Chapter 1

Zero-Coupon Bonds

This chapter is an excerpt from *A Morgan Stanley Guide to Fixed Income Analysis* by Andrew R. Young,

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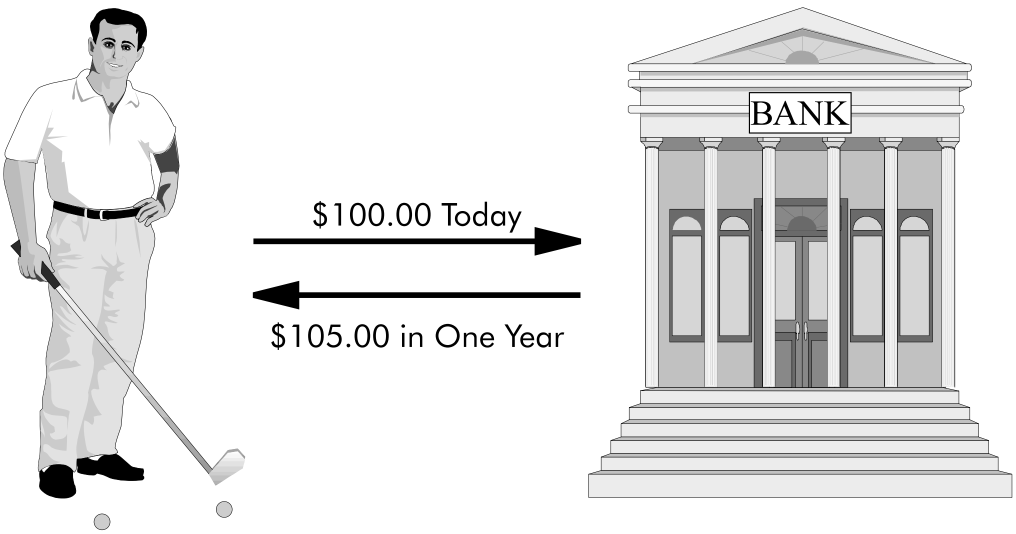
In This Chapter, You Will Learn...

- How to Calculate Present Value and Future Value of Single Cash Flows (Zero-Coupon Bonds)
- How to Compound Yields
- How Prices Change When Yields Change
- How to Estimate Price Changes Using
  - Duration
  - Convexity

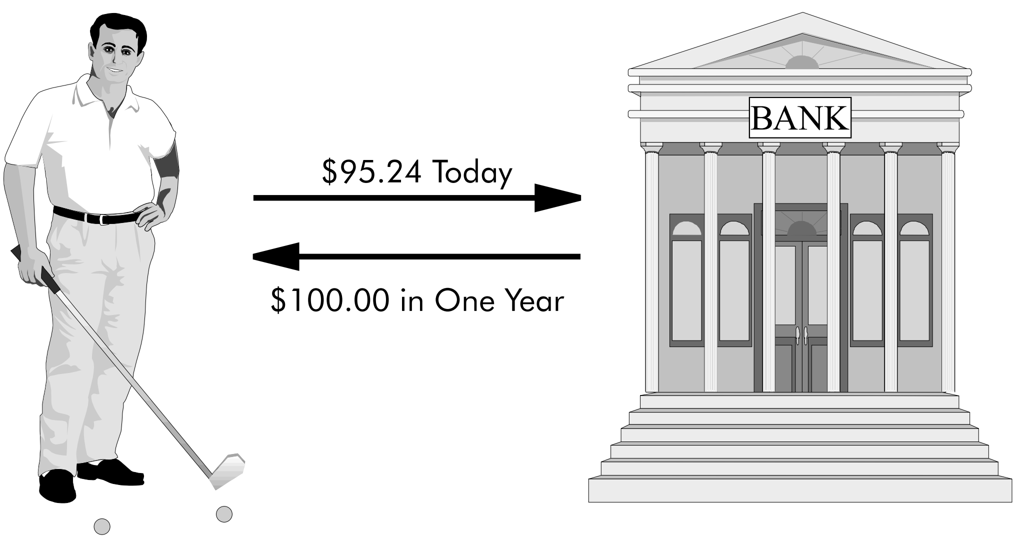
Future Value and Present Value

5% Per Annum (Simple Interest)

Future Value



Present Value



Present value and future value both address the "time value of money"

The basic concept of future value is "How much will I receive in the future for a fixed investment today?"

The basic concept of present value is "How much do I have to invest today for a fixed future cash amount?"

The present value is also described as the discounted value of the future cash flows (bond payments)

