Induced Linear Transformation

From now, we have learned three concepts

- Matrices
- Tuple of vectors
- Linear Transformation

In order to link all those three concepts together, we would like to bring all object in one world: Linear Transformation.

In the next part. We will give the way to translate each object as a linear transformation.

- $m \times n$ Matrices $\iff$ Linear transformations $F^n \longrightarrow F^m$
- n Tuple of vectors $\iff$ Linear transformations $F^n \longrightarrow V$

Induced Linear Transformation

There is also a way to translate subspaces into Linear Transformations. We would like to keep that part later because we would like you to be familiar with it first.

After those translation. All linear algebra are essentially studying linear transformations. The goal for this part is to put the linear transformation into a central point of all this subject.

Induced Linear Transformation

After we translate a tuple or a matrix into a linear transformation. We call it **induced linear transformation**.[1]

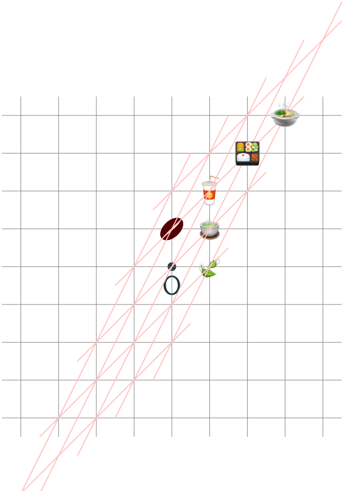
Before talking about this, let's continue our story of making meals.

[1] This concept is borrowed from Prof. Jason Siefken's Book

Induced Linear Transformation

Afterwards, Shinchan start using pictures to represents everything. Until one day, an old Chef complained.

Old Chef: *I want a clear number table! The recipe is not a doodle!*

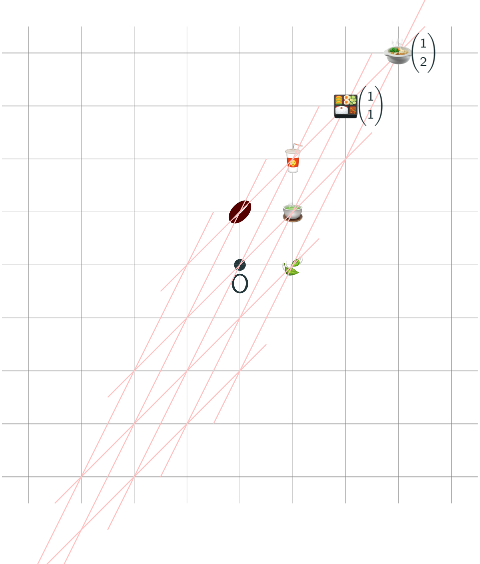


Old Chef wants

	🍱	🍲
🥤	1	1
🍵	1	2

Induced Linear Transformation

But still Shinchan wants to keep pictures, so he just cut the table and labeled it to each corresponding meals.



He did not put the row header 🥤,🍵on it. And those label depends on what it the row header.

To prevent the old chief complaining, he will label all future meals.

Induced Linear Transformation

Call the combination space of meals as V. Let $\mathcal{E} = \begin{pmatrix} \text{🥤} & \text{🍵} \end{pmatrix}$ be the tuple of materials in the row header. The way Shinchan label the meals, is a map

$$L_{\mathcal{E}} : F^2 \longrightarrow V, \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x\,\text{🥤} + y\,\text{🍵}$$

By the meaning of matrix multiplication, this map can also be written as

$$L_{\mathcal{E}} : F^2 \longrightarrow V, \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \text{🥤} & \text{🍵} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In other words, this is just the map of left multiplication by $\begin{pmatrix} \text{🥤} & \text{🍵} \end{pmatrix}$

Viewing tuple of vectors as a linear transformation

Definition 1

Let $\mathcal{E} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ be n-tuples of vectors in V, the **induced linear transformation** of $\mathcal{E}$ is a linear transformation $L_{\mathcal{E}}$ with the domain F^n, codomain V, by the rules of assiging coefficient list to corresponding linear combinations of $\mathcal{E}$. Explicitely,

$$L_{\mathcal{E}} : F^n \longrightarrow V, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

Viewing tuple of vectors as a linear transformation

Note that the linear combination can be written as a matrix multiplication.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Viewing tuple of vectors as a linear transformation

Let $\mathcal{E} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$, the induced linear transformation $L_{\mathcal{E}}$ is the same as **left multiplying** the list of vectors $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ as a matrix.

