1 Introduction to Dynamic Programming.

The vast majority of the interesting problems in economics are dynamic (have many periods) and involve taking optimal decisions across those periods. The solution to such problems is technically called dynamic optimization. But to start our course we will first drop the optimization part and focus on the dynamic concept to introduce a variety of concepts and a framework we will use extensively.

1.1 Example 1. A Bellman equation.

Consider an asset that pays x units every period forever. If the risk free interest rate is given by r what is the price of this asset today? It should be the simple discounted (**geometric**) **sum** of cash payments from this asset. Let $\beta = \frac{1}{1+r}$

$$S = x + \beta x + \beta^2 x \quad = x\left(1 + \beta + \beta^2 \quad \right)$$

and we can find this sum by subtracting $S \quad \beta S$

$$\begin{aligned} S \quad \beta S &= S(1 \quad \beta) = x \\ S &= \frac{x}{1 \quad \beta} = \frac{1+r}{r} x \end{aligned}$$

Now this equation can also be stated in the following form:

$$S = x + \beta S$$

and this is a trivial form of what is called a "**Bellman**" equation. It is a simple dynamic equation which says that *today's sum is just today's payment plus tomorrow's sum discounted back to today.* In antecipation of what we will study later we can also solve this equation using an **algorithm**: we are looking for the sum S so we start by giving it an initial value $S_0 = k$ and any value will do and then we iterate

$$\begin{aligned} S_1 &= x + \beta S_0 \\ S_0 &= S_1 \end{aligned}$$

and due to properties we will discuss later (**contraction** property) this algorithm will converge to the solution ($S = x/(1 \quad \beta)$).[1]

We can rearrange our sum in yet another way:

$$\begin{aligned} S &= x + \frac{1}{1+r} S \\ S(1+r) &= x(1+r) + S \\ rS &= x(1+r) + 0 \end{aligned}$$

[1] If you do not believe it try it yourself by hand and see what happens to s as you replace the previous S in the Bellman equation.

or simply that the return on the asset (rS) is equivalent to the dividend (x(1+r))[2] plus the capital gain which in this case is zero because nothing changes between two periods and the asset is a perpetuity. S is therefore unique! This equation is called an **arbitrage** *(or asset pricing) equation*.

Computer Exercise: write a simple matlab program with a loop to compute S by a simple sum and a Bellman equation.[3]

1.2 Example 2: x as a state variable deterministic.

Suppose now that x instead of being a constant actually increased over time.

$$x_t = x_0 + \alpha t$$

Consider now the problem of computing our sum at time zero.

$$S(x_0) = x_0 + \beta\left(x_0 + \alpha\right) + \beta^2\left(x_0 + 2\alpha\right) + \beta^3\left(x_0 + 3\alpha\right) +$$

Now this problem has solution (show this)

$$S(x_0) = \frac{x_0}{1 \quad \beta} + \frac{\beta\alpha}{(1 \quad \beta)^2}$$

But how do we write our Bellman equation?

Well here we start by defining $x_1 = x_0 + \alpha$ and rewrite the sum

$$\begin{aligned} S(x_0) &= x_0 + \beta\left(x_0 + \alpha\right) + \beta^2\left(x_0 + 2\alpha\right) + \beta^3\left(x_0 + 3\alpha\right) + \\ S(x_0) &= x_0 + \beta\left(x_1\right) + \beta^2\left(x_1 + \alpha\right) + \beta^3\left(x_1 + 2\alpha\right) + \\ S(x_0) &= x_0 + \beta\left[x_1 + \beta\left(x_1 + \alpha\right) + \beta\left(x_1 + 2\alpha\right) + \quad\right] \\ S(x_0) &= x_0 + \beta S(x_1) \end{aligned}$$

and we can do this because what are computing is not just a *number* S as above but in reality a **function** S(x). This function is here a straight line a+bx.

The crucial fact is that we do NOT write the Bellman equation as

$$S_0(x_0) = x_0 + \beta S_1(x_1)$$

that is **the function S does not have itself an index**. We are still trying to find a unique function but now with an argument that may change.

