Time Series Analysis

4. Model free methods

Andrew Lesniewski

Baruch College
New York

Fall 2017

Outline

Time series in frequency domain

- So far, we have discussed various models within the parametric approach to time series analysis. The key element of this approach is to specify a time series model with a number of free parameters which are determined via estimation from a data set.
- While this approach will remain the focus of these lectures, we will now take a brief side trip into the non-parametric (or model free) approach to time series analysis. In particular, we will focus of analyzing time series by means of expansion in various basis functions.
- The first approach that we discuss, namely *time series analysis in frequency domain* (in contrast to the *time domain* approach taken so far), is reminiscent of Fourier transform approach in signal processing. The idea is to decompose the underlying time series into components, each of which corresponds to evolution *cycles* of different frequencies.
- The appropriate basis functions are the trigonometric functions $\cos(\omega t)$ and $\sin(\omega t)$ or, equivalently, the complex exponential function $e^{i\omega t}$.

Spectral density function

- Let X_t be a covariance stationary time series, such that

Xt: univariate

$$\sum_{t=-\infty}^{\infty} |\Gamma_t| < \infty. \tag{1}$$

- The *spectral density function* (or *population spectrum*) of X_t is defined as

$$s_X(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} \Gamma_t e^{-i\omega t}. \tag{2}$$

It is essentially the Fourier transform of X_t.

- From the trigonometric representation of complex numbers, and the fact that $\Gamma_{-t} = \Gamma_t$, we can write this in terms of purely real valued quantities:

$$s_X(\omega) = \frac{1}{2\pi} \left(\Gamma(0) + 2 \sum_{t=1}^{\infty} \Gamma_t \cos(\omega t) \right). \tag{3}$$

Spectral density function for white noise

- The easiest example is that of a white noise, $X_t = \varepsilon_t$. In this case,

$$\Gamma_t = \begin{cases} \sigma^2, & \text{if } t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- As a consequence, the SDF is constant,

$$s_X(\omega) = \frac{\sigma^2}{2\pi}\,. \tag{4}$$

Spectral density function for $AR(1)$

- As a next example, let us determine the spectral density function of the $AR(1)$ process. From equation (14) in Lecture Notes #1,

