Theoretical Questions Chapter 5

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December 14, 2022

I : We need to prove that the following equations statifies axioms of inner product space over $\mathbb{C}$.

$$\langle u, v\rangle = \int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t. \tag{1}$$

(1) real positivity:

$$\forall u \in \mathcal{C}[a,b], \quad \langle u, u\rangle = \int_a^b \rho(t)u(t)\overline{u(t)}\mathrm{d}t = \int_a^b \rho(t)|u(t)|^2\mathrm{d}t \geq 0. \tag{2}$$

(2) definiteness:

$$\langle u, u\rangle = 0 \Leftrightarrow \rho(t)|u(t)|^2 = 0 \Leftrightarrow |u(t)|^2 = 0 \Leftrightarrow u = 0 \tag{3}$$

(3) additivity in the first slot:

$$\forall u, v, w \in \mathcal{C}[a,b], \quad \langle u+w, v\rangle = \int_a^b \rho(t)(u(t)+w(t))\overline{v(x)}\mathrm{d}t = \int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t + \int_a^b \rho(t)w(t)\overline{w(t)}\mathrm{d}t = \langle u, v\rangle \tag{4}$$

(4) homogeneity in the first slot:

$$\forall c \in \mathcal{C}, \quad \forall u, v \in \mathcal{C}[a,b], \quad \langle cu, v\rangle = c\int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t = c\langle u, v\rangle \tag{5}$$

(5) conjugate symmetry :

$$\forall u, v \in \mathcal{C}[a,b] \quad \langle u, v\rangle = \int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t = \overline{\int_a^b \overline{\rho(x)u(x)\overline{v(x)}}\mathrm{d}x} = \overline{\int_a^b \rho(x)v(x)\overline{u(x)}\mathrm{d}x} = \overline{\langle v, u\rangle} \tag{6}$$

We also need to prove the axioms of norm, (1) real positivity:

$$\|u\|_2 = \left(\int_a^b \rho(t)|u(t)|^2\mathrm{d}t\right)^{1/2} \geq 0 \tag{7}$$

(2) definiteness:

$$\|u\|_2 = 0 \Leftrightarrow \rho(t)|u(t)|^2 = 0 \Leftrightarrow |u(t)|^2 = 0 \Leftrightarrow u = 0 \tag{8}$$

(3) homogeneity

$$\forall c \in \mathbb{C}, \quad \|cu\|_2 = \left(\int_a^b \rho(t)|cu(t)|^2 \mathrm{d}t\right)^{1/2} = |c| \left(\int_a^b \rho(t)|u(t)|^2 \mathrm{d}t\right)^{1/2} = |c| \, \|u\|_2 \tag{9}$$

(4) triangle inequality:

$$\forall u, v \in \mathcal{C}[a, b], \quad \|u + v\|_2 = \left(\int_a^b \rho(x)|u(x) + v(x)|^2 \mathrm{d}x\right)^{1/2} \tag{10}$$

$$\leq \left(\int_a^b \rho(x)|u(x)|^2 \mathrm{d}x\right)^{1/2} + \left(\int_a^b \rho(x)|v(x)|^2 \mathrm{d}x\right)^{1/2} = \|u\|_2 + \|v\|_2 \tag{11}$$

II:By Definition 2.41, $T_n(x) = \cos(n \arccos(x))$,
(a): For $\forall m, n \, \langle Tm, Tn \rangle = \int_{-1}^{1} \rho(t) T_n(t) \overline{T_m(t)} dt$
$= \int_{-1}^{1} \frac{\cos(n \arccos t) \cos(m \arccos t)}{\sqrt{1-t^2}} dt$
$= \int_0^{\pi} \cos(m\theta) \cos(n\theta) d\theta$
$= \int_0^{\pi} \frac{\cos(m\theta + n\theta)}{\cos(m\theta - n\theta)} d\theta$
$= \begin{cases} \frac{\pi}{2} \quad when \quad m = n \neq 0 \\ 0, \quad when \quad m \neq n \\ \pi \quad when \quad m = n = 0 \end{cases}$
Therefore, T_n are orthogonal.

(bi): We have $T_0(x) = 1$, $T_1(x) = x$,$T_2(x) = 2x^2 - 1$, after normalized , we get $T_0^*(x) = \frac{1}{\sqrt{\pi}}$, $T_1^*(x) = \sqrt{\frac{\pi}{2}} x$ and $T_2^*(x) = \sqrt{\frac{2}{\pi}}(2x^2 - 1)$.

III(a): With the basis (T_0^*, T_1^*, T_2^*),the Fourier coefficients are $\langle y, T_0^* \rangle = \frac{2}{\sqrt{\pi}}$, $\langle y, T_1^* \rangle = 0$, $\langle y, T_2^* \rangle = -\frac{2}{3}\sqrt{\frac{2}{\pi}}$, the approximate function is $\hat{\phi}(x) = \frac{2}{\sqrt{\pi}} T_0^* + 0T_1^* + -\frac{2}{3}\sqrt{\frac{2}{\pi}} T_2^* = \frac{10}{3\pi} - \frac{8}{3\pi} x^2$

(b):

$$G(1, x, x^2) = \begin{bmatrix} \langle 1,1 \rangle & \langle 1,x \rangle & \langle 1,x^2 \rangle \\ \langle 1,1 \rangle & \langle 1,x \rangle & \langle 1,x^2 \rangle \\ \langle 1,1 \rangle & \langle 1,x \rangle & \langle 1,x^2 \rangle \end{bmatrix} = \begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{2} \end{bmatrix}$$

$$c = (\langle y, 1 \rangle, \langle y, x \rangle, \langle y, x^2 \rangle)^T = (2, 0, 3)^T$$

We can solve the equation $G^T a = c$, then we can get $a = (\frac{10}{3\pi}, 0, -\frac{8}{3\pi})$,thus the approximate function$\hat{\phi}(x) = \frac{10}{3\pi} - \frac{8}{3\pi} x^2$

IV (a): Using the monomials $(1, x, x^2)$,with inner product$\langle u, v \rangle = \sum_i^{12} u(t_i)v(t_i)$,then we have

$$u_1 = v_1 = 1, \|v_1\| = \sqrt{12}, u_1^* = \frac{1}{2\sqrt{3}}$$

,

$$v_2 = u_2 - \langle u_2, u_1^* \rangle u_1^* = x - \frac{13}{2}, u_1^* = \frac{1}{\sqrt{143}}(x - \frac{13}{2})$$

,

$$v_3 = u_3 - \langle u_3, u_1^* \rangle u_1^* - \langle u_3, u_2^* \rangle u_2^* = x^2 - 13x + \frac{91}{3}, u_3^* = \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3})$$

(b): The best approximate function is

$$\hat{\varphi}$$

$\hat{\varphi}(x) = \langle y, u_1^* \rangle u_1^* + \langle y, u_2^* \rangle u_2^* + \langle y, u_3^* \rangle u_3^*$
$= \frac{831}{\sqrt{3}} u_1^* + \frac{589}{\sqrt{143}} u_2^* + \frac{12068\sqrt{3}}{\sqrt{4004}} u_3^*$
$\approx 9.042x^2 - 113.4266x + 386.0013$

(c): The orthonormal polynomials can be reused but the normal equation cannot be reused. Due to we need to recalculated G and solving equation but the previous method just renew index of basis,therefore orthonormal polynomials has advantage over normal equations.