What has happened? Well now the problem is no longer the same whether we wake up today or tomorrow and try to solve it. But the solution is the same *function*. However it has an argument inside and that may be different every period. It summarizes here the state of the world outside your window when you first wake up in the morning. It is called a **state variable** of the system.[4]

Computer exercise: write a loop to build a vector with each period s payoff. At the end sum that vector (sv=sum(v)).

[2] We receive x at the beggining of the period, and so capitalize it by 1+r.

[3] x0=1;beta=0.9; vv= x0]; for i=1:10, vv= vv ; x0*(beta ^i)]; end, disp('vector vv'), vv

[4] "Groundhog Day" with Bill Murray is a great movie about dynamic programming where the state variable does not change.

1.3 Example 3. Time as a state variable.

Take the first problem but now the asset is valid for only a finite number T of periods. As an example consider T=3.

$$S = x + \beta x + \beta^2 x = x\left(1 + \beta + \beta^2\right)$$

This is immediately completely determined. For large but finite T we have

$$\begin{aligned} S &= x + \beta x + \beta^2 x + \quad \beta^{T \ \ 1} x \\ S \quad \beta S &= x \quad \beta^T x \\ S &= \frac{x}{1 \quad \beta}\left(1 \quad \beta^T\right) \end{aligned}$$

Lets try now to turn this into a **Bellman** equation. This time it is trickyer because S is indexed by the current date since we do not have a perpetuity anymore. Let the index be the number of periods *left to go* T T-1 etc:

$$\begin{aligned} S(T) &= x + \beta x + \beta^2 x + \quad \beta^{T \ \ 1} x \\ S(T \quad 1) &= x + \beta x + \beta^2 x + \quad \beta^{T \ \ 2} x \\ S(T) &= x + \beta S(T \quad 1) \end{aligned}$$

and note then that we do not have a unique sum S but instead we have as many sums as periods in the problem. That means: S is a function of T S(T).

Lets write the asset pricing equation then:

$$rS(T) = x(1 + r) + S(T \quad 1) \qquad S(T) = x(1 + r) + dS(T)$$

and now the alternative return on the asset (rS) equals the dividend plus the capital gain which in this case is not zero but negative.

Computer exercise: write a matlab program to compute the sequence of sums S(T). Plot this sequence in a graph and label the axis (use the commands "plot(object)" and "title(object)" and "xlabel(object)" where of course you have defined the object (a vector) before.

1.4 Example 4. x as a state variable but stochastic.

In the first problem the solution was very simple because we had a perpetuity and nothing changed from period to period. Each time we tried to solve the problem it would be the same. It is useful now to introduce a little variation of a new kind.

Suppose that x can take two values high and low with some probability. Suppose also that this can happen every period with the same probabilities. We call x an iid random variable.

$$x = \begin{cases} x_h & q \\ x_l & 1 \quad q \end{cases}$$

and the mean is given by

$$x = qx_h + (1 \quad q)x_l$$

Now consider the problem in any given period. But we must be careful because now the problem only makes sense if we **take expectations**.

1) Suppose today the value of x is given by $x_0 = x_h$. The expected value of the sum is then given by

$$\begin{aligned} S(x_h) &= x_h + E\left\{\beta x_1 + \beta^2 x_2 + \beta^3 x_3 \quad \right\} \\ &= x_h + \beta E(x_1) + \beta^2 E(x_2) \\ &= x_h + x\left(\beta + \beta^2 + \beta^3 \quad \right) \\ &= x_h + \frac{\beta x}{1 \quad \beta} \end{aligned}$$

and we can see that the solution is equally easy if we start the day with the low x.

