Linear Algebra Lectures

Column/Row operations

Learning Objectives

Matrix Multiplication(1.4) $\longrightarrow$ Row/Column Operations(1.3)

- What **three** kind of row/column operations are there?
- For the product $AB = C$,
  - Which two matrices is allowed to play **simutaneous row operation** without changing the equality?
  - Which two matrices is allowed to play **simutaneous column operation** without changing the equality?
- Will simutaneous (row/column) operation changes $A^{-1}B$ or AB^{-1}?
- Let E be a matrix obtained by applying some row operations from I, what does EA mean?
- Let E be a matrix obtained by applying some column operations from I, what does AE mean?
- Why matrix equation $Ax = b$ is a system of linear equations?
- How row operation help with solving system of linear equations?
  - What kind of matrix are we reducing to?
- How to use row/column opetations to find **inverse**?

A special request

A custormer requests for a special **new drink** need the following ingradients

Old Drinks ingradients:

	🥛	🍲	☕	🍵
🍃	0	2	0	2
🍋	0	1	0	1
🫘	0	4	2	0
🐄	1	1	0	0

New Drink requirement:

	🥤
🍃	4
🍋	2
🫘	2
🐄	4

Problem:

	🥤
🥛	?
🍲	?
☕	?
🍵	?

But the chef only have 🥛, 🍲, ☕, 🍵 at the hand, can he produce 🥤 by those materials?

A special request

The chef thought this problem is the same as a matrix product equation, indeed, replace those questionmarks by x, y, z, he need the following equation to be true

Old Drinks ingradients:

	🥛	🍲	☕	🍵
🍃	0	2	0	2
🍋	0	1	0	1
	0	4	2	0
🐄	1	1	0	0

$\times$

Problem:

	🥤
🥛	x
🍲	y
☕	z
🍵	w

$=$

New Drink requirement:

	🥤
🍃	4
🍋	2
☕	2
🐄	4

A special request

Mathematically, this equation is writting as

$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

A special request

By understanding by columns, solving it is the same as asking for

$$\begin{array}{|c|}\hline 4\\\hline 2\\\hline 2\\\hline 4\\\hline\end{array} = \begin{array}{|c|}\hline 0\\\hline 0\\\hline 0\\\hline 1\\\hline\end{array}\,\mathbf{x} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline 4\\\hline 1\\\hline\end{array}\,\mathbf{y} + \begin{array}{|c|}\hline 0\\\hline 0\\\hline 2\\\hline 0\\\hline\end{array}\,\mathbf{z} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline 0\\\hline 0\\\hline\end{array}\,\mathbf{w}$$

This can be write as the following and we call it the **Linear equation**

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 4y + 2z + 0w = 2 \\ 1x + 1y + 0z + 0w = 4 \end{cases}$$

Changing materials

Observation: The question is only asking for the meal′s demand for semi-product meals. It does not asking anything related to the raw material.

	🥤
🥛	x
🍲	y
☕	z
🍵	w

No 🍋, 🍃,  , 🐄 appeared in this question. Therefore we can change materials to **simplify** the problem.

Row reduction

The clever chef changes the **material** so the ingradients table is easier

	🥛	🍲	☕	🍵	🥤
🍃	0	2	0	2	4
🍋	0	1	0	1	2
🫘	0	4	2	0	2
🐄	1	1	0	0	4

	🥛	🍲	☕	🍵	🥤
🐄	1	1	0	0	4
🫘 🫘	0	2	1	0	1
🍋+🍃 🍃	0	1	0	1	2
🍃	0	0	0	0	0

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 4y + 2z + 0w = 2 \\ 1x + 1y + 0z + 0w = 4 \end{cases} \implies \begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 0y + 0z + 0w = 0 \end{cases}$$

Row Multiplying

Doubling the material  $\mapsto$  will **multiply 3rd row by $\frac{1}{2}$**.

