Spectural decomposition

We summarize the method. Suppose A is a matrix with

$$(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_m I) = 0,$$

to find $g(A)$ for any polynomial A, we know it only depends on the value of $g(\lambda_1), ..., g(\lambda_m)$.

Theorem 1

Let A be matrix with $(A - \lambda_1 I) \cdots (A - \lambda_m I) = 0$ then for any polynomial $g(x)$ and polynomial $h(x)$ such that $h(\lambda_i) \neq 0$

$$\frac{g(A)}{h(A)} = \frac{g(\lambda_1)}{h(\lambda_1)} f_{\lambda_1}(A) + \cdots + \frac{g(\lambda_m)}{h(\lambda_m)} f_{\lambda_m}(A)$$

Spectural decomposition

We wanna prove

$$g(A) = h(A)\left(\frac{g(\lambda_1)}{h(\lambda_1)}f_{\lambda_1}(A) + \cdots + \frac{g(\lambda_m)}{h(\lambda_m)}f_{\lambda_m}(A)\right)$$

For this purpose, we compare the two polynomial

$$g(t) \text{ v.s. } h(t)\left(\frac{g(\lambda_1)}{h(\lambda_1)}f_{\lambda_1}(t) + \cdots + \frac{g(\lambda_m)}{h(\lambda_m)}f_{\lambda_m}(t)\right)$$

Note that this two polynomial have the same value when $t = \lambda_1, t = \lambda_2$ until $t = \lambda_m$. Therefore the theorem is true.

Some more details for spectural decomposition

Proposition 1

Suppose $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_m I)=0$. Let $f_{\lambda_1}, f_{\lambda_2}, \cdots, f_{\lambda_m}$ be corresponding interpolation polynomials as before, denote by

$$P_{\lambda_i} := f_{\lambda_i}(A),$$

it is a projection matrix.

Proof: Remember the polynomial f_{λ_i} is defined by

$$f_{\lambda_i}(\lambda_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

No matter what value it takes, they all satisfies $f_{\lambda_i}(\lambda_j)^2 = f_{\lambda_i}(\lambda_j)$ for any j. Therefore

$$f_{\lambda_i}(A)^2 = f_{\lambda_i}(A) \implies P_{\lambda_i}^2 = P_{\lambda_i}$$

is a projection matrix.

Some more details for spectural decomposition

Proposition 2

We have

$$AP_{\lambda_i} = P_{\lambda_i}A = \lambda_i P_{\lambda_i}$$

and

$$A = \lambda_1 P_{\lambda_1} + \cdots + \lambda_m P_{\lambda_m}$$

$$I = P_{\lambda_1} + \cdots + P_{\lambda_m}$$

Note that either AP_{λ_i} or $P_{\lambda_i}A$ is obtained by plug in $t = A$ to the polynomial $tf_{\lambda_i}(t) = f_{\lambda_i}(t)t$. And the matrix $\lambda_i P_{\lambda_i}$ is obtained by plug in $t = A$ into polynomial $\lambda_i f_{\lambda_i}(t)$.

	$t = \lambda_1$	$\cdots$	$t = \lambda_{i-1}$	$t = \lambda_i$	$t = \lambda_{i+1}$	$\cdots$	$t = \lambda_n$
$\lambda_i f_{\lambda_i}(t)$	0	$\cdots$	0	λ_i	0	$\cdots$	0
$f_{\lambda_i}(t)t$	0	$\cdots$	0	λ_i	0	$\cdots$	0

Some more details for spectural decomposition

Therefore, the two polynomials $\lambda_i f_{\lambda_i}(t)$ and $f_{\lambda_i}(t)t$ have the same value for all $\lambda_1, \lambda_2, ..., \lambda_m$. Then we have $AP_{\lambda_i} = \lambda_i P_{\lambda_i}$ and $P_{\lambda_i} A = \lambda_i P_{\lambda_i}$

To prove $A = \lambda_1 P_{\lambda_1} + \cdots + \lambda_m P_{\lambda_m}$ and $I = P_{\lambda_1} + \cdots + P_{\lambda_m}$, we apply $h(t) = 1$, $g(t) = t$ and $g(t) = 1$ to the spectural decomposition

$$\frac{g(A)}{h(A)} = \frac{g(\lambda_1)}{h(\lambda_1)} P_{\lambda_1} + \cdots + \frac{g(\lambda_m)}{h(\lambda_m)} P_{\lambda_m}$$

Some more details for spectural decomposition

Another proof that P_{λ_i} saties fies $AP = \lambda P$ is easy. Note that the construction of Lagurange interpolation polynomial

$$f_{\lambda_i}(t) = \frac{(t-\lambda_1)\cdots(t-\lambda_{i-1}) \qquad (t-\lambda_{i+1})\cdots(t-\lambda_m)}{\text{Constant}}$$

$$P_{\lambda_i} = \frac{(A-\lambda_1)\cdots(A-\lambda_{i-1}) \qquad (A-\lambda_{i+1})\cdots(A-\lambda_m)}{\text{Constant}}$$

