LDU decomposition by row operation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\overset{\substack{r_2=r_2-r_1 \\ r_3=r_3-r_1}}{=\!=\!=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\overset{r_3=r_3-r_2}{=\!=\!=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{c_2=c_2+c_3}{=\!=\!=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{c_1=c_1+c_2}{=\!=\!=} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Forwarded Row Operation v.s. Cross Filling

One step cross-filling

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{1} & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ \mathbf{0} & 1 & 1 \\ \mathbf{0} & 1 & 2 \end{pmatrix}$$

One step cross-filling = forward row operation

Cross-center = pivot

Forwarded Row Operation v.s. Cross Filling

Look at the following cross-filling decomposition

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 3 & 3 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

This decompose the matrix into the product

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Cross-center = pivot

Forwarded Row Operation v.s. Cross Filling

Don't forgot that you can also look column by column

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The left factor is in fact obtained by **forward column operation.**

Pro, Cons of two method.

Cross-Filling	Row/Col Operation
Symmetric on rows/cols	Only emphasis in one of row/columns
Useful for projection operator	Easy for calculating inverse

Algebraic expression for cross-filling

Can you write first summand into a product of matrices?

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 4 & 4 \\ 2 & 2 & 4 & 4 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Algebraic expression for cross-filling

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \frac{1}{\begin{pmatrix} 2 \end{pmatrix}} \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$= Ae_1({e_1}'Ae_1)^{-1}{e_1}'A$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad {e_1}' = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The expression $Ae_1({e_1}'Ae_1)^{-1}{e_1}'A$ is algebraic expression for cross-filling.

Algebraic expression for cross-filling

Excercise. A is $m \times n$ matrix, let e_i be ith column of I_n, and $e_j{}'$ be j'th row of I_m. The corss-filling matrix obtained from the center at j'th row i'th column is given by

Answer: $Ae_i(e_j{}'Ae_i)^{-1}e_j{}'A$.

Algebraic expression is useful when writting proof.

Algebraic expression for cross-filling

Excercise.A **projection matrix** P is a square matrix with $P^2 = P$. Suppose P is a projection matrix, and P' is a matrix obtained from P by **cross filling**. Is P' again a projection matrix?

Solution: Suppose P is $n \times n$ matrix. Let e_i be i′th column of I_n, then

$$P' = Pe_i(e_j^T Pe_i)^{-1}e_j^T P$$

By calculation

$$\begin{aligned} P'P' &= Pe_i(e_j^T Pe_i)^{-1}e_j^T PPe_i(e_j^T Pe_i)^{-1}e_j^T P \\ &= Pe_i(e_j^T Pe_i)^{-1}e_j^T Pe_i(e_j^T Pe_i)^{-1}e_j^T P \\ &= Pe_i(e_j^T Pe_i)^{-1}e_j^T P = P'. \end{aligned}$$

In the future class, the cross-filling method is **extremely important for projection matrix**

Transpose of matrices

Recall that when we try explaining rows of matrices,

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

DO NOT MEAN

$$🍋 = 1 \cdot 🍱 + 1 \cdot 🍲 \text{(Wrong)}$$

Actual meaning: $[🍋] = 1 \cdot [🍱] + 1 \cdot [🍲]$(Correct)

The number of 🍋 $= 1 \cdot$ The number of 🍱 $+ 1 \cdot$ The number of 🍲.

Transpose of matrices

Therefore, a matrix of **making meals**, can therefore give a matrix of **making counters**.

	🍱	🍲
🍃	2	2
🍋	1	1
🫘	0	4
🐄	2	1

	[🍃]	[🍋]	[🫘]	[🐄]
[🍱]	2	1	0	4
[🍲]	2	1	4	1

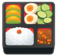
This is the transpose of the matrix.

Since the transpose exchanges the position of the material list and compund list, the order of the matrix multiplication should change.

