MTH 9893 Time Series HW 2

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3.
Consider the $AR(1)$ model with a linear drift and let

$$\widehat{\epsilon_t} = X_t - \beta X_{t-1} - \alpha - \delta t$$

for $t = 1, 2, \ldots, T$ be the disturbances implied from the data. According to the model specification, each $\hat{\epsilon}_t$ is independently drawn from $N\left(0, \sigma^2\right)$, and thus

$$p\left(x_1, \ldots, x_T \mid x_0, \theta\right) = \frac{1}{\left(2\pi\sigma^2\right)^{T/2}} exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\epsilon_t^2\right)$$
$$\mathcal{L}\left(\theta \mid x_0, x_1, \ldots, x_T\right) = \frac{1}{\left(2\pi\sigma^2\right)^{T/2}} exp\left(-\frac{1}{2\sigma^2}\sum_{t=0}^{T-1}\left(X_{t+1} - \beta X_t - \alpha - \delta t\right)^2\right)$$

The log-likelihood function is given as

$$-log\mathcal{L}\left(\theta \mid x_0, x_1, \ldots, x_T\right) = \frac{T}{2}log\sigma^2 + \frac{1}{2\sigma^2}\sum_{t=0}^{T-1}\left(X_{t+1} - \beta X_t - \alpha - \delta t\right)^2 + const.$$

Taking derivative of σ and set it to 0

$$\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=0}^{T-1}\left(X_{t+1} - \widehat{\beta}X_t - \widehat{\alpha} - \widehat{\delta}t\right)^2$$

To figure out $\widehat{\alpha}, \widehat{\beta}, \widehat{\delta}$, we need minimize $-log\mathcal{L}\left(\theta \mid x_0, x_1, \ldots, x_T\right)$. It could be written as

$$\begin{aligned}
-log\mathcal{L}\left(\theta \mid x_0, x_1, \ldots, x_T\right) &= \frac{T}{2}log\sigma^2 + \frac{1}{2\sigma^2}\sum_{t=0}^{T-1}\left(X_{t+1} - \beta X_t - \alpha - \delta t\right)^2 + const \\
&= \frac{T}{2}log\sigma^2 + \frac{1}{2\sigma^2}\left(X - Yv\right)^T\left(X - Yv\right) + const
\end{aligned}$$

where $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_T \end{bmatrix}$, $v = \begin{bmatrix} \alpha \\ \beta \\ \delta \end{bmatrix}$, $Y = \begin{bmatrix} 1 & X_0 & 0 \\ 1 & X_1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & X_{T-1} & T-1 \end{bmatrix}$.

Take derivatives of v,

$$\begin{aligned}
f^{'}\left(v\right) &= CY^T\left(X - Yv\right) = 0 \qquad \text{C is a constant} \\
v &= \left(Y^TY\right)^{-1}Y^TX \\
\begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\delta} \end{pmatrix} &= \begin{pmatrix} T & \sum_{t=0}^{T-1} X_t & \frac{T(T-1)}{2} \\ \sum_{t=0}^{T-1} X_t & \sum_{t=0}^{T-1} X_t^2 & \sum_{t=0}^{T-1} tX_t \\ \frac{T(T-1)}{2} & \sum_{t=0}^{T-1} tX_t & \frac{T(T-1)(2T-1)}{6} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=0}^{T-1} X_{t+1} \\ \sum_{t=0}^{T-1} X_tX_{t+1} \\ \sum_{t=0}^{T-1} tX_{t+1} \end{pmatrix}
\end{aligned}$$

and

$$\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=0}^{T-1}\left(X_{t+1} - \widehat{\beta}X_t - \widehat{\alpha} - \widehat{\delta}t\right)^2$$

4.
$E\left(\epsilon_t^4\right) = E\left(\sigma_t^4\right)E\left(z_t^4\right) = 3E\left(\sigma_t^4\right)$, since we know $E\left(z_t^4\right) = 3$.

Then,

$$
\begin{aligned}
E\left(\sigma_t^4\right) &= \kappa^2 + \eta^2 E\left(\sigma_{t-1}^4\right) + \zeta^2 E\left(\epsilon_{t-1}^4\right) + 2\kappa\eta E\left(\sigma_{t-1}^2\right) + 2\kappa\zeta E\left(\epsilon_{t-1}^2\right) + 2\eta\zeta E\left(\sigma_{t-1}^2\epsilon_{t-1}^2\right) \\
&= \kappa^2 + \eta^2 E\left(\sigma_t^4\right) + 3\zeta^2 E\left(\sigma_{t-1}^4\right) + 2\kappa\eta\sigma^2 + 2\kappa\zeta E\left(\sigma_{t-1}^2\right) + 2\eta\zeta E\left(z_{t-1}^2\right) E\left(\sigma_{t-1}^4\right) \\
&= \kappa^2 + \eta^2 E\left(\sigma_t^4\right) + 3\zeta^2 E\left(\sigma_t^4\right) + 2\kappa\eta\sigma^2 + 2\kappa\zeta\sigma^2 + 2\eta\zeta E\left(\sigma_t^4\right) \\
E\left(\sigma_t^4\right) &= \frac{\kappa^2 + 2\kappa\left(\eta+\zeta\right)\frac{\kappa}{1-\eta-\zeta}}{1-\eta^2-3\zeta^2-2\eta\zeta}
\end{aligned}
$$

Also

$$
\begin{aligned}
E\left(\epsilon_t^2\right)^2 &= \left(E\left(\sigma_t^2\right)E\left(z_t^2\right)\right)^2 \\
&= \left(\sigma^2\right)^2 \\
&= \frac{\kappa^2}{\left(1-\eta-\zeta\right)^2}
\end{aligned}
$$

The kurtosis in the GARCH(1,1) model is

$$
\begin{aligned}
\frac{E\left(\epsilon_t^4\right)}{E\left(\epsilon_t^2\right)^2} &= \frac{3\frac{\kappa^2+2\kappa(\eta+\zeta)\frac{\kappa}{1-\eta-\zeta}}{1-\eta^2-3\zeta^2-2\eta\zeta}}{\frac{\kappa^2}{(1-\eta-\zeta)^2}} \\
&= 3\frac{\left(1-\eta-\zeta\right)^2 + 2\left(1-\eta-\zeta\right)\left(\eta+\zeta\right)}{1-\eta^2-3\zeta^2-2\eta\zeta} \\
&= 3\frac{\left(1-\eta-\zeta\right)\left(1-\eta-\zeta+2\left(\eta+\zeta\right)\right)}{1-\left(\zeta+\eta\right)^2-2\zeta^2} \\
&= 3\frac{\left(1-\eta-\zeta\right)\left(1+\eta+\zeta\right)}{1-\left(\zeta+\eta\right)^2-2\zeta^2} \\
&= 3\frac{1-\left(\eta+\zeta\right)^2}{1-\left(\zeta+\eta\right)^2-2\zeta^2}
\end{aligned}
$$