Warm up exercises

Excercise. Which of the following argument is true?

1. $\text{Col}(AB) \subset \text{Col}(A)$
2. $\text{Col}(AB) \subset \text{Col}(B)$
3. $\text{Col}(AB) \supset \text{Col}(A)$
4. $\text{Col}(AB) \supset \text{Col}(B)$

Warm up exercises

Excercise. Which of the following codintion implies $\mathrm{Col}(A) = \mathrm{Col}(AB)$?

1. A has left inverse;
2. A has right inverse;
3. B has left inverse;
4. B has right inverse;

Warm up exercises

Excercise. Which of the following argument is true?

1. $\mathrm{Null}(AB) \subset \mathrm{Null}(A)$
2. $\mathrm{Null}(AB) \subset \mathrm{Null}(B)$
3. $\mathrm{Null}(AB) \supset \mathrm{Null}(A)$
4. $\mathrm{Null}(AB) \supset \mathrm{Null}(B)$

Warm up exercises

Excercise. Which of the following codintion implies $\text{Null}(B) = \text{Null}(AB)$?

1. A has left inverse;
2. A has right inverse;
3. B has left inverse;
4. B has right inverse;

Warm up exercises

Excercise. The product $A^{-1}B$ won′t change if we apply

1. Simutaneous row operation on A and B;
2. Simutaneous column operation on A and B;

Warm up exercises

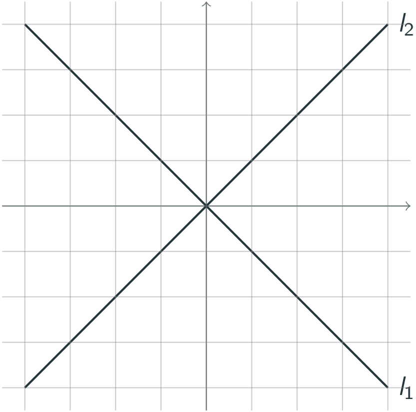
Excercise. The product AB^{-1} won't change if we apply

1. Simutaneous row operation on A and B;
2. Simutaneous column operation on A and B;

Mutually orthogonal projections

Excercise. Given two lines in a plane $l_1 : x + y = 0$, $l_2 : x - y = 0$. Find projection operators P_{l_1} and P_{l_2} such that

- $\mathrm{Ker} P_{l_1} = l_2$, $\mathrm{Im} P_{l_1} = l_1$
- $\mathrm{Ker} P_{l_2} = l_1$, $\mathrm{Im} P_{l_2} = l_2$



Mutually orthogonal projections

$B_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a matrix with column space l_1

$B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a matrix with column space l_2

$A_1 := \begin{pmatrix} 1 & 1 \end{pmatrix}$ is a matrix with null space l_1

$A_2 := \begin{pmatrix} 1 & -1 \end{pmatrix}$ is a matrix with null space l_2

Note that

$$l_2 \cap l_1 = \{0\} \iff A_2 B_1 \text{ invertible}$$

$$l_1 \cap l_2 = \{0\} \iff A_1 B_2 \text{ invertible}$$

Mutually orthogonal projections

Projection P_1 with $\mathrm{col}(P_1) = l_1 = \mathrm{col}(B_1)$ and $\mathrm{null}(P_1) = l_2 = \mathrm{null}(A_2)$ is given by

$$P_1 = B_1(A_2 B_1)^{-1} A_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}$$

Projection P_2 with $\mathrm{col}(P_2) = l_2 = \mathrm{col}(B_2)$ and $\mathrm{null}(P_2) = l_1 = \mathrm{null}(A_1)$ is given by

$$P_2 = B_2(A_1 B_2)^{-1} A_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{1} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Mutually orthogonal projections

The method can be generalized, in general , when given two spaces $W, V \subset \mathbb{R}^n$ such that

