Annihilating polynomial of matrices

Let A be an $n \times n$ matrix, it is an element in the set

$$\mathbb{R}^{n\times n} := \{M : \qquad M \text{ is an } n \times n \text{ matrix } \}$$

which is $n \times n$ dimensiona. Therefore, the following $n^2 + 1$ elements

$$I = A^0, A = A^1, A^2, A^3, ..., A^{n^2}$$

must be **linearly dependent**, and therefore gives some non-zero coefficients

$$c_0 + c_1 A + c_2 A^2 + ... + c^{n^2} A^{n^2} = 0.$$

This implies that there exists a polynomial

$$f(t) = c_0 + c_1 t + c_2 t^2 + ... + c^{n^2} t^{n^2} = 0.$$

with $f(A) = 0$.

Annihilating polynomial of matrices

Definition 1

A polynomial $f(t)$ with $f(A) = 0$ is called an **annilating polynomial** of an $\mathrm{n} \times \mathrm{n}$ matrix A.

Annihilating polynomial of matrices

Excercise. Try to find an annihilating polynomial of the matrxi

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

We calculate

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

We saw that $A^2 - 5A - 2I = 0$. Is there any other possible method to find the polynomial $x^2 - 5x - 2$ other than solving equations?

Determinant and polynomial

Definition 2

For any $n \times n$ matrix A, the formula

$$\det(tI - A)$$

gives a polynomial of t. Call this **the characteristic polynomial** of A.

Theorem 1

(Calay Hamilton Theorem)For any n $\times$ n matrix A, the charactiersitc polynomial $\det(tI - A)$ is an annilating polynomial of it.

False proof $\det(tI - A)|_{t=A} = \det(AI - A) = \det 0 = 0$

Instead, we should expand $\det(tI - A)$ first before plug in the matrix A.

Determinant and polynomial

Let t be a variable.

Why $\det(tI - A)$ is a polynomial?

Determinant and polynomial

$$\det\begin{pmatrix} t-1 & 3 & 4 \\ 1 & t-2 & 3 \\ 1 & 1 & t-1 \end{pmatrix}$$

have at least 3 position with t variables.

Expand it by a column, then for each determinant there have at most 2 position with t variables.

$$(t-1)\det\begin{pmatrix} 1 & 3 & 4 \\ 0 & t-2 & 3 \\ 0 & 1 & t-1 \end{pmatrix} + 1\det\begin{pmatrix} 0 & 3 & 4 \\ 1 & t-2 & 3 \\ 0 & 1 & t-1 \end{pmatrix} + 1\det\begin{pmatrix} 0 & 3 & 4 \\ 0 & t-2 & 3 \\ 1 & 1 & t-1 \end{pmatrix}$$

Expanding a column containing variable t reduces the number of variables inside determinant.

Determinant and polynomial

Excercise. Find the characteristic polynomial of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\det(tI - A) = \det \begin{pmatrix} t-1 & -2 \\ -3 & t-4 \end{pmatrix} = (t-1)(t-4) - (-2)(-3) = t^2 - 5t - 2.$$

Polynomial division

Then why do we have

Theorem 2

(Calay Hamilton Theorem)For any n $\times$ n matrix A, the charactier-sitc polynomial $\det(tI - A)$ is an annilating polynomial of it.

False proof $\det(tI-A)|_{t=A} = \det(AI - A) = \det 0 = 0$ (You are only allowed to plug in numbers when writting this)

!!

In scalar coeffient polynomial, you are allowed to plug in matrics. In matrix coefficient polynomial, you can only plug in scalars! **Never plug in matrix to matrix coefficient polynomials!**

About the False proof

For example,

$$\det(tI_2) = \det\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = t^2$$

How do you plug in a matrix $t = B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$?

Is that $\det(BI_2)$??

Is that

$$\det\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}?$$

The above is closer, but it is not B^2, it is equal to $\det B^2$

You can only first calculate out the polynomial before plug that in.