Compound Interest

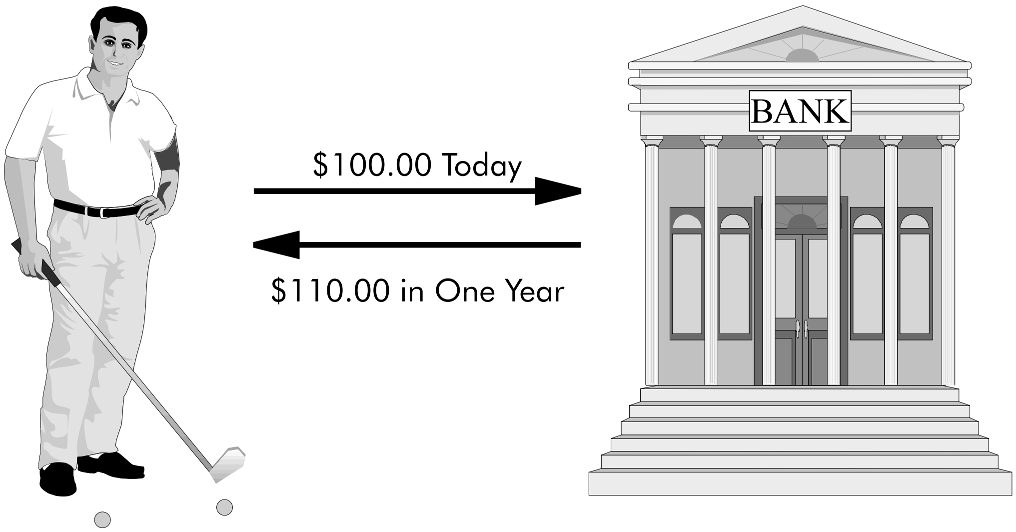
10% Per Annum

Compounding means that interest earns interest

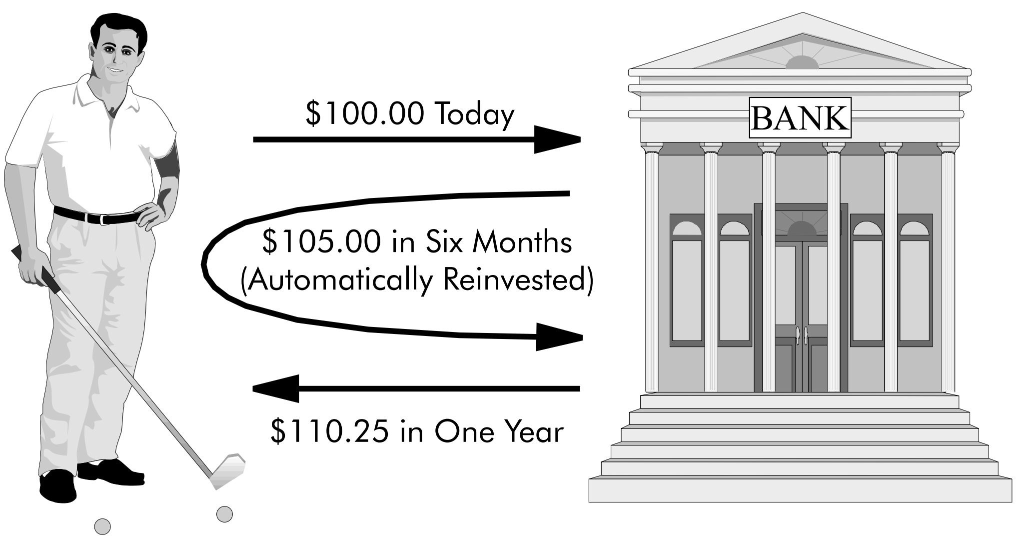
A 10% interest rate compounded semi-annually implies two six-month 5% interest periods per year; $100 would be worth $105 after six months and $110.25 after one year (the $105 earns 5%)

This is a higher effective rate of interest than the same 10% rate quotation with no compounding

No Compounding



Semi-Annual Compounding



Compounding $100 at 10% Interest

Compounding Frequency	After One Year	After 10 Years
Never	$110.00	$200.00
Annually	$110.00	$259.37
Semi-Annually	$110.25	$265.33
Quarterly	$110.38	$268.51
Monthly	$110.47	$270.70
Weekly	$110.51	$271.57
Daily	$110.52	$271.79
Continuously	$110.52	$271.83

The greater the time until maturity, the greater the difference compounding makes. For example, over one year, the continuous interpretation adds only $0.52 over the annual interpretation; over 10 years, it adds $12.46.

Although more frequent compounding increases return, it makes less incremental difference as the compounding divisions get finer. For example, over 10 years, the quarterly interpretation of a 10% rate would produce $9.14 more than the annual interpretation; the continuous interpretation only adds another $3.32.

Compounding makes a larger difference over a longer period of time

Given an investment term, successive divisions of the compounding frequency make less and less difference

Mathematics of Compounding

For every compounding frequency f, there is a different annualized yield quotation y_f that corresponds to a given annual yield

The term *annual yield* means that the yield compounds on an annual basis (once per year), as opposed to *annualized*, which can apply to any compounding frequency

Most securities follow the example set by the most prevalent securities in the market: Treasury notes and bonds. Their yields are quoted on a semi-annual compounding basis.

The fundamental formula for converting an annualized yield y_f — compounding f times per year — to an annually compounded yield is

$$1 + \textit{Annual Yield} = \left(1 + \frac{y_f}{f}\right)^f$$

Note that more frequent yield compounding results in a higher annual yield. Alternatively, given an annual yield, more frequent compounding results in a lower y_f.

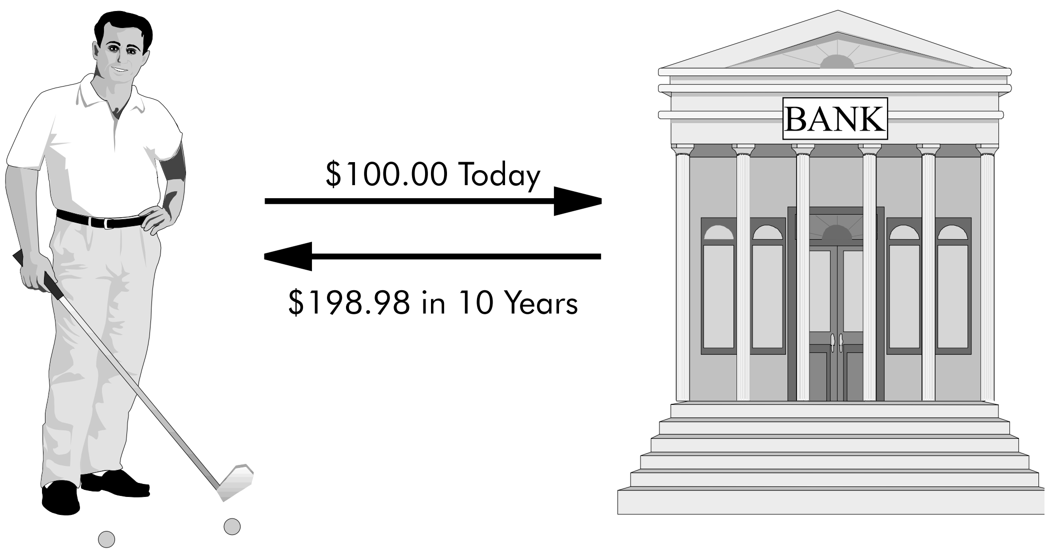
Q: If a bond's semi-annual yield is 7%, what is its quarterly yield? Its monthly yield?