$W \cap V = \{\vec{0}\}$ and $\dim(W) + \dim(V) = n$, we may find matrices A, B and write

$$W = \mathrm{Null}(A), \qquad V = \mathrm{Col}(B).$$

Then, the above condition would implies AB invertible. A projection can be constructed by

$$P = B(AB)^{-1}A.$$

$$\mathrm{Col}(B) \supset \mathrm{Col}(B{\color{red}(AB)^{-1}A}) \supset \mathrm{Col}(B(AB)^{-1}A{\color{red}B}) = \mathrm{Col}(B)$$

$$\mathrm{Null}(A) \supset \mathrm{Null}({\color{red}B(AB)^{-1}}A) \supset \mathrm{Null}({\color{red}A}B(AB)^{-1}A) = \mathrm{Null}(A)$$

Then

$$\mathrm{Col}(P) = \mathrm{Col}(B) \qquad \mathrm{Null}(P) = \mathrm{Null}(A).$$

Mutually orthogonal projections

Proposition 1

Let $P = P^2$ and $Q = Q^2$ be two projection matrices such that $\mathrm{Col}(P) = \mathrm{Col}(Q)$ and $\mathrm{Null}(P) = \mathrm{Null}(Q)$, then $P = Q$

Note that $\mathrm{Null}(I - P) = \mathrm{Col}(P) = \mathrm{Col}(Q)$, so $\mathrm{Col}(Q) \subset \mathrm{Null}(I - P)$, which implies that

$$(I - P)Q = 0.$$

Similarly $\mathrm{Col}(I - Q) = \mathrm{Null}(Q) = \mathrm{Null}(P)$, so $\mathrm{Col}(I - Q) \subset \mathrm{Null}(P)$

$$P(I - Q) = 0.$$

So $Q = PQ = P$.

Mutually orthogonal projections

!!

Therefore projection are completely determined by its null(kernel) and column space(image)

Mutually orthogonal projections

Excercise. Suppose W is a vector space spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and U is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Please find projection operator $P^2 = P$ such that $\mathrm{Ker}(P) = W$ and $\mathrm{Im}(P) = U$.

Mutually orthogonal projections

Solution. Note that $W = \text{Null}\underbrace{\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A}$ and $U = \text{Col}\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B}$

The formula $B(AB)^{-1}A$ give projection with such null and col space

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Mutually orthogonal projections

Excercise. For same W is a vector space spanned by

$$\begin{pmatrix}1\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

and U is spanned by

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Please find projection operator $P^2 = P$ such that $\mathrm{Ker}(P) = U$ and $\mathrm{Im}(P) = W$.

Mutually orthogonal projections

Solution.(Write your own solution)

Mutually orthogonal projections

Another method:

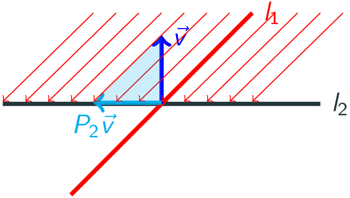
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

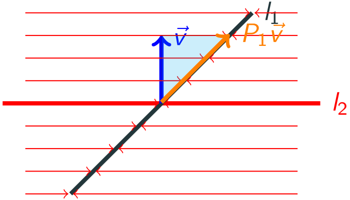
Mutually orthogonal projections

In all the above exercises, P_1 and P_2 is a pair **interchanging** kernel(null space) and images(column space)

Mutually orthogonal projections

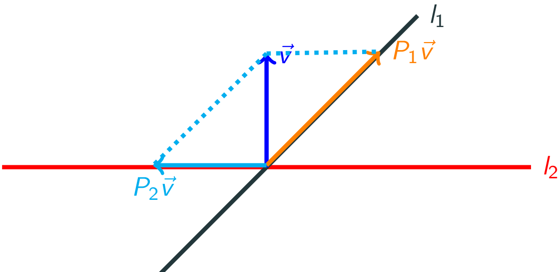
The pair of projection interchanging null space and column space





The following picture shows

$$\vec{v} = P_1\vec{v} + P_2\vec{v} \qquad \text{for any } \vec{v} \implies P_1 + P_2 = I$$



Mutually orthogonal projections

For any projection operator P, the operator $I - P$ is interchaging kernel and image

$$\mathrm{Ker}(P) = \mathrm{Im}(I - P); \qquad \mathrm{Im}(P) = \mathrm{Ker}(I - P)$$

This is a first example of **mutually orthogonal projections**

Mutually orthogonal projections

!!