	🥛	🍲	☕	🍵	🥤
🍃	0	2	0	2	4
🍋	0	1	0	1	2
	0	4	2	0	2
🐄	1	1	0	0	4

	🥛	🍲	☕	🍵	🥤
🍃	0	2	0	2	4
🍋	0	1	0	1	2
	0	2	1	0	1
🐄	1	1	0	0	4

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ \mathbf{0}x + \mathbf{4}y + \mathbf{2}z + \mathbf{0}w = \mathbf{2} \\ 1x + 1y + 0z + 0w = 4 \end{cases} \Longrightarrow \begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ \mathbf{0}x + \mathbf{2}y + \mathbf{1}z + \mathbf{0}w = \mathbf{1} \\ 1x + 1y + 0z + 0w = 4 \end{cases}$$

On equation, the third equation has been divided by 2. This is called **row multiplying**

Switching the order would not change problem,

	🥛	🍲	☕	🍵	🥤
🍃	0	2	0	2	4
🍋	0	1	0	1	2
🫘🫘	0	2	1	0	1
🐄	1	1	0	0	4

	🥛	🍲	☕	🍵	🥤
🐄	1	1	0	0	4
🫘🫘	0	2	1	0	1
🍋	0	1	0	1	2
🍃	0	2	0	2	4

$$\begin{cases} 0x + 2y + 0z + 2w = 4 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 1z + 0w = 1 \\ 1x + 1y + 0z + 0w = 4 \end{cases} \implies \begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 0z + 2w = 4 \end{cases}$$

This is called **row switching**

Row Adding

Replacing 🍋 by 🍋🍃🍃. Each time I use the package 🍋🍃🍃 I save 🍃🍃. Then the row for 🍃 is reduced by 2 times the row for 🍋.

	🥛	🍲	☕	🍵	🥤
🐄	1	1	0	0	4
🫘🫘	0	2	1	0	1
🍋	0	1	0	1	2
🍃	0	2	0	2	4

	🥛	🍲	☕	🍵	🥤
🐄	1	1	0	0	4
🫘🫘	0	2	1	0	1
🍋🍃🍃	0	1	0	1	2
🍃	0	0	0	0	0

$$\begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ 0x + 2y + 0z + 2w = 4 \end{cases} \implies \begin{cases} 1x + 1y + 0z + 0w = 4 \\ 0x + 2y + 1z + 0w = 1 \\ 0x + 1y + 0z + 1w = 2 \\ \mathbf{0}x + \mathbf{0}y + \mathbf{0}z + \mathbf{0}w = \mathbf{0} \end{cases}$$

On equation, $4'$th equation has been subtracted by 3rd equation. This is called **row adding**

When should row reduction stop?

In one words, row operation is **updating the raw ingradient list**.

We should stop if the matrix is **simple enough** for us to solve equations. So what is simple? There are many discussions.

Theory 1: We should stop if

- Each non-zero row has an entry 1, such that this 1 is the only non-zero entry on its columns.

					
	**1**	1	0	0	4
	0	2	**1**	0	1
 + 	0	1	0	**1**	2
	0	0	0	0	0

When should row reduction stop?

Explaination to Theory 1: If the matrix has been reduced that way, we obtain an equation

$$\begin{cases} 1\mathbf{x} + 1y + 0z + 0w = 4 \\ 0x + 2y + 1\mathbf{z} + 0w = 1 \\ 0x + 1y + 0z + 1\mathbf{w} = 2 \end{cases}$$

such that each equation has a variable, such that this variable does not appear in other equations. When this happens, we may move these variable to one side of equation. Assigning other variables arbitrary values would end up with a solutoin.

$$\begin{cases} x = 4 - y \\ z = 1 - 2y \\ w = 2 - y \end{cases}$$

Put $y = 0$, then $(x, y, z, w) = (4, 0, 1, 2)$ is a solution.

When should row reduction stop?

Explaination to Theory 1 without equation: A column with a single 1 and 0 elsewhere can help us to replace a material with some compunds. For example,

	☕
🐄	0
🫘 🫘	**1**
🍋+🍃🍃	0
🍃	0

$\Longrightarrow$ ☕ $=$ **0** 🐄 $+$**1** 🫘 🫘 $+$**0**(🍋🍃🍃)$+$**0**🍃

which means 🫘 🫘 $=$ ☕

When should row reduction stop?