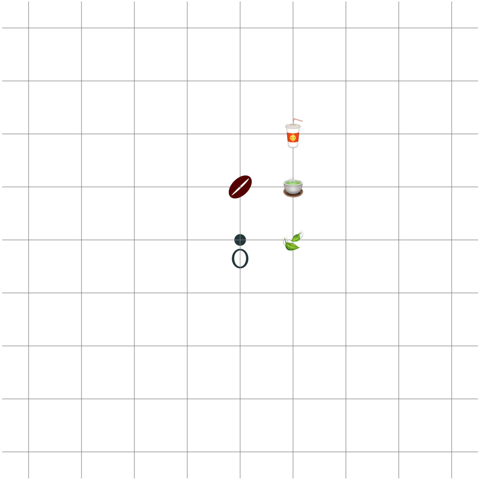
$$L_{\mathcal{E}} : F^n \longrightarrow V, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

The letter L refers to the **L***eft multiplication*

Viewing tuple of vectors as a linear transformation

The induced transformation is defined for **any finite tuple of vectors.**

The induced transformation may map two coefficient list to the same element. For example,



We consider the induced transformation $L_{\mathcal{E}} : F^3 \longrightarrow V$ with $\mathcal{E} = \big($ 🫘 🍃 🍵 $\big)$, then

🍵 $= 0$🫘 $+ 0$🍃 $+ 1$🍵

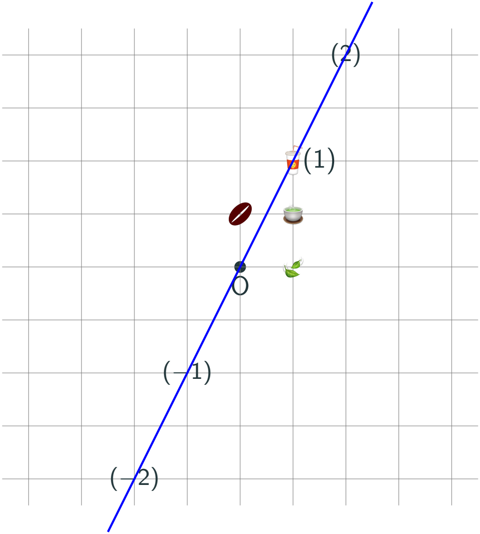
🍵 $= 1$🫘 $+ 1$🍃 $+ 0$🍵

Both coefficients list maps to the same vector

$L_{\mathcal{E}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = L_{\mathcal{E}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} =$ 🍵

Viewing tuple of vectors as a linear transformation

Some element in the codomain may not be associated with a coefficient list, for example



We consider the induced transformation

$$L_{\mathcal{E}} : F \longrightarrow V$$

with $\mathcal{E} = ($ 🥤 $)$.

The image of this map is as shown. Clear, 🍃can not be writen as any kinds of linear combination of 🥤, so 🍃 do not have an coefficient list corresponds to it.

Viewing tuple of vectors as a linear transformation

Summary

A tuple of vectors $\mathcal{E} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{pmatrix}$ in V can be viewed as a linear transformation.

$$L_{\mathcal{E}} : F^m \longrightarrow V$$

Any linear transformation $T : F^m \longrightarrow V$ must be an induced linear transformation by an m-tuples in V. This tuple is given by

$$\left(T\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad T\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

.

Viewing matrices as linear transformations

In the previous part we have understand how m tuples corresponds to linear transformations from domain F^m. Previously we give the link between tuple of vectors and linear transformations.

$$\text{Tuple of vectors} \longleftrightarrow \text{Linear Transformations}$$

Matrices

Now we will draw our road map to matrices.

Viewing matrices as tuples

If we look at a $n \times m$ matrix column by column. Each column is representing an element in F^n.

