv b\vec{w}$. Therefore, it is more accurate to say unique $a\vec{w}$ instead of unique a.

For examle,

$$(3+\epsilon)\binom{\infty}{2} \equiv (3+\epsilon+5\epsilon^2)\binom{\infty}{2}$$

but as a formal scalar $3+\epsilon \neq 3+\epsilon+5\epsilon^2$.

Review classical decomposition

To obtain such a basis, we apply cross-filling to the infinite matrix.

Review classical decomposition

Definition 14

Let $\vec{v}_1, \cdots, \vec{v}_n$ be a list of **non-zero** infinite vectors. We say they are **linearly independent** if any linear combination of formal scalars

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \equiv \vec{0}$$

implies

$$a_1\vec{v}_1 \equiv a_2\vec{v}_2 \equiv \cdots \equiv a_n\vec{v}_n \equiv 0.$$

Review classical decomposition

when cross-filling infinite matrix, the cross-center must choose at the largest degree

$$\begin{pmatrix} 2\infty - 1 & 2\infty & \infty \\ \infty^2 & \infty^2 & \infty^2 \\ \infty & \infty & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \infty & \infty & \infty \\ \infty^2 & \infty^2 & \infty^2 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \infty - 1 & \infty & 0 \\ 0 & 0 & 0 \\ \infty - 2 & \infty - 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Review classical decomposition

The cross cOlumns must be lInEARly iNDEpenDENt

$$a_1 \begin{pmatrix} \infty \\ \infty^2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} \infty \\ 0 \\ \infty - 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv 0$$

by looking at the rows from the first cross center,

$$a_1 \infty^2 \equiv 0 \implies a_1 \; O(\epsilon^2) \implies a_1 \begin{pmatrix} \infty \\ \infty^2 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Review classical decomposition

Therefore

$$a_2 \begin{pmatrix} \infty \\ 0 \\ \infty - 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv 0$$

Looking at the first cross center $a_2 \infty = 0$ so

$$a_2 \begin{pmatrix} \infty \\ 0 \\ \infty - 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Inductively, we also show

$$a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \equiv 0.$$

This inductively shows that **the cross-filling will always produce linearly independent vectors.** In fact, it produces basis for the column space of the matrix.

Review classical decomposition

Definition 15

For an infinite vector

$$\vec{v} = \vec{v}_0 + \vec{v}_1\infty + \vec{v}_2\infty^2 + \cdots + \vec{v}_m\infty^m$$

we call $\vec{v}_0$, $\vec{v}_1$, ..., $\vec{v}_m$ its components.

Review classical decomposition

Lemma 2

Suppose infinite vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ is a basis of $\mathrm{Im}(\mathscr{P}_\lambda)$, then the collection of its non-zero components is a (classical) basis of $\mathrm{Im}(P_\lambda)$.

Review classical decomposition

Proof: Since

$$\epsilon^j \vec{w}_i \in \mathrm{Im}(\mathscr{P})$$

$$\mathrm{Const}(\epsilon^j \vec{w}_i) \in \mathrm{Im}(P).$$

Let e_k be k′th column of I_n

$$\mathscr{P} e_k = \sum_i \sum_j a_{i,j} \epsilon^j \vec{w}_i$$

$$P e_k = \sum_i \sum_j a_{i,j} \mathrm{Const}\left(\epsilon^j \vec{w}_i\right)$$

Therefore, $\mathrm{Const}\left(\epsilon^j \vec{w}_i\right)$ spans $\mathrm{Im}(P)$.

Review classical decomposition

Test of linearly independency. Assume $a_{i,j}$ are constant scalars with

$$\sum_i \sum_j a_{i,j} \mathrm{Const}\left(\epsilon^j \vec{w}_i\right) = \vec{0}$$

$$\implies \mathrm{Const}\left(\underbrace{\sum_i \sum_j a_{i,j} \epsilon^j \vec{w}_i}_{\substack{\text{infinite eigenvector} \\ \text{eigenvalue } \lambda+\epsilon}} \right) = \vec{0}$$

$$\implies \sum_i \sum_j a_{i,j} \epsilon^j \vec{w}_i = \vec{0}$$

$$\implies a_{i,j} \epsilon^j \vec{w}_i = \vec{0} \text{ by linearly independence of } \vec{w}_i$$

$$\implies a_{i,j} \mathrm{Const}\left(\epsilon^j \vec{w}_i\right) = \vec{0}$$

This implies non-zero vectors in set $\{\mathrm{Const}(\epsilon^j \vec{w}_i)\}_{i,j}$ is linearly independent.

