Image

Image of a linear transformation describes how far away a linear transformation is from being **surjective** .

Image

There are three type of elements in codomain W for a linear transformation $T : V \longrightarrow W$

- *lonely* element : there is **no** element in V corresponds to it.
- *faithful, loyal element*: there is only one element in V corresponds to it.
- *unfaithful, unloyal element*: there is two or more element in V corresponds to it.

The $\mathrm{Im}\, T$ is the subset of all element of the second and third type.

Any element outside of $\mathrm{Im}\, T$ are *lonely*, do not have elements in V to corresponds to it.

Image

surjective means no *lonely* element, so we use image to describe how close it is from being a **surjective** .

Definition 1

$\boxed{L_S}$ We define the Image of a map $T : V \longrightarrow W$ as

$$\mathrm{Im}\, T = \{\vec{w} \in W : T(\vec{v}) = \vec{w} \text{ for some } \vec{v} \in V\}$$

if there is no *lonely* element, which is $\mathrm{Im}\, T = W$, the map become a **surjective** .

Proposition 1

$\boxed{S}$ $\boxed{L_S}$ A linear map $T : V \longrightarrow W$ is **surjective** if and only if

$$\mathrm{Im}\, T = W.$$

Proposition 2

For any map $S : V \longrightarrow W$ and $T : W \longrightarrow U$, we have

$$\mathrm{Im}(T \circ S) = T\left(\mathrm{Im}S\right).$$

Proof.

Firstly we prove $\mathrm{Im}(T \circ S) \subset T\left(\mathrm{Im}S\right)$. For any $\vec{u} \in \mathrm{Im}(T \circ S)$, there exists $\vec{v} \in V$ with $\vec{u} = T \circ S(\vec{v})$. So $\vec{u} = T(\vec{w})$ with $\vec{w} = S(\vec{v}) \in \mathrm{Im}S$, therefore $\vec{u} \in T(\mathrm{Im}S)$.

Then we prove $\mathrm{Im}(T \circ S) \supset T\left(\mathrm{Im}S\right)$. For any $\vec{u} \in T\left(\mathrm{Im}S\right)$, one can write $\vec{u} = T(\vec{w})$ for some $\vec{w} \in \mathrm{Im}S$, so one can write $\vec{w} = S(\vec{v})$ for some $\vec{v} \in V$. As total, $\vec{u} = T(S(\vec{v}))$ for some $\vec{v}$, then $\vec{u} \in \mathrm{Im}(T \circ S)$. $\square$

Image

Proposition 3

For any map $S : V \longrightarrow W$ and $T : W \longrightarrow U$, we have

$$\mathrm{Im}(T \circ S) \subset \mathrm{Im}\, T.$$

Proof.

If $\vec{u} \in \mathrm{Im}(T \circ S)$, this implies $\vec{u} = T \circ S(\vec{v})$ for some $\vec{v} \in V$, let $\vec{w} = S(\vec{v})$ then $\vec{u} = T(\vec{w})$. Therefore $\vec{u} \in \mathrm{Im}(T)$ we proved

$$\mathrm{Im}(T \circ S) \subset \mathrm{Im}(T).$$

$\square$

Image

Proposition 4

If $S : V \longrightarrow W$ is an **surjective** , for any map $T : W \longrightarrow U$, we have

$$\mathrm{Im}(T \circ S) = \mathrm{Im}\, T$$

Proof.

It is clear from previous proposition $\mathrm{Im}(T \circ S) \subset \mathrm{Im}\, T$.

To show inclusion in the other way, if $\vec{u} \in \mathrm{Im}(T)$, then $\vec{u} = T(\vec{w})$ for some $\vec{w} \in W$, since S is a **surjective** ,$\vec{w} = S(\vec{v})$ for some $\vec{v} \in V$. so $\vec{u} = T \circ S(\vec{v})$ so $\vec{u} \in \mathrm{Im}(T \circ S)$. So

$$\mathrm{Im}(T \circ S) \supset \mathrm{Im}(T).$$

$\square$

Subspace by constructive language

Previous arguments are all for maps. When they a linear transformation ,

Proposition 5

If $T : V \longrightarrow W$ is a linear transformation . Then $\operatorname{Im} T$ is a subspace.

Proof.

For any $\vec{w}_1, \vec{w}_2 \in \operatorname{Im} T$ and any scalar $\lambda \in F$. Since $\vec{w}_1, \vec{w}_2$ are in image, we can find $\vec{v}_1, \vec{v}_2 \in V$ such that

$$\vec{w}_1 = T(\vec{v}_1) \qquad \vec{w}_2 = T(\vec{v}_2)$$

Then because T is linear

$$\lambda \vec{w}_1 + \vec{w}_2 = \lambda T(\vec{v}_1) + T(\vec{v}_2) = T(\lambda \vec{v}_1 + \vec{v}_2) \in \operatorname{Im} T.$$

Therefore $\operatorname{Im} T$ is a subspace. $\square$

Subspace by constructive language

The way we describe Image is **constructive** . That is, we listed elements of the set one by one with a parametrization, where parameters are in domain.

$$\mathrm{Im}\, T = \{\vec{w} \in W : T(\vec{v}) = \vec{w} \textbf{ for some } \vec{v} \in V\}$$

In fact, any subspace described by **constructive language** should essentially related to image of a linear transformation.