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} \iff \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right)$$

Therefore, $n \times m$ matrices are equivalent to m-tuples in F^n. This gives a interpretation of matrices in tuples.

$$\begin{tikzcd}
\text{Tuple of vectors} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Linear Transformations} \\
& \text{Matrices} &
\end{tikzcd}$$

Viewing matrices as tuples

Since matrices can be viewed as tuples, $n \times m$ matrix is equivalent to choosen m vectors in F^n. This tuple induces a map of

$$L_{\mathcal{E}} : F^m \longrightarrow F^n$$

This map is obtained by left multiplying a matrix with columns by those selected vectors $\mathcal{E}$ in F^n. Since this is an actual matrix(every entry of this matrix are numbers $\in F$). This is called **the induced linear transformation by a matrix**.

This gives a interpretation of matrices in linear transformations.

$$\begin{tikzcd}
\text{Tuple of vectors} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Linear Transformations} \arrow[dl, leftrightarrow] \\
& \text{Matrices} &
\end{tikzcd}$$

This viewpoint tell us we can view a matrix as a linear transformation from F^m to F^n.

Viewing matrices as tuples

Now we replace the abstract vector space V by F^n. This implies we can natrually understand m-tuples in F^n as a linear transformation.

Viewing matrices as tuples

Excercise. Select two vectors in F^3 as a 2-tuple as follows

$$\mathcal{E} = \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

Write down the induced linear tranformation.

Solution.

$$L_{\mathcal{E}} : F^2 \longrightarrow F^3, \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto a \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

One can also write this as

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Viewing matrices as tuples

Conversely, **any linear tranformation** : $F^m \longrightarrow F^n$ **is induced by a matrix**. By looking at the image of natural basis.

Excercise. Consider a linear transformation $T : F^2 \longrightarrow F^3$ with

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix}, \qquad T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\3\\5\end{pmatrix}$$

Find a tuple of vectors $\mathcal{E}$, with $T = L_{\mathcal{E}}$.

Viewing matrices as tuples

Solution. We only need to know $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$. Those are easy

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\begin{pmatrix}2\\2\end{pmatrix} - T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\6\\4\end{pmatrix} - \begin{pmatrix}1\\3\\5\end{pmatrix} = \begin{pmatrix}1\\3\\-1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\begin{pmatrix}1\\2\end{pmatrix} - T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix}$$

This implies T is a linear tranformation induced by

$$\begin{pmatrix}1 & 0\\3 & 0\\-1 & 3\end{pmatrix}$$

Viewing matrices as tuples

Now let's summarise the induced linear transformation.

Viewing matrices as tuples

For any $n \times m$ matrix P, its induced linear transformation is defined by

$$L_P : F^m \longrightarrow F^n, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \longmapsto P \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

Remember here P is a **Matrix**, L_P is a **Linear Transformation**, L_P and P has the exactly the same information, but different meaning. L_P means how we use P to creat a map.(by left multiplication)

Excercise. What′s the differene between *you* as a person and *you eat* as an action?

Viewing matrices as tuples

For any m-tuple of vectors $\mathcal{G} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{pmatrix}$ in V, its induced linear transformation is defined by

$$L_{\mathcal{G}} : F^m \longrightarrow V, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \longmapsto \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

Remember here $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{pmatrix}$ is a **m-tuple of vectors**, $L_{\mathcal{G}}$ is a **Linear Transformation**, $L_{\mathcal{G}}$ and $\mathcal{G}$ has the exactly the same information, but different meaning. $L_{\mathcal{G}}$ means how we use $\mathcal{G}$ to creat a map. (by left multiplication)

Composition of induced transformation

From very begining we mentioned that we could write

$$\begin{cases} 3\vec{e}_1 + \vec{e}_2 & = \vec{w}_1 \\ \vec{e}_1 + \vec{e}_2 & = \vec{w}_2 \end{cases}$$

as

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

The reason of writing this way can be stated in the viewpoint of induced linear transformation.

Composition of induced transformation

Label each part with a symbol.

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 \end{pmatrix}}_{\mathcal{F}} = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 \end{pmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}_{A}$$

We have associated induced linear transformation $L_{\mathcal{F}}, L_{\mathcal{E}}, L_A$.

