CS545 – Machine Learning for Signal Processing

Matrix Factorizations and Beyond

25 October 2023

Today's lecture

- Latent Variable Models
  - Matrix and tensor decompositions
- Topic models
  - Probabilistic versions and signals
- Convolutive basis models

A non-signal divergence

- Latent Semantic Indexing/Analysis
- Deriving structure from documents
- Identifying topics and document classes

How text works*

** pre-deep learning*

- Large collections of text
  - Varying topics
- Bag of words (BOW) concept
  - Similar documents share common words
- Make statistical tools to extract structure

A simple example

- Consider these *documents*:
  - “Cats are kinda cute”
  - “Wars are very cruel”
  - “Inlaws are very cruel”
  - “Babies are kinda cute”
  - “Exams are very cruel”
  - “Dogs are kinda cute”

Bag of words representation

- Term-document matrix
- Keep track of all the words in all of the documents
- Represent each document as a histogram of words

Term-document matrix

	1	2	3	4	5	6
very	0	0	0	**1**	**1**	**1**
kinda	**1**	**1**	**1**	0	0	0
cute	**1**	**1**	**1**	0	0	0
cruel	0	0	0	**1**	**1**	**1**
are	**1**	**1**	**1**	**1**	**1**	**1**
wars	0	0	0	**1**	0	0
inlaws	0	0	0	0	**1**	0
exams	0	0	0	0	0	**1**
dogs	0	**1**	0	0	0	0
cats	**1**	0	0	0	0	0
babies	0	0	**1**	0	0	0

Document

Finding topics

- With this representation we want to find *topics*
  - i.e. common word collections that imply a subject

- Can you relate this to anything we know?

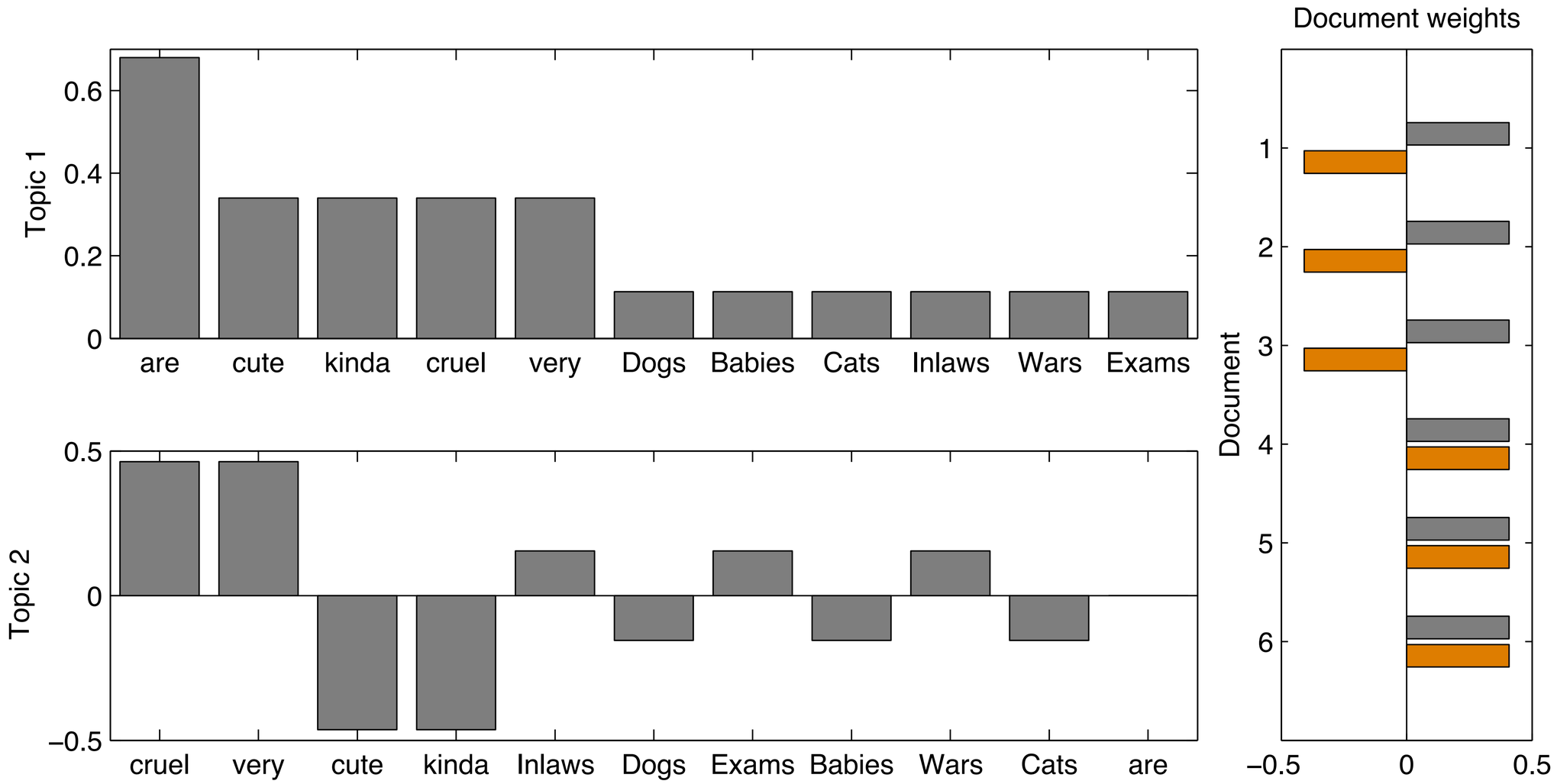
Finding eigen-BOWs

- We perform PCA (once again!)
- We SVD the term/doc matrix $\mathbf{C}$:

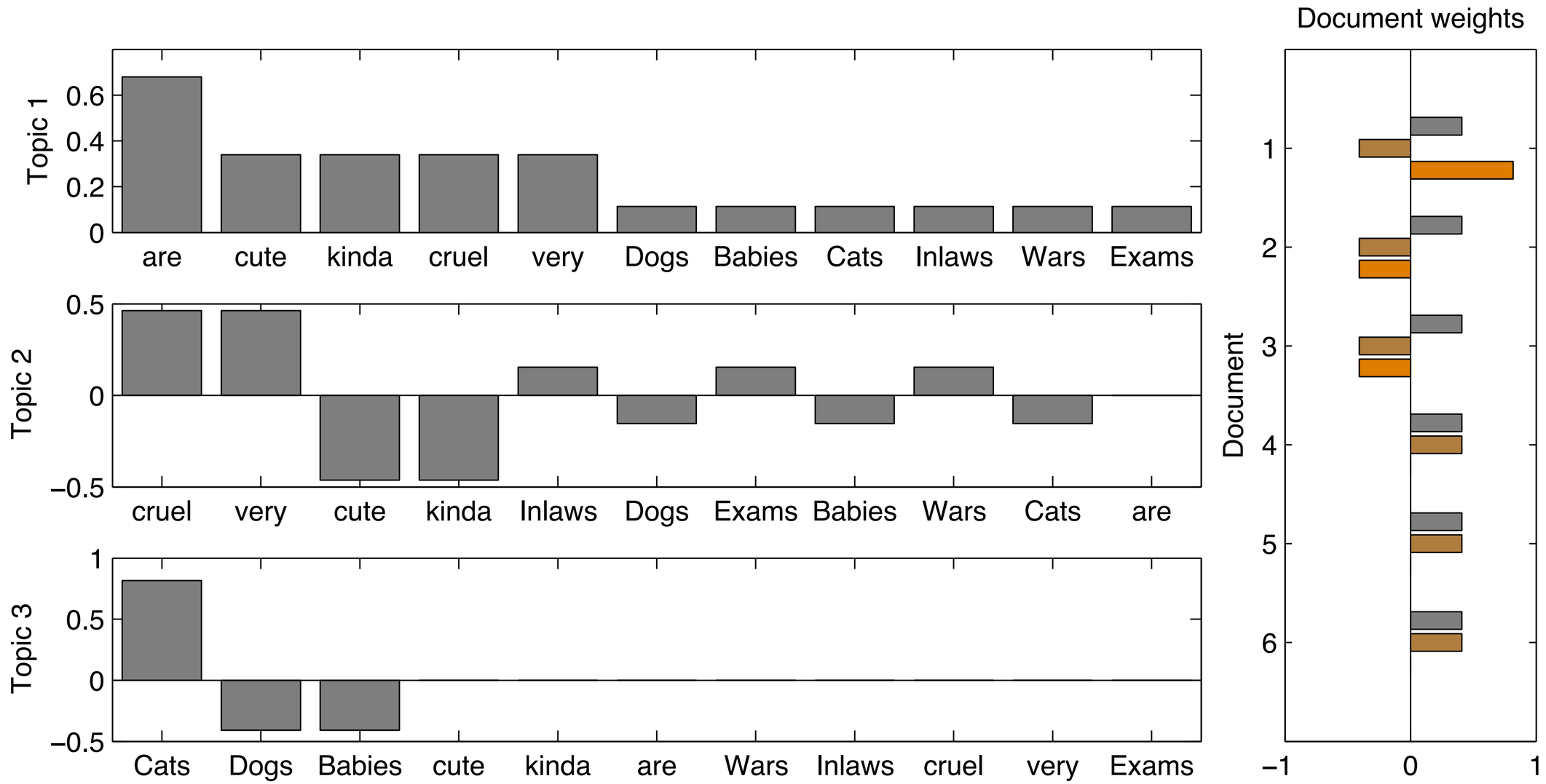
$$\mathbf{C} = \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{V}^\top$$

  - Columns of $\mathbf{U}$ are the eigen-BOWs
  - Rows of $\mathbf{V}$ are the document projections

Getting two eigen-BOWs



Getting three eigen-BOWs



Interpreting the data

- Eigen-BOWs
  - Are templates for “topics”
- Document projection
  - Mix of “topics” that document includes

Things we can do

- Characterize a document's semantic content
  - A mix of topics x, y, z, …
- Describe document similarity
  - Compare projections

$$S(\mathbf{v}_i, \mathbf{v}_j) = \frac{\mathbf{v}_i^\top \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \|\mathbf{v}_j\|}$$

But there’s a problem

- There is something fundamentally wrong
- What is it?

It’s a somewhat inappropriate model

- Negative words? Really?
- Why should the topics be orthogonal?

