MTH 9893 Time Series HW 1

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2. Ornstein-Uhlenbeck process:

$$dX_t = \lambda\left(\mu - X_t\right)dt + \gamma dW_t$$

where $\lambda, \gamma > 0$.
This process could be discretized at times $n\Delta t$, $n = 1, 2, 3, \ldots, \infty$ using Euler–Maruyama method.
Let's set $t = m\Delta t$

$$X_{m+1} - X_m = \lambda\left(\mu - X_m\right)\Delta t + \gamma\Delta W_m$$

where $\Delta W_m = W_{\tau_{m+1}} - W_{\tau_m}$. It is a normal random variables with expected value zero and variance Δt.
Then we have

$$\begin{aligned} X_{m+1} &= \lambda\mu\Delta t + \left(1 - \lambda\Delta t\right)X_m + \gamma\Delta W_m \\ &= \lambda\mu\Delta t + \left(1 - \lambda\Delta t\right)X_m + \gamma\sqrt{\Delta t}Z \end{aligned}$$

where Z is a normal random variables with expected value zero and variance 1.
By mapping between the parameters of this and AR(1) model,

$$X_t = \alpha + \beta X_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma^2)$$

with $0 < \beta < 1$.
we have
$\alpha = \lambda\mu\Delta t,\ \beta = 1 - \lambda\Delta t$, which should be between 0 and 1. And also $\gamma\sqrt{\Delta t} = \sigma$.
3.
$AR\left(2\right)$ model is specified as

$$X_t = \alpha + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma^2)$$

From the requirement that $E\left(X_t\right) = \mu$,

$$\begin{aligned} E\left(X_t\right) &= \alpha + \beta_1 E\left(X_{t-1}\right) + \beta_2 E\left(X_{t-2}\right) + E\left(\epsilon_t\right) \\ \mu &= \alpha + \beta_1\mu + \beta_2\mu + 0 \\ \mu &= \frac{\alpha}{1 - \beta_1 - \beta_2} \end{aligned}$$

where μ is a constant $\Longrightarrow \beta_1 + \beta_2 \neq 1$
Furthermore, we require that the roots of the characteristic polynomial $\psi\left(z\right) = 1 - \beta_1 z - \beta_2 z^2$ lie outside of the unit circle.
If we examine the reverse characteristic equation $z^2 - \beta_1 z - \beta_2 = 0$, the roots should lie inside the unit circle.
$\lambda_i = |\frac{\beta_1 \pm \sqrt{\beta_1^2 + 4\beta_2}}{2}| < 1$
If $\beta_1^2 + 4\beta_2 < 0$, the roots will be complex, which indicates a stationary status.

$$-1 < \frac{\beta_1 - \sqrt{\beta_1^2 + 4\beta_2}}{2} < \frac{\beta_1 + \sqrt{\beta_1^2 + 4\beta_2}}{2} < 1$$

From left side we have

$$\begin{aligned} \beta_1 - \sqrt{\beta_1^2 + 4\beta_2} &> -2 \\ \sqrt{\beta_1^2 + 4\beta_2} &< \beta_1 + 2 \\ \beta_2 - \beta_1 &< 1 \end{aligned}$$

or from the right side

$$\beta_1 + \sqrt{\beta_1^2 + 4\beta_2} < 2$$
$$\sqrt{\beta_1^2 + 4\beta_2} < -\beta_1 + 2$$
$$\beta_2 + \beta_1 < 1$$

Also we have $\lambda_1\lambda_2 = |\frac{\beta_1^2 - \beta_1^2 - 4\beta_2}{4\beta_2^2}| > 1 \Longrightarrow 1 > |\beta_2|$
In conclusion, we have $1 > |\beta_2|, \quad 1 > \beta_2 + \beta_1$ and $1 > \beta_2 - \beta_1$.