Review Lagurange Interpolation Polynomial construction

Consider polynomial $F(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

If a matrix A satisfies $F(A) = 0$, then

$$g(A) = g(\lambda_1)P_{\lambda_1} + g(\lambda_2)P_{\lambda_2} + g(\lambda_3)P_{\lambda_3}$$

where each projection $P_{\lambda_1} = f_{\lambda_1}(A)$ is obtained by plugging A into the interpolation at λ_1:

$$f_{\lambda_1} := \frac{(x - \lambda_2)(x - \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}.$$

The value $g(A)$ only depends on $g(\lambda_1), g(\lambda_2), g(\lambda_3)$.

Review Lagurange Interpolation Polynomial construction

Construction of f_{λ_1} follow the value table

	$x = \lambda_1$	$x = \lambda_2$	$x = \lambda_3$
$f_{\lambda_1}(x)$	1	0	0

Construction of f_{λ_1} by 3 steps.

First, consider $F(x)$, giving value 0 at the three specific points

	$x = \lambda_1$	$x = \lambda_2$	$x = \lambda_3$
$F_{\lambda_1}(x)$	0	0	0

Review Lagurange Interpolation Polynomial construction

Then, realizing that we want $f_{\lambda_1}(\lambda_1) \neq 0$, we consider the product $F(x) \cdot \frac{1}{x-\lambda_1}$

	$x = \lambda_1$	$x = \lambda_2$	$x = \lambda_3$
$F(x) \cdot \frac{1}{x-\lambda_1}$	$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \neq 0$	0	0

In this slides, we call $\frac{1}{x-\lambda_1}$ the **λ_1-factor replacer**. It replaces the factor $(x - \lambda_1)$ in F by 1.(Name only for this slides, not general terminology.)

$$F(x) \cdot \frac{1}{x - \lambda_1} = (x - \lambda_2)(x - \lambda_3)$$

Some side notes for simplification

Some side notes: when evaluating at $x = \lambda_1$

$$F(x) \cdot \frac{1}{x - \lambda_1}\bigg|_{x=\lambda_1} = (x - \lambda_2)(x - \lambda_3)|_{x=\lambda_1} = \underbrace{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}_{\text{looks complicated}}$$

Since direct evaluation $F(x) \cdot \frac{1}{x-\lambda_1}\Big|_{x=\lambda_1} = \frac{0}{0}$ is an undefined form, we may also use L$'$hospital rule,

$$\lim_{x \longrightarrow \lambda_1} F(x) \cdot \frac{1}{x - \lambda_1} = \lim_{x \longrightarrow \lambda_1} \frac{d}{dx} F_{\lambda_1}(x) \cdot \frac{1}{\frac{d}{dx}(x - \lambda_1)} = F'(\lambda_1)$$

So one may write

$$F(x) \cdot \frac{1}{x - \lambda_1}\bigg|_{x=\lambda_1} = \underbrace{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}_{\text{looks complicated}} = F'(\lambda_1)$$

Some side notes for simplification

The table from last step

	$x = \lambda_1$	$x = \lambda_2$	$x = \lambda_3$
$F(x) \cdot \frac{1}{x-\lambda_1}$	$F'(\lambda_1) \neq 0$	0	0

To get desired result, we divide $F'(\lambda_1)$

	$x = \lambda_1$	$x = \lambda_2$	$x = \lambda_3$
$F_{\lambda_1}(x) \cdot \frac{1}{x-\lambda_1} \cdot \frac{1}{F'(\lambda_1)}$	1	0	0

To sumarize, the interpolation polynomial has the following parts

$$f_{\lambda_1}(x) = F_{\lambda_1}(x) \cdot \underbrace{\frac{1}{F'(\lambda_1)}}_{\text{normalizer}} \cdot \underbrace{\frac{1}{x - \lambda_1}}_{\text{replacer}}.$$

Some side notes for simplification

In the interpolation theorem, for any polynomial $g(x)$, recall that we have

$$g(x) = Q(x)F(x) + \sum_{i=1}^{3} \underbrace{g(\lambda_i)}_{\text{interpolator}} \frac{F(x)}{(x-\lambda_i)F'(\lambda_i)}$$

.

We call the scalar $g(\lambda_i)$ **interpolator**,

	$x = \lambda_1$	$x = \lambda_2$	$x = \lambda_3$
$g(\lambda_1) \cdot F_{\lambda_1}(x) \cdot \frac{1}{x-\lambda_1} \cdot \frac{1}{F'(\lambda_1)}$	$g(\lambda_1)$	0	0
$g(\lambda_2) \cdot F_{\lambda_2}(x) \cdot \frac{1}{x-\lambda_2} \cdot \frac{1}{F'(\lambda_2)}$	0	$g(\lambda_2)$	0
$g(\lambda_3) \cdot F_{\lambda_3}(x) \cdot \frac{1}{x-\lambda_3} \cdot \frac{1}{F'(\lambda_3)}$	0	0	$g(\lambda_3)$

Some side notes for simplification

$$g(x) = Q(x)F(x) + \sum_{i=1}^{3} g(\lambda_i)\frac{F(x)}{(x-\lambda_i)F'(\lambda_i)}$$

.