The word **mutually orthogonal** DO NOT mean $\mathrm{Im}(P_1)$ othogonal to $\mathrm{Im}(P_2)$, instead, it means

$$\mathrm{Im}(P_1) \subset \mathrm{Ker}(P_2) \text{ and } \mathrm{Im}(P_2) \subset \mathrm{Ker}(P_1).$$

Historically speaking, people are most interested in discussing **orthogonal projections**($P = P^T$) where in their world $\mathrm{Im}(P) \perp \mathrm{Ker}(P)$, only in that case, we have

$$\mathrm{Im}(P_1) \subset \mathrm{Ker}(P_2) \implies \mathrm{Im}(P_1) \perp \mathrm{Im}(P_2).$$

Equivalent definitions of mutually orthogonal

Definition 1 Equivalent definition

Say a set of projections $\{P_1, P_2, ..., P_n\}$ **mutually orthogonal** if $P_1 + P_2 + ... + P_n$ is a projection operator.

Definition 2 Equivalent definition - more computational

Say a set of projections $\{P_1, P_2, ..., P_n\}$ **mutually orthogonal** if $P_i P_j = 0$ for any $i \neq j$.

Definition 3 Equivalent definition - more geometrical

Say a set of projections $\{P_1, P_2, ..., P_n\}$ **mutually orthogonal** if $\mathrm{Im}(P_i) \subset \mathrm{Ker}(P_j)$ for any $i \neq j$.

Equivalent definitions of mutually orthogonal

To show these difinition equivalent , we need to show that if $P_i^2 = P_i$ for $\forall$ i, then

$$
\begin{aligned}
&(P_1 + ... + P_n)^2 = P_1 + ... + P_n \\
\iff \quad & P_i P_j = 0 \forall i \neq j \\
\iff \quad & \mathrm{Im}(P_i) \subset \mathrm{Ker}(P_j) \forall i \neq j
\end{aligned}
$$

The proof for all part is easy except the part $(P_1 + ... + P_n)^2 = P_1 + ... + P_n \implies P_i P_j = 0 \forall i \neq j$ is hard.

Equivalent definitions of mutually orthogonal

Proof of the hardest part, Schetch: Firstly, we observe that all columns of $P_1 + ... + P_n$ is the sum of columns of each P_i, so

$$\mathrm{Col}(P_1 + ... + P_n) \subset \mathrm{span}(\mathrm{Col}(P_1) \cup \mathrm{Col}(P_2) \cup ... \cup \mathrm{Col}(P_n))$$

Since all of them, and the sum is projection, the rank and trace are the same, so we have

$$N := rank(P_1 + ... + P_n) = \mathrm{tr}(P_1 + ... + P_n) = \mathrm{tr}(P_1) + ... + \mathrm{tr}(P_n)$$

$$= \mathrm{rank}(P_1) + ... + \mathrm{rank}(P_n)$$

However, Each $\mathrm{Col}(P_i)$ has $\mathrm{tr}(P_i)$-many vectors as basis. Choosing for each, we have a total of N-vectors, spanning the entire $\mathrm{span}(\mathrm{Col}(P_1) \cup \mathrm{Col}(P_2) \cup ... \cup \mathrm{Col}(P_n))$. So

$$\dim(\mathrm{span}(\mathrm{Col}(P_1) \cup \mathrm{Col}(P_2) \cup ... \cup \mathrm{Col}(P_n))) \leq N$$

Equivalent definitions of mutually orthogonal

However, since

$$\dim(\mathrm{Col}(P_1 + ... + P_n)) = N$$

we must have

$$\mathrm{Col}(P_1 + ... + P_n) = \mathrm{span}(\mathrm{Col}(P_1) \cup \mathrm{Col}(P_2) \cup ... \cup \mathrm{Col}(P_n))$$

This implies

$$\mathrm{Col}(P_i) \subset \mathrm{Col}(P_1 + ... + P_n)$$

Therefore

$$P_1(P_1 + ... + P_i + ... + P_n) = P_1$$

Note that $P_1^T, ..., P_n^T$ is also mutually orthogonal projection, we also have

$$P_n^T(P_n + ... + P_n)^T = P_n^T \implies (P_1 + ... + P_n)P_n = P_n$$

The above equation implies $(P_1 + ... + P_{n-1})^2 = P_1 + ... + P_{n-1}$ is a projection as well. Then using induction implies the result.