Thus

$$(A-\lambda_i)P_{\lambda_i} = \frac{(A-\lambda_1)\cdots(A-\lambda_{i-1})(A-\lambda_i)(A-\lambda_{i+1})\cdots(A-\lambda_m)}{\text{Constant}} = 0$$

Some more details for spectural decomposition

The matrix P_{λ_i} has something to do with

Definition 1

A **non-zero** column vector $\vec{v}$ is called an eigenvector of eigenvalue λ if $A\vec{v} = \lambda\vec{v}$ for some saclar λ.

Definition 2

A **non-zero** row vector $\vec{w}^T$ is called a left eigenvector of eigenvalue λ if $\vec{w}^T A = \lambda\vec{w}^T$ for some saclar λ.

Some more details for spectural decomposition

Corollary 1

All non-zero columns of P_{λ_i} is a eigenvector of eigenvalue λ_i of A.
All non-zero rows of P_{λ_i} is a left-eigenvector of eigenvalue λ_i of A.

Some more details for spectural decomposition

If we write

$$P_{\lambda_i} = \begin{pmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{pmatrix}$$

Then

$$AP_{\lambda_i} = \begin{pmatrix} | & | & \cdots & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & \cdots & | \end{pmatrix}$$

and

$$\lambda_i P_{\lambda_i} = \begin{pmatrix} | & | & \cdots & | \\ \lambda_i \vec{v}_1 & \lambda_i \vec{v}_2 & \cdots & \lambda_i \vec{v}_n \\ | & | & \cdots & | \end{pmatrix}$$

then $AP_{\lambda_i} = \lambda_i P_{\lambda_i}$ implies $A\vec{v}_j = \lambda_i \vec{v}_j$.

Therefore all columns are eigenvectors.

Some more details for spectural decomposition

For rows, we write

$$P_{\lambda_i} = \begin{pmatrix} - & \vec{w}_1^T & - \\ - & \vec{w}_2 & - \\ \vdots & \vdots & \vdots \\ - & \vec{w}_n & - \end{pmatrix}$$

Note that

$$P_{\lambda_i} A = \begin{pmatrix} - & \vec{w}_1^T A & - \\ - & \vec{w}_2 A & - \\ \vdots & \vdots & \vdots \\ - & \vec{w}_n A & - \end{pmatrix}$$

and that

$$P_{\lambda_i} \lambda_i = \begin{pmatrix} - & \lambda_i \vec{w}_1^T & - \\ - & \lambda_i \vec{w}_2 & - \\ \vdots & \vdots & \vdots \\ - & \lambda_i \vec{w}_n & - \end{pmatrix}$$

Excercise. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_m$ are distinct numbers. A is a matrix with

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$$

$$I = P_1 + P_2 + \cdots + P_m$$

How many different projection matrices can be written as polynomial $g(A)$ of A?

	$t = \lambda_1$	$t = \lambda_2$	$\cdots$	$t = \lambda_m$
$g(t)$				

Some application of spectural decomposition

Theorem 2

If $\det(tI - A)$ is of simple roots, then A satisfies a polynomial with simple roots.

This is because any matrix satiesfies its characteristic polynomial.

Some application of spectural decomposition

However, even if $\det(tI - A)$ is not of simple roots, it is still possible that A satisfies a polynomial of simple roots. For example, for

$$A = \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}$$

it has characteristic polynomial $(x-1)(x-2)^2(x-3)^2$. However, it satisfies the polynomial

$$(A - I)(A - 2I)(A - 3I) = 0$$

Criterion for matrices satisfying polynomial of simple roots

For any matrix A, we have its characteristic polynomial $f(t) = \det(tI - A)$. If $f(t)$ is of simple roots, then $f(A) = 0$ implies A satisfying polynomial of simple roots as well.

However, if $f(t)$ is not of simple roots, how do we know if or not A satisfying polynomial of simple roots?

Criterion for matrices satisfying polynomial of simple roots

Theorem 3

Suppose

$$\det(tI - A) = (t-\lambda_1)^{n_1}\cdots(t-\lambda_k)^{n_k},$$

then A satisfies a polynomial of simple roots if and only if

$$(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_k I) = 0$$

In other words, we only need to check one polynomial $(t-\lambda_1)\cdots(t-\lambda_k)$, instead of checking for all polynomials of simple roots.