Can use NMF, but …

- Our inputs are histograms, not just non-negative data, they mean something
- We should analyze them appropriately

A probabilistic approach

- Moving from counts to probabilities

- Occurrence matrix seen as a distribution

$$C(\text{word}, \text{doc}) \propto P(\text{word}, \text{doc}) = \frac{\#\text{ word in doc}}{\#\text{ words in docs}}$$

Probabilistic decomposition

- We define the PLSA model:

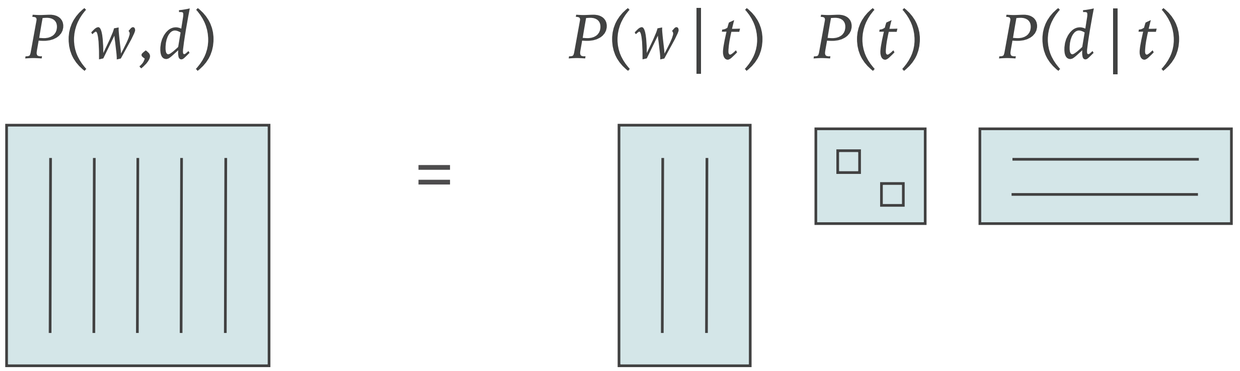
$$P(w,d) = \sum_t P(t)P(d \mid t)P(w \mid t)$$

- Which is a probabilistic version of the SVD

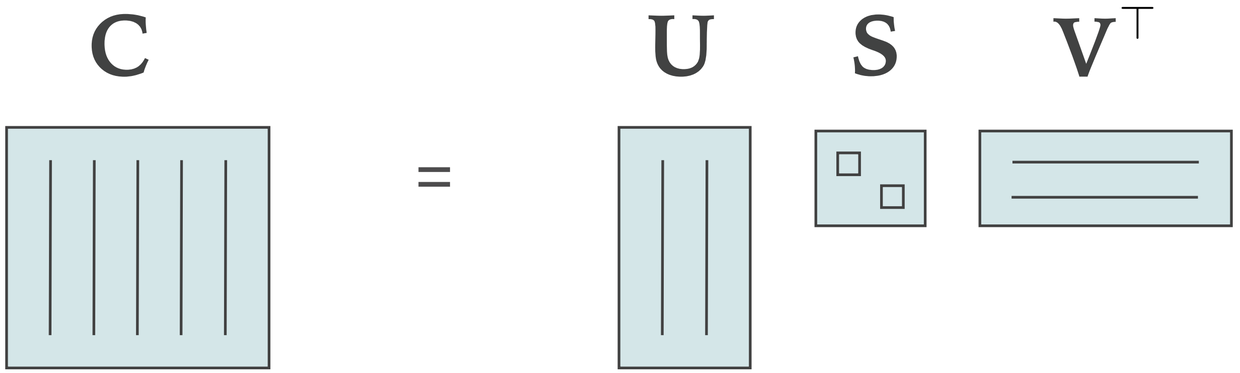
$$\mathbf{C}_{i,j} = \sum_k \mathbf{U}_{i,k}\mathbf{S}_{k,k}\mathbf{V}_{j,k}$$

Model correspondence

- PLSA

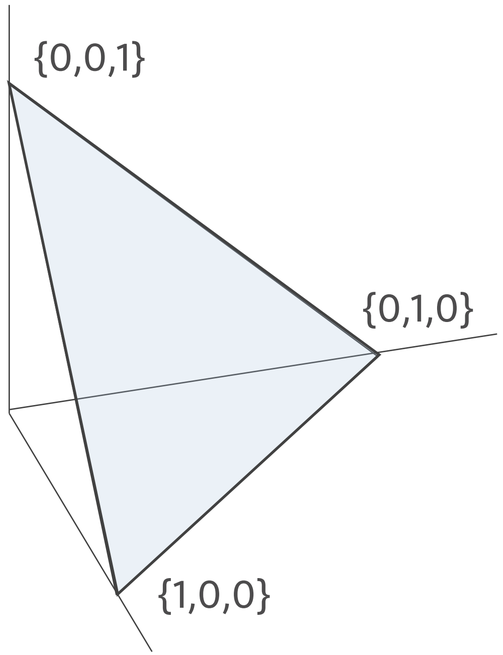


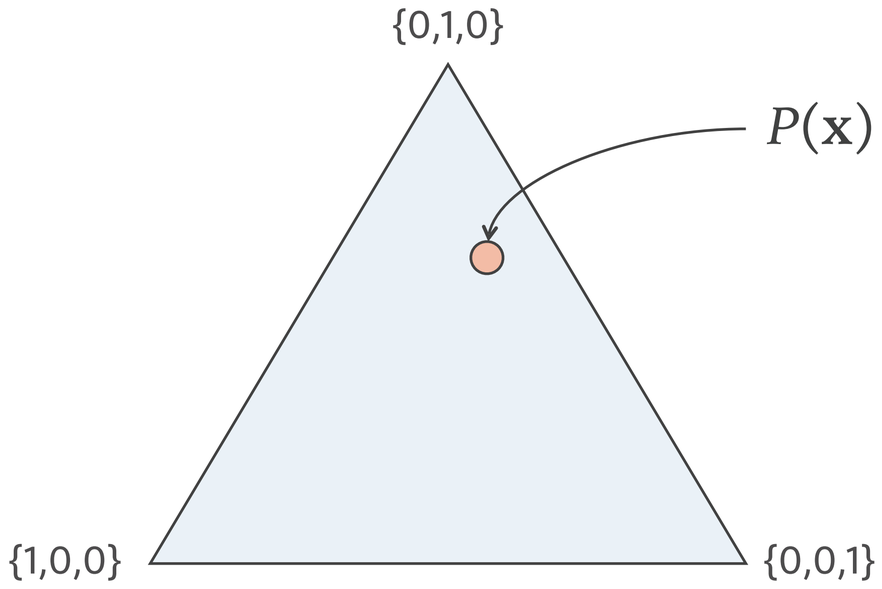
- SVD



Being in a new space

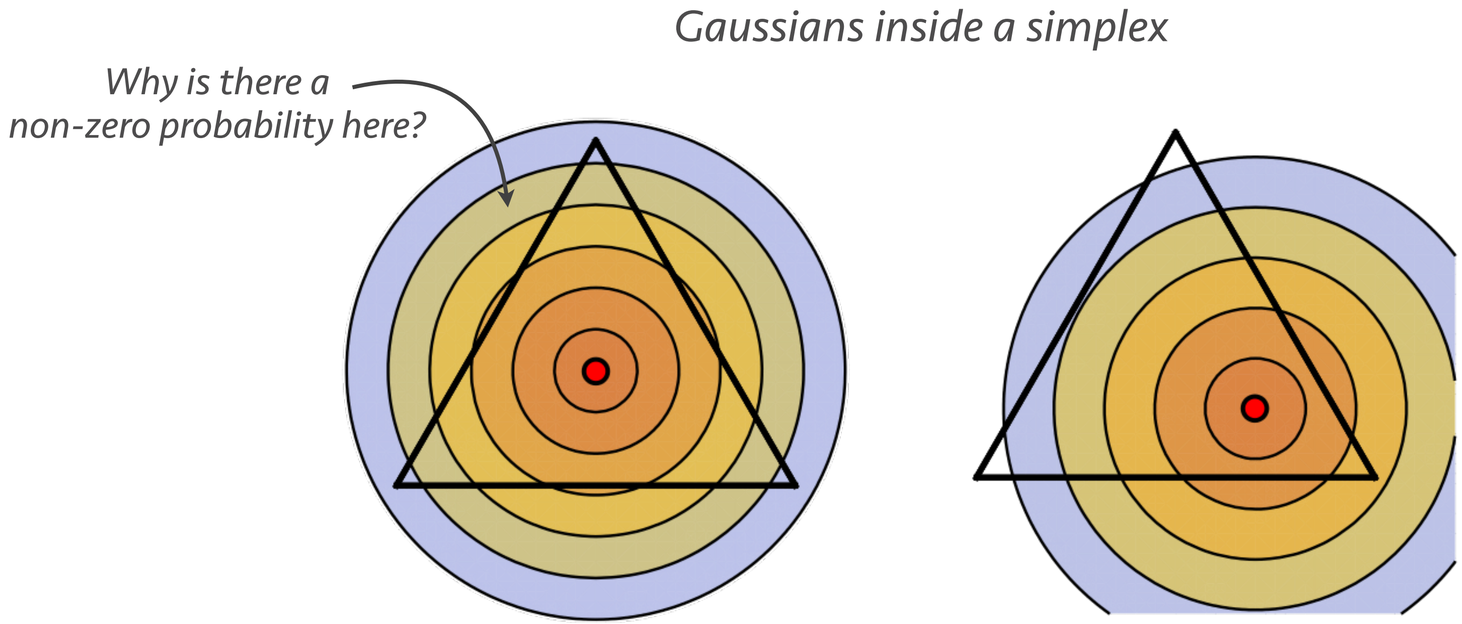
- The quantities involved sum to 1 (they are probabilities)
  - They therefore live inside a simplex





Why the SVD is bad here

- The SVD makes Gaussian assumptions
  - Euclidean distance, orthogonality, etc. …
- This is a poor choice in this space

