Mimamsa Number Theroy

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CONTENTS

CHAPTER 1

FUNCTIONS

1.1 READING

- Sets and Functions (as previously assigned) to be done from Prameya-Vismaya.

1.2 PROBLEMS

Problem 1.2.1. Show that there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(S) \neq S \tag{1.2.1}$$

for any proper non-empty subset S of $\mathbb{N}$.

CHAPTER 2

INDUCTION

2.1 READING

Following is the requisite reading.

- Read §1.1 from Burton's *Elementary Number Theory* (7th Edition).
- Why is mathematical induction a valid proof technique?
- Difference between strong and weak induction?

2.2 INDUCTION

Problem 2.2.1. Prove that $n! > n^2$ for every integer $n \geq 4$, whereas $n! > n^3$ for every integer $n \geq 6$.

Problem 2.2.2. Use mathematical induction to derive the following formula for all $n \geq 1$:

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

Problem 2.2.3. a) Verify that for all $n \geq 1$,

$$2 \cdot 6 \cdot 10 \cdot 14 \cdot \cdots \cdot (4n-2) = \frac{(2n)!}{n!}$$

b) Use part (a) to obtain the inequality $2^n(n!)^2 \leq (2n)\,!$ for all $n \geq 1$.

Problem 2.2.4. Establish the Bernoulli inequality: If $1 + a > 0$, then

$$(1+a)^n \geq 1 + na$$

for all $n \geq 1$.

Problem 2.2.5. For all $n \geq 1$, prove the following by mathematical induction:

a) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$.

b) $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{n+2}{2^n}$.

Problem 2.2.6. Show that the expression $(2n)!/2^n n!$ is an integer for all $n \geq 0$.

Problem 2.2.7. Suppose that the numbers a_n are defined inductively by $a_1 = 1, a_2 = 2, a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Show that $a_n < 2^n$ for every positive integer n.

Problem 2.2.8. If the numbers a_n are defined by $a_1 = 11, a_2 = 21$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$, prove that

$$a_n = 5 \cdot 2^n + 1, \quad n \geq 1$$

CHAPTER 3

DIVISION ALGORITHM

3.1 READING

- Read §2.2 from Burton's *Elementary Number Theory* (7th Edition).

3.2 PROBLEMS

Problem 3.2.1. Prove that if a and b are integers, with $b > 0$, then there exist unique integers q and r satisfying $a = qb + r$, where $2b \leq r < 3b$.

Problem 3.2.2. Show that any integer of the form $6k + 5$ is also of the form $3j + 2$, but not conversely.

Problem 3.2.3. Use the Division Algorithm to establish the following:

a) The square of any integer is either of the form $3k$ or $3k + 1$.
b) The cube of any integer has one of the forms: $9k, 9k + 1$, or $9k + 8$.
c) The fourth power of any integer is either of the form $5k$ or $5k + 1$.

Problem 3.2.4. Prove that $3a^2 - 1$ is never a perfect square.

Problem 3.2.5. For $n \geq 1$, prove that $n(n + 1)(2n + 1)/6$ is an integer.

Problem 3.2.6. Show that the cube of any integer is of the form $7k$ or $7k \pm 1$.

Problem 3.2.7. Prove that no integer in the following sequence is a perfect square:

$$11, 111, 1111, 11111, \ldots$$

Problem 3.2.8. Verify that if an integer is simultaneously a square and a cube (as is the case with $64 = 8^2 = 4^3$), then it must be either of the form $7k$ or $7k + 1$.