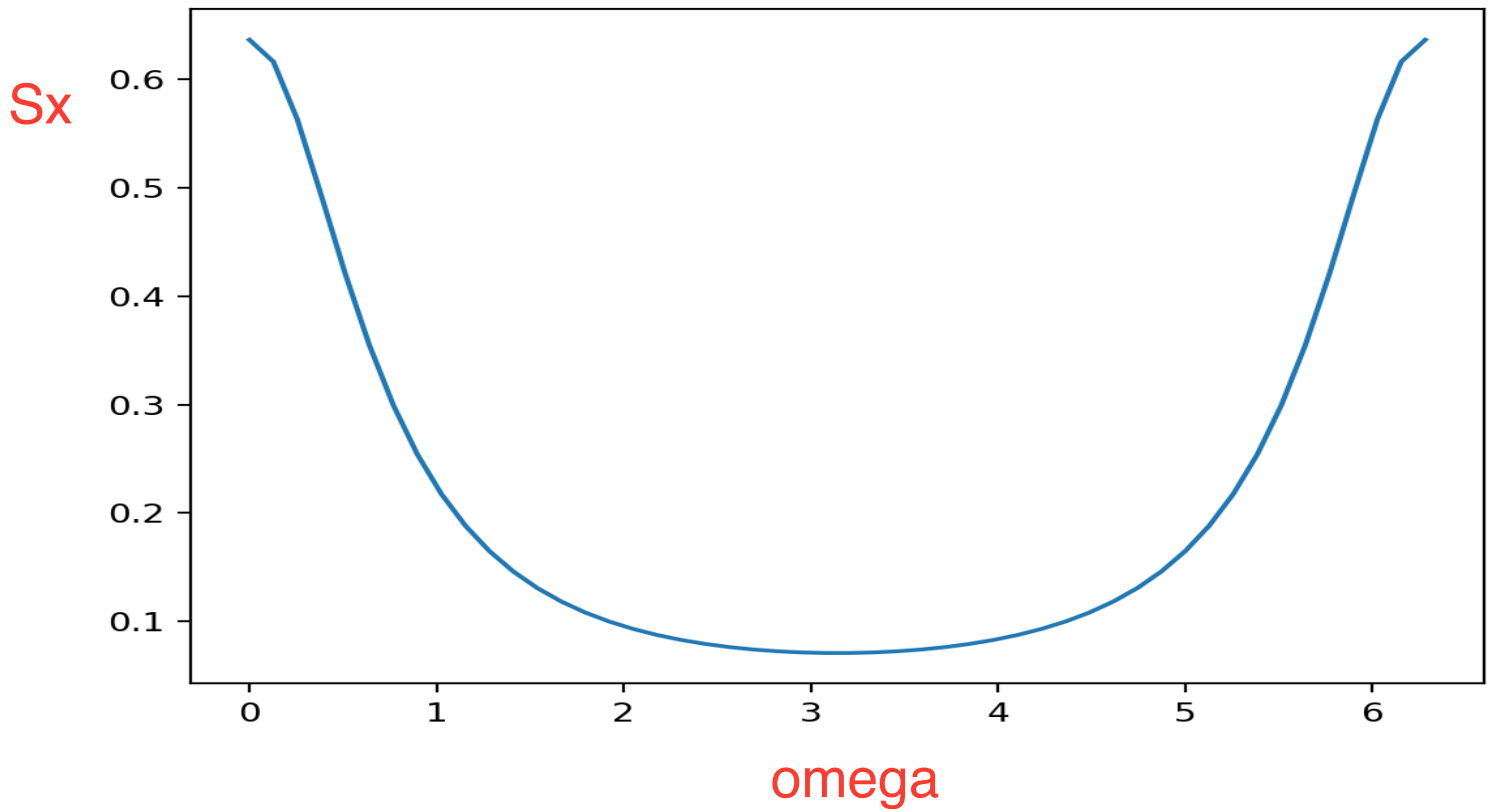
$$\begin{aligned} s_X(\omega) &= \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} \Gamma_0 \beta^{|t|} e^{-i\omega t} \\ &= \frac{\Gamma_0}{2\pi} \left(1 + \sum_{t=1}^{\infty} \beta^t e^{i\omega t} + \sum_{t=1}^{\infty} \beta^t e^{-i\omega t} \right) \\ &= \frac{\Gamma_0}{2\pi} \left(1 + \frac{\beta e^{i\omega}}{1 - \beta e^{i\omega}} + \frac{\beta e^{-i\omega}}{1 - \beta e^{-i\omega}} \right). \end{aligned}$$

- As a result,

$$s_X(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{1 - 2\beta \cos\omega + \beta^2}. \tag{5}$$

Spectral density function for $AR(1)$

- Below is the plot of (5) with $\beta = 0.5$ and $\sigma = 1$.

