Conjugate Transpose

The vector space $\mathbb{R}^n$ has an inner product, where we can define the length of a vector by

$$||\vec{v}|| = \sqrt{\vec{v}^T \vec{v}}$$

Conjugate Transpose

We would like to generalize the concept of length and angle for vectors in $\mathbb{C}^n$, but we have a problem:

What is the length of the vector

$$\begin{pmatrix} 1 \\ i \end{pmatrix}$$

After calculation

$$\begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 1 - 1 = 0.$$

The length of this vector is zero. This could be meaning ful, but, ...

Conjugate Transpose

Generalization of inner product of $\mathbb{R}^n$ to $\mathbb{C}^n$: 2 approach

1. First generalization: $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$;
2. Second generaliztion: $\vec{v} \cdot \vec{w} = \overline{\vec{v}^T} \vec{w}$.

The second generalization has been used by god to create our universe, it appears repeatedly in the quantum machanics.

In this class, we will study the second generalization, the inner product over $\mathbb{C}^n$ defined by $\vec{v} \cdot \vec{w} = \overline{\vec{v}^T} \vec{w}$. This is called **Hermitian inner product**

To simplify the notation, we also denote by

$$\vec{v}^H := \overline{\vec{v}^T}.$$

The letter H refers to Hermitian.

Conjugate Transpose

Definition 1

For a matrix A, we define

$$A^H := \overline{A^T}$$

and call it the **Hermitian transpose**

The Hermitian transpose is a direct generalization of the transpose for real number matrices.

Proposition 1

$A^H = A^T$ if and only if A is a real-coefficeint matrix.

Conjugate Transpose

Proposition 2

The Hermitian transpose clearly have the following properties.

- For any A, B of the same size, $(A + B)^H = A^H + B^H$
- For any A, B, we have $(AB)^H = B^H A^H$

Generalization of positivity lemma

Recall that the properties of inner product

Theorem 1

For any **non-zero vector** $\vec{0} \neq \vec{v} \in \mathbb{R}^n$, we have $\vec{v}^T \vec{v} > 0$.

This is because if we write $\vec{v} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$, then $\vec{v}^T \vec{v} = x_1^2 + \ldots + x_n^2 > 0$. The result directly generalizes to matrices

Lemma 1

For any **non-zero** matrix M, we have $M^T M \neq 0$.

Generalization of positivity lemma

Clear, the above positivity property no longer holds for the **first generalization** of inner product to $\vec{v}^T\vec{w}$ for $\mathbb{C}^n$, because

$$\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{v}^T\vec{v} = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 1^2 + i^2 = 0$$

A non-zero vector of length 0? A non-zero vector perpendicular to itself?

Generalization of positivity lemma

For the second generalization, the length of $\begin{pmatrix} 1 & i \end{pmatrix}^T$ is no longer zero

$$\vec{v}^H \vec{v} = \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 1^2 - i^2 = 2 = 0.$$

Generalization of positivity lemma

The positivity lemma holds. Idea:

$$\begin{pmatrix} \overline{x_1} & \overline{x_2} & \dots & \overline{x_n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \overline{x}_1 x_1 + \overline{x}_2 x_2 + \ldots + \overline{x}_n x_n$$

$$= \underbrace{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}_{\text{all non-negative real numbers}} > 0$$

Positivity for Hermitian Inner Product

Theorem 2

For any **non-zero vector** $\vec{0} \neq \vec{v} \in \mathbb{C}^n$, we have $\vec{v}^H \vec{v} > 0$.

This is because if we write $\vec{v} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$, then $\vec{v}^H \vec{v} = |x_1|^2 + \ldots + |x_n|^2 > 0$. The result directly generalizes to matrices

Lemma 2

For any **non-zero** complex coefficient matrix M, we have $M^H M \neq 0$.

Positivity for Hermitian Inner Product

Definition 2

A matrix M is called **(Hermitian) Normal** if

$$M^H M = MM^H$$

Positivity for Hermitian Inner Product

Next, we show a **general philosophy** in linear algebra, roughly speaking, that

$$\text{Normal under an } \textbf{involution} \text{ with positivity lemma} \implies \text{Diagonalizable}$$

Positivity for Hermitian Inner Product

To work on a more general settings, we introduce

Definition 3

An **involution** ι is an action associate each $n \times n$ matrix M with another matrix M^ι, such that

- $(A+B)^\iota = A^\iota + B^\iota$
- $(AB)^\iota = A^\iota B^\iota$ or $(AB)^\iota = B^\iota A^\iota$
- For any scalar, λ, there is a scalar λ^ι, such that $(\lambda I_n)^\iota = \lambda^\iota I_n$

Any involution satisfies the above property would satisfy

$$p(A^\iota) = (p(A))^\iota$$

for any polynomial p.