Analyse each term in the interpolation

$$F(x) \cdot \underbrace{g(\lambda_i)}_{\text{Interpolator}} \cdot \underbrace{\frac{1}{(x-\lambda_i)}}_{\text{Replacer}} \underbrace{\frac{1}{F'(\lambda_i)}}_{\text{Normalizer}}$$

we give the following names.

- $F(x)$: Make sure the value at λ_j is 0 for $j \neq i$.
- **Replacer**: Make sure the value at λ_i is non-zero
- **Normalizer**: Make the value at λ_i to be 1
- **Interpolator**: Make sure the value at λ_i agrees with $g(\lambda_i)$

Interpolation and Partial Fraction Decomposition

For a better elaboration, we introduce **partial fraction decomposition** and its close connection between interpolation.

The original interpolation formula

$$g(x) = Q(x)F(x) + \sum_{i=1}^{3} g(\lambda_i)\frac{F(x)}{(x-\lambda_i)F'(\lambda_i)}$$

can be simplified into:

$$\frac{g(x)}{F(x)} = Q(x) + \sum_{i=1}^{3} \frac{g(\lambda_i)}{F'(\lambda_i)} \cdot \frac{1}{x-\lambda_i}$$

.

This form is called **Partial fraction decomposition**.

Interpolation and Partial Fraction Decomposition

Note the equivalence between

$$\textbf{Lagurange interpolation} \iff \textbf{Partial fraction decomposition}$$

Partial fraction decomposition provides another point of view for Lagurange interpolation.

Interpolation and Partial Fraction Decomposition

General simple partial fraction decomposition takes the form

$$\frac{g(x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)} = Q(x) + \frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \cdots + \frac{a_n}{x-\lambda_n}.$$

where $a_1, \cdots, a_n$ are **scalars**.

$$\text{Lagurange interpolation} \implies \text{Partial Fraction Decomposition}$$

The Lagurange interpolation **proves** the existence of such a formula, and **gives a precise formula for each coefficient** $a_i = \frac{g(\lambda_i)}{F'(\lambda_i)}$.

Interpolation and Partial Fraction Decomposition

$$\frac{g(x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)} = Q(x) + \frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \cdots + \frac{a_n}{x-\lambda_n}.$$

However, sometimes, it might be **easier** and **more understandable** to **determine the coefficient a_i directly** rather than using Lagurange interpolation formula.

For this purpose, we introduce the notion of infinity ∞ and infinitesimal symbol ϵ.

Infinitesimals and infinities

The symbol ϵ is a variable, we are using it to repsent a scalar that it **too small to be a number**, ideally speaking,

$$\epsilon = 0.\underbrace{00\cdots\cdots\cdots\cdots01}_{\text{infintitly many}}.$$

We may introduce expressions like

$$3 + \epsilon = 3.00\cdots\cdots\cdots\cdots03$$

$$1 + 2\epsilon + 5\epsilon^2 = 1.00\cdots\cdots\cdots\cdots020\cdots\cdots\cdots\cdots05.$$

The expression can have infinitely many terms of ϵ^i, for example

$$1+2\epsilon+3\epsilon^2+4\epsilon^3+\cdots = 1.0\cdots\cdots\cdots\cdots020\cdots\cdots\cdots\cdots030\cdots\cdots\cdots\cdots040\cdots$$

Our ϵ is too ideal to be a number, so we would only represent it as a symbol ϵ.

Infinitesimals and infinities

However, when thinking of ϵ, we may try to use number to approach it, the smaller number you choose, the better it behaves.

For example, choosing

$$\epsilon \approx 0.01$$

You may understand the expression

$$\frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$$

In parctice, you may find

$$\frac{1}{0.99} = 1.0101010101010101\cdots$$

When choosing $\epsilon \approx 0.001$

$$\frac{1}{0.999} = 1.001001001001\cdots$$

Infiniteies

We define

$$\infty = \frac{1}{\epsilon}$$

You may understand that

$$\infty = 100 \cdots\cdots 0.$$

We may have expressions like

$$\infty + 2 + 3\epsilon + 4\epsilon^2$$

to represents, something ideally like

$$100 \cdots\cdots 2\ .\ 0 \cdots\cdots 030 \cdots\cdots 04$$

Terminologies

We introduce some terminalogies. The following terminology is mathematics terminology where you can find on Wikipedia

Polynomials: Finite sum, must be non-negative index

$$\sum_{i=0}^{k} a_i x^i$$

Formal Power series: Possibly infinite sum with non-negative index

$$\sum_{i=0}^{\infty} a_i x^i$$

Laurent series: Possibly infinite sum, allow finitely many negative index

$$\sum_{i=-N}^{\infty} a_i x^i$$

Terminologies

We give some name to our new-introduced number systems.
(Terminologies only in our slides!)

Laurant scalar: Could have infinitly many terms involving ϵ^i, but only finitly many terms involving ∞^i

$$\underbrace{a_{-N}\infty^N + a_{-N+1}\infty^{N-1} + \cdots + a_{-1}\infty}_{\text{infinite part}} + \underbrace{a_0}_{\text{constant}} + \underbrace{a_1\epsilon + a_2\epsilon^2 + \cdots}_{\text{infinitesimal part}}$$

Formal scalar: A special Laurant scalar with infinite part equal to 0.

$$a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

Note: All coefficients a_i here are classical scalars $a_i \in \mathbb{C}$.

