QR decomposition

Calculation Strategy :

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}$$

Find orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ such that

$$\vec{v}_1 \in \text{span}\{\vec{u}_1\}$$

$$\vec{v}_2 \in \text{span}\{\vec{u}_1, \vec{u}_2\}$$

$$\vec{v}_3 \in \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$$

Then

$$A = QR$$

where $Q = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}$

QR decomposition

The method: Find orthogonal basis first, then normalize it. Do not compute with unit vector.

Key formula:

Orthogonal Projection induced by a vector v: $\frac{vv^T}{v^Tv}$.

For simplicity, in test we will only test you about QR of square matrices.

QR decomposition

Excercise.QR the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Solution.: Assume $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix}$ We first find orthogonal vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$

$$\vec{w}_1 := \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}; \qquad \vec{w}_1^{\,T} \vec{w}_1 = 2.$$

Now directly calculate $\vec{w}_2$. A vector perpendicular to $\vec{w}_1$, we consider

$$\vec{v}_2 - \frac{\vec{w}_1^{\,T} \vec{v}_2}{\vec{w}_1^{\,T} \vec{w}_1} \vec{w}_1$$

QR decomposition

$$\vec{v}_2 - \frac{\vec{w}_1^T \vec{v}_2}{\vec{w}_1^T \vec{w}_1} \vec{w}_1$$

equals to

$$\begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 + \frac{3}{2} \vec{w}_1 = \frac{1}{2} \underbrace{\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}}_{w_2}$$

QR decomposition

So we obtain

$$\vec{v}_2 = -\frac{3}{2}\vec{w}_1 + \frac{1}{2}\vec{w}_2$$

With

$$\vec{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}; \qquad \vec{w}_2^T \vec{w}_2 = 6.$$

QR decomposition

Now dealing with $\vec{v}_3$, Consider

$$\vec{v}_3 - \frac{\vec{w}_1^{\,T}\vec{v}_3}{\vec{w}_1^{\,T}\vec{w}_1}\vec{w}_1 - \frac{\vec{w}_2^{\,T}\vec{v}_3}{\vec{w}_2^{\,T}\vec{w}_2}\vec{w}_2$$

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \frac{5}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \\ -6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

So we may take

$$w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \qquad w_3^T w_3 = 12.$$

$$\vec{v}_3 = \frac{1}{2}w_1 - \frac{5}{6}w_2 + \frac{1}{3}w_3.$$

QR decomposition

Use the above information fill in to the matrix, you got QR decomposition.

$$\underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix}}_{A} =$$

$$\underbrace{\begin{pmatrix} \frac{\vec{w}_1}{||\vec{w}_1||} & \frac{\vec{w}_2}{||\vec{w}_2||} & \frac{\vec{w}_3}{||\vec{w}_3||} & \frac{\vec{w}_4}{||\vec{w}_4||} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} ||\vec{w}_1|| & 0 & 0 & 0 \\ 0 & ||\vec{w}_2|| & 0 & 0 \\ 0 & 0 & ||\vec{w}_3|| & 0 \\ 0 & 0 & 0 & ||\vec{w}_4|| \end{pmatrix} \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}}_{R}$$

Determining the Jordan Canonical Form

Recall the notion of Jordan canonical form

$$A \underbrace{\begin{pmatrix} 1 & 2 & 3 & 1 & 4 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 \\ 4 & 1 & 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 1 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 1 & 4 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 \\ 4 & 1 & 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 1 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}}_{P} J$$

where

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Determining the Jordan Canonical Form

We use colors to make it more clear

$$A\begin{pmatrix}\vec{v}_1 & \vec{u}_1 & \vec{u}_2 & \vec{w}_1 & \vec{w}_2 & \vec{w}_3\end{pmatrix}$$

$$=\begin{pmatrix}\vec{v}_1 & \vec{u}_1 & \vec{u}_2 & \vec{w}_1 & \vec{w}_2 & \vec{w}_3\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 0 & 2\end{pmatrix}$$