Equivalent definitions of mutually orthogonal

The following matrices are projection matrices.

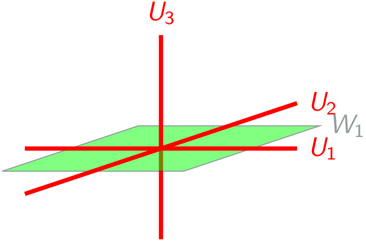
0. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 2. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 3. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 5. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 7. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find all pairs of mutually orthogonal projections.

Equivalent definitions of mutually orthogonal

The geometric intuition of mutually orthogonal projection



For this, there is a mutually orthogonal projection

$$\mathrm{Ker}(P_1) = \mathsf{Span}(U_3 \cup U_2); \qquad \mathrm{Im}(P_1) = U_1$$

$$\mathrm{Ker}(P_2) = \mathsf{Span}(U_3 \cup U_1); \qquad \mathrm{Im}(P_2) = U_2$$

$$\mathrm{Ker}(P_3) = \mathsf{Span}(U_1 \cup U_2); \qquad \mathrm{Im}(P_3) = U_3$$

Computation of mutually (orthogonal) projections.

Suppose $P_1, \ldots, P_n$ are mutually orthogonal projections.

Suppose

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n,$$

$$B = \mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_n P_n.$$

What is AB? (note that $P_i P_j = 0$ for $i \neq j$)

$$AB = (\lambda_1 \mu_1) P_1 + (\lambda_2 \mu_2) P_2 + \ldots + (\lambda_n \mu_n) P_n$$

!!

The decomposition of a matrix into a linear combination of mutually orthogonal projections is called **the spectrual decomposition**.

Computation of mutually (orthogonal) projections.

Example

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1\mu_1 & 0 & 0 \\ 0 & \lambda_2\mu_2 & 0 \\ 0 & 0 & \lambda_3\mu_3 \end{pmatrix} = \lambda_1\mu_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_2\mu_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_3\mu_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Computation of mutually (orthogonal) projections.

If $P_1 = P_1^2, \qquad P_2 = P_2^2, \qquad ..., P_n = P_n^2$, $\left(\sum_i P_i\right)^2 = \sum_i P_i$, and

$$A = \lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_n P_n,$$

for what is A^k?

$$A^k = \lambda_1^k P_1 + \lambda_2^k P_2 + ... + \lambda_n^k P_n,$$

Computation of mutually (orthogonal) projections.

The decomposition of a matrix into mutually orthogonal projections is useful for calculation. For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} + 3 \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Then we obtain a formula

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} + 3^n \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

So

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} \frac{1+3^n}{2} & \frac{-1+3^n}{2} \\ \frac{-1+3^n}{2} & \frac{1+3^n}{2} \end{pmatrix}$$

This is the main topic for the chapter of eigenvalue and eigenvectors.

Cross-Filling of projection operators

Proposition 2

Let $P = P^2$ be a $n \times n$ projection matrix, the cross-filling decomposition:

$$P = Q + (P - Q); \qquad Q = Pe_j(e_i^T Pe_j)^{-1} e_i^T P$$

make both $P - Q$ and Q projection operators.

$$\begin{aligned} Q^2 &= Pe_j(e_i^T Pe_j)^{-1} e_i^T PPe_j(e_i^T Pe_j)^{-1} e_i^T P \\ &= Pe_j(e_i^T Pe_j)^{-1} \underbrace{e_i^T Pe_j}(e_i^T Pe_j)^{-1} e_i^T P \\ &= Pe_j(e_i^T Pe_j)^{-1} e_i^T P = Q. \end{aligned}$$

It is easy to see $PQ = Q = QP$, so
$(P - Q)^2 = P^2 + Q^2 - PQ - QP = P + Q - Q - Q = P - Q.$

Cross-Filling of projection operators

!!