			
	0	0	2
	0	0	1
	0	2	0
	1	0	0

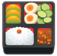
A

$\cdot$

		
	2	1
	0	2
	1	1

B

$=$

		
	2	2
	1	1
	0	4
	2	1

C

$AB = C$

Transpose of matrices

	[🥛]	[☕]	[🍵]
[🍱]	2	0	1
[🍲]	1	2	1

B^T

$\cdot$

	[🍃]	[🍋]	[🫘]	[🐄]
[🥛]	0	0	0	1
[☕]	0	0	2	0
[🍵]	2	1	0	0

A^T

$=$

	[🍃]	[🍋]	[🫘]	[🐄]
[🍱]	2	1	0	2
[🍲]	2	1	4	1

C^T

$B^T A^T = C^T.$

Transpose of matrices

Therefore

Theorem 1

$$AB = C \iff B^T A^T = C^T$$

Inverse of matrices

Excercise. Recall the way we introduce inverse is to flip the materials and compunds each other

		
	0	2
	1	0

		
	0	1
	0.5	0

Please use the same logic, try to understand why

$$AB = C \iff B^{-1}A^{-1} = C^{-1}.$$

Inverse of matrices

The expression can be deduced formally.

$$AB = C \iff B = A^{-1}C \iff BC^{-1} = A^{-1} \iff C^{-1} = B^{-1}A^{-1}.$$

Another story to understand this:

You put socks on then shoes on ; But when take them off, you put shoes off and then socks off.

LDU decomposition for symmetric matrix

If A is symmetric, we have $A = A^T$.

Suppose

$$A = LDU.$$

Therefore $A^T = U^T D^T L^T$. If A is symmetric, we have $A = A^T$, so

$$U^T D^T L^T = LDU$$

Furthermore, if A is invertible, its LDU decomposition is unique, then

$$U = L^T$$

We can write LDU decomposition of invertible symmetric matrix A as LDL^T.

LDU decomposition for symmetric matrix

Warning: Not all matrix is LDU decomposible (even invertible matrices). For example

$$\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

is symemtric but not LDU-decomposable.

Positive definite matrix

However, things are nice for so-called **positive definite matrix**

Definition 1

An $n \times n$ matrix A is called **positive definite** if

$$v^T A v > 0$$

for any $v \neq 0$.

YOU MUST REMEMBE THE FOLLOWING IMPORTANT THEOREM

Theorem 2

A positive definite matrix A is both **invertible** and **LDL^T**-decomposable, all diagonal entries of D are positive numbers!

Positive definite matrix

Recall: Suppose A is $n \times n$ matrix.

LDU-decomposable $\iff$ cross-fillable with center on **diagonal**

invertible $\iff$ decomposing into **n** matrices with cross-filling

In other words, the theorem is equivalent as saying.

positive definite matrix cross-fillable along diagonals one by one.

Positive definite matrix

Let A be a positive definite matrix, we need to start by proving its first element non-zero.

First diagonal element $e_1^T A e_1 > 0$ (recall e_1 first column of I_n). So is **cross-fillable**!

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

The process will finish if

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

is again a positive definite matrix, then an induction can be use.

Positive definite matrix

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

So why

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

is positive definite? That is why

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} > 0$$

for $\begin{pmatrix} x & y \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$?

Positive definite matrix

The idea is simple, firstly, realize that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} * & x & y \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{2} \end{pmatrix} \begin{pmatrix} * \\ x \\ y \end{pmatrix}$$

Then we may simply choose $*$ to make sure

$$\begin{pmatrix} * & x & y \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} * \\ x \\ y \end{pmatrix} = 0$$

In our case, $* = -(x+y)$. This implies that

$$\begin{pmatrix} x & y \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{A_1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{-(x+y)} & x & y \end{pmatrix} A \begin{pmatrix} \mathbf{-(x+y)} \\ x \\ y \end{pmatrix} > 0$$

So A_1 is positive definite, and therefore the cross-filling process may continue.

Connection between two ways of representing subspaces

Cross-Filling and LDU decomposition

We explain the connection between cross-filling and matrix LDU decomposition.

