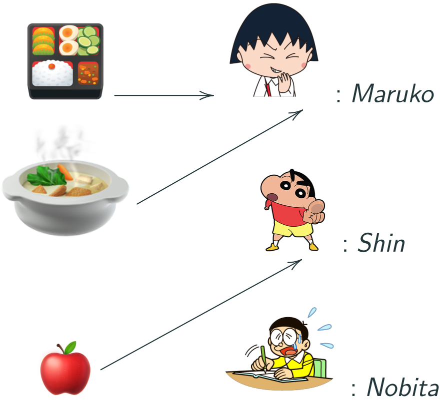
Problems while taking the inverse

When there is a map $T : V \longrightarrow W$. We would like to know is that possible to have an inverse of the map

$$T^{-1} : W \longrightarrow V$$

Problems while taking the inverse

Let′s consider the example of the restaurant before



The map is assign each food to a customer. Its inverse is to assign customer to a food. As a map, the assignment must be exists and unique.

Problems while taking the inverse

Let's consider if that is possible to assign in the inverse way. You are working in this food shop. One day I come and ask you: Give me **the** food that  ordered, which of the following question would you ask me:

A. **No Problem** . It will be ready in 1 munites.

B. **uniqueness problem** : She ordered multiple foods, which one do you refer? The food she order is not **unique** .

C. **existence problem** : He never order food with us? Would you chek it again? The food he order does not **exists** .

Problems while taking the inverse

I change my request to **the** food that custormer  or  ordered, fill in the following table of your answer

I wanna food of			
Your Question:	B.**uniqueness problem**		

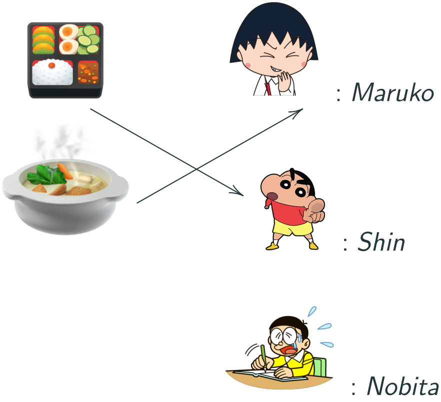
Concepts of injective and surjective

The previous example shows that it is generally impossible to define an inverse of a map. It might have **uniqueness problem** or **existence problem** . We want to focus on maps that **do not** has those problems.

Therefore we define **injective** and **surjective** maps.

Injective Maps

For maps that defining the inverse of it **does not** have **uniqueness problem** , we call it an **injective**



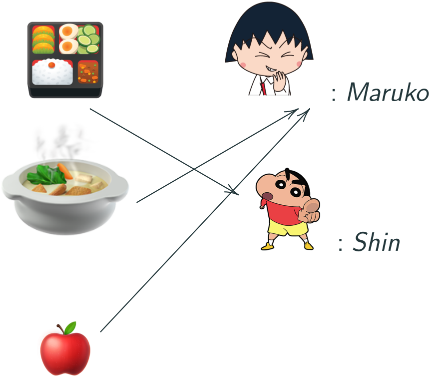
Injective Maps

For above situation. Fill in the following table when I request foods.

I wanna food of			
Your Question:	A.**No Problem**		

Surjective Maps

For maps that that defining the inverse of it **does not** have **existence problem** for all element, we call it an **surjective**



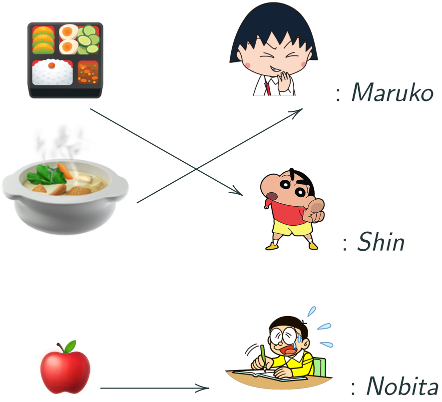
Surjective Maps

For above situation. Fill in the following table when I request foods.

