Real Quadratic Form and symmetric matrices

Suppose you are working with polynomials $\{a + bx : a, b \in \mathbb{R}\}$, and you see a note on the table claiming that they have defined an inner product to this space by

$$\langle 1, 1 \rangle = 2; \qquad \langle 1, x \rangle = 3; \qquad \langle x, x \rangle = 5$$

You wonder how this inner product being defined?

Real Quadratic Form and symmetric matrices

Your frinend tells you that their method is let

$$f(x) \longmapsto \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$$

so that they define $\langle f, g \rangle = f(1)g(1) + f(2)g(2)$.

Why not $f(1)g(1) + f(3)g(3)$??? Why not $f(7)g(7) + f(2)g(2)$???

Real Quadratic Form and symmetric matrices

In $\mathbb{R}^n$, We have inner product defined by

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$$

Explicitly,

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Is this the only possible inner product?

Try to classify inner product

Inner product satisfies

$$\langle \lambda \vec{v} + \mu \vec{u}, \vec{w} \rangle = \lambda \langle \vec{v}, \vec{w} \rangle + \mu \langle \vec{u}, \vec{w} \rangle.$$

$$\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle$$

Definition 1

Call a function $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ a **bilinear symmetric form** if it is

1. symmetry: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$;
2. bilinear:
   - $\langle \lambda \vec{v} + \mu \vec{u}, \vec{w} \rangle = \lambda \langle \vec{v}, \vec{w} \rangle + \mu \langle \vec{u}, \vec{w} \rangle$
   - and $\langle \vec{v}, \lambda \vec{w} + \mu \vec{u} \rangle = \lambda \langle \vec{v}, \vec{u} \rangle + \mu \langle \vec{v}, \vec{w} \rangle$

Try to classify inner product

Furthermore, as an inner product, we must have $\langle v, v\rangle > 0$ for any $v \neq 0$;

Definition 2

Call a function $\langle -, -\rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ an **inner product** if it is

1. symmetry: $\langle \vec{v}, \vec{w}\rangle = \langle \vec{w}, \vec{v}\rangle$;
2. bilinear:
    - $\langle \lambda\vec{v} + \mu\vec{u}, \vec{w}\rangle = \lambda\langle \vec{v}, \vec{w}\rangle + \mu\langle \vec{u}, \vec{w}\rangle$
    - and $\langle \vec{v}, \lambda\vec{w} + \mu\vec{u}\rangle = \lambda\langle \vec{v}, \vec{u}\rangle + \mu\langle \vec{v}, \vec{w}\rangle$
3. **positive definite**: For any non-zero vector $\vec{v}$, we have $\langle \vec{v}, \vec{v}\rangle > 0$.

In other words, an inner product is a **positive deifnite** bilinear symmetric form.

Classification of bilinear form

We have a way to construct bilinear form:

$$\langle \vec{v}, \vec{w} \rangle_A := \vec{v}^T A \vec{w}$$

Our inner product is a special case when $A = I_n$.

Note that $\langle e_i, e_j \rangle_A = e_i^T A e_j$ is the entry of A at i'th row and j'th column, we may view A as the matrix

$$A = \begin{pmatrix} \langle e_1, e_1 \rangle_A & \langle e_1, e_2 \rangle_A & \cdots & \langle e_1, e_n \rangle_A \\ \langle e_2, e_1 \rangle_A & \langle e_2, e_2 \rangle_A & \cdots & \langle e_2, e_n \rangle_A \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_n, e_1 \rangle_A & \langle e_n, e_2 \rangle_A & \cdots & \langle e_n, e_n \rangle_A \end{pmatrix}$$

The matrix A is the value table of inner products for vectors in natural basis. **Every bilinear form can be written as** $\langle \vec{v}, \vec{w} \rangle_A := \vec{v}^T A \vec{w}$

Classification of bilinear form

We have natural way of constructing positive definite bilinear form

$$\langle \vec{v}, \vec{w} \rangle_{M^T M} = \langle Mv, Mw \rangle$$

If M is **linealy independent** , then $v \neq 0 \iff Mv \neq 0$. Therefore, for any $v \neq 0$, we have

$$\langle Mv, Mv \rangle > 0$$

If M is a general matrix with columns might not linearly independent, we have

$$\langle Mv, Mv \rangle \geq 0.$$

Classification of bilinear form

Definition 3

A bilinear form $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is called **positive semi-definite** if

$$\langle v, v \rangle \geq 0 \text{ for all } v \in \mathbb{R}^n$$

furthremore it is called **positive definite** if

$$\langle v, v \rangle > 0 \text{ for all } 0 \neq v \in \mathbb{R}^n$$

Proposition 1

The form $\langle -, - \rangle_{M^TM}$ defined by $\langle \vec{v}, \vec{w} \rangle_{M^TM} := \langle Mv, Mw \rangle = v^T M^T M w$ is always **positive semi-definite**, and