Hint: What is its annual yield?

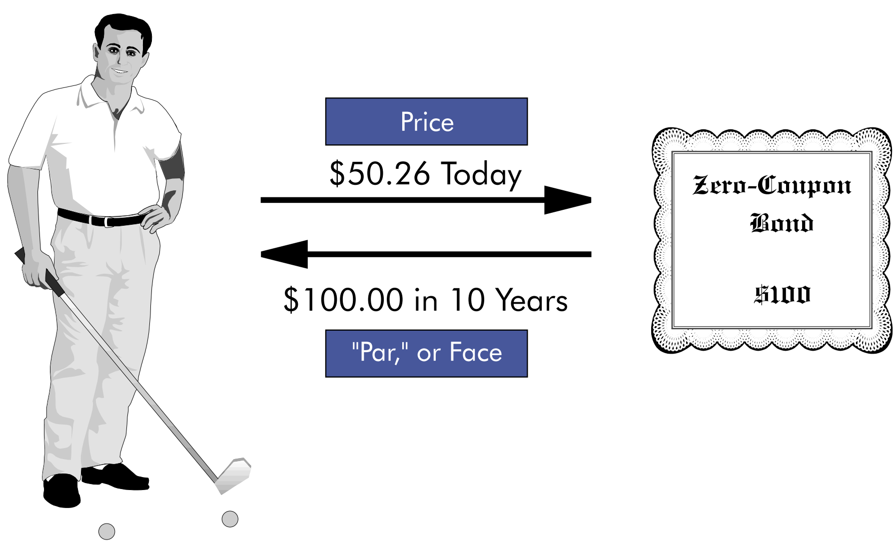
Future Value and Present Value

7% Per Annum, Semi-Annual Compounding

Future Value



Present Value



$100 invested at 7%, compounded semi-annually, would return $198.98 in 10 years; the amount invested today is 50.26% of the final value

Likewise, the price of a 10-year zero-coupon investment (paying $100 at maturity) is $50.26 (50.26% of par) at a 7% semi-annually compounded yield

$$Price = \frac{v}{\left(1 + \frac{y}{f}\right)^{n}}$$

where

- v **is par (100%)**
- y **is yield (7%)**
- f **is the compounding frequency (2)**
- n **is the number of compounding periods (20)**

Pricing a Zero-Coupon Bond

v **is the par amount (usually 100%)**

y **is the yield**

f **is the compounding frequency**

n **is the number of compounding periods;** n/f **is the number of years until maturity**

For zero-coupon bonds, the price is the present value—"How much do I have to invest today to get the face value of the bond at maturity?" The price can be calculated from the yield, and the yield can be calculated from the price:

Price (Given Yield)

$$Price = \frac{v}{\left(1+\frac{y}{f}\right)^{n}}$$

Yield (Given Price)

$$Yield = f \times \left[\left(\frac{v}{Price}\right)^{\frac{1}{n}} - 1\right]$$

Example: A 10-year U.S. Treasury zero-coupon bond yielding 6%, compounded semi-annually, has a price of:

$$\frac{100\%}{\left(1+\frac{6\%}{2}\right)^{20}} = \frac{100\%}{\left(1+\frac{0.06}{2}\right)^{20}} = \frac{100\%}{(1.03)^{20}} = 55.368\%$$

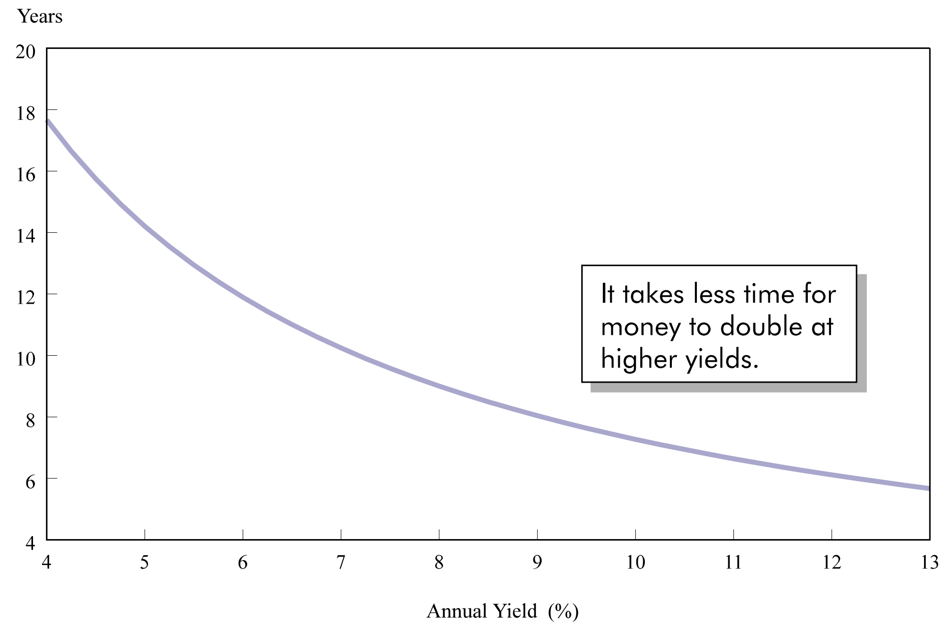
Note: Price is generally quoted as a percent of par (face value). For example, a zero-coupon bond with a price of 50 means that the security costs 50% of par. Since the decimal representation of 50% is 0.50, the cost of $1,000,000 face amount of the bond would be