Subspace by constructive language

Excercise. Let V be the following vector spaces over F, verify the following subsets $W \subset V$ is a **subspace** of V.

- $V = P_{2,x} = \{ax^2 + bx + c, \text{ where } a, b, c \in F\}$,
  $W = P_{1,x} = \{ax + b, \text{ where } a, b \in F\} \subset V$

(See some footnote[1])

[1] In fact, W can be described as a image of a linear transformation $T : F^2 \longrightarrow V$ with $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto ax + b$. If you can check this map is linear, you automatically know it is a subspace without further proof.

Kernel

Now let us turn to subspaces by **descriptive language** . With the rise of kernel, you will undestand such a subspaces always can be described by a kernel of a linear transformation.

Three type of vectors in codomain – Once again

There are three type of elements in codomain W for a linear transformation $T : V \longrightarrow W$

- *lonely* element : there is **no** element in V corresponds to it.
- *faithful, loyal element*: there is only one element in V corresponds to it.
- *unfaithful, unloyal element*: there is two or more element in V corresponds to it.

To measure unfaithful elements. Review the definition of preimage

Definition 2 Reminder from previous slides

$\boxed{L_I}$ For any map $f : X \longrightarrow Y$ and any subset $S \subset Y$, the **preimage** of S is defined by

$$f^{-1}(S) := \{x \in X : f(x) \in S\}$$

Three type of vectors in codomain – Once again

Now we can use preimage to describe those three type of elements: Let $f : X \longrightarrow Y$ be a map

- $y \in Y$ is *lonely* element : $f^{-1}(\{y\}) = \emptyset$.
- $y \in Y$ is *faithful, loyal element*: $f^{-1}(\{y\}) = \{x\}$ for some $x \in X$.
- $y \in Y$ is *unfaithful, unloyal element*: $f^{-1}(\{y\})$ has more than 1 element.

Superposition Principal

injective map means there is no *unfaithful* element. If a map is **injective** , it means preimage of any singleton is a singleton or empty set.

For a linear transformation, the preimage of two different singletons have a relation described by **Superposition Principal**. It concluds that for any linear transformation, faithful and unfaithful element can not appear simutaneously in the codomain of the same map.

Superposition Principal

Lemma 1

Let $T : V \longrightarrow W$ be a linear transformation . Suppose $\vec{v}_1 \in V$ and $\vec{v}_2 \in V$ with $\vec{v}_1 \in T^{-1}(\{\vec{w}_1\})$ and $\vec{v}_2 = T^{-1}(\{\vec{w}_2\})$, then for any $\lambda, \mu \in F$,

$$\lambda\vec{v}_1 + \mu\vec{v}_2 \in T^{-1}(\{\lambda\vec{w}_1 + \mu\vec{w}_2\})$$

Proof.

$\vec{v}_1 \in T^{-1}(\{\vec{w}_1\})$ and $\vec{v}_2 = T^{-1}(\{\vec{w}_2\})$ implies $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$.

Since T is a linear transformation , So

$$\lambda\vec{w}_1 + \mu\vec{w}_2 = \lambda T(\vec{v}_1) + \mu T(\vec{v}_2) = T(\lambda\vec{v}_1 + \mu\vec{v}_2)$$

This implies $\lambda\vec{v}_1 + \mu\vec{v}_2 \in T^{-1}(\{\lambda\vec{w}_1 + \mu\vec{w}_2\})$ □

Proposition 6 Superposition Principal–Addition

Let $T : V \longrightarrow W$ be a linear transformation . For any $\vec{w}_1, \vec{w}_2 \in \operatorname{Im} T$, let $\vec{v}_1, \vec{v}_2 \in V$ be such that $\vec{v}_1 \in T^{-1}(\{\vec{w}_1\})$ and $\vec{v}_2 = T^{-1}(\{\vec{w}_2\})$, we have an isomorphism between their preimages

$$+_{\vec{v}_2 - \vec{v}_1} : T^{-1}\left(\{\vec{w}_1\}\right) \longrightarrow T^{-1}\left(\{\vec{w}_2\}\right), \vec{x} \mapsto \vec{x} + \left(\vec{v}_2 - \vec{v}_1\right)$$

