Trace and determinant in terms of eigenvalues

Definition 1

An **eigenvalue** λ of matrix A is a value with

$$\det(\lambda I - A) = 0.$$

Suppose one can factorize characteristic polynomial into

$$\det(tI - A) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n).$$

Then we call the list $\lambda_1, \lambda_2, ..., \lambda_n$ the list of its eigenvalues. (Note,

the element can be repeated in the list.)

Trace and determinant in terms of eigenvalues

Excercise. Suppose matrix A has

$$\det(tI - A) = (t-1)^3(t-2)^2(t-3)^5$$

what is a list of its eigenvalues?

Solution.

$$1, 1, 1, 2, 2, 3, 3, 3, 3, 3.$$

Trace and determinant in terms of eigenvalues

Proposition 1

If $\det(tI - A) = \det(tI - B)$, then the sum of **principal minors** of size k in A and in B are the same, because they are all given by the coefficient of $(-1)^{n-k}t^k$.

Do you remember the definition of principal minor?? they are submatrices with diagonal on diagonal of its father...

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

Trace and determinant in terms of eigenvalues

Therefore suppose A is a matrix with

$$\det(tI - A) = (t-1)^2(t-2)^3$$

Then we consider a diagonal matrix with the same chacracteristic polynomial with it, which can be written

$$\Lambda = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}$$

Trace and determinant in terms of eigenvalues

$$\det(tI - \Lambda)$$

$$= \begin{pmatrix} t-1 & & & & \\ & t-1 & & & \\ & & t-2 & & \\ & & & t-2 & \\ & & & & t-2 \end{pmatrix} = (t-1)(t-1)(t-2)(t-2)(t-2)$$

Note, Λ has nothing to do with A except sharing the same characteristic polynomial with A. Then we have

sum of principal minor of size $1 = \mathrm{tr}(A) = \mathrm{tr}(\Lambda) =$ sum of eigenvalues of A

sum of principal minor of size n $= \det(A) = \det(\Lambda) =$ product of eigenvalues A

Trace and determinant in terms of eigenvalues

Proposition 2

$n \times n$ matrix has n eigenvalues. The trace of a matrix, equals to the sum of all its eigenvalues, the product of a matrix, equals to the product of all its eigenvalues.

Trace and determinant in terms of eigenvalues

Excercise. Given one eigenvalue of the following matrix, are you able to find the other eigenvalue? and the determinant?

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \qquad \text{Eigenvalue}: \ 2, \ \_\_, \qquad \det:$$

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \qquad \text{Eigenvalue}: \ 4, \ \_\_ \qquad \det:$$

$$\begin{pmatrix} 5 & 0 \\ 1 & 5 \end{pmatrix} \qquad \text{Eigenvalue}: \ 5, \ \_\_ \qquad \det:$$