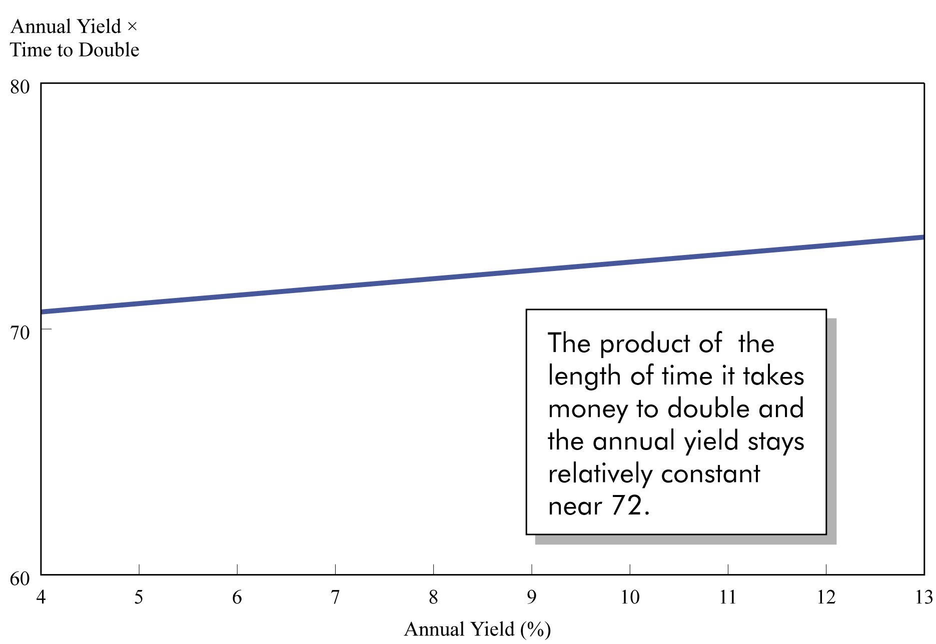
$$\$1,000,000 \times 50\% = \$1,000,000 \times 0.50 = \$500,000$$

Frequently, the percent designation is not quoted. There are two ways to reconcile this with economic reality: Take the percent designation as implied, so that a price of 50 would really mean a price of 50% of par; alternatively, the price could represent a cost of $50 for $100 face amount of the bond.

A Quick Valuation of Future Cash Flows

Length of Time for Money to Double × Annual Yield ≅ 72





The *Rule of 72*[1] provides a quick way to estimate the value of future cash flows

It states that over a wide range of interest rates, the length of time it takes money to double is approximately

$$\frac{72}{y_{Annual}} \approx \frac{71}{y_{Semi\text{-}Annual}}$$

For example, given a 7% annual yield,

$$\frac{72}{7} \cong 10$$

Thus a dollar in 10 years is worth approximately 50 cents today; this is consistent with the price of a 10-year zero at 7% (50.26%)

[1] Do not confuse the Rule of 72 with the *Rule of 78*, which applies to proration of interest on some consumer loan contracts.

Determinants of Market Yields and Prices

Market yields are an imperfect, but easily quoted, measure of the nominal rate of return an investor can earn by purchasing a security

The primary components of yield are "real" yield and expected inflation

Secondary components include the expected value of credit losses, the value of any options embedded in the security, tax effects, and compensation for accepting additional risk or illiquidity

When the market moves, prices and yields move simultaneously because they are mathematically related. A market decline can thus be thought of as either 1) investors demanding to pay less for fixed future cash flows or 2) investors demanding to earn a higher rate on their investment.

The primary determinant of a change in yields is a change in the expected rate of inflation. Investors ultimately care about the purchasing power of their future cash flow receipts; in an inflationary environment, their purchasing power decreases. Therefore, when inflation expectations rise, fixed-income prices decline and yields rise. Inflation expectations can be influenced by the current rate of inflation, the rate of growth of the economy, various industrial-capacity constraints, and governmental policy, amidst a host of other potential factors.

Another cause of a change in yields is a change in "real" yields. Real yields are what investors would demand to earn (risk-free) if there were no prospect of inflation. Depending on changes in supply and demand, investors may be able to command higher or lower real yields.

Investors also demand higher yields for taking additional risk. As the market's perception of risk changes, yields and prices for the affected bonds will also change. These additional risks may arise through extending credit to riskier borrowers, accepting payments that are not fixed, participating in a less-liquid market, or granting rights to issuers of bonds (embedded options).

10-Year U.S. Treasury Zero-Coupon Bond
(Base Yield 7%, Compounded Semi-Annually)

Calculate the price of this zero-coupon bond at the given yields (answers on following page):

Yield (%)	Price (%)
6.99	?
7.00	50.257
7.01	?

Remember:

$$Price = \frac{v}{\left(1 + \frac{y}{f}\right)^n}$$

where

v is the par amount (usually 100%)

y is the yield

f is the compounding frequency (2 in this case)

n is the number of compounding periods (20 in this case)

As markets move, prices and yields change

For a bond, the yield defines a price, and the price also defines a yield

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Estimating Price Changes

10-Year U.S. Treasury Zero-Coupon Bond
(Base Yield 7%, Compounded Semi-Annually)

A basis point (bp) is one-hundredth of a percent (0.01%)

The price change for a 10-basis-point (0.10%) change in yield is roughly 10 times as big as the price change for a one-basis-point (0.01%) change in yield

The change in value for a one-basis-point change in yield is also known as the *dollar value of a basis point.* (*DV01*) or the *present value of a basis point* (*PV01*)

For most fixed-income securities, prices decline as yields rise

We can estimate the price change of a bond when interest rates change by extrapolating from the price change of the bond over small changes in interest rates.

Yield (%)	Linearly Extrapolated Price Estimate (%)
6.80	?
6.90	?
6.99	50.305
7.00	50.257
7.01	50.208
7.10	?
7.20	?

Note that prices decline when yields rise. This truism of fixed income follows directly from the formula for converting a bond's yield to a price. Phrased differently, when yields rise, an investor would need to invest less to produce a fixed future value.