Arithmetic of Laurent scalars

Any two Laurent scalars can add, subtract , and multiply together.

A Laurent scalar can divide another non-zero Laurent scalar.

The process of calculating expressions involving ϵ or ∞ and representing it to standard form like $a_{-N}\infty^N + \cdots + a_0 + a_1\epsilon + \cdots$ is called **Laurent expansion** (or **Taylor expansion** if there is no ∞ involves.)

$$\frac{1}{(1-\epsilon)^2} = 1 + 2\epsilon + 3\epsilon^2 + 4\epsilon^3 + \cdots$$

Some calculation strategy

In some cases, we have a complicated formal scalar

$$\frac{(1+\epsilon)^2}{(1-2\epsilon)^3} = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

and **suppose we only want** a_0, and don't care about infinitesimal part $a_1, \cdots, a_n$, then we may just evaluate at $\epsilon = 0$ and obtain

$$a_0 = \frac{1^2}{1^3} = 1.$$

Partial fraction decomposition

Excercise.: Decompose the function $\frac{g(x)}{(x-1)(x-2)(x-3)}$ into partial fractions.

Solution. Express $\frac{g(x)}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions:

$$Q(x) + \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

where A, B, and C are constants to be determined.

Partial fraction decomposition

$$Q(x) + \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = \frac{g(x)}{(x-1)(x-2)(x-3)}$$

Determine A, B, and C by:

Let $x = 1 + \epsilon$, then the this equation specialized to

$$Q(1+\epsilon) + A\infty + \frac{B}{-1+\epsilon} + \frac{C}{-2+\epsilon} = \frac{g(1+\epsilon)}{(-1+\epsilon)(-2+\epsilon)}\infty$$

Compare the coefficient at ∞, we obtain

$$A = \frac{g(1)}{(-1)(-2)} = \frac{g(1)}{2}.$$

Letting $x = 2 + \epsilon$, we may find B, for $x = 3 + \epsilon$, we may find C.

Partial fraction decomposition

Therefore, from the partial fraction decomposition, we realized that

$$\frac{g(x)}{(x-1)(x-2)(x-3)}$$

$$= Q(x) + \underbrace{g(1)}_{\text{Interpolator}} \underbrace{\frac{1}{(1-2)(1-3)}}_{\text{Normalizer}} \underbrace{\frac{1}{x-1}}_{\text{replacer}} + \underbrace{g(2)}_{\text{Interpolator}} \underbrace{\frac{1}{(2-1)(2-3)}}_{\text{Normalizer}} \underbrace{\frac{1}{x-2}}_{\text{replacer}}$$

$$+ \underbrace{g(3)}_{\text{Interpolator}} \underbrace{\frac{1}{(3-1)(3-2)}}_{\text{Normalizer}} \underbrace{\frac{1}{x-3}}_{\text{replacer}}$$

To obtain the Lagurange interpolation, we only need to multiply $F(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ on both sides.

Repeated roots

Next we consider a new problem. When we obtain a matrix A with

$$A^2(A - I) = 0$$

and suppose we want to compute $g(A)$, then the following decomposition is crutial for our first step

Excercise. Decompose

$$\frac{g(x)}{x^2(x-1)}.$$

Repeated roots

However, the decomposition of

$$\frac{g(x)}{x^2(x-1)}$$

is unclear since the denominator **have repeated roots**. The factor x shows twice. If we keep using the original formula, we would face some problem.

$$\frac{g(x)}{(x-0)(x-0)(x-1)}$$

$$= Q(x) + \overbrace{\underbrace{g(0)}_{\text{Interpolator}} \underbrace{\frac{1}{(0-0)(0-1)}}_{\text{Normalizer}} \underbrace{\frac{1}{x-0}}_{\text{replacer}}}^{\infty} + \overbrace{\underbrace{g(0)}_{\text{Interpolator}} \underbrace{\frac{1}{(0-0)(0-1)}}_{\text{Normalizer}} \underbrace{\frac{1}{x-0}}_{\text{replacer}}}^{\infty}$$

$$+ \underbrace{g(3)}_{\text{Interpolator}} \underbrace{\frac{1}{(3-1)(3-2)}}_{\text{Normalizer}} \underbrace{\frac{1}{x-3}}_{\text{replacer}}$$

Repeated roots

Idea: since we would have ∞-problem when dealing with repeated roots. We may slightly change the denomenator, and finally use limit

$$\frac{g(x)}{x^2(x-1)} = \lim_{\epsilon \longrightarrow 0} \frac{g(x)}{(x^2-\epsilon^2)(x-1)} = \lim_{\epsilon \longrightarrow 0} \frac{g(x)}{(x-\epsilon)(x+\epsilon)(x-1)}.$$

In our formalism of ϵ,∞, the Partial fraction decomposition of

$$\frac{g(x)}{x^2(x-1)}$$

is **the constant term** of the partial fraction decomposition of

$$\frac{g(x)}{(x-\epsilon)(x+\epsilon)(x-1)}.$$

Example of two term

Introduction: Decompose $\frac{g(x)}{(x-1)x^2}$ as $\varepsilon \to 0$ into $\frac{g(x)}{(x-1)(x^2-\varepsilon^2)}$.

