# Projection Matrices

The importance of projection matrix comes from the following observation

**Definition 1**

An $n \times n$ square matrix $P$ is a **projection matrix** if

$$P^2 = P.$$

**Proposition 1**

If $AB = I_m$, then $BA$ is a projection matrix.

$$(BA)^2 = BABA = B(AB)A = BI_mA = BA$$

# Projection Matrices

It is very easy to verify if a vector is in $\mathrm{Col}(P)$ or $\mathrm{Null}(P)$ since

**Proposition 2**

Let $P$ be an $n \times n$ projection matrix. For any vector $x \in \mathbb{R}^n$, we have

$$y \in \mathrm{Col}(P) \iff Py = y, \qquad x \in \mathrm{Null}(P) \iff Px = 0.$$

If $y \in \mathrm{Col}(P)$, then $y = Px$ for some $x$. Then $Py = PPx = Px = y$. Conversely, if $Py = y$, then $y \in \mathrm{Col}(P)$ by definition.

Please finish the proof of $x \in \mathrm{Null}(P) \iff Px = 0$ by yourself!

# Projection Matrices

**Excercise.** The following matrix is a projection matrix

$$P = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Calculate $\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{P} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ Is that true that $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \in \mathrm{Col}(P)$?

(Hint: Since $P$ is a projection, $\mathrm{Col}(P) = \{y | Py = y\}$ )

# Projection Matrices

**Proposition 3**

If $P$ is a $n \times n$ projection matrix, then $I_n - P$ is also a projection matrix.

(Homework, you only need to check $(I_n - P)^2 = I_n - P$)

# Projection Matrices

**Proposition 4**

If $P$ is a $n \times n$ projection matrix, then

$$\mathrm{Col}(P) = \mathrm{Null}(I_n - P) \qquad \mathrm{Col}(I_n - P) = \mathrm{Null}(P)$$

(Homework)

!!

Therefore $P$ and $I - P$ **interchanges** their columns space and null spaces! What a beautiful phenomenon!

# Decomposition of Projection

Recall that just from $AB = I_m$, we create a projection matrix

$$P = BA.$$

Are all projection matrices arises this way?

# Decomposition of Projection

Giving a projection matrix $P$, recall that we might use cross-filling method

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{P} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

This decompsoes

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{A}$$

Last time, we have learned that corss-filling process automatially give **linealy independent** rows and columns, therefore, $A$ must have **right inverse** and $B$ must have **left inverse**.

But is that true that $AB = I_m$? if true, then all projection matrices arises from $P = BA$ with ivnerse pairs $AB = I_m$.

# Decomposition of Projection

Cross-Filling decomposes $n \times n$ rank $m$ projection matrix $P$ into

$$P = BA$$

where $B$ of size $n \times m$ and $A$ of size $m \times n$, and the cross-filling garantees that

- Columns of $B$ **linealy independent**
- Rows of $A$ **linealy independent** .

So $B$ has left inverse, $A$ has right inverse

**Our question**: Is that true $AB = I_m$?

# Decomposition of Projection

Yes! It is true, for simple reasons!!!

$$P^2 = P \iff BABA = BA$$

Since $A$ has right inverse

$$BAB = B$$

Since $B$ has left inverse

$$AB = I_m.$$

# Decomposition of Projection

**Proposition 5**

If $P$ is a projection matrix, then

$$\operatorname{rank}(P) = \operatorname{tr}(P).$$

Let $m = \operatorname{rank}(P)$. Let $P = BA$ be obtained from cross-filling process. Then we must have $AB = I_m$.

$$\operatorname{rank}(P) = m = \operatorname{tr}(I_m) = \operatorname{tr}(AB) = \operatorname{tr}(BA) = \operatorname{tr}(P).$$

# Decomposition of Projection

**Excercise.** The following matrix is a projeciton matrix $P = P^2$, directly find out the rank of the matrix!

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

**Excercise.** The following matrix is a projection matrix, what is its rank?

$$\begin{pmatrix} -4 & -9 & 5 & 4 & -3 \\ 4 & 9 & -5 & -4 & 3 \\ -8 & -18 & 10 & 8 & -6 \\ 8 & 18 & -10 & -8 & 6 \\ -8 & -18 & 10 & 8 & -6 \end{pmatrix}$$

# Cross-Filling for projection matrices

This implies important observations. Suppose one can decompose projection matrix by cross-filling

$$P = c_1 r_1^T + c_2 r_2^T + \cdots + c_m r_m^T.$$

The cross-filling process garantees that both $\begin{pmatrix} c_1 & \cdots & c_m \end{pmatrix}$ and $\begin{pmatrix} r_1 & \cdots & r_m \end{pmatrix}$ **linealy independent** . Then we write

$$P = \underbrace{\begin{pmatrix} c_1 & \cdots & c_m \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \end{pmatrix}}_{A}$$

By what we learned before, using left and right cancelations, we must have $AB = I_m$!

# Cross-Filling for projection matrices

But $I_m = AB$ gives this result

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} r_1^T \\ \vdots \\ r_m^T \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_1 & \cdots & c_m \end{pmatrix}}_{B} = \begin{pmatrix} r_1^T c_1 & r_1^T c_2 & \cdots & r_1^T c_n \\ r_2^T c_1 & r_2^T c_2 & \cdots & r_2^T c_n \\ \vdots & \vdots & \ddots & \vdots \\ r_2^T c_1 & r_2^T c_2 & \cdots & r_2^T c_n \end{pmatrix}$$

This means

$$r_i^T c_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

# Cross-Filling for projection matrices

**Theorem 1**

Let $P = P^2$ be a $n \times n$ projection martrix. Let $c_1, c_2, \dots, c_m$ and $r_1, r_2, \dots, r_m$ be two lists of linearly independent $n \times 1$ matrices such that

$$P = \sum_{i=1}^{m} c_i r_i^T$$

Let $P_i := c_i r_i^T$, so $P = P_1 + P_2 + \cdots + P_m$, and we have

- $\operatorname{rank}(P) = m$
- $P_i$ are rank 1 projection matrices in the sense that $P_i^2 = P_i$.
- $P_i P_j = 0$ if $i \neq j$,

**Conclusion**: The cross-filling decomposes projection matrix into good rank-1 projections. We will address the importance of this in the future.

# Rank 0 matrix

Next, we finish our discussion of square matrices. Recall the question. If $A$ is a square matrix, does its right inverse equal to its left inverse?

# Rank 0 matrix

**Proposition 6**

A rank 0 matrix is zero matrix.

Because zero dimensional space can only have zero vector. A matrix with zero columns is a zero matrix.

# Full rank projection matrix

**Proposition 7**

An $n \times n$ invertible perjection matrix $P$ must be $P = I_n$

Because $I_n - P$ is a projection matrix of trace 0, so is of rank 0, therefore $I_n - P = 0$.

# Full rank projection matrix

**Excercise.**Please fill in the blanks so that the following matrix can be a projection matrix

$$P = \begin{pmatrix} 0.5 & \square \\ 0.5 & \square \end{pmatrix}$$

# Inverse for square matrices

**Proposition 8**

Supose $A$ is $n \times n$ matrix with right inverse $B$ in the sense $AB = I_n$, then we have $BA = I_n$ as well.

Because $BA$ must be a projection matrix with $\mathrm{tr}(AB) = \mathrm{tr}(BA) = n$, so $BA = I_n$.