Chapter 2

Coupon Bonds

This chapter is an excerpt from *A Morgan Stanley Guide to Fixed Income Analysis* by Andrew R. Young, ©2003 Morgan Stanley & Co. Incorporated.

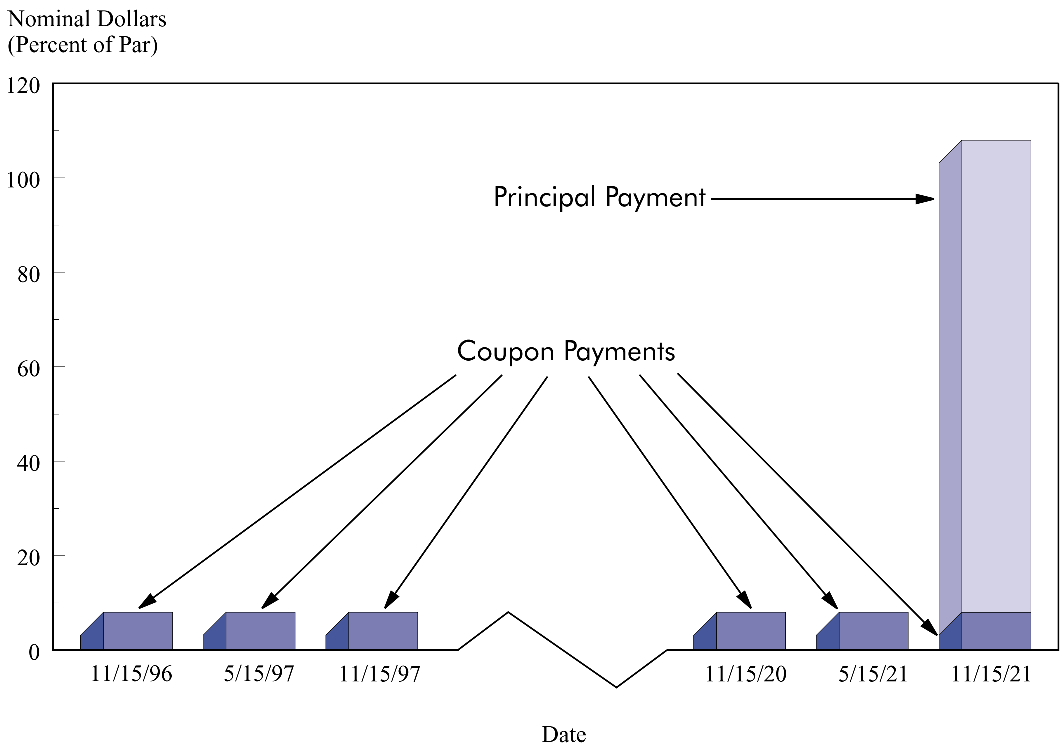
Morgan Stanley

In This Chapter, You Will Learn...

- About Accrued Interest
- Fixed-Income Calendar Conventions
- How to Price a Coupon Bond
- How to Value Annuities
- How to Calculate a Yield
- How to Amortize a Premium or Discount
- Different Methods of Quoting Duration
- The Durations of Coupon Bonds with Different Maturities
- The Value of Convexity

U.S. Treasury Bond Structure

U.S. Treasury 8% Due November 15, 2021



The Treasury bond is the most common type of coupon bond

A U.S. Treasury bond pays interest semi-annually (in arrears)

Each coupon payment is half the nominal rate of interest: 4% of face value on this 8% coupon bond

The present value of the bond—how much the buyer must pay now to get all the bond's future cash flows—is the sum of the present values of the individual cash flows

Morgan Stanley

Valuing a Coupon Bond

A Whole Number of Coupon Periods Remaining

Bearing in mind that different cash flows may have different yields, we can write a formula for the present value of a coupon bond as the sum of the values of the individual cash flows

Alternatively, we can value all the cash flows at the same rate, called the *yield-to-maturity*

In the example, the yield-to-maturity is 6.28% and the present value of the bond equals 105.038%; if the yield-to-maturity had been equal to the coupon (in this case 9%), the present value of the bond would have been 100% (par)

$$PV = \frac{\frac{c}{f}}{\left(1+\frac{y_1}{f}\right)} + \frac{\frac{c}{f}}{\left(1+\frac{y_2}{f}\right)^2} + \cdots + \frac{\frac{c}{f}}{\left(1+\frac{y_{n-1}}{f}\right)^{n-1}} + \frac{\frac{c}{f}}{\left(1+\frac{y_n}{f}\right)^n} + \frac{v}{\left(1+\frac{y_n}{f}\right)^n}$$

$$= \frac{c}{f}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{y_i}{f}\right)^i} + \frac{v}{\left(1+\frac{y_n}{f}\right)^n}$$

c is the annual coupon rate,

v is the redemption value,

y_i is the yield for an i-period zero-coupon bond, quoted on a compound basis,

f is the payment and compounding frequency,

n is the number of whole coupon periods between settlement and maturity, and

$\frac{n}{f}$ is the number of years remaining until maturity.

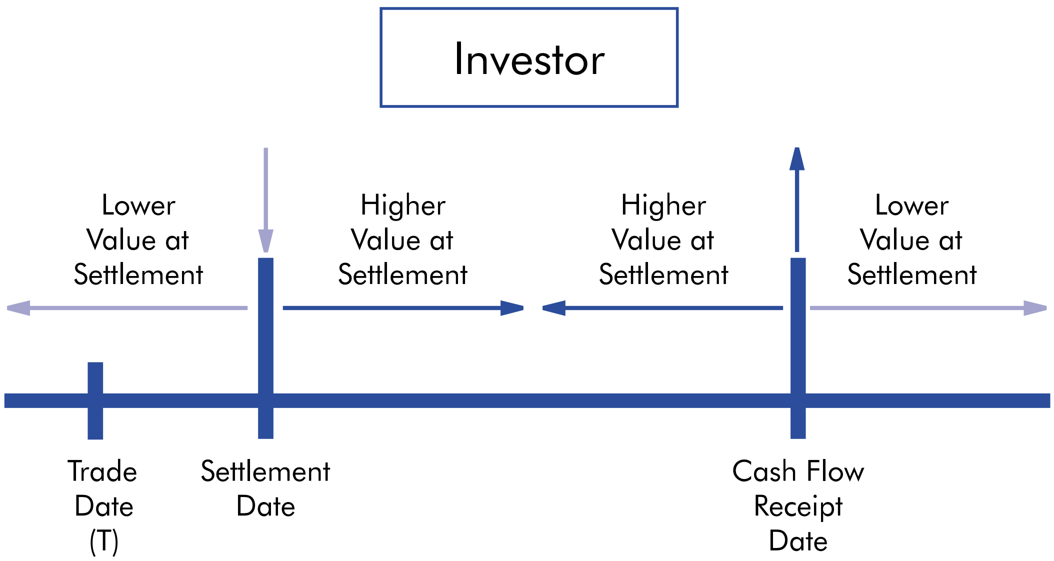
If all the y_i's are the same, y is equal to the *yield-to-maturity*.

Example: U.S. Treasury 9% due May 15, 1998, with a yield-to-maturity of 6.28% and a settlement (funds-bond transfer) date of May 15, 1996:

$$PV = \frac{\frac{9\%}{2}}{\left(1+\frac{6.28\%}{2}\right)} + \frac{\frac{9\%}{2}}{\left(1+\frac{6.28\%}{2}\right)^2} + \frac{\frac{9\%}{2}}{\left(1+\frac{6.28\%}{2}\right)^3} + \frac{100\% + \frac{9\%}{2}}{\left(1+\frac{6.28\%}{2}\right)^4}$$

$$= 105.038\%$$

Cash Flow Timeline



Because of the time value of money, future cash flows are more valuable the closer they are to settlement (the day when they are paid for). The earlier the settlement date, the farther away the future cash flows, and the lower the value of the bond (unless the earlier settlement entitles the buyer to additional cash flows). The later the settlement date, the nearer the future cash flows and the higher the value of the bond. Conversely, the earlier the cash flows, the higher the value of the bond; and the later the cash flows, the lower the value of the bond.

It is important to understand exactly when cash changes hands (the settlement date and the future cash flow payment dates)

Every market has a *regular* settlement schedule; currently, regular settlement for Treasuries is T (trade) + 1 (next business day), and most other domestic products settle T+3

For Treasuries, *skip-day* means T+2, and *cash* or *same-day* means T+0

Quoting Bonds: Price and Present Value

For zero-coupon bonds, price and present value are identical; however, this is not true for coupon bonds (except on a coupon payment date)

For a zero-coupon bond, price and present value are identical, and we have used them interchangeably. For a coupon bond, price and present value are the same only on a coupon payment date.

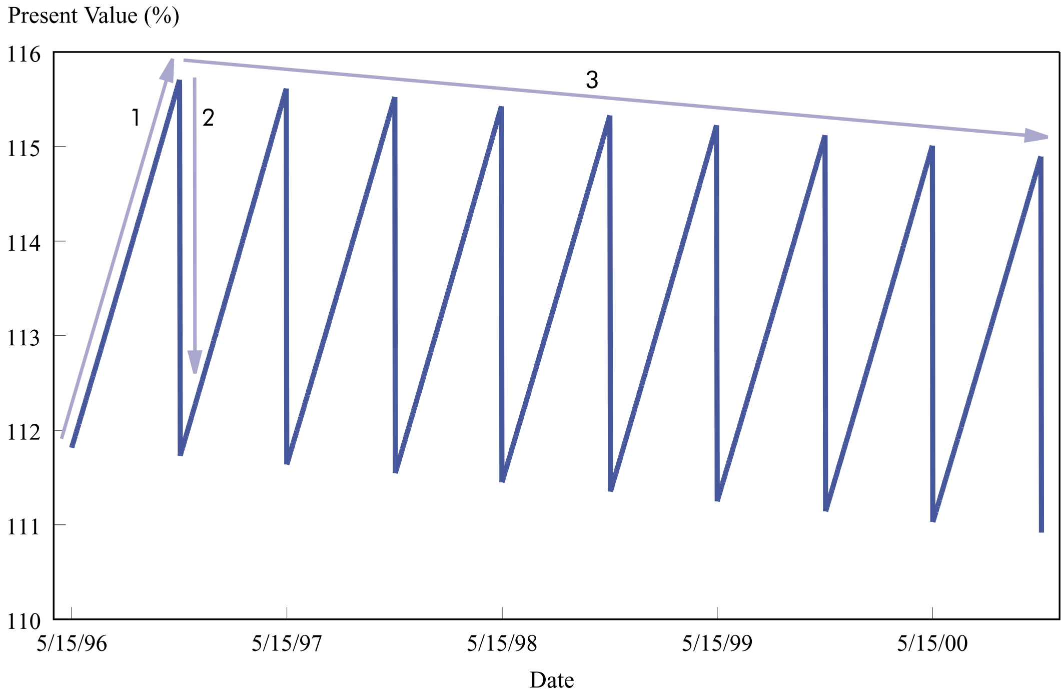
A bond's present value (market value) identifies its total cost — or the amount of money which must be given to the seller as compensation for delivering the security. Present value is, therefore, the fundamental measure of value for a bond.