Decomposition Strategy: Express as $\frac{g(x)}{(x-1)(x-\varepsilon)(x+\varepsilon)}$.

$$\frac{g(x)}{(x-1)(x-\varepsilon)(x+\epsilon)} = Q(x) + \frac{A}{x-1} + \frac{B}{x-\epsilon} + \frac{C}{x+\epsilon}$$

This partial fraction decomposition results

$$\underbrace{\frac{g(1)}{(1-\epsilon)(1+\epsilon)}}_{A} \cdot \frac{1}{x-1} + \underbrace{\frac{g(\epsilon)}{(\epsilon-1)(\epsilon+\epsilon)}}_{B} \cdot \frac{1}{x-\epsilon} + \underbrace{\frac{g(-\epsilon)}{(-\epsilon-1)(-\epsilon-\epsilon)}}_{C} \cdot \frac{1}{x+\epsilon}$$

Example of two term

What pattern between the two terms:

$$\underbrace{\frac{g(\epsilon)}{(\epsilon-1)(\epsilon+\epsilon)}}_{B} \cdot \frac{1}{x-\epsilon} \qquad \underbrace{\frac{g(-\epsilon)}{(-\epsilon-1)(-\epsilon-\epsilon)}}_{C} \cdot \frac{1}{x+\epsilon}$$

Observation: Substituting $\epsilon \leftrightarrow -\epsilon$ one obtain the other.

Example of two term

$$\underbrace{g(\epsilon)}_{\text{interpolator}} \overbrace{\frac{1}{\underbrace{(\epsilon - 1)}_{\substack{\text{outside}\\\text{normalizer}}} \underbrace{(\epsilon + \epsilon)}_{\substack{\text{inside}\\\text{normalizer}}}}}^{\text{normalizer}} \cdot \frac{1}{\underbrace{x - \epsilon}_{\text{replacer}}}$$

We seperate the normalizer into two groups.

Outside Normalizer: This part would not go to infinity when $\epsilon \longrightarrow 0$

Inside normalizer: This are all factors that contributes as ∞ when $\epsilon \longrightarrow 0$.

How can we calculate

$$\frac{1}{x-\epsilon} \qquad \frac{1}{\epsilon - 1}?$$

We use the following formula

$$\frac{1}{x-\epsilon} = \frac{1}{x} + \frac{\epsilon}{x^2} + \frac{\epsilon^2}{x^3} + \cdots$$

Geometric Series

Proof: Let $S = \frac{1}{x} + \frac{\varepsilon}{x^2} + \frac{\varepsilon^2}{x^3} + \cdots$. We need to show that $S \times (x - \varepsilon) = 1$.

First, compute $S \times x$ and $S \times \varepsilon$:

$$\begin{aligned} S \times x &= \left(\frac{1}{x} + \frac{\varepsilon}{x^2} + \frac{\varepsilon^2}{x^3} + \cdots\right) \times x \\ &= 1 + \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \cdots \\ S \times \varepsilon &= \left(\frac{1}{x} + \frac{\varepsilon}{x^2} + \frac{\varepsilon^2}{x^3} + \cdots\right) \times \varepsilon \\ &= \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \frac{\varepsilon^3}{x^3} + \cdots \end{aligned}$$

Now, subtract the second equation from the first:

$$\begin{aligned} (S \times x) - (S \times \varepsilon) &= \left(1 + \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \cdots\right) - \left(\frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \frac{\varepsilon^3}{x^3} + \cdots\right) \\ &= 1 \end{aligned}$$

Therefore, $S \times (x - \varepsilon) = 1$.

Calculation

Now let us go back to the summand in partial fraction decomposition

$$\underbrace{g(\epsilon)}_{\text{interpolator}} \quad \frac{1}{\underbrace{(\epsilon - 1)}_{\substack{\text{outside}\\ \text{normalizer}}} \; \underbrace{(\epsilon + \epsilon)}_{\substack{\text{inside}\\ \text{normalizer}}}} \cdot \frac{1}{x - \epsilon}$$

$$= \frac{1}{(\epsilon + \epsilon)} \underbrace{\left(a_0 + a_1\epsilon + \cdots\right)}_{g(\epsilon)} \underbrace{\left(-1 - \epsilon - \epsilon^2 - \cdots\right)}_{\frac{1}{\epsilon - 1}} \underbrace{\left(\frac{1}{x} + \epsilon\frac{1}{x^2} + \epsilon^2\frac{1}{x^3} + \cdots\right)}_{\frac{1}{x-\epsilon}}$$

$$= \frac{1}{2} \cdot \frac{1}{\epsilon}(* + *\epsilon + *\epsilon^2 + \cdots)$$

$$= \frac{1}{2} \cdot \left(\frac{*}{\epsilon} + * + *\epsilon + *\epsilon^2 + \cdots\right)$$

Question: Does the limit exists when $\epsilon \longrightarrow 0$? what happens?

