Orthogonal Projections

Definition 1 equivalent definitions

A projection $P = P^2$ is called **orthogonal projection** if $P^T = P$

Definition 2 equivalent definitions

A projection $P = P^2$ is called **orthogonal projection** if $\mathrm{Ker}(P) \perp \mathrm{Im}(P)$

We need to show for $P^2 = P$, that $P^T = P \iff \mathrm{Ker}(P) \perp \mathrm{Im}(P)$.

Orthogonal Projections

Since always $\mathrm{col}(P^T) \perp \mathrm{null}(P)$, if $P^T = P$, then $\mathrm{col}(P) = \mathrm{col}(P^T) \perp \mathrm{null}(P)$.

On the other hand, if $\mathrm{Ker}(P) \perp \mathrm{Im}(P)$, then

$$\mathrm{Im}(I - P) = \mathrm{Ker}(P) \perp \mathrm{Im}(P),$$

so

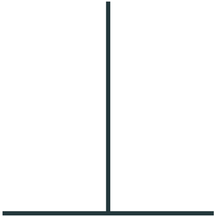
$$(I - P)^T P = 0.$$

This implies $P = P^T P$, transpose this expression we get

$$P^T = (P^T P)^T = P^T P = P.$$

Fake proof of $P^T = P \iff \mathrm{Ker}(P) \perp \mathrm{Im}(P)$

"being orthogonal" $=$ "being symmetric"



Orthogonal

Symmetric

Uniqueness of Orthogonal Complement

Lemma 1

For any vector $\vec{v}$, $\vec{v}^T\vec{v} = 0$ implies $\vec{v} = \vec{0}$

$$x_1^2 + ... + x_n^2 = 0 \implies x_1 = x_2 = ... = x_n = 0.$$

Uniqueness of Orthogonal Complement

Theorem 1 Extremely important theorem

For any real number matrix A, if $A^T A = 0$, then $A = 0$.

$$(\forall i, e_i^T A^T A e_i = 0 \implies A e_i = \vec{0}) \implies A = 0.$$

Proposition 1

For any real number matrix A, $\text{Null}(A^T A) = \text{Null}(A)$

For sure $\text{Null}(A^T A) \supset \text{Null}(A)$,

If $A^T A \vec{v} = 0$, then $\vec{v}^T A^T A \vec{v} = 0$, then $A\vec{v} = 0$ then $\vec{v} \in \text{Null}(A)$.

Uniqueness of Orthogonal Complement

Theorem 2 Orthogonal Complement is unique

For any subspace W, there is only one **orthogonal** projection P with $\mathrm{Im}(P) = W$.

Let $P = P^T = P^2$ and $Q = Q^T = Q^2$ be projections such that $W = \mathrm{Im}(P) = \mathrm{Im}(Q)$.

$$\mathrm{Im}(P) \supset \mathrm{Im}(Q) \implies PQ = Q, \qquad \mathrm{Im}(Q) \supset \mathrm{Im}(P) \implies QP = P.$$

$$\begin{aligned} &(P-Q)^T(P-Q) \\ &= (P-Q)(P-Q) \\ &= P^2 - PQ - QP - Q^2 \\ &= P - Q - P - Q = 0. \end{aligned}$$

So $P - Q = 0$.

Uniqueness of Orthogonal Complement

Corollary 1

The orthogonal complement has to be unique.

There is a unique orthogonal projection P for $W \subset \mathbb{R}^n$.

If U is orthogonal compment of W, then $U \perp W$ implies $U \subset \ker(\mathrm{P})$.

U has to have dimension $\dim(U) = n - \dim(W) = \dim \ker(P)$.

So $U = \ker(P)$

P is unique, $\ker(P)$ is unique, so U is unique.

Notation for Orthogonal Complement

Definition 3

For a vector space $W \subset \mathbb{R}^n$, we denote its orthogonal complement by $W^\perp$.

Let P be the orthogonal projection that is uniquely determined by W, for any vector $\vec{v}$, we can always decompose

$$\vec{v} = \underbrace{P\vec{v}}_{\in W} + \underbrace{(I-P)\vec{v}}_{\in W^\perp}$$

Notation for Orthogonal Complement

Proposition 2

For any vector $\vec{v}$ such that $\vec{v}^T\vec{w} = 0$ for all $\vec{w} \in W$, then $\vec{v} \in W^{\perp}$

Proof: For any vector $\vec{e}$, we have

$$\vec{e}^T P\vec{v} = \vec{e}^T P^T \vec{v} = \underbrace{(P\vec{e})^T}_{\in W} \vec{v} = 0.$$

So $P\vec{v} = 0$. Therefore, $(I - P)\vec{v} = \vec{v}$, then $\vec{v} \in \mathrm{Im}(I - P) = W^{\perp}$.

Notation for Orthogonal Complement

!!