Each color represents an infinite eigenvector, and those infinite eigen vectors are linearly independent.

$$A\vec{v}_1 = (1+\epsilon)\vec{v}_1$$

$$A(\vec{u}_1\infty + \vec{u}_2) = (1+\epsilon)(\vec{u}_1\infty + \vec{u}_2)$$

$$A(\vec{w}_1\infty^2 + \vec{w}_2\infty + \vec{w}_3) = (2+\epsilon)(\vec{w}_1\infty^2 + \vec{w}_2\infty + \vec{w}_3)$$

Determining the Jordan Canonical Form

This is called Yang Tableau, it represents an infinite vector by its components, for example

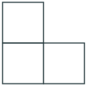
$$\begin{array}{|c|c|}\hline \vec{u}_2 & \vec{u}_1 \\ \hline \end{array} = u_2 + u_1\infty$$

We collect vectors by putting them row by row.

$$\begin{array}{|c|c}\cline{1-1} \vec{v}_1 & \\ \hline \vec{u}_2 & \multicolumn{1}{c|}{\vec{u}_1} \\ \hline \end{array} = \vec{v}_1\ ,\ \vec{u}_2 + \vec{u}_1\infty$$

Determining the Jordan Canonical Form

InfiniteEigenvectors of eigenvalue $1+\epsilon$



InfiniteEigenvectors of eigenvalue $2+\epsilon$

Each row represents an infinite eigenvector, and corresponds to a **Jordan Block**

Determining the Jordan Canonical Form

Now let us try to multiply formal scalar $\lambda + \epsilon$ on the infinite vector. If $\lambda \neq 0$, then

$$(\lambda)\boxed{v_0}\boxed{v_1}\boxed{v_2} = \boxed{\lambda v_0}\boxed{\lambda v_1}\boxed{\lambda v_2}$$

However, for the case $\lambda = 0$

$$0\boxed{v_0}\boxed{v_1}\boxed{v_2} = \text{Empty}$$

$$\epsilon\boxed{v_0}\boxed{v_1}\boxed{v_2} = \boxed{v_1}\boxed{v_2}.$$

One box will vanish.

If $\lambda \neq 0$, then

$$(\lambda+\epsilon)\boxed{v_0}\boxed{v_1}\boxed{v_2} = \boxed{\lambda v_0 + v_1}\boxed{\lambda v_1 + v_2}\boxed{\lambda v_2}$$

Determining the Jordan Canonical Form

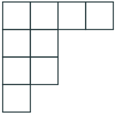
Roughly speaking,

$$\begin{cases} (\lambda+\epsilon)\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} & \text{if}\lambda \neq 0 \\ \\ (\lambda+\epsilon)\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} = \begin{array}{|c|c|}\hline & \\ \hline \end{array} & \text{if}\lambda = 0 \end{cases}$$

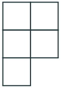
Determining the Jordan Canonical Form

Excercise. Suppose A is a 17×17 matrix of eigenvalue $1, 2, 3$ and one have the following shape of infinite-eigenbasis of the space

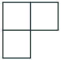
Eigenvalue 1



Eigenvalue 2



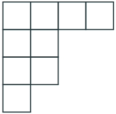
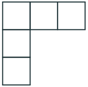
Eigenvalue3



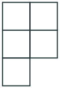
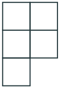
Use this diagram, describe how to calculate $\text{rank}(p(A))$ for any polynomial p.