However, present value is not the most convenient (or common) way to quote bonds. As the next page shows, a bond's present value fluctuates dramatically over time, even when its yield remains constant.

So, for convenience, market participants quote a *price* that is more stable than present value over time. The quoted price is slightly less than the present value of a bond, but they are precisely related. All market participants know how to transform a price into present value to determine the cost of an acquisition or the proceeds of a sale.

Present Value Over Time (at Constant Yield)

U.S. Treasury 8% Due November 15, 2021, Priced to Yield 7%



It would be nice if the value of a bond over time were smoother so we could isolate the change in value due to a change in rates!

Even when a bond's yield does not change, its present value changes over time because:

1) Value increases as a coupon date approaches,

2) Value decreases after a coupon is paid to the bondholder, and

3) As the number of coupons remaining decreases, the value of the bond drifts toward par

Quoting Bonds: Price and Present Value (Continued)

When securities are traded, the money exchanged is the present value of the securities

However, transactions are usually agreed upon based on a quoted price, which is the present value reduced for a somewhat arbitrary *accrued interest*

Since present value is the critical quantity, the precise methodology for calculating accrued interest is irrelevant, as long as all market participants calculate it the same way

Price	**=**	**Present Value**	**–**	**Accrued Interest**
Convenient for quotations		Value of the bond (exchanged at sale)		Just a definition (to smooth price quotations)

Equivalent terminology used in the marketplace:

- *Flat* Price = *Full* Price – Accrued Interest
- *Clean* Price = *Dirty* Price – Accrued Interest
- Principal = Net – Accrued Interest

Alternatively,

- Present Value = Price + Accrued Interest
- *Full* Price = *Flat* Price + Accrued Interest
- *Dirty* Price = *Clean* Price + Accrued Interest
- Net = Principal + Accrued Interest

Elements of Accrued Interest

Accrued interest represents the value of interest earned since the last coupon payment date

Up until now, we have computed the present value of bonds with a whole number of compounding periods until maturity. For the traditional bond, whose compounding frequency equals its payment frequency, there are a whole number of coupon periods remaining only when settlement lies on a coupon payment date. In that situation, the bond has no accrued interest, and the seller receives the coupon paid on the settlement date.

Between coupon payment dates, a bond will have accrued interest. The mechanics of computing accrued interest depend on the calendar conventions that hold for that particular type of security.

The following are the fundamental elements of accrued interest:

- The size of the next coupon, usually c/f (if the coupon is irregular, the size of the next coupon may be larger or smaller and could depend on the calendar conventions);
- The days on which coupons are paid. Most types of securities pay every coupon at the end of the month, if the bond matures at the end of the month. For example, a Treasury maturing on November 30, 1996 pays coupons on May 31 and November 30. An exception is bonds issued by the Federal Home Loan Bank (FHLB);
- The amount of time a (numerator) in the accrual period that has elapsed since the last coupon date (or interest-accrual date, if the settlement date falls prior to the first coupon payment of the bond). The coupon date is determined without regard to business days, even though a coupon scheduled to be paid on a weekend would be paid on the following business day. The measurement of a depends on the calendar; and
- The amount of time b (denominator) in the full coupon accrual period. This measurement also depends on the calendar.

The accrued interest would then be calculated as $\frac{c}{f} \times \frac{a}{b}$.

Actual/Actual Calendar

U.S. Treasury notes, bonds, and STRIPS use the *actual/actual* calendar convention

Because of the predominance of Treasury securities, the actual/actual calendar will sometimes be applied to other securities with different calendar conventions so they can be compared to Treasuries on an equal footing

- The *actual/actual* calendar applies to U.S. Treasury notes, bonds and STRIPS.

- The "actual" number of days of interest in the accrual period a is the number of calendar days that have elapsed since the last coupon date, not including that date, up to and including the settlement date.

- The "actual" number of days b is the number of days in the complete coupon period containing the settlement date. A full six-month period can have only 181, 182, 183, or 184 calendar days.

- **Example 1**: With a settlement date of August 1, 1996, the actual/actual accrued interest on the 8% due November 15, 2021 would be:

$$\frac{8\%}{2} \times \frac{\text{Days Between May 15, 1996 and August 1, 1996}}{\text{Days Between May 15, 1996 and November 15, 1996}} = 4\% \times \frac{78}{184} = 1.696\%$$

- **Example 2**: With a settlement date of September 30, 1996, the actual/actual accrued interest on a 6% due January 31, 2007 would be:

$$\frac{6\%}{2} \times \frac{\text{Days Between July 31, 1996 and September 30, 1996}}{\text{Days Between July 31, 1996 and January 31, 1997}} = 3\% \times \frac{61}{184} = 0.994\%$$

- Leap day (February 29) counts as a calendar day.

30/360 Calendar

SIA Convention[1]

- To determine the number of 30/360 days between two dates, $Date_1$ (prior coupon date) and $Date_2$ (settlement), where $Date_1$ is earlier:

$$
\begin{array}{l}
\quad 360 \times (Year_2 - Year_1) \\
+\ 30 \times (Month_2 - Month_1) \\
+\ \Delta Days \text{ (from the following table)} \\
\hline
=\ 30/360 \text{ days between } Date_1 \text{ and } Date_2
\end{array}
$$

Day_1	Day_2	$\Delta Days$
Not End of Month		$Day_2 - Day_1$
End of Month	Not End of Month	$Day_2 - 30$
End of Month	End of Month	0
Except:		
End of Month (Excluding February)	End of February	$Day_2 - 30$

- The denominator always has 180 days for a semi-annual bond. More generally, it has $360/f$ days.

- **Example 1**: With a settlement date of August 1, 1996 and a 30/360-day calendar, accrued interest on an 8% due November 15, 2021 would be:

$$\frac{8\%}{2} \times \frac{30/360 \text{ Days Between May 15, 1996 and August 1, 1996}}{30/360 \text{ Days Between May 15, 1996 and November 15, 1996}} = 4\% \times \frac{76}{180} = 1.689\%$$

- **Example 2**: With a settlement date of September 30, 1996 and a 30/360-day calendar, accrued interest on a 6% due January 31, 2007 would be:

$$\frac{6\%}{2} \times \frac{30/360 \text{ Days Between July 31, 1996 and September 30, 1996}}{30/360 \text{ Days Between July 31, 1996 and January 31, 1997}} = 3\% \times \frac{60}{180} = 1.000\%$$

[1] Jan Mayle, *Standard Securities Calculation Methods,* vol. 1, Third Edition. New York: Securities Industry Association, 1993.

For corporates, municipals, and agencies, the market uses a 30/360-day calendar, where every year is composed of 12 30-day months

Using the 30/360 calendar, any bond that matures at the end of a month accrues no interest on the 31st day of any month

Q: What is the impact of this non-accrual on corporate bond prices?

Morgan Stanley

Price Over Time
(at Constant Yield)

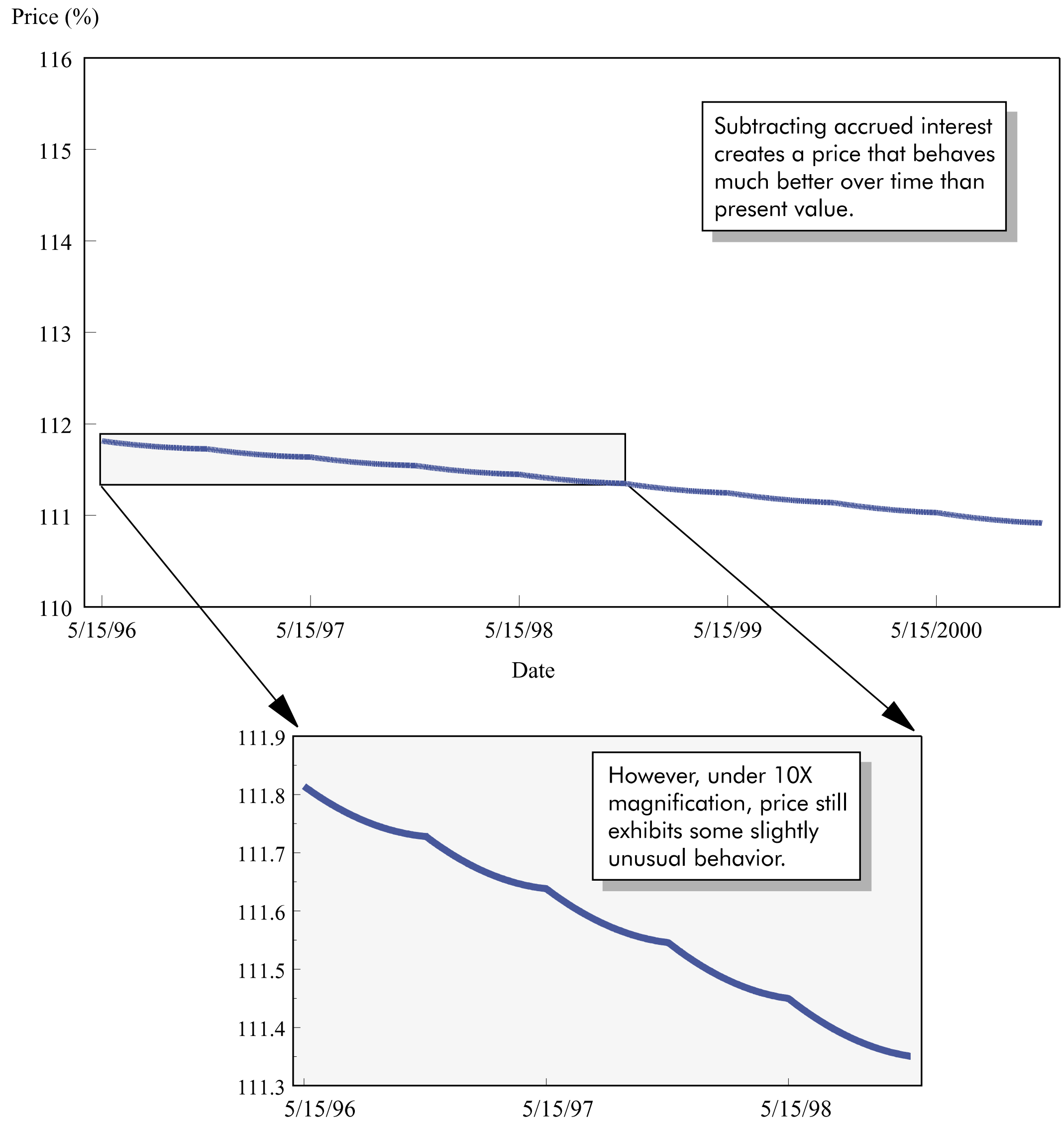
U.S. Treasury 8% Due November 15, 2021, Priced to Yield 7%

