Projection operators

Projection operators are the most simple and intuitive, yet also the most important linear transformations.

Definition 1

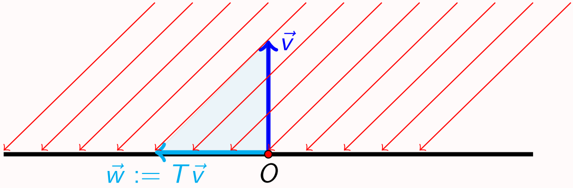
For a vector space V over F, a projection operator $T : V \longrightarrow V$(or an idempotent) is an operator such that

$$T^2 = T.$$

Geometric explaination of projection operator

We will firstly introduce the idea of the projection operator by **sun light model** before explaining this definition.

A vector standing straight at the origin (like a tree). The sunlight shines on it through some direction. The projection T maps any vector $\vec{v}$ to its shadow $\vec{w} := T\vec{v}$ on the floor.



The sunlight direction **needs not** to be perpendicular to the floor.

Geometric explaination of projection operator

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In the definition, we have required $T^2 = T$. Algebraically, this means that $T^2\vec{v} = T\vec{v}$ for any vector $\vec{v}$. Geometrically, it means that the shadow of a shadow is the shadow itself. Here is the explaination: Since $\vec{w} := T\vec{v}$ represents the shadow of $\vec{v}$, the $\vec{w}$ lies on the floor. The shadow of a vector on the floor is itself. Therefore $T\vec{w} = \vec{w}$.

Why a projection operator?

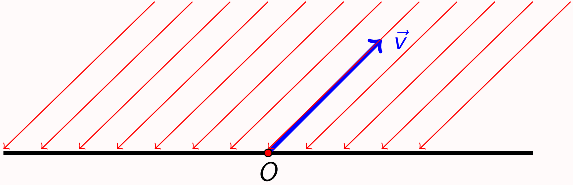
Intuitively, a projection needs to specify the floor and the sunlight. But we define them only by $T^2 = T$. Therefore, we need to answer the following questions:

1. If $T^2 = T$, how do we define the floor?
2. If $T^2 = T$, how do we define the direction of sunlight?
3. After specifying the floor and the direction of sunlight, we have enough information to draw the projection geometrically. But why $T\vec{v}$ is the shadow of $\vec{v}$?

Why a projection operator?

To answer this question, we need an example of an actural projection.

Let′s assume T is an actural projection. If a vector $\vec{v}$ is pointing to the sun, what does its shadow looks like? What is $\vec{w} = T\vec{v}$ in this case?



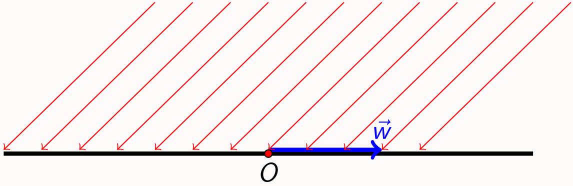
Should $\vec{v}$ be an element in the following set?

$$\{\vec{v} \in V : T\vec{v} = \vec{0}\}.$$

Why this set describes the direction of the sunlight?

Why a projection operator?

Let′s assume T is an actural projection. If a vector $\vec{w}$ is lying on the ground, what does its shadow looks like? What is $T\vec{w}$ in this case? Is $\vec{w}$ the same as $T\vec{w}$?



Should $\vec{w}$ be an element of the following set?

$$\{\vec{w} : \vec{w} = T\vec{v} \text{ for some } \vec{v} \in V\}.$$

Why a projection operator?

Furthermore, when T is a projection, is the following statement equivalent?

$$\vec{w} = T\vec{v} \text{ for some } \vec{v} \in V \iff T\vec{w} = \vec{w}.$$

Which of the following set describes the floor? or both of them do?

$$\{\vec{w} : \vec{w} = T\vec{v} \text{ for some } \vec{v} \in V\}, \qquad \{\vec{w} \in V : \vec{w} = T\vec{w}\}.$$

Why a projection operator?