This implies that $W^\perp = \{\vec{v} : \vec{v}^T\vec{w} = 0 \text{for all } \vec{w} \in W\}$ Put

$$P = \begin{pmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{pmatrix}$$

$$W^\perp = \mathrm{Ker}(P) \overset{P=P^T}{=\!=} \mathrm{Ker}(P^T) = \{\vec{v} : P^T\vec{v} = \vec{0}\}$$

$$=\!= \{\vec{v} : (\vec{w}_i)^T\vec{v} = 0 \text{ for all } 1 \le i \le n\}$$

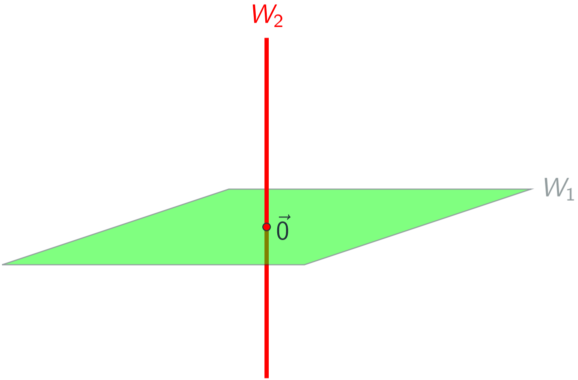
$$=\!= \{\vec{v} : \vec{w}^T\vec{v} = 0 \text{ for all } \vec{w} \in \mathrm{Im}(P)\}$$

$$=\!= \{\vec{v} : \vec{w}^T\vec{v} = 0 \text{ for all } \vec{w} \in W\}$$

In one words, $W^\perp$ is the subset of **all** vectors orthogonal to W.

Notation for Orthogonal Complement

The intuitive picture for orthogonal complement



Formula for orthogonal projection

Proposition 3

Let $W = \mathrm{Col}(A)$ for some matrix A with **linealy independent** columns. Then $W^\perp = \mathrm{Null}(A^T)$.

$$P := A(A^TA)^{-1}A^T$$

is an **orthogonal projection** with $\mathrm{Im}(P) = \mathrm{Im}(A)$ and $\mathrm{Null}(P) = \mathrm{Null}(A^T) = W^\perp$.

- (being projection $P^2 = P$)A more general formula: $P = A(BA)^{-1}B$ is a projection with $\mathrm{Null}(P) = \mathrm{Null}(B)$ and $\mathrm{Col}(P) = \mathrm{Col}(A)$.
- (being orthogonal)$P^T = (A(BA)^{-1}B)^T = B^T(A^TB^T)A^T$. If $B = A^T$, then $P = P^T$.
- A^TA is invertible since $\mathrm{Null}(A^TA) = \mathrm{Null(A)} = 0$

Formula for orthogonal projection

A special case for one-dimensional space $\mathrm{Im}(A) = \mathrm{Col}(\vec{v})$ for one vector $\vec{v}$.

There is a unique rank-1 orthogonal projection given by

$$P = v(v^T v)^{-1} v^T \qquad = \frac{vv^T}{v^T v}$$

all rank-1 orthogoal proejctions arises in this way.

Formula for orthogonal projection

Excercise. Suppose W is a subspace in $\mathbb{R}^3$ spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Find the orthogonal projection to this space, and find a basis of its orthogonal complement.

Formula for orthogonal projection

Solution. The orthogonal complement is just the kernel of the orthogonal projection to it. We find orthogonal projection using formula. Note that

$$W = \text{Col}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}}_{A}$$

An orthogonal projection can be formulated as

$$P = \underbrace{A}_{\text{for column space}} (A^T A)^{-1} \underbrace{A^T}_{\text{for null space}}$$

Formula for orthogonal projection

A calculation implies that

$$
\begin{aligned}
A(A^TA)^{-1}A^T &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}
\end{aligned}
$$

This is the orthogonal projection we are looking for.

Formula for orthogonal projection

Now that we have find

$$P = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

The orthogonal complement is $\mathrm{Ker}(P) = \mathrm{Im}(I - P)$. Just calculate

$$I - P = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

Since it is trace(rank) 1 projection, any column is a basis for its image. So a basis of $W^\perp$ can be

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Formula for orthogonal projection

Solution. Yet there is another solution. We may find orthogonal complement first before determine the orthogonal projeciton.