For $\iota = T$, $\iota = H$ they are all involutions. But $A \longmapsto -A$ or $A \longmapsto A^{-1}$ is not an involution.

Positivity for Hermitian Inner Product

Definition 4

Let ι be an involution, a matrix is called **ι-normal** if $AA^{\iota} = A^{\iota}A$.

Positivity for Hermitian Inner Product

Theorem 3

Let ι be an involution on n $\times$ n matrices, if ι **satisfies the positivity lemma** $A^\iota A = 0 \implies A = 0$ for all matrices A, then any ι-normal matrices A with

$$\det(tI - A) = \prod_{i=1}^{k} (t - \lambda_i)^{m_i}$$

are **diagonalizable**

Proof: Without loss of generality we may say $\lambda_1, \ldots, \lambda_k$ are distinct roots, let

$$N = \prod_{i=1}^{k} (A - \lambda_i I_n)$$

Then $A$is diagonalizable $\iff N = 0$. By Caylay Hamilton Theroem, we know $N^n = 0$.

Positivity for Hermitian Inner Product

Since A is ι-normal, $A^\iota A = AA^\iota$, N must be normal $N^\iota N = NN^\iota$.

Now

$$\underbrace{NN\dots N}_{n-\text{many}} = 0 \implies \underbrace{NN\dots N}_{n-\text{many}}\underbrace{N^\iota N^\iota \dots N^\iota}_{n-\text{many}} = 0$$

Since N commute with N^ι, we have

$$\underbrace{NN^\iota NN^\iota NN^\iota \dots NN^\iota}_{n-\text{many } NN^\iota} = 0$$

If n is odd, we replace n by $n+1$, we separate

$$\underbrace{NN^\iota \dots NN^\iota}_{\frac{n}{2}-\text{many}}\underbrace{NN^\iota \dots NN^\iota}_{\frac{n}{2}-\text{many}} = 0$$

However, this equals to

$$\underbrace{NN^\iota \dots NN^\iota}_{\frac{n}{2}-\text{many}}(\underbrace{NN^\iota \dots NN^\iota}_{\frac{n}{2}-\text{many}})^\iota = 0$$

Therefore $\underbrace{NN^\iota \dots NN^\iota}_{\frac{n}{2}-\text{many}} = 0$

Positivity for Hermitian Inner Product

Use this strategy again and again, we finally got

$$NN^{\iota} = 0$$

Therefore $N = 0$. This implies that the matrix A is diagonalizable.

Positivity for Hermitian Inner Product

In particular, if A is a matrix with complex numbers, any complex-coefficient polynomial can be written in the form

$$\prod_{i=1}^{k}(t-\lambda_i)^{m_i}$$

Corollary 1

Let A be a Hermitian Normal matrix in the sense that $A^H A = AA^H$, then A is diagonalizable.

Positivity for Hermitian Inner Product

We are not satisfied.

Definition 5

A matrix A is called **Hermitian-symmetric** if $A = A^H$

Definition 6

A matrix A is called **ι-symmetric** if $A = A^\iota$

An example of Hermitian symmetric matrix

$$\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Positivity for Hermitian Inner Product

Definition 7

A matrix A is called **unitary** if $AA^H = A^H A = I$

Definition 8

A matrix A is called **ι-orthogonal** if $AA^\iota = A^\iota A = I$

An example of Hermitian orthogonal matrix

$$\begin{pmatrix} \sqrt{0.5} & i\sqrt{0.5} \\ \sqrt{0.5} & -i\sqrt{0.5} \end{pmatrix}$$

Positivity for Hermitian Inner Product

In our lecture, we will prove this theroem

Theorem 4

Let A be a **Hermitian normal** matrix, then there exists **unitary** matrix $\Omega = {\Omega^{-1}}^H$, such that

$$\Omega^{-1} A \Omega = \Omega^H A \Omega = \Lambda$$

is a diagonal matrix!

Positivity for Hermitian Inner Product

Note that in spectural decomposition

$$g(A) = g(\lambda_1)P_1 + g(\lambda_2)P_2 + \ldots + g(\lambda_k)P_k$$

All matrices P_i are obtained by plugging A into interpolation polynomials, so if A is Hermitian-normal, then P_i are normal as well!

Positivity for Hermitian Inner Product

Lemma 3

If $P = P^2$ is a projection and normal

$$P^H P = PP^H$$

Then P must be Hermitian symmetric $P^H = P$.

