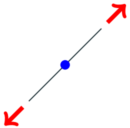
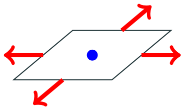
Vector Spaces

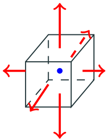
Geometrically, vector spaces are **infinitely extended**,**flat** spaces **with a selected origin**.



1-dim linear spaces



2-dim linear spaces



3-dim linear spaces

Here the red arrow means the actual object is extending, not only what you saw in the picture.

Vector Spaces

Definition 1

An $\mathbb{R}$-**vector space** is a set V, equipped with an operator "$+$" and scalar multiplication from $\mathbb{R}$ satisfying

1.0) For any $\vec{v}, \vec{w} \in V$, $\vec{v} + \vec{w}$ is an element in V;

1.2) For any $\vec{v}, \vec{w}, \vec{u} \in V$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$;

1.3) There exists $\vec{0} \in V$ such that for any $\vec{v} \in V$, $\vec{0} + \vec{v} = \vec{v}$;

1.4) For any $\vec{v} \in V$, There is $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = 0$;

2.0) For any $\vec{v} \in V$, $\lambda \in \mathbb{R}$, $\vec{v} \cdot \lambda$ defines an element in V;

2.1) For any $\vec{v} \in V$, $\vec{v} \cdot 1 = \vec{v}$;

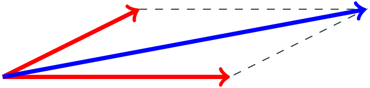
2.2) For any $\vec{v} \in V$, $\lambda, \mu \in \mathbb{R}$,$\vec{v} \cdot (\lambda\mu) = (\vec{v} \cdot \lambda) \cdot \mu$;

2.3) For any $\vec{v} \in V$, $\lambda, \mu \in \mathbb{R}$,$\vec{v} \cdot (\lambda + \mu) = \vec{v} \cdot \lambda + \vec{v} \cdot \mu$;

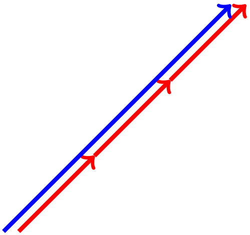
2.4) For any $\vec{v}, \vec{w} \in V$, $\lambda \in \mathbb{R}$,$(\vec{v} + \vec{w}) \cdot \lambda = \vec{v} \cdot \lambda + \vec{w} \cdot \lambda$.

Examples of vector spaces–Geometry

The flat, infinitely extended lines or planes with selected origins are the most common vector spaces, where each point is identified with the end point of each vector starting from the origin. The addition is defined by the parallelogram law and the scalar multiplication is defined by the usual scaling.



parallelogram law



scalar multiplication

Question: What is zero vector in this space?

Abstraction

Not all vector spaces can be described in such a geometric and intuitive way, but this is the only way for data visualization.

Shinchan has a coffee shop. He wants to treat the recipe of each dish as a vector intuitively, what should he do?

Abstraction

 : Are they vectors also? , , .

: There are no arrows like $\longrightarrow$.

 : But I'd still say — vectors!

: If you like.

 : Actually, you can, as long as you define the addition and the scalar multiplication properly! Everything can be a vector.

Abstraction

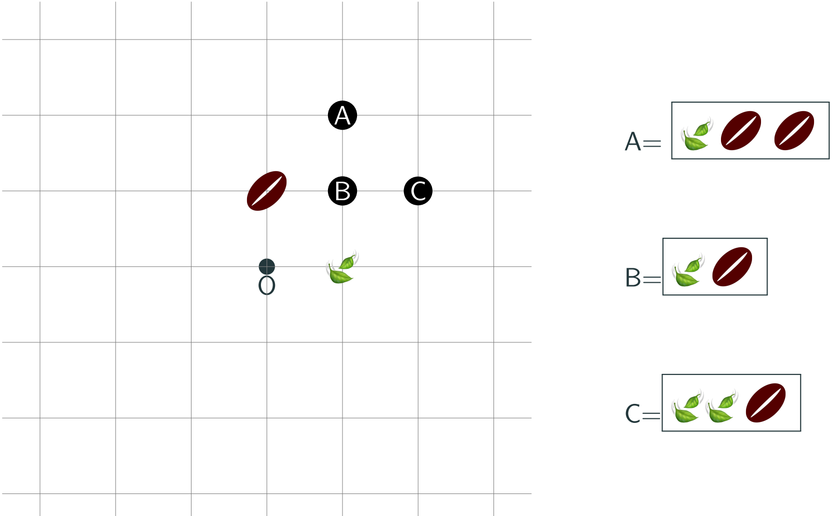
 : I don′t believe, how? Show me arrows!

