Determinat and linearly independence

For determinant,

if one column is a linear combination of others, then the determinant is 0

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & 3v_1 + 2v_2 \\ | & | & | \end{pmatrix} = 0$$

Determinat and linearly independence

For determinant, we may add/subtract a column by linear combination of other columns, the determinant would not change.

For any $\lambda_1, ..., \lambda_n$

$$\det \begin{pmatrix} | & | & \cdots & | & \cdots & | \\ v_1 & v_2 & \cdots & v_i & \cdots & v_n \\ | & | & \cdots & | & \cdots & | \end{pmatrix}$$

$$= \det \begin{pmatrix} | & | & \cdots & | & \cdots & | \\ v_1 - \lambda_1 v_i & v_2 - \lambda_2 v_i & \cdots & v_i & \cdots & v_n - \lambda_n v_i \\ | & | & \cdots & | & \cdots & | \end{pmatrix}$$

Determinat and linearly independence

Explaination: expand by columns

$$\det\begin{pmatrix} | & | \\ v_1+3v_2 & v_2 \\ | & | \end{pmatrix} = \det\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \underbrace{+\det\begin{pmatrix} | & | \\ 3v_2 & v_2 \\ | & | \end{pmatrix}}_{0}$$

$$\det\begin{pmatrix} | & | & | \\ v_1+3v_2+5v_3 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \det\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} + \det\begin{pmatrix} | & | & | \\ 3v_2+5v_3 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

Determinat and linearly independence

Suppose cross-filling decomposes

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} = \begin{pmatrix} 0.5 & 1 & 1.5 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0.5 & 0 & -0.5 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} = \det \begin{pmatrix} 0.5 & 1 & -0.5 \\ 0 & 2 & 0 \\ -1 & 4 & 3 \end{pmatrix} = \det \begin{pmatrix} 0.5 & 0 & -0.5 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Determinat and linearly independence

Why

$$\det\begin{pmatrix} 0.5 & 1 & -0.5 \\ 0 & 2 & 0 \\ -1 & 4 & 3 \end{pmatrix} = \det\begin{pmatrix} 0.5 & 0 & -0.5 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}?$$

Think about it

$$\det\begin{pmatrix} 0.5 & 1 & -0.5 \\ 0 & 2 & 0 \\ -1 & 4 & 3 \end{pmatrix} = \det\begin{pmatrix} 0.5 & 0 & -0.5 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} + \det\begin{pmatrix} 0.5 & 1 & -0.5 \\ 0 & 0 & 0 \\ -1 & 4 & 3 \end{pmatrix}$$

determinant by cross-filling

Theorem 1

Let A be an $n \times n$ matrix and one have a one-step cross-filling decomposition

$$A = A_1 + R$$

where A_1 is a rank 1 matrix with cross-center a_1, R has a zero cross. Let $\widetilde{R}$ be the R relacing the cross center by 1. Then

$$\det A = a_1 \det \widetilde{R}.$$

Example

Excercise. Calculating determinant by cross-filling

$$\det \underbrace{\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}}_{A}$$

$$A = \begin{pmatrix} 2 & 6 & 2 \\ 1 & 3 & 1 \\ 2 & 6 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -5 & 1 \\ 0 & 0 & 0 \\ 0 & -5 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\det A = 1(-5)(-1) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Determinant of Switching Matrix

The calculation of determinant by cross-filling gives us the question of determining the value of the following determinant

Definition 1

A switching matrix is a matrix with each row and column a unique non-zero entry valued 1

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

zigzag method

The formula for determinant of switching matrix can be summarized as follows

- Find all horizontal and vertical segment inking 1 and diagonal entrise(no matter what that is).

- Count the number m of **connected loops**

- Determinant is $(-1)^{m+n}$.

Calculation example

$$
\begin{tikzcd}
0 \arrow[rr] & 0 & 1 \arrow[dd] & 0 \\
0 & 0 \arrow[rr] & 0 & 1 \arrow[dd] \\
0 & 1 \arrow[u] & 0 \arrow[l] & 0 \\
1 \arrow[uuu] & 0 & 0 & 0 \arrow[lll]
\end{tikzcd}
$$

There are only one **loop** linking all one.

$$
\det \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = (-1)^{4+1} = -1.
$$

zigzag method

To see why . This is because that each swithcing inside path will break a loop into two.

$$\begin{tikzcd}
0 \arrow[rr] & 0 & 1 \arrow[dd] & 0 \\
1 \arrow[u] & 0 \arrow[l] & 0 & 0 \\
0 & 1 \arrow[u] & 0 \arrow[l] & 0 \\
0 & 0 & 0 & 1 \arrow[loop left]
\end{tikzcd}$$

Note that in this picture, 1 itself is a loop.

Determinant of Transpose

Theorem 2

We have $\det(A) = \det(A^T)$

Because the cross filling is symmetric on rows and columns.

$$A = P_1 + ... + P_n$$

is a cross-filling for A, then $A^T = P_1^T + ... + P_n^T$ is cross-filling for A^T with center value the same.

Left $\det S = \det S^T$ for switching matrix. S with path denoted, the transpose is still a path. So number of path not changed.

Determinant of Block Triangular Matrices

Theorem 3

We have

$$\det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det A \det D.$$

and similarly

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D.$$

Determinant of Block Triangular Matrices

Proof, this is because

When performing cross filling for A, the matrix B or D has automatically be deleted, and the paths for the switching matrix has been constrained inside each diagonal block. Let us demonstrade this by examples.