Now, we have a convenient way of quoting a price that "behaves" better than present value

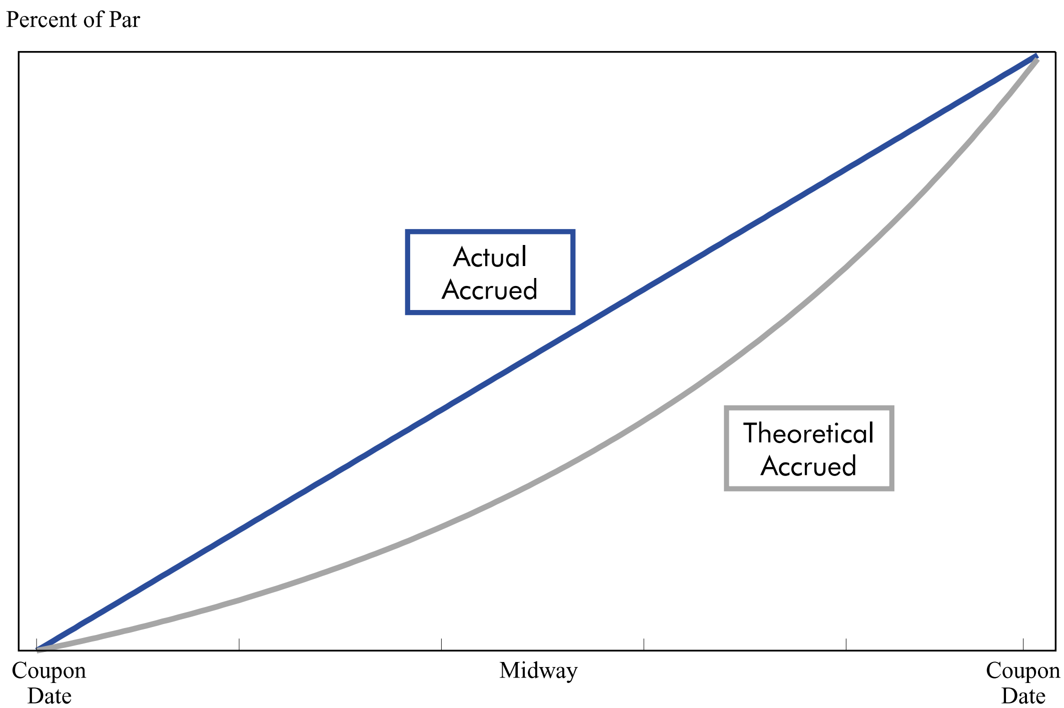
Price = PV – Accrued

To a first approximation, when yields remain constant, the quoted price of a bond only changes as it drifts toward par

Q: Why isn't this line smooth?



Actual vs Theoretical Accrued Interest



There is a "theoretical" accrued interest that would cause the price to drift smoothly towards par over time.

However, accrued interest is actually computed using the methodology on the prior pages. Market participants always use the "actual" calculation for accrued interest, so that is all you really need to know.

Accrued interest is actually calculated using the linear rule on the prior pages

The theoretical accrued interest grows according to the rules of compound interest, the same way the bond's present value grows

The actual accrued interest is always higher than the theoretical accrued interest

The difference between actual and theoretical has been accentuated here for presentation purposes

Sample Confirm

Different securities have different conventions for rounding

Treasury securities, including STRIPS, round price to seven decimal places, and round accrued to eight decimal places

Corporate bonds round price to three decimal places

Note that U.S. Treasury transactions of more than $50 million face amount are broken into lots no bigger than $50 million; this is the maximum size for the Fed-wire system

You	**Sold**		
Trade Date	07/05/1996	U.S. Treasury Note,	
Settle Date	07/08/1996	5 3/8 11/30/1997	
Cusip	912827V90	DTD 11/30/1995	
Symbol	Note	Price 98 25/32	
Acct Type	C.O.D.	Principal	49,390,625.00
Qty	50,000,000	Interest	279,030.05
		Net Due (You)	49,669,655.05

You	**Sold**		
Trade Date	07/05/1996	U.S. Treasury Note,	
Settle Date	07/08/1996	5 3/8 11/30/1997	
Cusip	912827V90	DTD 11/30/1995	
Symbol	Note	Price 98 25/32	
Acct Type	C.O.D.	Principal	49,390,625.00
Qty	50,000,000	Interest	279,030.05
		Net Due (You)	49,669,655.05

You	**Bought**		
Trade Date	07/05/1996	U.S. Treasury Note,	
Settle Date	07/08/1996	6 5/8 due 06/30/2001	
Cusip	912827V48	DTD 07/01/1996	
Symbol	Note	Price 99 12/32	
Acct Type	C.O.D.	Principal	49,687,500.00
Qty	50,000,000	Interest	69,009.51
		Net Due (Us)	49,750,509.51

You	**Bought**		
Trade Date	07/05/1996	U.S. Treasury Note,	
Settle Date	07/08/1996	6 5/8 due 06/30/2001	
Cusip	912827V48	DTD 07/01/1996	
Symbol	Note	Price 99 12/32	
Acct Type	C.O.D.	Principal	49,687,500.00
Qty	50,000,000	Interest	63,009.51
		Net Due (Us)	49,750,509.51

Coupon Bonds: Pricing

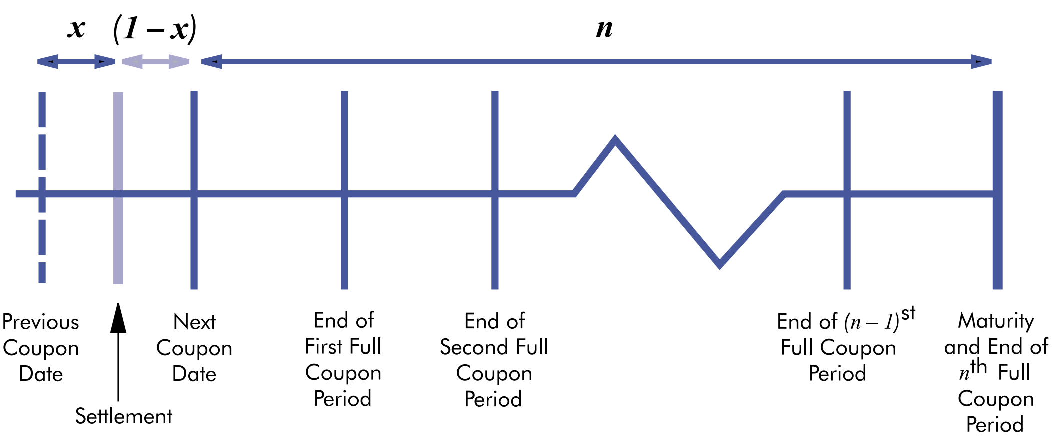
Basic Bond Pricing Timeline

The bond and its cost (present value) are transferred on the settlement date

The new owner receives all future cash flows

The time-dependent quantities n and x are critical for pricing the future cash flows of a bond

The calculations are identical for zero-coupon bonds, even though there are no actual "coupon" dates



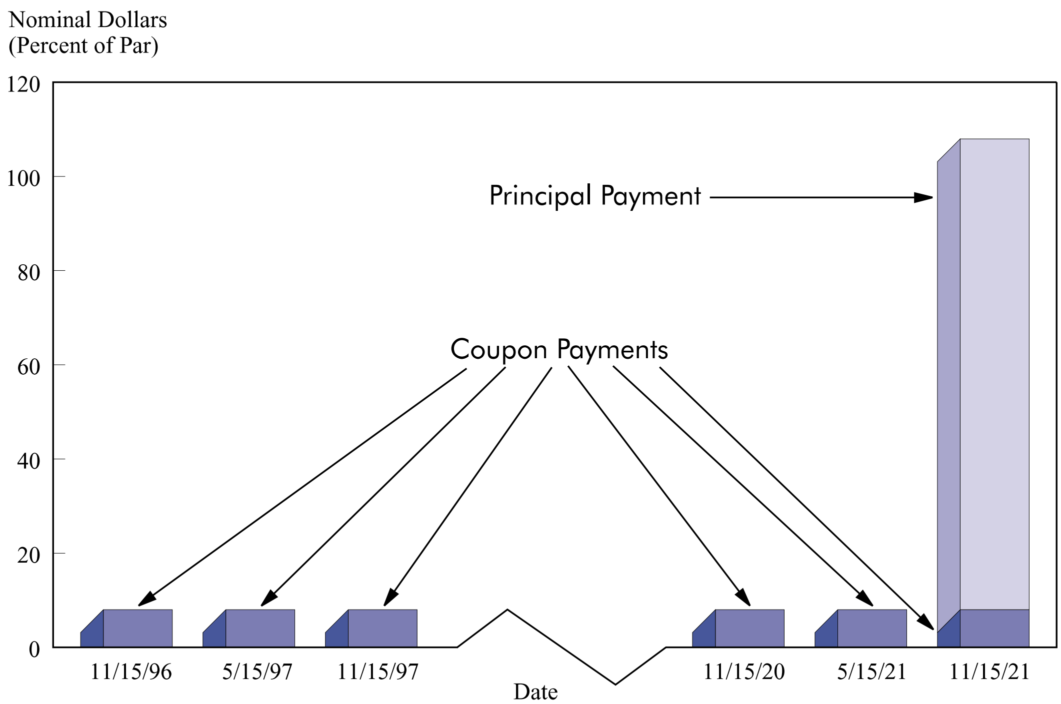
where

n is the number of whole coupon periods between the next coupon date and maturity.

x is the length of the accrual period (measured in units of whole coupon periods).