 : I can do it! Let me show you!

Vector space: Package of Materials

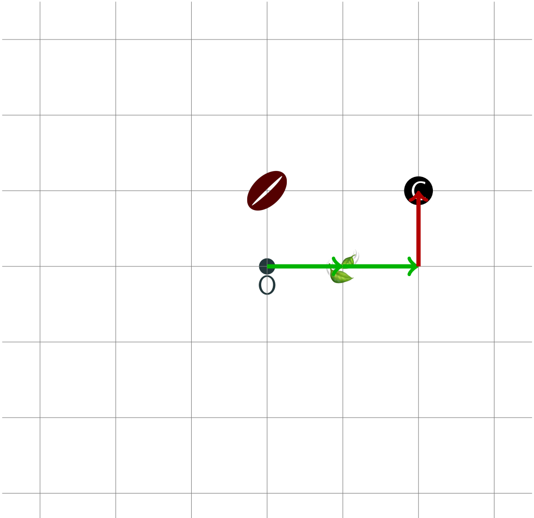
Abstract vector spaces can be visualized by linear combinations.

Here is the way how Shinchan represent each of the vegetable package intuitively in the space. How?



Vector space: Package of Materials

As long as your definition of vector spaces satiesfy these axioms, it is always valid to substitute your data by some arrows. For example, position C is obtained by replacing 🍃 by $\rightarrow$, and 🫘 by $\uparrow$.



C= $\boxed{🍃🍃🫘}$

🍃:$\rightarrow$ 🫘:$\uparrow$

C: $\boxed{\rightarrow\rightarrow\uparrow}$

Abstract vector space v.s. intuitive vector space

You can always visualize a vector space by an intuitive vector space by this method. But in general, **they are not the same**. It is simply because you can only visualize it after a choice of replacement by arrows, but the element in a vector space itself should not be always arrows — it could be vegetables, human, airplane or whatever as long as you can define addition and scalar multiplication that satisfy axioms.

Examples of vector spaces–$n \times 1$ matrices

The set of all $n \times 1$ matrices over $\mathbb{R}$ is denoted by $\mathbb{R}^n$, which is a vector space by the following structure: The sum of two matrices is just the normal sums. the scalar multiplication is naturally obtained by scaling each entries. For example.

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \text{ vector addition}$$

$$\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} 5 = \begin{pmatrix} 25 \\ 5 \\ 10 \end{pmatrix} \text{ scalar multiplication,}$$

Question: What is zero vector in this space?

Examples of vector spaces–Polynomials

Let $P_{2,x} = \{ax^2 + bx + c, \text{ where } a, b, c \in F\}$ be the set of polynomials of degree at most 2 with real number coefficients. It is a vector space over $\mathbb{R}$ in the following sense: The addition structure is the sum of two polynomials, the scalar multiplication is multiplying a constant.

$$(x^2 + 1) + x = x^2 + x + 1 \text{ vector addition}$$

$$3(x^2 + 1) = 3x^2 + 3 \text{ scalar multiplication}$$

Question: What is zero vector in this space?

Example of vector spces – Rational functions

A rational function is a quotient of two polynomials. For example, the set

$$\left\{ f : f(x) = \frac{x^2 a + xb + c}{(x+1)(x+2)(x+3)} \text{ for some } a, b, c \in \mathbb{R} \right\}$$

is a vector space over $\mathbb{R}$ in the following sense: The addition of two vectors is the sum of two functions,

$$\frac{1}{x+1} + \frac{1}{x+2} = \frac{2x+3}{(x+1)(x+2)};$$

The scaling is defined by multyplying functions with constants,

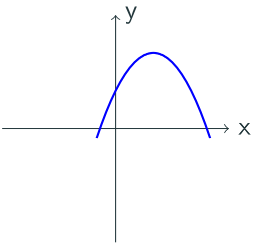
$$3 \cdot \frac{x}{x+1} = \frac{3x}{x+1}.$$

Question: What is zero vector in this space?