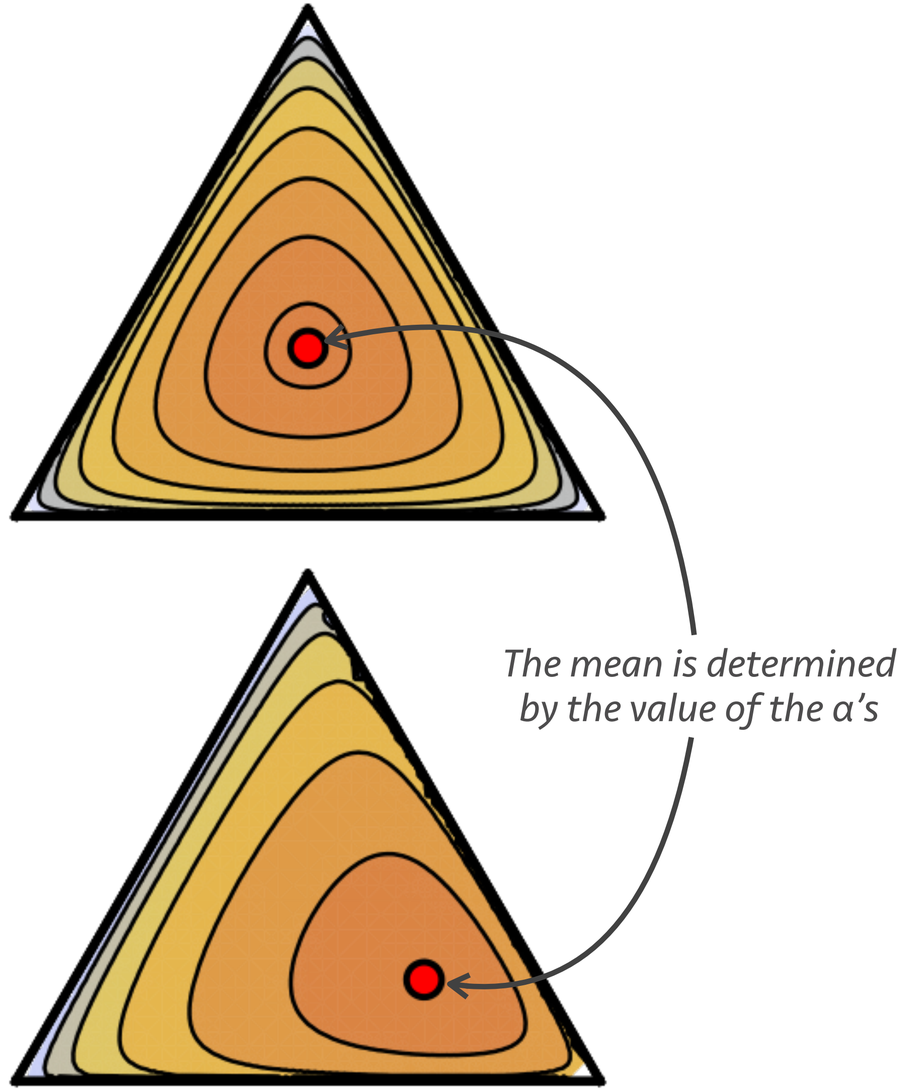


A proper distribution on the simplex

- The Dirichlet distribution

$$\mathcal{D}(x;\alpha) = \frac{\Gamma\left(\sum \alpha_i\right)}{\prod \Gamma\left(\alpha_i\right)} \prod x_i^{\alpha_i - 1}$$

- Respects the constraints of this new space



Implied cost function

- The Gaussian assumption means that we optimize the Euclidean distance

$$D(\mathbf{x},\mathbf{y}) \propto \log \mathcal{N}(\mathbf{x};\mu=\mathbf{y}) \propto \sum \left(x_i - y_i\right)^2$$

- The Dirichlet assumption results in optimizing the *cross-entropy* instead:

$$D(\mathbf{x},\mathbf{y}) \propto \log \mathcal{D}(\mathbf{x};\alpha-1=\mathbf{y}) \propto \sum x_i \log(y_i)$$

Relation to NMF

- In NMF we optimize a form of the Kullback-Leibler divergence

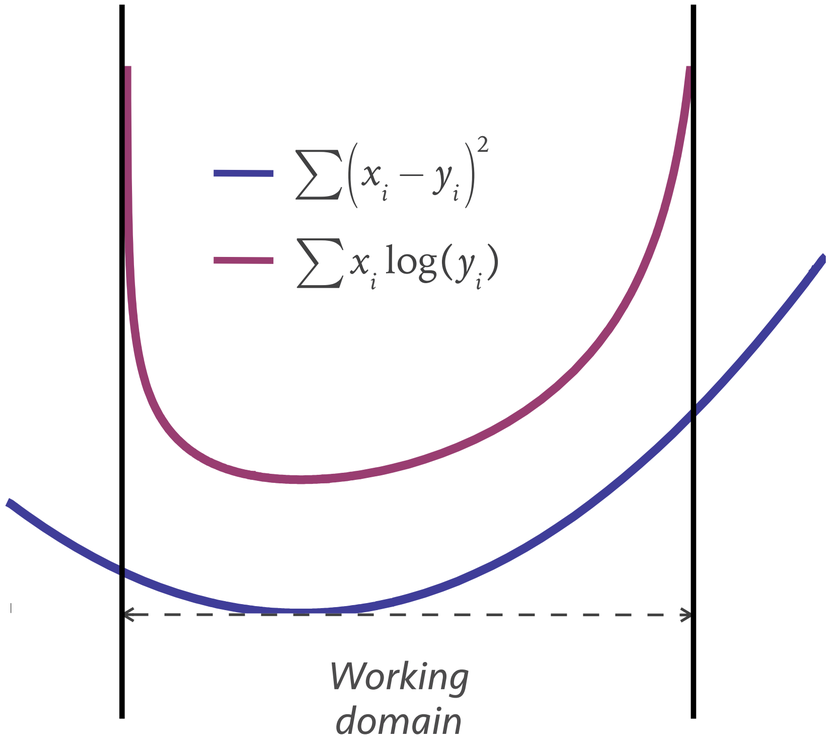
$$KL(\mathbf{x},\mathbf{y}) = \sum x_i \log \frac{x_i}{y_i}$$

- The cross entropy is just an offset of that

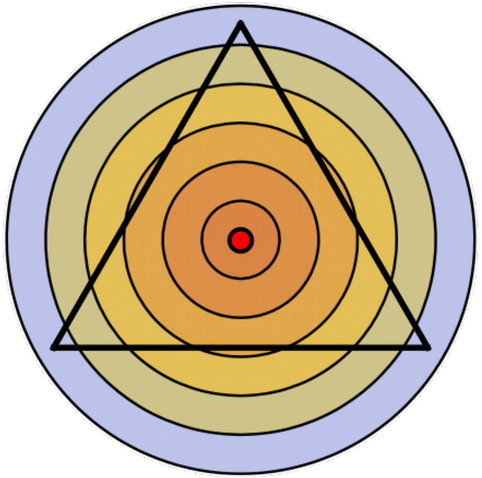
$$C(\mathbf{x},\mathbf{y}) = KL(\mathbf{x},\mathbf{y}) - H(\mathbf{x})$$

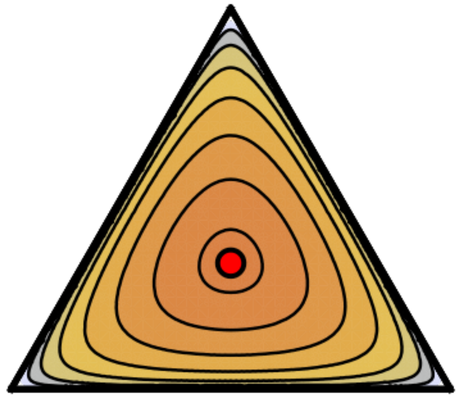
Comparing the costs

{2-1}-D

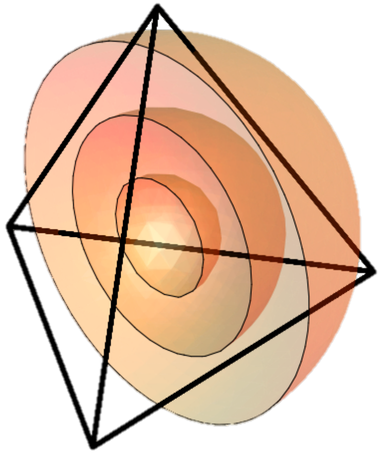


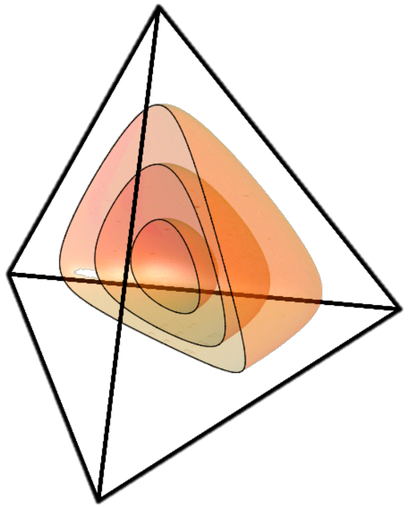
{3-1}-D





{4-1}-D





Estimating the parameters

- We can use Expectation Maximization
  - This is a mixture model (mixing topics!)

- E-step (find each topic's contribution)

$$P(t \mid d, w) = \frac{P(t)P(d \mid t)P(w \mid t)}{\sum_{t'} P(t')P(d \mid t') \mid P(w \mid t')}$$

Estimating the parameters

- M-step (use E-step weights to re-estimate)

$$P(w \mid t) \propto \sum_{d} P(w,d)P(t \mid d,w)$$

$$P(d \mid t) \propto \sum_{w} P(w,d)P(t \mid d,w)$$

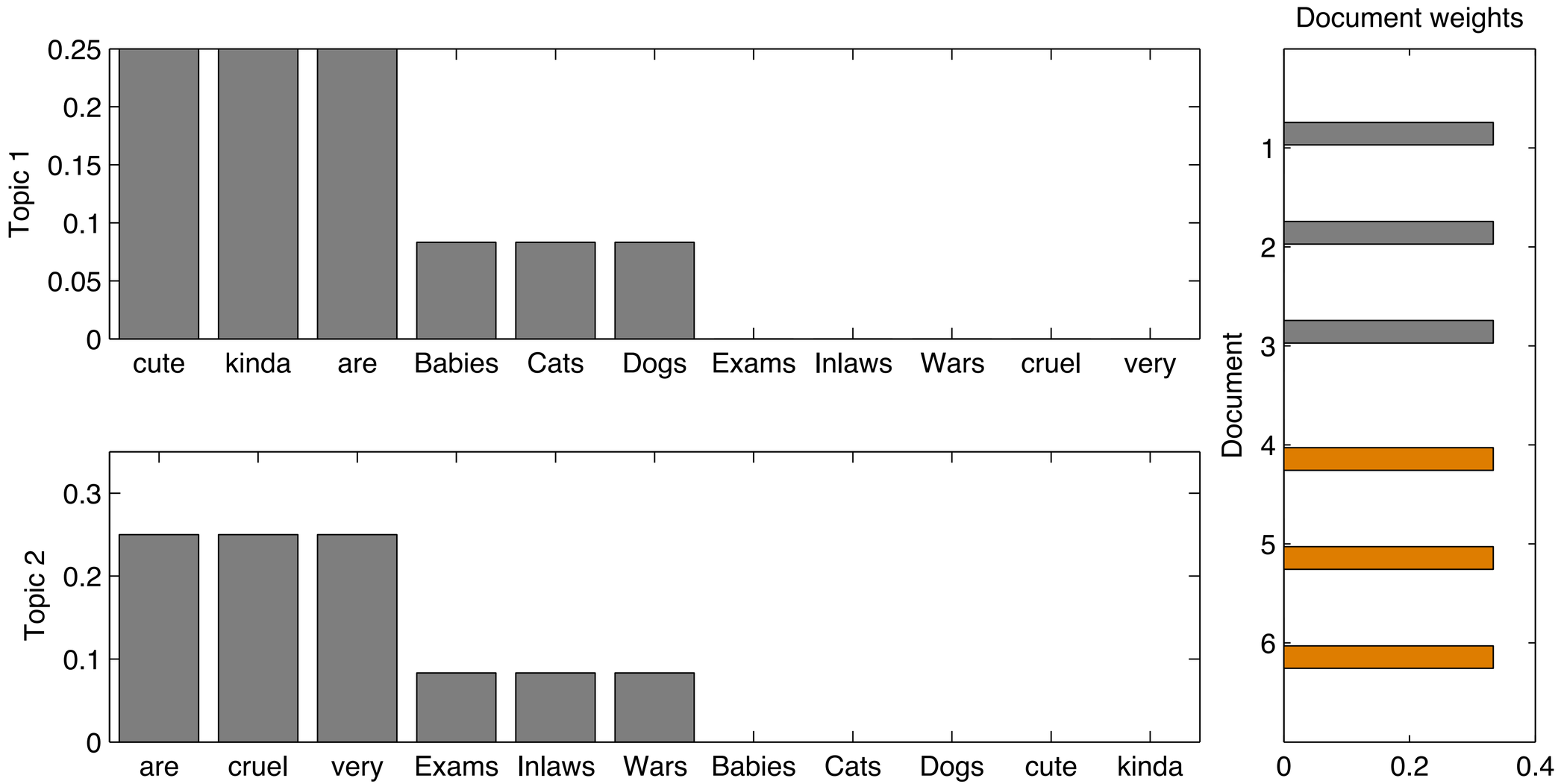
$$P(t) \propto \sum_{d,w} P(w,d)P(t \mid d,w)$$

  - Which is pretty much the same as the NMF update!