Review classical decomposition

Let $\vec{e}_j$ be j'th column of I_n. We have

$$\mathscr{P}_{\lambda_i}\vec{e}_j = a_1\vec{w}_1 + \ldots + a_l\vec{w}_l.$$

$$\text{Const}(\mathscr{P}_{\lambda_i}\vec{e}_j) = P_{\lambda_i}\vec{e}_j = \underbrace{\text{Const}(a_1\vec{w}_1)}_{\substack{\text{linear combination of} \\ \text{components of } \vec{w}_1}} + \ldots + \text{Const}(a_l\vec{w}_l)$$

Therefore, components of $\{\vec{w}_i\}$ forms a basis of P_{λ_i}

Review classical decomposition

Since P_{λ_i} is projection, a basis of $\text{Im}(P_{\lambda_i})$ has $\text{tr}(P_{\lambda_i})$ many element, denote these elements by

$$v_{\lambda_i,1}, \ldots, v_{\lambda_i,\text{tr}(P_{\lambda_i})}$$

It is basis, any vector of the form $P_{\lambda_i}\vec{v}$ can be written into linear combinations of them.

Collecting all these basis across all λ_i,

$$\mathscr{E} := \left\{v_{\lambda_1,1}, \ldots, v_{\lambda_1,\text{tr}(P_{\lambda_1})} v_{\lambda_i,1}, \ldots, v_{\lambda_i,\text{tr}(P_{\lambda_i})} \cdots v_{\lambda_k,1}, \ldots, v_{\lambda_k,\text{tr}(P_{\lambda_k})}\right\}$$

Then any vector $\vec{v} = P_{\lambda_1}\vec{v} + P_{\lambda_2}\vec{v} + \ldots + P_{\lambda_k}\vec{v}$ can be written as linear combination of them, then $\mathscr{E}$ **span the whole space** .

Counting the element, $\mathscr{E}$ has $\text{tr}(P_{\lambda_1}) + \text{tr}(P_{\lambda_2}) + \ldots + \text{tr}(P_{\lambda_k}) = \text{tr}(I_n) = n$ many element, the dimension of the whole space. So $\mathscr{E}$ is **basis** . This basis is called the **Canonical basis**.

Review classical decomposition

Using canonical basis , the matrix A is similar to Jordan canonical form. Jordan canonical form is a **block-digaonal matrix** with Jordan block

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

Review classical decomposition

The Following is a spctural decoMpositION oF JordAn cAnOnIcAL ForM

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

characteristic polynomial $(t-1)^5(t-2)^2(t-3)$

Review classical decomposition

$$\mathscr{P}_1 = \begin{pmatrix} 1 & \infty & \infty^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \infty & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \infty & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} \mathbf{0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

Review classical decomposition

$$\mathscr{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \infty & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathscr{P}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Review classical decomposition

Excercise. Find the matrix P such that $P^{-1}AP$ is a Jordan Canonical form, and find this Jordan canonical form.

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ -2 & 2 & 2 & 5 \\ -2 & 0 & 4 & 3 \\ 1 & 0 & -1 & 1 \end{pmatrix}$$

By calculation, the characteritic polynomial is.

$$\det(tI_4 - A) = (t-2)^3(t-3)$$

Note , the index of $t-3$ implies that $\mathscr{P}_3$ is a classical projection. We would first caluclate that by interpolation.

$$\mathscr{P}_3 = P_3 = \frac{(A-2I)^3}{(3-2)^3} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

This is already a rank 1 matrix.

Review classical decomposition

We have no idea what $\mathscr{P}_2$ is, let us assume

$$\mathscr{P}_2 = P_2 + N_2\infty + N_2^2\infty^2.$$

Use the fact that $P_2 + P_3 = I_4$, we know

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix}$$

Review classical decomposition

To find the nilpotent part, use that

$$A = \mathrm{Const}\left((3+\epsilon)\mathscr{P}_3 + (2+\epsilon)\mathscr{P}_2\right) = 3P_3 + 2P_2 + N_2$$

So

$$N_2 = A - 3P_3 - 2P_2 = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -3 & 0 & 3 & 6 \\ -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By a simple calculation, we see

$$N_2^2 = 0$$

Therefore

$$\mathscr{P}_2 = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix}}_{P_2} + \infty \underbrace{\begin{pmatrix} -1 & 0 & 1 & 2 \\ -3 & 0 & 3 & 6 \\ -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{N_2}$$

Review classical decomposition

Or we can write

$$\mathscr{P}_2 = \begin{pmatrix} -\infty & 0 & \infty + 1 & 2\infty + 1 \\ -3\infty - 1 & 1 & 3\infty + 1 & 6\infty + 1 \\ -\infty + 1 & 0 & \infty & 2\infty - 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}$$

