HIGHER GROSS–ZAGIER FORMULA FOR TWISTED GL_2

QIRUI LI, ANDREAS MIHATSCH, WEI ZHANG, AND QIRUI LI, ANDREAS MIHATSCH AND WEI ZHANG

1. Introduction

1.1. **Set up and groups.** Let F'/F be a quadratic extension of global fields, and define

$$U(2) := \left\{ g \in \mathrm{GL}_2(F') : g^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

where $g^\dagger$ denotes the transpose conjugate (i.e., g^T if F'/F is unramified).

Note that for any $g \in \mathrm{GL}_2(F)$,

$$g^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \det(g),$$

so for any $u \in U(2)$, and $k \in F^\times$ such that $\mathrm{Nm}_{F'/F}(k) = \det(g)$, the element guk^{-1} still lies in $U(2)$.

We define

$$(\mathrm{GL}_2 \times \mathrm{Res}_{F'/F}\mathrm{GL}_1)^\circ := \{(g,u) \in \mathrm{GL}_2(F') \times \mathrm{Res}_{F'/F}\mathrm{GL}_1(F) : \det(g) = \mathrm{Nm}(u)\},$$

where the superscript $\circ$ indicates the condition $\det(g) = \mathrm{Nm}(u)$. This group acts transitively on $U(2)$, with kernel given by the diagonal embedding of $F^\times$. Thus, we may identify

$$U(2) \cong (\mathrm{GL}_2 \times \mathrm{Res}_{F'/F}\mathrm{GL}_1)^\circ / \Delta(F^\times).$$

Formulation: The RTF will compare

$$G = U(2), \quad H_1 = H_2 = U(1)$$

with

$$G' = \mathrm{Res}_{F'/F}\,\mathrm{GL}_2, \quad {H_1}' = \mathrm{Res}_{F'/F}\,\mathrm{GL}_1, {H_2}' = \mathrm{GL}_{2,F}.$$

1.2. **RTF on geometric side.** Let $\nu : X' \to X$ be unramified, quadratic and non-split. (I allow geometrically split for now.) Let $\widetilde{T} = \mathrm{Res}_{F'/F}(\mathbb{G}_m)$. The RTF on the geometric side is for the triple (G, T, T) where

$$\begin{aligned} G &= (GL_2 \times \widetilde{T})^\circ / \Delta\mathbb{G}_m \cong U(V) \\ T &= \widetilde{T}/\mathbb{G}_m \cong \widetilde{T}^{\mathrm{Nm}=1}. \end{aligned} \tag{1.2.1}$$

The hermitian space V here is defined as follows. Fix an embedding $F' \to M_2(F)$ to consider

$$V = M_2(F) \cong \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} | a, b \in F' \}$$

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as a 2-dimensional F'-vector space via left-multiplication. As F'/F is étale, there is a decomposition

$$V = F' \oplus W = \{\begin{pmatrix} a & \\ & \overline{a} \end{pmatrix}\} \oplus \{\begin{pmatrix} & b \\ \overline{b} & \end{pmatrix}\}$$

where W denotes the σ-linear endomorphisms. Let $* : M_2(F) \to M_2(F)$ be the main involution defined by sending a matrix to its adjugate:

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}^* = \begin{pmatrix} \overline{a} & -b \\ -\overline{b} & a \end{pmatrix}$$

. It preserves both F' and W, so

$$((x_1, y_1), (x_2, y_2)) := x_1 x_2^* + y_1 y_2^* \in F'$$

defines a hermitian form on V. The group G acts faithfully on the right of V by

$$(g, c) \cdot x := c^{-1} x g.$$

Since $gg^* = \det(g)$, this action preserves the hermitian form up to scalar and then identifies $G \cong GU(V)$.

The embedding $T \subset G$ is given by identifying T with the stabilizer of the vector $1 \in V$. In particular, we may identify

$$\big(T \circlearrowright G/T\big) \cong \big(F' \oplus (T \circlearrowright W)\big) \setminus \{0\}$$

where the second action is multiplication on W along $T \subset \widetilde{T}$.

Good orbits: the orbits under the twisted conjugation action by $\mathrm{Res}_{F'/F}\,\mathrm{GL}_1$ on S_2:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(F_0).$$

the invariant is $d \in F'$. It is regular semisimple when $d \notin F'^1$.

Similarly, the invariant for $H \backslash G / H$ is given by $d = \langle gu, u \rangle$, where we let $u \in V$ be a vector of length 1.

Compare with GGP: $G = U(1) \times U(2), H = U(1)$ and Jacquet–Rallis $G' = \mathrm{Res}_{F'/F}(\mathrm{GL}_2 \times \mathrm{GL}_1), {H_1}' = \mathrm{Res}_{F'/F}\,\mathrm{GL}_1, {H_2}' = \mathrm{GL}_1 \times \mathrm{GL}_{2,F}$.

1.3. The Twisted higher Gross–Zagier formula.

Qirui:Let L-function, state higher derivative of L-function **Qirui**:The goal is to give an proof of twisted and original formula without using perverse sheaf.

1.4. The Twisted higher linear Arithmetic Fundamental Lemma.

1.5. Strategy of proof.

Qirui:Only states local strategy

2. Introduction

2.1. RTF on analytic side.

2.2. Relation with other RTFs.

3. Introduction

3.1. **Background.** The higher linear AFL is important because people need it for a better life...
Qirui:TBA: Background.