Cross-Filling and LDU decomposition

Suppose a cross filling decomposition has been made by **choosing cross center on diagonals**

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Cross-Filling and LDU decomposition

Observation, (think from perspective of column actions)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 1 & * & * \\ -1 & * & * \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or (row actions)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}$$

Cross-Filling and LDU decomposition

Now you can see

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 1 & * & * \\ -1 & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}$$

Cross-Filling and LDU decomposition

Question: Why the following is true?

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{2} & 2 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} * & 0 & * \\ * & 2 & * \\ * & 2 & * \end{pmatrix} \begin{pmatrix} 0 & & \\ & \frac{1}{\mathbf{2}} & \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & 2 & 2 \\ * & * & * \end{pmatrix}$$

A **scaled** version:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{2} & 2 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} * & \frac{0}{\mathbf{2}} & * \\ * & \frac{2}{\mathbf{2}} & * \\ * & \frac{2}{\mathbf{2}} & * \end{pmatrix} \begin{pmatrix} 0 & & \\ & \mathbf{2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ \frac{0}{\mathbf{2}} & \frac{2}{\mathbf{2}} & \frac{2}{\mathbf{2}} \\ * & * & * \end{pmatrix}$$

Cross-Filling and LDU decomposition

$$\begin{pmatrix} \mathbf{1} & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{2} & 2 \\ 0 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{3} \end{pmatrix}$$

Cross-Filling and LDU decomposition

$$\begin{pmatrix} \mathbf{1} & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{1}}{\mathbf{1}} & * & * \\ \frac{1}{\mathbf{1}} & * & * \\ \frac{-1}{\mathbf{1}} & * & * \end{pmatrix} \begin{pmatrix} \mathbf{1} & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \frac{\mathbf{1}}{\mathbf{1}} & \frac{1}{\mathbf{1}} & \frac{1}{\mathbf{1}} \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{2} & 2 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} * & 0 & * \\ * & \frac{\mathbf{2}}{\mathbf{2}} & * \\ * & \frac{2}{\mathbf{2}} & * \end{pmatrix} \begin{pmatrix} 0 & & \\ & \mathbf{2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & \frac{\mathbf{2}}{\mathbf{2}} & \frac{2}{\mathbf{2}} \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{3} \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & \frac{\mathbf{3}}{\mathbf{3}} \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & \mathbf{3} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \frac{\mathbf{3}}{\mathbf{3}} \end{pmatrix}$$

Cross-Filling and LDU decomposition

$$\begin{pmatrix} \boxed{1} & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 1 & * & * \\ -1 & * & * \end{pmatrix} \begin{pmatrix} \boxed{1} & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \boxed{2} & 2 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} * & 0 & * \\ * & 1 & * \\ * & 1 & * \end{pmatrix} \begin{pmatrix} 0 & & \\ & \boxed{2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & 1 & 1 \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{3} \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & \boxed{3} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$$

Cross-Filling and LDU decomposition

$$\begin{pmatrix} \mathbf{1} & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{2} & 2 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & \mathbf{2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & \mathbf{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Cross-Filling and LDU decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ -1 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \mathbf{1} & & \\ & \mathbf{2} & \\ & & \mathbf{3} \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}}_{U}$$

Cross-Filling and LDU decomposition

Proposition 1

Let $A = P_1 + P_2 + \cdots + P_m$ be a cross-filling decomposition of $n \times n$ matrix. Suppose $A = LDU$ for some lower triangular L, upper U an diagonal D such that diagonals of L and U are all 1. Assume

$$D = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_i & & \\ & & & \ddots & \\ & & & & d_n \end{pmatrix} \qquad \underbrace{D_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & d_i & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}}_{\text{only keep ith entry}}$$

Then $P_i = LD_iU$.

Cross-Filling and LDU decomposition

This proposition has the following conclusions

1. The i'th diagonal entry d_i of D is the **cross-filling center** of P_i
2. Let $\vec{c}_i$ be i'th column of L, then the scaled column $\vec{c}_i \cdot d_i$ is the **column of the cross** of P_i
3. Let r_i be i'th row of U, then the scaled row $d_i \cdot r_i$ is the **row of the cross** of P_i.

