# Thoeretical Questions Chapter 3

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**I**: By substitution, $p(0) = s(0) = 0$ , $p(1) = s(1) = (2-1)^3$ , $p'(1) = s'(1) = -3(2-1)^3 = -3$ , $p''(1) = s''(1) = 6(2-1) = 6$. The following table:

| x | | | | |
|---|---|---|---|---|
| 0 | 0 | | | |
| 1 | 1 | 1 | | |
| 1 | 1 | -3 | -4 | |
| 1 | 1 | -3 | 6 | 7 |

The cubic polynomial can be written as

$$s(x) = 0 + x - 4x(x-1) + 7(x-1)^2 = 7x^3 - 18x^2 + 12x.$$

Since $s''(2) = -36$, then $s(x)$ is not a natural cubic spline.

**II (a)**: Denote $p_i(x) = s|_{[x_i,x_{i+1}]}, i = 0, 1, 2..., n-1$. The splines are given by $s_i(x) = a_i x^2 + b_i x + c_i fori = 1, 2, 3..., n-1$, which has totally $3(n-1)$ variables. Each quadratic spine goes through two consecutive data points,

$$p_i(x_i) = f_i, p_i(x_{i+1}) = f_{i+1}, i = 1, 2..., n-1$$

these conditions give 2(n-1) equations. The first derivatives of two quadratic splines are continuing at the interior points, thus have

$$p_i'(x_i) = p_{i+1}'(x_{i+1}), i = 1, 2..., n-2.$$

By the property of $s(x)$ , totally construct $2(n-1) + (n-2) = 3n-4$ equations. Since the the number of variables are more than the number o equations, a additional condition is required to determine uniquely.

**II (b)** By the property of $s(x)$ , we have $p_i(x_i) = f_i, p_i(x_{i+1}) = f_{i+1}$,then the following table of divided difference

| x | | | |
|---|---|---|---|
| $x_i$ | $f_i$ | | |
| $x_i$ | $f_i$ | $m_i$ | |
| $x_{i+1}$ | $f_{i+1}$ | $\frac{f_{i+1}-f_i}{x_{i+1}-x_i}$ | $\frac{f_{i+1}-f_i-(x_{i+1}-x_i)m_i}{(x_{i+1}-x_i)^2}$ |

The Newton's Formula yields

$$p_i(x) = f_i + m_i(x-x_i) + \frac{f_{i+1} - f_i - (x_{i+1} - x_i)m_i}{(x_{i+1} - x_i)^2}(x-x_i)^2 . i = 1, 2...n-1 \quad (1)$$

**II (c)** Differentiate (1), let $x = x_{i+1}$, we have

$$p_i'(x_{i+1}) = m_i + \frac{2[f_{i+1} - f_i - (x_{i+1} - x_i)m_i]}{x_{i+1} - x_i}.$$

Since $p_i/(x_{i+1}) = m_{i+1}$, we have

$$m_{i+1} = \frac{m_i(x_{i+1} - x_i) + 2[f_{i+1} - f_i - 2(x_{i+1} - x_i)m_i]}{x_{i+1} - x_i}$$

$$\Rightarrow m_{i+1} = \frac{2[f_{i+1} - f_i]}{x_{i+1} - x_i} - m_i,$$

$$\begin{cases} m_i = f'(a), \\ m_{i+1} = 2f[x_i, x_{i+1}] - m_i i = 1, 2..., n-1. \end{cases}$$

We get general formula $m_i = 2\sum_{k=1}^{i}[(-1)^{k+1}f[x_k, x_{k+1}]] + f'(a)$