Calculation

Conclusion, the limit

$$\lim_{\epsilon \longrightarrow 0} \frac{g(\epsilon)}{(\epsilon - 1)(\epsilon + \epsilon)} \cdot \frac{1}{x - \epsilon}$$

does not exist in general

Calculation

Then, how about

$$\lim_{\epsilon \longrightarrow 0} \left(\frac{g(\epsilon)}{(\epsilon - 1)(\epsilon + \epsilon)} \cdot \frac{1}{x - \epsilon} + \frac{g(-\epsilon)}{(-\epsilon - 1)(-\epsilon - \epsilon)} \cdot \frac{1}{x + \epsilon} \right)$$

Calculation

Let us say

$$\frac{g(\epsilon)}{(\epsilon - 1)(\epsilon + \epsilon)} \cdot \frac{1}{x - \epsilon} = \frac{1}{2} \cdot \left(\frac{c_{-1}(x)}{\epsilon} + c_0(x) + c_1(x)\epsilon + c_2(x)\epsilon^2 + \cdots \right)$$

Then

$$\frac{g(-\epsilon)}{(-\epsilon - 1)(-\epsilon - \epsilon)} \cdot \frac{1}{x + \epsilon} = \frac{1}{2} \cdot \left(\frac{c_{-1}(x)}{-\epsilon} + c_0(x) + c_1(x)(-\epsilon) + c_2(x)(-\epsilon)^2 + \cdots \right.$$

Calculation

$$\left(\frac{g(\epsilon)}{(\epsilon-1)(\epsilon+\epsilon)}\cdot\frac{1}{x-\epsilon}+\frac{g(-\epsilon)}{(-\epsilon-1)(-\epsilon-\epsilon)}\cdot\frac{1}{x+\epsilon}\right)$$

$$= c_0(x)+c_2(x)\epsilon^2+c_4(x)\epsilon^4+\cdots$$

Calculation

$$\lim_{\epsilon \longrightarrow 0} \left(\frac{g(\epsilon)}{(\epsilon - 1)(\epsilon + \epsilon)} \cdot \frac{1}{x - \epsilon} + \frac{g(-\epsilon)}{(-\epsilon - 1)(-\epsilon - \epsilon)} \cdot \frac{1}{x + \epsilon} \right) = c_0(x)$$

$$= \mathrm{Const}_{\epsilon} \left(\frac{g(\epsilon)}{(\epsilon - 1)(\epsilon + \epsilon)} \cdot \frac{1}{x - \epsilon} \right)$$

Example of general term

$$\frac{g(x)}{x^n(x-1)^2}$$

We consider it as when $\epsilon \longrightarrow 0$

$$\frac{g(x)}{(x^n-\epsilon^n)(x-1)^2}$$

To factorize $(x^n - \epsilon^n)$, we need n'th root of unity ζ_n,

Example of general term

The n-th roots of unity are given by:

$$\zeta_n^k = \exp\left(\frac{2\pi i k}{n}\right) = i \sin\frac{2\pi k}{n} + \cos\frac{2\pi k}{n}, \quad k = 0, 1, \ldots, n-1.$$

where $i = \sqrt{-1}$ is the imaginary unit and exp denotes the exponential function.

The polynomial $x^n - \epsilon^n$ has the following factorization

$$x^n - \epsilon^n = (x - \epsilon)(x - \zeta_n \epsilon)(x - \zeta_n^2 \epsilon) \cdots (x - \zeta_n^{n-1} \epsilon).$$

Example of general term

For simplicity, we start with an example of $n = 3$

$$\frac{g(x)}{(x^3 - \epsilon^3)(x-1)^2}$$

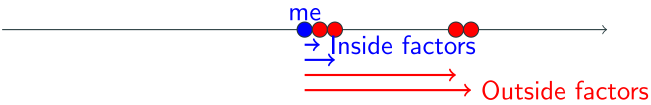
Let ζ_3 to denote the 3rd root of unity. $\zeta_3^3 = 1$ but $\zeta_3 \neq 1$, $\zeta_3^2 \neq 1$.

$$\frac{g(x)}{(x-\epsilon)(x-\zeta_3\epsilon)(x-\zeta_3^2\epsilon)(x-1)^2}$$

Example of general term

We structurize the normalizer by calling the following

$$\cdots + \underbrace{g(\epsilon)}_{\text{interpolator}} \cdot \overbrace{\underbrace{\frac{1}{(\epsilon-\zeta_3\epsilon)(\epsilon-\zeta_3^2\epsilon)}}_{\text{Inside Normalizer}} \cdot \underbrace{\frac{1}{(\epsilon-1)^2}}_{\text{Outside Normalizer}}}^{\text{Normalizer}} \cdot \underbrace{\frac{1}{x-\epsilon}}_{\text{Replacer}} + \cdots$$



Example of general term

The summand associated to the factor $x - \epsilon$

$$\cdots + g(\epsilon) \cdot \frac{1}{(\epsilon - \zeta_3\epsilon)(\epsilon - \zeta_3^2\epsilon)} \cdot \frac{1}{(\epsilon - 1)^2} \cdot \frac{1}{x - \epsilon} + \cdots$$

Question 1: What is the summand associated to other factors $x - \zeta_3\epsilon$ and $x - \zeta_3^2\epsilon$?

Example of general term

We factor out ϵ

$$\cdots + g(\epsilon) \cdot \frac{1}{\epsilon^2(1-\zeta_3)(1-\zeta_3^2)} \cdot \frac{1}{(\epsilon-1)^2} \cdot \frac{1}{x-\epsilon} + \cdots$$

Question 2: What is the product $(1-\zeta_3)(1-\zeta_3^2)$ equal?

