Linear Algebra Lectures

Linear Combination and Matrix Multiplication

Learning Objectives

Matrix Multiplication(1.4)

- How to multiply two matrices?
  - what size of matrices can be multiply.
- How to represent a matrix by math symbol?
- Matrix multiplication through linear combination of columns
- Matrix multiplication through linear combination of rows
- Doing algebra with matrices
  - Is $AB = BA$?
  - How to expand $(A+B)(A-B)$ or $(A+B)^2$?
- Some special matrices
  - Upper triangular, lower triangular matrices.
  - Diagonal matrices, symmetric matrices.

Explain matrix multiplication to elementary school students!

Matrix Multiplication

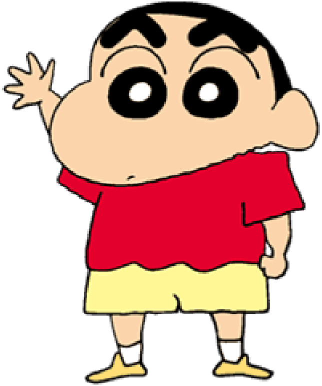
Shinchan is operating a coffee shop, making various drinks. For each drink he need the following ingredients.

Milk 🥛 : need 🐄;
Lemon Tea🍵 : need 🍃🍃🍋;
Coffee☕ : need 🫘 🫘

Matrix Multiplication

To make it clear, he made it into a table

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
	0	2	0
🐄	1	0	0



Matrix Multiplication

People like those drinks, to sale it better, Shinchan designed the following meal plan

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$$\begin{cases} \text{Meal 1🍱: 2 milks and 1 tea;} \\ \text{Meal 2🍲: 1 milk 2 coffee and 1 tea;} \end{cases}$$

Let us call 🍱, 🍲 **compunds** and 🥛, 🍵☕ **materials**.

Matrix Multiplication

To prepare for each meal, Shinchan need to know how much material is needed, can you combine those two table for him?

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍃		
🍋		
🫘		
🐄		

Matrix Multiplication

Mathematically, we call the table of the ingradients to produce something as the matrix. The combination of two ingradients is called the matrix multiplication.

left factor

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐮	1	0	0

right factor

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

product

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐮	2	1

(1)

Matrix notation

For the product

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

=

	🍱	🍲
🍃	2	2
🍋	1	1
	0	4
🐄	2	1

mathematically we write

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 0 & 4 \\ 2 & 1 \end{pmatrix}$$

Matrix notation

When representing matrices abstractly, the subindices are arranged

$$a_{\text{row number, column number}}$$

and whenever one write $A = (a_{ij})$, he means

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Exercise: If

$$A = (a_{ij}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Write down the value of a_{11}, a_{12}, a_{21} and a_{22}.

Size of a matrix

Definition 1

If a matrix A has m rows and n columns, we say that A is an $m \times n$ matrix.

We see that if A is $m \times n$ matrix,

1. $m =$ number of rows $=$ number of **materials**
2. $n =$ number of columns $=$ number of **compunds**

Size of a matrix

There are only 3 material lists playing the role in $C = AB$. Put

1. $m =$ Number of materials of A;
2. $n =$ Number of compunds of $A =$ Number of materials of B;
3. $p =$ Number of compunds of B ;
4. Materials of $A =$ Matrials of C
5. Compunds of $B =$ Compunds of C

A

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

B

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

C

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

Size of a matrix

Proposition 1

The matrix multiplication AB makes sense only when

the number of **columns** in $A =$ the number of **rows** in B

The size is given by

$$\underbrace{A}_{m\times n}\underbrace{B}_{n\times p} = \underbrace{C}_{m\times p}$$

Size of a matrix

Exercise: Without knowing entries. Let

$$A = \begin{pmatrix} ? & ? \\ ? & ? \\ ? & ? \end{pmatrix} \qquad B = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

Which product is definable? AB or BA?

Size of a matrix

Next we will study matrix multiplications from the perspective of entries, columns and rows. You will see its relation with linear combination.

Dot Product of rows and columns

Each entry of the ingradients corresponds to how much the material is needed for a single good.

	🍱	🍲
🍃	2	2
🍋	1	1
	0	4
🐄	2	1

🍲need 1 🍋.

Dot Product of rows and columns

To know how many 🍋 is needed for 🍲. Notice that 🍲 are made of semi-finished meals 🥛, ☕ and 🍵. It is sufficient to know how many 🍋 is needed by those semi-finished meals.