I wanna food of		
Your Question:	B.**uniqueness problem**	

Invertible Maps

For maps that that defining the inverse of has **No Problem** , we call it **invertible**



Invertible Maps

For above situation. Fill in the following table when I request foods.

I wanna food of			
Your Question:	A.**No Problem**		

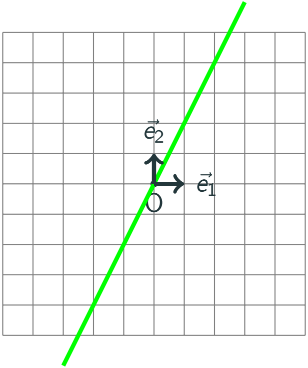
In this case, find the food that each custormer ordered.

I wanna food of			
Food:	A. 		

The above table is the **inverse** of this map.

Injective linear transformation

Let $T : V \longrightarrow V$ be a linear map defined by the orthogonal projection to $\mathrm{span}\,\{\vec{e}_1 + 2\vec{e}_2\}$.

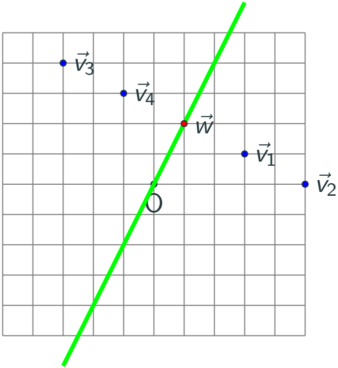


Why T is not invertible?

Injective linear transformation

There is an **uniqueness problem** . we don′t know which value should be assigned to $T^{-1}(\vec{w})$. **All blue points seems wants to project on $\vec{w}$.**

$$T(\vec{v}_1) = T(\vec{v}_2) = T(\vec{v}_3) = \cdots = \vec{w}$$



$$T^{-1}(\vec{w}) = \vec{v}_1? \qquad \vec{v}_2? \qquad \vec{v}_3?$$

Injective linear transformation

We call a linear map that does not have **uniqueness problem** for defining inverse as **injective**

Definition 1

A linear map $T : V \longrightarrow W$ is called an **injective** if $T(\vec{v}_1) = T(\vec{v}_2)$ implies $\vec{v}_1 = \vec{v}_2$ for any $\vec{v}_1, \vec{v}_2 \in V$.

In this situation, if $T(\vec{v}_1) = T(\vec{v}_2) = T(\vec{v}_3) = \cdots = \vec{w}$, we have $\vec{v}_1 = \vec{v}_2 = \cdots$ and we could **possibly** just define the inverse $T^{-1}(\vec{w})$ by $\vec{v}_1$.

But remember, even if for **injective** maps there is **NO** **uniqueness problem** , we can define $T^{-1}(\vec{w})$ for some $\vec{w}$, we may not able to define T^{-1} for other vectors, since it **MAY** have **existence problem** .

An alternative definition for injective maps

Proposition 1

A linear map $T : V \longrightarrow W$ is an **injective** if and only if $T(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$ for all $\vec{v} \in V$.

Proof.

If $T(\vec{v}) = \vec{0}$ implies $\vec{v} = 0$ for all $\vec{v} \in V$, then for any $\vec{v_1}, \vec{v_2} \in V$ if $T(\vec{v_1}) = T(\vec{v_2})$, then $T(\vec{v_1} - \vec{v_2}) = \vec{0}$ therefore $\vec{v_1} - \vec{v_2} = \vec{0}$ so $\vec{v_1} = \vec{v_2}$.

On the contary, if $T(\vec{v_1}) = T(\vec{v_2})$ implies $\vec{v_1} = \vec{v_2}$ for any $\vec{v_1}, \vec{v_2} \in V$. Then if $T(\vec{v}) = \vec{0}$, then we have $T(\vec{v}) = T(\vec{0})$, this implies $\vec{v} = \vec{0}$.

