Change of basis and commutative diagram

From now, we have related the concept of linear transformation , tuple of vectors, matrices together.

Now we would start to talk about different perspective in linear algebra. For abstract vector spaces, we can only take a coordinate of an element after choosing a basis. Basis is a perspective, coordinates are phinomenon you have seen through this perspective. Different perspective will give different phinomenon. But all of them are describing the same truth.

Commutative Diagram

In math, we use **commutative diagram** to discribe different phinomenon of the same truth in different perspective.

Definition 1

A commutative diagram is a collection of maps, in which all map compositions starting from the same set and ending with the same set give the same result.

Commutative Diagram

For example, when we say the following diagram commutes

$$
\begin{tikzcd}
A \arrow[rrr, "\alpha"] \arrow[dr, "\gamma"] \arrow[dd, "\beta"'] & & & B \arrow[dl, "\epsilon"'] \arrow[dd, "\delta"] \\
& C \arrow[r, "\zeta"] & D & \\
E \arrow[ur, "\iota"] \arrow[rrr, "\theta"] & & & F \arrow[ul, "\mu"']
\end{tikzcd}
$$

we mean that we have $\epsilon \circ \alpha = \zeta \circ \gamma$, and $\epsilon = \mu \circ \delta$ and so on

Excercise. Write more equations of map compositions from this diagram.

Commutative Diagram

The definition of a commutative diagram itself does not reveal any purpose of interpreting a different phinomenon in different perspective. We need to understand it by specific examples. In this class, we are trying to understand commutative diagrams of the following shape.

Translation of Language:

$$\begin{tikzcd} {} \arrow[rr] \arrow[dr] & & {} \arrow[dl] \\ & {} & \end{tikzcd}$$

Phinomenons in different perspective:

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

Commutative diagram of triangular shape

We study the commutative diagram of triangular shape, the philosophy is listed as following

$$\begin{tikzcd} A^{P_1}_{T} \arrow[rr, "\text{translation}"] \arrow[dr, "\text{parametrization 1}"'] & & B_{P_2} \arrow[dl, "\text{parametrization 2}"] \\ & C & \end{tikzcd}$$

As a map, this means $P_1 = P_2 \circ T$

Typically, such type of commutative diagram represents a **Translation of parameters**. Here A, B are both paramter sets, C is the set of objects.

Commutative diagram of triangular shape

Intuitive examples from Language. Consider **commutative diagram**

$$
\begin{tikzcd}
\text{Japanese words of fruit} \arrow[rr, "\text{translate}"] \arrow[dr, "\text{tell in japanese}"'] & & \text{English words of fruit} \arrow[dl, "\text{tell in English}"] \\
& \text{Set of Fruits} &
\end{tikzcd}
$$

For example, 🍎=tell in Japanese(lingo) = tell in English(apple)

The lingo and apple are two different parameter for the same object 🍎in two different parametrization.

Then two words apple and lingo are related by translation

$$\text{apple} = \text{translate}(\text{lingo}).$$

Commutative diagram of triangular shape

Another example, as we all know, grandfather is the dad of dad. We have the following diagram

$$\begin{tikzcd} \text{Creatures existed} \arrow[rr, "\text{daddy}"] \arrow[dr, "\text{grandfather}"] & & \text{Creatures existed} \arrow[dl, "\text{daddy}"'] \\ & \text{Creatures existed} & \end{tikzcd}$$

Commutative diagram of triangular shape

$$\begin{tikzcd}
\text{Creatures existed} \arrow[rr, "\text{daddy}"] \arrow[dr, "\color{blue}\text{grandfather}"'] & & \text{Creatures existed} \arrow[dl, "\color{blue}\text{daddy}"] \\
& \text{Creatures existed} &
\end{tikzcd}$$

We understand the blue maps as parametrizations.

In parametrization of ‘grandfater‘, we describe every objects by

Jonh′s grandfater Amy′s grandfater ⋯

In parametrization of ‘daddy‘, we describe every objects by

Ricky′s daddy Speedy′s daddy

Question: How do we translate between those langrage?

Commutative diagram of triangular shape

To see relation between two laguages, compare if

$$\text{Ricky}'\text{s daddy} = \text{Jonh}'\text{s grandfater}$$

What is the relation betweeen parameter ‘Ricky‘ and ‘John‘?

