林修弘 3200300602 数值理论作业

1.8.1 Theoretical questions

I. Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the nth step?
- What is the 上限至 supremum of the distance between the root r and the midpoint of the interval?

I: $W_0 = 3.5 - 1.5 = 2$

At the nth step, the width $W_n = \frac{W_0}{2^n} = \frac{1}{2^{n-1}}$

Supremum of the distance bewteen the root r and the midpoint of the interval is 1.

II. In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its *relative* error no greater than ϵ. Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1.$$

Denote the root be $x \in [a_0, b_0]$,

Using Bisection Method: $c_0 = b_0 - a_0$ $\quad c_n = \frac{1}{2^n}(b_0 - a_0)$

relative error: $\frac{|x - c_n|}{x} \leq \frac{\frac{1}{2} c_n}{x} \leq \frac{\frac{1}{2} c_n}{a_0} = \frac{b_0 - a_0}{2^{n+1} a_0}$

when the relative error is not greater than ε,

then $\frac{b_0 - a_0}{2^{n+1}\, a_0} \leq \varepsilon$

$\Rightarrow \quad 2^n \geq \frac{b_0 - a_0}{2\varepsilon a_0}$

$\log 2^n \geq \log \frac{b_0 - a_0}{2\varepsilon a_0}$

$n \log 2 \geq \log(b_0 - a_0) - \log\varepsilon - \log a_0 - \log 2$

$\therefore \quad n \geq \frac{\log(b_0 - a_0) - \log\varepsilon - \log a_0}{\log 2} - 1$

III. Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

$p(x) = 4x^3 - 2x^2 + 3 = 0 \qquad p'(x) = 12x^2 - 4x$

Newton Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$p(x_0) = -3$

$p'(x_0) = 16$

$x_1 = -1 - \frac{-3}{16} = -0.8125$

$p(x_1) = -0.4658$

$p'(x_1) = 11.1719$

$x_2 = -0.8125 - \frac{-0.4658}{11.1719} = -0.7708$

$p(x_2) = -0.0201$

$p'(x_2) = 10.2128$

$x_3 = -0.7708 - \frac{-0.0201}{10.2128} = -0.7688$

$p(x_3) = 2.86 \times 10^{-4}$

$p'(x_3) = 10.1678$

$$x_4 = -0.7688 - \frac{2.86 \times 10^{-4}}{10.1678} = -0.7688$$

n	x_n	$p(x_n)$	$p'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	-3	16	-0.8125
1	-0.8125	-0.4658	11.1719	-0.7708
2	-0.7708	-0.0201	10.2136	-0.7688
3	-0.7688	2.86×10^{-4}	10.1678	-0.7688
4	-0.7688			

IV. Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

Find C and s such that

$$e_{n+1} = Ce_n^s,$$

where e_n is the error of Newton's method at step n, s is a constant, and C may depend on x_n, the given function f and its derivatives.

let the root be α, $e_n = x_n - \alpha$

By Taylor Theorem, we know that $f(\alpha) = f(x_n) + (\alpha - x_n) f'(\xi)$ $\quad \xi \in [x_n, \alpha]$

$$\Rightarrow \quad 0 = f(x_n) + (\alpha - x_n) f'(\xi)$$

$$f(x_n) = f'(\xi)(x_n - \alpha)$$

$$= f'(\xi) e_n \quad \text{——} \quad ①$$

$$e_{n+1} = x_{n+1} - \alpha$$

$$e_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} - \alpha = e_n - \frac{f(x_n)}{f'(x_0)} \quad \text{——} \quad ②$$

substitute ① into ②,

$$e_{n+1} = e_n - \frac{e_n f'(\xi)}{f'(x_0)}$$

$$= e_n \left(1 - \frac{f'(\xi)}{f'(x_0)}\right)$$

Comparing with $e_{n+1} = C e_n^s$

$\therefore \quad s = 1 \qquad C = \left(1 - \frac{f'(\xi)}{f'(x_0)}\right)$

V. Within $(-\frac{\pi}{2}, \frac{\pi}{2})$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

① when $x_0 = 0$, it is obviously that iteration converges.

② if $0 < x_0 < \frac{\pi}{2}$, since we have $0 < \tan^{-1} x < x$

$0 < x_{n+1} = \tan^{-1} x_n < x_n$

Since $\{x_n\}$ is stricly decreasing sequence with lower bound 0

By Monotonic sequence theorem (Thm 1.(2)), $\{x_n\}$ converges

③ if $-\frac{\pi}{2} < x_0' < 0 \xRightarrow{\text{let } x_0 = -x_0'} 0 < x_0 < \frac{\pi}{2}$,

from ②, we know that $\{x_n'\}$ converges

$\therefore$ Iteration is convergent.

VI. Let $p > 1$. What is the value of the following continued fraction?

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \dots}}}$$

Prove that the sequence of values converges. (Hint: this can be interpreted as $x = \lim_{n\to\infty} x_n$, where $x_1 = \frac{1}{p}$, $x_2 = \frac{1}{p+\frac{1}{p}}$, $x_3 = \frac{1}{p+\frac{1}{p+\frac{1}{p}}}$, ..., and so forth. Formulate x as a fixed point of some function.)

If $x_1 = \frac{1}{p}$, then we have $x_n = \frac{1}{p + x_{n-1}}$

as $x = \lim_{n\to\infty} x_n$

Clearly that for $p > 1$, $x \in (0,1)$

let $f(x) = \frac{1}{p+x}$, $f(x) \in (0,1)$

$$\lambda = \max_{x \in [0,1]} |f'(x)| = \max_{x \in [0,1]} \left| -\frac{1}{(x+p)^2} \right| < \frac{1}{p^2} < 1$$

By Theorem 1.39, f has a unique fixed point α in $[0,1]$

$$f(\alpha) = \alpha$$

$$\frac{1}{p+\alpha} = \alpha$$

$$\alpha^2 + p\alpha - 1 = 0$$

$$\alpha = \frac{-p \pm \sqrt{p^2+4}}{2}$$

$\therefore$ The sequence $\{x_n\}$ is convergent, the fixed point $\alpha \in [0,1]$, the value of

$$x = \alpha = \frac{-p \pm \sqrt{p^2+4}}{2}$$

VII. What happens in problem II if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in II. In this case, is the relative error still an appropriate measure?

Denote the root be $\alpha \in [a_0, b_0]$,

Using Bisection Method : $c_0 = b_0 - a_0$ $\quad c_n = \frac{1}{2^n}(b_0 - a_0)$

relative error : $\frac{|\alpha - c_n|}{|\alpha|} \leq \frac{\frac{1}{2} c_n}{|\alpha|} = \frac{\frac{1}{2^{n+1}}(b_0 - a_0)}{|\alpha|}$

when the relative error is not greater than ε,

then $\frac{b_0 - a_0}{2^{n+1}|\alpha|} \leq \varepsilon$

$$\Rightarrow \quad 2^n \geq \frac{b_0 - a_0}{2\varepsilon|\alpha|}$$

$$\log 2^n \geq \log \frac{b_0 - a_0}{2\varepsilon|\alpha|}$$

$$n \log 2 \geq \log(b_0 - a_0) - \log\varepsilon - \log|\alpha| - \log 2$$

$$n \geq \frac{\log(b_0 - a_0) - \log\varepsilon - \log|\alpha|}{\log 2} - 1$$

However, $\alpha \to 0$, $\quad n \geq +\infty$, it is contradiction.

$\therefore$ The relative error can't be an appropriate measure.