Computing polynomials of a matrix

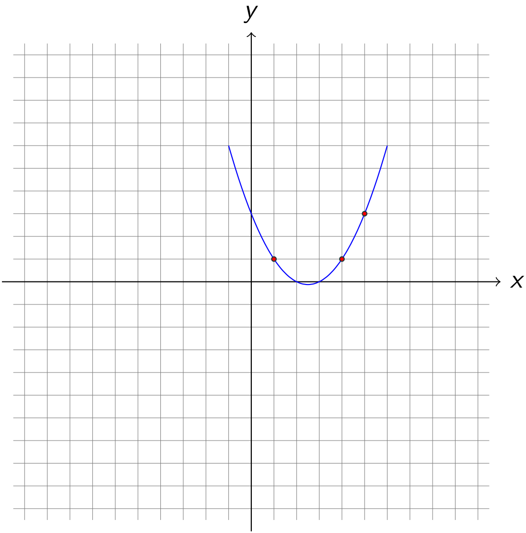
In previous slides, we leared that $\text{char}_A(t) = \det(tI - A)$ is an annihilating polynomial, and we call the list of its roots the list of **eigenvalues**

The imporatance of the characteristic polynomial is that $\text{char}_A(A) = 0$, the Caylay Hamilton theorem.

In this part, we introduce an easier method to calculate $g(A)$ for general polynomial g in this section. This enables us to calculate formulas like A^n or e^A for various purposes. This motivate us to define **eigenvectors** nand **eigenspaces**

Review Lagurange Interpolation Polynomial

Interpolation means to find a function with its graph passing through certain points,



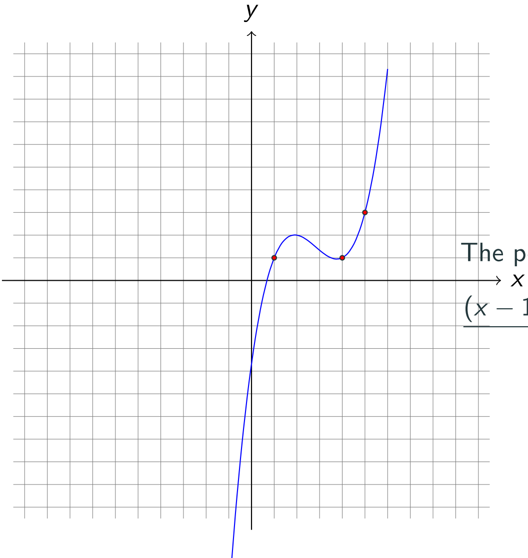
the left is a quadratic polynomial interpolation of

x	1	4	5
y	1	1	3

The interpolation is given by $f(x) = \frac{x^2-5x+6}{2}$

Review Lagurange Interpolation Polynomial

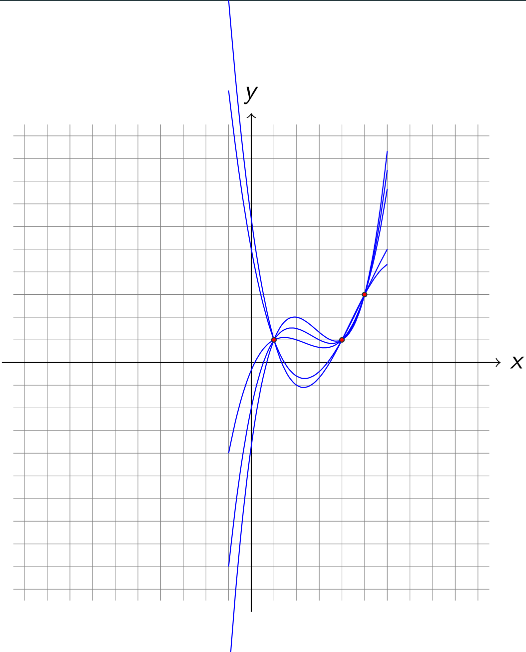
Note that the **interpolation is not unique**, for example, the following is also a polynomial interpolating of $(x, y) = (1, 1), (4, 1), (5, 3)$



The polynomial used here is

$$\frac{(x-1)(x-4)(x-5)}{3} + \frac{(x-2)(x-3)}{2}.$$

Review Lagurange Interpolation Polynomial



We have all arbitrary choices of interpolation polynomials for $(x, y) = (1, 1), (4, 1), (5, 3)$. What is the difference between any two of the interpolation polynomial?

Review Lagurange Interpolation Polynomial

Proposition 1

Suppose $x_1, ..., x_k$ are distinct numbers. Let $g(x)$ and $h(x)$ interpolating the same data set in the sense that

$$g(x_i) = h(x_i) = y_i$$

for

$$(x_i, y_i) \in \{(x_1, y_1), (x_2, y_2), \cdots (x_k, y_k)\}.$$

Then $g(x) - h(x)$ is divisible by $(x - x_1) \cdots (x - x_k)$.