Comparing Estimated and Actual Price Changes

10-Year U.S. Treasury Zero-Coupon Bond (Base Yield 7%, Compounded Semi-Annually)

Yield (%)	Linearly Extrapolated Price Estimate (%)	Actual Price (%)	Difference in Price (%)
6.80	51.228	51.238	0.010
6.90	50.742	50.745	0.002
6.99	50.305	50.305	0.000
7.00	50.257	50.257	0.000
7.01	50.208	50.208	0.000
7.10	49.771	49.773	0.002
7.20	49.285	49.295	0.010

For small changes in interest rates, the linear method of estimating price changes is very accurate

Note that, for this security, the actual price is always higher than the estimated price

The price fell slightly more when the yield rose one basis point (to 7.01%) than it rose when the yield fell one basis point (to 6.99%); this table shows extrapolation using the average of the change for a one-basis-point increase and decrease in yield

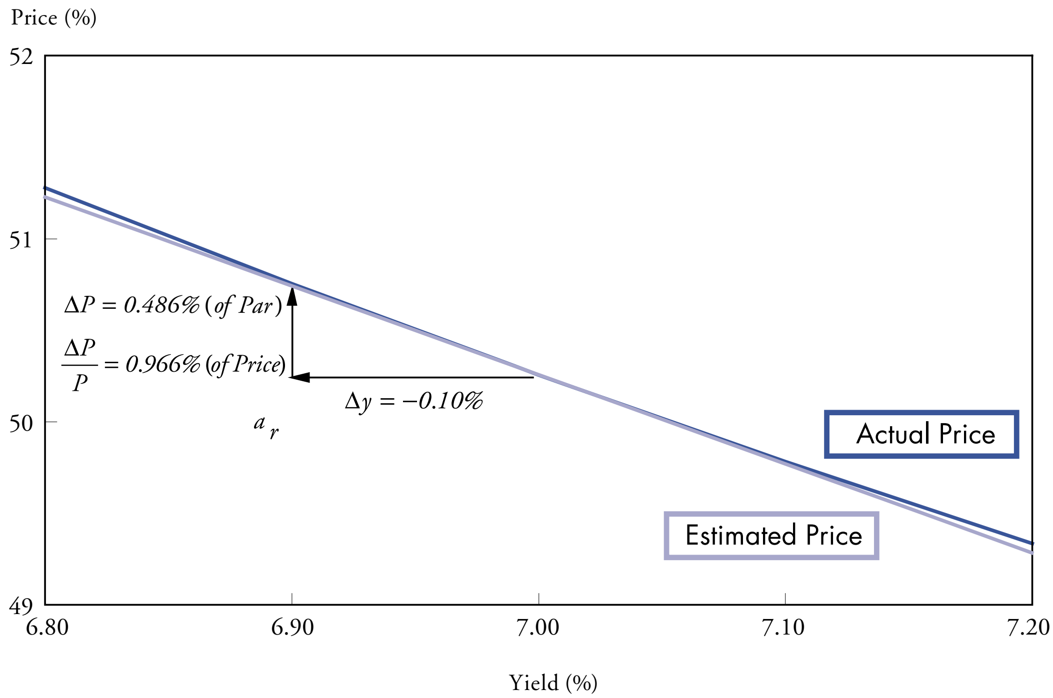
Estimated Prices vs Actual Prices

10-Year U.S. Treasury Zero-Coupon Bond
(Base Yield 7%, Compounded Semi-Annually)

Our extrapolation is simply a tangential approximation of the price/yield curve at the given (base) yield

The slope of the price estimation line, the change in price for a change in yield, is also known as *dollar duration*

The related quantity, *modified duration*, also known as *duration*, shows the same price change as a percent of the current price



The price of the 10-year zero-coupon bond rose by 0.486% of par when yields fell by 10 bp (0.10%). The dollar duration is then:

$$Duration_{Dollar} = -\frac{dP}{dy} \cong -\frac{\Delta P}{\Delta y} = -\frac{0.486\%}{-0.10\%} = 486\%$$

To estimate the price change if yields fall, for example 1%, multiply the dollar duration by the yield change:

$$\Delta P \cong -Duration_{Dollar} \times \Delta y = -486\% \times -1\% = 4.86\%$$

As a percent of initial price, the price of the zero rose by 0.966%. The modified duration is then:

$$Duration_{Modified} = Duration = -\frac{dP/P}{dy} \cong -\frac{\Delta P/P}{\Delta y} = -\frac{0.966\%}{-0.10\%} = 9.66$$

Dollar duration estimates how much a security's value changes for a given change in interest rates. If the dollar duration is quoted as a percent of face, then it can be used for an estimation of price in a different rate environment. If it is quoted in dollars, then it illustrates how the dollar value of a position or portfolio changes when rates change.

Modified duration estimates how much the present value changes *as a percent of the current present value*, and so it is more useful for comparing the interest rate sensitivity of the value of different securities or portfolios.

Mathematically, dollar duration is the slope of the line tangent to the price/yield curve at the current yield (with the sign changed to produce a positive number). Dollar duration, therefore, is given by the negative of the first derivative of the price function with respect to yield.

For a zero-coupon bond:

$$Price = \frac{v}{\left(1+\frac{y}{f}\right)^{n}}$$

$$Duration_{Dollar} = -\frac{dP}{dy} = \frac{\frac{vn}{f}}{\left(1+\frac{y}{f}\right)^{n+1}} \cong -\frac{\Delta P}{\Delta y}$$

$$(\textit{for small } \Delta y)$$

$$Duration = -\frac{dP/P}{dy} = \frac{\frac{n}{f}}{\left(1+\frac{y}{f}\right)} \cong -\frac{\Delta P/P}{\Delta y}$$

$$(\textit{for small } \Delta y)$$

Q: Calculate the dollar and modified duration of a 10-year zero-coupon bond, using a semi-annual yield of 7%. How does your answer compare to the duration on the graph on the preceding page? How does the modified duration compare to the term of the zero?