Back to text

- We now have a more satisfying representation of the documents/topics
  - $P(d|t)$ is the link between document and topic
  - $P(w|t)$ is the link between word and topic
  - $P(t)$ is the prior of a topic
- More interpretable quantities

On our mini document data



Much more satisfying results

- Probabilistic “eigen-BOWs”
  - No orthogonality constraint
    - Sensible topics that can share words
  - Non-negative words probabilities in topics
- Document “weights”
  - Probabilistic mix of topics
- This is known as the PLSI/PLSA model

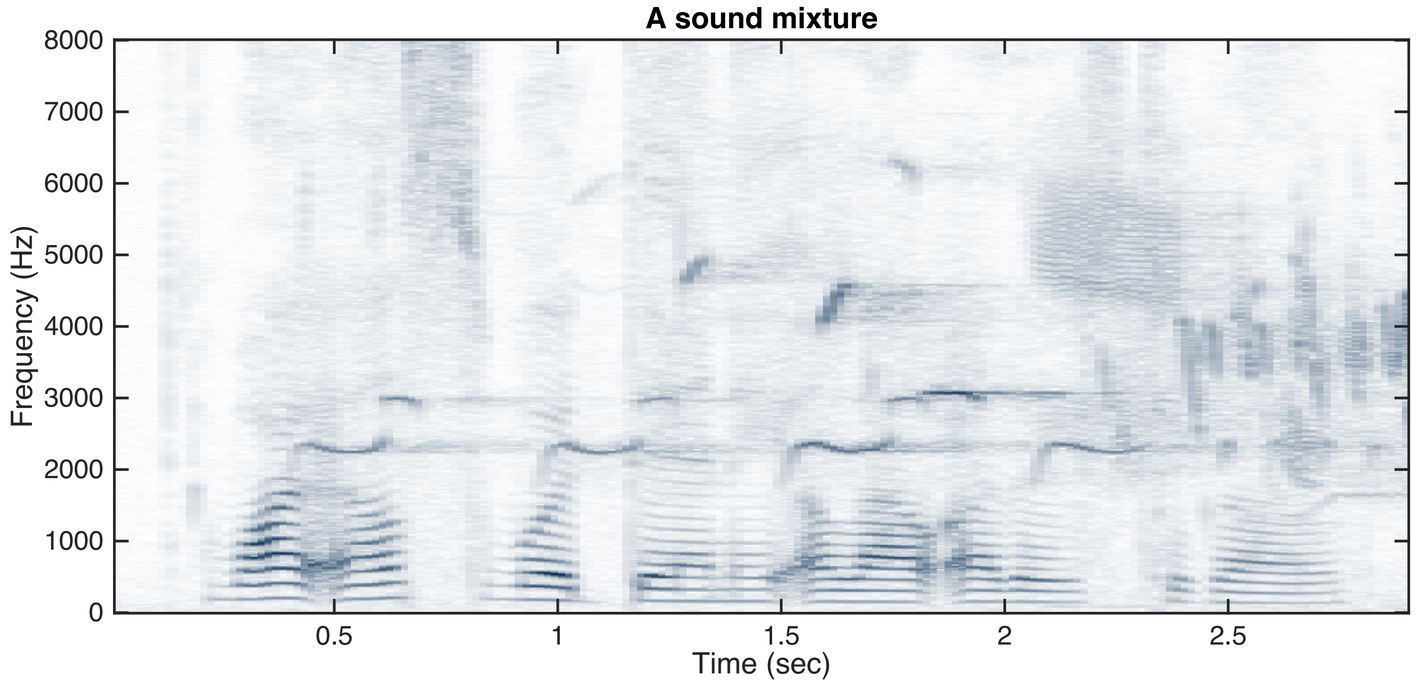
Back to signals

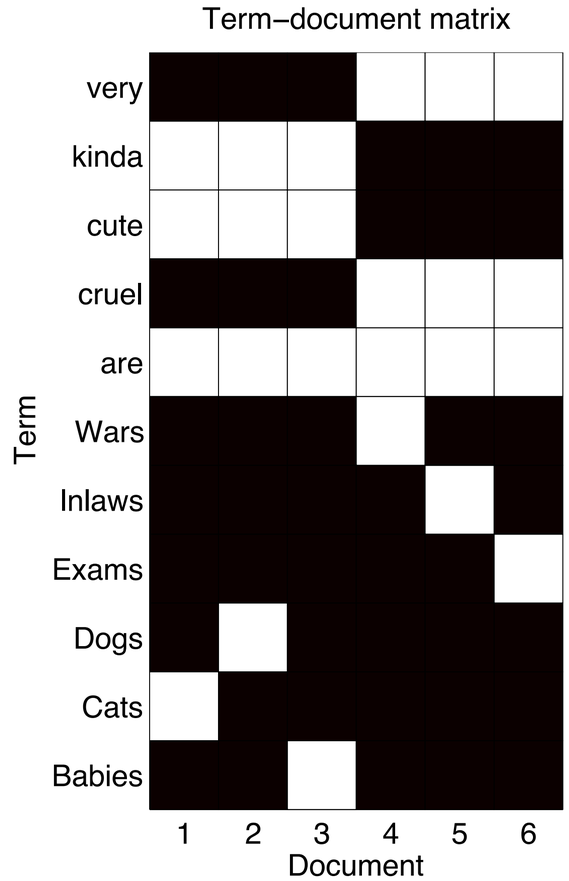
- Taking that lesson to the signal domain

- Analysis of any non-negative data
  - Counts, energies, variances, probabilities

Incorporating time

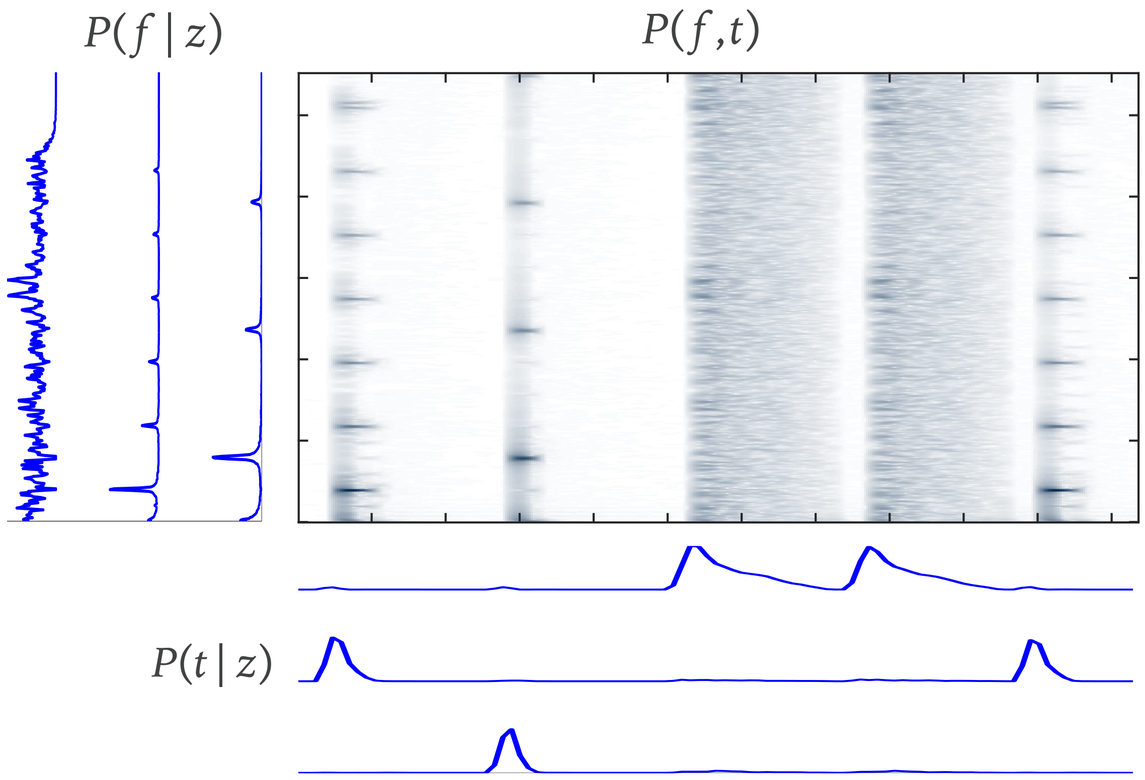
- PLSA doesn't explicitly use time information
  - But the model is flexible enough to use it
  - E.g. $P(f,t) = \sum_z P(z)P(f|z)P(t|z)$