3.2. **Notations.** In this paper we carry the similar setting and notation as in [?]. Let $k = \mathbb{F}_q$ be a finite field of characteristic $p \neq 0$. We expand our adaptability to allow $p = 2$ in this paper. Let X be a geometrically connected smooth projective curve over k and consider

$$\nu : X' \to X$$

a finite étale cover of degree 2. The curve X' does not have to be geometrically connected. Let $\sigma \in \mathrm{Gal}(X'/X)$ be the non-trivial involution. The set of closed points on X (resp. X') are denoted by $|X|$ (resp. $|X'|$). Let $F = k(X)$ (resp.$F' = k(X')$) be their function fields. For $x \in |X|$, denote by $\mathcal{O}_x$ the local ring of X at x and F_x its fraction field. Let $\mathbb{A}_X = \prod_{x\in|X|}{}' F_x$ be the ring of adèles with ring of integers $\mathbb{O}_X = \prod_{x\in|X|} \mathcal{O}_x$. Define $\mathbb{A}_{X'}, \mathbb{O}_{X'}$ similarly. Let

$$\eta_{F'/F} : F^\times \backslash \mathbb{A}_X^\times / \mathbb{O}_X^\times \longrightarrow \{\pm 1\}$$

be the character via class field theory corresponding to étale double cover $\nu : X' \longrightarrow X$. Denote the divisor groups and Picard groups of X by $\mathrm{Div}(X)$ and $\mathrm{Pic}(X)$ respectively.

Let n be an integer. The spherical Hecke algebra $\mathcal{H}_X = \mathcal{C}_c^\infty(\mathrm{GL}_{2n}(\mathbb{O}_X)\backslash \mathrm{GL}_{2n}(\mathbb{A}_X)/\mathrm{GL}_{2n}(\mathbb{O}_X), \mathbb{Q})$ is the $\mathbb{Q}$-algebra of bi-K-invairant functions with the product given by convolution. For any divisor $D = \sum_{w\in|X|} D_w \in \mathrm{Div}(X)$ where $D_w = n_w(D) \cdot w \in \mathrm{Div}(X_w)$, let $h_D \in \mathcal{H}_X$ be the characteristic function of the image of

$$\{g \in \mathrm{Mat}_2(\mathbb{O}_X) : \mathrm{div}(\det g) = D\} \longrightarrow G(\mathbb{A}_X).$$

In fact, h_D generates $\mathcal{H}_X$ as a vector space over $\mathbb{Q}$. Therefore, it is sufficient to only work with elements of the form h_D. Let $T = \mathrm{Res}_{X'/X}\,\mathrm{GL}_1/\mathrm{GL}_1$.

3.3. **Global Orbital Integrals.** Let $\widetilde{G} := \mathrm{GL}_2$ and let $N(F) \subset \widetilde{G}(F)$ be the normalizer of $T(F) \subset \widetilde{G}(F)$. We call an element $g \in \widetilde{G}(F)$ *regular semi-simple* if one can decompose $g = g_+ + g_-$ with $g_+ \in T(F)$ and $g_- \in N(F) \smallsetminus T(F)$. If g is regular semisimple, then so is g^{-1}. The regular semisimple orbits $T(\mathbb{A}_F) \cdot g \cdot T(\mathbb{A}_F)$ is classfied by the invariant defined by $\mathbf{s}_g := (g^{-1})_+ g_+ \in T(F)$. In fact, we have $\mathbf{s}_g \in F^\times \subset T(F)$. For any $t = (t_1, t_2) \in T(\mathbb{A}_F)$, put $\eta_{F'/F}(t) := \eta_{F'/F}(t_1^{-1} \cdot t_2)$ and $|t|^s := |t_1^{-1} \cdot t_2|^s_{\mathbb{A}_X}$. Define $\eta_{F'_w/F_w}$ and $|\cdot|_w$ for elements of $T(F_w)$ in a similar way. For any divisor $D \in \mathrm{Div}(X)$, the global orbital integral is defined by

$$\mathrm{Orb}(h_D, g, s) = \int_{\frac{T(\mathbb{A}_X)\times T(\mathbb{A}_X)}{I_g}} h_D(t_1^{-1} g t_2) |t_1 t_2|^s \eta_{F'/F}(t_2)\, dt_1\, dt_2$$

where $I_g = \{(t_1, t_2) : t_1^{-1} g t_2 = g\}$ is the stablizer of g.