2) how do we write the Bellman equation now? Well it includes an expectation. We observe the current value of x so we write

$$\begin{aligned} S(x_h) &= x_h + E\left\{\beta x_1 + \beta^2 x_2 + \beta^3 x_3 \quad \right\} \\ S(x_h) &= x_h + \beta\left\{x + \beta x + \beta^2 x \quad \right\} \\ S(x_h) &= x_h + \beta\left\{q x_h + (1 \quad q)x_l + \beta\left(qx + (1 \quad q)x\right) + \beta^2\left(qx + (1 \quad q)x\right) \quad \right\} \\ S(x_h) &= x_h + \beta q\left\{x_h + \beta x + \beta^2 x \quad \right\} + \beta(1 \quad q)\left\{x_l + \beta x + \beta^2 x \quad \right\} \\ S(x_h) &= x_h + \beta q S(x_h) + \beta(1 \quad q)S(x_l) \\ S(x_h) &= x_h + \beta E(S) \end{aligned}$$

But this is a fundamental transformation because instead of using the expected value on x we are using it on the sum of the x s.

Computer exercise: try to write a loop where instead of having two separate values for x you have a vector with 2 elements X=[xh xl] and look for a solution function which is also a vector with two elements. Your probability distribution is also a vector Q=[q 1-q].

1.5 Example 5. Dynamic Optimization (almost).

In these exercises we just do a little algebra. Now it is the time to solve a simple optimization problem. Sometimes these dynamic problems involve choices.

One of the basic ideas of these dynamic choices can be looked at (curiously) with a static problem. This is the notion of **value function** which can be understood using the **indirect utility** function.

Example

Consider the following one period problem where income is I and prices and the utility function coefficient γ are parameters of the problem.

$$\max_{(c_1, c_2)} c_1^{\gamma} c_2^{1 \quad \gamma} \qquad s\ t \qquad I = p_1 c_1 + p_2 c_2$$

with Lagrangean

$$\pounds = c_1^{\gamma} c_2^{1 \;\; \gamma} + \lambda(I \quad p_1c_1 + p_2c_2)$$

The solution to this problem starts with the first order conditions in $c_1 \;\; c_2$

$$\begin{aligned} u_1 &= \gamma c_1^{\gamma \;\; 1} c_2^{1 \;\; \gamma} = \lambda p_1 \\ u_2 &= (1 \quad \gamma) c_1^{\gamma} c_2^{\;\; \gamma} = \lambda p_2 \end{aligned}$$

and we use their ratio $\frac{c_2}{c_1} = \frac{1 \;\; \gamma}{\gamma} \frac{p_1}{p_2}$ and replace in the budget constraint

$$p_1c_1 + p_2c_1 \frac{1 \quad \gamma}{\gamma} \frac{p_1}{p_2} = I = p_1c_1 \frac{1}{\gamma}$$

so that $c_1^* = \frac{\gamma I}{p_1}$ and $c_2^* = \frac{(1 \;\; \gamma)I}{p_2}$ and we can now replace these values back in the utility function to obtain **indirect utility** or utility measured in terms of income or in the terminology we want measured in terms of the **state variable** of the problem[5]

$$V(I) = U(c_1^* \;\; c_2^*) \equiv (c_1^*)^{\gamma} (c_2^*)^{1 \;\; \gamma} = \left[\left(\frac{\gamma}{p_1} \right)^{\gamma} \left(\frac{1 \quad \gamma}{p_2} \right)^{1 \;\; \gamma} \right] I = \Delta I$$

Here $\frac{\partial V}{\partial I}$ describes the change in the **maximum** utility attainable when we change income. That is the true marginal utility of income measured above by the Lagrange multiplier.[6]

This problem is important because it can be considered the last stage of a larger problem where we first had to decide what optimal income I should be and in order to determine that we had to take into account what the best possible use of that income is. So we reduce the larger problem to the determination of I since we have the optimal function V(I) that is V(I) embodies the all the optimal decisions that come after income I is determined. V(I) embodies all the information of the problem after we determined I. We can read V(I) as the value of reaching the current period when the value of the state variable is I.