Review Lagurange Interpolation Polynomial

Therefore,

$$Q(x) = \frac{g(x) - h(x)}{(x - x_1) \cdots (x - x_k)}$$

is a polynomial. We may write

$$g(x) = Q(x)(x - x_1) \cdots (x - x_k) + h(x).$$

for any two interpolation of the data at $x_1, ..., x_k$.

Review Lagurange Interpolation Polynomial

For any polynomial $g(x)$, it interpolate itself at

x_1	x_2	$\cdots$	x_n
$g(x_1)$	$g(x_2)$	$\cdots$	$g(x_n)$

Using Lagurange interpolation polynomial, we may also construct

$$h(x) = g(x_1)\frac{(x - x_2)\cdots(x - x_n)}{(x_1 - x_2)\cdots(x_1 - x_n)} + \cdots + g(x_n)\frac{(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_1)\cdots(x_n - x_{n-1})}$$

Note that $h(x)$ also interpolates the same data, we have $g(x_1) = h(x_1)$, ..., $g(x_n) = h(x_n)$. We have

$$g(x) = Q(x)(x - x_1)\cdots(x - x_n) + h(x).$$

Review Lagurange Interpolation Polynomial

Excercise. Write down an interpolation of t^5 at point $t = 1$, $t = 2$ and $t = 3$.

Solution. We have the following table

	$t = 1$	$t = 2$	$t = 3$
t^5	**1**	**32**	**243**

The Lagurange interpolation at these 3 points is given by the following Lagurange interpolation.

$$\boxed{\mathbf{1}}\ \underbrace{\frac{(t-2)(t-3)}{(1-2)(1-3)}} + \boxed{\mathbf{32}}\ \underbrace{\frac{(t-1)(t-3)}{(2-1)(2-3)}} + \boxed{\mathbf{243}}\ \underbrace{\frac{(t-1)(t-2)}{(3-1)(3-2)}}$$

t=1	t=2	t=3
1	0	0

t=1	t=2	t=3
0	1	0

t=1	t=2	t=3
0	0	1

Then we can write

$$t^5 = Q(t)(t-1)(t-2)(t-3)$$
$$+\frac{(t-2)(t-3)}{(1-2)(1-3)} + 32\frac{(t-1)(t-3)}{(2-1)(2-3)} + 243\frac{(t-1)(t-2)}{(3-1)(3-2)}$$

Review Lagurange Interpolation Polynomial

Let us write things more straightforward for comfortable.

$$g(t) = t^5$$

and

$$h(t) = \frac{(t-2)(t-3)}{(1-2)(1-3)} + 32\frac{(t-1)(t-3)}{(2-1)(2-3)} + 243\frac{(t-1)(t-2)}{(3-1)(3-2)}$$

$$= 90t^2 - 239t + 150.$$

In fact

$$t^5 = \underbrace{(t^2+6t+25)}_{Q(t)}(t-1)(t-2)(t-3) + \underbrace{90t^2 - 239t + 150}_{h(t)}$$

Review Lagurange Interpolation Polynomial

Excercise. Suppose A is a matrix with

$$(A - I)(A - 2I)(A - 3I) = 0,$$

write A^5 as a linear combination of I, A, A^2.

Hint: use the Lagurange interpolation polynomial

$$t^5 = \underbrace{(t^2 + 6t + 25)}_{Q(t)}(t-1)(t-2)(t-3) + \underbrace{90t^2 - 239t + 150}_{h(t)}$$

Review Lagurange Interpolation Polynomial

Answer: $A^5 = 90A^2 - 239A + 150$

Review Lagurange Interpolation Polynomial

Definition 1

We say a polynomial $f(t)$ is of **simple roots** if one can write

$$f(t) = (t - x_1)\cdots(t - x_k)$$

for **distinct** $x_1, ..., x_k$.

Here distinct means $x_i \neq x_j$ for any $i \neq j$.

Definition 2

We say a matrix A **satisfies a polynomial of simple roots** if A satiesfies

$$(A - x_1 I)\cdots(A - x_m I) = 0$$

for some **distinct** $x_1, ..., x_m$.

Review Lagurange Interpolation Polynomial

Summary

Suppose $t_1, ..., t_m$ are **distinct** poimts, amd A satisfies a polynomial with simple roots

$$(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_m I) = 0.$$

Let $g(t)$ and $h(t)$ be any polynomials, then

$$\begin{cases} g(\lambda_1) = h(\lambda_1) \\ g(\lambda_2) = h(\lambda_2) \\ \cdots \\ g(\lambda_m) = h(\lambda_m) \end{cases} \implies g(t) - h(t) = Q(t)(t - \lambda_1) \cdots (t - \lambda_m)$$

$$t = A \implies g(A) - h(A) = 0 \implies g(A) = h(A)$$

So $g(A)$ only depends on the value $g(x_1), \cdots, g(x_n)$.