We may define local Hecke algebra $\mathcal{H}_{X_w} = \mathcal{C}_c^\infty(G(F_w)//K_w, \mathbb{Q})$ and elements $h_{D_w} \in \mathcal{H}_{X_w}$ in a similar way. Then we have a decomposition

$$h_D = \bigotimes_{w\in|X|} h_{D_w}$$

allowing us to decompose the orbital integral into

orb-product (3.3.1)
$$\mathrm{Orb}(h_D, g, s) = \prod_{w\in|X|} \mathrm{Orb}(h_{D_w}, g_w, s)$$

with local orbital integrals defined by

$$\mathrm{Orb}(h_{D_w}, g_w, s) = \Omega_X(g_w, s)\cdot \int_{\frac{T(F_w)\times T(F_w)}{I_g}} h_{\underline{D}_w}(t_1^{-1} g_w t_2)\cdot |t_1 t_2|^s \cdot \eta_{F_w'/F_w}(t_2)\, dt_1\, dt_2$$

with $I_g = \{(t_1, t_2) : t_1^{-1} g t_2 = g\}$ the stablizer of g as usual and transfer factor defined by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \quad \Omega_X(g, s) := \eta_{F'_w/F_w}(d^{-1}c)\cdot |bc^{-1}|_w^s.$$

The formula (3.3.1) is well defined since almost all factors are 1 by Lemma **??**.

3.4. **Intersection of CM cycles.** Let $\mathrm{Sht}_G^{\underline{\mu}}$ (resp. $\mathrm{Sht}_T^{\underline{\mu}}$) be the stack of G (resp. T)-shtukas of $r \in 2\mathbb{Z}$ legs of type $\underline{\mu} \in \{\mu_+, \mu_-\}^r$ (See Definition 5.2.4,7.2.1). The natural embedding

$$\mathrm{Sht}_T^{\underline{\mu}} \longrightarrow \mathrm{Sht}_G^{\underline{\mu}}$$

defined by pushing forward shtukas via $\nu : X' \to X$ give rise to a Heegner—Drinfeld cycle. Let $\mathcal{N}_D^{\underline{\mu}}$ be the intersection stack defined by the following Catesian diagram of stacks

$$\begin{tikzcd}
\mathcal{N}_D^{\underline{\mu}} \arrow[r] \arrow[d] & \Gamma_D \arrow[d] \\
\mathrm{Sht}_{T'}^{\underline{\mu}} \times_{X'^r} \mathrm{Sht}_{T'}^{\underline{\mu}} \arrow[r] & \mathrm{Sht}_{G'}^{\underline{\mu}} \times_{X^r} \mathrm{Sht}_{G'}^{\underline{\mu}} \times_{X^r} X'^r,
\end{tikzcd}$$

where Γ_D is the graph of Hecke correspondence induced by D (see Definition **??**). We refer subsection **??** for details of $\mathcal{N}_{n+1}$.

3.5. **Main results.** In section **??**, we define invariants for shtukas, which give rise to a map $\mathcal{N}_{n+1} \longrightarrow F$ classifying orbits on geometric side. Let

$$\mathcal{N}_{\mathcal{D},\mathbf{s}}^{\underline{\mu}} \longrightarrow \mathcal{N}_{n+1}$$

be the fiber at $\mathbf{s} \in F \smallsetminus \{0, 1\}$, our result is the global linear AFL for regular semi-simple orbits.

globalAFL **Theorem 3.5.1** (Global linear AFL for PGL_2)**.** *For any $\mathbf{s} \in F^\times \smallsetminus \{0, 1\}$, we have*

$$\left.\frac{d^r}{ds^r}\right|_{s=0} \mathrm{Orb}(h_D, g, s) = (\ln q)^r \cdot \mathrm{length}\left(\mathcal{N}_{\mathcal{D},\mathbf{s}}^{\underline{\mu}}\right)$$

if $\mathbf{s}_g = \mathbf{s}$. In particular, $\mathcal{N}_{\mathcal{D},\mathbf{s}}^{\underline{\mu}}$ is Artinian and the right hand side is independent of the choice of $\underline{\mu}$.

We use local method to prove our theorem. The product formula 3.3.1 implies the following decomposition of derivatives

derivative (3.5.1)
$$\frac{d^r}{ds^r}\mathrm{Orb}(h_D, g, s) = \sum_{\underline{v}\in|X|^r} \prod_{w\in|X|} \frac{d^{r(\underline{v})_w}}{ds^{r(\underline{v})_w}} \mathrm{Orb}_w(h_{D_w}, g_w, s).$$

where $r(\underline{v})_w$ is the number of elements in $\underline{v}$ coincide with w. On the geometric side, we have some similar decomposition

cderivative (3.5.2)
$$\mathcal{N}^{\underline{\mu}}_{\mathcal{D},\mathbf{s}} = \coprod_{\underline{v}\in|X|^r} \prod_{w\in|X|} \left(\mathcal{N}^{\underline{\mu}_w}_{\mathcal{D}_w,\mathbf{s},w}\right)$$

where $\underline{\mu}_w = (\mu_{i_1}, \cdots, \mu_{i_{r(\underline{v})_w}})_{i_k\in I}$ for $I = \{i : v_i = w\}$ and $\mathcal{N}_n$ is the similar local intersection problem where $U_w \subset X$ is some affine open around w and $U_w{}' = U_w \times_X X'$. Here $\mathrm{Sht}^{\underline{\mu}}_{T,w}$ and $\mathrm{Sht}^{\underline{\mu}}_{G,w}$ are $(T, U_w{}')$ and (G, U_w)-shtukas of type $\underline{\mu}$ with modification factoring through on X_w and $X'{}_w := X_w \times_X X'$. And $\mathcal{N}_n$ are locus of $\mathcal{N}_n$ for invariants $\mathbf{s}$. The global linear AFL conjecture follows from the local conjecture, which is our main theorem in this paper

localAFL **Theorem 3.5.2.** *Let $\underline{\mu} \in \{\mu_+, \mu_-\}^r$, we have*
$$\left.\frac{d^r}{ds^r}\right|_{s=0} \mathrm{Orb}_{\eta_w}(h_{\underline{m}}, g, s) = (\ln q_w)^r \cdot \mathrm{length}(\mathcal{N}_n).$$
In particular, $\mathcal{N}_n$ is Artinian and the right hand side is independent of the choice of $\underline{\mu}$.