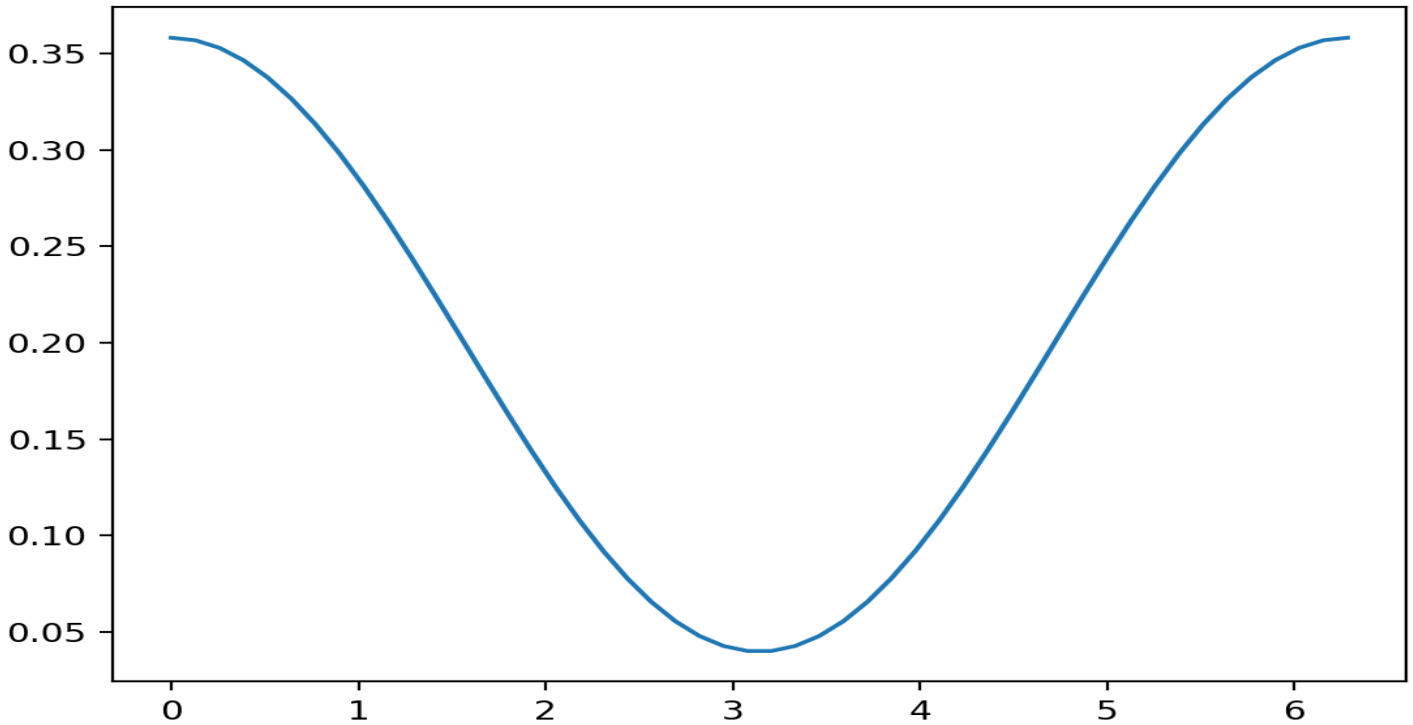


Spectral density function for $MA(1)$

- Let us now consider an $MA(1)$ model. Using equation (62) in Lecture Notes #1, we see that
$$s_X(\omega) = \frac{1}{2\pi}\left((1+\theta^2)\sigma^2 + \theta\sigma^2 e^{i\omega} + \theta\sigma^2 e^{-i\omega}\right).$$
- This implies that the spectral density function of an $MA(1)$ process is
$$s_X(\omega) = \frac{\sigma^2}{2\pi}\left(1 + 2\theta\cos\omega + \theta^2\right). \tag{6}$$

Spectral density function for $MA(1)$

- Below is the plot of (6) with $\theta = 0.5$ and $\sigma = 1$.



