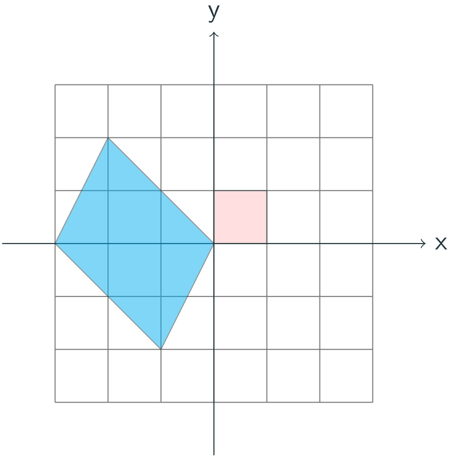
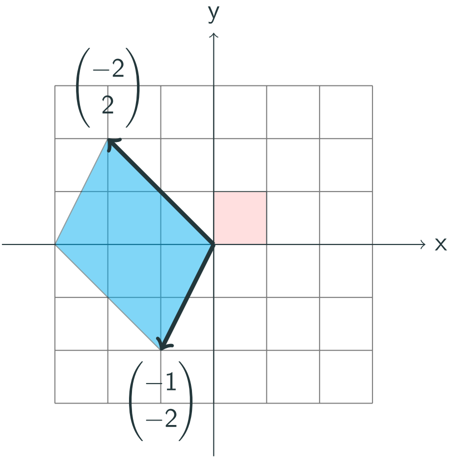
Determinant

Warm up: Suppose the area of each small rectangle (for example, the pink colored one) is 1. How to calculate the blue area?



Determinant

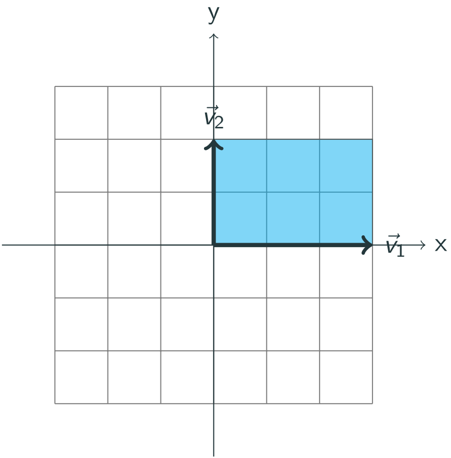
The blue area is determined by two vectors, which describe the sides of the parallelogram. We put the function **determinant to describe the area of this part**



$$\det \begin{pmatrix} -2 & -1 \\ 2 & -2 \end{pmatrix} = \text{Blue area}$$

Positive area v.s. negative area

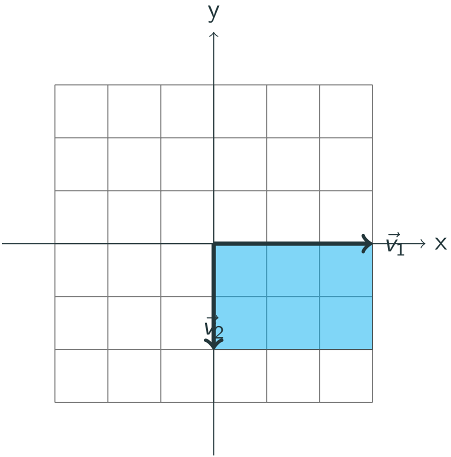
What is the area of the rectangle enclosed by the vector $\vec{v}_1$ and $\vec{v}_2$?



Since the area is base times height, the area is $3 \times 2 = 6$.

Positive area v.s. negative area

However, if I put the height in another direction, it makes more sense to define the height to be a negative number



Since the area is base times height, the area is $3 \times (-2) = -6$.

Positive area v.s. negative area

This signed area is called the **oriented area**. We call $\vec{v}_1$ the base vector, and $\vec{v}_2$ the height vector.

If the height vector is on the counterclockwise side of the base vector, the area is positive. Otherwise, the area is negative.

Intended properties of determinant

Before defining determinant, we should study what are the properties we are expecting for areas. Let's start with the easiest 2×2 matrices.

