

Note: Preview of slides from (compositionOfLinearTransformations.tex) by Qirui Li (<https://orcid.org/0000-0002-6042-1291>). For educational and non-commercial use only. Any unlawful use will be prosecuted.

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# Composition of linear transformations

Next part, we will see that the matrix multiplication interprets the composition of linear transformations.

# Composition of linear transformations

## Proposition 1

Let  $V, W, U$  be linear spaces with bases

$\mathcal{E}$  basis for  $V$     $\mathcal{F}$  basis for  $W$     $\mathcal{H}$  basis for  $U$

Consider the following linear transformation

$$V \xrightarrow{T} W \xrightarrow{S} U$$

Then the matrix representation of  $S \circ T$  is the product of matrix representations of  $S$  and  $T$  in corresponding bases.

# Composition of linear transformations

We suppose the linear transformation has the following matrix representations

$$T \left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) = \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) A$$

$$S \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) = \left( \underbrace{\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_p}_{\mathcal{H}} \right) B$$

The proof is straightforward when looking at the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{S} & U \\ L_{\mathcal{E}} \uparrow & & L_{\mathcal{F}} \uparrow & & L_{\mathcal{H}} \uparrow \\ F^n & \xrightarrow{L_A} & F^m & \xrightarrow{L_B} & F^p \end{array}$$

# Composition of linear transformations

Induced transformation proof:

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{S} & U \\ \uparrow L_{\mathcal{E}} & & \uparrow L_{\mathcal{F}} & & \uparrow L_{\mathcal{H}} \\ F^n & \xrightarrow{L_A} & F^m & \xrightarrow{L_B} & F^p \end{array}$$

Let  $C$  be the matrix of  $S \circ T$  on the basis  $\mathcal{E}$  and  $\mathcal{H}$ . We want to show  $C = BA$ . To do so, we only need to prove

$$L_C = L_B \circ L_A$$

Since  $L_C = L_{\mathcal{H}}^{-1} \circ S \circ T \circ L_{\mathcal{E}}$ ,  $L_B = L_{\mathcal{H}}^{-1} \circ S \circ L_{\mathcal{F}}$ ,  $L_A = L_{\mathcal{F}}^{-1} \circ T \circ L_{\mathcal{E}}$ . Then

$$L_B \circ L_A = L_{\mathcal{H}}^{-1} \circ S \circ L_{\mathcal{F}} \circ L_{\mathcal{F}}^{-1} \circ T \circ L_{\mathcal{E}} = L_{\mathcal{H}}^{-1} \circ S \circ T \circ L_{\mathcal{E}} = L_C.$$

# Composition of linear transformations

**Standard Proof:** Since we have

$$\begin{aligned} T \left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) &= \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) A \\ S \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) &= \left( \underbrace{\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_p}_{\mathcal{H}} \right) B \\ S \circ T \left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) &= S \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) A \\ &= \left( \underbrace{\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_p}_{\mathcal{H}} \right) BA \end{aligned} \tag{1}$$

This implies

$$S \circ T \left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) = \left( \underbrace{\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_p}_{\mathcal{H}} \right) BA$$

# Composition of linear transformations

## Corollary 1

Let  $V, W$  be linear spaces with bases

$\mathcal{E}$  basis for  $V$     $\mathcal{F}$  basis for  $W$

Suppose  $T : V \longrightarrow W$  is **invertible** and represented by matrix  $A$  on those bases, then its inverse  $T^{-1} : W \longrightarrow V$  is represented by  $A^{-1}$

# Composition of linear transformations

**Standard Proof:** By what given, we have

$$T \left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) = \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) A$$

Apply  $T^{-1}$  on both hand, we have

$$\left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) = T^{-1} \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right) A$$

Right multiplying  $A^{-1}$ , we have

$$\left( \underbrace{\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n}_{\mathcal{E}} \right) A^{-1} = T^{-1} \left( \underbrace{\vec{w}_1 \quad \vec{w}_2 \quad \cdots \quad \vec{w}_m}_{\mathcal{F}} \right)$$

This proves the corollary. ■