The modified duration, or simply duration, is defined as:

$$-\frac{\Delta P/P}{\Delta y}$$

for small changes in y and estimates the percentage change in price for an instantaneous, parallel change in yield

Estimating Price Changes (Revisited)

10-Year U.S. Treasury Zero-Coupon Bond
(Base Yield 7%, Compounded Semi-Annually)

We can use dollar duration to estimate prices at different yields

We can quickly estimate the price change of a bond when interest rates change by using the dollar duration of the bond.

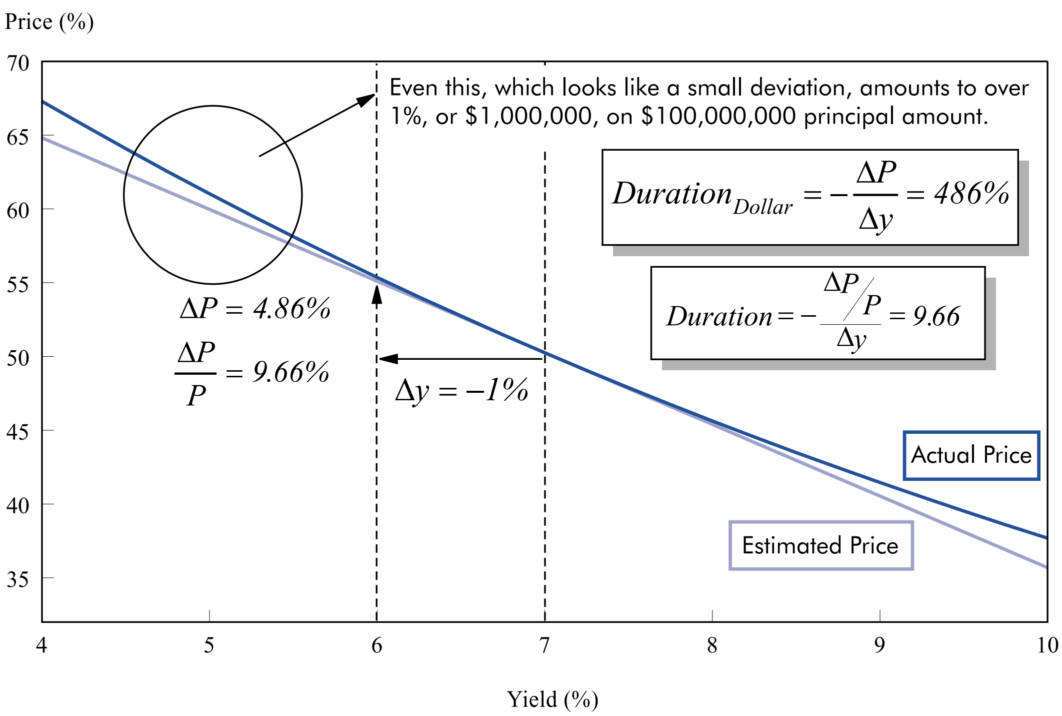
The dollar duration is estimated by $-\dfrac{\Delta P}{\Delta y}$ for small changes in y.

Yield (%)	Linearly Extrapolated Price Estimate (%)
5.00	?
6.00	?
6.99	50.305
7.00	50.257
7.01	50.208
8.00	?
9.00	?

Comparing Estimated and Actual Price Changes

10-Year U.S. Treasury Zero-Coupon Bond (Base Yield 7%)

Yield (%)	Linearly Extrapolated Price Estimate (%)	Actual Price (%)	Difference in Price (%)
5.00	59.968	61.027	1.059
6.00	55.112	55.368	0.255
6.99	50.305	50.305	0.000
7.00	50.257	50.257	0.000
7.01	50.208	50.208	0.000
8.00	45.401	45.639	0.238
9.00	40.545	41.464	0.919



Duration is the linear estimate of how price changes when yield changes

The duration-based estimates are always lower than the actual prices (for bonds with no embedded options)

The error in estimation grows larger with the *square* of the change in interest rates: it quadruples when the change in interest rate doubles

There is a second-order correction called convexity that explains the majority of the difference between the linear estimate and the actual price

Q: What happens to dollar duration as yields change?

Convexity

Second-Order Interest Rate Sensitivity of a Security

Convexity **estimates the difference between the actual price and the price estimate obtained using duration**

Noncallable bonds have positive convexity; the actual price is always higher than the duration-based estimate

Trick question: Why do zero-coupon bonds have positive convexity if their duration always equals their maturity?

Convexity measures the degree of curvature in a security's price/yield relationship, i.e., the rate of change in dollar duration. When the price/yield relationship is curved, the linear estimate (using constant dollar duration) will always have error.

Just as there were two related quantities for expressing a linear estimate of price change, dollar duration and duration, there are two related quantities for expressing the error: *dollar convexity* and *convexity*. Dollar convexity estimates both the additional change in price and the change in dollar duration for a given change in rates and is useful for refining price estimates in a different interest rate environment. Convexity estimates the same changes, but as a percent of price, and so it is more useful for comparing which security or portfolio can expect a greater percentage price boost above the duration estimate when rates change.

Mathematically, dollar convexity is the second derivative of the price function with respect to yield:

$$Convexity_{Dollar} = \frac{d^2P}{dy^2} = \frac{-d(Duration_{Dollar})}{dy} \cong \frac{-\Delta Duration_{Dollar}}{\Delta y} \quad \text{(for small } \Delta y\text{)}$$

For a zero-coupon bond:

$$Price = \frac{v}{\left(1+\frac{y}{f}\right)^n}$$

$$Convexity_{Dollar} = \frac{d^2P}{dy^2} = \frac{vn(n+1)}{f^2\left(1+\frac{y}{f}\right)^{n+2}} \qquad Convexity = \frac{d^2P/P}{dy^2} = \frac{n(n+1)}{f^2\left(1+\frac{y}{f}\right)^2}$$

Note that for zero-coupon bonds, convexity goes up approximately with the square of maturity n/f.