3.6. **Strategy of proof.** Let $w_+, w_- \in |X'{}_w|$ be two places above w. A (T, U_w) shtukas of CM type $\underline{\phi} \in \{\pm\}^r$ is the one whose i'th leg is located at w_{ϕ_i}. Based on the CM type, the space $\mathcal{N}^{\underline{\mu}_w}_{\mathcal{D}_w,\mathbf{s},w}$ has further decompositions
$$\mathcal{N}^{\underline{\mu}_w}_{\mathcal{D}_w,\mathbf{s},w} = \coprod_{\underline{\phi}\in\{\pm\}^r} \mathcal{N}^{\underline{\mu}_w,\underline{\phi}}_{\mathcal{D}_w,\mathbf{s},w}.$$

4. Boon shakalaks

By an abuse of notation we identify $\mu_+ = +1$ and $\mu_- = -1$. The termwise product $\underline{\mu}_w \cdot \underline{\phi}$ is again an element of $\{\pm\}^r$. Write $\underline{\mu} = \underline{\mu}_w$ for simplicity. Let $\mathcal{D}_w = m \cdot w$ and $\alpha = m + \mathrm{ord}_w(\mathbf{s})$, $\beta = m + \mathrm{ord}_w(1 - \mathbf{s})$. Let $\mathbb{V}^+_\alpha$ be the abelian group of $\mathbb{Z}$-coefficient homogeneous polynomial of degree α in variables A, D. Let $\mathbb{V}^-_\beta$ be the one for degree β in variables B, C (see Definition **??**). Let W be the operator acting on $\mathbb{V}^+_\alpha \otimes \mathbb{V}^-_\beta$ by interchanging variables $B \leftrightarrow C$ and $A \leftrightarrow D$ when w innert, or trivial action when w split. Let
$$\mathscr{D}_+ = C \cdot \frac{d}{dB} \qquad \mathscr{D}_- = B \cdot \frac{d}{dC}.$$
The CM type $\underline{\phi}$ give rise to an operator (see Definition **??**) on $\mathbb{V}^+_\alpha \otimes \mathbb{V}^-_\beta$ defined by
$$\mathscr{D}_{\underline{\phi}\cdot\underline{\mu}} W := \mathscr{D}_{\phi_1\cdot\mu_1} \circ \mathscr{D}_{\phi_2\cdot\mu_2} \circ \cdots \circ \mathscr{D}_{\phi_r\cdot\mu_r} \circ W.$$
In Theorem **??**, **??**, we proved that the connected components of $\mathcal{N}^{\underline{\mu}_w,\underline{\phi}}_{\mathcal{D}_w,\mathbf{s},w}$ corresponds to 1-dimensional eigensubspaces of $\mathbb{V}^+_\alpha \otimes \mathbb{V}^-_\beta$ under the linear operator $\mathscr{D}_{\underline{\phi}\cdot\underline{\mu}} W$ with the length of each connected component equal to its corresponding eigenvalue. Therefore, when summing up all possible CM types $\underline{\phi} \in \{\pm\}^r$, the result is given by the trace
$$\mathrm{length}\left(\mathcal{N}^{\underline{\mu}_w}_{\mathcal{D}_w,\mathbf{s},w}\right) = \mathrm{tr}\left((\mathscr{D}_+ + \mathscr{D}_-)^r \circ W | \, \mathbb{V}^+_\alpha \otimes \mathbb{V}^-_\beta\right).$$
On the analytic side, suppose $\mathbf{s}_g = \mathbf{s}$. By associating an element to a polynomial
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (A + D)^{\mathrm{ord}_w(a)}(B + C)^{\mathrm{ord}_w(b)}(B - C)^{\mathrm{ord}_w(c)}(A - D)^{\mathrm{ord}_w(d)},$$

we are able to prove an identity for orbital integral

$$Orb_w(h_D, g, s) = Tr(W \cdot q_w^{s\cdot(\mathscr{D}_+ + \mathscr{D}_-)} \Big| \mathbb{V}_\alpha^+ \otimes \mathbb{V}_\beta^-),$$

which completes the proof of the main theorem by taking r'th derivative.

We introduce Moduli space of Shtukas in Section 5. The relation between global and local orbital and functional equations are proved in Section **??**, where we compare the definition of invariant in different literature and give a detailed discussion on convergence. Section **??** is a geometric analogue of the local-global relation in the analytic side, where we proved the uniformization theorem and reduce the global higher linear AFL to local case. Section **??** is the technical core of this paper, which constructs operators for CM types and relates connected components of intersection locus to eigenspace and eigenvalues. We finish the proof of local higher linear AFL in Section **??**.