We proved this two defitition are equivalent. $\square$

Properties of injective maps

From this section we demonstrate three important properties of injective maps.

Left Cancellation for Injective linear maps

Three important properties of injective map: No.1

Proposition 2

Let $T : W \longrightarrow U$ and $S : V \longrightarrow W$ $R : V \longrightarrow W$ be linear maps, if T is an **injective** map, then **left cancellation** rule holds for T.

$$T \circ R = T \circ S \implies R = S$$

Proof.

We need to show for any $\vec{v} \in V$, $R(\vec{v}) = S(\vec{v})$, indeed, let $\vec{w}$ be the element

$$\vec{w} = T(R(\vec{v})) = T(S(\vec{v})).$$

Since T is an **injective** , we have $R(\vec{v}) = S(\vec{v})$. Our proof does not depends on the choice of $\vec{v}$, therefore $R = S$. $\square$

Left Cancellation for Injective linear maps

We understand Left cancellation in plain words, in the following diagram.

$$V \xrightarrow{S} W \xrightarrow{T} U$$

One may ask if one can choose a different S but keep the composition $T \circ S$. Let us say for some $\vec{u}$ we have $T \circ S(\vec{v}) = \vec{u}$, then the freedom of the choice of S should garantee $S(\vec{v})$ in the preimage $T^{-1}(\{\vec{u}\})$

$$S(\vec{v}) \in T^{-1}(\{\vec{u}\})$$

In general, $T^{-1}(\{\vec{u}\})$ might have more than 2 elements so we can choose different S without influencing $T \circ S$. But when T is **injective** , every preimage of a point is **unique** . So there is no other way to choose a different S.

Composition of Injective linear maps

Three important properties of injective map: No.2

Proposition 3

Let $T : V \longrightarrow U$, $S : U \longrightarrow W$ be two **injective** linear maps, then $S \circ T$ is also an **injective** map.

Proof.

We assume

$$S(T(\vec{v}_1)) = S(T(\vec{v}_2))$$

Since S is an **injective** this implies

$$T(\vec{v}_1) = T(\vec{v}_2).$$

Since T is an **injective** this implies

$$\vec{v}_1 = \vec{v}_2.$$

Composition of Injective linear maps

In plain words, in the following diagram.

$$V \xrightarrow{S} W \xrightarrow{T} U$$

If both maps are **injective** , then for any two different element in V, it keep different after each step. The total effect is they are still different element in U.

Right Factor of injective composition

Three important properties of injective map: No.3

Proposition 4

Let $T : V \longrightarrow U$, $S : U \longrightarrow W$ be two linear maps, suppose $S \circ T$ is **injective** map, then the **right factor** T must be **injective** .

Proof.

Assume $T(\vec{v}_1) = T(\vec{v}_2)$, Left compose with S we have

$$S \circ T(\vec{v}_1) = S \circ T(\vec{v}_2)$$

Since $S \circ T$ is an **injective** , therefore

$$\vec{v}_1 = \vec{v}_2.$$

We proved $T(\vec{v}_1) = T(\vec{v}_2)$ implies $\vec{v}_1 = \vec{v}_2$, therefore T is an **injective** . $\square$

Right Factor of injective composition

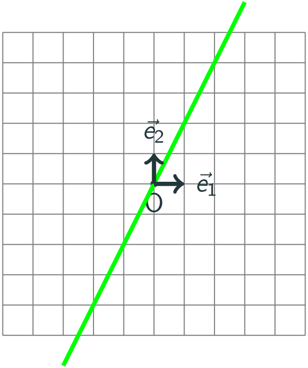
In plain words, in the following diagram.

$$V \overset{S}{\rightarrowtail} W \overset{T}{\rightarrowtail} U$$

If the composition of two maps are **injective** , then any two different element in V has to keep different until it reaches the destination U, to do so, they must still keep different in W, otherwise if they stick together in W, they will never become different in U.