Determining the Jordan Canonical Form

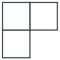
If we multiply $A - I$, then this is the effect: Eigenvalue 1


$\times \, ((1+\epsilon) - 1) =$


Eigenvalue 2

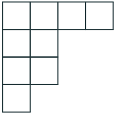
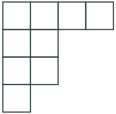

$\times \, (2+\epsilon) - 1) =$


Eigenvalue3

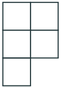

$\times \, (3+\epsilon) - 1) =$


Determining the Jordan Canonical Form

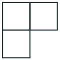
If we multiply $A - 2I$, then this is the effect: Eigenvalue 1

 $\times\ ((1 + \epsilon) - 2) =$ 

Eigenvalue 2

 $\times\ (2 + \epsilon) - 2) =$ 

Eigenvalue3

 $\times\ (3 + \epsilon) - 2) =$ 

Determining the Jordan Canonical Form

If we multiply $A - 3I$, then this is the effect: Eigenvalue 1

$$\ydiagram{4,2,2,1} \times ((1+\epsilon)-3) = \ydiagram{4,2,2,1}$$

Eigenvalue 2

$$\ydiagram{2,2,1} \times (2+\epsilon)-3) = \ydiagram{2,2,1}$$

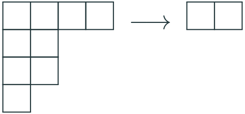
Eigenvalue3

$$\ydiagram{2,1} \times (3+\epsilon)-3) = \ydiagram{1}$$

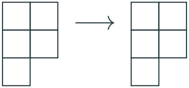
Determining the Jordan Canonical Form

If we multiply $(A-I)^2(A-3I)$, then this is the effect:

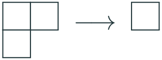
Eigenvalue $1+\epsilon$



Eigenvalue $2+\epsilon$



Eigenvalue$3+\epsilon$



Determining the Jordan Canonical Form

Excercise. Find the Jordan form of the following matrix

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

Power and exponential of matrices

Recall if

$$\det(tI - A) = (t - \lambda_1)^{n_1} \dots (t - \lambda_k)^{n_k}$$

Then

$$g(A) = \sum_{i=1}^{k} \mathrm{Const}(g(\lambda_i + \epsilon)\mathscr{P}_{\lambda_i})$$

with

$$\mathscr{P}_{\lambda_i} = P_{\lambda_i} + \infty N_{\lambda_i} + \infty^2 N_{\lambda_i}^2 + \dots + \infty^{n_i - 1} N_{\lambda_i}^{n_i - 1}$$

In particular, we have

$$P_{\lambda_1} + P_{\lambda_2} + \dots P_{\lambda_k} = I \qquad ; \qquad A = \sum_{i=1}^{k} (\lambda_i P_{\lambda_i} + N_{\lambda_i})$$

and

$$\mathrm{tr}(P_{\lambda_i}) = n_i.$$

A is diagonalizable $\iff$ $N_{\lambda_i} = 0$ for all λ_i.

Power and exponential of matrices

The best way to compute A^n and e^A is by spectural decomposition! We will only test you when A is at most 3×3 matrix.

Power and exponential of matrices

For the following matrix

$$A = \begin{pmatrix} 1 & 1 \\ -6 & 6 \end{pmatrix}$$

Compute a formula for A^n and e^A.

We use spectural decomposition. Note that $\mathrm{tr}(A) = 7$ and $\det(A) = 12$.
The characteristic polynomial is $\det(tI - A) = t^2 - 7t + 12$

We have

$$\det(tI - A) = (t-3)(t-4).$$

By spectural decomposition

$$A = 3P_3 + 4P_4; \qquad P_3 + P_4 = I.$$

Therefore

$$P_4 = (3P_3 + 4P_4) - 3(P_3 + P_4) = A - 3I = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$$

Power and exponential of matrices

Now

$$P_3 = I - P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$$

Therefore

$$A^n = 3^n P_3 + 4^n P_4 = 3^n \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} + 4^n \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$$

$$e^A = e^3 \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} + e^4 \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$$

Power and exponential of matrices

Excercise. Solving the equation

$$\frac{dy}{dt} = \underbrace{\begin{pmatrix} 1 & 1 \\ -6 & 6 \end{pmatrix}}_{A} y, \qquad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

By **definition** of e^{At}, we directly have

$$y(t) = e^{At} y(0)$$

$$\left(e^{3t} \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} + e^{4t} \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{4t} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Power and exponential of matrices

Excercise. Solving the following diffrential equation

$$\frac{dy}{dt} = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} y(t) \qquad y(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Look at this matrix, it is skew symmetric. So all its eigenvalues are purly imaginary!