Polynomial division

Example. When simplifiying fractions,

$$\frac{t^3+2}{t-1}$$

$$\begin{array}{rlllll}
 & t^2 & & +t & +1 \\
\cline{2-5}
t-1) & t^3 & & & +2 \\
 & t^3 & -t^2 & & \\
\cline{2-4}
 & & t^2 & & \\
 & & t^2 & -t & \\
\cline{3-5}
 & & & t & +2 \\
 & & & t & -1 \\
\cline{4-5}
 & & & & 3
\end{array}$$

$$\frac{t^3+2}{t-1} = t^2+t+1+\frac{3}{t-1}$$

3 is the value of pluging $t=1$ to t^3+2

Polynomial division

$$\frac{t^3+2}{t-1} = t^2+t+1+\frac{3}{t-1}$$

$$\iff$$

$$t^3+2 \qquad = \qquad \underbrace{(t-1)(t^2+t+1)}_{\text{This part vanishes when plugin} t=1} \qquad + \qquad 3$$

Polynomial division

Instead of pluging matrix to determinant. We first calculate out determinant, and then apply polynomial division.

For example

$$\det(tI - A) = \det \begin{pmatrix} t-1 & -2 \\ -3 & t-4 \end{pmatrix} = t^2 - 5t - 2$$

$$\begin{array}{r|lll} & & t & +(A-5) \\ \hline t - A) & t^2 & -5t & -2 \\ & t^2 & -At & \\ \hline & & (A-5)t & -2 \\ & & (A-5)t & A^2 - 5A \\ \hline & & & A^2 - 5A + 2 \end{array}$$

The remainder is exactly we plug in $t = A$ to the polynomial.

$$\frac{t^2 - 5t - 2}{t - A} = t + (A - 5) + \frac{A^2 - 5A + 2}{t - A}$$

Polynomial division

We already have

$$\frac{t^2 - 5t - 2}{t - A} = t + (A - 5) + \frac{A^2 - 5A + 2}{t - A}$$

Is that possible to have another decomposition??

$$\frac{t^2 - 5t - 2}{t - A} = Bt + C + \frac{D}{t - A}$$

Polynomial division

No, the polynomial division is uniquely determined, say, once

$$t + (A-5) + \frac{A^2 - 5A + 2}{t - A} = Et^2 + Bt + C + \frac{D}{t - A}$$

Then $E = 0$, $B = I_2$, $C = (A-5)$ and $D = A^2 - 5A + 2$.

$$\frac{t^2 - 5t - 2}{t - A} = t + (A-5) + \frac{A^2 - 5A + 2}{t - A}$$

Polynomial division

Theorem 3

Let $C_{-1}, C_0, ..., C_n$ and $D_{-1}, D_0, ..., D_n$ be matrices that satisfying the following expression

$$C_n t^n + C_{n-1} t^{n-1} + ... C_0 + C_{-1}(t - A)^{-1}$$

$$= D_n t^n + D_{n-1} t^{n-1} + ... D_0 + D_{-1}(t - A)^{-1}$$

Then $C_i = D_i$.

Proof: Dividing both side by t^n, we have

$$C_n + \frac{C_{n-1}}{t} + ... + \frac{C_0}{t^n} + \frac{C_{-1}(I - \frac{A}{t})^{-1}}{t^{n+1}}$$

$$= D_n + \frac{D_{n-1}}{t} + ... + \frac{D_0}{t^n} + \frac{D_{-1}(I - \frac{A}{t})^{-1}}{t^{n+1}}$$

Comparing this equality at limit $t \longrightarrow \infty$, we have $C_n = D_n$.

Polynomial division

Going back to the proof of Calay Hamilton theroem, letting

$$f(t) := \det(tI - A).$$

on one hand, we may decompose the polynomial generally

$$\frac{\det(tI - A)}{tI - A} = \frac{f(t)}{tI - A} = B_k t^k + ... + B_0 + \frac{f(A)}{tI - A}$$

Polynomial division

However, we have the adjugate formula.

$$(tI - A)^*(tI - A) = \det(tI - A)I,$$

which gives us another decomposition

$$\frac{\det(tI - A)}{tI - A} = (tI - A)^*$$

We claim that $(tI - A)^*$ MUST BE a polynomial!

$$(tI - A)^* = C_m t^m + ... + C_0 \qquad \text{for some } C_i.$$

Once we done this, then by comparing coefficients,

$$\begin{aligned} &C_m t^m + ... + C_0 \\ &= B_k t^k + ... + B_0 + \frac{f(A)}{tI - A} \end{aligned}$$

we obtain $f(A) = 0$, the Calay–Hamilton theorem.