Surjective Maps

Let $T : V \longrightarrow V$ be a linear map defined by the orthogonal projection to $\operatorname{span}\{\vec{e}_1 + 2\vec{e}_2\}$.

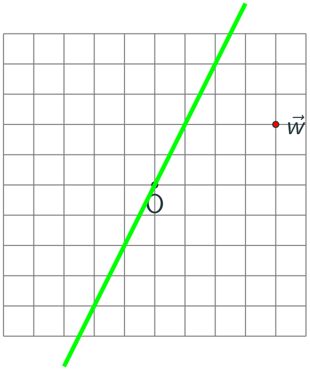


Why T is not invertible?

Surjective Maps

There is an **existence problem** . **There is no $\vec{v}$ such that $T(\vec{v}) = \vec{w}$.** we can't assign any value to $T^{-1}(\vec{w})$.

$$T(?) = \vec{w}$$



$$T^{-1}(\vec{w}) =?$$

Surjective Maps

We call a linear map that does not have **existence problem** for defining inverse as **surjective**

Definition 2

A linear map $T : V \longrightarrow W$ is called an **surjective** if for all $\vec{w} \in W$, there always **exists** $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

In this situation, we could **possibly** define the inverse $T^{-1}(\vec{w})$ by some element.

But remember, even if for **surjective** maps there is **NO existence problem** , we might not able to define $T^{-1}(\vec{w})$ since it **MAY** have **uniqueness problem** .

Three important properties of surjective map: No.1

Proposition 5

Let $T : V \longrightarrow W$ and $S : W \longrightarrow U$ $R : W \longrightarrow U$ be linear maps, if T is an **surjective** map, then **right cancellation** rule holds for T.

$$R \circ T = S \circ T \implies R = S$$

Proof.

We need to show for any $\vec{w} \in W$, $R(\vec{w}) = S(\vec{w})$. Since T is an **surjective** map, **there exists** $\vec{v} \in V$ such that $\vec{w} = T(\vec{v})$, since $R \circ T = S \circ T$, we have

$$R(T(\vec{v})) = S(T(\vec{v})).$$

Therefore $R(\vec{w}) = S(\vec{w})$.. $\square$

Three properties of surjective maps

In plain words, in the following diagram.

$$V \overset{T}{\twoheadrightarrow} W \overset{S}{\twoheadrightarrow} U$$

One may ask if one can choose a different S but keep the composition $S \circ T$. Since $S \circ T$ only see the correspondance between V and U, this correspondance is transfered by W. So there might be some *lonely* element in W which does not corresponds by any $\vec{v} \in V$. For those *lonely* element $\vec{w} \in W$, no matter how we define $S(\vec{w})$, it does not affect the composition $S \circ T$.

In the case of **surjective** , there is no *lonely* element, therefore we have no freedom to change S without $S \circ T$ changed. This is the main sense of the right cancellation.

Composition of Surjective linear maps

Three important properties of surjective map: No.2

Proposition 6

Let $T : V \longrightarrow U$, $S : U \longrightarrow W$ be two **surjective** linear maps, then $S \circ T$ is also an **surjective** map.

Proof.

For any $\vec{w} \in W$, since S is an **surjective** , **there exists** $\vec{u} \in U$ such that

$$\vec{w} = S(\vec{u})$$

For this $\vec{u}$, **there exists** $\vec{v} \in V$ such that $T(\vec{v}) = \vec{u}$.

The whole proof indicates for any $\vec{w} \in W$ we could found $\vec{v} \in V$ such that $\vec{w} = S \circ T(\vec{v})$. Therefore $S \circ T$ is a **surjective** . $\square$

Composition of Surjective linear maps

In plain words, in the following diagram.

$$V \overset{T}{\twoheadrightarrow} W \overset{S}{\twoheadrightarrow} U$$

If both are **surjective** , then all element in V maps on all element in W, then all element in W maps on all element in U.

Totally, all elements in V maps on all elements in U, by $S \circ T$, so $S \circ T$ is a **surjective** .