Proof: Consdier

$$(P - P^H P)(P - P^H P)^H = PP^H - PP^H P - P^H P^H P + (P^H P)^2$$
$$= PP^H - 2P^H P + P^H P = 0$$

By **positivity lemma** for ^{H}, we have $P - P^H P = 0$, this implies $P = P^H P$. So

$$P = P^H P = (P^H P)^H = P^H.$$

Positivity for Hermitian Inner Product

!!

The above proof essentially only uses positivity. Therefore, if ι is an involution with positivity lemma, any ι-normal projection is ι-symmetric.

Positivity for Hermitian Inner Product

Corollary 2

Suppose A is a normal matrix over $\mathbb{C}$, then A is diagonalizable and its spectural decomposition

$$g(A) = g(\lambda_1)P_1 + \ldots + g(\lambda_k)P_k$$

all eigenspace projection are Hermitian symmetric.

Recall previously, we leaned a projection P is symmetric iff $\ker(P) \perp \mathrm{Im}(P)$. In real number matrices, we have learned that if $P_1 + \ldots + P_k = I$ and $P_i^T = P_i$, then all $\mathrm{Im}(P_i)$ are orthogonal to each other.

Positivity for Hermitian Inner Product

Definition 9

Two vector $\vec{v}, \vec{w}$ is called **Hermitian orthogonal**, if

$$\vec{v}^H \vec{w} = 0$$

Definition 10

Two subspaces W_1, W_2 are called **Hermitian orthogonal** if and only if for any $\vec{v} \in W_1$ and $\vec{w} \in W_2$, we have $\vec{v}^H \vec{w} = 0$. In this case, we denote it by $W_1 \perp W_2$.

Positivity for Hermitian Inner Product

Theorem 5

A matrix A is normal if and only if it is diagonalizable with **mutually Hermitian-orthogonal eigenspaces**.

Positivity for Hermitian Inner Product

To obtain an unitary diagonalization, we simply use diagonal cross-filling.

Positivity for Hermitian Inner Product

Excercise. Find an unitary diagonalization of the following matrix

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Eigenvalues $0, 3$.

Special types of normal operators and eigenvalues

We have the following special type of normal matrices,

- Hermitian symmetric matrices $A^H = A$;
- skew-Hermitian symmetric matrices $A^H = -A$;
- Unitary matrices $A^H = A^{-1}$;

They are all normal operators!!

How can we visualize them in the context of spectural decomposition???

$$g(A) = g(\lambda_1)P_1 + \ldots + g(\lambda_k)P_k \qquad P_i^H = P_i$$

Special types of normal operators and eigenvalues

Note that complex conjugation are the one compatible with ^{H}

$$(\lambda I_n)^H = \overline{\lambda} I_n$$

Special types of normal operators and eigenvalues

Theorem 6

A matrix A is **Hermitian symmetric** $A = A^H$ if and only if A is diagonalizable, eigenspaces mutually Hermitian orthogonal **with real eigenvalues**.

Proof:

$$A = A^H$$

$$\iff$$

$$\lambda_1 P_1 + \ldots + \lambda_k P_k = \overline{\lambda}_1 P_1 + \ldots + \overline{\lambda}_k P_k, \qquad P_k^H = P_K$$

Note that $\lambda = \overline{\lambda}$ means $\lambda \in \mathbb{R}$

Theorem 7

A matrix A is **skew-Hermitian symmetric** $A = -A^H$ if and only if A is diagonalizable, eigenspaces mutually Hermitian orthogonal **with purly imaginary eigenvalues**.

Proof:

$$A = A^H$$

$\iff$

$$\lambda_1 P_1 + \ldots + \lambda_k P_k = -\overline{\lambda}_1 P_1 + \ldots + -\overline{\lambda}_k P_k, \qquad P_k^H = P_K$$

Note that $\lambda = -\overline{\lambda}$ means $\lambda = bi$

Special types of normal operators and eigenvalues

Theorem 8

A matrix A is **unitary** $A^H = A^{-1}$ if and only if A is diagonalizable, eigenspaces mutually Hermitian orthogonal **with all eigenvalues of absolute value 1**.

Proof:

$$A = A^H$$

$$\iff$$

$$\lambda_1 P_1 + \ldots + \lambda_k P_k = \overline{\lambda}_1^{-1} P_1 + \ldots + \overline{\lambda}_k^{-1} P_k, \qquad P_k^H = P_K$$

Note that $\lambda = \overline{\lambda}^{-1} \iff \lambda\overline{\lambda} = 1 \iff |\lambda| = 1.$

Special types of normal operators and eigenvalues

Over complex plane, the distribution of eigenvalues determines the property of matrix