$\langle -, - \rangle_{M^TM}$ positive definite $\iff$ columns of M linearly independent

Classification of bilinear form

Definition 4

We call a symmetric matrix A positive semi-definite (resp. definite) if its bilinear form $\langle -, - \rangle_A$ is semi-definite (resp. definite). In other words

$$A \text{ positive semi-definite} \quad \Longleftrightarrow \quad v^T A v \geq 0 \text{ for all } v$$

$$A \text{ positive definite} \quad \Longleftrightarrow \quad v^T A v > 0 \text{ for all } v \neq 0$$

Unit ball of bilinear form

Call the set

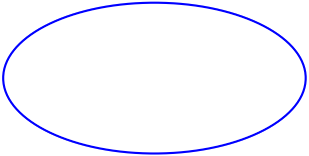
$$\{v \in \mathbb{R}^n : \langle v, v \rangle = 1\}$$

the unit ball of the bilinear form $\langle -, - \rangle$.

Unit ball of bilinear form

Let′s visualise unit balls in $\mathbb{R}^2$.

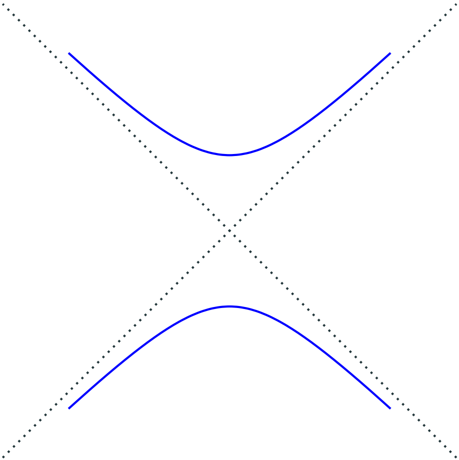
Unit ball of positive definite form $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$ defines an eclipse.



Unit ball of semi-positive definite form $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Unit ball of bilinear form

Unit ball of indefinite form $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Vectors on the asymptotes have length 0.



Unit ball of bilinear form

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In general references, whenever they say **positive definite**, **positive semidefinite**, or **indefinite**, we automatically refers to symmetric matrices.

Properties of Positive Semi-definite matrices

Proposition 2

The **sum** of two symmetric **positive semidefinite** matrix is again **symmetric positive semi definite**, if one of them is **definite**, then the **sum is definite** as well.

Proof:Let A and B be semi-definite. For any v,

$$v^T(A+B)v = v^TAv + v^TBv \geq 0 + 0 = 0.$$

If one of them is definite, then

$$v^T(A+B)v = v^TAv + v^TBv > 0 + 0 = 0.$$

Properties of Positive Semi-definite matrices

Last lecture, every symmetric matrix A can be written as

$$A = \Omega^H \Lambda \Omega.$$

with $\Omega^H \Omega = I_n$. We have

$$A = A^T \implies \Lambda = \Lambda^H \implies \Lambda = \overline{\Lambda} \implies A \text{ has real eigenvalues.}$$

Properties of Positive Semi-definite matrices

Lemma 1

A diagonal matrix is **positive definite** if and only if all diagonal entries is **positive**, and **semi-definite** if and only if all diagonal entries is **non-negative**.