Multilinearity: We should have

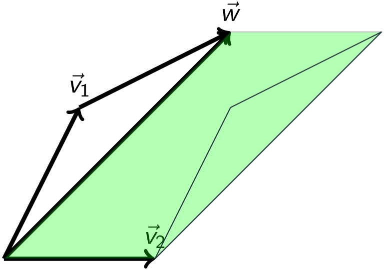
$$\det \begin{pmatrix} \vec{v}_1 + \vec{w} & \vec{v}_2 \end{pmatrix} = \det \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix} + \det \begin{pmatrix} \vec{w} & \vec{v}_2 \end{pmatrix}$$

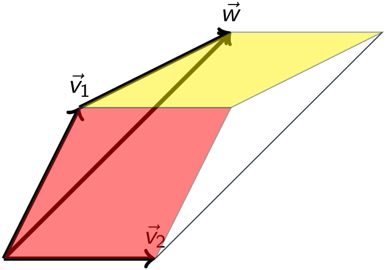
and

$$\det \begin{pmatrix} \lambda \vec{v}_1 & \vec{v}_2 \end{pmatrix} = \lambda \det \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}.$$

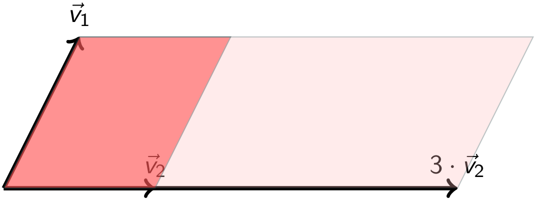
Explaination of multiplinearity

The area of parallelogram expanded by $\vec{v}_1 + \vec{w}$ and $\vec{v}_2$ should have the same area as the sum of the one expanded by $\vec{v}_1, \vec{v}_2$ and $\vec{w}, \vec{v}_2$. The reason is explained in the following graph.





Explaination of multiplinearity



Explaination of multiplinearity

The preceding properties yeilds another important property, the **column swapping property**:

$$\det\begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix} = -\det\begin{pmatrix}\vec{v}_2 & \vec{v}_1\end{pmatrix}$$

This property can be deduced by the following steps

$$0 = \det\begin{pmatrix}\vec{v}_1 + \vec{v}_2 & \vec{v}_1 + \vec{v}_2\end{pmatrix}$$

$$0 = \det\begin{pmatrix}\vec{v}_1 & \vec{v}_1\end{pmatrix}$$

$$0 = \det\begin{pmatrix}\vec{v}_2 & \vec{v}_2\end{pmatrix}$$

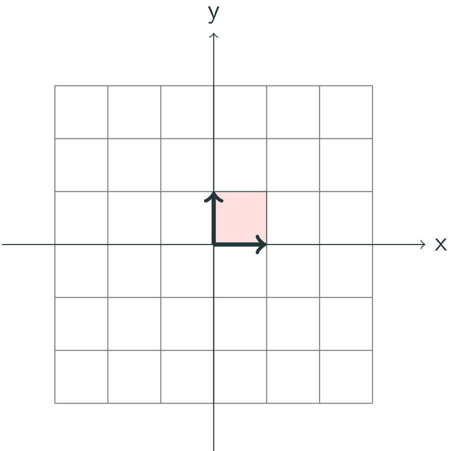
Then the expansion of $\det\begin{pmatrix}\vec{v}_1 + \vec{v}_2 & \vec{v}_1 + \vec{v}_2\end{pmatrix}$ gives

$$\det\begin{pmatrix}\vec{v}_1 & \vec{v}_1\end{pmatrix} + \det\begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix} + \det\begin{pmatrix}\vec{v}_2 & \vec{v}_1\end{pmatrix} + \det\begin{pmatrix}\vec{v}_2 & \vec{v}_2\end{pmatrix}$$

Normalization

Don′t forget that we are assuming the basic square box has area one. Therefore, we should have

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$



In general, we will denote $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We can denote $\det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} = 1$.