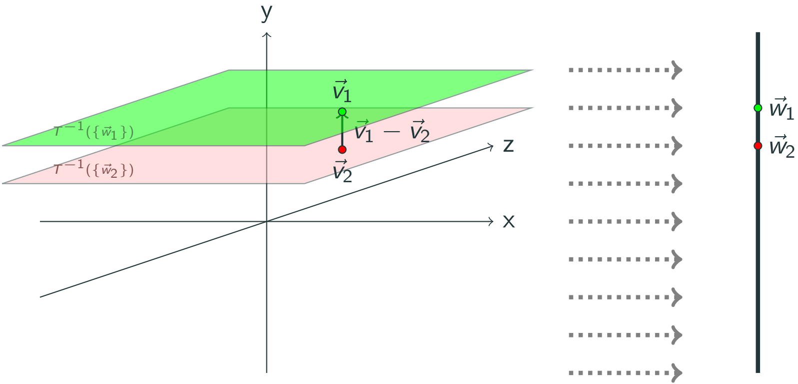
Proof.

This is indded a map, if $\vec{x} \in T^{-1}(\{\vec{w}_1\})$, since $\vec{v}_2 - \vec{v}_1 \in T^{-1}(\vec{w}_2 - \vec{w}_1)$, this implies $\vec{x} + (\vec{v}_2 - \vec{v}_1) \in T^{-1}(\{\vec{w}_1 + \vec{w}_2 - \vec{w}_1\}) = T^{-1}(\{\vec{w}_1\})$.

The inverse is given by $\vec{x} \mapsto \vec{x} - (\vec{v}_2 - \vec{v}_1)$. So is an isomorphism. □

Superposition Principal

In plain words, it means the preimage of $\{\vec{w}_1\}$ is obtained from preimage of $\{\vec{w}_2\}$ by translation of $\vec{v}_2 - \vec{v}_1$. Use the following visualize **example of projection** to understand superposition principal.



Please note the preimages in this picture are **not** subspaces as it does not contain the origin.

Superposition Principal

Excercise. Try to state the superposition principal for saclar multiplication. In other words, when comparing the preimage of $T^{-1}(\{\vec{w}\})$ and $T^{-1}(\{\lambda\vec{w}\})$ for a scalar $\lambda \in F$, how to construct a natural isomorphism between those two sets?

Uniqueness problem and the kernel

All preimages of a singleton in a linear transformation is either empty or can be translated to each other.

We pick up the most representative preimage, $T^{-1}(\{\vec{0}\})$, to measure how large is the preimage for every singleton, which describe how for a linear transformation is from being an **injective** .

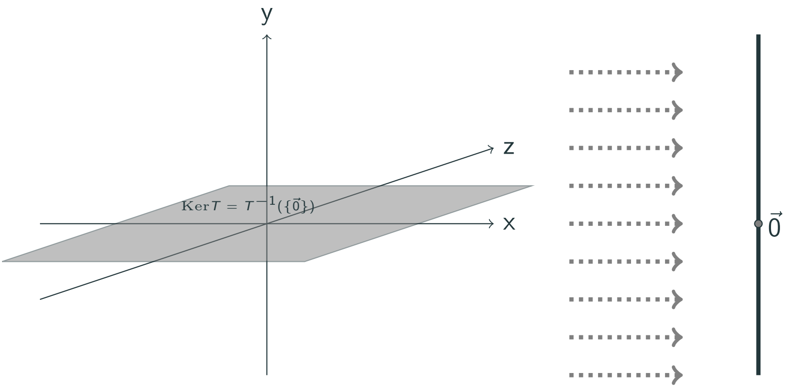
Definition 3 kernel

$\boxed{L_I}$ We define the **Kernel** of a linear map $T : V \longrightarrow W$ as

$$\mathrm{Ker}\, T = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$$

Uniqueness problem and the kernel

Picture of kernel in the previous example of projection.



Superposition Principal and kernel

Proposition 7

Suppose $T : V \longrightarrow W$ is a linear transformation . Let $\vec{v} \in V$ and $T(\vec{v}) = \vec{w}$. Then the preimage of $\{\vec{w}\}$ is given by

$$T^{-1}(\{\vec{w}\}) = \{\vec{v}\} + \mathrm{Ker}\, T := \{\vec{v} + \vec{x} : \vec{x} \in \mathrm{Ker}\, T\}.$$

Proof.

Since $\mathrm{Ker}\, T = T^{-1}(\{\vec{0}\})$, this is clear from previous arguments $\square$

Superposition Principal and kernel

No **uniqueness problem** for every element make it to be an **injective** .

Proposition 8

$\boxed{I}\boxed{L_I}$A linear map $T : V \longrightarrow W$ is **injective** if and only if

$$\operatorname{Ker} T = \{\vec{0}\}.$$

Proof.