Pricing Coupon Bonds

U.S. Treasury 8% Due November 15, 2021
(7% Yield; May 15, 1996 Settlement)



$$\frac{4\%}{\left(1+\frac{7\%}{2}\right)}+\frac{4\%}{\left(1+\frac{7\%}{2}\right)^{2}}+\frac{4\%}{\left(1+\frac{7\%}{2}\right)^{3}}+\cdots\cdots+\frac{4\%}{\left(1+\frac{7\%}{2}\right)^{49}}+\frac{4\%}{\left(1+\frac{7\%}{2}\right)^{50}}+\frac{104\%}{\left(1+\frac{7\%}{2}\right)^{51}}$$

The present value of a bond is the sum of the present values of its individual cash flows

The present value of a bond is also the present value of the principal payment at redemption plus the present value of an annuity representing all the coupon payments

Valuing a Coupon Annuity

Valuing the Coupon Stream (S) on the *Next* Coupon Date

There is a mathematical trick for valuing a coupon annuity with a (relatively) simple formula

$$S = \frac{c}{f} + \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)} + \cdots + \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n-1}} + \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n}}$$

c = annual coupon rate
y = yield, quoted on a compound basis
f = payment and compounding frequency
n = number of whole coupon periods remaining until maturity

$$\frac{S}{\left(1+\frac{y}{f}\right)} = \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)} + \cdots + \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n}} + \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n+1}}$$

Dividing each side by $1+\frac{y}{f}$

$$S - \frac{S}{\left(1+\frac{y}{f}\right)} = \frac{c}{f} - \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n+1}}$$

Taking the difference between the two allows us to reduce the stream to a simpler form.

$$S - \frac{S}{\left(1+\frac{y}{f}\right)} = S \times \left[\frac{\frac{y}{f}}{1+\frac{y}{f}}\right] = \frac{c}{f} - \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n+1}}$$

Simplifying the left-hand side.

$$S = \frac{\frac{c}{f}\left(1+\frac{y}{f}\right) - \frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{n}}}{\frac{y}{f}}$$

Solving for S.

$$= \frac{c\left(1+\frac{y}{f}\right) - \frac{c}{\left(1+\frac{y}{f}\right)^{n}}}{y}$$

Simplifying.

c is the annual coupon rate,

v is the redemption value,

y is the yield, quoted on a compound basis,

f is the payment and compounding frequency,

n is the number of whole coupon periods between the next coupon date and maturity,

S is the value of the coupon stream on the next coupon date, and

x is the length of the accrual period (measured in units of whole coupon periods), using the appropriate calendar, $0 \le x < 1$

First, compute the value of the bond *on the next coupon date* by adding the value of the principal to the value of the coupon annuity:

$$PV_{\text{Next Coupon Date}} = S + \frac{v}{\left(1+\frac{y}{f}\right)^n} = \frac{c\left(1+\frac{y}{f}\right) - \frac{c}{\left(1+\frac{y}{f}\right)^n}}{y} + \frac{\frac{vy}{\left(1+\frac{y}{f}\right)^n}}{y} = \frac{c\left(1+\frac{y}{f}\right) + \frac{vy-c}{\left(1+\frac{y}{f}\right)^n}}{y}$$

Then discount this value back to settlement using the fraction of a period between settlement and the next coupon date (the complement of the accrual period) according to the appropriate calendar:

$$\boxed{PV = \frac{c\left(1+\frac{y}{f}\right) + \frac{vy-c}{\left(1+\frac{y}{f}\right)^n}}{y\left(1+\frac{y}{f}\right)^{1-x}}} \quad \boxed{Price = \frac{c\left(1+\frac{y}{f}\right) + \frac{vy-c}{\left(1+\frac{y}{f}\right)^n}}{y\left(1+\frac{y}{f}\right)^{1-x}} - x \times \frac{c}{f}}$$

The present value of a bond is the present value of the coupons (S) plus the present value of the principal

The price of a bond is the present value less the accrued interest

Securities with less than one coupon period until maturity (i.e., $n=0$) are valued using a *simple-interest* methodology, described next

Q: Which inputs are calendar-dependent?

Compound- vs Simple-Interest Yield

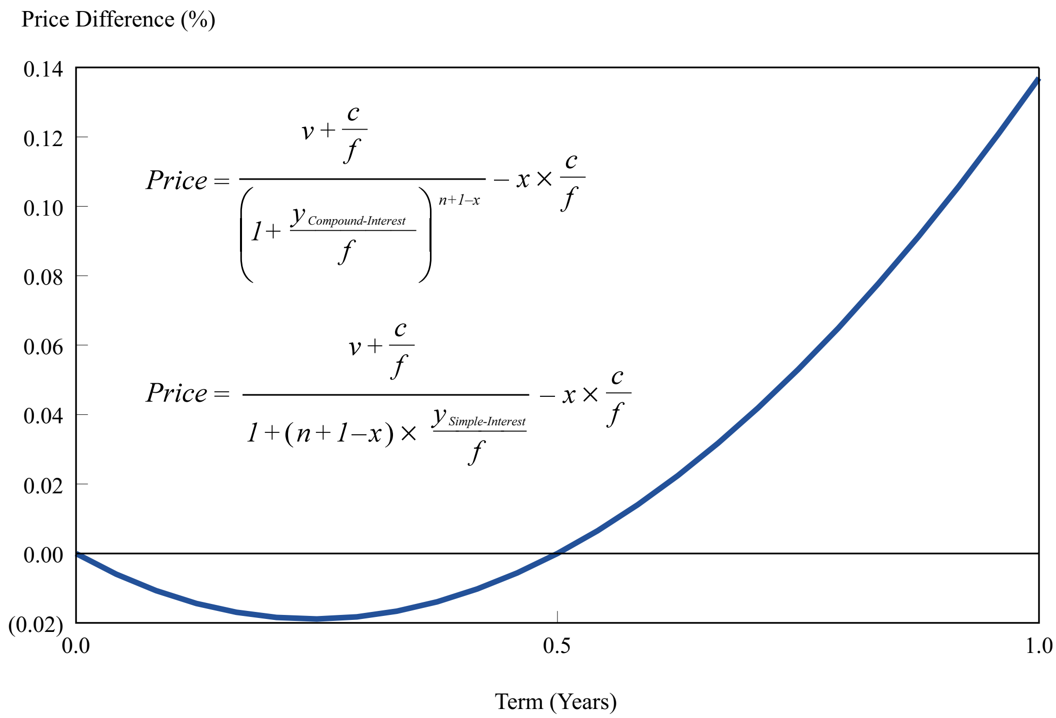
The price difference between a compound- and a simple-interest interpretation of a yield is less than a 32nd for periods less than seven months

By convention, securities in their last coupon period ($n=0$) are quoted on a simple-interest basis

The difference is small because of the mathematical approximation: $(1+y)^t \approx 1+t \times y$ for small t

This graph illustrates the difference between the price computed using a compound- and simple-interest interpretation of yield. During the first compounding period, the price using compound-interest yield is higher. After one compounding period, the price using simple-interest yield is higher, because the yield does not compound.

The following graph shows the difference between the price calculated using a simple-interest yield interpretation and the price calculated using a compound-interest yield interpretation.



Pricing a Bond When the Yield and Coupon Are Equal

From the previous page,

$$Price = \frac{c\left(1+\frac{y}{f}\right) + \frac{vy-c}{\left(1+\frac{y}{f}\right)^{n}}}{y\left(1+\frac{y}{f}\right)^{1-x}} - x \times \frac{c}{f}$$

If $v = 100\%$ and $c = y$, this formula reduces to:

$$Price = \left(1+\frac{y}{f}\right)^{x} - x \times \frac{y}{f}$$

If $x = 0$, implying that the settlement date is a coupon date for the security, then

$$Price = \left(1+\frac{y}{f}\right)^{0} - 0 \times \frac{y}{f} = 100\%$$

Otherwise, the price will be near, but slightly below, par.

Using the bond price formula, it can be proven that the price of a bond whose coupon equals its yield is par (100%) on a coupon payment date

Pricing a Bond Using the HP-17B II

U.S. Treasury 8% Due November 15, 2021
(7.252% Yield; June 26, 1996 Settlement)

Whenever you use a tool to do financial computations, you have the added responsibility of understanding all the settings so you can ensure they are correct

Spreadsheets have the advantage of retaining inputs and assumptions for later verification

- Display the BOND menu: press **“FIN” “BOND”**
- Press **“CLEAR DATA”**
- Define the type of the bond. If the message in the display does not match Treasury conventions, press **“TYPE”**
  - Press **“A/A”** to set the calendar basis to actual/actual
  - Press **“SEMI”** to set the coupon payment frequency to semi-annual
  - Press **“EXIT”** to restore the BOND menu
- Enter the bond’s settlement date: **“06.261996 SETT”**
- Enter the bond’s maturity date: **“11.152021 MAT”**
- Enter the bond’s coupon (actually, the coupon × 100): **“8 CPN%”**
- Move to the next screen: **“MORE”**
- Enter the yield (actually, the yield × 100): **“7.252 YLD%”**
- Request the price: **“PRICE”**; the calculator should respond **“108.611177”**
- Request the accrued interest: **“ACCRU”**; the calculator should respond **“0.913043”**
- For the present value, request **“PRICE,” “+,” “ACCRU,” “=”**; the calculator should respond **109.524221**. (Unless your calculator is set for “Reverse Polish Notation”).
- Did your calculator show enough significant digits?
- How much would $100,000,000 bonds cost?