**III**: We have $s_2 = (0) = s_1(0) = 1 + c$, $s_2' = (0) = s_1'(0) = 3c$ , $s_2'' = (0) = s_1''(0) = 6c.s(x)$ is a natural cubic spline, thus$s''(1) = s''(-1) = 0$. Let $x_1 = -1$,$x_2 = 0$,$x_3 = 1$ ,$M_1 = s''(-1) = 0$ ,$M_2 = s''(0) = 6c$, $M_3 = s''(1) = 0$.So that,

$$s_2'''(0) = \frac{M_3 - M_2}{x_3 - x_2} = \frac{0 - 6c}{1 - 0} = -6c.$$

Taylor expansion of $s_2(x)$ at $x_2 = 0$ yields

$$s_2(x) = s(0) + s'(0)x + \frac{M_2}{2}x^2 + \frac{s'''(0)}{6}x^3 = 1 + c + 3cx + 3cx^2 - cx^3.$$

When $s(-1) = -1$, then$-1 = 1 + c + 3c + 3c - c \Rightarrow c = \frac{1}{3}$ .

**IV (a)**: let$x_1 = -1, x_2 = 0, x_3 = 1$, we know that $M_1 = s''(-1) = 0, M_3 = s''(-1) = 0, \mu_2 = \lambda_2 = \frac{1}{2}$. From Lemma 3.4, $\frac{1}{2}M_1 + 2M_2 + \frac{1}{2}M_3 = -12 \Rightarrow M_3 = -3$.

The following table of divided difference of f :

| x | | | |
|---|---|---|---|
| -1 | 0 | | |
| 0 | 1 | 1 | |
| 1 | 0 | -1 | -1 |

$$s_1'(-1) = f[-1, 0] - \frac{1}{6}(M_2 + 2M_1)(x_2 - x_1) = \frac{3}{2}$$

$$s_2'(-1) = f[0, 1] - \frac{1}{6}(M_3 + 2M_2)(x_3 - x_2) = 0$$

Taylor expansion of s(x) at $x_i$:

$$s(x) = \begin{cases} s_1(x) = \frac{3}{2}(x+1) - \frac{1}{2}(x+1)^3 = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1, x \in [-1, 0], \\ s_2(x) = 1 + 0x - \frac{3}{2}x^2 + \frac{3}{6}x^3 = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1, x \in [0, 1]. \end{cases}$$

**IV(b)-i**:By Newton formula,$g(x) = (x+1) - x(x+1) = 1 - x^2$,we have $g''(x) = -2$,$s_1''(x) = -3(x+1), s_2''(x) = 3(x-1)$.

$$\int_{-1}^{1} [g''(x)]^2 dx = \int_{-1}^{1} 4dx = 8$$

$$\int_{-1}^{1} [s''(x)]^2 dx = 9[\int_{-1}^{0} (x-1)^2 dx + \int_{0}^{1} (x-1)^2] = 6$$

Therefore, $\int_{-1}^{1} [g''(x)]^2 dx > \int_{-1}^{1} [s''(x)]^2 dx$

**IV(b)-ii**: We have $g(x) = f(x) = cos(\frac{\pi}{2}x)$, then $g''(x) = -\frac{\pi^2}{4} cos(\frac{\pi}{2}x)$

$$\int_{-1}^{1} [g''(x)]^2 dx = \frac{\pi^4}{16} \int_{-1}^{1} cos^2(\frac{\pi}{2}x) dx = \frac{\pi^4}{16}$$

Therefore, $\int_{-1}^{1} [g''(x)]^2 dx > \int_{-1}^{1} [s''(x)]^2 dx$

**V(a)**:By definition 3.23 and the hat function , we have

$$B_i^1(x) = \frac{x - t_{i-1}}{t_i - t_{i-1}} B_i^0(x) + \frac{t_{i+1} - x}{t_{i+1} - t_i} B_{i+1}^0(x).$$

$$B_i^1(x) = \hat{B}_i = \begin{cases} \frac{x - t_{i-1}}{t_i - t_{i-1}} & , x \in (t_{i-1}, t_i] \\ \frac{t_{i+1} - x}{t_{i+1} - t_i} & , x \in (t_i, t_{i+1}] \\ 0 & , otherwise. \end{cases}$$