1

5. Multi-legged moduli spaces of Shtukas

For a reductive group G over k (or a parahoric group scheme $\mathcal{G}$ over X), we have a moduli space of Shtukas $\mathrm{Sht}_G^{N,\underline{\mu}}$ with auxiliary structure (i.e., bounds by an r-tuple of co-weights $\underline{\mu}$, N-level structure for a (finite) closed sub-scheme $N \subset X$). This was the space introduced by Drinfeld when $r = 2$ and $G = \mathrm{GL}_n$ (D87 [?]), and in general by Varshavsky (Va [?]). In this paper, we treat the case $G = \mathrm{PGL}_2$ and $T = \mathrm{Res}_{X'/X}\,\mathrm{GL}_1/\mathrm{GL}_1$.

5.1. **Notations.** In this section, we fix X a smooth geometrically connected curve over $\mathbb{F}_q$. For any scheme S over $\overline{\mathbb{F}_q}$, we write $X \times S$ for the fiber product over the base field of X. Consider relative Frobenius map

$$\tau := \mathrm{id} \times \mathrm{Frob}_S : X \times S \longrightarrow X \times S \qquad (x, s) \longmapsto (x, s^q).$$

We denote $\tau^*\mathcal{E}$ by $\mathcal{E}^\tau$ for any sheaf $\mathcal{E}$ over $X \times S$.

5.2. **Definition of shtukas.**

5.2.1. *Modification of vector bundles.*

Definition 5.2.2. Let

$$\mu_+ = (1, 0, ..., 0), \quad \mu_- = (0, ..., 0, -1).$$

be dominant coweights of GL_n. A type μ_+ (resp. μ_-), leg $x : S \longrightarrow X$ modification $\mathcal{E} \dashrightarrow \mathcal{F}$ of rank n vector bundles over $X \times S$ is a morphism $\mathcal{E} \longrightarrow \mathcal{F}$ (resp. $\mathcal{E} \longleftarrow \mathcal{F}$) with quotient isomorphic to the push forward of a line bundle through the graph of x

$$S \longrightarrow X \times S.$$

5.2.3. *Definition of G-shtukas.*

shtukas **Definition 5.2.4.** Let r be an even integer and let $\underline{\mu} \in \{\mu_+, \mu_-\}^r$ be a sequence with half μ_+ and half μ_-. A *type* $\underline{\mu}$ (PGL_n, X)*-shtukas* is a collection type μ_i, leg $x_i : S \longrightarrow X$ modifications

$$\mathcal{E}_{i-1} \dashrightarrow \mathcal{E}_i$$

of rank n vector bundles together with an isomorphism

$$\mathcal{E}_r \xrightarrow{\cong} \mathcal{E}_0^\tau.$$

Two such shtukas are considered equivalent if they differ by simutaneous twist of line bundles $\mathscr{L}$ over $S \times X$

$$\left(\mathcal{E}_{i-1} \dashrightarrow \mathcal{E}_i, \mathcal{E}_r \xrightarrow{\cong} \mathcal{E}_0^\tau\right) = \left(\mathcal{E}_{i-1} \otimes \mathscr{L} \dashrightarrow \mathcal{E}_i \otimes \mathscr{L}, \mathcal{E}_r \otimes \mathscr{L} \xrightarrow{\cong} \mathcal{E}_0^\tau \otimes \mathscr{L}\right)$$

A morphism between two type $\underline{\mu}$ (PGL_n, X)-shtukas $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ is a choice of two line bundles $\mathscr{L}, \mathscr{F}$ over $S \times X$, together with a family of morphisms

$$f_i : \mathcal{E}_i \otimes \mathscr{L} \longrightarrow \mathcal{F}_i \otimes \mathscr{F}$$

for $0 \le i \le r$ and a section

$$l : \mathscr{L} \otimes \mathscr{L}^{\tau\vee} \longrightarrow \mathscr{F} \otimes \mathscr{F}^{\tau\vee}$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{E}_{i-1} \otimes \mathscr{L} \arrow[r, dashed] \arrow[d, "f_{i-1}"'] & \mathcal{E}_i \otimes \mathscr{L} \arrow[d, "f_i"] & \mathcal{E}_r \otimes \mathscr{L} \arrow[r, dashed] \arrow[d, "f_r"'] & \mathcal{E}_0^\tau \otimes \mathscr{L} \arrow[d, "f_0^\tau \otimes l"] \\
\mathcal{F}_{i-1} \otimes \mathscr{F} \arrow[r, dashed] & \mathcal{F}_i \otimes \mathscr{F} & \mathcal{F}_r \otimes \mathscr{F} \arrow[r, dashed] & \mathcal{F}_0^\tau \otimes \mathscr{F}.
\end{tikzcd}$$

Let $\mathrm{Sht}_G^{\underline{\mu}}$ be the moduli stack parametrizing type $\underline{\mu}$, (G, X)-shtukas. There is a natural map $\mathrm{Sht}_G^{\underline{\mu}} \longrightarrow X^r$ parametrizing legs of type $\underline{\mu}$ shtukas.

bugpatch **Lemma 5.2.5.** *If X is a smooth projective curve , let*

$$\mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_r \xrightarrow{\cong} \mathcal{E}_0^\tau$$

be a (PGL_n, X)-shtukas over S over $\overline{\mathbb{F}_q}$, then for any $\mathscr{F} \in \mathrm{Pic}^0(X \times S)$,

$$\mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_r \xrightarrow{\cong} \mathcal{E}_0^\tau \otimes \mathscr{F}$$

is also a (PGL_n, X)-shtukas over S.