In one word, cross-filling decompose projection matrix into projection matrices.

Cross-Filling of projection operators

Excercise. The following matrix is a projection

$$P = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find a basis of $\mathrm{Col}(P)$ and $\mathrm{Null}(P)$. For each vector

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Write down a formula for decomposition of this vector in your basis.

Cross-Filling of projection operators

Finding basis of Col(P) we use cross-filling

$$P\begin{pmatrix}1 & 2 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 2 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}}_{A_1} + \underbrace{\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}}_{A_2}$$

So a basis of Col(P) is

$$\vec{w_1} = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix}, \vec{w_2} = \begin{pmatrix}0\\ 0\\ 0\\ 1\end{pmatrix}$$

Cross-Filling of projection operators

Since P is a projection matrix, we have $\mathrm{Null}(P) = \mathrm{Col}(I - P)$. Finding a basis of it

$$I - P = \begin{pmatrix} 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -2 & -3 & 0 \\ 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_3} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_4}$$

A basis is given by

$$\vec{w}_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{w}_4 = \begin{pmatrix} 0 \\ -1.5 \\ 1 \\ 0 \end{pmatrix}$$

Cross-Filling of projection operators

Now to get a decomposition of an arbitrary vector $\vec{v}$ into vectors $\vec{w}_1, \cdots,$ $\vec{w}_4$, we note

$$\vec{v} = I\vec{v} = P\vec{v} + (I - P)\vec{v} = A_1\vec{v} + A_2\vec{v} + A_3\vec{v} + A_4\vec{v}.$$

Since

$$A_1\vec{v} = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{\vec{v}} = (x + 2y + 3z)\vec{w}_1$$

$$A_2\vec{v} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A_2} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{\vec{v}} = (w)\vec{w}_2$$

Cross-Filling of projection operators

We continue

$$A_3\vec{v} = \underbrace{\begin{pmatrix} 0 & -2 & -3 & 0 \\ 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_3} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{\vec{v}} = (y + 1.5z)\vec{w}_3$$

and

$$A_4\vec{v} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{A_4} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{\vec{v}} = (z)\vec{w}_4$$

So $\vec{v} = (x + 2y + 3z)\vec{w}_1 + (w)\vec{w}_2 + (y + 1.5z)\vec{w}_3 + (z)\vec{w}_4$.

Linearly Independency

Proposition 3

If $P_1, P_2, ..., P_n$ is a mutually orthogonal family of projection operators, then for any **non-zero** vector $\vec{0} \neq \vec{v}_i \in \mathrm{Im}(P_i)$, the list

$$\vec{v}_1, ..., \vec{v}_n$$

is automatically linearly independent.

Suppose $a_1\vec{v}_1 + ... + a_n\vec{v}_n = \vec{0}$.

We can write

$$a_1 P_1 \vec{v}_1 + ... + a_n P_n \vec{v}_n = \vec{0}$$

Multiply P_1 on both side

$$a_1 P_1 \vec{v}_1 = \vec{0} \implies a_1 \vec{v}_1 = \vec{0} \implies a_1 = 0.$$

Linearly Independency

- Mutually orthogonal is a **pairwise**-condition (good!)
- Linearly independent is **not a pairwise**-condition (bad!)

Excercise. If

- $\{\vec{v}_1, \vec{v}_2\}$ **linealy independent** ,
- $\{\vec{v}_3, \vec{v}_2\}$ **linealy independent** ,
- $\{\vec{v}_1, \vec{v}_3\}$ **linealy independent** ,

is that true $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ **linealy independent** ?

Vector only has information of its direction

Linearly Independency

Excercise. If

- $\{P_1, P_2\}$ mutually orthogonal,
- $\{P_3, P_2\}$ mutually orthogonal,
- $\{P_1, P_3\}$ mutually orthogonal,

is that true $\{P_1, P_2, P_3\}$ mutually orthogonal?

Projections not only have information of its direction $\mathrm{Im}(P)(\mathrm{Col}(P))$, but also has $\mathrm{Ker}(P)(\mathrm{null}(P))$.