Criterion for matrices satisfying polynomial of simple roots

Let's suppose A satisfies a polynomial of simple roots.

$$(A - x_1 I) \cdots (A - x_m I) = 0.$$

For any $x_i \notin \{\lambda_1, ..., \lambda_k\}$, we have $x_i - \lambda_j \neq 0$ and therefore $\det(x_i I - A) = (x_i - \lambda_1)^{n_1} \cdots (x_i - \lambda_k)^{n_k} \neq 0$, this means $x_i I - A$ is invertible, therefore

$$(A - x_1 I) \cdots (\widehat{A - x_i} I) \cdots (A - x_m I) = 0.$$

where

$$(\widehat{A - x_i} I)$$

means deleting the factor. Since all factor with root outside $\{\lambda_1, ..., \lambda_k\}$ can be deleted, it only left with factors with root in $\{\lambda_1, ..., \lambda_k\}$.

Criterion for matrices satisfying polynomial of simple roots

Excercise. Computer the characteristic polynomial of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Will this matrix satisfies a polynomial of simple roots? why?

Criterion for matrices satisfying polynomial of simple roots

The characteristic polynoimal is

$$(t-4)(t-1)^2$$

Therefore, to see if it satisfies a polynomial of simple roots, we only need to check if $(A-4I)(A-I)=0$. Indeed.

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A-I} \underbrace{\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}}_{A-4I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Criterion for matrices satisfying polynomial of simple roots

We see some applications of the spectural decomposition.

Criterion for matrices satisfying polynomial of simple roots

Excercise. Find a formula for

$$A^n := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^n$$

Solution. Firstly, we calculate characteristic polynomial of A

$$\det \begin{pmatrix} t-1 & & -1 \\ & t-1 & \\ -1 & & t-1 \end{pmatrix} = t(t-1)(t-2)$$

By Calay – Hamilton theorem, we have

$$A(A-I)(A-2I) = 0$$

so it is diagonalizable! Based on the information, we need interpolation polynomials at point $0, 1, 2$. Now construct some.

Criterion for matrices satisfying polynomial of simple roots

	$\frac{(t-1)(t-2)}{2}$	$-t(t-2)$	$\frac{t(t-1)}{2}$
t=0	1	0	0
t=1	0	1	0
t=2	0	0	1

Therefore, by Spectural Decomposition, we have

$$A^n = 0^n \cdot \frac{(A-I)(A-2I)}{2} - 1^n \cdot A(A-2I) + 2^n \cdot \frac{A(A-I)}{2}.$$

For $n \geq 1$, this can be calculated as

$$-A(A-2I) + 2^{n-1}A(A-I).$$

Criterion for matrices satisfying polynomial of simple roots

Let us finish this exercise

$$A(A-I) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A(A-2I) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So **for** $n \geq 1$, we have

$$A^n = \begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & 1 & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix}$$

Criterion for matrices satisfying polynomial of simple roots

Excercise. Suppose $a_0 = 0$, $a_1 = 1$ and

$$a_{n+1} = 3a_n - 2a_{n-1}$$

Find a formula for a_n.

Solution. We may complete this equation as

$$\begin{cases} a_{n+1} & = 3a_n - 2a_{n-1} \\ a_n & = a_n \end{cases}$$

So we can write it as

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$$

Since this formula is true for all n, we have an induction formula

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Criterion for matrices satisfying polynomial of simple roots

To calculate

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n,$$

we need an annihilating polynomial. For this ,

$$\det \begin{pmatrix} t-3 & 2 \\ -1 & t \end{pmatrix} = t(t-3)+2 = t^2-3t+2 = (t-1)(t-2).$$

This motivate us to write t^n via interpolation at 1 and 2.

	-(t-2)	t-1
t=1	1	0
t=2	0	1

$$t^n = Q(t)(t-1)(t-2) - 1^n(t-2) + 2^n(t-1)$$

Criterion for matrices satisfying polynomial of simple roots

Plug in $t = A$, we have

$$A^n = -(A - 2I) + 2^n (A - I)$$

$$= \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} + 2^n \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + 2^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^{n+1} - 1 \\ 2^n - 1 \end{pmatrix}$$

This implies that $a_n = 2^n - 1$.

Criterion for matrices satisfying polynomial of simple roots

Excercise. Rolling a coin, if face up, proceed 3 meters, if not, proceed 2 meters. Calculate the possibility that the man has stopped at n meters.

Solution. To n meters, the man has to reach n-3 meterand proceed 3 meter, or reach n-2 meter and proceed 2 meters. Let p_n be such probability. We have

$$\left\{ p_n = \tfrac{1}{2} p_{n-3} + \tfrac{1}{2} p_{n-2}. \right.$$

$$\begin{pmatrix} 0 & .5 & .5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

characteristic polynomial $t^3 - .5t - .5 = (t-1)(t^2 + 0.5t + 1)$