Determinant of Block Triangular Matrices

For finding the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 8 & 9 & 7 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 3 & 1 \end{pmatrix}$$

we notice the block makes it to upper triangular block matrix.

$$\begin{pmatrix} 1 & 1 & 8 & 9 & 7 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 3 & 1 \end{pmatrix}$$

Determinant of Block Triangular Matrices

while the cross filling for the first block has been performed, it automatically cleared non-diagonal blocks.

$$\begin{pmatrix} 1 & 1 & 8 & 9 & 7 \\ 1 & 1 & 8 & 9 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -7 & -7 & -6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 3 & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 3 & 1 \end{pmatrix}}_{\text{remainder block}}$$

Then one perform the corss-filling for remainder block.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{product of cross center} = \overbrace{\boxed{1} \times \boxed{1}}^{\text{first block}} \times \overbrace{\boxed{1} \times \boxed{1} \times \boxed{1}}^{\text{second blcok}}$$

Determinant of Block Triangular Matrices

Now we trying to figure out the sign. Using zigzag method, Explain why

$$\det \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is

$$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Determinant of Block Triangular Matrices

Inductively

Theorem 4

The determinant of block upper or lower triangular matrix is the product of determinant of blocks.

$$\det \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{pmatrix} = \det(A_{11})\det(A_{22})\cdots\det(A_{nn})$$

Determinant of Block Triangular Matrices

The formula is also true for block lower triangular matrix and block diagonal matrix

$$\det \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{12} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} = \det(A_{11})\det(A_{22})\cdots\det(A_{nn})$$

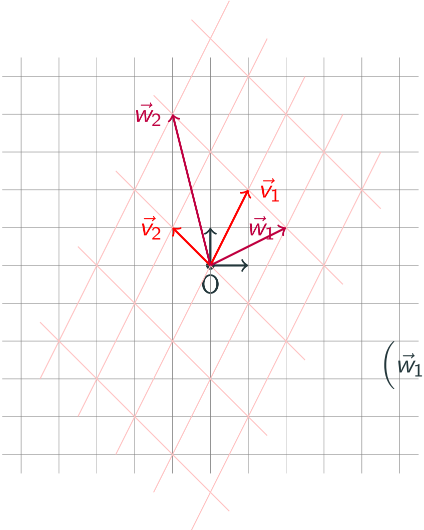
$$\det \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{pmatrix} = \det(A_{11})\det(A_{22})\cdots\det(A_{nn})$$

Determinant of Block Triangular Matrices

In particular, the determinant of usual upper triangular, diagonal, and lower triangular matrices is the product of diagonal entries.

Determinant of product of matrices

Look at the following graph, explain the matrix product



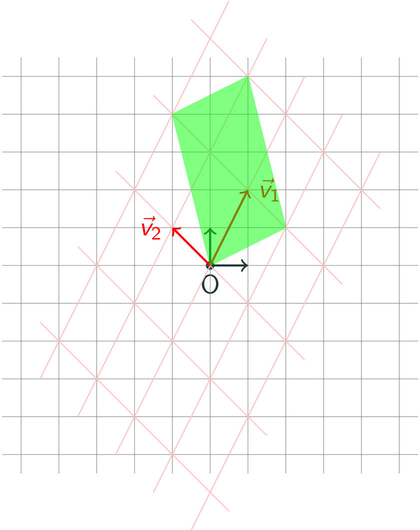
$$\begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix} = \begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}$$

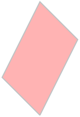
$$\begin{pmatrix}\vec{w}_1 & \vec{w}_2\end{pmatrix} = \begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$$

$$\begin{pmatrix}\vec{w}_1 & \vec{w}_2\end{pmatrix} = \begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$$

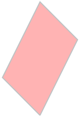
Determinant of product of matrices

How to calculate the deteminant without calculating matrix product?





: $\square = \det\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}$

: 

$= \det\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$

Determinant of product of matrices

Theorem 5

$$\det(AB) = \det(A)\det(B)$$

We have a geometrical understanding of the theorem, but we do not yet have a mathematical proof.

Determinant of product of matrices

$$\begin{pmatrix} B \\ AB \end{pmatrix} = \begin{pmatrix} I \\ A \end{pmatrix} B$$

Each column of $\begin{pmatrix} B \\ AB \end{pmatrix}$ is a linear combination of columns of $\begin{pmatrix} I \\ A \end{pmatrix}$ by coefficients listed in B

Determinant of product of matrices

Here illustrates mathematical proof.

$$\det \begin{pmatrix} I & \\ A & AB \end{pmatrix} = \det \begin{pmatrix} I & -B \\ A & 0 \end{pmatrix} = \det \begin{pmatrix} I & -B + I \\ A & A \end{pmatrix} = \det \begin{pmatrix} B & -B + I \\ 0 & A \end{pmatrix}$$

Determinant of product of matrices

Proposition 1

A matrix A is invertible if and only if $\det(A) \neq 0$

If A has inverse, then $\det(A)\det(A^{-1}) = \det(I_n) = 1$, so $\det(A) \neq 0$

If $\det(A) \neq 0$, then we can construct A^{-1} by $A^* / \det(A)$

Determinant of product of matrices

We have

$$\det(A^k) = \det(A)^k$$

$$\det(\lambda A) = \lambda^n \det(A)$$