Since it has 3 eigenvalues, and if z is an eignevalue, so is $\overline{z}$, therefore, 0 must be an eigenvalue of it! Therefore, its characteristic polynomial must be of the form

$$\det(tI - A) = (t - bi)(t + bi)t = t^3 + b^2 t.$$

We have

$$b^2 = \det \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = 4 + 1 + 4 = 9$$

Power and exponential of matrices

Therefore, its eigenvalues are $0, 3i, -3i$. Note that skew Hermitian matrices are normal, and therefore always diagonalizable. So, we have

$$P_0 + P_{3i} + P_{-3i} = I. \qquad 3iP_{3i} + (-3i)P_{-3i} = A.$$

Since A is real matrix, we must have $P_{-3i} = \overline{P_{3i}}$. We may write

$$P_{3i} = X + Yi, \qquad P_{-3i} = X - Yi$$

Therefore

$$3i(2Yi) = A \implies Y = -\frac{1}{6}A = -\frac{1}{6}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$$

Power and exponential of matrices

To determine X, we may calculate A^2. Note that it must be a symmetric matrix

$$A^2 = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix}$$

Note that $A^2 = (3i)^2 P_{3i} + (-3i)^2 P_{-3i} = (-9)(P_{3i} + P_{-3i})$

So

$$X = -\frac{1}{18} \begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix}$$

Power and exponential of matrices

Now $P_0 + P_{3i} + P_{3i} = I$, we obtain

$$P_0 = I - 2X = \frac{1}{9} \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}$$

Power and exponential of matrices

So the spectral decomposition is

$$g\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$= g(0)\frac{1}{9}\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}$$

$$+g(3i)\left(-\frac{1}{18}\begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} - \frac{i}{6}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}\right)$$

$$+g(-3i)\left(-\frac{1}{18}\begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} + \frac{i}{6}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}\right)$$

Power and exponential of matrices

Don't be scared, if g is a real polynomial, then we may write

$$= g(0)\frac{1}{9}\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}$$

$$+\mathrm{Re}\left(g(3i)\left(-\frac{1}{9}\begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} - \frac{i}{3}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}\right)\right)$$

Power and exponential of matrices

Therefore

$$e^{At} = \frac{1}{9}\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix} +$$

$$\mathrm{Re}\left((\cos(3t) + i\sin(3t))\left(-\frac{1}{9}\begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} - \frac{i}{3}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}\right)\right)$$

This equals to e^{At}

$$\frac{1}{9}\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix} - \frac{\cos(3t)}{9}\begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}$$

Power and exponential of matrices

The solution is given by

$$y(t) = e^{At}y(0)$$

$$= \frac{1}{9}\begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$-\frac{\cos(3t)}{9}\begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} + \frac{\cos(3t)}{3}\begin{pmatrix} -7 \\ -4 \\ -5 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} 1 \\ -8 \\ 5 \end{pmatrix}$$

Power and exponential of matrices

!!

In general, if $AB = BA$, then $e^A e^B = e^{A+B}$.
If A is a skew Hermitian matrix, then e^A is always unitary, indeed.

$$e^A (e^A)^H = e^A e^{A^H} = e^{A+A^H} = e^0 = I.$$

Therefore, for skew symmetric real matrices $A = -A^T$, the system $y' = Ay$ always represents rotation.

For 3×3 matrix. It is easier if it is diagonalizable and have repeated roots.

