Theoretical Questions Chapter 6

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December 28, 2022

I(a) : As using interpolation polynomial, the following table of divided difference

$$
\begin{array}{c|cccc}
-1 & y(-1) & & & \\
0 & y(0) & y(0)-y(-1) & & \\
0 & y(0) & y'(0) & y'(0)-y(0)+y(-1) & \\
1 & y(1) & y(1)-y(0) & y(1)-y(0)-y'(0) & \frac{y(1)-2y'(0)-y(-1)}{2}
\end{array}.
$$

Then the interpolation polynomial

$$
\begin{aligned}
&p_3(y;-1,0,0,1;x) \\
&= y(-1)+(y(0)-y(-1))(x+1)+(y'(0)-y(0)+y(-1))x(x+1)+\frac{(y(1)-2y'(0)-y(-1)}{2}x^2(x+1),
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&\int_{-1}^{1} p_3(y;-1,0,0,1;t)dt \\
&= \int_{-1}^{1} [y(-1)+(y(0)-y(-1))(t+1)+(y'(0)-(y(0)-y(-1)))t(t+1)x \\
&\quad + \frac{y(1)-2y'(0)-y(-1)}{2}t^2(t+1)]dt \\
&= 2y(-1)+2(y(0)-y(-1))+\frac{2}{3}(y'(0)-(y(0)+y(-1)))+\frac{2}{3}\frac{y(1)-2y'(0)-y(-1)}{2} \\
&= \frac{y(-1)+4y(0)+y(1)}{3} \\
&= \int_{-1}^{1} y(t)dt - E^S(y),
\end{aligned}
$$

the proof is shown.

I(b) : From the previous question, we have

$$
\begin{aligned}
&E^s(y)\\
&= \int_{-1}^{1} [y(t) - p_3(y; -1, 0, 0, 1; t)]dt\\
&= \int_{-1}^{1} \frac{y^{(4)}(\xi)}{4!} t^2(t+1)(t-1)dt \quad (Theorem 2.35)\\
&= \frac{y^{(4)}(\zeta)}{4!} \int_{-1}^{1} t^2(t+1)(t-1)dt \quad (Integration\ mean\ value\ theorem)\\
&= -\frac{y^{(4)}(\zeta)}{90}
\end{aligned}
$$

where $\xi, \zeta \in (-1, 1)$.

I(c) : For function f on general interval $[a, b]$,by substitute $x = \frac{b-a}{2}t + \frac{b+a}{2}$, $y(t) := f(\frac{b-a}{2}t + \frac{b+a}{2})$, then we can get

$$
\begin{aligned}
\int_a^b f(x)dx &= \int_{-1}^{1} f(\frac{b-a}{2}t + \frac{b+a}{2})\frac{b-a}{2}dt\\
&= \frac{b-a}{2}\int_{-1}^{1} [p_3(y; -1, 0, 0, 1; t)dt + E^(y)]dt\\
&= \frac{b-a}{2}[\frac{1}{3}(y(-1) + 4(y(0)) + y(1) + E^(y)]\\
&= \frac{b-a}{2}[\frac{1}{3}(f(a) + 4f(\frac{a+b}{2}) + f(b)) - \frac{y^{(4)}(\zeta)}{90}] \quad \zeta \in (-1, 1).
\end{aligned}
$$

Therefore, the error estimation is

$$
\begin{aligned}
E^S(f) &= -\frac{b-a}{180} f^{(4)}(\frac{b-a}{2}t + \frac{b+a}{2})|_\zeta\\
&= -\frac{(b-a)^5}{2880} f^{(4)}\xi,
\end{aligned}
$$

where $\xi \in (a, b)$. Denote $a = x_0 < x_1 < \cdots < x_n = b, \quad n = 2m \in \mathbb{N}^+$,and $h = \frac{a+b}{n}$ we have the partitions of the interval $[a, b]$ into m sub-intervals , which $[x_0, x_2], \cdots, [x_{n-2}, x_n]$, therefore