Commutative diagram of triangular shape

$$\text{Ricky's daddy} = \text{John's grandfater}$$

implies

$$\text{Ricky} = \text{John's daddy}$$

Therefore, daddy plays the role of Translation of parameters

$$\begin{tikzcd}
\text{Creatures existed} \arrow[rr, "\text{daddy}"] \arrow[dr, "\text{grandfather}"] & & \text{Creatures existed} \arrow[dl, "\text{daddy}"'] \\
& \text{Creatures existed} &
\end{tikzcd}$$

Commutative diagram of triangular shape

Suppose we have a vector space V with two bases and the change of basis matrix P

$$\underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix} P}_{\mathcal{F}}$$

From the perspective of induced transformations this means

$$L_{\mathcal{E}} = L_{\mathcal{F}} \circ L_P$$

We can write this equation into the follwing commutative diagram of induced transformations.

$$\begin{tikzcd} \mathrm{F}^n \arrow[rr, "L_P"] \arrow[dr, "L_{\mathcal{E}}"'] & & \mathrm{F}^m \arrow[dl, "L_{\mathcal{F}}"] \\ & V & \end{tikzcd}$$

Commutative diagram of triangular shape

$$\begin{tikzcd} \mathbb{F}^n \arrow[rr, "L_P"] \arrow[dr, "L_{\mathcal{E}}"'] & & \mathbb{F}^m \arrow[dl, "L_{\mathcal{F}}"] \\ & V & \end{tikzcd}$$

This commutative diagram can be interprete as that, **left multiplying the change of basis matrix** P on **coordinates in** $\mathcal{E}$-basis will give the coordinate of the vector in $\mathcal{F}$-basis.

Let′s verify: Since $L_{\mathcal{E}} = L_{\mathcal{F}} \circ L_P$, we have $L_P \circ L_{\mathcal{E}}^{-1} = L_{\mathcal{F}}^{-1}$. Then

$$\overbrace{L_P \circ \underbrace{L_{\mathcal{E}}^{-1}(\vec{v})}_{\text{coordinate in } \mathcal{E}-\text{basis}}}^{\text{multiplying the coordinate by } P} = \underbrace{L_{\mathcal{F}}^{-1}(\vec{v})}_{\text{coordinate in } \mathcal{F}-\text{basis}} \tag{1}$$

Commutative diagram of square shape

A commutative diagram of square shape, is **describing different phenomena for the same truth in different points of view**

Experiment: Face to face with your partner, put your cellphone in the middle of your views. and rotate it clockwise. Ask your partner about which direction of the rotation in her point of view.

$$\begin{tikzcd}
\text{Images you saw} \arrow[r, "\text{rotation clockwise}"] \arrow[d, "\text{translation}"'] & \text{Images you saw} \arrow[d, "\text{translation}"] \\
\text{Images she saw} \arrow[r, "\text{rotation counterclockwise}"] & \text{Images she saw}
\end{tikzcd}$$

Commutative diagram of square shape

$$\begin{tikzcd}
\text{Images you saw} \arrow[r, "\text{rotation clockwise}"] \arrow[d, "\text{flip over}"'] & \text{Images you saw} \arrow[d, "\text{flip over}"] \\
\text{Images she saw} \arrow[r, "\text{rotation counterclockwise}"] & \text{Images she saw}
\end{tikzcd}$$

Question: What is the translation map from what you saw to what she saw? Without changing seats, can you do some thing to your cellphone to visualize what she saw?

Answer: Flip over the cell phone, what you saw is exactly what she saw before. This means the translation map is flipping over.