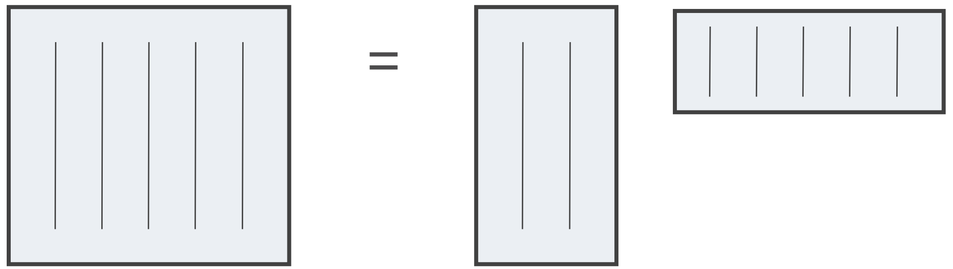
Now we can do NMF-like things

- Like spectrogram factorizations
  - And components are now interpretable

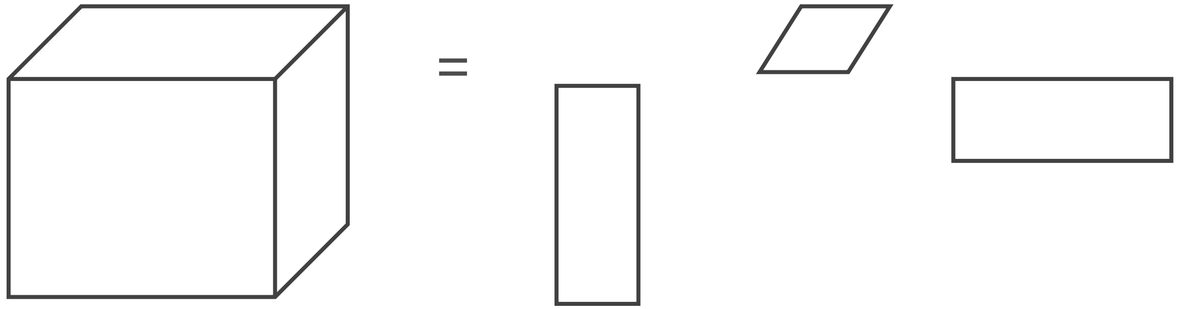


Moving up: Tensor decompositions

- SVD/PCA model was for matrices

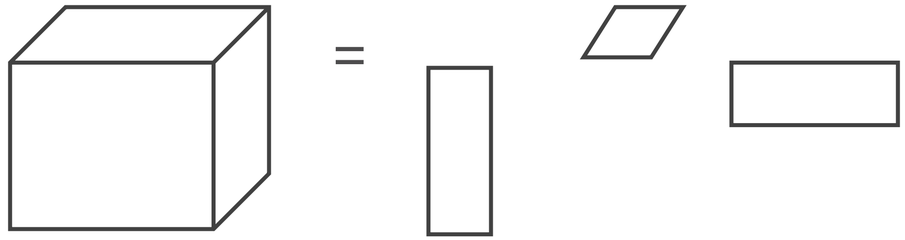


- Similar models exist for tensors



PARAFAC model

- Factor in terms of outer matrix products



- Easy notation: $x_{i,j,k} = \sum_l a_{i,l} b_{j,l} c_{k,l}$

- Matrix notation: $\mathbf{X}^{(I \times JK)} = \sum_l \mathbf{a}_l \left(\mathbf{b}_l^\top \otimes \mathbf{c}_l^\top \right)$

$$= \mathbf{A} \cdot (\mathbf{B}^\top \otimes \mathbf{C}^\top)$$

Solving for PARAFAC

- Treat as three problems:

$$\mathbf{X}_{:,:,k} = \mathbf{A} \cdot \text{diag}(\mathbf{c}_k) \cdot \mathbf{B}^\top$$

$$\mathbf{X}_{i,:,:} = \mathbf{B} \cdot \text{diag}(\mathbf{a}_i) \cdot \mathbf{C}^\top$$

$$\mathbf{X}_{:,j,:} = \mathbf{C} \cdot \text{diag}(\mathbf{b}_j) \cdot \mathbf{A}^\top$$

- And solve for them by alternating

$$\hat{\mathbf{A}} = \left(\sum_k \mathbf{X}_{:,:,k} \cdot \mathbf{B} \cdot \text{diag}(\mathbf{c}_k) \right) \cdot \left(\left(\mathbf{B}^\top \cdot \mathbf{B}^\top \right) \odot \left(\mathbf{C}^\top \cdot \mathbf{C}^\top \right) \right)^{-1}$$

$$\hat{\mathbf{B}} = \left(\sum_i \mathbf{X}_{i,:,:} \cdot \mathbf{C} \cdot \text{diag}(\hat{\mathbf{a}}_i) \right) \cdot \left(\left(\mathbf{C}^\top \cdot \mathbf{C}^\top \right) \odot \left(\hat{\mathbf{A}}^\top \cdot \hat{\mathbf{A}}^\top \right) \right)^{-1}$$

$$\hat{\mathbf{C}} = \left(\sum_j \mathbf{X}_{:,j,:} \cdot \hat{\mathbf{A}} \cdot \text{diag}(\hat{\mathbf{b}}_j) \right) \cdot \left(\left(\hat{\mathbf{A}}^\top \cdot \hat{\mathbf{A}}^\top \right) \odot \left(\hat{\mathbf{B}}^\top \cdot \hat{\mathbf{B}}^\top \right) \right)^{-1}$$

Variations

- Extension to N-way tensors
  - Corresponding to N factors
- Non-negative PARAFAC
  - Same as before, factors are non-negative
- Tucker decompositions
  - More involved, mixes components between dimensions

Probabilistic version

- We can recast this as the PLCA model

$$P(\mathbf{x}) = \sum_z P(z) \prod_j P(x_j \mid z)$$

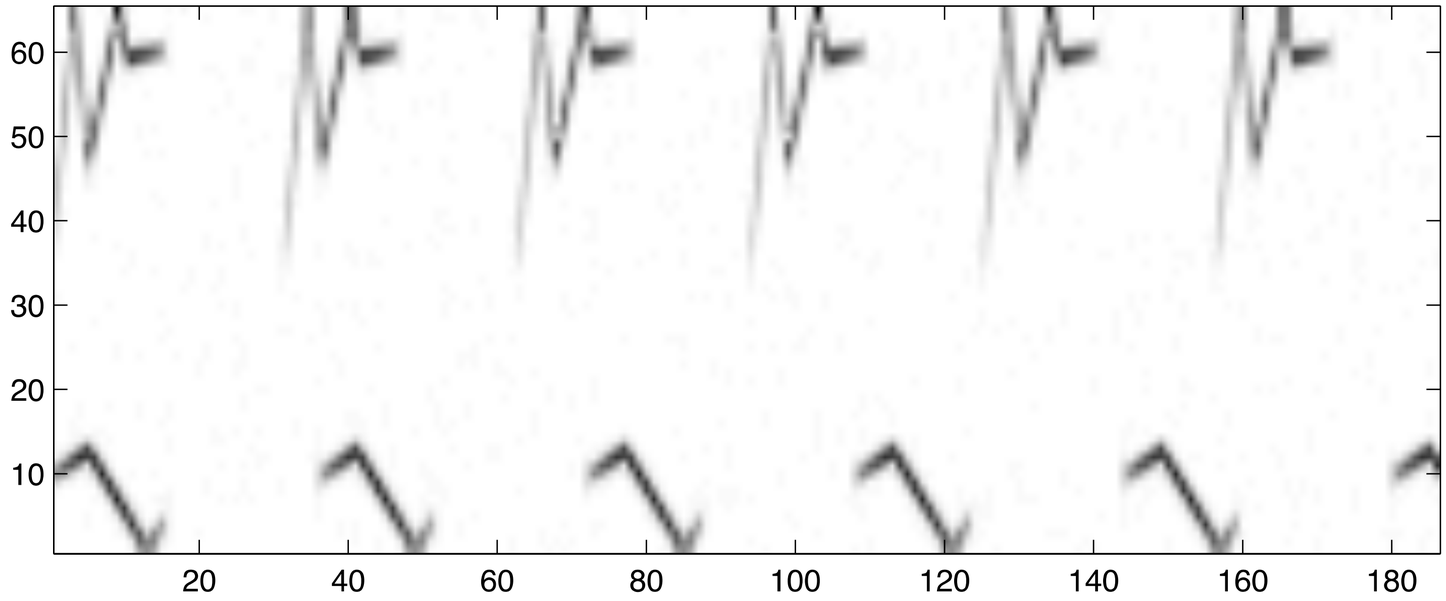
  - $\mathbf{x}$ can be of arbitrary dimensions
    - And seen as a non-negative tensor

- Same EM estimation equations as with 2-D case
  - So we don't have to worry about more dimensions

Adding one more dimension in the mix

- Example input with temporal structure
  - What would PLCA give us?

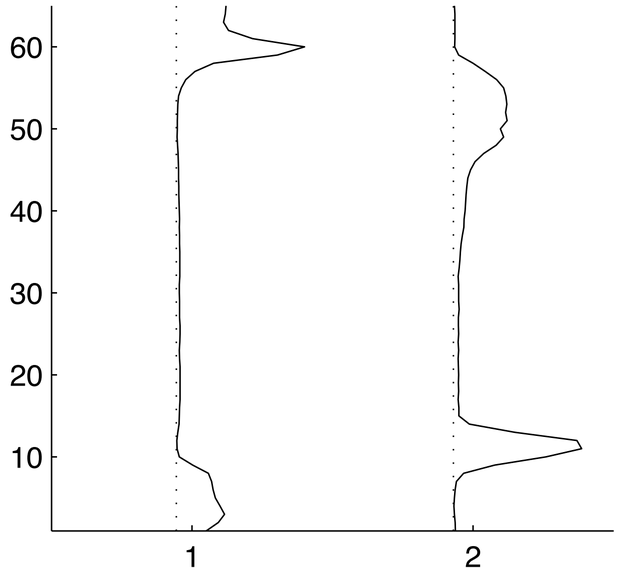
$P(x, y)$



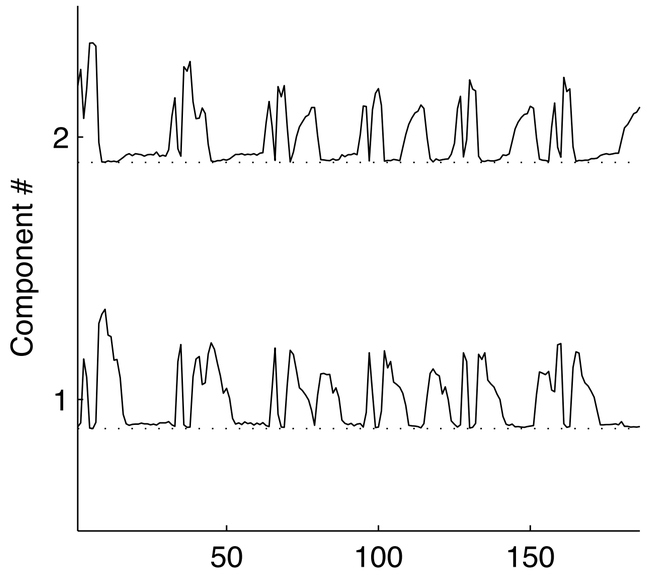
PLCA on temporal example

- Temporal structure is obscured
  - Components time-average the "objects"

$P(y \mid z)$



$P(x \mid z)$



Rethinking PLCA

- Convolutive component models
  - Substitute the multiplications with convolutions

$$P(x,y) = \sum_{z} P(z) \sum_{\tau} P(x,\tau \mid z) P(y-\tau \mid z)$$

- Each component is now 2-d
  - x-axis and y-axis presence
  - We shift these around along the x-axis

