MTH 9893 Time Series HW 5

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1.

The innovations in the local level model are denoted as $v_t = Y_t - E\left(X_t \mid Y_{1:t-1}\right) = Y_t - \mu_t$.

First, consider the joint density of $Y_1, \cdots, Y_t$, which is

$$\begin{aligned} p\left(Y_1, \cdots, Y_t\right) &= p\left(Y_1\right) p\left(Y_2 \mid Y_1\right) \cdots p\left(Y_t \mid Y_{1:t-1}\right) \\ &= p\left(Y_1\right) \prod_{i=2}^{t} p\left(Y_i \mid Y_{1:i-1}\right) \end{aligned}$$

Then we transform from $Y_1, \cdots, Y_t$ to $v_1, \cdots, v_t$. We could see that

$$\begin{aligned} v_1 &= Y_1 - \mu_1 \\ v_2 &= Y_2 - \mu_2 = Y_1 - \mu_1 - K_1 v_1 = Y_1 - \mu_1 - K_1\left(Y_1 - \mu_1\right) \\ v_3 &= Y_3 - \mu_1 - K_2\left(Y_2 - \mu_1\right) - K_1\left(1 - K_2\right)\left(Y_1 - \mu_1\right) \end{aligned}$$

and so on.

Put them in matrix, we would have $\boldsymbol{v} = \boldsymbol{C}\left(\boldsymbol{Y} - \mathbf{1}_t \mu_1\right)$, where $\boldsymbol{v} = \left(v_1, \cdots, v_t\right)^T$, $\boldsymbol{Y} = \left(Y_1, \cdots, Y_t\right)^T$,$\mathbf{1}_t$ is t-dimensional vector of ones, and $\boldsymbol{C}$ is a lower trangular matrix defined as following:

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_{21} & 1 & 0 & \cdots & 0 \\ c_{31} & c_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{t1} & c_{t2} & c_{t3} & \cdots & 1 \end{bmatrix}$$

$$\begin{aligned} c_{i,i-1} &= -K_{i-1} \\ c_{ij} &= -\left(1 - K_{i-1}\right)\left(1 - K_{i-2}\right) \cdots \left(1 - K_{j+1}\right) K_j \end{aligned}$$

From this we would see the Jacobian matrix is a lower triangular matrix with diagonal entries one and the determinant is one, so $p\left(v_1, \cdots, v_t\right) = p\left(Y_1, \cdots, Y_t\right)$.

Furthermore, since μ_1 is given,

$$\begin{aligned} p\left(v_1\right) &= p\left(Y_1 - \mu_1\right) \\ &= p\left(Y_1\right) \end{aligned}$$

Also for each $p\left(v_i\right)$ for $i = 2, \cdots, t$,

$$\begin{aligned} p\left(v_i\right) &= p\left(Y_i - \mu_i\right) \\ &= p\left(Y_i - \mu_1 - c_{i1}\left(Y_1 - \mu_1\right) - \cdots c_{i,i-1}\left(Y_{i-1} - \mu_1\right)\right) \end{aligned}$$

If $Y_1, \cdots, Y_{i-1}$ is given, we would see

$$\begin{aligned} p\left(v_i\right) &= p\left(Y_i - \mu_1 - c_{i1}\left(Y_1 - \mu_1\right) - \cdots c_{i,i-1}\left(Y_{i-1} - \mu_1\right)\right) \\ &= p\left(Y_i \mid Y_{1:i-1}\right) \end{aligned}$$

for $i = 2, \cdots, t$.

Consequently the joint probability density function of $v_1, \cdots, v_t$ would be

$$\begin{aligned} p\left(v_1, \cdots, v_t\right) &= p\left(Y_1, \cdots, Y_t\right) \\ &= p\left(Y_1\right) \prod_{i=2}^{t} p\left(Y_i \mid Y_{1:i-1}\right) \\ &= p\left(v_1\right) \prod_{i=2}^{t} p\left(v_i\right) \\ &= \prod_{i=1}^{t} p\left(v_i\right) \end{aligned}$$

Thus we proved that the innovations $v_t = Y_t - E\left(X_t \mid Y_{1:t-1}\right)$ in the local level model are mutually independent for different values of t.