Example of general term

Answer to question 2:

$$x^3 - 1 = (x-1)(x-\zeta_3)(x-\zeta_3^2)$$

Take derivative on both sides

$$3x^2 = (x-1)(x-\zeta_3) + (x-\zeta_3)(x-\zeta_3^2) + (x-1)(x-\zeta_3^2)$$

Let $x = 1$, we have

$$3 = (1-\zeta_3)(1-\zeta_3^2)$$

!!

In general, for $\zeta_n = i \sin \frac{2\pi}{n} + \cos \frac{2\pi}{n}$, we have

$$(1-\zeta_n)(1-\zeta_n^2)\cdots(1-\zeta_n^{n-1}) = n$$

Example of general term

Now we have arrived to the stage

$$\cdots + g(\epsilon) \cdot \underbrace{\frac{1}{\epsilon^2(1-\zeta_3)(1-\zeta_3^2)}}_{\frac{1}{3}\cdot\frac{1}{\epsilon^2}} \cdot \frac{1}{(\epsilon-1)^2} \cdot \frac{1}{x-\epsilon} + \cdots$$

Finally, this equals to

$$\underbrace{\frac{1}{3}\cdot\frac{1}{\epsilon^2}}_{\text{Inside normalizer}} \cdot \underbrace{\left(* + *\epsilon + *\epsilon^2 + \cdots\right)}_{\text{Outside normalizer } \times \text{ Replacer } \times \text{ Interpolator}}$$

We can write it to

$$\frac{1}{3}\left(\frac{c_{-2}(x)}{\epsilon^2} + \frac{c_{-1}(x)}{\epsilon} + c_0(x) + c_1(x)\epsilon + \cdots\right)$$

where $c_i(x)$ are some polynomials of x.

Example of general term

The summand associated to the factor $x - \epsilon$ gives

$$\frac{1}{3}\left(\frac{c_{-2}(x)}{\epsilon^2} + \frac{c_{-1}(x)}{\epsilon} + c_0(x) + c_1(x)\epsilon + \cdots\right)$$

The summand associated to the factor $x - \zeta_3\epsilon$ gives

$$\frac{1}{3}\left(\frac{c_{-2}(x)}{(\zeta_3\epsilon)^2} + \frac{c_{-1}(x)}{\zeta_3\epsilon} + c_0(x) + c_1(x)(\zeta_3\epsilon) + \cdots\right)$$

The summand associated to the factor $x - \zeta_3^2\epsilon$ gives

$$\frac{1}{3}\left(\frac{c_{-2}(x)}{(\zeta_3^2\epsilon)^2} + \frac{c_{-1}(x)}{\zeta_3^2\epsilon} + c_0(x) + c_1(x)(\zeta_3^2\epsilon) + \cdots\right)$$

Example of general term

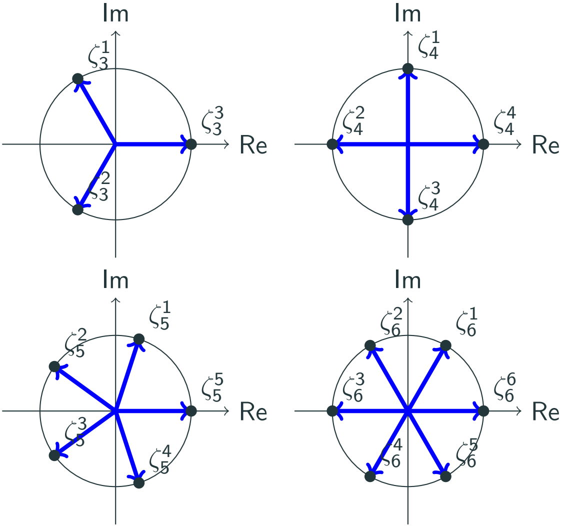
The sum of the above three summand equals

$$c_0(x) + c_3(x)\epsilon^3 + c_6(x)\epsilon^6 + c_9(x)\epsilon^9 + \cdots$$

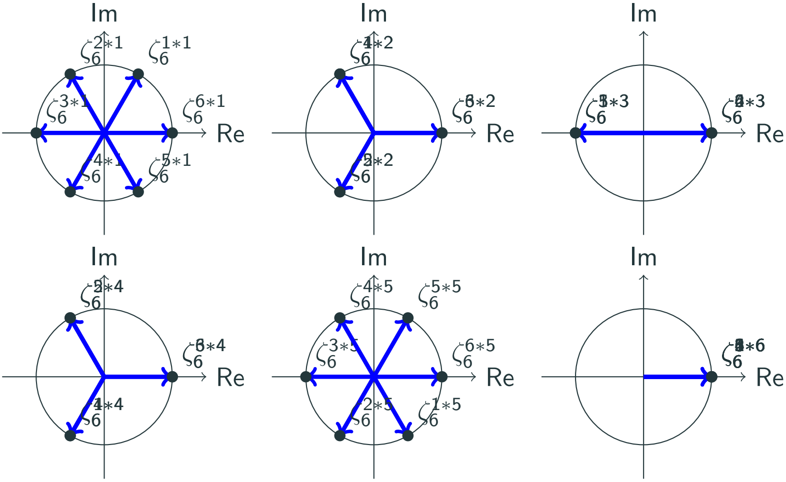
which using the fact that

$$1^n + \zeta_3^n + \zeta_3^{2n} = \begin{cases} 3 & n \in 3\mathbb{Z} \\ 0 & n \notin 3\mathbb{Z} \end{cases}$$