Suppose

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let e_i be the i'th column of identity matrix, then

$$e_i^T \Lambda e_i = \lambda_i$$

So Λ positive semi-definite $\implies \lambda_i \geq 0$. Λ positive definite $\implies \lambda_i > 0$

Properties of Positive Semi-definite matrices

On the contary, if $\lambda_i \geq 0$ for all i, then for any non-zero vector $\vec{v} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$,

$$\vec{v}^T \Lambda \vec{v} = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 \geq 0$$

Furthermore, if $\lambda_i > 0$, then we have

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 > 0.$$

Properties of Positive Semi-definite matrices

Lemma 2

Suppose Ω is an invertible matrix and $A = \Omega^T \Lambda \Omega$, then A is positive semi-definite or definite if and only if Λ is.

$$v^T A v = v^T \Omega^T \Lambda \Omega v = (\Omega v)^T \Lambda (\Omega v).$$

$$\Lambda \text{ positive semi-definite} \implies A \text{ positive semi-definite}$$

Because $v \neq 0 \implies \Omega v \neq 0$, then

$$\Lambda \text{ positive definite} \implies A \text{ positive definite}$$

$$v^T \Lambda v = v^T (\Omega^{-1T} \Lambda \Omega^{-1}) v = (\Omega^{-1} v)^T A (\Omega^{-1} v).$$

$$A \text{ positive semi-definite} \implies \Lambda \text{ positive semi-definite}$$

similarly, because $v \neq 0 \implies \Omega^{-1} v \neq 0$, then

$$A \text{ positive definite} \implies \Lambda \text{ positive definite}$$

Properties of Positive Semi-definite matrices

Theorem 1

A symmetric matrix A is positive semi-definite if and only if all its eigenvalue are non-nogative, it is positive definite if and only if all its eigenvalue are positive.

Theorem 2

A semi-positive definite matrix A is positive definite if and only if and only if it is **invertible**.

Positive semi-definite form and Cauchy Inequality

Let A be a semi-positive definite symmetric matrix, it defines an inner product

$$\langle x, y \rangle = x^T A y.$$

Theorem 3

For any semi-positive definite symmetric matrix A, we have Cauchy Inequality

$$\langle x, y \rangle_A^2 \leq \langle x, x \rangle_A \langle y, y \rangle_A$$

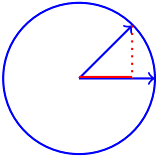
Positive semi-definite form and Cauchy Inequality

The geometric intuition of Cauchy inequality can be viewed as follows. Let us consider the case where $A = I$. Take square root, the Cauchy inequality can be written as $\langle x, y\rangle \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$ Then this inequality can be written as

$$\langle \frac{x}{\sqrt{\langle x, x\rangle}}, \frac{y}{\sqrt{\langle y, y\rangle}}\rangle \leq 1.$$

In orther words, it means the dot product of two unit vector must be less than 1.

Geometrically, the dot product of a vector with another unit vector is the length of projection



Positive semi-definite form and Cauchy Inequality

Proof of Cauchy inequality If $\langle x,x\rangle_A \neq 0$, then $\langle x, x\rangle_A > 0$ consider the vector defined by

$$w = \langle x, x\rangle_A y - \langle x, y\rangle_A x$$

We have

$$\langle w, w\rangle_A = \langle\langle x, x\rangle_A y, \langle x, x\rangle_A y\rangle - \langle\langle x, x\rangle_A y, \langle x, y\rangle_A x\rangle$$

$$-\langle\langle x, y\rangle_A x, \langle x, x\rangle_A y\rangle + \langle\langle x, y\rangle_A x, \langle x, y\rangle_A x\rangle$$

Simplify this expression we have

$$0 \leq \langle w, w\rangle_A = \langle x, x\rangle_A^2 \langle y, y\rangle_A - \langle x, y\rangle_A^2 \langle x, x\rangle_A$$

Divide both sides by $\langle x, x\rangle_A$, we have

$$0 \leq \langle x, x\rangle_A \langle y, y\rangle_A - \langle x, y\rangle_A^2$$

Positive semi-definite form and Cauchy Inequality

The Cauchy inequality also holds for $\langle$y,y$\rangle_A \neq 0$ by symmetry. We have left over a case if $\langle x, x\rangle_A = \langle y, y\rangle_A = 0$. In this case,

$$0 \leq \langle x + ay, x + ay\rangle = 2a\langle x, y\rangle$$

for all a, therefore, we must have $\langle x, y\rangle = 0$, the inequality holds.

Cauchy Inequality and Cross Filling

Now we study methods of verifying positive semi-deinity.

Cauchy Inequality and Cross Filling

Proposition 3

If A is a symmetric positive semi-definite matrix, then all its diagonal entries must be non-negative.

This is because that the diagonal entry is given by $e_i^T A e_i$.

Cauchy Inequality and Cross Filling

Proposition 4

If A is a positive semi-definite matrix and there is a number 0 on diagonal of A, then the entire row and column containing that diagonal 0 will be 0.