Bond-Equivalent Yield (BEY)

Bond-equivalent yield is defined as the yield that equates the discounted value of a bond's actual future cash flows (determined using the conventions and methodologies of that bond's market) with the bond's present value in the market. The actual cash flows are discounted using a semi-annual rate, and the lengths of the discounting periods are determined using the actual/actual calendar to put the bond-equivalent yield on the same footing as Treasury yields.

Bond-equivalent yield is a useful measure for comparing securities from different markets with different payment frequencies and conventions

Because the formula for determining a bond's price from its yield is not "invertible," there is no closed-form expression for determining a bond's yield from its price (except for bonds with only one future payment). The yield is, therefore, found by trial and error. One algorithm begins with an estimate of the yield (call it y_0) and then computes the price and dollar duration of the security. The algorithm calls for taking the difference between two prices, which is equivalent to the difference between two present values since they both have the same accrued interest. Since dollar duration provides an estimate of the price's absolute sensitivity to yield changes, it can provide an estimate of the yield change required to match the price of the bond, as follows:

Bond-equivalent yield is calculated using actual cash flows, semi-annual compounding, and an actual/actual calendar

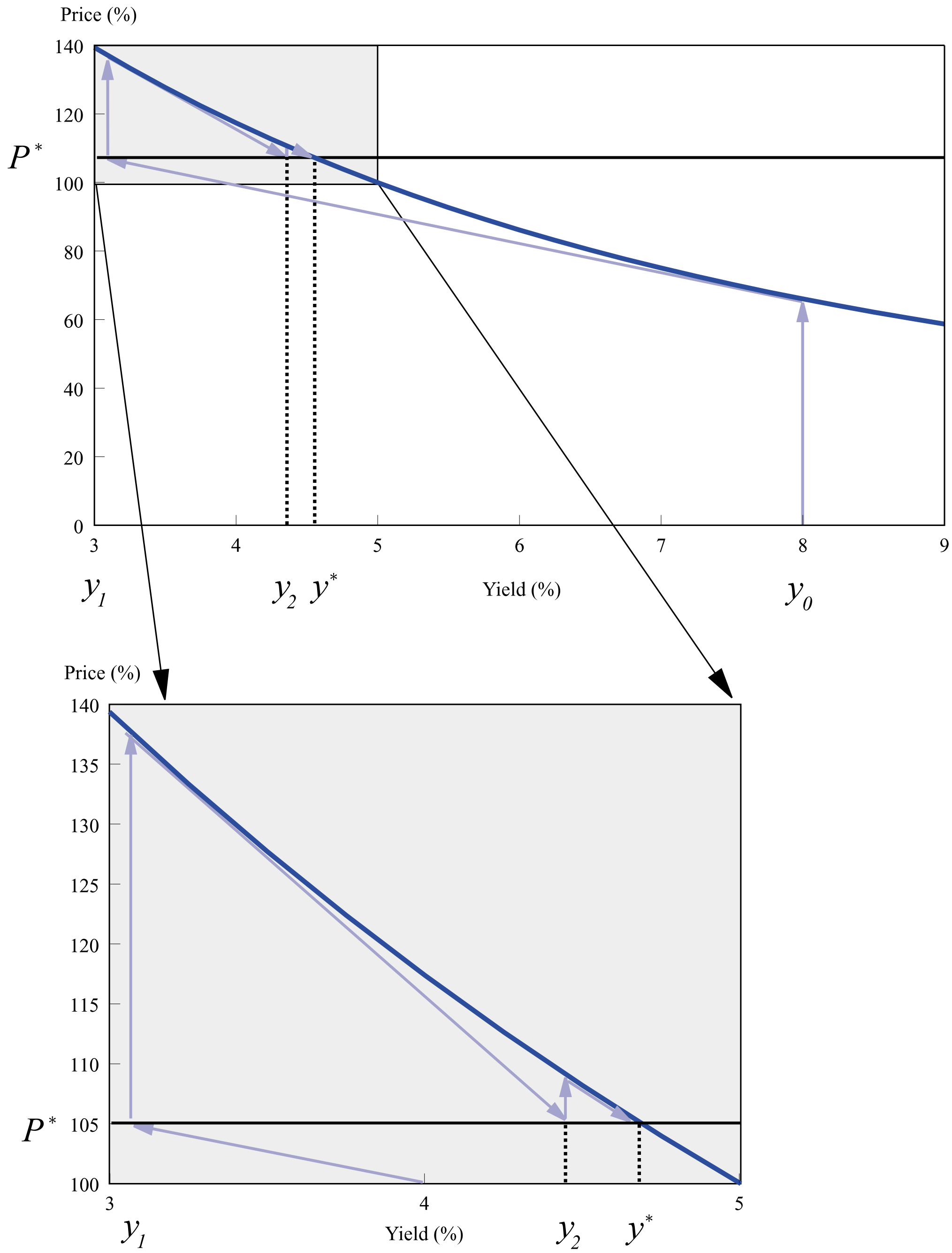
$$y_{i+1} = y_i + \frac{Price_i - Price_{Actual}}{Duration_{Dollar,i}}$$

The new yield is used as the starting point for the next iteration. When the price is accurate enough, the algorithm stops. This is called the *Newton–Raphson* method of equation solving.

Newton–Raphson Equation Solving

The Newton–Raphson method seeks to solve an equation by iterating from an initial guess y_0 and refining the guess (y_1, y_2, y^*) based upon the slope of the curve at each successive point

For a bond, the slope of the curve is the dollar duration



*Actual

Accretion and Amortization

Under the *effective-interest* method, many investors record income that is different from actual cash received. Bonds are frequently carried on a company's books at a *carrying value* derived from the acquisition price and unrelated to the current market value of the security. The carrying value starts at the acquisition price and drifts toward par over the life of the bond. The income reported by the investor would be the bond's carrying value multiplied by the yield at acquisition (compounded for the reporting period).

Accretion refers to a growing carrying value, while amortization refers to a declining carrying value

Many investors account for bonds using the *effective-interest* method, which amortizes or accretes principal toward par over time

Neither accretion nor amortization changes actual cash flows

There are two equivalent methods for calculating income and updating the carrying value according to the effective-interest method:

- Under the effective-interest method, the carrying value on any date can be determined by calculating the price of the bond at the acquisition yield for settlement on that date. The income is then defined as actual cash received plus the change in carrying value.

- Alternatively, the income can be calculated as the carrying value at the beginning of the period multiplied by the acquisition yield (compounded appropriately). The change in carrying value is then income less actual cash received.

Use the second method to amortize the 8% due November 15, 2021, with an acquisition yield of 7%.

Period Start Date	Beginning Carrying Value (%)	Actual Cash Received (%)	Income (%)	Amortization (%)	Ending Carrying Value (%)
5/15/96	111.814	4.000			
11/15/96		4.000			
5/15/97		4.000			

Accretion and Amortization (Continued)

The remaining principal balance on a bond that makes level principal and interest payments (like a mortgage) can be calculated in similar fashion

8% due November 15, 2021 with an acquisition yield of 7%:

Period Start Date	Beginning Carrying Value (%)	Actual Cash Received (%)	Income (%)	Amortization (%)	Ending Carrying Value (%)
5/15/96	111.814	4.000	3.913	(0.087)	111.727
11/15/96	111.727	4.000	3.910	(0.090)	111.638
5/15/97	111.638	4.000	3.907	(0.093)	111.545
⋮	⋮	⋮	⋮	⋮	⋮
11/15/10	107.584	4.000	3.765	(0.235)	107.349
5/15/11	107.349	4.000	3.757	(0.243)	107.106
11/15/11	107.106	4.000	3.749	(0.251)	106.855
⋮	⋮	⋮	⋮	⋮	⋮
5/15/20	101.401	4.000	3.549	(0.451)	100.950
11/15/20	100.950	4.000	3.533	(0.467)	100.483
5/15/21	100.483	4.000	3.517	(0.483)	100.000

Note that the bond amortized to par on its maturity date. As this example illustrates, bonds always accrete or amortize toward par more quickly as they approach maturity.

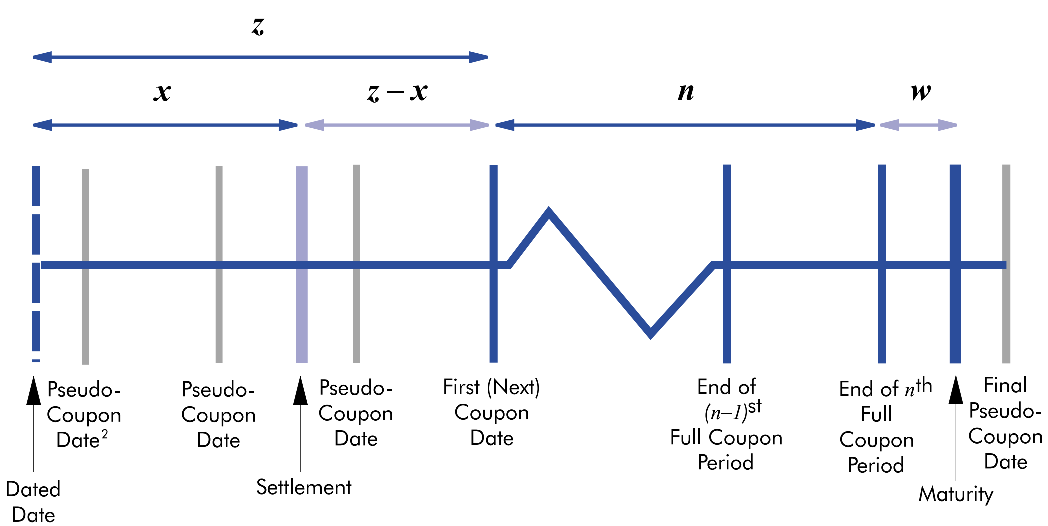
The same methodology can be used to allocate payments on a bond that makes level payments of principal and interest. The interest is the coupon on the security. Any difference between actual cash and interest is a principal payment. The ending principal balance is the beginning principal balance less the principal paid during the period.