Spectral density for $ARMA(p, q)$

- The calculations above can be generalized to produce an expression for the $ARMA(p, q)$ model:
$$\psi(L)X_t = \alpha + \varphi(L)\varepsilon_t, \tag{7}$$
where our notation follows Lecture Notes #1.
- Namely, as you will establish in Homework Assignment #4, the spectral density is then given by
$$s_X(\omega) = \frac{\sigma^2}{2\pi}\left|\frac{\varphi(e^{i\omega})}{\psi(e^{i\omega})}\right|^2. \tag{8}$$
- If we factorize the polynomials $\psi(z)$ and $\varphi(z)$,
$$\psi(z) = (1 - \lambda_1 z)\ldots(1 - \lambda_p z),$$
$$\varphi(z) = (1 - \mu_1 z)\ldots(1 - \mu_q z),$$
then
$$s_X(\omega) = \frac{\sigma^2}{2\pi}\frac{(1 - 2\mu_1\cos\omega + \mu_1^2)\ldots(1 - 2\mu_q\cos\omega + \mu_q^2)}{1 - 2\lambda_1\cos\omega + \lambda_1^2)\ldots(1 - 2\lambda_p\cos\omega + \lambda_p^2)}. \tag{9}$$

Spectral density function

- In general, the spectral density function $s_X(\omega)$ has the following properties:
  - (i) It is non-negative.
  - (ii) It is a periodic function of ω with period 2π (assuming $h = 1$).
  - (iii) It is continuous in ω.
- The autocovariance can be calculated from the population spectrum by means of

$$\Gamma_t = \int_{-\pi}^{\pi} s_X(\omega) e^{i\omega t} d\omega. \tag{10}$$

- This is an immediate consequence of the fact that

$$\int_{-\pi}^{\pi} e^{i\omega(t-s)} d\omega = \begin{cases} 2\pi, & \text{if } t = s' \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

- Alternatively,

$$\Gamma_t = \int_{-\pi}^{\pi} s_X(\omega) \cos(\omega t) d\omega. \tag{12}$$

Spectral density function

- In particular,

$$\Gamma_0 = \int_{-\pi}^{\pi} s_X(\omega) d\omega, \tag{13}$$

  i.e. the variance of X_t is equal to the area under the population spectrum between $-\pi$ and π.
- This also leads to the interpretation of $s_X(\omega)$ as the fraction of the variance that is attributable to cycles of frequency ω.

Spectral representation theorem

- There is a general result that states that any covariance-stationary time series process can be expressed in terms of its spectral data.
- Namely, there exists a unique complex valued stochastic function $z_X(\omega)$, such that

$$X_t = \mu + \int_{-\pi}^{\pi} e^{i\omega t} z_X(\omega) d\omega, \tag{14}$$

where $\mu = \mathrm{E}(X_t)$.

- Since X_t is real valued, the random function $z_X(\omega)$ must have the following symmetry property: bar: complex contribute

$$\overline{z_X(\omega)} = z_X(-\omega). \tag{15}$$

- Furthermore, $z_X(\omega)$ has the following properties:
  - (i) For all ω,

$$\mathrm{E}\big(z_X(\omega)\big) = 0. \tag{16}$$

  - (ii) For all ω, ω',

$$\mathrm{E}\big(z_X(\omega)\overline{z_X(\omega')}\big) = s_X(\omega)\delta(\omega - \omega'), \tag{17}$$

    where $\delta(\omega - \omega')$ denotes Dirac's delta function.
- This result is known as the *spectral representation theorem* or *Cramer's theorem*.

Spectral representation theorem

- The spectral representation theorem can also be written in terms of real quantities only. Namely, we define

$$\begin{aligned} a_X(\omega) &= \mathrm{Re}\, z_X(\omega), \\ b_X(\omega) &= -\mathrm{Im}\, z_X(\omega) \end{aligned} \qquad \begin{matrix} \text{Re: real part} \\ \text{IM: imagine part} \end{matrix} \tag{18}$$

  (the negative sign is just for convenience).

- Note that the random functions $a_X(\omega)$ and $b_X(\omega)$ have the following properties:

  (i)

$$\begin{aligned} a_X(-\omega) &= a_X(\omega), \\ b_X(-\omega) &= -b_X(\omega). \end{aligned} \tag{19}$$

  This is simply a conseqence of (15).