Proof Since

$$\mathrm{Pic}^0(X) \xrightarrow{\mathrm{id}-\mathrm{Frob}_q} \mathrm{Pic}^0(X)$$

is an isogeny of abelian varieties and therefore a surjective. Therefore, there exists a line budle $\mathscr{L}$ over $X \times S$ such that $\mathscr{F} = \mathscr{L}^\vee \otimes \mathscr{L}^\tau$. Therefore,

$$\mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_r \xrightarrow{\cong} \mathcal{E}_0^\tau \otimes \mathscr{L}^\vee \otimes \mathscr{L}^\tau$$

is equivalent to

$$\mathcal{E}_0 \otimes \mathscr{L} \dashrightarrow \mathcal{E}_1 \otimes \mathscr{L} \dashrightarrow \cdots \dashrightarrow \mathcal{E}_r \otimes \mathscr{L} \xrightarrow{\cong} (\mathcal{E}_0 \otimes \mathscr{L})^\tau$$

as desired. □

Theorem 5.2.6 (Drinfel′d $r = 2$, ? $r > 2$)**.** *The stack $\mathrm{Sht}_G^{\underline{\mu}}$ is an algebraic Deligne–Mumford stack and the characteristic map*

$$(x_1, ..., x_r) : \mathrm{Sht}_G^{\underline{\mu}} \longrightarrow X^r$$

is smooth of relative dimension $r(n-1)$, and locally of finite type.

Qirui:Please attach reference of the above Theorem.

6. Isoshtukas

For the global-to-local reduction, we need to see that a certain global obstruction for non-emptyness of $\mathcal{I}(D_0, D_1, \gamma_+, \delta)$ is equivalent to the non-emptyness of all local obstructions. (This vague sentence will become clearer later.)

In this section, we consider the following setting.

- Let $x'_\bullet \in X'(\bar{\mathbb{F}}_q)^r$ be a fixed tuple of legs. Set $P = \sum_{i=1}^r \mu_i[x'_i] \in \mathrm{Div}(\bar{\mathbb{F}}_q \otimes X')$.
- Let $(L_\bullet, \phi_\bullet) \in \mathcal{S}_T(\bar{\mathbb{F}}_q)$ lie above $x'_\bullet$. Existence is ensured by the property that $\mathcal{S}_T \to (X')^r$ is finite étale.
- Let $L = L_0$ and let $\phi : L \to {}^\tau L$ be the meromorphic map obtained by composing the r modifications of $\phi_\bullet$. Its divisor is P.
- Consider the F'-vector space

$$V(L, \phi) := \left\{ f : \sigma L \dashrightarrow L \mid \bar{\mathbb{F}}_q \otimes_{\mathbb{F}_q} F'\text{-linear and s.th. } {}^\tau f \circ {}^\sigma \phi = \phi \circ f \right\}. \tag{6.0.1}$$

Note that $V(L, \phi)$ is either 0 or a 1-dimensional F'-vector space. Our aim is to prove the following theorem.

hm:Drinfeld **Theorem 6.0.1.** *The space $V(L, \phi)$ is non-zero if and only if there exists $m \geq 1$ such that*

obstruction
$$\sum_{i=0}^{m-1} \tau^i (P - \sigma P) = 0. \tag{6.0.2}$$

In other words, we require that for every split place $\nu^{-1}(w) = \{v\} \sqcup \{\sigma(v)\}$,

$$\deg_v(P) = \deg_{\sigma(v)}(P).$$

Proof Choose $m \geq 1$ such that $x'_\bullet \in X'(\mathbb{F}_{q^m})^r$ and such that there exists a line bundle $\widetilde{L}$ over $\mathbb{F}_{q^m} \otimes X'$ together with a linear map $\widetilde{\phi} : \widetilde{L} \to \widetilde{L}$ and an isomorphism

$$(L, \phi^{(m)}) \cong \bar{\mathbb{F}}_q \otimes_{\mathbb{F}_{q^m}} (\widetilde{L}, \widetilde{\phi}).$$

Assume that there exists some $0 \neq f \in V(L, \phi)$. Enlarging m if necessary, we may assume that f comes by scalar extension from some $\widetilde{f} \in \mathrm{Hom}(\sigma^* \widetilde{L}, L)$. Then $\widetilde{f}$ satisfies

$$\widetilde{f} \circ {}^\sigma \widetilde{\phi} = \phi \circ \widetilde{f}$$

which implies

$$\mathrm{div}(\widetilde{f}) + \sum_{i=0}^{m-1} \tau^i(\sigma P) = \mathrm{div}(\widetilde{f}) + \sum_{i=0}^{m-1} \tau^i P$$

and we obtain identity (6.0.2) . Assume conversely that (6.0.2) holds. Add Galois descent argument. □