More General Pricing Timeline

Every security has a *dated date*, from which interest begins accruing. The dated date is not necessarily the issue or first settlement date. If the dated date lies on the coupon payment cycle, then the first coupon will usually be regular. If the dated date is not on the coupon cycle, the first coupon may be larger than usual ("long" first coupon) or smaller than usual ("short" first coupon). The size of the first coupon will be scaled by the length of the first coupon period, measured as a number of regular coupon periods plus a partial period using the appropriate calendar. Some securities also have an irregular coupon at maturity.

The basic timeline can be extended to account for bonds with an odd first coupon period and a coupon cycle that does not coincide with maturity

If the bond has already paid a coupon, then its first coupon period is regular (*z=1*)



z measures the length of the first coupon period and, therefore, the size of the first coupon. x is the length of the accrual period, $0 \leq x < z$. n is the number of full coupon periods between the next coupon and the final regular coupon. w measures the length of the final coupon period and, therefore, the size of the final coupon. z, x and w are all measured in units of whole coupon periods.

[2] A *pseudo-coupon* date is a date on which a generic coupon bond with the same maturity and conventions would pay a coupon, but on which the specific bond does not pay a coupon.

General Coupon Bond Pricing Formula

Here is a more general formula that can price bonds with odd first and last coupons, including medium-term notes (MTNs)

c is the annual coupon rate,

v is the redemption value,

y is the yield, quoted on a compound basis,

f is the payment and compounding frequency,

n is the number of whole coupon periods between the next coupon date (not including that coupon) and the final regular coupon,

x is the length of the accrual period, using the appropriate calendar, $0 \le x < z$,

w is the length of the partial last coupon period, if any, using the appropriate calendar, and

z is the length of the first coupon period (from the dated date), using the appropriate calendar (if the first coupon is regular, $z=1$)

$$PV = \frac{\dfrac{cy(z-1)}{f} + c\left(1+\dfrac{y}{f}\right) + \dfrac{\dfrac{y\left(v+\dfrac{wc}{f}\right)}{\left(1+\dfrac{y}{f}\right)^{w}} - c}{\left(1+\dfrac{y}{f}\right)^{n}}}{y\left(1+\dfrac{y}{f}\right)^{z-x}} \qquad Price = PV - x \times \frac{c}{f}$$

Coupon Bonds: Duration and Convexity

Modified Duration

When duration is quoted, it is usually quoted as modified duration

The *modified duration* of a bond can be quoted as either *present-value* or *price duration*, all relating to the same *dollar duration;* it is important to be clear about the quoting convention

We generally use modified present-value duration, which estimates the percentage change in price for an instantaneous, parallel change in yield

Modified duration is so named to differentiate it from *Macaulay duration* (to be covered later). Modified duration is generally the only type of duration that we use because it shows the sensitivity of a bond's value to changes in interest rates. There are three different methods of quoting the same modified duration: dollar duration, present-value duration, and price duration. The distinction is necessary because price and present value are different for coupon bonds. For zero-coupon bonds, there is no difference between present-value and price duration. The difference between the three different methods is in how they express the *same* price sensitivity.

Dollar duration estimates the price impact of a change in yield *as a percent of par* and is the result of differentiating the formula for price with respect to yield. Dollar duration is also sometimes quoted as an absolute number: how much the dollar value of a security or portfolio will change when yields change.

$$Duration_{Dollar} = -\frac{dP}{dy} = -\frac{dPV}{dy}$$

The *present-value duration* of a bond estimates the price impact *as a percent of dollars invested (present value).* Present-value duration can be useful for evaluating the relative riskiness of a fixed-dollar investment in different securities. This is the most common way of quoting duration and is usually what is being described when "duration" or "modified duration" is quoted without further description.

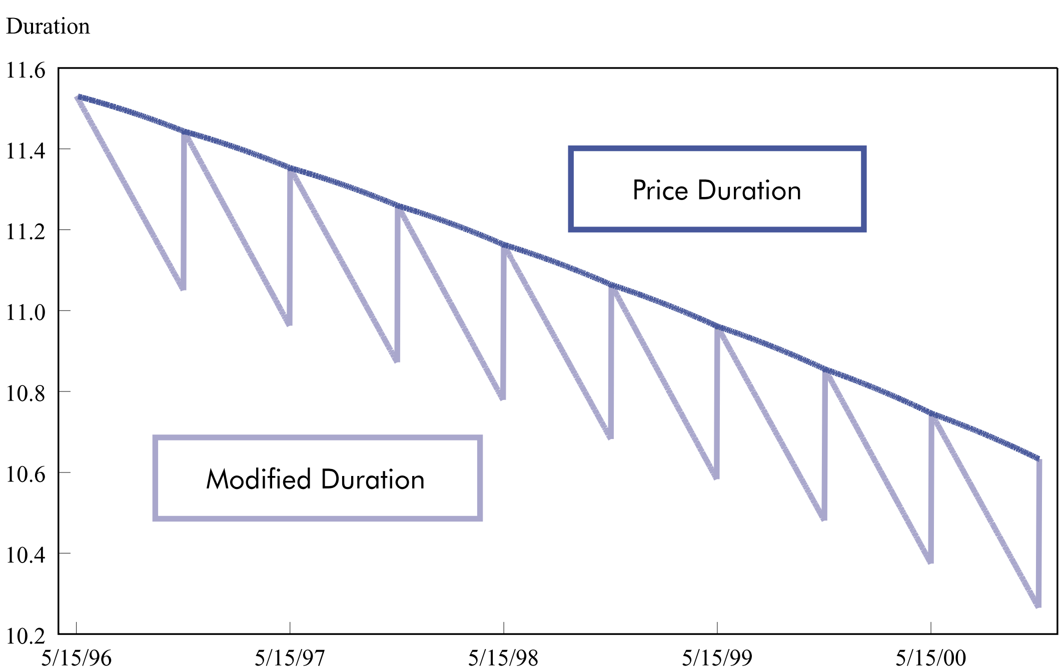
$$Duration_{PV} = \frac{Duration_{Dollar}}{Present\ Value} = -\frac{dPV/PV}{dy} = -\frac{dP/PV}{dy}$$

Modified Duration (Continued)

The *price duration* of a bond estimates the same price impact *as a percent of price*. Price duration can be multiplied by quoted price to compute dollar duration without ever calculating accrued interest or present value. Therefore, it can be useful for estimating the price change of a security when interest rates change. Price duration is also more stable than present-value duration over time because price is more stable than present value under constant interest rates. Because the price is always less than the present value, a security's price duration is always greater than its present-value duration.

$$Duration_{Price} = \frac{Duration_{Dollar}}{Price} = -\frac{dP/P}{dy} = -\frac{dPV/P}{dy}$$

Modified and Price Durations of 8% Due November 15, 2021, Priced to Yield 7% Over Time



Weighted Averages

The duration of a portfolio is the average of the durations of the securities in the portfolio, weighted by market value

Likewise, the duration of a security is the average of the durations of the security's individual cash flows, weighted by each cash flow's contribution to the security's market value (its present value at the bond-equivalent yield)

There are many other security and portfolio characteristics that can be calculated as weighted averages

The average of an attribute x_i weighted by w_i is defined as:

$$\frac{1}{\sum_i w_i} \times \sum_i w_i x_i$$

An alternative definition would be to define normalized weights y_i as:

$$y_i = \frac{w_i}{\sum_i w_i} \text{ so that } \sum_i y_i = 1$$

Then the weighted average would be defined as:

$$\sum_i y_i x_i$$

For example, as discussed on the next page, the duration of a security is the average of the durations of the individual cash flows, weighted by their respective present values:

$$Duration_{Modified} = \frac{1}{\sum_i PV_i} \times \sum_i PV_i \times Duration_i$$

$$= \frac{1}{PV} \times \sum_i PV_i \times Duration_i$$

This approach can provide worthwhile insight into how a security's individual cash flows affect the duration of the security. It works because absolute dollar duration is additive: if we double our holdings in a security, we will have twice the absolute market risk.

Weighting Duration and Convexity

Because the duration of a security is the present-value-weighted duration of the individual cash flows, we can write an intuitive formula for the duration of a coupon bond (with no irregular coupons):

$$\begin{aligned} Duration_{Modified} &= \frac{1}{PV} \times \left(\overbrace{\frac{v}{\left(1+\frac{y}{f}\right)^{n+1-x}}}^{Par\ PV} \times \overbrace{\frac{\frac{n+1-x}{f}}{\left(1+\frac{y}{f}\right)}}^{Par\ Dur} + \sum_{i=0}^{n} \overbrace{\frac{\frac{c}{f}}{\left(1+\frac{y}{f}\right)^{i+1-x}}}^{Cpn\ PV} \times \overbrace{\frac{\frac{i+1-x}{f}}{\left(1+\frac{y}{f}\right)}}^{Cpn\ Dur} \right) \\ &= \frac{1}{PV} \times \left(\frac{v}{f} \times \frac{(n+1-x)}{\left(1+\frac{y}{f}\right)^{n+2-x}} + \frac{c}{f^2} \times \sum_{i=0}^{n} \frac{i+1-x}{\left(1+\frac{y}{f}\right)^{i+2-x}} \right) \end{aligned}$$

Where x is defined as the accrual period, $0 \le x < 1$, and n is the number of whole coupon periods between the next coupon date and maturity.

