Transformation addition and matrix addition

Given linear transformation from V to W, we can construct new linear transformation by taking linear combinations of those linear transformation . This is corresponding to linear combination of matrices.

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Definition 1

For two linear transformation $T : V \longrightarrow W$ and $S : V \longrightarrow W$, we can define their linear combination for any scalar $\lambda, \mu \in F$ by

$$\lambda T + \mu S : V \longrightarrow W, \vec{v} \mapsto \lambda T(\vec{v}) + \mu S(\vec{v})$$

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Proposition 1 Distribution law

Let V, W, U be vector spaces, then for the following linear transformation and arbitrary scalar $\lambda, \mu \in F$

$$V \underset{S}{\overset{T}{\rightrightarrows}} W \overset{R}{\to} U$$

We have

$$R \circ (\lambda S + \mu T) = \lambda(R \circ S) + \mu(R \circ T);$$

For

$$U \overset{R}{\to} V \underset{S}{\overset{T}{\rightrightarrows}} W$$

We have

$$(\lambda S + \mu T) \circ R = \lambda(S \circ R) + \mu(T \circ R);$$

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We leave the above proof as an Excercise since it is straightforward.

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Proposition 2

Let V, W be vector spaces,

$$\mathcal{E} \text{ basis for } V \quad \mathcal{F} \text{ basis for } W$$

For a pair of linear transformation

$$\begin{tikzcd} V \arrow[r, shift left, "T"] \arrow[r, shift right, "S"'] & W \end{tikzcd}$$

Suppose A is the matrix for T under bases $\mathcal{E}, \mathcal{F}$ and B is the matrix for S under bases $\mathcal{E}, \mathcal{F}$. Then for any scalar $\mu, \lambda \in F$, the matrix for $\lambda T + \mu S$ under bases $\mathcal{E}, \mathcal{F}$ is given by

$$\lambda A + \mu B.$$

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Proof: By what given, we have

$$T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} A$$

$$S\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} B$$

So

$$\lambda T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} \lambda A$$

$$\mu S\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} \mu B$$

Then

$$(\lambda T + \mu S)\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}}(\lambda A + \mu B)$$