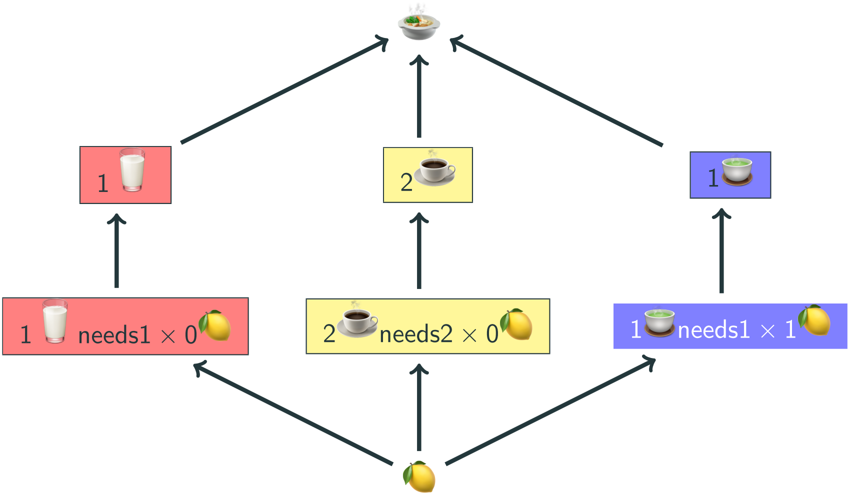
	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

Product by entries

Product by entries

We represents the need by the following graph



Total demand: $\boxed{1} \times \boxed{0} + \boxed{2} \times \boxed{0} + \boxed{1} \times \boxed{1} = 1$

Product by entries

Proposition 2

In the matrix product $C = AB$. Each entry of C is given by the corresponding inner product of a row of A and a column of B

A

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

B

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

C

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

$$1 = \boxed{1} \times \boxed{0} + \boxed{2} \times \boxed{0} + \boxed{1} \times \boxed{1}$$

Product by entries

Come back to serious math. This method is the common method to compute matrix product, try it now.

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

The column of a matrix

Let's look at the product column by column.

Each column of the ingradients corresponds to how to produce meals by materials

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

🍱 $= \mathbf{2} \times$ 🍃 $+ \mathbf{1} \times$ 🍋 $+ \mathbf{0} \times$ 🫘 $+ \mathbf{2} \times$ 🐄.

This kind of expression is called Linear combinations.

Linear Combination of Column

Ingradients demand of making a meal comes form demand of making semi-finished meals.

Leftfactor

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

Rightfactor

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

=

Product

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

🍱 :

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \underbrace{2 \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{2\times \text{🥛}} \underbrace{+ 0 \times \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{0\times \text{☕}} \underbrace{+ 1 \times \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{1\times \text{🍵}}$$

Linear Combination of Column

Proposition 3

In the matrix product $C = AB$. Each column of C is linear combination of columns of A,with coefficient given by the corresponding column of B.

A

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

B

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

C

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

Linear Combination of Column

Excercise: Find the ingradients list of soup, which coefficent do you use?

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

🍲 :

🍃	
🍋	
🫘	
🐄	

$$= \overbrace{\square \times \begin{bmatrix}0\\0\\0\\1\end{bmatrix}}^{\square \times \text{🥛}} + \overbrace{\square \times \begin{bmatrix}0\\0\\2\\0\end{bmatrix}}^{\square \times \text{☕}} + \overbrace{\square \times \begin{bmatrix}2\\1\\0\\0\end{bmatrix}}^{\square \times \text{🍵}}$$

Linear Combination of Column

omit the header, and leave only the middle numbers. This is the way to write matrix multiplication in mathematics. For example, the equation **??** can be written as

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 0 & 4 \\ 2 & 1 \end{pmatrix}$$

Linear Combination of Column

In the following expression, some element is missing, can you find it out?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} = \quad \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ \square & \square & \square \end{pmatrix}$$

Hint: Each column of the product is a linear combination of columns of the left factor, with coefficient coming from the corresponding column on the right factor.

The row of a matrix

The row,

	🍱	🍲
🍃	2	2
🍋	1	1
	0	4
🐄	2	1

DO NOT MEAN

🍋 $= 1 \cdot$ 🍱 $+ 1 \cdot$ 🍲 (Wrong)

The row of a matrix

Actual meaning:

	🍱	🍲
🍃	2	2
🍋	1	1
	0	4
🐄	2	1

The number of 🍋 $= 1 \cdot$ The number of 🍱 $+ 1 \cdot$ The number of 🍲.