Likewise, the convexity of a security is the present-value-weighted convexity of the individual cash flows.

$$Convexity = \frac{1}{PV} \times \left(\frac{v}{f^2} \times \frac{(n+1-x)\times(n+2-x)}{\left(1+\frac{y}{f}\right)^{n+3-x}} + \frac{c}{f^3} \times \sum_{i=0}^{n} \frac{(i+1-x)\times(i+2-x)}{\left(1+\frac{y}{f}\right)^{i+3-x}} \right)$$

These equations for duration and convexity can also be obtained as the derivative of the summation expression for present value with respect to yield.

We can apply the principles of calculating weighted averages to construct intuitive formulas for duration and convexity

We will later construct more complicated, but more computationally efficient, formulas by differentiating the closed-form equation for price as a function of yield

Modified Durations vs Maturity

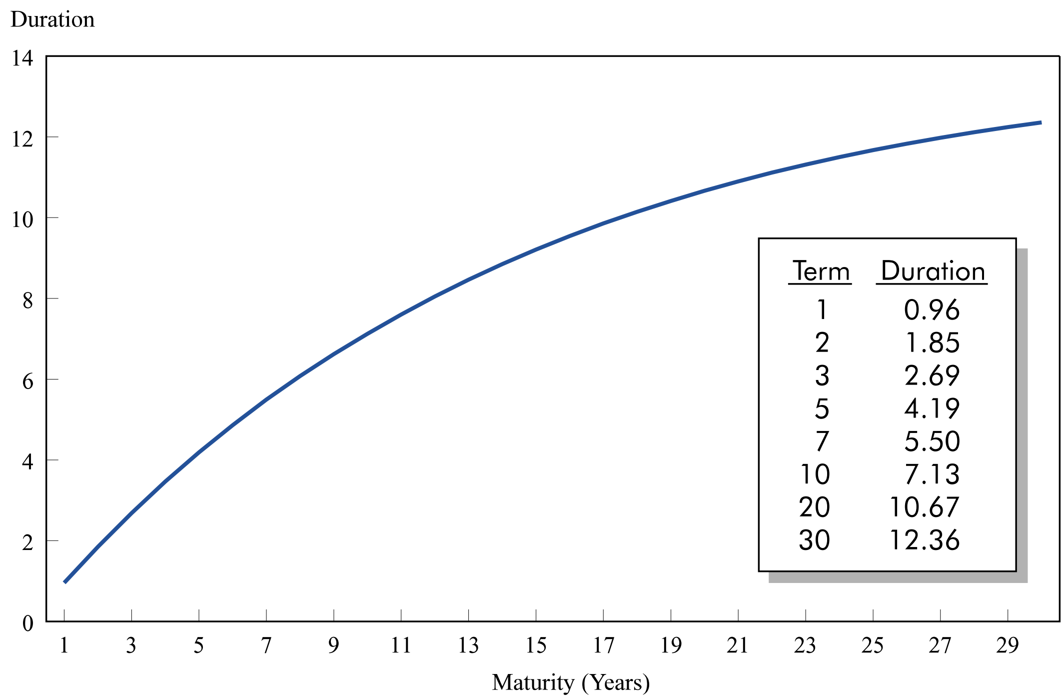
Par Bonds Priced as of June 24, 1996:
30-Year Bond Yielding 7.09%

The modified duration of a par bond increases with maturity, but at a diminishing rate

It is critical to build an intuitive sense for these durations

This curve shows durations for bonds priced at par

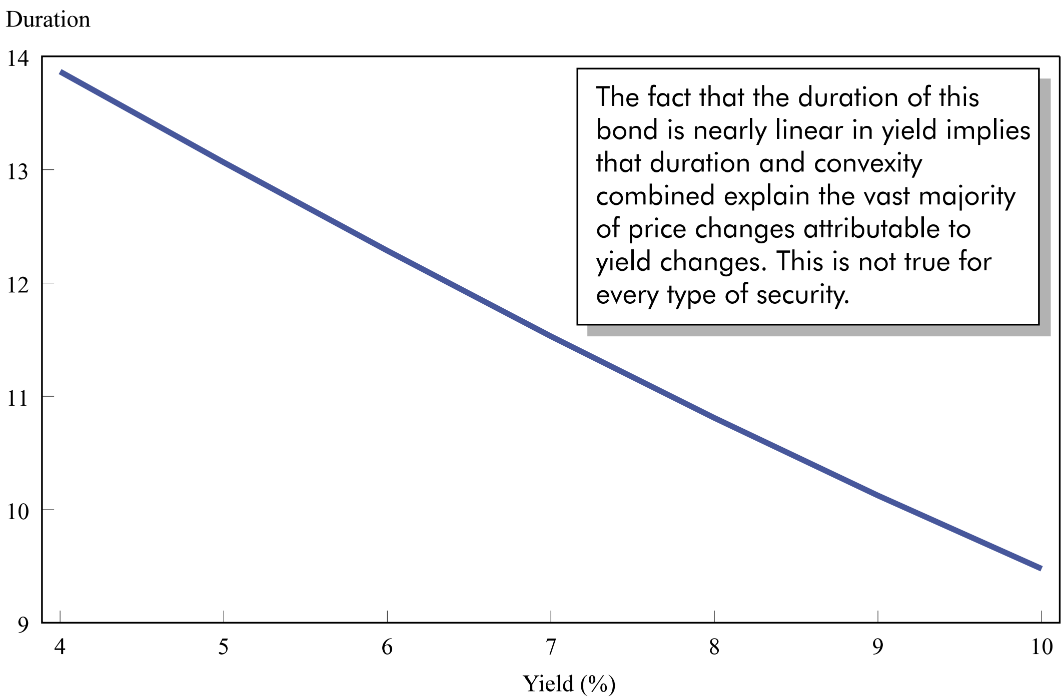
The *par-bond* construct is necessary to avoid comparing bonds with similar maturities but different coupons and, therefore, different durations



Term	Duration
1	0.96
2	1.85
3	2.69
5	4.19
7	5.50
10	7.13
20	10.67
30	12.36

Modified Duration vs Yield

U.S. Treasury 8% Due November 15, 2021 for Settlement May 15, 1996



Because of convexity, duration increases when yield declines (for this bond)

A Closed-Form Expression for Dollar Duration

Dollar duration can be calculated simply by taking the first derivative of the price formula

$Duration_{Dollar}$ = $Duration_{PV} \times PV$

Alternatively,

$Duration_{Dollar}$ = $Duration_{Price} \times Price$

It is usually easier and more reliable to estimate dollar duration as the change in price for a small change in yield:

$$Duration_{Dollar} \cong -\frac{\Delta P}{\Delta y}$$

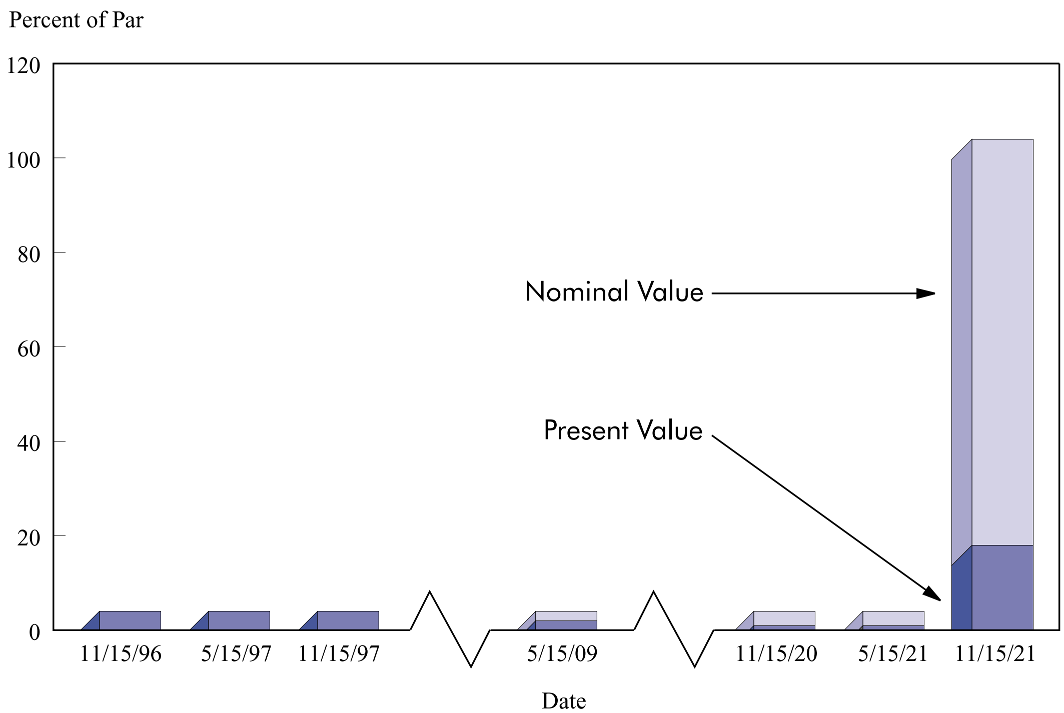
$$Duration_{Dollar} = -\frac{dPV}{dy} = -\frac{dP}{dy} \cong -\frac{\Delta P}{\Delta y}$$

$$= \frac{1}{y^2\left(1+\frac{y}{f}\right)^{z-x}} \times \left[\frac{\dfrac{y^2\left(v+\frac{wc}{f}\right)(n+w+z-x)}{f\left(1+\frac{y}{f}\right)^{w}} - c\left(1+\frac{y}{f}(n+z-x+1)\right)}{\left(1+\frac{y}{f}\right)^{n+1}} + \frac{cy^2(z-1)(z-x)}{f^2\left(1+\frac{y}{f}\right)} + c\left(1+\frac{y}{f}(z-x)\right) \right]$$

Where the variables are defined as follows:

- c is the annual coupon rate,
- v is the redemption value,
- y is the yield, quoted on a compound basis,
- f is the payment and compounding frequency,
- n is the number of whole coupon periods between the next coupon date (not including that coupon) and the final regular coupon,
- x is the length of the accrual period, using the appropriate calendar, $0 \le x < z$,
- w is the length of the last coupon period, if any, using the appropriate calendar, and
- z is the length of the first coupon period (from the dated date), using the appropriate calendar (if the first coupon is regular, $z = 1$).