Excercise. Consider the following matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}$$

Find a unitary matrix U such that $U^H AU$ is diagonal.

Suppose one find the characteristic polynomial $\det(tI - A) = (t-1)^2(t-15)$

Power and exponential of matrices

The eigenvalue $1+\epsilon$ is suspicious, We find

$$\operatorname{rank}(A-I) = \operatorname{rank}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = 1$$

This implies the Yang tableau of eigenvalue $1+\epsilon$ looks like

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

This means all its eigenvectors of $1+\epsilon$ are finite. Therefore A is diagonalizable. By spectural decomposition,

$$A = P_1 + 15P_{15} \qquad , \qquad P_1 + P_{15} = I$$

$$\operatorname{tr}(P_1) = 2 \qquad , \qquad \operatorname{tr}(P_{15}) = 1$$

Power and exponential of matrices

Therefore,

$$A - I = (P_1 + 15P_1) - (P_1 + P_{15}) = 14P_{15}$$

So

$$P_{15} = \frac{1}{14}(A - I) = \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

and

$$P_1 = I - P_{15} = \frac{1}{14}\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}$$

The eigenspace of eigenvalue 15 is 1-dimensional and for eigenvalue 1 is 2-dimensional.

, From here we already have

$$A^n = \frac{1}{14}\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix} + \frac{15^n}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

and

$$e^{At} = \frac{e^t}{14}\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix} + \frac{e^{15t}}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

The above formula enable us to do other exercises, but in this question we particularly need diagoanalization.

!!

In most applicational problems, **spectural decomposition is sufficient**, one need not to diagonalize a matrix. Diagonalization is useful only when you wanna decompse matrix.

Power and exponential of matrices

We already have $\mathrm{Col}(P_{15}) \perp \mathrm{Col}(P_1)$. But we need orthogonal basis in P_1 also. For this purpose, we use **diagonal cross filling**. It is up to you

$$\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix} = \begin{pmatrix} 0.4 & -2 & 1.2 \\ -2 & 10 & -6 \\ 1.2 & -6 & 3.6 \end{pmatrix} + \begin{pmatrix} 12.6 & 0 & -4.2 \\ 0 & 0 & 0 \\ -4.2 & 0 & 1.4 \end{pmatrix}$$

Using this, we obtain an **ortho**, but not yet normal eigenvectors!

$$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{15}, \underbrace{\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}}_{1}, \underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{1}$$

Power and exponential of matrices

!!

We only need corss-filling for eigenvalues shown as repeated roots. The difference between unitary diagonalization and typical diagonalization is that unitary diagonalization you choose **diagonal** as center, but general diagonalization you do not have to.

Power and exponential of matrices

Now the length of it is

$$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\sqrt{14}}, \underbrace{\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}}_{\sqrt{35}}, \underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{\sqrt{10}}$$

Collecting these as

$$U = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{35}} & \frac{-3}{\sqrt{10}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} & \frac{0}{\sqrt{10}} \\ \frac{3}{\sqrt{14}} & \frac{-3}{\sqrt{35}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$AU = U \begin{pmatrix} 15 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies U^H AU = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculating powers and exponentials for the non-diagonalizable matrices

The most important is to determine P_λ and N_λ for each eigenvalue λ.

Excercise. For the following matrix A, try to solve A^n and e^A

$$A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

We have

$$\det(\lambda I - A) = (\lambda - 1)^3.$$

So $P_1 = I$,

$$A = 1 \cdot P_1 + N_1$$

Therefore

$$N_1 = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Calculating powers and exponentials for the non-diagonalizable matrices