The order of our symbol is logical in the sense that we have

$$L_{\mathcal{F}} = L_{\mathcal{E}} \circ L_A$$

Composition of induced transformation

More generally, in this part, you would find all our symbol make sense because it is representing its meaning of induced transformations

We will prove the following equations in this part

Regular Symbol	Translate to Induced Transformation
$\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2\end{pmatrix}}_{\mathcal{F}} = \underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix}3 & 1 \\ 1 & 1\end{pmatrix}}_{A}$	$L_{\mathcal{F}} = L_{\mathcal{E}} \circ L_A$
$T\underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2\end{pmatrix}}_{\mathcal{F}} = \underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2\end{pmatrix}}_{\mathcal{E}}$	$T \circ L_{\mathcal{F}} = L_{\mathcal{E}}$
$C = AB$	$L_C = L_A \circ L_B$

Composition of induced transformation

Lemma 1

For any tuple $\mathcal{F} = \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix}$ and $\mathcal{E} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_m \end{pmatrix}$in V, suppose

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix}}_{\mathcal{F}} = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_m \end{pmatrix}}_{\mathcal{E}} A$$

Then

$$L_{\mathcal{F}} = L_{\mathcal{E}} \circ L_A$$

The following picture shows domain and codomain for each map.

$$\begin{tikzcd} F^n \arrow[r, "L_A"] \arrow[rr, bend right, "L_{\mathcal{F}}"'] & F^m \arrow[r, "L_{\mathcal{E}}"] & V \end{tikzcd}$$

Composition of induced transformation

Proof: We only need to prove $L_{\mathcal{F}}(\vec{v}) = L_{\mathcal{E}} \circ L_A(\vec{v})$ for any $\vec{v} \in F^n$. To see this, we see

$$L_{\mathcal{E}} \circ L_A(\vec{v}) = L_{\mathcal{E}}(L_A(\vec{v})) = L_{\mathcal{E}}(A\vec{v}) = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_m \end{pmatrix}}_{\mathcal{E}} A\vec{v}$$

Note that this equals to

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix}}_{\mathcal{F}} \vec{v} = L_{\mathcal{F}}(\vec{v}).$$

Therefore we proved this Lemma.

Composition of induced transformation

Lemma 2

For any tuple $\mathcal{E} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ in V, any linear transformation $T : V \longrightarrow W$. Let $\mathcal{F} = T\mathcal{E}$ be the corresponding output in W. In other words,

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix}}_{\mathcal{F}} = T \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}}_{\mathcal{E}}$$

Then

$$L_{\mathcal{F}} = T \circ L_{\mathcal{E}}$$

The following picture shows domain and codomain for each map.

$$\begin{tikzcd} F^n \arrow[r, "L_{\mathcal{E}}"] \arrow[rr, bend right, "L_{\mathcal{F}}"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

Composition of induced transformation

Proof.

By definition, we have

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix}}_{\mathcal{F}} = \begin{pmatrix} T\vec{v}_1 & T\vec{v}_2 & \cdots & T\vec{v}_n \end{pmatrix}$$

To prove $L_{\mathcal{F}} = T \circ L_{\mathcal{E}}$, we only need to show $L_{\mathcal{F}}(x) = T \circ L_{\mathcal{E}}(x)$ for any

$$x = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n,$$

Clear

$$L_{\mathcal{F}}(x) = a_1 T\vec{v}_1 + \cdots + a_n T\vec{v}_n = T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = T(L_{\mathcal{E}}(x)) = T \circ L_{\mathcal{E}}(x).$$

Therefore $L_{\mathcal{F}} = T \circ L_{\mathcal{E}}$.

$\square$

Composition of induced transformation

Lemma 3

Let A be an $n \times m$ matrix, B an $m \times p$ matrix, let C be the product of them

$$C = AB$$

Then we have

$$L_C = L_A \circ L_B$$

The following picture shows domain and codomain for each map.

$$\begin{tikzcd} F^p \arrow[r, "L_B"] \arrow[rr, bend right, "L_C"'] & F^m \arrow[r, "L_A"] & F^n \end{tikzcd}$$

Composition of induced transformation

Proof.

We only need to show for any $x \in F^p$, we have

$$L_C(x) = L_A \circ L_B(x).$$

To do so, we see

$$L_C(x) = Cx = ABx = L_A(Bx) = L_A(L_B(x)) = L_A \circ L_B(x).$$

$\square$

Invertible matrix and its induced transformation

In the world of matrices, we have identity matrix and invertible matrices.

In the world of linear transformation , we have identity linear transformation and invertible linear transformation .