U.S. Treasury 8% Due November 15, 2021
(7% Yield, May 15, 1996 Settlement)



$$Duration_{Macaulay} = \frac{1}{PV} \times \sum_{i=1}^{n} PV_i \times T_i$$

where T_i is the time (in years) until the i^{th} cash flow

$$Duration_{Macaulay} = Duration_{Modified} \times \left(1 + \frac{y}{f}\right)$$

$$Duration_{Modified} = \frac{Duration_{Macaulay}}{1 + \frac{y}{f}}$$

The Macaulay duration is defined as the present-value-weighted time to payment of a bond's cash flows

It happens to be related to modified duration by a simple formula

The Macaulay duration of a zero-coupon bond is its term

Morgan Stanley

Convexity

Convexity is the second-order correction to estimated price change given a change in yield

Convexity is on the order of the square of duration (for bonds without embedded options)

For bonds (without embedded options) having the same duration, the bond with the wider dispersion of cash flows will have the higher convexity; a zero-coupon bond, with the lowest possible dispersion, has the lowest possible convexity for a given duration

We have already noted that convexity goes up with the square of maturity for a zero-coupon bond. More generally, convexity is on the order of the square of duration.

For bonds with the same duration, the bond with the wider dispersion of cash flows will have the higher convexity.

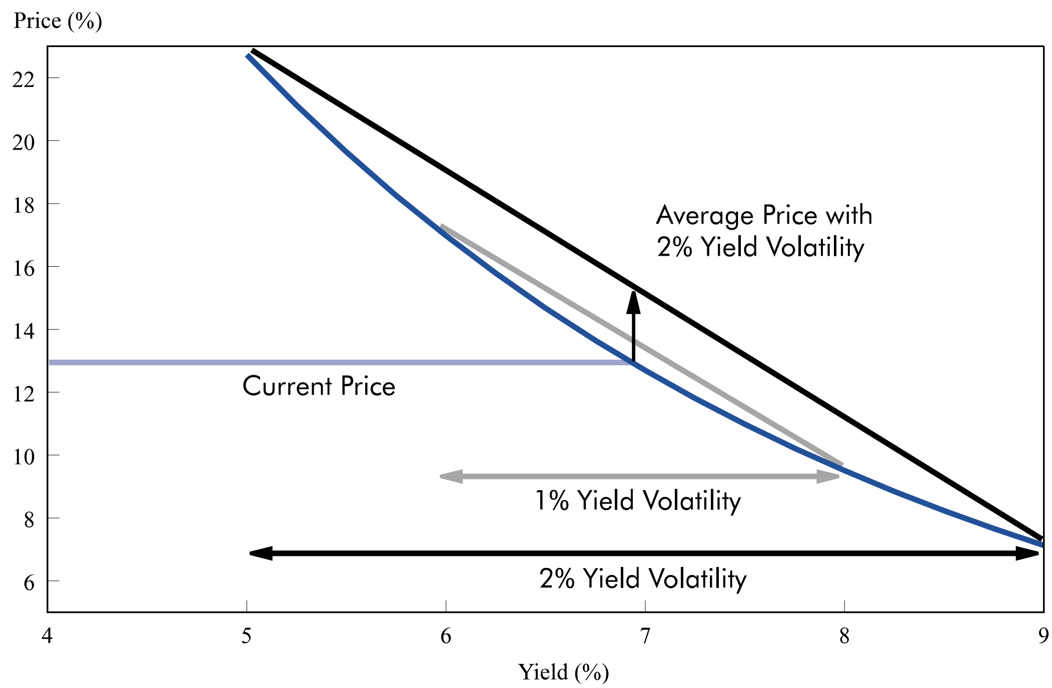
Example: A 10-year STRIPS has approximately the same duration as a portfolio of 50% cash and 50% 20-year STRIPS. Because convexity increases with the square of duration, the 20-year STRIPS has four times the convexity of the 10-year STRIPS. The cash and STRIPS portfolio, therefore, has twice the convexity of the 10-year STRIPS portfolio. Likewise, a portfolio of two-thirds cash and one-third STRIPS also has a duration of 10, but has a convexity three times as great as for the 10-year STRIPS.

The "value of convexity" lies in the fact that the higher the convexity, the more the expected rate of return exceeds the yield. This is because the average of the price of a portfolio in both a "down" and "up" interest rate scenario will be higher than the current price. The higher the convexity, the more the average price will exceed the current price.

Since convexity has value, we should expect the more convex portfolio of cash and STRIPS to have a lower yield. In fact, it does: a 10-year STRIPS (May 15, 2006) has a yield of 7.13%, and a portfolio of cash (yielding 5.25%) and 20-year STRIPS (May 15, 2016, yielding 7.459%) has a market-value-weighted-average yield of 6.355%, significantly lower.

Value of Convexity

30-Year U.S. Treasury Zero-Coupon Bond



When there is anticipated yield volatility, a convex portfolio has a short-term expected return that is greater than its yield. The greater the expected volatility and the greater the convexity, the greater this effect.

When there is volatility in yields, positive convexity implies that a portfolio's expected return is greater than its yield

Q: Why would anyone choose to buy a bullet (10-year STRIPS) rather than a barbell (50% cash and 50% 20-year STRIPS) portfolio with the same duration and higher convexity?

A Closed-Form Expression for Dollar Convexity

Dollar convexity can be calculated simply by taking the second derivative of the price formula

Convexity would then be calculated as:

$$Convexity = \frac{d^2PV \big/ dy^2}{PV}$$

Convexity is divided by two in the Taylor expansion for price; some firms quote it already divided by two

$$Convexity_{Dollar} = \frac{d^2P}{dy^2} \cong -\frac{\Delta Duration_{Dollar}}{\Delta y}$$

$$= \frac{1}{y^3\left(1+\frac{y}{f}\right)^{z-x+1}} \times \left[\frac{\dfrac{y^3\left(v+\frac{wc}{f}\right)(n+w+z-x)(n+w+z-x+1)}{f^2\left(1+\frac{y}{f}\right)^{w}} - c\left(2+\frac{y}{f}(n+z-x+2)\left(2+\frac{y}{f}(n+z-x+1)\right)\right)}{\left(1+\frac{y}{f}\right)^{n+1}} + \frac{cy^3(z-1)(z-x)(z-x+1)}{f^3\left(1+\frac{y}{f}\right)} + c\left(2+\frac{y}{f}(z-x+1)\left(2+\frac{y}{f}(z-x)\right)\right) \right]$$

where the variables are determined as follows:

c is the annual coupon rate,

v is the redemption value,

y is the yield, quoted on a compound basis,

f is the payment and compounding frequency,

n is the number of whole coupon periods between the next coupon date (not including that coupon) and the final regular coupon,

x is the length of the accrual period using the appropriate calendar, $0 \leq x < z$,

w is the length of the last coupon period, if any, using the appropriate calendar, and

z is the length of the first coupon period (from the dated date), using the appropriate calendar (if the first coupon is regular, $z=1$).

Chapter 2 Exercises

1. Calculate the present value, modified duration, dollar duration, and convexity of these Treasury STRIPS for settlement on June 26, 1996.

Maturity	Bond-Equivalent Yield (%)	Present Value (%)	Modified Duration	Dollar Duration	Convexity
11/15/99	6.63				
11/15/22	7.38				
02/15/23	7.36				

2. Using the bond price formula, what is the price of a 10-year 7% coupon bond at an 8% bond-equivalent yield?

3. What is the price of an 8% semi-annual-pay coupon bond that matures in exactly 15 years if the required bond-equivalent yield-to-maturity is 6%?

4. Many bonds pay interest twice per year, but their coupons are quoted on an annual basis. That is, an 8% 2-year U.S. Treasury note pays a 4% coupon twice per year. What is the bond's annual yield-to-maturity if it is priced at par on a coupon date?

5. If a 10-year Treasury bond with a 7% coupon is issued today at a price of 99-24 (99.750%), what is its bond-equivalent yield-to-maturity? Its annual yield-to-maturity?

6. For settlement on June 26, 1996, the price of the February 15, 1997 STRIPS was 96.444%. The yield is quoted as the yield to the stated maturity date, but that day is a Saturday and the cash is not delivered until the following Monday. What is the difference between the quoted yield and the yield actually earned by the investor?

7. Is the price of a bond above or below par if its yield is less than its coupon?
8. Which has a longer duration, a 7-year zero-coupon bond yielding 7.20% (BEY) or a 10-year 7.25% coupon bond yielding 7.20% (BEY)?
9. As long as you can safely stuff cash under your mattress (non-negative interest rates), what is the most you would ever pay for a bond that matures in eight years and has a 7% coupon paid annually? What if the bond paid a semi-annual coupon? Could interest rates ever become negative?
10. A bond issued by company A has a 6% coupon and matures February 15, 2026. The U.S. Treasury bond that matures the same date also has a coupon of 6% and is priced at 86-18+ (86.578125%). Is the price of company A's bond greater or less than 86-18+?
11. If three bonds promise the following cash flows, which is worth the most? *Estimate* the duration of each at a 7% semi-annual yield.

Years from Now	Cash Flow A ($)	Cash Flow B ($)	Cash Flow C ($)
1	1,000	400	
2		500	1,000
3	1,000	600	1,000
4	1,000	700	
5		800	1,000

12. A perpetual bond pays coupons forever, but never matures. If a perpetual bond pays a 7% coupon annually and is priced at 95%, what is its yield? What is its duration? What is its convexity? How does its convexity compare to a zero-coupon bond with the same duration?

Chapter 2 Exercises (Continued)

13. One year ago, a bank loaned you enough to purchase a home with a 30-year fixed-rate mortgage requiring a payment of $1,000 per month. Mortgage payments are level across the life of the note, so each payment comprises both interest and principal. The monthly interest rate on the mortgage is 8%. What was its original face value? What is the balance today? What is the BEY? Who is the issuer?

Morgan Stanley