You observe that

$$\operatorname{rank}(N_1) = \operatorname{rank}(A - I) = 1$$

Therefore, its Yang tableau must be the form

$$\ydiagram{2,1}$$

This implies $N_1^2 = 0$ Therefore

$$g(A) = \text{Const}\left(g(1+\epsilon)\left(I_3 + \infty N_1\right)\right)$$

For A^n, we have $(1 + \epsilon + O(\epsilon))^n = 1 + n\epsilon + O(\epsilon)$

$$A^n = \text{Const}\left((1+n\epsilon)\left(I_3 + \infty N_1\right)\right)$$

$$A^n = I_3 + nN_1$$

Calculating powers and exponentials for the non-diagonalizable matrices

And for e^{At}, we note that

$$e^{(1+\epsilon)t} = e^t e^{\epsilon t} = e^t(1 + \epsilon t + O(\epsilon))$$

Therefore

$$e^{At} = e^t P_1 + te^t N_1$$

$$= e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + te^t \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Calculating powers and exponentials for the non-diagonalizable matrices

Do the same problem with

$$A = \begin{pmatrix} -6 & -5 & -1 \\ 10 & 8 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

Suppose you know eigenvalue are 1 and 2.

Solution. From here we know the characteristic polynomial is given by

$$\det(tI - A) = (t-1)^2(t-2)$$

Therefore, the spectural decomposition is

$$g(A) = \mathrm{Const}(g(1+\epsilon)(P_1 + N_1\infty)) + g(2)P_2.$$

Calculating powers and exponentials for the non-diagonalizable matrices

To detect each matrix. Note that

	$1+\epsilon$	2
$(t-1)^2$	$0+O(\epsilon)$	1

Therefore

$$P_2 = (A - I)^2$$

Note that $\mathrm{tr}(P_2) = \mathrm{rank}(P_2)$ is the multiplicity of 2, so $\mathrm{rank}(P_2) = 1$. So $(A - I)^2$ will be a rank 1 trace 1 matrix

$$A - I = \begin{pmatrix} -7 & -5 & -1 \\ 10 & 7 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

$$(A - I)^2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 2 & 2 & 2 \end{pmatrix}$$

Calculating powers and exponentials for the non-diagonalizable matrices

This implies that

$$P_1 = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & 2 \\ -2 & -2 & -1 \end{pmatrix}$$

Note that

$$A = P_1 + N_1 + 2P_2 = I + P_2 + N_1$$

This implies that

$$\begin{aligned} N_1 &= A - I - P_2 \\ &= \begin{pmatrix} -7 & -5 & -1 \\ 10 & 7 & 1 \\ -2 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 2 & 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -8 & -6 & -2 \\ 12 & 9 & 3 \\ -4 & -3 & -1 \end{pmatrix} \neq 0 \end{aligned}$$

Calculating powers and exponentials for the non-diagonalizable matrices

Therefore, its Yang tableau must be of this form

Eigenvalue $1+\epsilon$:

$\square\square$

Eigenvalue $2+\epsilon$:

$\square$

This means all columns of $\mathscr{P}_1$ must be colinear to each other, we do not need cross-filling

$$\mathscr{P}_1 = \begin{pmatrix} -8\infty & -6\infty-1 & -2\infty-1 \\ 12\infty+2 & 9\infty+3 & 3\infty+2 \\ -4\infty-2 & -3\infty-2 & -\infty-1 \end{pmatrix}$$

Calculating powers and exponentials for the non-diagonalizable matrices

So we may take any column of $\mathscr{P}_1$ to be infinite eigenvector, we take half of the first column

$$\begin{pmatrix} -4\infty \\ 6\infty + 1 \\ -2\infty - 1 \end{pmatrix}$$

We can write it as

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} \infty$$

$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$

Calculating powers and exponentials for the non-diagonalizable matrices

Now

$$P_2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 2 & 2 & 2 \end{pmatrix}$$

Take any problem of P_2 we obtain eigenvector

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

The Jordan canonical form, is

$$A = \begin{pmatrix} 1 & -4 & 0 \\ -2 & 6 & 1 \\ 2 & -2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -4 & 0 \\ -2 & 6 & 1 \\ 2 & -2 & -1 \end{pmatrix}}_{P}$$