Using Duration and Convexity

The Taylor series expansion for price P_1 given an initial price P_0 and a change in yield from y_0 to y_1 is

$$P_1 = P_0 + \frac{dP}{dy} \times (y_1 - y_0) + \frac{1}{2} \times \frac{d^2P}{dy^2} \times (y_1 - y_0)^2 + \cdots$$

So,

$$P_1 \cong P_0 - Duration_{Dollar} \times (y_1 - y_0) + \frac{1}{2} \times Convexity_{Dollar} \times (y_1 - y_0)^2$$

Alternatively,

$$P_1 \cong P_0 - P_0 \times Duration \times (y_1 - y_0) + \frac{1}{2} \times P_0 \times Convexity \times (y_1 - y_0)^2$$

$$\cong P_0 \times \left(1 - Duration \times (y_1 - y_0) + \frac{1}{2} \times Convexity \times (y_1 - y_0)^2 \right)$$

Since $Duration_{Dollar} = -\dfrac{dP}{dy} = P \times Duration$ and

$$Convexity_{Dollar} = \frac{d^2P}{dy^2} = P \times Convexity$$

Some firms quote a *gain from convexity*, which is defined as:

$$Convexity_{Gain} = \frac{Convexity}{2}$$

Then,

$$P_1 \cong P_0 - P_0 \times Duration \times (y_1 - y_0) + P_0 \times Convexity_{Gain} \times (y_1 - y_0)^2$$

The Taylor series expansion for price shows how to use duration and convexity to estimate the price for a given change in yield

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Using Duration and Convexity (Continued)

Example: 10-Year U.S. Treasury Zero-Coupon Bond

Use the following formulas to answer the questions below:

$$Price = \frac{v}{\left(1+\frac{y}{f}\right)^{n}} \qquad Duration = \frac{\frac{n}{f}}{\left(1+\frac{y}{f}\right)} \qquad Convexity = \frac{n(n+1)}{f^{2}\left(1+\frac{y}{f}\right)^{2}}$$

Q1: At a yield of 7%, what is the price of the bond?

Q2: What is the duration?

Q3: What is the convexity?

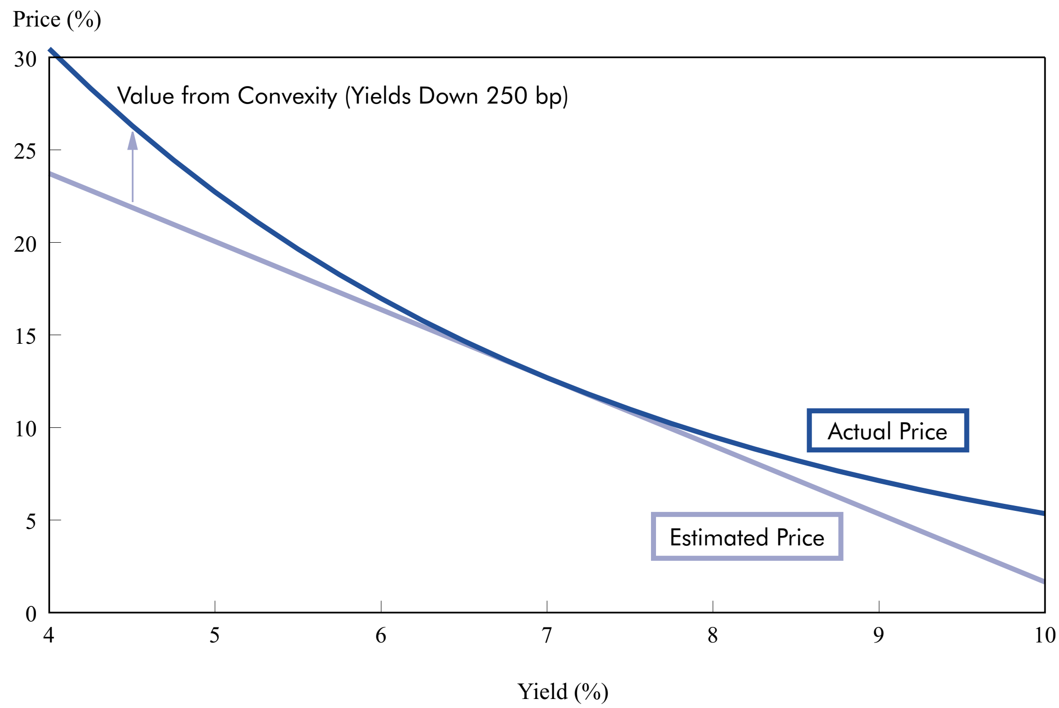
Q4: Estimate the price if yields fall 200 bp using the following formula:

$$P_1 \cong P_0 - P_0 \times Duration \times (y_1 - y_0) + \frac{1}{2} \times P_0 \times Convexity \times (y_1 - y_0)^2$$

Q5: How does your estimate compare to the actual price of 61.027%?

Duration and Convexity
(for a Longer-Duration Security)

30-Year U.S. Treasury Zero-Coupon Bond (Base Yield 7%)



	10-Year	30-Year
Price	50.26%	12.69%
Slope of Estimator Line	– 4.86	?
Dollar Duration	486%	?
Duration	9.66	?
Dollar Convexity	4926%	?
Convexity	98.02	?

The longer the duration, the more significant the gain from convexity

Q1: What is the slope of the dotted estimator line for the 30-year vs. the 10-year zero-coupon bond?

Q2: What are the dollar convexity and convexity of the 30-year U.S. Treasury zero-coupon bond?

Q3: How do these compare to the dollar convexity and convexity of the 10-year zero-coupon bond?