Polynomial division

So why $(tI - A)$ is a polynomial?

Illustrate idea with an example: $$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Think about how you find $(tI - A)^*$

$$\underbrace{\begin{pmatrix} t-1 & 0 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t-1 \end{pmatrix}^*}_{(tI-A)^*} = \begin{pmatrix} (t-1)^2 & 0 & t-1 \\ 0 & t^2-2t & 0 \\ t-1 & 0 & (t-1)^2 \end{pmatrix}$$

Polynomial division

Therefore, we may always write $(tI - A)^*$ as matrix polynomial of t.

$$\begin{pmatrix} (t-1)^2 & 0 & t-1 \\ 0 & t^2-2t & 0 \\ t-1 & 0 & (t-1)^2 \end{pmatrix} = \begin{pmatrix} t^2-2t+1 & 0 & t-1 \\ 0 & t^2-2t & 0 \\ t-1 & 0 & t^2-2t+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} t^2 + \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} t + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Multiplinear expansion of determinant

Having proved that the characteristic polynomial $f(t) = \det(tI - A)$ is annihilating polynomial $f(A) = 0$, we want methods for calculating $\det(tI - A)$.

Multiplinear expansion of determinant

Determinant can be expanded once on one column

$$\det \begin{pmatrix} v + w & u \end{pmatrix} = \det \begin{pmatrix} v & u \end{pmatrix} + \det \begin{pmatrix} w & u \end{pmatrix}$$

Multiplinear expansion of determinant

If wanna expand in two columns, we expand one by one

$$\det\begin{pmatrix} v+w & u+k \end{pmatrix}$$

$$= \det\begin{pmatrix} v & u+k \end{pmatrix} + \det\begin{pmatrix} w & u+k \end{pmatrix}$$

$$= \det\begin{pmatrix} v & u \end{pmatrix} + \det\begin{pmatrix} v & k \end{pmatrix} + \det\begin{pmatrix} w & u \end{pmatrix} + \det\begin{pmatrix} w & k \end{pmatrix}$$

That is, to expand determinant from multiple columns, we add up all possible combinations of choosing a summand inside each factor.

Multiplinear expansion of determinant

This applies to the calculation of charcteristic polynomial.

Let $e_1, ..., e_n$ be natural basis and let $A = \begin{pmatrix} v_1 & ... & v_n \end{pmatrix}$

$$\det(tI - A) = \det \begin{pmatrix} te_1 - v_1 & te_2 - v_2 & ... & te_n - v_n \end{pmatrix}$$

Multiplinear expansion of determinant

Example of expansion $\det(tI_3 - A)$ for 3×3 matrix.

$$\det \begin{pmatrix} te_1 - v_1 & te_2 - v_2 & te_3 - v_3 \end{pmatrix}$$

$$=$$

$$\det \begin{pmatrix} te_1 & te_2 & te_3 \end{pmatrix}$$

$$+ \det \begin{pmatrix} -v_1 & te_2 & te_3 \end{pmatrix} + \det \begin{pmatrix} te_1 & -v_2 & te_3 \end{pmatrix} + \det \begin{pmatrix} te_1 & te_2 & -v_3 \end{pmatrix}$$

$$\det \begin{pmatrix} -v_1 & te_2 & e_3 \end{pmatrix} + \det \begin{pmatrix} te_1 & -v_2 & e_3 \end{pmatrix} + \det \begin{pmatrix} -v_1 & -v_2 & te_3 \end{pmatrix}$$