We define kernel and image for linear operators, this is just an analogue of null space and column space (if you think T as a matrix)

Definition 2

For any linear operator $T : V \longrightarrow V$, the kernel and image of T are defined by

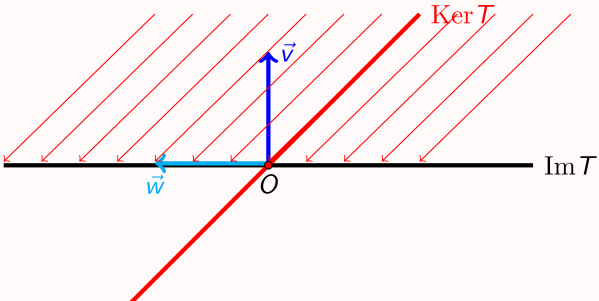
$$\mathrm{Ker}(T) := \{\vec{v} : T\vec{v} = \vec{0}\},$$

$$\mathrm{Im}(T) := \{\vec{w} : \vec{w} = T\vec{v} \text{ for some } \vec{v} \in V\}.$$

Why a projection operator?

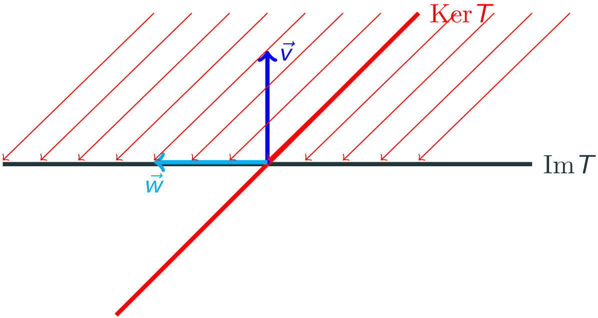
Come back to our abstract definition $T^2 = T$. We may draw a line from the vector head along $\mathrm{Ker}(T)$ until it hit $\mathrm{Im}(T)$ to get its shadow, but why this is $T\vec{v}$?

Let T be an operator with $T^2 = T$. By drawing $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$, the sunlight and the ground have been determined. Along the direction of the sunlight, one finds the shadow $\vec{w}$ of $\vec{v}$. Why $\vec{w} = T\vec{v}$?



Why a projection operator?

There are many operators with the same $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ with T.



$T\vec{v}$ has to be a vector in $\mathrm{Im}(T)$, but there are many choices. But only one choice makes

$$T\vec{v} - \vec{v} \in \mathrm{Ker}(T).$$

Note that

$$T^2 = T \iff T(T - I) = 0 \iff T\vec{v} - \vec{v} \in \mathrm{Ker}(T) \text{ for all } \vec{v} \in V.$$

Why a projection operator?

Summary

As long as an operator $T : V \longrightarrow V$ satisfies $T^2 = T$, it is a projection such that

- For any vector $\vec{v} \in V$, $T\vec{v}$ represents the shadow and $\vec{v} - T\vec{v} = (I - T)\vec{v}$ is a vector pointing to the direction of the sun.
- $\{\vec{v} : T\vec{v} = \vec{0}\}$ describes the direction of the sunlight, it consists of vectors $\vec{v}$ with $T\vec{v} = \vec{0}$.
- $\{\vec{v} : T\vec{v} = \vec{v}\}$ describes the floor, it consists of all possible shadows $\vec{w} = T\vec{v}$. In particular, for any $\vec{w} \in \mathrm{Im}(T)$, we have $T\vec{w} = \vec{w}$.

Why a projection operator?

Projection operator also has other equivalent definition.

Proposition 1

Suppose $T : V \longrightarrow V$ is a linear operator, the following are equivalent

- $T^2 = T$;
- For any $\vec{v} \in \mathrm{Im}(T)$, we have $T\vec{v} = \vec{v}$;
- For any $\vec{v} \in V$, we have $T\vec{v} - \vec{v} \in \mathrm{Ker}(T)$.

Proof: Exercise.