Left Factor of injective composition

Three important properties of surjective map: No.3

Proposition 7

Let $T : V \longrightarrow U$, $S : U \longrightarrow W$ be two linear maps, suppose $S \circ T$ is a **surjective** map, then the **left factor** S must be **surjective** .

Proof.

For $\vec{w} \in W$, since $S \circ T$ is surjective, **there exists** $\vec{v} \in V$ such that

$$\vec{w} = S(T(\vec{v})).$$

Let $\vec{u} = T(\vec{v})$, so $\vec{w} = S(\vec{u})$. We proved for any $\vec{w} \in W$, **there exists** $\vec{u} \in U$, such that $\vec{w} = S(\vec{u})$, therefore S is a **surjective** . $\square$

Left Factor of injective composition

In plain words, in the following diagram.

$$V \xrightarrow{T} W \xrightarrow{S} U$$

If $S \circ T$ is a **surjective** , then all elemnts in V maps on all elemnts in U, since element in V arrives U throught W, **think element in W as a car**. All elements in W that carrying an element in V must maps to all elements in U. Therefore all non-empty cars maps onto U.(S is a **surjective**)

Non-empty cars has already occupied all element in U, no matter where those empty car go, All cars occupied all elements in U.

Invertible linear transformation

We call a linear map that have **No Problem** for defining inverse as **isomorphism** .

Definition 3

A linear map $T : V \longrightarrow W$ is called an **isomorphism** if for all $\vec{w} \in W$, **there exists an unique** $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

In this situation, we could define the inverse $T^{-1}(\vec{w}) = \vec{w}$. This linear transformation saitiesfies

$$T^{-1} \circ T = \mathrm{id}_V; \qquad T \circ T^{-1} = \mathrm{id}_W.$$

Three important properties for isomorphisms

Proposition 8

An **isomorphism** map have both **left cancellation** and **right cancellation** rule.

Proposition 9

A compostition of two **isomorphism** must be an **isomorphism** .

Proposition 10

Let $T : V \longrightarrow U$, $S : U \longrightarrow W$ be two linear maps, suppose $S \circ T$ is a **isomorphism** , then the **left factor** S must be **surjective** and **right factor** T must be **injective** .

Three important properties for isomorphisms

Excercise. The last proposition didn′t say both T, S be **isomorphism** .
Find a counter example that two linear maps $T : V \longrightarrow U$, $S : U \longrightarrow W$ with $S \circ T$ an **isomorphism** , but both of them are not **isomorphism** .

Solution. Consider maps $T : F \longrightarrow F^3$ $S : F^3 \longrightarrow F$ defined by

$$T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Then

$$S \circ T = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

is invertible. But each S or T is not. This is also an example, the product of two matrices is invertible but each of them may not.

Properties of Isomorphisms

We call a linear map that have **No Problem** for defining inverse as **isomorphism** .

Definition 4

A linear map $T : V \longrightarrow W$ is called an **isomorphism** if for all $\vec{w} \in W$, **there exists an unique** $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

In this situation, we could define the inverse $T^{-1}(\vec{w}) = \vec{w}$. This linear transformation saitiesfies

$$T^{-1} \circ T = \mathrm{id}_V; \qquad T \circ T^{-1} = \mathrm{id}_W.$$

Three important properties for isomorphisms

Proposition 11

An **isomorphism** map have both **left cancellation** and **right cancellation** rule.

Proposition 12

A compostition of two **isomorphism** must be an **isomorphism** .

Proposition 13

Let $T : V \longrightarrow U$, $S : U \longrightarrow W$ be two linear maps, suppose $S \circ T$ is a **isomorphism** , then the **left factor** S must be **surjective** and **right factor** T must be **injective** .

Three important properties for isomorphisms

Excercise. The last proposition didn′t say both T, S be **isomorphism** . Find a counter example that two linear maps $T : V \longrightarrow U$, $S : U \longrightarrow W$ with $S \circ T$ an **isomorphism** , but both of them are not **isomorphism** .

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