The row of a matrix

Write [🍋] for the **Counter macine** of lemon, measuring how much 🍋inside. For example,

$$\begin{cases} [🍋](🍃) & = 0; \\ [🍋](🐄) & = 0; \\ [🍋](🍋) & = 1; \\ [🍋](🍱) & = 1; \\ [🍋](2🍱 + 3🍲) & = 5 \end{cases}$$

The row of a matrix

Therefore, we can write

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

$$[🍋] = 1 \cdot [🍱] + 1 \cdot [🍲]$$

This is an instruction that to produce a 🍋counter machine, one just need to add numbers showing on the screen from 🍱-counter machine and 🍲-counter machine.

Linear Combination of Rows

In the matrix product,

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

The left factor indicates a method to produce machines.

[🍋] $= 0 \times$ [🥛] $+ 0 \times$ [☕] $+ 1 \times$ [🍵].

Linear Combination of Rows

Now we replace each counter in terms of [🍱] and [🍲].

$$\begin{cases} [🥛] = & 2 \times [🍱] + 1 \times [🍲] \\ [☕] = & 0 \times [🍱] + 2 \times [🍲] \\ [🍵] = & 1 \times [🍱] + 1 \times [🍲] \end{cases}$$

White all machine in terms of [🍱] and [🍲], we obtain

$$\underbrace{\boxed{1 \mid 1}}_{[🍋]} = 0 \times \underbrace{\boxed{2 \mid 1}}_{[🥛]} + 0 \times \underbrace{\boxed{0 \mid 2}}_{[☕]} + 1 \times \underbrace{\boxed{1 \mid 1}}_{[🍵]}$$

Linear Combination of Rows

Proposition 4

In the matrix product $C = AB$. Each rows of C is linear combination of rows of B,with coefficient given by the corresponding rows of A.

A

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

B

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$=$

C

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

Excercise.: Find how to produce the counter machine [🍃] from [🍱] and [🍲]. which coefficent do you use?

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍃		

$= \square \times \overbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}^{[🥛]} + \square \times \overbrace{\begin{bmatrix} 0 & 2 \end{bmatrix}}^{[☕]} + \square \times \overbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}^{[🍵]}$

Linear Combination of Rows

Back to the serious math, some element is missing, can you find it out?

$$\begin{pmatrix} \square & \square \\ \square & \square \\ 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 5 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \quad \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ \square & \square & \square \end{pmatrix}$$

Hint: Each row of the product is a linear combination of rows of the right factor, with coefficient coming from the corresponding row on the left factor.

Matrix addition

Definition 2

Suppose M and N are matrices of the same size, we define $M + N$ to be a new matrix by adding them entriwise.

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

$+$

	🍹	🥤
🥛	0	3
☕	1	2
🍵	1	1

$=$

	🍱 + 🍹	🍲 + 🥤
🥛	2	4
☕	1	4
🍵	2	2

Matrix addition

Proposition 5

We have distributive law for matrix multiplication and addition

1. $A(M+N) = AM + AN$;
2. $(M+N)B = MB + NB$.

We have associativity for matrix multiplication

1. $(AB)C = A(BC)$;

Matrix multiplication not commutative

Note that: Matrix multiplication is **not commutative** in general!

Calculate

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Matrix multiplication not commutative

Expanding expressions should keep the orders

$$(A + B)(C + D) = AC + AD + BC + BD.$$

Question: Why does the following formula is **WRONG** for matrices? How to correct it?

$$(A + B)^2 = A^2 + 2AB + B^2 \qquad (A + B)(A - B) = A^2 - B^2.$$

Question: Is that true that $(A^2 + 1)(A^3 + 1) = (A^3 + 1)(A^2 + 1)$? Why?

Question: More generaly, is that true that polynomials of A would commute with A?

Some special square matrices

Square matrix: those $n \times n$ matriecs with same number of rows and columns.

Diagonal: It only refers to the diagonal from top left corner to right bottom corner. We only talk about diagonal for square matrices.

$$\begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix}$$

Upper Triangular Matrix: A square matrix that all entries under diagonal is 0.

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Some special square matrices

Lower Triangular Matrix: A square matrix that all entries above diagonal is 0.

$$\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$

Diagonal Matrix: A square matrix that all entries off diagonal is 0.

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Symmetric Matrix:A matrix which entries are symmetric along diagonal

$$\begin{pmatrix} a & b & c \\ b & e & f \\ c & f & g \end{pmatrix}$$