With this observation, we can replace **certain materials** by **certain meals**

	🥛	🍲	☕	🍵	🥤
🐄	1	1	0	0	4
🫘 🫘	0	2	1	0	1
🍋+🍃🍃	0	1	0	1	2
🍃	0	0	0	0	0

	🥛	🍲	☕	🍵	🥤
🥛	1	1	0	0	4
☕	0	2	1	0	1
🍵	0	1	0	1	2
🍃	0	0	0	0	0

When should row reduction stop?

Note that the leaves 🍃 is no longer needed for those packaged materials, we can delete it.

	🥛	🍲	☕	🍵	🥤
🥛	1	1	0	0	4
☕	0	2	1	0	1
🍵	0	1	0	1	2

When should row reduction stop?

which tell us directly the list we want, lets compare the original question

Output

	🥤
🥛	4
☕	1
🍵	2

Original Question

	🥤
🥛	x
🍲	y
☕	z
🍵	w

So $x = 4, y = 0, z = 1, w = 2$.(🍲have not been used.) This process is equivalent as setting $y = 0$ to obtain a solution.

$$\begin{cases} x & = 4 \\ y & = 0 \\ z & = 1 \\ w & = 2 \end{cases}$$

When should row reduction stop?

Theory 2: We should stop if

- After deleting zero rows, there are columns that can be rearranged into a triangular matrix with non-zero diagonal.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 6 \\ 2 & 0 & 3 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

When should row reduction stop?

We beifly explain why , firstly, circle the entry that corresponding to the diagonal of the triangular matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 6 \\ 2 & 0 & 3 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We read equations from bottom to top, one by one

$$x + 0y + 0z + 2w + 1u = 0 \implies x = -2w - 1u.$$

The circled variable would appear as a new variabl never appeared before

$$2x + 0y + 3z + 2w + 1u = 1 \implies z = \frac{-2x - 2w - u}{3}.$$

Then the next new variable is y.

$$x + 2y + 3z + 4u + 6w = 6 \implies y = \frac{6 - 6w - 4u - 3z - x}{2}.$$

Therefore, the equation can be solved. We call this process **backwards substituting**.

Identity matrix

Filling the following blanks.

	🍵	🥛	☕
?	1	0	0
?	0	1	0
?	0	0	1

	?	?	?
🍵	1	0	0
🥛	0	1	0
☕	0	0	1

Identity matrix

Suppose 🥛 ☕🍵, any two can not blend to the third drink (called **linealy independent**). Filling the following blanks.

	🍵	🥛	☕
🍵	?	?	?
🥛	?	?	?
☕	?	?	?

Identity matrix

	🍵	🥛	☕
🍵	1	0	0
🥛	0	1	0
☕	0	0	1

This matrix is the **ingradient list of making ingradient**, i.e. do nothing.

Definition 1

The **identity matrix** is a $n \times n$ square matrix with 1 on the diagonal and 0 elsewhere.

Proposition 1

For any $n \times m$ matrix P, $I_n P = P I_m = P$.

Inverse Matrix

The chef is wondering if another guest coming with a special request, so he would like a list to produce the ingradient out of meals.

He has a list

	🥛	☕
🫘	0	2
🐄	1	0

How could he make another list?

	🫘	🐄
🥛	?	?
☕	?	?

He realize this list should have a property, combinging them should be.

	🥛	☕
🫘	0	2
🐄	1	0

	🫘	🐄
🥛	?	?
☕	?	?

=

	🫘	🐄
🫘	1	0
🐄	0	1

Inverse Matrix

Definition 2

For a $n \times n$ matrix A, an inverse is a matrix B, such that

$$AB = BA = I_n.$$

If such a B exists, A is called **invertible** and denote the inverse as A^{-1}.

Simutaneous Row Operation

For the product $C = AB$, changing raw materials only affect rows of A and C by certain simutaneous row operations. the matrix B would not change since it does not depend on raw materials. Therefore,

Proposition 2

The equality $C = AB$ will still be true if we perform arbitrary simutaneous row operation on A and C

Simutaneous Row Operation

left factor

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
	0	2	0
🐄	1	0	0

right factor

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

=

product

	🍱	🍲
🍃	2	2
🍋	1	1
	0	4
🐄	2	1

Simutaneous Row Operation

If A is invertible, we can write $B = A^{-1}C$, this actually tells us that

Corollary 1

If A is invertible, the product $A^{-1}C$ does not change if we perform simutaneous row operation on A and C.