1.6 Example 6. Dynamic Optimization (at last).

We now extend the previous example to two periods. Besides making clear the concept of value function in a dynamic environment the next important concept

[5]When we label a variable a "state variable", it means that its value describes the state of the world. A classic example is what happens when you get up in the morning. First you look out of the window and observe the state of the world. If it is raining you will then take your optimal decision regarding your clothes and your umbrella, conditional on the state of the world. For this decision to be truly dynamic, you only have to consider the state of your laundry. Do you really have rain clothes for tomorrow? What should you do? You had better take this course, or you are likely to get your clothes wrong for the rest of your life!

[6]The envelope theorem states that for a function $f(x, q)$, the solution $v(x^*(q), q)$ of $\max_x f(x, q)$, obeys $\frac{dv(q)}{dq} = \frac{df(x^*(q), q)}{dq}$. This has the obvious application that at the op timum, the derivative of the value function is simply the derivative of the utility function.

to understand here is **backward induction**. The idea is simple: even though life moves forward we must solve our problems backward by considering first what we will do tomorrow and only then will we make our decision today.[7]

Suppose that the problem you just solved was the problem in the last period of life. But life has a first period. In the first period there is only one consumption good. The overall problem when you start your life with wage w is

$$\begin{aligned} V(w) &= \max_{c, c_1, c_2} \left[u(c) + \frac{1}{1+\rho} U(c_1 \;\; c_2) \right] \\ & \quad st \\ I &= p_1c_1 + p_2c_2 \\ I &= (w \quad c)(1+r) \end{aligned}$$

with $u(c) = \log(c)$ and the interest rate (r) and discount rate (ρ) both constant and exogenous. The last budget constraint says that the income you do not consume in the first period you put in the bank and earn interest on it.

How do we solve this?

We can solve for all variables at the same time explicitly. But we can do a much easier thing.

1) Solve the last problem first to get V(I). this we did above.

2) Solve a simple two period problem of savings:

$$\begin{aligned} V(w) &= \max_{I} \left[u(c) + \frac{1}{1+\rho} V(I) \right] \\ & \quad st \\ I &= (w \quad c)(1+r) \end{aligned}$$

and this has first order condition $u_c + \frac{1}{1+\rho} V_I I_c = 0$ or

$$\begin{aligned} c &= \frac{1}{\Delta} \frac{1+\rho}{1+r} \\ I &= w \quad c = w \quad \frac{1}{\Delta} \frac{1+\rho}{1+r} \end{aligned}$$

and we just need to make sure the parameters are such that $I > 0$ is verified. One condition is that the interest rate (which gives you an incentive to save) is bigger than your discount rate (which gives you an incentive to consume).

1.7 Example 7. Computer exercise

This is an adaptation of the previous two period problem for you to solve in the computer.

[7]There is another fundamental concept in dynamic programming. It is so important that it has a name: **the optimality principle**. The idea is that we do not have to solve big long problems, we just have to solve a simple problem between today and tomorrow, because tomorrow we will make all the necessary optimal decisions. We will sstudy it later.

Let $p_1 = 1$ $p_2 = 0\ 5$ $\gamma = 0\ 5$ $\rho = 0\ 05$ $r = 0\ 07$.

Define then the value w to make sure I will obey $0 < I < w$ from the equation above.

Then define a vector with 3 possibilities for I all of them interior $IV = [I_1\ I_2\ I_3]$. Then solve the problem by stages.

First write the formula for the solution of the second period problem for each value of I.

Then write the code for the solution of the first period problem where you have three possibilities for the second period $V(I_1)$ $V(I_2)$ and $V(I_3)$.

Finally and *only if you have done the steps above without much problem* then you can define a vector with many elements

$$IV = 0\ 001 : step : w \quad 0\ 001$$

where you define a step size (say step=w/20) and try to write a code where you solve the consumption problem in vector form that is you create a vector for the second period with

$$V(I) = [V(I_1)\ V(I_2) \quad V(I_n)]$$

where you write only one command for all the elements in the vector.

Note: in matlab you write commands exactly as you write them here so do not be intimidated. read the easy tutorial.