Example of general term



Example of general term



Example of general term

$$\zeta_6 + \zeta_6^2 + \zeta_6^3 + \zeta_6^4 + \zeta_6^5 + \zeta_6^6 = 0$$

$$(\zeta_6^2) + (\zeta_6^2)^2 + (\zeta_6^2)^3 + (\zeta_6^2)^4 + (\zeta_6^2)^5 + (\zeta_6^2)^6 = 0$$

$$(\zeta_6^3) + (\zeta_6^3)^2 + (\zeta_6^3)^3 + (\zeta_6^3)^4 + (\zeta_6^3)^5 + (\zeta_6^3)^6 = 0$$

$$(\zeta_6^4) + (\zeta_6^4)^2 + (\zeta_6^4)^3 + (\zeta_6^4)^4 + (\zeta_6^4)^5 + (\zeta_6^4)^6 = 0$$

$$(\zeta_6^5) + (\zeta_6^5)^2 + (\zeta_6^5)^3 + (\zeta_6^5)^4 + (\zeta_6^5)^5 + (\zeta_6^5)^6 = 0$$

$$(\zeta_6^6) + (\zeta_6^6)^2 + (\zeta_6^6)^3 + (\zeta_6^6)^4 + (\zeta_6^6)^5 + (\zeta_6^6)^6 = 6$$

In general

$$1^n + \zeta_n^k + (\zeta_n^2)^k + \cdots + (\zeta_n^{n-1})^k = \begin{cases} n & k \in n\mathbb{Z} \\ 0 & k \notin n\mathbb{Z} \end{cases}$$

Partial fraction decomposition for repeated roots

The general form of partial fraction decomposition for with repeated roots.

Theorem 1

Suppose

$$F(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\cdots(x-\lambda_k)^{n_k}$$

and let

$$K_i(x) = \frac{F(x)}{(x-\lambda_i)^{n_i}}$$

Then we have **partial fraction decomposition**

$$\frac{g(x)}{F(x)} = Q(x) + \sum_{i=1}^{k} \mathrm{Const}_{\epsilon}\left(g(\lambda_i+\epsilon)\cdot\frac{1}{\epsilon^{n_i-1}}\cdot\frac{1}{K_i(\lambda_i+\epsilon)}\cdot\frac{1}{x-\lambda_i-\epsilon}\right)$$

Partial fraction decomposition for repeated roots

The formula is resonable but hard to memorize. We wanna simplify it by writting each term as Laurant series of ϵ and focus on its coefficients

$$\underbrace{g(\lambda_i + \epsilon) \cdot \frac{1}{\epsilon^{n_i - 1}} \cdot \frac{1}{K_i(\lambda_i + \epsilon)}}_{\text{interpolator}\cdot\text{normalizer}} \cdot \underbrace{\frac{1}{x - \lambda_i - \epsilon}}_{\text{replacer}}$$

Partial fraction decomposition for repeated roots

Now we define some notion to describe how large it would be **around** $x = \lambda_i$. Define

$$\lambda_i - \text{infinite-degree}\left(\frac{\infty^a}{(x-\lambda_i)^b}\right) = a + b$$

This describes that when $x \longrightarrow \lambda_i$, the result is around ∞^{a+b}, in other words, it is describing the value at $x = \lambda_i + \epsilon$

$$\frac{\infty^a}{((\lambda_i+\epsilon)-\lambda_i)^b} = \frac{\infty^a}{\epsilon^b} = \infty^a\infty^b = \infty^{a+b}.$$

Let $A(x, \epsilon)$ be of λ_i-inf-deg a and $B(x, \epsilon)$ be of λ_i-inf-deg b, then $A(x, \epsilon)B(x, \epsilon)$ has λ_i-inf-deg $a + b$.

Partial fraction decomposition for repeated roots

The coeffficeint for the replacer using geometric expansion

$$\frac{1}{x-\lambda_i-\epsilon} = \frac{1}{x-\lambda_i} + \frac{\epsilon}{(x-\lambda_i)^2} + \cdots$$

Each of the above summand **is of infinite degree -1.**

When multiplying by ∞^{n_i-1},

$$\frac{1}{\epsilon^{n_i-1}} \cdot \frac{1}{x-\lambda_i-\epsilon} = \frac{\infty^{n_i-1}}{x-\lambda_i} + \frac{\infty^{n_i-2}}{(x-\lambda_i)^2} + \cdots + \frac{1}{(x-\lambda_i)^{n_i}} + \frac{\epsilon}{(x-\lambda_i)^{n_i+1}} + \cdots$$

Observation: **Each term is of infinite degree n_i.**

Partial fraction decomposition for repeated roots

The coefficeint for the normalizer and interpolator, since they do not contain variable x, we can write

$$g(\lambda_i + \epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots \qquad \frac{1}{K_i(\lambda_i + \epsilon)} = b_0 + b_1\epsilon + b_2\epsilon^2 + \cdots$$

for $a_i, b_i \in \mathbb{C}$. Note all their coefficient are **scalars**!