Since $B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_{i+1}^1(x)$ , thus we have

$$B_i^2(x) = \begin{cases} \frac{(x - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})} & , x \in (t_{i-1}, t_i] \\ \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \frac{t_{i+1} - x}{t_{i+1} - t_i} + \frac{t_{i+2} - x}{t_{i+2} - t_i} \frac{x - t_i}{t_{i+1} - t_i} & , x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} & x \in (t_{i+1}, t_{i+2}] \end{cases}$$

原本上星期，以为四题想用latex作答 结果这星期的四题latex有点繁琐，后半段就改为手写了

V-b: When $x = t_i$, we have

$$\lim_{x \to t_i^+} \frac{d}{dx} B_i^2(x) = \lim_{x \to t_i^+} \frac{2(x - t_{i-1})}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})} = \frac{2}{t_{i+1} - t_{i-1}}$$

$$\lim_{x \to t_i^-} \frac{d}{dx} B_i^2(x) = \lim_{x \to t_i^-} \left[ \frac{t_{i+1} + t_{i-1} - 2x}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{t_{i+2} + t_i - 2x}{(t_{i+2} - t_i)(t_{i+1} - t_i)} \right] = \frac{2}{t_{i+1} - t_{i-1}}$$

Since $\lim_{x \to t_i^+} \frac{d}{dx} B_i^2(x) = \lim_{x \to t_i^-} \frac{d}{dx} B_i^2(x)$, $\frac{d}{dx} B_i^2(x)$ is continuous at $t_i$

When $x = t_{i+1}$, we have

$$\lim_{x \to t_{i+1}^+} \frac{d}{dx} B_i^2(x) = \lim_{x \to t_{i+1}^+} \frac{-2(t_{i+2} - x)}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} = -\frac{2}{t_{i+2} - t_i}$$

$$\lim_{x \to t_{i+1}^-} \frac{d}{dx} B_i^2(x) = \lim_{x \to t_{i+1}^-} \left[ \frac{t_{i+1} + t_{i-1} - 2x}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{t_{i+2} + t_i - 2x}{(t_{i+2} - t_i)(t_{i+1} - t_i)} \right] = -\frac{2}{t_{i+2} - t_i}$$

Since $\lim_{x \to t_{i+1}^+} \frac{d}{dx} B_i^2(x) = \lim_{x \to t_{i+1}^-} \frac{d}{dx} B_i^2(x)$, $\frac{d}{dx} B_i^2(x)$ is continuous at $t_{i+1}$

V-c: $\forall x \in (t_{i-1}, t_i]$, $\frac{d}{dx} B_i^2(x) > 0$ and $\forall x \in (t_i, t_{i+1}]$, $\frac{d}{dx} B_i^2(x) < 0$

With (b), then $\exists! \, x^* \in (t_{i-1}, t_{i+1})$ such that $\frac{d}{dx} B_i^2(x^*) = 0$

$$\Rightarrow \frac{t_{i+1} + t_{i-1} - 2x^*}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{t_{i+2} + t_i - 2x^*}{(t_{i+2} - t_i)(t_{i+1} - t_i)} = 0$$

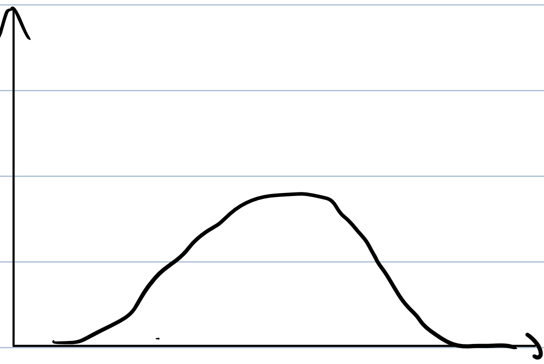
$$x^* = \frac{t_{i+1} t_{i+2} - t_{i-1} t_i}{t_{i+2} + t_{i+1} - t_i - t_{i-1}}$$