Estimating the parameters

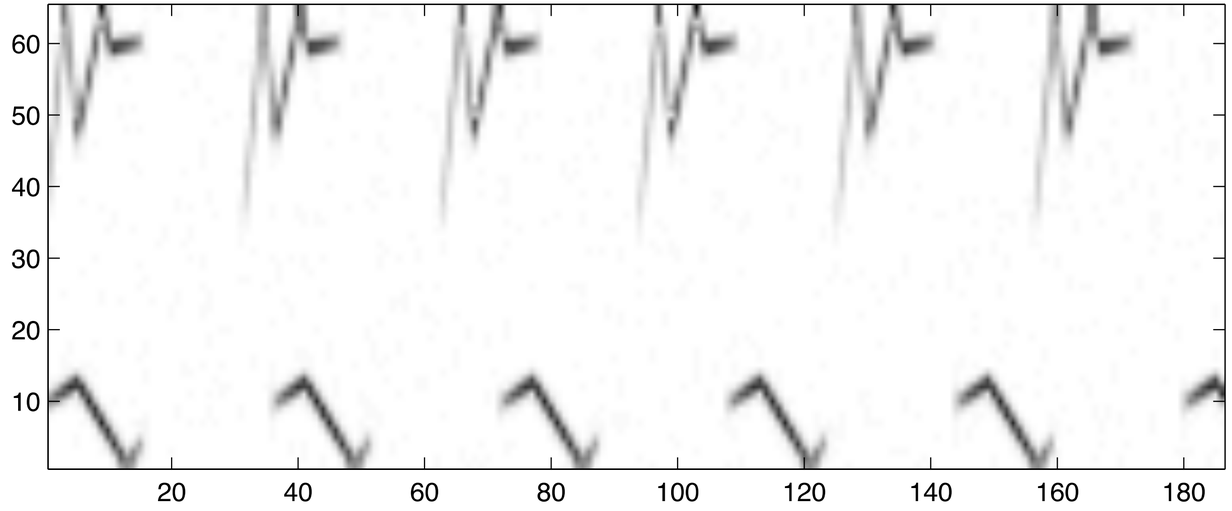
- EM again, this time over two parameters
- E-step: $$P(\tau, z \mid x, y) = \frac{P(z)P(x, \tau \mid z)P(y - \tau \mid z)}{\sum_{z'} P(z') \sum_{\tau'} P(x, \tau' \mid z')P(y - \tau' \mid z')}$$
- M-step: $$P(z) = \sum_x \sum_y \sum_z P(x, y)P(\tau, z \mid x, y)$$

$$P(x, \tau \mid z) \propto \sum_y P(x, y)P(\tau, z \mid x, y)$$

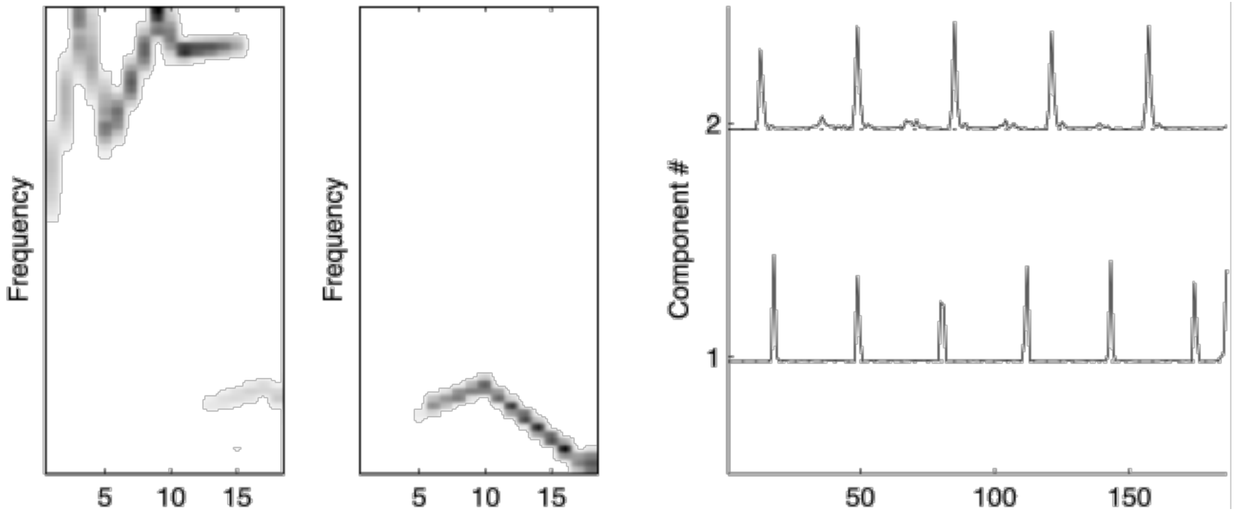
$$P(y, z) \propto \sum_x \sum_\tau P(x, y + \tau)P(\tau, z \mid x, y + \tau)$$

Shift-invariant components

$P(x,z)$



$P(x,\tau|z)$

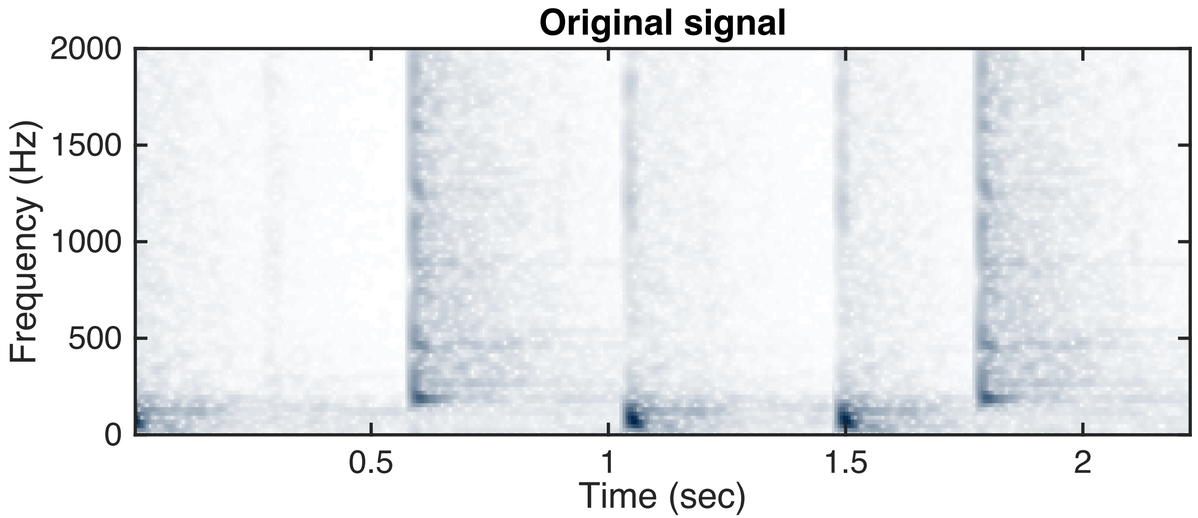


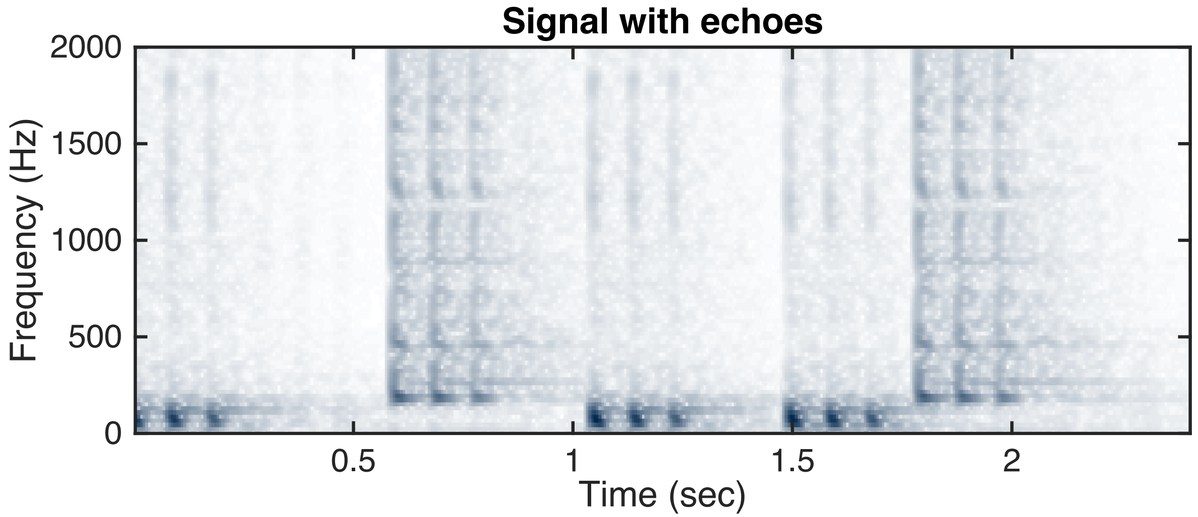
$P(y|z)$

An application

- Deconvolution
  - Echoes are repetitions of the same pattern in time
- Decompose input using a 1-component analysis
  - Component is signal
  - Weight is echo pattern