Formula for determinant

Proposition 1

We have

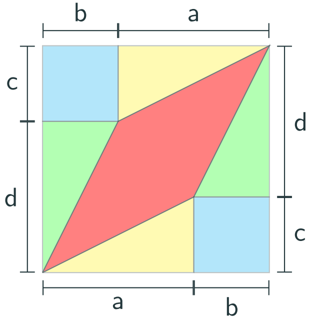
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Formula for determinant

This formula can be well-explained by the folloging graph. The red area of the parallelogram enclosed by the vector

$$\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$$

can be cut off from a rectangle with length $a + b$, height $c + d$ by the following way.



$$(a + b)(c + d) - bc - bd - ac = ?$$

Formula for determinant

Excercise. Using the formula of the determinant, calculate the following

1. $\det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
2. $\det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$
3. $\det \begin{pmatrix} 1 & 0 \\ 8 & 0 \end{pmatrix}$

Summary

- Axiom 1:

  $$\det(\vec{v}_1, ..., \vec{v}_i + \vec{z}_i, ..., \vec{v}_n) = \det(\vec{v}_1, ..., \vec{v}_i, ..., \vec{v}_n) + \det(\vec{v}_1, ..., \vec{z}_i, ..., \vec{v}_n)$$

- Axiom 2: For any $\lambda \in \mathbb{R}$,
  $\det(\vec{v}_1, \vec{v}_2, \cdots, \lambda \cdot \vec{v}_i, \cdots, \vec{v}_n) = \lambda \cdot \det(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_i, \cdots, \vec{v}_n)$
  .
- Axiom 3: $\det(\cdots, \vec{v}, \cdots, \vec{v}, \cdots) = 0$ for any $\vec{v} \in V$.
- Axiom 4: $\det I_n = \det(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n) = 1$.

Here $\vec{e}_i$ are the i'th column in identity matrix I_n.

These axiomizes the notion of volumn in that space.

We require the matrix for determinant calculation to be a square matrix!

Common Mistakes

Axiom 1:

$$\det(\vec{v}_1, ..., \vec{v}_i + \vec{z}_i, ..., \vec{v}_n) = \det(\vec{v}_1, ..., \vec{v}_i, ..., \vec{v}_n) + \det(\vec{v}_1, ..., \vec{z}_i, ..., \vec{v}_n)$$

It states only for one column, other columns has to be fixed.

What is wrong with the following calculation???

WRONG!

$$\det\begin{pmatrix} 1+2 & 1+3 \\ 2+1 & 3+2 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + \det\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

Common Mistakes

The determinant is pretty like calculation of factors In general, we have

$$(x+y)(a+b)(c+d) = x(a+b)(c+d) + y(a+b)(c+d)$$

but you can not do

$$\textbf{WRONG!}(\mathbf{x}+y)(\mathbf{a}+b)(c+d) = \mathbf{xa}(c+d) + yb(c+d)$$

Instead, there are intersection terms

$$(x+y)(a+b)(c+d) = xa(c+d) + xb(c+d) + ya(c+d) + yb(c+d)$$

Common Mistakes

For determinant, similar situation happens, if you wanna expand multiple columns, you have to take care of intersection terms

$$\det\begin{pmatrix} v_1 + w_1 & v_2 + w_2 & v_3 \end{pmatrix}$$

$$= \det\begin{pmatrix} v_1 & w_1 & v_3 \end{pmatrix} + \det\begin{pmatrix} v_2 & w_1 & v_3 \end{pmatrix} + \det\begin{pmatrix} v_1 & w_2 & v_3 \end{pmatrix} + \det\begin{pmatrix} v_2 & w_2 & v_3 \end{pmatrix}$$

Common Mistakes

Axiom 2: For any $\lambda \in \mathbb{R}$,
$\det(\vec{v}_1, \vec{v}_2, \cdots, \lambda \cdot \vec{v}_i, \cdots, \vec{v}_n) = \lambda \cdot \det(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_i, \cdots, \vec{v}_n)$.

What is wrong for the following?

WRONG

$$\det(2A) = 2 \det A?$$

Common Mistakes

If fact, if $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$, then $2A = \begin{pmatrix} 2v_1 & 2v_2 & 2v_3 \end{pmatrix}$ We have

$$\det 2A = \det \begin{pmatrix} 2v_1 & 2v_2 & 2v_3 \end{pmatrix} = 2 \det \begin{pmatrix} v_1 & 2v_2 & 2v_3 \end{pmatrix}$$

$$= 4 \det \begin{pmatrix} v_1 & v_2 & 2v_3 \end{pmatrix} = 8 \det \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = 2^3 \det A.$$

Algebraic cofactor

$$A := \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{1}$$

Define **algebraic cofactors** by the **scalar**

$$A_{ij} = \det \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & (\text{replace } \vec{v}_j \text{ by } \vec{e}_i) & \cdots & \vec{v}_n \end{pmatrix}.$$

A quickway to remember A_{ij} is to replace the element at ith row jth column by 1 and put 0 to else where in its columns.