$$+ \det \begin{pmatrix} -v_1 & -v_2 & -v_3 \end{pmatrix}$$

Multiplinear expansion of determinant

We may factor out the sign and t

$$
\begin{gathered}
\det\begin{pmatrix} te_1 - v_1 & te_2 - v_2 & te_3 - v_3 \end{pmatrix} \\
= \\
\det\begin{pmatrix} te_1 & te_2 & te_3 \end{pmatrix} {\color{blue} t^3} \\
{\color{red} -} \left(\det\begin{pmatrix} v_1 & e_2 & e_3 \end{pmatrix} + \det\begin{pmatrix} e_1 & v_2 & e_3 \end{pmatrix} + \det\begin{pmatrix} e_1 & e_2 & v_3 \end{pmatrix}\right) {\color{blue} t^2} \\
{\color{red} +} \left(\det\begin{pmatrix} v_1 & e_2 & e_3 \end{pmatrix} + \det\begin{pmatrix} e_1 & v_2 & e_3 \end{pmatrix} + \det\begin{pmatrix} v_1 & v_2 & e_3 \end{pmatrix}\right) {\color{blue} t} \\
{\color{red} -} \left(\det\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\right)
\end{gathered}
$$

Multiplinear expansion of determinant

If

$$A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

as an example, then,

$$\det \begin{pmatrix} e_1 & v_2 & e_3 & v_4 \end{pmatrix} = \det \begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & 1 & * \\ 0 & * & 0 & * \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & * & 0 & * \end{pmatrix}$$

Strategy of calculating such determinant

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & * & 0 & * \end{pmatrix} = \det \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

This submatrix has its diagonal lies in the same diagonal of original matrix.

Multiplinear expansion of determinant

Definition 3

A principle minor of size k is the **determinant** of a $k \times k$ submatrix, whose diagonal coincide with the diagonal of its father.

Location of submatrices for principal minor of size 2 of a 3×3 matrix.

$$\begin{pmatrix} \boxed{* \quad *} & * \\ \boxed{* \quad *} & * \\ * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & \boxed{* \quad *} \\ * & \boxed{* \quad *} \end{pmatrix} \qquad \begin{pmatrix} \boxed{*} & * & \boxed{*} \\ * & * & * \\ \boxed{*} & * & \boxed{*} \end{pmatrix}$$

Location of submatrices for principal minor of size 1 of a 3×3 matrix.

$$\begin{pmatrix} \boxed{*} & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & \boxed{*} & * \\ * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & \boxed{*} \end{pmatrix}$$

Multiplinear expansion of determinant

Theorem 4

The characteristic polynomial of n $\times$ n matrix A has formula

$$\det(tI - A) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \ldots + (-1)^n a_n$$

where a_i is the **sum** of principal minors of size i. In particular $a_1 = \mathrm{tr}(A)$ and $a_n = \det(A)$

Multiplinear expansion of determinant

Excercise. Calculate the charcteristic polynomial for the following matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Lagurange interpolation polynomial

Excercise. Suppose $f(x) = x - c$. Given that $f(2) = 0$, determine the number c.

Lagurange interpolation polynomial

Excercise. Let f(x)=(x-a)(x-b). Determine the value of a,b such that

$$f(3) = 0 \qquad f(9) = 0$$

Lagurange interpolation polynomial

Excercise. Consdier the following exercises

Suppose

$$f(x) = \frac{(x-2)(x-5)(x-7)(x-9)}{(11-2)(11-5)(11-7)(11-9)}.$$

Please calculate the value of $f(x)$ to fill into the following value table

$x =$	$f(x) =$
2	
5	
7	
9	
11	

Lagurange interpolation polynomial

Using the idea from previous exercise. Please write down a polynomial $f(x)$ such that

$$f(2) = f(3) = f(5) = f(8) = 0, \qquad f(10) = 1.$$

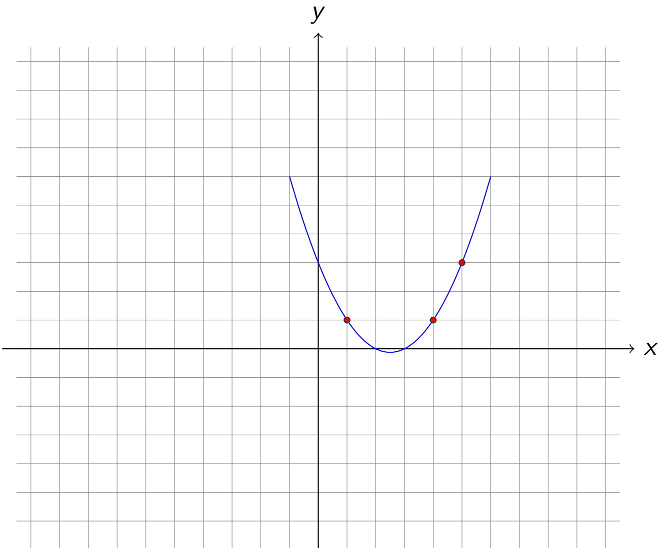
Lagurange interpolation polynomial

Write down polynomials $f_2(x), f_3(x), f_5(x), f_7(x)$ with the following value table