(T **injective** $\implies \operatorname{Ker} T = \{\vec{0}\}$)Since $T(\vec{0}) = \vec{0}$ so $\vec{0} \in \operatorname{Ker}(T)$, so $\operatorname{Ker}(T) \neq \emptyset$. If $\vec{v} \in \operatorname{Ker} T$, then $T(\vec{v}) = \vec{0} = T(\vec{0})$, since T is an **injective** , so $\vec{v} = \vec{0}$.

($\operatorname{Ker} T = \{\vec{0}\} \implies T$ **injective**)Suppose $T(\vec{v}_1) = T(\vec{v}_2)$, then $T(\vec{v}_1 - \vec{v}_2) = \vec{0}$(footnote[2]), so $\vec{v}_1 - \vec{v}_2 \in \operatorname{Ker} T$ then $\vec{v}_1 = \vec{v}_2$.

$\square$

[2]This step essentially use the idea of superposition principal.

Properties of kernel

Proposition 9

For any linear transformation $T : V \longrightarrow W$ and linear transformation $S : U \longrightarrow V$, we have

$$\operatorname{Ker}(T \circ S) = S^{-1}(\operatorname{Ker} T)$$

Proof.

$$\vec{v} \in \operatorname{Ker}(T \circ S) \iff T(S(\vec{v})) = \vec{0} \iff S(\vec{v}) \in \operatorname{Ker} T \iff \vec{v} \in S^{-1}(\operatorname{Ker} T)$$

$\square$

Properties of kernel

Proposition 10

For any linear transformation $T : V \longrightarrow W$ and linear transformation $S : U \longrightarrow V$, we have

$$\mathrm{Ker}(T \circ S) \supset \mathrm{Ker} S$$

Proof.

Since $\mathrm{Ker}(T \circ S) = S^{-1}(\mathrm{Ker}\, T)$ and $\mathrm{Ker}\, T \supset \{\vec{0}\}$, so

$$S^{-1}(\mathrm{Ker}\, T) \supset S^{-1}(\{\vec{0}\}) = \mathrm{Ker} S.$$

$\square$

Properties of kernel

Proposition 11

If $T : V \longrightarrow W$ is an **injective** , for any linear transformation $S : U \longrightarrow V$, we have

$$\mathrm{Ker}(T \circ S) = \mathrm{Ker} S$$

Proof.

Since $\mathrm{Ker}(T \circ S) = S^{-1}(\mathrm{Ker}\, T)$, if T is **injective** then $\mathrm{Ker}\, T = \{\vec{0}\}$, so in this case

$$\mathrm{Ker}(T \circ S) = S^{-1}(\mathrm{Ker}\, T) = S^{-1}(\{\vec{0}\}) = \mathrm{Ker} S.$$

□

We choose the preimage of $\vec{0}$ because it is the only subset who become a subspace.

Corollary 1

For any linear transformation , $T : V \longrightarrow W$, $\mathrm{Ker}\, T$ is a subspace.

Standard Proof.

For any $\vec{v}_1, \vec{v}_2 \in \mathrm{Ker}\, T$ and any scalar $\lambda \in F$, we have

$$T(\lambda\vec{v}_1 + \vec{v}_2) = \lambda T(\vec{v}_1) + T(\vec{v}_2) = \lambda 0 + 0 = 0$$

So $\lambda\vec{v}_1 + \vec{v}_2 \in \mathrm{Ker}\, T$, which means it is a subspace. $\square$

Subspaces described by descriptive language

We can also understand it is a subspace from the superposition principal.

We have learned adding a vector in $T^{-1}(\{\vec{w}_1\})$ with a vector in $T^{-1}(\{\vec{w}_2\})$ will result a vector in $T^{-1}(\{\vec{w}_1 + \vec{w}_2\})$. Therefore, if $\vec{w}_1 = \vec{w}_2 = \vec{0}$, then it means $T^{-1}(\{\vec{0}\})$ is closed under addition (and also in the same way closed under scalar multiplication).

Therefore $\mathrm{Ker}\, T = T^{-1}(\{\vec{0}\})$ is a subspace.

Subspaces described by descriptive language

The way we describe Kernel is **descriptive** . That is, we describe it to be set of all element that have the property of mapping to 0.

$$\operatorname{Ker} T = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$$

In fact, any subspace described by **descriptive language** should essentially related to kernel of a linear transformation.

Subspaces described by descriptive language

Excercise. Let V be the following vector spaces over F, verify the following subsets $W \subset V$ is a **subspace** of V.

- $V = P_{2,x} = \{ax^2 + bx + c, \text{ where } a, b, c \in F\}$,
  $W = \{f \in V : f(1) = 0\}$

(See some footnote[3])

[3] In fact, W can be described as the kernel of a linear transformation of evaluation map $T : V \longrightarrow F^2$ with $f \mapsto f(1)$. If you can check this map is linear, you automatically know it is a subspace without further proof.