Sufficient linearly independent vectors give a basis

Proposition 1

If V is a vector space of $\dim(V) = n$, then any n many **linealy independent** vectors is a **basis** .

Let

$$B = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$$

be the $n \times n$ matrix collecting those **linealy independent** vectors. Then B has a left inverse $AB = I_n$. Since B is a square matrix, $BA = I_n$

But this means B also a right inverse, then columns of B **span the whole space** , which means it is a **basis** .

Sufficient linearly independent vectors give a basis

The following is application of above theorem

Corollary 1

Suppose $W \subset V$ and $\dim(W) = \dim(V)$, then $W = V$.

A **basis** $\begin{pmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{pmatrix}$ of W is **linealy independent** in V, but since there are $\dim(V)$-many of them, it is a **basis** of V. So

$$V = \operatorname{span}\begin{pmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{pmatrix} = W$$

Extension of linearly indepdent basis

Another application is extension of **linealy independent** to a **basis**

Extension of linearly indepdent basis

B linearly independent, B has left inverse $AB = I_m$.

BA is projection matrix, then $I_n - BA$ is also proejction matrix.

Use cross filling, we may write

$$I_n - BA = DC \qquad CD = I_?$$

how to determine the size of CD?

$$\mathrm{tr}(CD) = \mathrm{tr}(DC) = \mathrm{tr}(I_n - BA) = n - \mathrm{tr}(BA) = n - \mathrm{tr}(AB) = n - m$$

So $CD = I_{n-m}$.

Extension of linearly indepdent basis

Then

$$\begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} B & D \end{pmatrix} = \begin{pmatrix} AB & AD \\ CB & CD \end{pmatrix}$$

$$(BA)^2 = BA \iff BA(I_n - BA) = 0 \iff BADC = 0 \iff AD = 0$$

$$(BA)^2 = BA \iff (I_n - BA)BA = 0 \iff DCBA = 0 \iff CB = 0$$

$$\begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} B & D \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix} = I_n.$$

Extension of linearly indepdent basis

Proposition 2

Any linearly indepdent list of vectors can be extended to a **basis**

Extension of linearly indepdent basis

Proposition 3

If $W \subset V$, then $\dim(W) \leq \dim(V)$