7. CM cycles and intersection problem

7.1. **Drinfeld—Heegner cycles.** We now restrict to the case $G = \mathrm{PGL}_2$. Let $\nu : X' \longrightarrow X$ be an étale double covering of X. Let $T = (\mathrm{Res}_{X'/X} \mathrm{GL}_1)/\mathrm{GL}_1$. Let $\sigma : X' \to X'$ be the non-trivial involution. Let $\underline{\mu} = (\mu_1, ..., \mu_r)$ where $\mu_i \in \{\mu_+, \mu_-\}$ as usual.

tshtukas **Definition 7.1.1.** A type $\underline{\mu}$ (T, X')-shtukas is collection type μ_i, leg $x_i : S \longrightarrow X'$ modifications

$$\mathcal{E}_{i-1} \dashrightarrow \mathcal{E}_i$$

of line bundles together with an isomorphism

$$\mathcal{E}_r \xrightarrow{\cong} \mathcal{E}_0^\tau.$$

Two such shtukas are considered equivalent if they differ by simutaneous twist of $\nu^*\mathscr{L}$ for some line bundles $\mathscr{L}$ over $S \times X$. Morphisms are defined similarly as in Definition 5.2.4. Denote its moduli stack by $\mathrm{Sht}_T^{\underline{\mu}}$.

The association $\mathcal{E}_\bullet \mapsto \nu_*\mathcal{E}_\bullet$ give rise to an embedding $\mathrm{Sht}_T^{\underline{\mu}} \longrightarrow \mathrm{Sht}_G^{\underline{\mu}}$.

7.2. **Drinfeld—Heegner cycles.** We now restrict to the case $G = \mathrm{PGL}_2$. Let $\nu : X' \longrightarrow X$ be an étale double covering of X. Let $T = (\mathrm{Res}_{X'/X}\mathrm{GL}_1)/\mathrm{GL}_1$. Let $\sigma : X' \to X'$ be the non-trivial involution. Let $\underline{\mu} = (\mu_1, ..., \mu_r)$ where $\mu_i \in \{\mu_+, \mu_-\}$ as usual.

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7.3. **Recap of Yun–Zhang.** Let $k = \mathbb{F}_q$ and let X/k be a ge ’ ometrically connected proper smooth curve. Let $\nu : X' \longrightarrow X$ be a geometrically connected étale double cover with non-trivial automorphism σ. Denote by F'/F the corresponding extension of function fields.

Consider th’e following two algebraic groups over F,

$$T = \mathrm{Res}_{F'/F}(\mathbb{G}_m), \quad G = T \times GL_2.$$

Given an even integer $r \geq 0$ and a tuple $\mu \in \{\pm\}^r$ precisely half of which are $+$, Yun–Zhang have defined moduli of shtuka with r legs,

$$\mathrm{Sht}_T^{r,\mu} \longrightarrow (X')^r, \quad \mathrm{Sht}_2^{r,\mu} \longrightarrow X^r.$$

Push forward along ν provides a morphism of stacks over X^r

$$\mathrm{Sht}_T^{r,\mu} \longrightarrow \mathrm{Sht}_2^{r,\mu}, \quad \big((x'_\bullet), \mathcal{E}_\bullet, \phi_\bullet\big) \longmapsto \big((\nu(x'_\bullet)), \nu_*\mathcal{E}_\bullet, \nu_*\phi_\bullet\big).$$

Taking its graph, we obtain a morphism

$$\mathrm{Sht}_T^{r,\mu} \longrightarrow \mathrm{Sht}_G^{r,\mu} := \mathrm{Sht}_T^{r,\mu} \times_{X^r} \mathrm{Sht}_2^{r,\mu}.$$

This is the so-called Heegner–Drinfeld cycles. Given a pair (D_0, D_1) of divisors $D_0 \in \mathrm{Div}(X')$ and $D_1 \in \mathrm{Div}(X)$, there are Hecke correspondences

$$\mathrm{Hk}(D_0) \longrightarrow \mathrm{Sht}_T^{r,\mu} \times_{(X')^r} \mathrm{Sht}_T^{r,\mu}, \quad \mathrm{Hk}(D_1) \longrightarrow \mathrm{Sht}_2^{r,\mu} \times_{X^r} \mathrm{Sht}_2^{r,\mu}.$$

We define

$$\mathrm{Hk}(D_0, D_1) := \mathrm{Hk}(D_0) \times_{X^r} \mathrm{Hk}(D_1) \longrightarrow \mathrm{Sht}_G^{r,\mu} \times_{(X')^r} \mathrm{Sht}_G^{r,\mu}.$$

7.4. **CM cycles(by Andreas).** Let $k = \mathbb{F}_q$ and let X/k be a geometrically connected proper smooth curve. Let $\nu : X' \to X$ be a connected (but not necessarily geometrically connected) étale double cover with non-trivial automorphism σ. Denote by F'/F the corresponding extension of function fields.