In this part, we demonstrate the concept of being identity and invertible in matrix world is a kind of translation of the same concept from linear transformation world.

Invertible matrix and its induced transformation

Lemma 4

Let I_n be an $n \times n$ identity matrix, then

$$L_{I_n} = \mathrm{id}_{F^n}$$

The following picture shows domain and codomain for each map.

$$F^n \xrightarrow{L_{I_n}} F^n$$

Proof.

For any $x \in F^n$, $L_{I_n}(x) = I_n x = x$. So $L_{I_n} = \mathrm{id}_{F^n}$. $\square$

Invertible matrix and its induced transformation

Lemma 5

Let A be an $n \times n$ invertible matrix and let A^{-1} be its inverse, then L_A is an invertible linear transformation and its inverse is $L_{A^{-1}}$. The following picture shows domain and codomain for each map.

$$F^n \underset{L_{A-1}}{\overset{L_A}{\longleftrightarrow}} F^n$$

Proof.

We have $L_A \circ L_{A^{-1}} = L_{AA^{-1}} = L_{I_n} = \mathrm{id}_{F^n}$, and $L_{A^{-1}} \circ L_A = L_{A^{-1}A} = L_{I_n} = \mathrm{id}_{F^n}$. This implies the lemma. $\square$

Basis and its induced transformation

In the world of tuples in V, we have discussed how to verify certain tuples are **linealy independent** , **span the whole space** , or being a **basis** .

In this part, we will show being a **basis** translates to the property of being **invertible** in linear transformation world.

The inverse of the induced linear transformation of a **basis** is called **the coordinate map**.

Basis and its induced transformation

Lemma 6

A tuple $\mathcal{E} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ in V is a **basis** if and only if its induced linear transformation $L_{\mathcal{E}}$ is **invertible** .

The following picture shows domain and codomain for each map.

$$F^n \xrightarrow{L_{\mathcal{E}}} V$$

We will prove this lemma by directly giving its inverse.

Basis and its induced transformation

Proposition 1

For a basis $\mathcal{E} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$, let $L_{\mathcal{E}}$ be its induced linear transformation , then

$$[\vec{v}]^{\mathcal{E}} = L_{\mathcal{E}}^{-1}(\vec{v})$$

In other words, the linear transformation

$$V \xrightarrow{L_{\mathcal{E}}^{-1}} F^n$$

maps every vector to its coordinate. We call the map $L_{\mathcal{E}}^{-1}$ the **coordinate map** of the **basis** $\mathcal{E}$.

Basis and its induced transformation

Proof Consider the following map

$$R : V \longrightarrow F^n, \vec{v} \longmapsto [\vec{v}]^{\mathcal{E}}$$

We will prove this map is the inverse of $L_{\mathcal{E}}$. For any $\vec{v} \in V$

$$L_{\mathcal{E}} \circ R(\vec{v}) = L_{\mathcal{E}}(R(\vec{v})) = L_{\mathcal{E}}([\vec{v}]^{\mathcal{E}}) = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} [\vec{v}]^{\mathcal{E}} = \vec{v}.$$

Basis and its induced transformation

Continue from previous

And for any coefficient list

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n,$$

we have

$$R \circ L_{\mathcal{E}}(x) = R(x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x$$

This implies the coordinate map R is the inverse of the linear transformation $L_{\mathcal{E}}$.

Basis and its induced transformation

Proposition 2

Let $\mathcal{E} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ be a basis of V. For any vectors $\vec{v}, \vec{w} \in V$, we have

$$[\lambda \vec{v} + \mu \vec{w}]^{\mathcal{E}} = \lambda [\vec{v}]^{\mathcal{E}} + \mu [\vec{w}]^{\mathcal{E}}$$

Proof.

Since the inverse of a linear transformation must be a linear transformation . We know $L_{\mathcal{E}}^{-1}$ is a linear transformation . Therefore

$$[\lambda \vec{v} + \mu \vec{w}]^{\mathcal{E}} = L_{\mathcal{E}}^{-1}(\lambda \vec{v} + \mu \vec{w}) = \lambda L_{\mathcal{E}}^{-1}(\vec{v}) + \mu L_{\mathcal{E}}^{-1}(\vec{w}) = \lambda [\vec{v}]^{\mathcal{E}} + \mu [\vec{w}]^{\mathcal{E}}$$

□