In other words, the decomposition $A = LDU$ stores all the data of the cross-filling **along diagonals**. Therefore all matrices L, D, U is **uniquely determined** by A.

Triangular system of linear equations

Our previous method of solving linear equations with cross-filling can be formally described by triangular systems.

Triangular system of linear equations

Consider previous linear system

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

The cross filling

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

yields to a LDU decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ -1 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \mathbf{1} & & \\ & \mathbf{2} & \\ & & \mathbf{3} \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}}_{U}$$

Triangular system of linear equations

Each row of DU is the **rows of each cross**,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ -1 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ & 2 & 2 \\ & & 3 \end{pmatrix}}_{DU}$$

Each rows of the cross is exactly each intermediate equations that helpped us solving equations before. Let

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} := \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ & 2 & 2 \\ & & 3 \end{pmatrix}}_{DU} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then u, v, w are those intermeidate equations.

Triangular system of linear equations

The original equation just have the form

$$\underbrace{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ -1 & 1 & 1 \end{pmatrix}}_{L} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ & 2 & 2 \\ & & 3 \end{pmatrix}}_{DU} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Solving triangular system is just by substitutions. For example $u = a$, $v = b - au = b - a^2$, and so on...

Not all matrices are LDU-decomposable

A matrix is LDU-decomposable if and only if it can be cross-filled from diagonal. But this is not always the case. For example

$$A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

It just fails from the first step, we got 0 on diagonal, preventing us from cross-filling from it.

Not all matrices are LDU-decomposable

Even if some matrix can be cross-filled from the first step, **the second step might fail** if we insists of picking diagonals

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{0} & 1 \\ 0 & 1 & 2 \end{pmatrix}}_{\substack{????\mathit{oh}!\mathit{NO}! \\ \text{a non-zero cross} \\ \text{with zero center..}}}$$

Not all matrices are LDU-decomposable

The problem comes from our restriction to ourselves.

WHY picking diagonals only?

A lot of trouble in this world come from the lack of freedom.

We ask for more freedom!

P-LDU decomposition

Now we are allow to do cross-filling following the order of columns, and lift the restriction on rows. This allows us to do more.

P-LDU decomposition

The following matrix is not LDU decomposable, but we may decompose it the following way

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For PLDU decomposition, every time we **must** choose the first non-zero column. But no restriction on rows.

P-LDU decomposition

The point is that the corss center can be distributed on diagonal after applying row operations on both side

$$\underbrace{\begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}}_{P}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 8 \end{pmatrix}}_{A}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

P-LDU decomposition

This gives the decomposition

$$PA = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U}.$$

Therefore, we can always find some row operator P, such that PA can be decomposed into LDU.

Comments

When we write it into LDU with one of the diagonal entry of D equal to 0, we can assign arbitray numbers in that corresponding col and ros of L and U

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U}.$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & * & 0 \\ 1 & 2 & * & 1 \end{pmatrix}}_{L} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U}.$$

In other words, we may always assume L and U are triangular matrices with 1 on diagonal.

Column and row switchings together

If we allow both column and row switchings, we have **complete freedom**

$$\underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 3 & 1 & -3 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 3 \end{pmatrix}}_{P_1} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 6 \\ 1 & 0 & -6 \end{pmatrix}}_{P_2} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{pmatrix}}_{P_3}$$

Then

$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} A \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & -1 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

which can be decomposed into LDU.

Observation:

Proposition 2

A triangular matrix is invertible if and only if all its diagonal entries are non-zero.

The invertibility of a matrix

Proposition 3

An $n \times n$ matrix A is invertible if and only if it can be decomposed into n many rank-1 matrices by cross-filling.

A cross-filling corresponds to a decomposition of the form $P_1 LDUP_2$, where all P_1, L, U, P_2 are invertible. The number of non-zero entries in D corresponds to the number of rank 1 matrices in the decomposition.