$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{k=1}^{m} \int_{2k-2}^{2k} f(x)dx\\
&= \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1} + f(x_n))]\\
&\quad + \sum_{k=1}^{m} E^S_{x_{2k-2}, x_{2k}}(f)
\end{aligned}
$$

. The composite Simpson's rule is $\frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{n-1}+f(x_n))]$,and the error estimation is

$$\begin{aligned}\sum_{k=1}^{m} E^S_{x_{2k-2},x_{2k}}(f) &= -\sum_{k=1}^{m}\frac{(2h)^5}{2880}f^{(4)}(\xi_k)\\ &= -\frac{b-a}{180}h^4\left(\frac{2}{n}\sum_{k=1}^{m}f^{(4)}(\xi_k)\right)\\ &= -\frac{b-a}{180}h^4 f^{(4)}(\xi), \quad where \quad \xi \quad \xi_k \in (a,b).\end{aligned}$$

II(a): Let $a=0, b=1, h=\frac{1}{n}$,

$$f''(x) = (4x^2-2)e^{-x^2} \qquad \max_{[0,1]}|f''(x)| = \frac{2}{e}.$$

By

$$\begin{aligned}|E^T_n(f)| = |\frac{b-a}{12}h^2 f''(\xi)| = |\frac{f''(\xi)}{12n^2}| \leq 0.5\times 10^{-6},\\ n \geq \sqrt{\frac{2}{e(12)(0.5)(10^{-6})}} \approx 350.18\end{aligned}$$

Hence, at least 351 sub-intervals are required.

II(b) : By

$$f^{(4)}(x) = (16x^4-48x^2+12)e^{-x^2} \qquad \max_{[0,1]} f^{(4)}(x) = 12,$$

and

$$\begin{aligned}|E^S_n(f)| = |\frac{1}{180n^4}f^{(4)}(\xi)| \leq \frac{1}{15n^4} \leq 0.5\times 10^{-6},\\ n \geq \sqrt[4]{\frac{12}{180(0.5)(10^{-6})}} \approx 19.11.\end{aligned}$$

Hence, at least 20 sub-intervals are required.

III(a): By $<1,\pi_2(t)>=<t,\pi_2(t)>=0$, we have

$$\begin{aligned}\int_0^{+\infty}(t^2+at+b)e^{-t}dt = 2+a+b=0,\\ \int_0^{+\infty}t(t^2+at+b)e^{-t}dt = 6+2a+b=0.\end{aligned}$$

We can get $a=-4, b=2$,then the polynomial is $\pi_2(t)=t^2-4t+2$.

III(b) : Let $\pi_2(t) = 0$,we can get $t_1 = 2 - \sqrt{2}, t_2 = 2 + \sqrt{2}$.By Corollary 6.26, we have

$$\begin{aligned} w_1 + w_2 &= \int_0^{+\infty} e^{-t} dt = 1, \\ t_1 w_1 + t_2 w_2 &= \int_0^{+\infty} t e^{-t} dt = 1. \end{aligned}$$

We can get $w_1 = \frac{2+\sqrt{2}}{4}$ and $w_2 = \frac{2-\sqrt{2}}{4}$, thus

$$I_2(f) = \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2}).$$

Therefore, the two-point Gauss-Laguerre quadrature formula is

$$\int_0^{+\infty} f(t) e^{-t} dt = \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2}) + E_2(f).$$

By Theorem 6.36,

$$\begin{aligned} E_2(f) &= \frac{f^{(4)}(\tau)}{4!} \int_0^{+\infty} (\pi_2(t))^2 e^{-t} dt \\ &= \frac{f^{(4)}(\tau)}{4!} \int_0^{+\infty} (t^4 - 8t^3 + 20t^2 - 16t + 4) e^{-t} dt \\ &= \frac{f^{(4)}(\tau)}{6} \end{aligned}$$

