Length and Orthogonality

Formula for length of vector

$$\vec{v} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

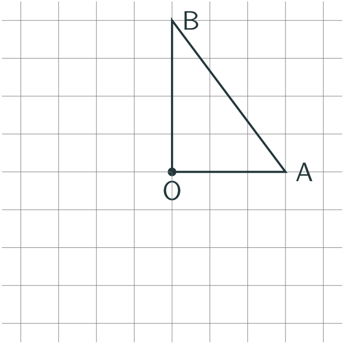
length

$$||\vec{v}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

Length and Orthogonality

How to verify two segment orthogonal to each other?

$$OA \perp OB \iff ||AB||^2 = ||OA||^2 + ||OB||^2$$



Length and Orthogonality

$$\vec{OA} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \vec{OB} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$||OA||^2 = x_1^2 + ... + x_n^2 \qquad ||OB||^2 = y_1^2 + ... + y_n^2$$

$$||AB||^2 = (x_1 - y_1)^2 + ... + (x_n - y_n)^2$$

$$||OA||^2 + ||OB||^2 - ||AB||^2 = (x_1^2 + y_1^2 - (x_1 - y_1)^2) + ... + (x_n^2 + y_n^2 - (x_n - y_n)^2)$$

$$= 2(x_1 y_1 + x_2 y_2 + ... + x_n y_n)$$

Length and Orthogonality

$$\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \vec{w} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\vec{v}^T \vec{w} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

$$\vec{v} \perp \vec{w} \iff \vec{v}^T \vec{w} = 0.$$

Definition 1

Say two vector perpendicular if

$$\vec{v}^T \vec{w} = 0.$$

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Zero vector perpendicular to all vectors

Length and Orthogonality

Definition 2

Let $V, W \subset \mathbb{R}^n$ be two subspaces, we say they are perpendicular $V \perp W$ if

$$\vec{v}^T \vec{w} = 0$$

for any $\vec{v} \in V$ and $\vec{w} \in W$.

Definition 3

For $V \perp W$, we say furthermore V an **orthogonal complement** if moreover

$$\dim(V) + \dim(W) = n.$$

Orthogonality of null and row space

Theorem 1

For any matrix A, the null space $\text{null}(A)$ and the row space $\text{col}(A^T)$ are **orthogonal complement** to each other.

Suppose A is $m \times n$ matrix, we have

$$\text{null}(A) \subset \mathbb{R}^n \qquad \text{col}(A^T) \subset \mathbb{R}^n$$

Dimension Check:

$$\dim(\text{null}(A)) + \dim(\text{col}(A^T)) = \dim(\text{null}(A)) + \dim(\text{col}(A)) = \dim(\mathbb{R}^n)$$

Perpendicular: For any $\vec{v} \in \text{null}(A)$, and for any $\vec{w} \in \mathbb{R}^m$, write $\vec{u} = A^T\vec{w} \in \text{col}(A)$

$$\vec{u}^T\vec{v} = \vec{w}^TA\vec{v} = \vec{w}^T\vec{0} = 0.$$

Orthogonality of null and row space

By definition of solution, the null space automatically perpendicular with row space, because the reason of matrix multiplication.

$$\begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ \dots & \dots & \dots \\ - & r_n & - \end{pmatrix} \begin{pmatrix} | \\ s \\ | \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$