Observation on degree: All summand has λ_i-inf-degree ≤ 0.

Partial fraction decomposition for repeated roots

Therefore, as a polynomial of ϵ, when calculating their product

$$\underbrace{g(\lambda_i+\epsilon)\cdot\frac{1}{\epsilon^{n_i-1}}\cdot\frac{1}{K_i(\lambda_i+\epsilon)}}_{\text{scalar coefficient}}\cdot\underbrace{\frac{1}{x-\lambda_i-\epsilon}}_{\frac{1}{(x-\lambda_i)^k}-\text{coefficient}}$$

Each coefficient of ∞^i or ϵ^i can only be taken as product and sum by elements from the set

$$\{\text{scalars}\}\cup\left\{\frac{1}{x-\lambda_i},\frac{1}{(x-\lambda_i)^2},\frac{1}{(x-\lambda_i)^3},\cdots\right\}$$

First Observation: Each coefficient of ∞^i or ϵ^i can only be in the form

$$\frac{a_1}{x-\lambda_i}+\frac{a_2}{(x-\lambda_i)^2}+\frac{a_3}{(x-\lambda_i)^3}+\cdots \qquad a_i\in\mathbb{C}$$

Partial fraction decomposition for repeated roots

Now we Consider the infinite degree at λ_i. The term for ∞^k

$$\left(\frac{a_1}{x-\lambda_i}+\frac{a_2}{(x-\lambda_i)^2}+\frac{a_3}{(x-\lambda_i)^3}+\cdots\right)\infty^k$$

Since all its terms can only have λ_i-inf-degree $\leq \mathrm{n}_i$, the coefficeint is at most sum up to $n_i - k$, like

$$\left(\frac{a_1}{x-\lambda_i}+\frac{a_2}{(x-\lambda_i)^2}+\frac{a_3}{(x-\lambda_i)^3}+\cdots+\frac{a_{n_i-k}}{(x-\lambda_i)^{n_i-k}}\right)\infty^k$$

Partial fraction decomposition for repeated roots

Therefore

$$\mathrm{Const}_{\epsilon}\left(g(\lambda_i+\epsilon)\cdot\frac{1}{\epsilon^{n_i-1}}\cdot\frac{1}{K_i(\lambda_i+\epsilon)}\cdot\frac{1}{x-\lambda_i-\epsilon}\right)$$

$$= \text{Coefficient of } \epsilon^0 intheexpansion$$

$$= \frac{a_1}{x-\lambda_i}+\frac{a_2}{(x-\lambda_i)^2}+\frac{a_3}{(x-\lambda_i)^3}+\cdots+\frac{a_{n_i}}{(x-\lambda_i)^{n_i}}$$

for some scalar $a_1, a_2, \cdots \in \mathbb{C}$ The sum is of degree up to n_i

Partial fraction decomposition for repeated roots

Therefore, although the formula

$$\frac{g(x)}{F(x)} = Q(x) + \sum_{i=1}^{k} \mathrm{Const}_{\epsilon} \left(g(\lambda_i + \epsilon) \cdot \frac{1}{\epsilon^{n_i - 1}} \cdot \frac{1}{K_i(\lambda_i + \epsilon)} \cdot \frac{1}{x - \lambda_i - \epsilon} \right)$$

is explicit and concrete, we would like to emphasis its

Theorem 2

The general partial fraction decomposition we can decompose

$$\frac{g(x)}{(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\cdots(x-\lambda_k)^{n_k}} = Q(x) + \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{a_{i,j}}{(x-\lambda_i)^j}$$

Interpolation for repeated root

Excercise. Find a function with the following property

$$g(1) = 2 \qquad g'(1) = 3 \qquad g(2) = 3 \qquad g'(2) = 1$$

This is $g(1+\epsilon) = 2 + 3\epsilon$ and $g(2+\epsilon) = 3 + \epsilon$.

Solution.

Using the decomposition, we may decompose this fraction into

$$\frac{g(x)}{(x-1)^2(x-2)^2} = Q(x) + \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}$$

Interpolation for repeated root

Plug in $x = 1 + \epsilon$, we have

$$\frac{g(1+\epsilon)}{((1+\epsilon)-1)^2((1+\epsilon)-2)^2}$$

$$= Q(1+\epsilon) + \frac{a}{(1+\epsilon)-1} + \frac{b}{((1+\epsilon)-1)^2} + \frac{c}{(1+\epsilon)-2} + \frac{d}{((1+\epsilon)-2)^2}$$

We only interested in coefficient of ∞ and ∞^2. now calculate left hand side

$$\frac{g(1+\epsilon)}{((1+\epsilon)-1)^2((1+\epsilon)-2)^2} = \frac{g(1+\epsilon)}{\epsilon^2(-1+\epsilon)^2} = \frac{2+3\epsilon}{(-1+\epsilon)^2}\infty^2$$

$$= (2 + 7\epsilon + *\epsilon^2 + ..)\infty^2 = 2\infty^2 + 7\infty + * + *\epsilon + \cdots$$

At the same time, the right hand side equals

$$* + a\infty + b\infty^2 + * + *$$

So $a = 7, \qquad b = 2$

We use the same method to find the value of c and d. By plug in