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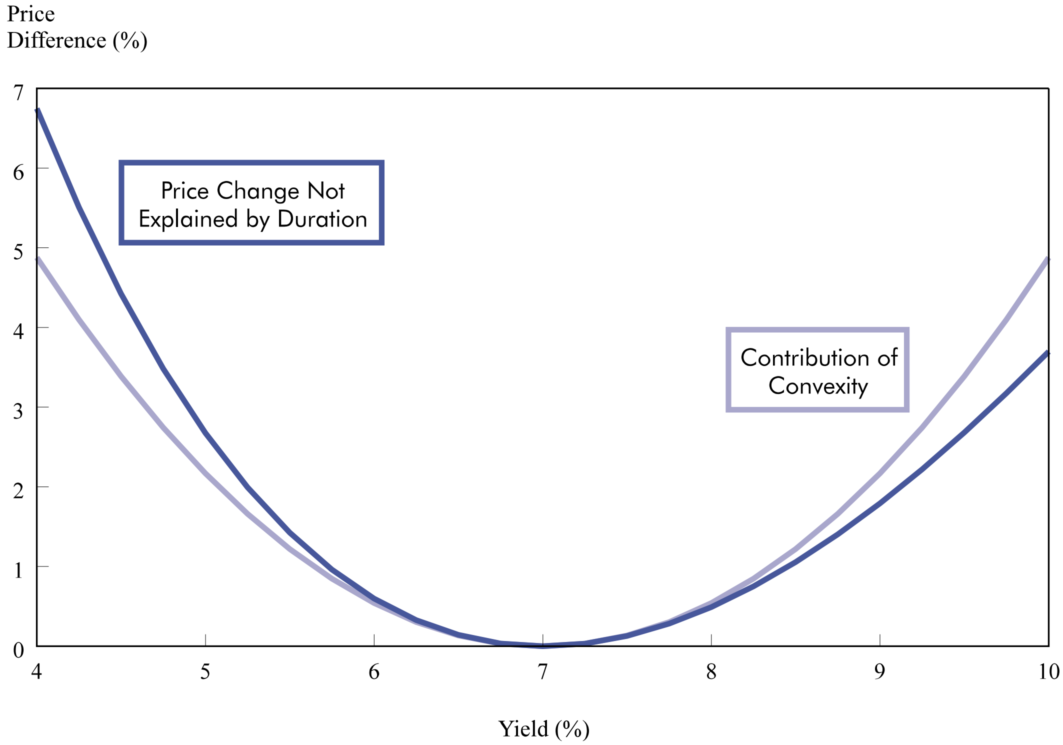
Contribution of Convexity to Estimate of Price

30-Year U.S. Treasury Zero-Coupon Bond (Base Yield 7%, Compounded Semi-Annually)

The contribution of convexity increases with the square of the change in interest rates

It underestimates the excess price appreciation in a declining-interest-rate environment and overestimates the appreciation in a rising-interest-rate environment



Continuous Compounding

Advanced

The formula for one year's worth of continuous compounding is

$$\lim_{f\to\infty}\left(1+\frac{y}{f}\right)^{f} = e^{y}$$

Therefore, the price of a 1-year zero is e^{-y} and the price of a t-year zero is e^{-yt}, where y is a continuously compounded yield. Note that t is measured in years because y is an annualized rate.

This formula has some interesting attributes:

$$Duration = -\frac{dP/dy}{P} = t \qquad Convexity = \frac{d^2P/dy^2}{P} = t^2$$

Why are the duration and convexity formulas simpler when the yield is continuously compounded than in the noncontinuous case? Define the number of periods as tf where t is the term measured in years. Then

$$Duration = \lim_{f\to\infty}\frac{\frac{tf}{f}}{\left(1+\frac{y}{f}\right)} = \lim_{f\to\infty}\frac{t}{\left(1+\frac{y}{f}\right)} = t$$

Note that for a given change in quoted (semi-annual) yield, the continuously compounded yield changes by less (because it compounds more frequently). The estimated price change is the yield change multiplied by the duration. Since the continuously compounded yield changes by less, the security's duration with respect to that yield must be longer to estimate the same price change.

Continuous compounding provides an economy of expression for theoretical analysis; no security is quoted using continuous compounding

Chapter 1 Exercises

1. Calculate the present value, modified duration, dollar duration, and convexity of these two Treasury STRIPS (zero-coupon bonds).

Maturity (Years)	Yield (%)	Present Value (%)	Modified Duration	Dollar Duration (%)	Convexity
5 Years	6.75				
25 Years	7.50				

2. What is the dollar duration of a 1-year STRIPS yielding 5%? What is its modified duration? What is the dollar duration of a 30-year STRIPS yielding 8%? What is its modified duration?

3. What are the price, modified duration, and convexity of a 30-year STRIPS at a 7% and a 7½% yield? How do these numbers all fit together?

4. A pension fund manager has a $23 million liability due in five years. How much needs to be invested today if the manager can lock in an annual interest rate of 6.75% for five years? How much if the rate compounds semi-annually?

5. What is the semi-annually compounded yield of a Treasury STRIPS that matures in 20 years and is priced at 23.111%?

6. If Manhattan was worth $24 in trade goods 360 years ago, what has been the annual total rate of return on the investment if the island is worth $100 billion today?

Chapter 1 Exercises (Continued)

7. If a corporation expects to pay $100 million in the year 2020 (24 years from now) to its pension beneficiaries, what is the present value of this liability at an annual discount rate of 7.25%? If rates decline by 100 bp, what is the new value of the liability? What is the error if we estimate the new liability value using duration?

8. A security that promises to pay $10,000 five years from now can be purchased for $7,175.38 today. What is its semi-annually compounded yield? If there is a secondary market for this security, how will its market yield change if the credit quality of the issuer deteriorates?

9. Should you pay $6 million today for a bond that promises to pay $9 million in five years if you need to earn an 8.00% annual return?

10. A municipality has a $10 million liability payable July 15, 2020. To satisfy the liability, the municipality must either set aside $10 million cash today (June 26, 1996) or buy U.S. Treasury securities disbursing $10 million to ensure that the debt will be paid. If the following zero-coupon Treasury securities are available, what must the municipality pay today to satisfy this liability, assuming short rates rarely fall below 3%?

Maturity	Price (%)	Yield (%)
2/15/20	17.828	7.43
5/15/20	17.507	7.43
8/15/20	17.269	7.41
11/15/20	17.040	7.39

11. Derive a simple formula for convexity of a zero-coupon bond in terms of its duration and yield.

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