III(c) : Substitute $f(t) = \frac{1}{1+t}$, the approximate

$$I^G(f) = \frac{2+\sqrt{2}}{4} \frac{1}{1+2-\sqrt{2}} + \frac{2-\sqrt{2}}{4} \frac{1}{1+2+\sqrt{2}} = \frac{4}{7}.$$

The estimation error is $E_2(f) = \frac{f^{(4)}(\tau)}{6} = \frac{4}{(1+\tau)^5}$. With $I = 0.596347361$,

$$\begin{aligned} E_2(F) &= I - I^G(f) \\ \frac{4}{(1+\tau)^5} &\approx 0.02491879. \end{aligned}$$

we can get $\tau = 1.761$.

Iv(a) : Consider h_m and q_m in the form $h_m(t) = (a_m + b_m t) l_m^2(t)$, $q_m(t) = (c_m + d_m t) l_m^2(t)$, we have $h'_m(t) = b_m l_m^2(t) + 2(a_m + b_m t) l_m(t) l'_m(t)$ and $q'_m(t) = d_m l_m^2(t) + 2(c_m + d_m t) l_m(t) l'_m(t)$,where the $l'_m(x_m) = \sum_{i=1, i \neq m}^{n} \frac{1}{x_m - x_i}$.

Therefore , we solve the following equations, $\forall i = 1, 2, \ldots, n$

$$\begin{aligned}
h_m(x_m) &= a_m + b_m x_m = 1, \\
h'_m(x_m) &= b_m + 2(a_m + b_m x_m) l'_m(x_m) = 0, \\
q_m(x_m) &= c_m + d_m x_m = 1, \\
q'_m(x_m) &= d_m + 2(c_m + d_m x_m) l'_m(x_m) = 0,
\end{aligned}$$

thus, $a_m = 1 + 2x_m l'_m(x_m)$, $b_m = -2l'_m(x_m)$, $c_m = -x_m$,$d_m = 1$.

IV(b):

$$\begin{aligned}
I_n(f) &= I_n(p(f)) \\
&= \int_a^b \rho(t) \sum_{k=1}^{n} (h_k f_k + q_k f'_k) dt \\
&= \sum_{k=1}^{n} [f(x_k) \int_a^b \rho(t) h_k(t) dt + f'(x_k) \int_a^b \rho(t) q_k(t) dt] \\
&:= \sum_{k=1}^{n} [w_k f(x_k) + \mu_k f'(x_k)]
\end{aligned}$$

where $w_k = \int_a^b \rho(t) h_k(t) dt$ and $\mu_k = \int_a^b \rho(t) q_k(t) dt$.

IV(c) : Let

$$\begin{aligned}
\mu_k &= \int \rho(t)(t - x_k) l_k^2(t) dt = 0 \\
&\Rightarrow \int \rho(t) v_n(t) l_k(t) = 0, \quad \forall k = 1, 2, \cdots, n.
\end{aligned}$$

As $\mathbf{P}_{n-1} = span\{l_1, l_2, \cdots, l_n\} = span\{1, t, \cdots, t^{n-1}\}$,thus the condition is $\int_a^b \rho(t) v_n(t) p(t) dt = 0$.

By Taylor Expansion,

$$u(\bar{x}-h) = u(\bar{x}) - hu'(\bar{x}) + \tfrac{1}{2}h^2u''(\bar{x}) - \tfrac{1}{6}h^3u'''(\bar{x}) + \tfrac{1}{24}h^4u^{(4)}(\xi_1) \qquad \xi_1 \in (\bar{x}-h, \bar{x})$$

$$u(\bar{x}+h) = u(\bar{x}) + hu'(\bar{x}) + \tfrac{1}{2}h^2u''(\bar{x}) + \tfrac{1}{6}h^3u'''(\bar{x}) + \tfrac{1}{24}h^4u^{(4)}(\xi_2) \qquad \xi_2 \in (\bar{x}, \bar{x}+h)$$