Review classical decomposition

Since this is not a rank 1 matrix, we can do cross-filling decomposition

$$\begin{pmatrix} -\infty & 0 & \infty+1 & 2\infty+1 \\ -3\infty-1 & 1 & 3\infty+1 & 6\infty+1 \\ -\infty+1 & 0 & \infty & 2\infty-1 \\ -1 & 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\infty & 0 & \infty+1 & 2\infty+1 \\ -3\infty+2 & 0 & 3\infty+1 & 6\infty-1 \\ -\infty+1 & 0 & \infty & 2\infty-1 \\ -1 & 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Review classical decomposition

Eigenvector of eigenvalue $3+\epsilon$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvector of eigenvalue $2+\epsilon$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \infty+1 \\ 3\infty+1 \\ \infty \\ 1 \end{pmatrix}$$

Review classical decomposition

All vectors can be decomposed as

$$a_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathrm{Const} \left((a_3 + a_4 \epsilon) \begin{pmatrix} \infty + 1 \\ 3\infty + 1 \\ \infty \\ 1 \end{pmatrix} \right)$$

Review classical decomposition

However, if we just trying to decompose vectors into only classical eigenvectors, we are making $a_3 = 0$

$$a_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (0 + a_4\epsilon) \begin{pmatrix} \infty + 1 \\ 3\infty + 1 \\ \infty \\ 1 \end{pmatrix}$$

$$a_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

Then there are not enough eigenvectors.

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$$A\begin{pmatrix}1&0&1&1\\1&1&3&1\\-1&0&1&0\\1&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&1&1\\1&1&3&1\\-1&0&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}3&&&\\&2&&\\&&2&1\\&&&2\end{pmatrix}$$

$$\begin{pmatrix}1&0&1&1\\1&1&3&1\\-1&0&1&0\\1&0&0&1\end{pmatrix}^{-1}A\begin{pmatrix}1&0&1&1\\1&1&3&1\\-1&0&1&0\\1&0&0&1\end{pmatrix}=\begin{pmatrix}3&&&\\&2&&\\&&2&1\\&&&2\end{pmatrix}$$

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Excercise. Solve the differential equation

$$y' = \begin{pmatrix} 2 & 0 & 0 & 1 \\ -2 & 2 & 2 & 5 \\ -2 & 0 & 4 & 3 \\ 1 & 0 & -1 & 1 \end{pmatrix} y$$

for

$$y(0) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Solution. The solution is give by

$$y(t) = exp \begin{pmatrix} 2 & 0 & 0 & 1 \\ -2 & 2 & 2 & 5 \\ -2 & 0 & 4 & 3 \\ 1 & 0 & -1 & 1 \end{pmatrix} t y(0).$$

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By spectural decomposition

$$g(A) =$$

$$g(3)\begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

$$+\mathrm{Const}\left(g(2+\epsilon)\begin{pmatrix} -\infty & 0 & \infty+1 & 2\infty+1 \\ -3\infty-1 & 1 & 3\infty+1 & 6\infty+1 \\ -\infty+1 & 0 & \infty & 2\infty-1 \\ -1 & 0 & 1 & 2 \end{pmatrix}\right)$$

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Therefore

$$e^{At} =$$

$$e^{3t}\begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

$$+\mathrm{Const}\left(e^{(2+\epsilon)t}\begin{pmatrix} -\infty & 0 & \infty + 1 & 2\infty + 1 \\ -3\infty - 1 & 1 & 3\infty + 1 & 6\infty + 1 \\ -\infty + 1 & 0 & \infty & 2\infty - 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}\right)$$

Note that $e^{2t+\epsilon t} = e^{2t}(1 + \epsilon t + O(\epsilon^2))$

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where

$$\mathrm{Const}\left(e^{(2+\epsilon)t} \begin{pmatrix} -\infty & 0 & \infty+1 & 2\infty+1 \\ -3\infty-1 & 1 & 3\infty+1 & 6\infty+1 \\ -\infty+1 & 0 & \infty & 2\infty-1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \right)$$

$$= e^{2t} \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix} + te^{2t} \begin{pmatrix} -1 & 0 & 1 & 2 \\ -3 & 0 & 3 & 6 \\ -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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So the complete solution e^{At} equals to

$$e^{3t}\begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix} + te^{2t}\begin{pmatrix} -1 & 0 & 1 & 2 \\ -3 & 0 & 3 & 6 \\ -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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