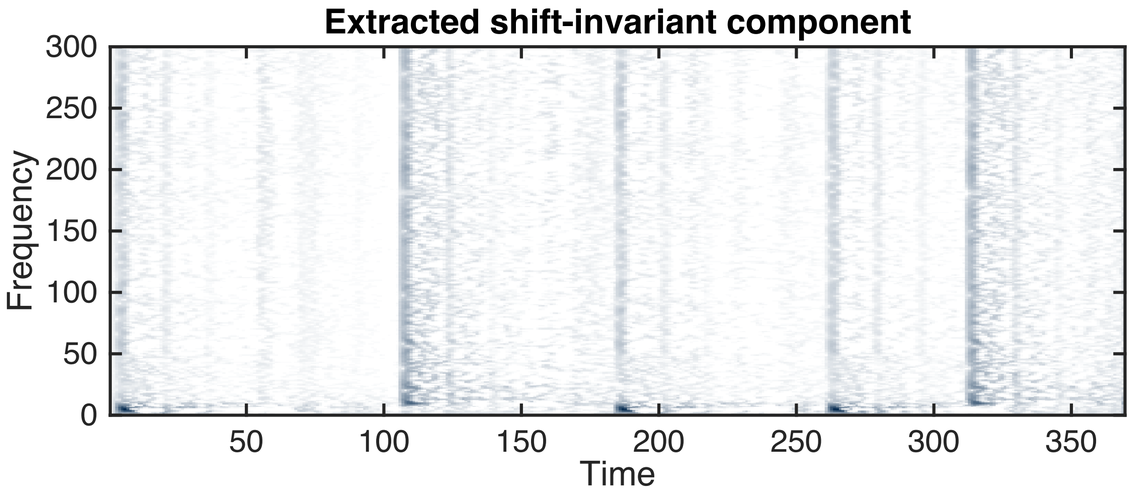
Original Sound  *Reverberant Sound* 

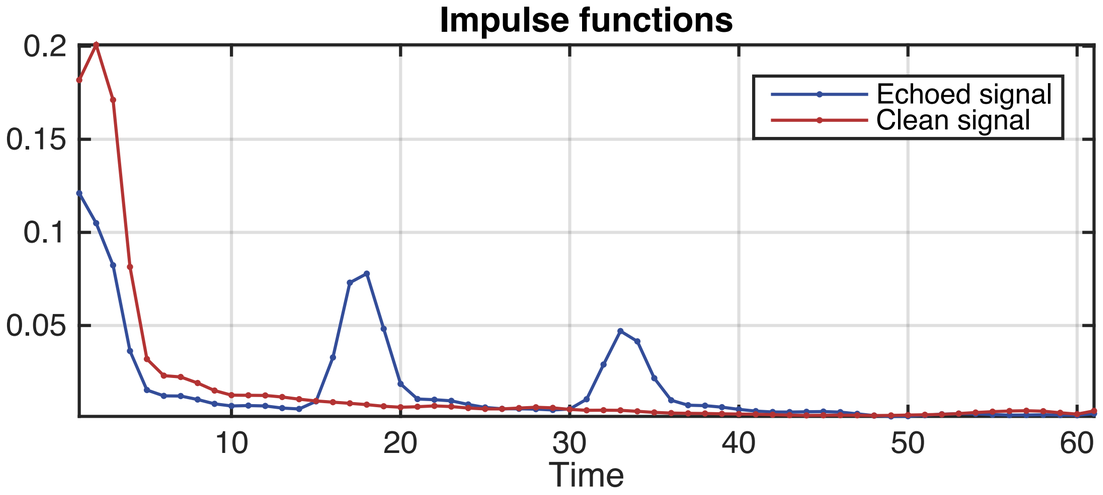




Analyzing this

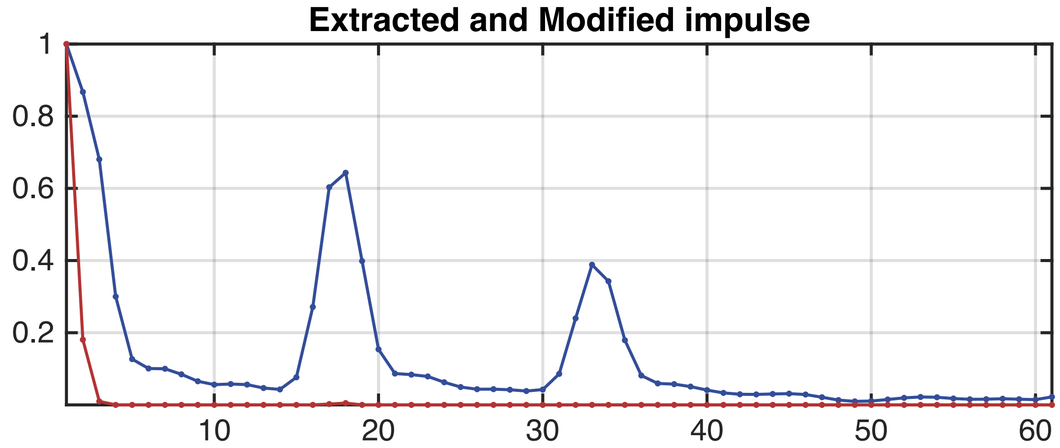
- We extract a time/frequency pattern (the one component)
  - Corresponds to the original sound
- Also get an impulse function
  - Corresponds to the echoes
- These two convolved approximate the input

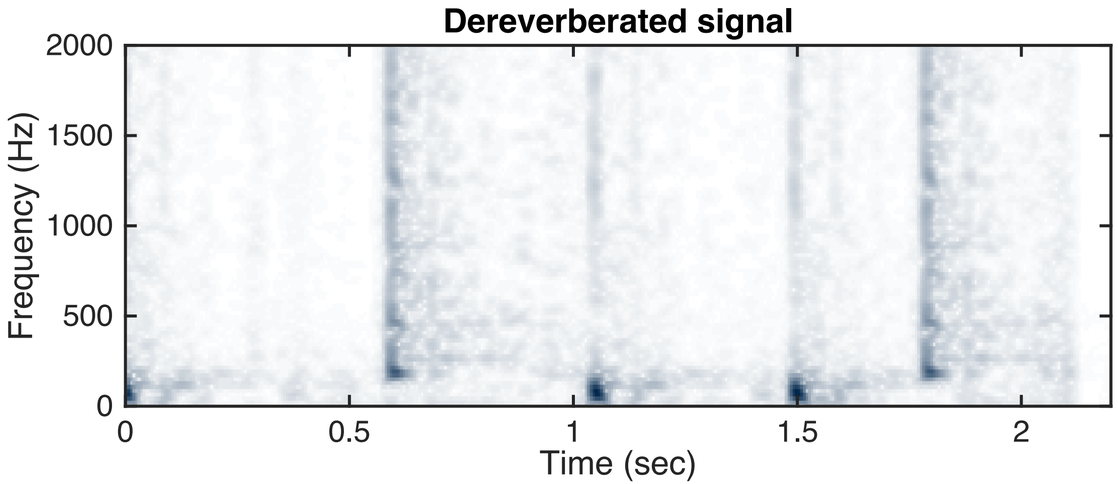




Removing the echoes

- We can suppress the echoes in the impulse and resynthesize
- Results in input without echoes





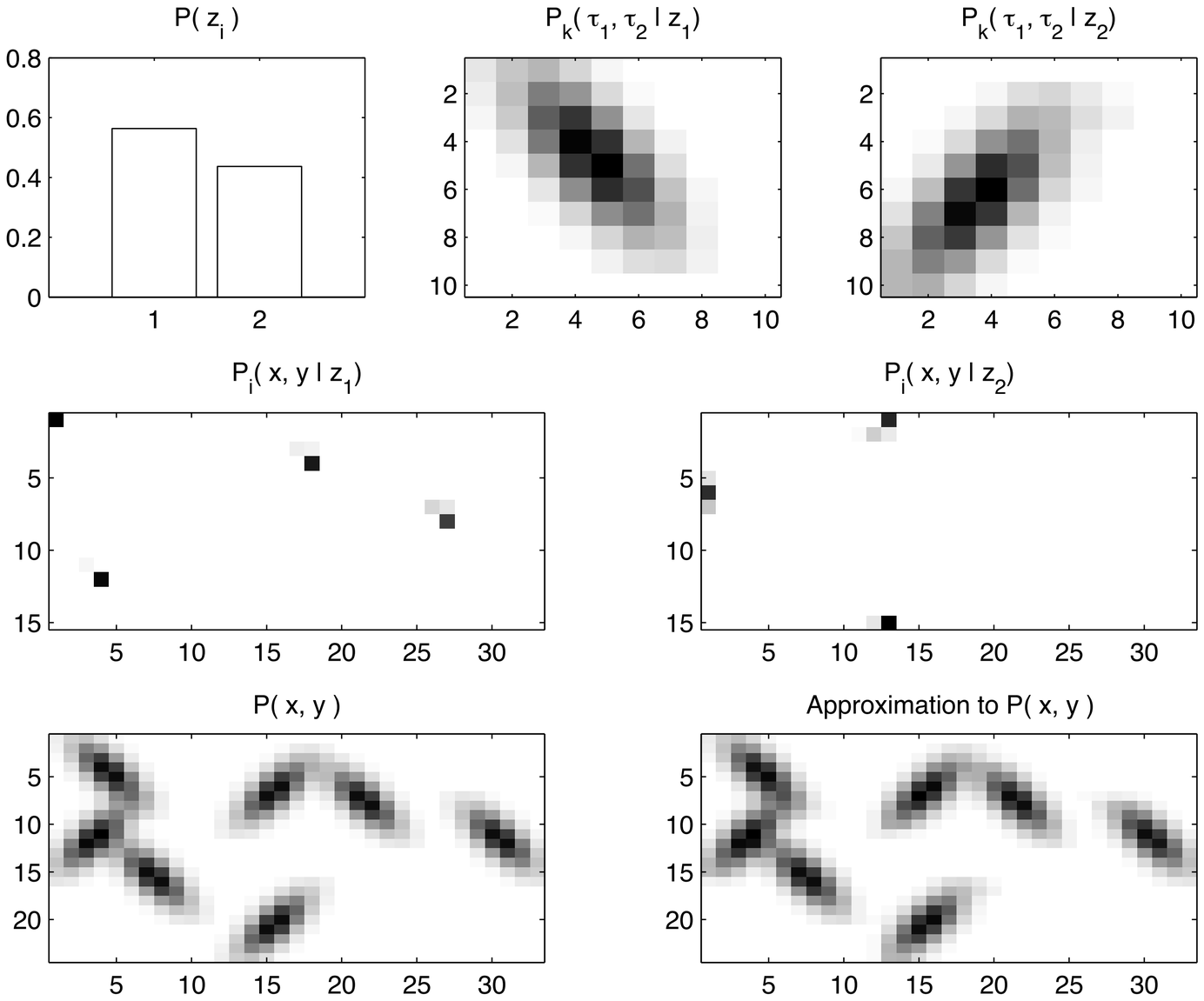
Cleaned up Sound

Generalizing to multiple axes

- We can also formulate impose shift-invariance over all dimensions

- In this case components are multi-dimensional and can shift everywhere

2-D example

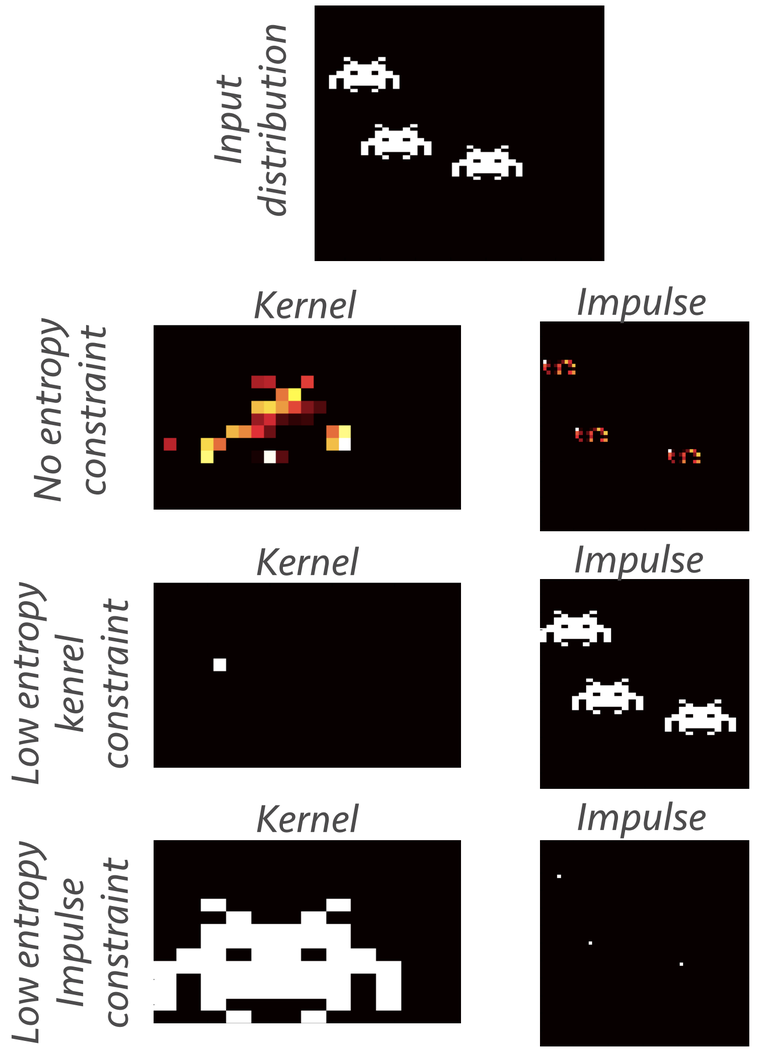

P(z_i)
P_k(τ_1, τ_2 | z_1)
P_k(τ_1, τ_2 | z_2)
P_i(x, y | z_1)
P_i(x, y | z_2)
P(x, y)
Approximation to P(x, y)