Commutative diagram of square shape

$$\begin{tikzcd}
\text{Images you saw} \arrow[r, "\text{rotation clockwise}"] \arrow[d, "\text{flip over}"'] & \text{Images you saw} \arrow[d, "\text{flip over}"] \\
\text{Images she saw} \arrow[r, "\text{rotation counterclockwise}"] & \text{Images she saw}
\end{tikzcd}$$

Now let′s verify it is a commutative diagram by yourself:

Rotation clockwise by 90 degree then **Flip Over**

= **Flip Over** then **Rotation counterclockwise by 90 degree**

Commutative diagram of square shape

If we have a lienar transformation $T : V \longrightarrow W$ and a basis $\mathcal{E} = \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}$ in V and a basis $\mathcal{F} = \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}$ in W. The matrix representation P of T is described by the following equation

$$T \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} P$$

In the perspective of induced linear transformation , this means

$$L_T \circ L_{\mathcal{E}} = L_{\mathcal{F}} \circ L_P$$

By drawing the domain and codomain for each linear transformation . We can draw this equation into a commutative diagram

$$\begin{tikzcd} F^n \arrow[r, "L_P"] \arrow[d, "L_{\mathcal{E}}"'] & F^m \arrow[d, "L_{\mathcal{F}}"] \\ V \arrow[r, "T"'] & W \end{tikzcd}$$

Commutative diagram of square shape

We can understand each row of the above diagram as a viewpoint, and vertical maps as translations(translate coordinate to actual vectors).

$$\begin{tikzcd} \text{View from coordinate:} & F^n \arrow[r, "L_P"] \arrow[d, "L_{\mathcal{E}}"'] & F^m \arrow[d, "L_{\mathcal{F}}"] \\ \text{View from actual map:} & V \arrow[r, "T"'] & W \end{tikzcd}$$

it is saying applying the linear transformation T, in the view of coordinates, is exactly like left multiplying the matrix P.

Indeed, since $T \circ L_{\mathcal{E}} = L_{\mathcal{F}} \circ L_P$, we have $L_{\mathcal{F}}^{-1} \circ T = L_P \circ L_{\mathcal{E}}^{-1}$. Then

$$\underbrace{L_{\mathcal{F}}^{-1}(T(\vec{v}))}_{\mathcal{F}-\text{coordinate of } T(\vec{v})} = \overbrace{L_P(\quad \underbrace{L_{\mathcal{E}}^{-1}(\vec{v})}_{\mathcal{E}-\text{coordinate of } \vec{v}} \quad)}^{\text{left multiplying } P \text{ on the } \mathcal{E}-\text{coordinate of } \vec{v}}$$

Change of basis for matrix representation

Let $T : V \longrightarrow W$ be a linear transformation . With selected basis $\mathcal{E}$ in V and $\mathcal{F}$ in W, we may write the matrix represenation of T

$$T \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} P$$

The matrix P depends on the choice of bases. Suppose we change another basis $\mathcal{G}$ in V and $\mathcal{H}$ in W, the matrix changes to

$$T \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}}_{\mathcal{H}} Q$$

What is the realtion between P and Q?

Change of basis for matrix representation

In fact, we can write the following expression

$$T \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} P$$

into a commutative diagram of induced transformations

View from coordinate:

View from actual map:

$$\begin{tikzcd} F^n \arrow[rrr, "L_P"] \arrow[dr, "L_{\mathcal{E}}"'] & & & F^m \arrow[dl, "L_{\mathcal{F}}"] \\ & V \arrow[r, "T"'] & W & \end{tikzcd}$$

Change of basis for matrix representation

Then consider the expression of Q

$$T\underbrace{\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix}}_{\mathcal{H}} Q$$

We may draw it into the previous commutative diagram

$$\begin{tikzcd}
\text{View from coordinate:} & F^n \arrow[dr, "L_{\mathcal{E}}"'] \arrow[rrr, "L_P"] & & & F^m \arrow[dl, "L_{\mathcal{F}}"] \\
\text{View from actual map:} & & V \arrow[r, "T"'] & W & \\
\text{View from coordinate:} & F^n \arrow[ur, "L_{\mathcal{G}}"] \arrow[rrr, "L_Q"] & & & F^m \arrow[ul, "L_{\mathcal{H}}"']
\end{tikzcd}$$

Change of basis for matrix representation

Since $\mathcal{E}, \mathcal{G}$ are bases of V, they must related by a change of basis matrix.

$$\underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} A$$

And $\mathcal{H}, \mathcal{F}$ are bases of W, we can find the change of basis matrix such that

$$\underbrace{\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}}_{\mathcal{H}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} B$$

Change of basis for matrix representation

By using A and B, we can complete the commutative diagram.