V- d : $\because$ $B_i^2(x) = B_i^2(t_{i+2}) = 0$ , $B_i^2(x) < 0$ $x \in (t_{i-1}, x^*)$ , $B_i^2(x) > 0$ $x \in (x^*, t_{i+2})$

$$\frac{d}{dx} B_i^2(x^*) = \frac{t_{i+2} - t_{i-1}}{t_{i+2} + t_{i-1} - t_{i-1} - t_i}$$

so that $B_i^2(x^*) < 1$, $\Rightarrow B_i^2(x) \in [0, 1)$

V- e : For $t_i = i$



VI : We have $B_i^0(x) = (t_i - t_{i-1})[t_{i-1}, t_i](t-x)_+^0$

As we know that $(t-x)_+^1 = (t-x)(t-x)_+^0$ $\quad (t-x)_+^2 = (t-x)(t-x)_+^1$

$$(t_{i+1} - t_{i-1})[t_{i-1}, t_i, t_{i+1}](t-x)_+^1 = (t_{i-1} - x)[t_i, t_{i+1}](t-x)_+^0 - (t_{i-1} - x)[t_{i-1}, t_i](t-x)_+^0 + (t_{i+1} - t_{i-1})[t_i, t_{i+1}](t-x)_+^1$$

$$= \frac{x - t_{i-1}}{t_i - t_{i-1}} B_i^0(x) + \frac{t_{i+1} - x}{t_{i+1} - t_i} B_i^0(x)$$

$$= B_i^1(x)$$

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = (t_{i-1} - x)\{[t_i, t_{i+1}, t_{i+2}](t-x)_+^1 - [t_{i-1}, t_i, t_{i+2}](t-x)_+^1\}$$

$$+ (t_{i+2} - t_{i-1})[t_i, t_{i+1}, t_{i+2}](t-x)_+^1$$

$$= \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_i^1(x)$$

$$= B_i^2(x)$$

In conclusion, $(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2(x)$ is proved.

VII : By theorem of derivatives of B-splines,

$$\frac{d}{dx} B_i^n(x) = \frac{n B_i^{n-1}(x)}{t_{i+n-1} - t_{i-1}} - \frac{n B_{i+1}^{n-1}(x)}{t_{i+n} - t_i}$$

$$\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n} - t_{i-1}} dx - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i} dx = \int_{t_{i-1}}^{t_{i+n+1}} \left( \frac{B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i} \right) dx$$

$$= \frac{1}{n} B_i^{(n+1)}(x) \Big|_{t_{i-1}}^{t_{i+n+1}}$$

$$= 0$$

$$\therefore \int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n} - t_{i-1}} dx = \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i} dx$$

then the statement is true.

VII (a) when $m=4$, $n=2$, we have divided difference table

| | | | |
|---|---|---|---|
| $x$ | | | |
| $x_1$ | $x_1^4$ | | |
| $x_2$ | $x_2^4$ | $x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$ | |
| $x_3$ | $x_3^4$ | $x_2^3 + x_2^2 x_3 + x_2 x_3^2 + x_3^3$ | $x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1$ |

$$T_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1 = [x_1, x_2, x_3]x^4$$

(b) $(x_{n+1} - x_0)\, \tau_k (x_0, \dots x_n, x_{n+1})$

$= \tau_{k+1} (x_0, \dots x_n, x_{n+1}) - \tau_{k+1} (x_0, \dots, x_n) - x_0 \tau_k (x_0, \dots, x_n, x_{n+1})$

$= \tau_{k+1} (x_1, \dots x_n, x_{n+1}) - \tau_{k+1} (x_0, \dots x_n)$

When $n=0$, for $\forall m$, we have $\tau_m (x_0) = [x_0] x^m$.

Suppose for $n<m$, the statement is true,

$\tau_{m-n-1} (x_0, \dots x_{n+1})$

$$= \frac{\tau_{m-n} (x_1, \dots x_n, x_{n+1}) - \tau_{m-n} (x_0, \dots x_n)}{x_{n+1} - x_0}$$

$$= \frac{[x_1, \dots x_{n+1}] x^m - [x_0, \dots x_n] x^m}{x_{n+1} - x_0}$$

$= [x_0, \dots x_{n+1}] x^m$