$x =$	$f_1(x) =$	$f_4(x) =$	$f_5(x) =$
1	1	0	0
4	0	1	0
5	0	0	1

Lagurange interpolation polynomial

Try to determine a polynomial with its graph passing through the following points



Lagurange interpolation polynomial

We are trying to find $g(x)$ with $g(1) = 1, g(4) = 1, g(5) = 3$.

$x =$	$f_1(x) =$	$f_4(x) =$	$f_5(x) =$	$g(x) =$
1	1	0	0	1
4	0	1	0	1
5	0	0	1	3

and suppose that

$$g(x) = a_1 \cdot f_1(x) + a_4 \cdot f_4(x) + a_5 \cdot f_5(x).$$

Determine the value of a_1, a_4, a_5.

$x =$	$f_1(x) =$	$f_4(x) =$	$f_5(x) =$	$a_1 \cdot f_1(x) + a_4 \cdot f_4(x) + a_5 \cdot f_5(x) =$
1	1	0	0	a_1
4	0	1	0	a_4
5	0	0	1	a_5

Lagurange Interpolation

Lemma 1

Let $g(t)$ be an arbitray polynomial, let $x_1, x_2, ..., x_n$ be distinct numbers. Let $f_{x_1}, ..., f_{x_n}$ be interpolation polynomials satisfying

$$f_{x_i}(x_j) = \begin{cases} 1 & x_i = x_j \\ 0 & x_i \neq x_j. \end{cases}$$

Then the polynomial

$$h(t) := g(t) - g(x_1)f_{x_1}(t) - g(x_2)f_{x_2}(t) - ... - g(x_n)f_{x_n}(t)$$

satisfies $h(x_1) = h(x_2) = ... = h(x_n) = 0$.

Lemma 2

If $h(x)$ is a polynomial satisfying $h(x_1) = h(x_2) = ... = h(x_n) = 0$ for distinct inputs $x_i \neq x_j$ for $i \neq j$, then $h(x)$ is divisible by $(x - x_1)...(x - x_n)$ in the sense that

$$h(t) = Q(t)(t - x_1)...(t - x_n)$$

for some polynomial Q.

Lagurange Interpolation

Explaination of the proof:

$$\frac{h(t)}{t-x_1} = \text{Some polynomial} + \frac{h(x_1)}{t-x_1}$$

Therefore

$$h(x_1) = 0 \qquad \Longrightarrow \qquad \frac{h(t)}{t-x_1} \text{ is a polynomial.}$$

Just write

$$h_2(t) = \frac{h(t)}{t-x_1},$$

then $h_2(x_2) = ... = h_2(x_n) = 0$. Use the method again and again, we know

$$\frac{h(t)}{(t-x_1)(t-x_2)\cdots(t-x_n)}$$

is a polynomial.

Theorem 5

Let $g(t)$ be arbitrary polynomials, given distinct points $x_1, ..., x_n$ and a choice of interpolation polynomials $f_{x_1}(t), ..., f_{x_n}(t)$ with $f_{x_j}(x_i) = 0$ for $i \neq j$ and $f_{x_i}(x_i) = 1$ for any $1 \leq i \leq n$, we can write

$$g(t) = Q(t)(t - x_1) \cdots (t - x_n) + g(x_1)f_{x_1}(t) + \cdots + g(x_n)f_{x_n}(t)$$

Lagurange Interpolation Theorem

Excercise. Write down an interpolation of t^5 at point $t = 1$ and $t = 2$.

Solution. First we pick interpolation polynomials.

$t =$	$2 - t$	$t - 1$	t^5
1	1	0	1
2	0	1	32

Then we can write

$$t^5 = Q(t)(t-1)(t-2) + (2-t) + 32(t-1)$$

for some polyonimal $Q(t)$.