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \Longrightarrow \begin{pmatrix} * & * & 0 & * & * \\ * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * \\ * & * & 0 & * & * \end{pmatrix}$$

This is to prove $\langle e_i, e_i \rangle_A = 0 \implies \langle e_i, e_j \rangle_A = 0$, this follows from Cauchy's inequality for semi-definite matrix.

Cauchy Inequality and Cross Filling

Explain why the following symmetric matrix is NOT positive semi-definite?

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & 2 \\ -3 & 2 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 2 \\ -3 & 2 & -1 \end{pmatrix}$$

Cauchy Inequality and Cross Filling

Corollary 1

Let A be a symmetric matrix with all diagonal entries equal to 0, then it is positive semi-definite if and only if $A = 0$

Cauchy Inequality and Cross Filling

Therefore, we always able to find positive entry in an non-zero positive semi-definite matrix!.

We may do diagonal-cross-filling for positive semi-definite matrix!.

Cauchy Inequality and Cross Filling

Theorem 4

Let A be a symmetric matrix with a non-zero diagonal entry valued a. Let $A = P + (A - P)$ be one step of diagonal cross-filling with this non-diagonal entry. We have

1. P (resp. $-P$)is positive semi-definite if and only if $a > 0$ (resp.$a < 0$).
2. A is positive semi-definite $\iff$ $a > 0$ and $A - P$ positive semi-definite.

Suppose the center is located at i′th row and i′th column. Let e_i be i'th column of identity matrix. Then $a = \langle e_i, e_i \rangle_A$. The cross filling decomposes

$$P = Ae_i(e_i^T Ae_i)^{-1} e_i^T A; \qquad A = P + (A - P)$$

Cauchy Inequality and Cross Filling

Now prove 1.

First we analyse P

$$v^T P v = v^T A e_i (e_i^T A e_i) e_i^T A v = \frac{\langle v, e_i \rangle_A^2}{\langle e_i, e_i \rangle_A} = \frac{\langle v, e_i \rangle_A^2}{a}$$

Therefore

$$P \text{ positive semi-definite} \quad \Longleftrightarrow \quad a > 0$$

$$-P \text{ positive semi-definite} \quad \Longleftrightarrow \quad a < 0$$

Cauchy Inequality and Cross Filling

Now prove 2. Suppose $a > 0$ and $A - P$ positive semi-definite, then

$$A = P + (A - P)$$

is the sum of two positive semi-definite matrix. Therefore A is positive semi-definite.

A is positive semi-definite $\quad\Longleftarrow\quad$ $a > 0$ and $A - P$ positive semi-definite.

Cauchy Inequality and Cross Filling

On the contary, suppose A positive semi-definite, then $a > 0$.

To show $A - P$ positive semi-definite, we only need to show

$$v^T A v \geq v^T P v,$$

that is

$$\langle v, v \rangle_A \geq \frac{\langle v, e_i \rangle_A^2}{\langle e_i, e_i \rangle_A}$$

Note that this directly follows from Cauchy inequality.

A is positive semi-definite $\implies$ $a > 0$ and $A - P$ positive semi-definite.

We finished the proof.

Cauchy Inequality and Cross Filling

Excercise. Verify if the following matrices are positive semi-definite

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 2 & 4 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 10 & 7 \\ 2 & 7 & 9 \end{pmatrix}$$

Cauchy Inequality and Cross Filling

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Is this positive semi-definite?

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 2 & 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

Is this positive semi-definite?

Cauchy Inequality and Cross Filling

$$\begin{pmatrix}1 & 3 & 2\\ 3 & 10 & 7\\ 2 & 7 & 9\end{pmatrix} = \begin{pmatrix}1 & 3 & 2\\ 3 & 9 & 6\\ 2 & 6 & 4\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 5\end{pmatrix}$$

$$\begin{pmatrix}1 & 3 & 2\\ 3 & 10 & 7\\ 2 & 7 & 9\end{pmatrix} = \begin{pmatrix}1 & 3 & 2\\ 3 & 9 & 6\\ 2 & 6 & 4\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{pmatrix}$$

Is this positive definite?

Cauchy Inequality and Cross Filling

Another significance of diagonal cross-filling is that we are able to decompose $A = M^T M$, therefore writing the inner product into classcial form $\langle x, y \rangle_A = \langle Mx, My \rangle$.

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 10 & 7 \\ 2 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}}_{M^T} \underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{M}$$