$$\Rightarrow \quad u''(\bar{x}) = \frac{u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)}{h^2} - \frac{1}{24}h^2\left[u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right]$$

$$D^2u(\bar{x}) = \frac{u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)}{h^2} = u''(\bar{x}) + \frac{1}{12}h^2u^{(4)}(\xi) \qquad \xi \in (\bar{x}-h, \bar{x}+h)$$

$$\therefore \quad D^2u(\bar{x}) = u''(\bar{x}) + O(h^2)$$

Thus, $D^2u(\bar{x})$ is second-order accurate.

Consider $\exists\, \varepsilon \in [-E, E)$.

let

$$\hat{u}(\bar{x}-h) = u(\bar{x}+h) + \varepsilon_1 = u(\bar{x}) - hu'(\bar{x}) + \tfrac{1}{2}h^2u''(\bar{x}) - \tfrac{1}{6}h^3u'''(\bar{x}) + \tfrac{1}{24}h^4u^{(4)}(\xi_1) + \varepsilon_1$$

$$\hat{u}(\bar{x}+h) \quad u(\bar{x}+h) + \varepsilon_2 = u(\bar{x}) + hu'(\bar{x}) + \tfrac{1}{2}h^2u''(\bar{x}) + \tfrac{1}{6}h^3u'''(\bar{x}) + \tfrac{1}{24}h^4u^{(4)}(\xi_2) + \varepsilon_2$$

$$\hat{u}(\bar{x}) = u(\bar{x}) + \varepsilon_3$$

$$\Rightarrow D^2u(\bar{x}) = u''(\bar{x}) + \frac{1}{12}h^2u^{(4)}(\xi) + \frac{\varepsilon_1 - 2\varepsilon_3 + \varepsilon_2}{h^2}$$

By triangle inequality, $\quad |u''(\bar{x}) - D^2u(\bar{x})| \leq \frac{h^2}{12}|u^{(4)}(\xi)| + \frac{4E}{h^2} \qquad \xi \in (\bar{x}-h, \bar{x}+h)$

We have

$$u(\bar{x}-2h) + u(\bar{x}+2h) = 2u(\bar{x}) + 4h^2u''(\bar{x}) + \frac{4}{3}h^4u^{(4)}(\bar{x}) + \frac{8}{45}h^6u^{(6)}(\xi_1) \qquad \xi_1 \in (\bar{x}-2h, \bar{x}+2h)$$

$$u(\bar{x}-h) + u(\bar{x}+h) = 2u(\bar{x}) + h^2u''(\bar{x}) + \frac{1}{12}h^4u^{(4)}(\bar{x}) + \frac{1}{360}h^6u^{(6)}(\xi_2) \qquad \xi_2 \in (\bar{x}-h, \bar{x}+h)$$

Thus, $u''(\bar{x}) = \dfrac{-u(\bar{x}-2h) + 16u(\bar{x}-h) - 30u(\bar{x}) + 16u(\bar{x}+h) - u(\bar{x}+2h)}{12h^2} + \dfrac{h^4}{540}\left(8u^{(6)}(\xi_1) - 2u^{(6)}(\xi_2)\right)$

$= D^4 u(\bar{x}) + O(h^4)$

$\Rightarrow \quad |u''(\bar{x}) - D^4 u(\bar{x})| \leq \dfrac{h^4}{54} M + \dfrac{16E}{3h^2}$, where $M = \max\limits_{[\bar{x}-2h,\, \bar{x}+2h]} |u^{(6)}(x)|$

let $\dfrac{h^4}{54} M + \dfrac{16E}{3h^2} = 0$

$$\Rightarrow h = \sqrt[6]{\frac{144E}{M}}$$

It is required three initial points for second order accurate and required five initial points for forth order accurate. As a conclusion, it is required more initial points to get the higher order accuracy.