Example of vector spces – Rational functions

Previously you might think vectors are like this $\longrightarrow$

Now I tell you a vector might look like this (the following is a graph of $4 - x^2$).



As long as we defined what is addition and scalar multiplication, everything can be a vector!

Matrix notation for linear combination

Vector space is a playground of linear combination of vectors. Those vectors might be arbitrary. We need **matrix** to organize them.

Matrix notation for linear combination

Come back to previous list

	🥛	☕	🍵	🥤
🍃	0	0	2	4
🍋	0	0	1	2
🫘	0	2	0	2
🐄	1	0	0	4

Why not think an object 🥛 as created out of 1? just think

	🍃	🍋	🫘	🐄
1	🍃	🍋	🫘	🐄

Indeed, 🍃 = 🍃 × 1; 🍋 = 🍋 × 1; ☕ = ☕ × 1; 🐄 = 🐄 × 1...

Matrix notation for linear combination

When write a thing made out of 1, it made by multiply 1 with the coefficient as itself. So we have

	🍃	🍋	🫘	🐄
1	🍃	🍋	🫘	🐄

$\times$

	🥛	☕	🍵	🥤
🍃	0	0	2	4
🍋	0	0	1	2
🫘	0	2	0	2
🐄	1	0	0	4

$=$

	🥛	☕	🍵	🥤
1	🥛	☕	🍵	🥤

This shows the following expression is **valid**

$$\begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} & \text{🥤} \end{pmatrix}$$

This is an important way to represent the ingradients mathematically.

Change of materials

The imporance of this symbol is not only because it shows the materials and products clear. It also represents in a natural way so that **Combination of ingradients is the same as play substitution to the factors**. Go back to our original example

	🥛	☕	🍵
🍃	0	0	2
🍋	0	0	1
🫘	0	2	0
🐄	1	0	0

	🍱	🍲
🥛	2	1
☕	0	2
🍵	1	1

	🍱	🍲
🍃		
🍋		
🫘		
🐄		

Change of materials

We can simply write this is a question

$$\begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix} = \begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} \text{🍱} & \text{🍲} \end{pmatrix} = \begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} \text{🍱} & \text{🍲} \end{pmatrix} = \begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \times ? \tag{3}$$

Change of materials

To get questionmark in (**??**) is easy, just use (**??**) to substitute $\begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix}$ part in (**??**), we got

$$
\begin{pmatrix} \text{🍱} & \text{🍲} \end{pmatrix} = \begin{pmatrix} \text{🥛} & \text{☕} & \text{🍵} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \text{🍃} & \text{🍋} & \text{🫘} & \text{🐄} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}
$$

Therefore, the questionmark is gien by the matrix product. This method of **substitution** is a very important strategy and will be **repeatedly used in our course**. Make sure you familiar with it.

Change of materials

We end up this lecture by showing a math example.

Excercise: Suppose we have the following expression

$$\begin{cases} \vec{v}_1 = \vec{e}_2 + 2\vec{e}_3 \\ \vec{v}_2 = \vec{e}_3 \\ \vec{v}_3 = \vec{e}_1 \end{cases} \qquad \begin{cases} \vec{w}_1 = 2\vec{e}_1 + \vec{e}_2 + 3\vec{e}_3 \\ \vec{w}_2 = 4\vec{e}_1 + \vec{e}_2 + \vec{e}_3 \\ \vec{w}_3 = 8\vec{e}_1 + \vec{e}_2 \end{cases}$$

Write $\vec{w}_1, \vec{w}_2, \vec{w}_3$ as linear combinations of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Change of materials

From what given, we write

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

Note that the right-side matrix is invertible, therefore

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \tag{4}$$

Also from the given equation, we write

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 8 \\ 1 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix} \tag{5}$$

Change of materials

We replace (**??**) into (**??**) for $\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix}$

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 4 & 8 \\ 1 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Doing simutaneous row reduction to factors of the form $A^{-1}B$, we have