Consider the following algebraic groups over F,

$$\begin{aligned} \widetilde{T} &= \mathrm{Res}_{F'/F}(\mathbb{G}_m), & \widetilde{G} &= \widetilde{T} \times GL_2 \\ T &= T/\mathbb{G}_m, & G &= \widetilde{G}/\Delta\mathbb{G}_m. \end{aligned}$$

Given an even integer $r \geq 0$ and a tuple $\mu \in \{\pm\}^r$ precisely half of which are $+$, Yun–Zhang [?] YZ17 have defined moduli spaces of shtuka with r legs,

$$\mathrm{Sht}_{\widetilde{T}}^{r,\mu} \longrightarrow (X')^r, \quad \mathrm{Sht}_2^{r,\mu} \longrightarrow X^r.$$

These are Deligne–Mumford stacks. Push forward along ν provides a morphism of stacks over X^r

hism-moduli (7.4.1)
$$\mathrm{Sht}_{\widetilde{T}}^{r,\mu} \longrightarrow \mathrm{Sht}_2^{r,\mu}, \quad \big((x'_\bullet), \mathcal{L}_\bullet, \phi_\bullet\big) \longmapsto \big((\nu(x'_\bullet)), \nu_*\mathcal{L}_\bullet, \nu_*\phi_\bullet\big).$$

Taking its graph, we obtain a morphism

$$\mathrm{Sht}_{\widetilde{T}}^{r,\mu} \longrightarrow \mathrm{Sht}_{\widetilde{G}}^{r,\mu} := \mathrm{Sht}_{\widetilde{T}}^{r,\mu} \times_{X^r} \mathrm{Sht}_2^{r,\mu}.$$

We consider r and μ fixed in the following and omit them from the notation. The group $\Xi = \mathrm{Pic}_X(k)$ acts on $\mathrm{Sht}_{\widetilde{T}}$ and Sht_2 by tensor product. We consider the diagonal action of Ξ on $\mathrm{Sht}_{\widetilde{G}}$ and define

tient-gamma (7.4.2)
$$\mathcal{S}_T = \Xi \backslash \mathrm{Sht}_T, \quad \mathcal{S}_G = \Xi \backslash \mathrm{Sht}_{\widetilde{G}}.$$

These are again Deligne–Mumford stacks. The morphism $\mathcal{S}_T \to (X')^r$ is finite étale. The morphism in (7.4.1) is Ξ-equivariant and hence descends to a morphism

eq:GZ-cycle (7.4.3)
$$\mathcal{S}_T \longrightarrow \mathcal{S}_G.$$

We call this the Gross–Zagier cycle. Let $D_0 \in \mathrm{Div}(X')$ and $D_1 \in \mathrm{Div}(X)$. Our next aim is to define a Hecke correspondence

$$\mathrm{Hk}(D_0, D_1) \longrightarrow \mathcal{S}_G \times_{(X')^r} \mathcal{S}_G.$$

It is modeled on the Hecke operator for $\widetilde{G}$ which parametrizes

$$(x'_\bullet, \mathcal{L}_\bullet, \mathcal{E}_\bullet, \mathcal{L}'_\bullet, \mathcal{E}'_\bullet) \in \mathrm{Sht}_{\widetilde{G}}^{r,\mu} \times_{(X')^r} \mathrm{Sht}_{\widetilde{G}}^{r,\mu}$$

together with morphisms

$$\iota : \mathcal{L}_\bullet \longrightarrow \mathcal{L}'_\bullet, \quad \gamma : \mathcal{E}_\bullet \longrightarrow \mathcal{E}'_\bullet$$

such that $\mathrm{div}(\iota) = D_0$ and $\mathrm{div}(\det(\gamma)) = D_1$. The first condition can also be reinterpreted as ι defining an isomorphism $\mathcal{L}'_\bullet \cong \mathcal{L}_\bullet(D_0)$.

Definition 7.4.1. Let $D_0 \in \mathrm{Div}(X')$ and $D_1 \in \mathrm{Div}(X)$. Define $\widetilde{\mathrm{Hk}}(D_0, D_1) \to (X')^r$ as the stack of tuples

eq:Hk-tuple

$$(x'_\bullet, \mathcal{L}_\bullet, \mathcal{E}_\bullet, \mathcal{L}'_\bullet, \mathcal{E}'_\bullet, \mathcal{K}, \iota, \gamma) \tag{7.4.4}$$

where $\mathcal{K} \in \Xi$ and where

$$\iota : \mathcal{L}_\bullet \longrightarrow \mathcal{K} \otimes \mathcal{L}'_\bullet, \quad \gamma : \mathcal{E}_\bullet \longrightarrow \mathcal{K} \otimes \mathcal{E}'_\bullet$$

satisfy $\mathrm{div}(\iota) = D_0$ and $\mathrm{div}(\det \gamma) = D_1$. Put $\mathrm{Hk}(D_0, D_1) := (\Xi \times \Xi)\backslash\widetilde{\mathrm{Hk}}(D_0, D_1)$ and let

$$\mathrm{Hk}(D_0, D_1) \longrightarrow \mathcal{S}_G \times_{(X')^r} \mathcal{S}_G$$

be given by projecting a tuple as in (7.4.4) to $((\mathcal{L}_\bullet, \mathcal{E}_\bullet),\ (\mathcal{L}'_\bullet, \mathcal{E}'_\bullet))$.

8. Analytic side: Relative trace formula

9. Analytic side: Orbital Integral of degenerate orbits

10. Geometric side: Uniformization of Shtukas around Artinian intersection

11. Geometric side: Calculating the self intersection

12. Proof of main theorem

13. Appendix: Proof of some key equations

Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany, and University of Maryland, Department of Mathematics, College Park, MD 20742, USA

Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Email address: weizhang@mit.edu