Algebraic cofactor

For example,

$$\text{put } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \implies A_{32} = \det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 9 \end{pmatrix} \tag{2}$$

For this A in (**??**), please write down A_{21}, A_{22} and A_{23} as well (you don't need to calculate determinant for the exact number).

Algebraic cofactor

For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A_{12} = \det \begin{pmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \\ 7 & 0 & 9 \end{pmatrix} \qquad A_{22} = \det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 6 \\ 7 & 0 & 9 \end{pmatrix} \qquad A_{32} = \det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 9 \end{pmatrix}$$

Algebraic cofactor

First important observation: Algebraic cofactors can be used for calculating the determinant with replaced columns.

$$A[\text{replace 2nd col by another vector}] = \begin{pmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 5 \\ 6 & x_3 & 9 \end{pmatrix}$$

Then

$$\det \begin{pmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 5 \\ 6 & x_3 & 9 \end{pmatrix} = \mathbf{x}_1 \underbrace{\det \begin{pmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{12}} + \mathbf{x}_2 \underbrace{\det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{22}} + \mathbf{x}_3 \underbrace{\det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 9 \end{pmatrix}}_{A_{32}}$$

This is called **Laplacian Expansion**.

Algebraic cofactor

Question: In the laplacian Expansion formula, which axiom did we used? and how did we use that?

$$\det\begin{pmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 5 \\ 6 & x_3 & 9 \end{pmatrix} = \mathbf{x}_1 \underbrace{\det\begin{pmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{12}} + \mathbf{x}_2 \underbrace{\det\begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{22}} + \mathbf{x}_3 \underbrace{\det\begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 9 \end{pmatrix}}_{A_{32}}$$

- Axiom 1:

  $$\det(\vec{v}_1, ..., \vec{v}_i + \vec{z}_i, ..., \vec{v}_n) = \det(\vec{v}_1, ..., \vec{v}_i, ..., \vec{v}_n) + \det(\vec{v}_1, ..., \vec{z}_i, ..., \vec{v}_n)$$

- Axiom 2: For any $\lambda \in \mathbb{R}$,
  $\det(\vec{v}_1, \vec{v}_2, \cdots, \lambda \cdot \vec{v}_i, \cdots, \vec{v}_n) = \lambda \cdot \det(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_i, \cdots, \vec{v}_n)$.
- Axiom 3:

  $$\det(\cdots, \vec{v}, \cdots, \vec{v}, \cdots) = 0 \qquad \text{for any} \qquad \vec{v} \in V.$$

Algebraic cofactor

Excercise. Look at the formula, think about the question. Originally

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 5 \\ 7 & x_3 & 9 \end{pmatrix} = \mathbf{x}_1 \underbrace{\det \begin{pmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{12}} + \mathbf{x}_2 \underbrace{\det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{22}} + \mathbf{x}_3 \underbrace{\det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 9 \end{pmatrix}}_{A_{32}}$$

which choicse of x_1, x_2, x_3 could make the following equation true?

$$\det A = \mathbf{x}_1 A_{12} + \mathbf{x}_2 A_{22} + \mathbf{x}_3 A_{32}$$

Algebraic cofactor

Excercise. Any repeated column will result 0 for the determinant, therefore we have

$$0 = \begin{pmatrix} 1 & 1 & 3 \\ 4 & 4 & 6 \\ 7 & 7 & 9 \end{pmatrix} \qquad 0 = \begin{pmatrix} 1 & 3 & 3 \\ 4 & 6 & 6 \\ 7 & 9 & 9 \end{pmatrix}$$

Recall the formula

$$\det \begin{pmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 5 \\ 7 & x_3 & 9 \end{pmatrix} = \mathbf{x}_1 \underbrace{\det \begin{pmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{12}} + \mathbf{x}_2 \underbrace{\det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 6 \\ 7 & 0 & 9 \end{pmatrix}}_{A_{22}} + \mathbf{x}_3 \underbrace{\det \begin{pmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 9 \end{pmatrix}}_{A_{32}}$$