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 2 & 4 & 8 \end{pmatrix}$$

This means

$$\begin{cases} \vec{w}_1 = \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 \\ \vec{w}_2 = \vec{v}_1 - \vec{v}_2 + 4\vec{v}_3 \\ \vec{w}_3 = \vec{v}_1 - 2\vec{v}_2 + 8\vec{v}_3 \end{cases}$$

Subspace

Definition 2

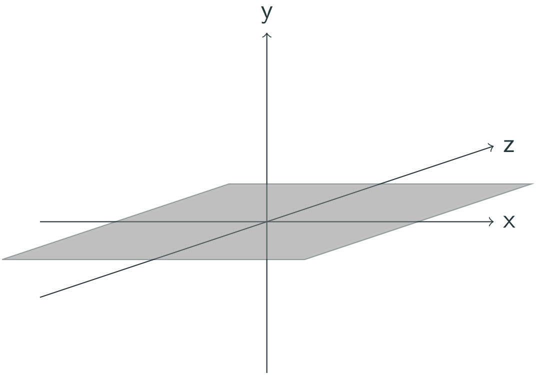
A subspace $W \subset V$ is a **non-empty** subset that is closed under addition and scalar multiplication. In other words,

$$\vec{v} + \vec{w} \in W \text{ for any } \vec{v}, \vec{w} \in W$$

$$\lambda\vec{v} \in W \text{ for any } \vec{v} \in W, \lambda \in \mathbb{R}.$$

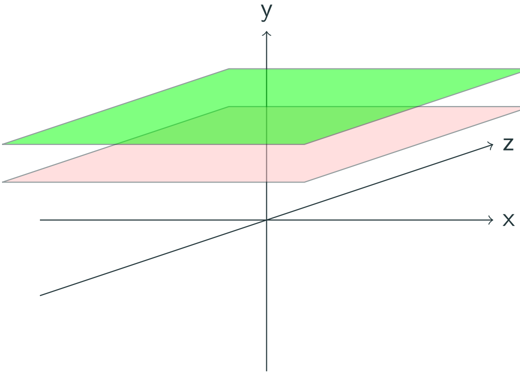
Subspace

Intuitively speaking, subspace is sub-space. Like a line inside a plane, or a plane inside the cube. But, **we require that subspaces must passes though origin**. Because linear spaces are spaces with origin chosen, and their origin should match.



Subspace

A subspace must pass through the origin, for example, the following subsets, altough they seems **flat** and **infinitely extended**, they are not **NOT** a subspace because they do notpass the origin.



Ways of constructing subspaces

We introduce two types of representing subspaces.

Now I give you an example of **constructive language** in our daily life.

Ways of constructing subspaces

Shinchan calls his mom to pick up some book for him.

Shinchan:*Mom, give me* **the** **third** *book* **at the second level of bookshelf!**

Misae:*Here you are* 📕*!*

In this situation, Shinchan is specifying this book by telling his mom a parameter to locate the book directly. We call this way of specifying an objects as **constructive language** $\boxed{L_S}$.

Subspace by parametric equations

In $\mathbb{R}^3$, we may represent a subspace of it in the following way

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} s, \text{for some } t, s \in \mathbb{R} \right\}$$

Here we are using t and s as parameters for

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

This subspace is constructed by using **constructive language** with **parameters** with parametric equation

$$\begin{cases} x = t + s \\ y = 2t + s \\ z = t. \end{cases}$$

Subspace by parametric equations

Note that the equation can be written into matrix form, we define A as in

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} s = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} t \\ s \end{pmatrix}$$

Therefore W is the set of **all possible linear combination of columns of** A, we call this the **Column space** of A. We can write

$$W = \left\{ A\vec{v} : \text{ for some } \vec{v} \in \mathbb{R}^2 \right\}.$$

Definition 3

$\boxed{L_S}$ The column space of a $n \times m$ matrix A is a subspace of $\mathbb{R}^n$ defined by $\{A\vec{v} : \vec{v} \in \mathbb{R}^m\}$.