$W = \text{col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$ So its orthogonal complement is

$$\text{null}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Solve the equation, we have

$$W^{\perp} = \text{null}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \text{Col}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Formula for orthogonal projection

To find orthogonal projection to W. We found orthogonal projection to $W^\perp$ first.

$$P_W = P_{W^\perp} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \left(\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 \end{pmatrix}$$

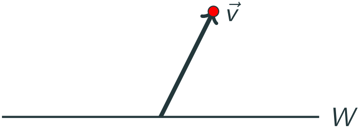
$$= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

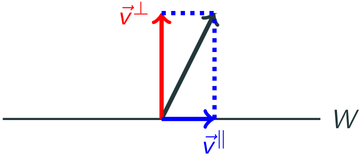
So the projection to W is

$$I - P_{W^\perp} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

Distance and Orthogonal Projections

In order to find the distance of a point to a subspace, we may decompose a vector into two component $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$, with $\vec{v}^{\parallel} \in W$ and $\vec{v}^{\perp} \in W^{\perp}$.





Let P be the orthogonal projection associated to W, we may write

$$\vec{v}^{\parallel} = P\vec{v}$$

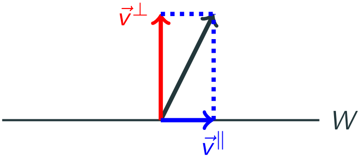
$$\vec{v}^{\perp} = (I - P)\vec{v}$$

For any $\vec{w} \in W$, we have $\vec{w} - \vec{v}^{\parallel} \in W$ therefore

$$(\vec{v}^{\perp})^T(\vec{w} - \vec{v}^{\parallel}) = 0.$$

$$(\vec{w} - \vec{v}^{\parallel})^T(\vec{v}^{\perp}) = 0.$$

Distance and Orthogonal Projections



$$
\begin{aligned}
&(\vec{v}-\vec{w})^T(\vec{v}-\vec{w}) \\
&= (\vec{v}^{\perp}-(\vec{w}-\vec{v}^{\parallel}))^T(\vec{v}^{\perp}-(\vec{w}-\vec{v}^{\parallel})) \\
&= (\vec{v}^{\perp})^T(\vec{v}^{\perp})+(\vec{w}-\vec{v}^{\parallel})^T(\vec{w}-\vec{v}^{\parallel})
\end{aligned}
$$

This implies that

$$||\vec{v}-\vec{w}||^2 = ||\vec{v}^{\parallel}-\vec{w}||^2 + ||\vec{v}^{\perp}||^2$$

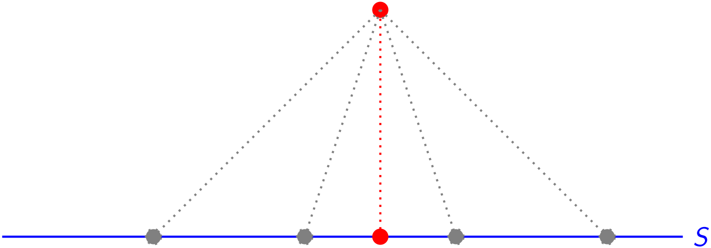
Distance and Orthogonal Projections

$$||\vec{v} - \vec{w}||^2 = \underbrace{||\vec{v}^{||} - \vec{w}||^2}_{\geq 0} + ||\vec{v}^{\perp}||^2$$

$$\min_{\vec{w} \in W} ||\vec{v} - \vec{w}||^2 = ||\vec{v}^{\perp}||^2$$

The minimal such choice is $\vec{w} = \vec{v}^{||}$.

In other words, the orthogonal projection is the point on the plane with minimal distance to the given point. We define this distance as distance of a point to the subspace.

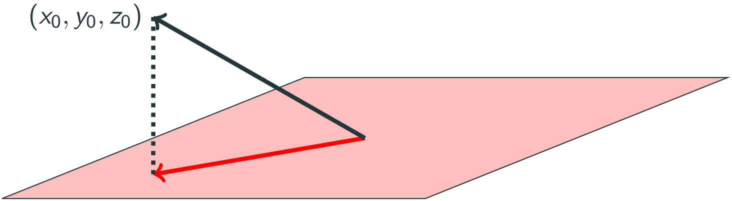


Distance and Orthogonal Projections

Excercise. Suppose $W \subset \mathbb{R}^3$ is given by the equation

$$x + 2y + 3z = 0$$

Please find the formula of the distance of the point (x_0, y_0, z_0) to the plane and find its closest point on the plane, in terms of x_0, y_0, z_0.



Solution.. First, we want to find out the projection to $W^{\perp}$. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Then $\mathrm{Col}(A^T) = W^{\perp}$ The unique orthogonal projection mapping to $W^{\perp}$ is given by

$$P = A^T(AA^T)^{-1}A = \frac{1}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$P\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \frac{x_0 + 2y_0 + 3z_0}{14} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Distance and Orthogonal Projections

Therefore, the distance is given by

$$||P\vec{v}|| = \frac{\sqrt{14}}{14} \cdot |x_0 + 2y_0 + 3z_0|.$$

Then , the orthogonal projection to W is given by $I - P$. So

$$\vec{v} \longmapsto (I-P)\vec{v} = \frac{1}{14}\begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

is the formula of the point in W that is closest to $\vec{v}$.