$$
\begin{tikzcd}
\text{View from coordinate:} & F^n \arrow[rrr, "L_P"] \arrow[dr, "L_{\mathcal{E}}"'] & & & F^m \arrow[dl, "L_{\mathcal{F}}"] \\
\text{View from actual map:} & & V \arrow[r, "T"'] & W & \\
\text{View from coordinate:} & F^n \arrow[rrr, "L_Q"] \arrow[ur, "L_{\mathcal{G}}"] \arrow[uu, "L_A"] & & & F^m \arrow[ul, "L_{\mathcal{H}}"'] \arrow[uu, "L_B"']
\end{tikzcd}
$$

Change of basis for matrix representation

From here we may easily see the relation between P, Q, A, B

$$Q = B^{-1}PA$$

$$\begin{tikzcd}
\text{View from coordinate:} & F^n \arrow[r, "L_P"] & F^m \\
\text{View from coordinate:} & F^n \arrow[u, "L_A"] \arrow[r, "L_Q"] & F^m \arrow[u, "L_B"']
\end{tikzcd}$$

Change of basis for matrix representation

To summarise, Let $T : V \longrightarrow W$ be a linear transformation . Suppose $\mathcal{E}, \mathcal{G}$ are bases in V with

$$\underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} A$$

Suppose $\mathcal{F}, \mathcal{H}$ are bases in W with

$$\underbrace{\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}}_{\mathcal{H}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} B$$

Let P, Q be matrices such that

$$T \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} P$$

$$T \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{pmatrix}}_{\mathcal{H}} Q$$

Proposition 1

With the settings of the previous page, we have

$$Q = B^{-1}PA$$

Proof.

It is clear from commutative diagram

$$
\begin{tikzcd}
\text{View from coordinate:} & F^n \arrow[rrr, "L_P"] \arrow[dr, "L_{\mathcal{E}}"] & & & F^m \arrow[dl, "L_{\mathcal{F}}"'] \\
\text{View from actual map:} & & V \arrow[r, "T"'] & W & \\
\text{View from coordinate:} & F^n \arrow[uu, "L_A"] \arrow[ur, "L_{\mathcal{G}}"'] \arrow[rrr, "L_Q"] & & & F^m \arrow[uu, "L_B"'] \arrow[ul, "L_{\mathcal{H}}"]
\end{tikzcd}
$$

Now let us give a direct proof. □

Change of basis for matrix representation

Direct Proof

Use the equation

$$T\underbrace{\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix}}_{\mathcal{H}} Q$$

Now we replace it by

$$\underbrace{\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} A$$

and

$$\underbrace{\begin{pmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m\end{pmatrix}}_{\mathcal{H}} = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} B$$

We have

$$T\underbrace{\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n\end{pmatrix}}_{\mathcal{E}} A = \underbrace{\begin{pmatrix}\vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m\end{pmatrix}}_{\mathcal{F}} BQ$$

Change of basis for matrix representation

Therefore, we have two equation

$$T \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} BQA^{-1}$$

$$T \underbrace{\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix}}_{\mathcal{E}} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} P$$

This implies

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} BQA^{-1} = \underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_m \end{pmatrix}}_{\mathcal{F}} P$$

Since $\mathcal{F}$ is a **basis** , it is **linealy independent** , so we apply **left cancellation** .

$$BQA^{-1} = P$$

So $Q = B^{-1}PA$. We proved this Proposition.

Change of basis for matrix representation

Nevertheless, the commutative diagram is the most clear way to show relative relations among objects in linear algebra. You will finally find out it is useful.

$$
\begin{tikzcd}
\text{View from coordinate:} & F^n \arrow[rrr, "L_P"] \arrow[dr, "L_{\mathcal{E}}"] & & & F^m \arrow[dl, "L_{\mathcal{F}}"'] \\
\text{View from actual map:} & & V \arrow[r, "T"'] & W & \\
\text{View from coordinate:} & F^n \arrow[uu, "L_A"] \arrow[ur, "L_{\mathcal{G}}"] \arrow[rrr, "L_Q"] & & & F^m \arrow[uu, "L_B"'] \arrow[ul, "L_{\mathcal{H}}"']
\end{tikzcd}
$$