Subspace by parametric equations

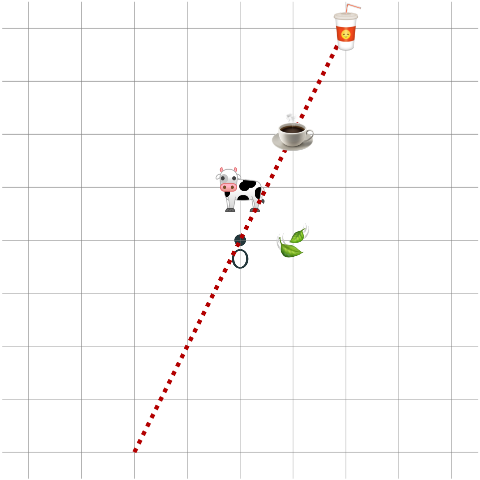
Shinchan is operating a coffee shop. He has the following Recipe Table.

	☕	🥤
🍃	1	2
🐄	2	4

What is the column space of this matrix corresponds to?

Subspace by parametric equations

It corresponds to all possible matrial combinations to make an arbitrary drink combinations! **Space for all possible drink!**



Subspaces spanned by vectors

Next, we will learn the first set of properties for vectors in the linear space. They are being **span the whole space** and being **linealy independent** , corresponding to **uniqueness** and **existence** .

Subspaces spanned by vectors

Definition 4

$\boxed{L_S}$ The subspace **spanned by** $\{v_1, v_2, \cdots, v_n\}$ is the subset of all possible linear combinations of $\{v_1, v_2, \cdots, v_n\}$. Denoted as $span\{v_1, v_2, \cdots, v_n\}$.

Matrix Form of Linear Combination

The linear combination

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

can be written by multiplication of two matrices:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Therefore we can write

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} = \left\{ \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathrm{F}^n \right\}$$

Matrix Form of Linear Combination

Excercise. Why $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a subspace of V?

Matrix Form of Linear Combination

Definition 5

$\boxed{S}\,\boxed{L_S}$ We call the subset $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \subset V$ **span the whole space** if

$$V = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$$

Proposition 1

$\boxed{S}$ If the set $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \subset V$ **span the whole space** , then for any vector $\vec{v} \in V$, **there exists** (**but may not unique**)a coefficient list

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathrm{F}^n$$

such that

$$\vec{v} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Strategies of proving spanning the whole space

- For any two sets X, Y, if we want to show $X = Y$, then it suffices to show $X \subset Y$ and $Y \subset X$.
- To show $X \subset Y$, it suffices to show that for any $x \in X$, one has $x \in Y$.
- If the set X is defined by

$$X = \{t : t \text{ has some perperty } p\},$$

then by assuming $x \in X$ it is automatically true that x satisfies the property p. Such strategies have been used everywhere in the proof.

Strategies of proving spanning the whole space

Proposition 2

Let V be a vector space and $U, W \subset V$ are two subsets of V. If $U \subset W$, then

$$\mathrm{span}\ U \subset \mathrm{span} W.$$

Proof.

For any $\vec{u} \in \mathrm{span} U$, there are some scalars $a_1, \cdots, a_n \in F$ such that

$$\vec{u} = a_1 \vec{u}_1 + \cdots + a_n \vec{u}_n \tag{6}$$

for some $\vec{u}_1, \cdots, \vec{u}_n \in U$. Since $U \subset W$, we also have $\vec{u}_1, \cdots, \vec{u}_n \in W$. Therefore (**??**) is also a linear combination of elements in W, and so $\vec{u} \in W$. Then we have $U \subset W$. □

Strategies of proving spanning the whole space

Excercise. Suppose V is a vector space and $U, W \subset V$ are subsets. Are the following statements true? Write down a proof if true, give a counter example if false.

- $\mathrm{span}\emptyset = \{\vec{0}\}$;
- $\mathrm{span}\ \mathrm{span}\ U = \mathrm{span}\ U$;
- $U = V \implies \mathrm{span}\ U = \mathrm{span}\ V$;
- $\mathrm{span}\ U = \mathrm{span}\ V \implies U = V$;
- $\mathrm{span}(U \cap V) \subset \mathrm{span}\ U \cap \mathrm{span}\ V$;
- $\mathrm{span}(U \cap V) \supset \mathrm{span}\ U \cap \mathrm{span}\ V$.

Strategies of proving spanning the whole space

Problem: Consider the following vector space

$$V = \{f : f(x) = \frac{ax+b}{x^2-1}, a, b \in \mathbb{R}\}.$$

Show that the following vectors

$$\frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1}$$

span the whole space .