Simutaneous Row Operation

Let me show you an application of this in calculation

$$\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}1 & 2 & 3\\1 & 2 & 1\\0 & 2 & 9\end{pmatrix}$$

$$\xlongequal{r_1\mapsto r_1-r_3}\begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}1 & 0 & -6\\1 & 2 & 1\\0 & 2 & 9\end{pmatrix}$$

$$\xlongequal{r_2\leftrightarrow r_3}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0 & -6\\0 & 2 & 9\\1 & 2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 & -6\\0 & 2 & 9\\1 & 2 & 1\end{pmatrix}$$

Simutaneous Column Operation

Similarly, Column operations corresponding to updating list when changing final meals by its order, amount, or packing them together. For the product $C = AB$, when final meals changes, the matrices affected is C and B. A is the list of making intermediates, it does not change.

Proposition 3

The equality $C = AB$ will still be true if we perform arbitrary simutaneous column operation on B and C

If B is invertible, we can write $A = CB^{-1}$, this actually tells us that

Corollary 2

If B is invertible, the product CB^{-1} does not change if we perform simutaneous column operation on B and C.

Elementary Matrices

We will introduce elementary matrices.

Let A be an $m \times n$ matrix, and let I_m be $m \times m$ identity matrix. We have a equation

$$A = I_m A$$

After some simutaneous row operation, A changes to A' and I_m changes to E, by what we discussed before, the following equation is true

$$A' = EA$$

We may view this fact by another perspective: The product EA is the same as applying row operations recorded in E to A.

Elementary Matrices

Experiment:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 3 \end{pmatrix}$$

Explaination from another perspective:

left factor	row operation recorded	product: apply row operation
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	add 2nd row to 1st row.	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{2} & \mathbf{4} \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{3} & \mathbf{7} \\ 1 & 3 \end{pmatrix}$

Elementary Matrices

Proposition 4

Suppose E is a matrix obtained by applying some row operations from I_n, then EA is exactly the matrix by applying the same row operations on A.

Definition 3

An elementary matrix E is a matrix after one-step row operation from identity matrix.

Elementary Matrices

Name	Elementary matrices example	Inverse
Switching	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
Multiplying	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Adding	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Elementary Matrices

Exercise: What is the following product? calculate in mind.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 6 & 8 \end{pmatrix}$$

Elementary Matrices

Look at the following matrix(maybe not elementary). By thinking how it changed from identity matrix, what kind of row operation does it record?

left-factor	row operation recorded.
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Switch two rows
$\begin{pmatrix} 0 & 1 \end{pmatrix}$	Delete the first row
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	
$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	

Elementary Matrices

The same logic is also for columns. Recall that $AB = C$ stays as an equation when applying column oprtations on B and C. **Experiment:**

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \implies \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$$

Explaination from another perspective:

right factor	row operation recorded	product: apply row operation
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	add 1st column to 2nd column.	$\begin{pmatrix} \mathbf{2} & 4 \\ \mathbf{1} & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & \mathbf{6} \\ 1 & \mathbf{4} \end{pmatrix}$

Elementary Matrices

Proposition 5

Suppose E is a matrix obtained by applying some column operations from I_n, then AE is exactly the matrix by applying the same column operations on A.

Elementary Matrices

right-factor	column operation recorded.
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Switch two columns
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	
$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	

Elementary Matrices

What is the meaning of the following product? From the perspective of column action?

$$\underbrace{A}_{3\times 4\text{Matrix}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

What is the meaning of the following product? From the perspective of column action?

$$\underbrace{A}_{3\times 4\text{Matrix}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Elementary Matrices

What is the meaning of the following matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{A}_{2\times 2\text{Matrix}}$$

How about the following

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \underbrace{A}_{2\times 2\text{Matrix}}$$

Is this product depends on the second row of A?

Elementary Matrices

Let A be a $m \times n$ matrix, and let l be the 2nd row of A, can you write l in terms of a matrix multiplication?

Let $\vec{v}$ be the 3rd column of A, can you write $\vec{v}$ into a matrix multiplication?