Applying Polynomial on Linear Operators

Remember the definition of linear operators

Definition 1

A linear operator $T : V \longrightarrow V$ is a linear transformation with domain identical to the codomain.

Applying Polynomial on Linear Operators

Since domain and codomain are identical, To obtain a matrix representation, we only need to take a basis in V.

$$T\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix} = \begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix} M$$

Onece we have another basis

$$\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{pmatrix} = \begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix} P$$

We obtain

$$T\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{pmatrix} = \begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{pmatrix} P^{-1}MP$$

Applying Polynomial on Linear Operators

Definition 2

We say two matrices A, B similar to each other if there is an **invertible** matrix P such that

$$B = P^{-1}AP$$

Similar matrices essentially could be a matrix representation for the same linear transformation.

If a property is essentially defined for linear transformation, they shares for similar matrices.

Polynomials on Linear Operators

Now we only consider one Linear Operator $T : V \longrightarrow V$. By

$$T^n := \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ many } T}$$

we define many new operators on V. We can also make linear combinations of them to define things like

$$T^2 + 2T + \mathrm{id}_V$$

You found it is exactly equals to

$$(T + \mathrm{id}_V)^2$$

For our convenience, we abbreviate id_V as I.

Polynomials on Linear Operators

This motivates us we can apply any degree polynomial

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$$

to the operator T by evaluating P at $x = T$ by defining

$$p(T) := a_m T^m + a_{m-1} T^{m-1} + \cdots + a_0$$

Corollary 1

Suppose $\mathcal{E} \subset V$ is a basis, we have

$$[p(T)]^{\mathcal{E}} = p([T]^{\mathcal{E}})$$

i.e. The matrix of applying polynomial is applying polynomial on the matrix of the linear operator.

Polynomials on Linear Operators

Excercise. Suppose T is a linear operator on V such that

$$(T - I)^2 = 0.$$

Show that T is invertible.

Proof. We have $T^2 - 2T + I = 0$, therefore $-T^2 + 2T = I$, so $T(2 - T) = I$. This implies T is invertible and

$$T^{-1} = 2 - T.$$

Power series on Nilpotent Operators

Definition 3

A linear operator $T : V \longrightarrow V$ is **nilpotent** if $T^n = 0$ for some $n \in \mathbb{Z}_+$.

At this time we should be careful for power series because it might cause convergence problem. But for a some operators T that

$$T^n = 0 \text{ for some } n \in \mathbb{Z}_+$$

Since $T^n = 0 \implies T^{n+1} = 0$. Apply power series to it would only result finitely many non-zero terms. Therefore it is the same as applyting a Polynomial to it.

Power series on Nilpotent Operators

Excercise. Let $V = P_{2,x} = \{ax^2 + bx + c, \text{ where } a, b, c \in F\}$, define $T(f(x)) = f'(x)$ Consider the power series

$$\exp(-x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

Evaluate $\exp(-T)(x^2)$

Solution. We have

$$\begin{cases} I(x^2) = x^2. \\ T(x^2) = 2x. \\ T^2(x^2) = 2. \\ T^k(x^2) = 0. \text{ for } k \geq 3. \end{cases}$$

Therefore

$$\exp(-T)(x^2) = I(x^2) - T(x^2) + \frac{T^2(x^2)}{2} = x^2 - 2x + 1 = (x-1)^2.$$

Power series on Nilpotent Operators

Suppose T is a linear operator on V such that

$$T^3 = 0.$$

Show that $I - T$ is invertible.

Idea: Let $p(x) = 1 - x$, we realize $I - T = p(T)$. We want to find it inverse so the idea is to apply $\frac{1}{p(x)}$ on it, which has the Talor Expansion

$$\frac{1}{p(x)} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Since $T^3 = 0$, so we think

$$\frac{1}{p(T)} = 1 + T + T^2 + 0 + 0 + \cdots = 1 + T + T^2.$$

So the idea is try $1 + T + T^2$.

Power series on Nilpotent Operators

Proof: Since

$$(I - T)(I + T + T^2) = (I + T + T^2) - T(I + T + T^2) = I - T^4 = I.$$

This implies $I - T$ is invertible and its inverse is given by $I + T + T^2$.