Please come up with more x_1, x_2, x_3 to keep the following equation true

$$0 = \mathbf{x}_1 A_{12} + \mathbf{x}_2 A_{22} + \mathbf{x}_3 A_{32}$$

Algebraic cofactor

Clear, we may write

$$\det \begin{pmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 5 \\ 7 & x_3 & 9 \end{pmatrix} = \mathbf{x}_1 A_{12} + \mathbf{x}_2 A_{22} + \mathbf{x}_3 A_{32} = \begin{pmatrix} A_{12} & A_{22} & A_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Would you please filling scalars in the following slot?

$$\begin{pmatrix} \square & \square & \square \end{pmatrix} = \begin{pmatrix} A_{12} & A_{22} & A_{32} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Algebraic cofactor

$$\begin{pmatrix} 0 & \det(A) & 0 \end{pmatrix} = \begin{pmatrix} A_{12} & A_{22} & A_{32} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Algebraic cofactor

Now let us do first and third columns

$$A_{11} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix} \qquad A_{21} = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix} \qquad A_{31} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 8 & 9 \end{pmatrix}$$

$$\det \begin{pmatrix} x_1 & 2 & 3 \\ x_2 & 5 & 6 \\ x_3 & 8 & 9 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} \square & \square & \square \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Algebraic cofactor

Let's do last column

$$A_{13} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 0 \\ 7 & 8 & 0 \end{pmatrix} \qquad A_{23} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 1 \\ 7 & 8 & 0 \end{pmatrix} \qquad A_{33} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 2 & x_1 \\ 4 & 5 & x_2 \\ 7 & 8 & x_3 \end{pmatrix} = \begin{pmatrix} A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} \square & \square & \square \end{pmatrix} = \begin{pmatrix} A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Algebraic cofactor

Put all these together, what is your discovery?

$$\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

Definition 1

Let A be $n \times n$ matrix, we call the matrix

$$A^* := \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

The **adjugate** of matrix A. We have

$$AA^* = (\det A) \cdot I_n.$$

Algebraic cofactor

Please be very careful on how position of the cofactors put into adjugate

$$\begin{pmatrix} A_{11} & \boxed{A_{21}} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \qquad A_{21} = \begin{pmatrix} 0 & 2 & 3 \\ \boxed{1} & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix}$$

Note:

$$A^* = \begin{pmatrix} A_{11} & \boxed{A_{21}} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ \boxed{A_{21}} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$$

Inverse matrix formula

Theorem 1

If $\det A \neq 0$, we have a formula for its inverse, given by

$$A^{-1} = \frac{A^*}{\det A}.$$

Inverse matrix formula

If A is 2×2 matrix, let's calculate the adjugate matrix of A

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \qquad A_{21} = \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}$$

Therefore

$$A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Cramer's rule

Now that we have inverse matrix formula

$$A^{-1} = \frac{1}{\det A} A^*.$$

Suppose we wanna solve an equation

$$\underbrace{A}_{n\times n \text{ matrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where we are in lucky situations that A invertible, then the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Cramer's rule

Therefore, we may write

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{A^*}{\det(A)} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Cramer′s rule

This gives a formula

$$x_i = \frac{1}{\det(A)} \begin{pmatrix} A_{1i} & A_{2i} & \cdots & A_{ni} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

However, we remembered that the expression

$$\begin{pmatrix} A_{1i} & A_{2i} & \cdots & A_{ni} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

is the determinant of replacing $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ to i'th column of A.

Cramer's rule

Theorem 2

Suppose

$$\underbrace{A}_{n\times n \text{ matrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\vec{b}}$$

is a system of linear equations with A invertible so that this equation has unique solution. Then x_i is given by the formula

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is the matrix by replacing ith column of A by constant $\vec{b}$

Cramer's rule

Excercise. Using Cramer's rule, write a formula for the component of the following solution in terms of determinant

$$\begin{pmatrix} 1 & 8 & 2 \\ 3 & 9 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$x = \frac{\det \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}}{\det \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}} \qquad y = \frac{\det \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}}{\det \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}} \qquad z = \frac{\det \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}}{\det \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}}$$