A bit of a complication

- Convolution is commutative
  - Component/weight confusion
- Using sparsity control
  - Impose entropic constraints
  - Apply as a prior

$$P(\mathbf{q}) = e^{-\beta H(\mathbf{q})}$$

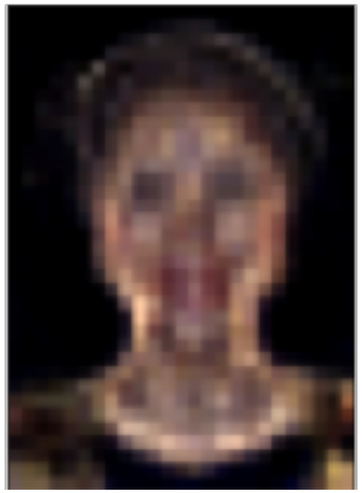
  - Or use simpler forms
    - More at next lecture



Pattern discovery

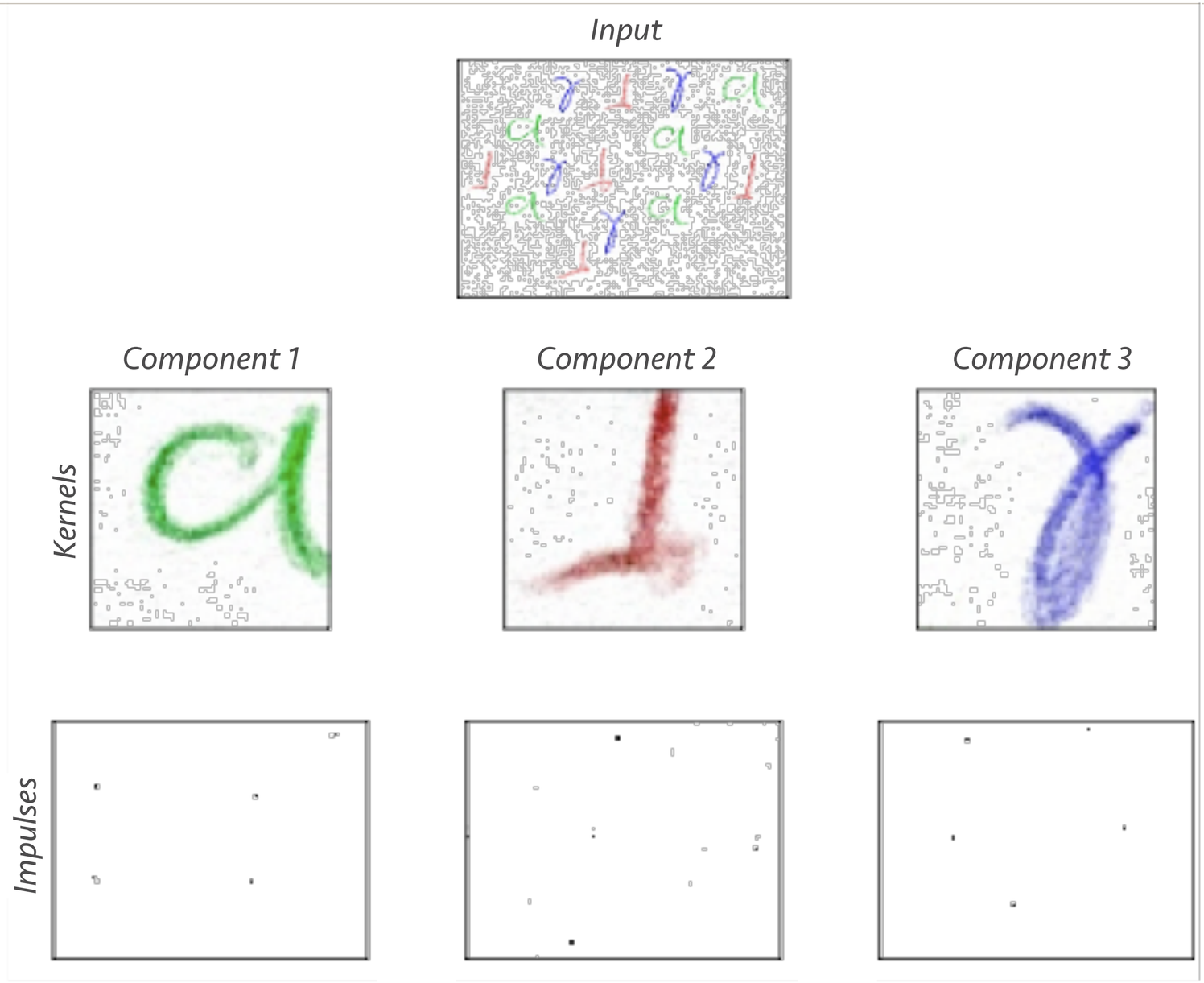
Input

Kernel distribution



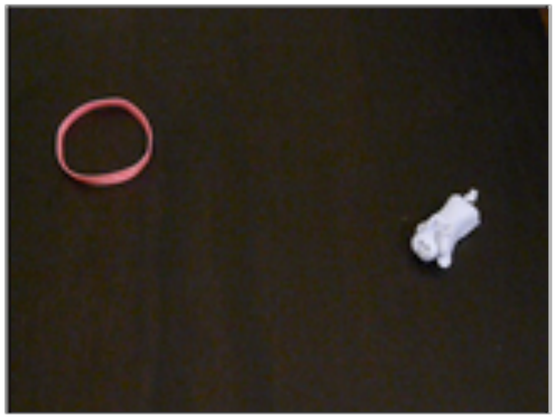
Impulse distribution

Shift-invariant features

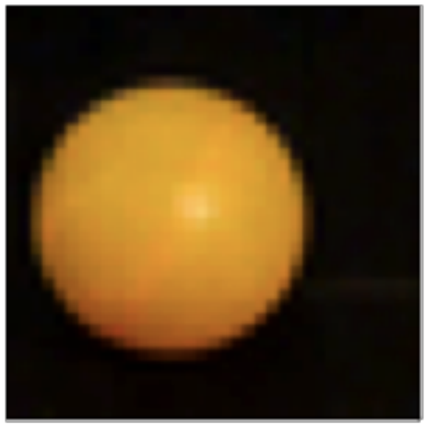


More shift-invariant features (in 4d)

Description of input



Kernel 1

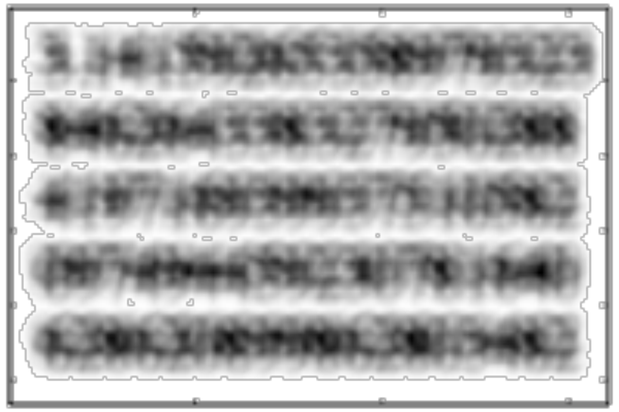


Kernel 2

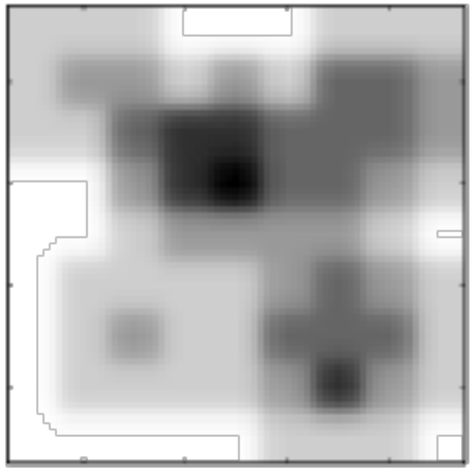
Kernel 3

Image deconvolution

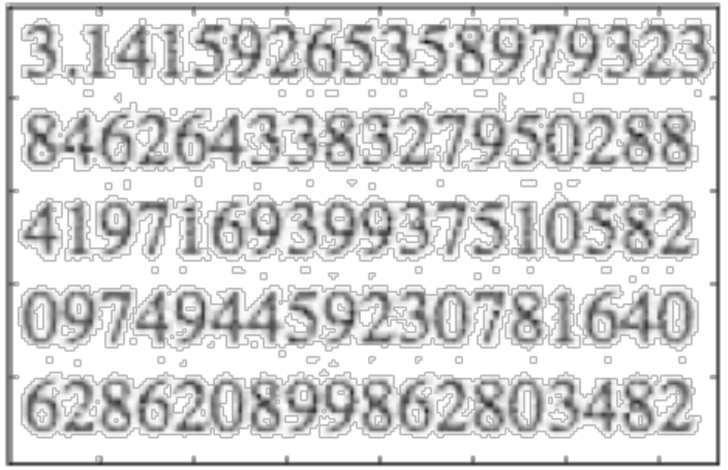
Blurred input



Blurring kernel



Deconvolution result



Many more models

- Probabilistic approach opens new doors
  - E.g. HMMs with PLCA state model
  - Mixtures of PLCA models
  - etc …

- Remember, probabilities integrate together nicely!

Recap

- Latent Variable Models
  - Matrix and tensor decompositions
- Probabilistic versions of the above
- Convolutive models

Reading

- LSA
  - http://openscienceasap.org/wp-content/uploads/2013/10/Mathematical-Foundations-for-LSA.pdf
- PLSI/PLSA/LDA
  - https://www.iro.umontreal.ca/~nie/IFT6255/Hofmann-UAI99.pdf
  - http://jmlr.csail.mit.edu/papers/volume3/blei03a/blei03a.pdf
- *N*-way decompositions
  - http://www.models.life.ku.dk/sites/default/files/brothesis_0.pdf
- Convolutive models
  - http://paris.cs.illinois.edu/pubs/smaragdis-icassp2008.pdf

Next Lecture

- More matrix fun with compressive sensing
  - Sparsity and all that business

- If you haven't, submit your project proposal
  - It can change a bit over time, that's fine
  - Final deliverables:
    - A 4-6 page paper describing your work
    - In class presentations