Strategies of proving spanning the whole space

Proof: By definition, we need to show all elements of V can be written as a linear combination of those three vectors. Note that

$$\frac{1}{x^2-1} = 0 \cdot \frac{1}{x+1} + 0 \cdot \frac{1}{x-1} + 1 \cdot \frac{1}{x^2-1}$$

and that

$$\frac{x}{x^2-1} = \frac{1}{2} \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{1}{x-1} + 0 \cdot \frac{1}{x^2-1}.$$

Then for any $\vec{v} \in V$, we can find two scalars $a, b \in \mathbb{R}$, such that

$$\begin{aligned} \vec{v} &= \frac{ax+b}{x^2-1} \\ &= a \cdot \frac{1}{x^2-1} + b \cdot \frac{x}{x^2-1} \\ &= \frac{b}{2} \cdot \frac{1}{x+1} + \frac{b}{2} \cdot \frac{1}{x-1} + a \cdot \frac{1}{x^2-1} \\ &\in \operatorname{span}\left\{ \frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1} \right\}. \end{aligned}$$

Strategies of proving spanning the whole space

Continue proof: Therefore $\vec{v} \in \mathrm{span}\left\{\frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1}\right\}$, which implies that

$$V \subset \mathrm{span}\left\{\frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1}\right\}.$$

Since

$$\frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1} \in V,$$

we also have

$$\mathrm{span}\left\{\frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1}\right\} \subset V.$$

Therefore

$$V = \mathrm{span}\left\{\frac{1}{x+1}, \frac{1}{x-1}, \frac{1}{x^2-1}\right\}.$$

Turning point

We want a set of vectors **span the whole space** because we want **there exists** a coefficient list to represent the vector. To **span the whole space** , the number of vectors should be as large as possible to guarantee **existence** of coefficients.

On the other hand, there might be too many possible ways to write down coefficient list to represent a vector. We need to save our time and choose the most efficient way to span the whole space. The number of vectors should be as smaller as possible to guarantee **uniqueness** of coefficients. 1

Subspace cutting out by equations

Our first goal in this lecture is discussing two ways to represent a subspace.

The first way of representing a subspace is writing equations for it. It gives the subspace by **describing certain properties**, then formulate a subset by collecting all points with such a property. In our slides, we call this language as **descriptive language** .

Now I give you an example of **descriptive language** in our daily life.

Subspace cutting out by equations

Shinchan′s mom, Misae, find a book 📘somewhere. She says

Misae: *Hey, what is this book* 📘*?*

Shinchan: *This is* **a** *book* **in my comics drawer***.*

In this situation, Shinchan is describing this book. We call this way of describing an objects as **descriptive language** $\boxed{L_l}$.

Subspace cutting out by equations

In F^3, we may represent a subspace of it in the following way

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x + y = 0, 2x + y + z = 0 \right\}$$

Here equations $2x + y = 0$ and $2x + y + z = 0$ are describing the properties of our desired point

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This subspace W is a subspace constructed by **descriptive language** . In this slides, we also call a subspace given by this form as a **Standard Equation Form**.

Subspace cutting out by equations

When working on an abstract subspace V, we need to choose a basis $\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$. Then we could represent a subspace of it by giving equations on its coorinate.

$$W = \left\{ \vec{v} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in V : P \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Excercise. Verify a subset defined by this way is actually a subspace.

Value Table and descriptive Language

In our slides, we often uses two kinds of tables. One of them is the **Value Table**

f	1	x	x^2
2	1	2	4
3	1	3	9

Describing objects using **Value Table** is a **descriptive language** . Each entry of the table is describing a property of the object.

Summary

Value Tables are **descriptive language** .

Value Table and descriptive Language

Shinchan is operating a coffee shop. He has the following Value Table.

	☕	🥤	🍵
Price	3	2	3
Weight	1	1	2

Shinchan would like to know all possible combinations of ☕, 🥤 ,🍵such that one has total price 0 and Weight 0. This gives a subspace in the space of linear combinations of ☕, 🥤 ,🍵. This subspace is described by **descriptive language** because Shinchan cut it out by describing its total weight and total price.

Excercise. Can you find out the subspace described by this page?