

Note: Preview of slides from (leftrightinverseoflineartransformations.tex) by Qirui Li (<https://orcid.org/0000-0002-6042-1291>). For educational and non-commercial use only. Any unlawful use will be prosecuted.

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# Left and right inverse for Linear Transformations

We add those pages for completeness, read those proofs by yourself as a homework. And try to prove it directly without using induced transformation and cancellation rule.

# Left Inverse for injective

## Proposition 1

$R_I$  Let  $V, W$  be finite dimensional vector spaces. If  $T : V \longrightarrow W$  is an **injective** linear transformation, then it has a **left inverse**  $R : W \longrightarrow V$  with

$$R \circ T = \text{id}_V$$

**Proof:** Let  $\mathcal{E}$  be a basis of  $V$ , since  $T$  is **injective**, then  $T\mathcal{E}$  is also **linearly independent** in  $W$  (By Prop.  $C_I$ ). Therefore there exists a linear transformation  $R : W \longrightarrow V$  with  $R(T\mathcal{E}) = \mathcal{E}$  (By Prop.  $I \mapsto S$ ), therefore,

$$R \circ T \circ L_{\mathcal{E}} = L_{\mathcal{E}} = \text{id}_V \circ L_{\mathcal{E}}$$

Since  $\mathcal{E}$  **span the whole space**,  $L_{\mathcal{E}}$  is a **surjective**, use the **right cancellation** rule we have

$$R \circ T = \text{id}_V.$$

## Right Inverse for surjective

### Proposition 2

$R_5$  Let  $V, W$  be finite dimensional vector spaces. If  $T : V \rightarrow W$  is an **surjective** linear transformation, then it has a **right inverse**  $R : W \rightarrow V$  with

$$T \circ R = \text{id}_W$$

**Proof:** Let  $\mathcal{E}$  be a basis of  $V$ , since  $T$  is **surjective**, then  $T\mathcal{E}$  **span the whole space**  $W$  (By Prop.  $C_5$ ). Let  $\mathcal{F}$  be a subtuple of  $\mathcal{E}$  such that  $T\mathcal{F}$  is a **basis** of  $W$ . Then there exists a linear transformation  $R : W \rightarrow V$  with  $R(T\mathcal{F}) = \mathcal{F}$  (By Prop.  $I \mapsto S$ ). So  $T(R(T\mathcal{F})) = T\mathcal{F}$

Now we have

$$T \circ R \circ L_{T\mathcal{F}} = L_{T\mathcal{F}} = \text{id}_W \circ L_{T\mathcal{F}}$$

Since  $T\mathcal{F}$  **span the whole space**,  $L_{T\mathcal{F}}$  is a **surjective**, by **right cancellation** rule we have  $T \circ R = \text{id}_W$ .