  (ii)

$$a_X(\omega)^2 + b_X(\omega)^2 = |z_X(\omega)|^2. \tag{20}$$

- As a result, we can write

$$X_t = \mu + \int_{-\pi}^{\pi} \left(cos(\omega t) a_X(\omega) + sin(\omega t) a_X(\omega) \right) d\omega. \tag{21}$$

Sample periodogram

- A complete proof of the spectral representation theorem is a bit technical, and can be found in specialized mathematical literature. Instead, we will interpret it in terms sample data.
- Let $x_1, \ldots, x_T$ be observations of X_t, and let $\widehat{\Gamma}_t$ denote the estimated autocovariance as defined by equation (5) in Lecture Notes #1. For any ω, the estimated sample spectral density function,

$$\widehat{s}_X(\omega) = \frac{1}{2\pi} \sum_{t=-(T-1)}^{T-1} \widehat{\Gamma}_t e^{-i\omega t}. \tag{22}$$

  is called the *sample periodogram*.
- We can then verify that

$$\widehat{\Gamma}_0 = \int_{-\pi}^{\pi} \widehat{s}_X(\omega) d\omega, \tag{23}$$

  i.e. the area under the periodogram is equal to the sample variance.

Sample periodogram

- In order to formulate the sample version of the spectral representation theorem, we assume that T is odd, and denote $\omega_j = 2\pi j/T$, for $j = -M, -M+1, \ldots, M$, where $M = (T-1)/2$.
- For each j, we define

If T is even, change T to T+1

$$\widehat{z}_X(\omega_j) = \frac{1}{T}\sum_{t=1}^{T} e^{-i\omega_j t} x_t - \widehat{\mu}. \tag{24}$$

Notice that

$$\widehat{z}_X(\omega_0) = 0. \tag{25}$$

- Then

$$x_t = \widehat{\mu} + \sum_{j=-M}^{M} e^{i\omega_j t}\, \widehat{z}_X(\omega_j). \tag{26}$$

Sample periodogram

- To see this, we multiply both sides of (24) by $e^{i\omega_j s}$ and sum over $j = 1, \ldots, M$, and notice that

$$\sum_{j=-M}^{M} e^{i\omega_j(s-t)} = \begin{cases} T, & \text{if } s = t, \\ 0, & \text{otherwise.} \end{cases}$$

- Finally, notice that

$$\sum_{j=1}^{T}(x_t - \widehat{\mu})^2 = \sum_{j=-M}^{M} |\widehat{z}_X(\omega_j)|^2. \tag{27}$$

Singular spectrum analysis

- *Singular spectrum analysis* (SSA) is a model free feature extraction methodology, which may be thought of as a variant of the principal component analysis (PCA).
- Its extension to multivariate time series (not discussed here) is referred to as *multi channel singular spectrum analysis* (M-SSA).
- We consider a sample from a time series $X_1, \ldots, X_T$, and let $1 < l < T$ be the length of the rolling window. Then $k = T - l + 1$ is the number of lagged vectors.
- The basic algorithm of SSA consists of two stages:
  (i) embedding,
  (ii) reconstruction.

Singular spectrum analysis

- Embedding is carried out in two steps. First, we form the *trajectory matrix*:

$$\mathcal{X} = \begin{pmatrix} X_1 & X_2 & \dots & X_k \\ X_2 & X_3 & \dots & X_{k+1} \\ \vdots & \vdots & & \vdots \\ X_l & X_{l+1} & \dots & X_T \end{pmatrix}. \tag{28}$$

  Note that $\mathcal{X}_{ij} = X_{i+j-1}$; matrices of this form are called *Hankel matrices*.
- The columns in the trajectory matrix correspond to the observations of the time series as the length l observation window slides forward.

Singular spectrum analysis

- Then, we perform the singular value decomposition (SVD) of the trajectory matrix $\mathcal{X}$:
  - (i) Let $\mathcal{S} = \mathcal{X}\mathcal{X}^{\mathrm{T}}$. Then $\mathcal{S}$ is positive definite; we denote its eigenvalues by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 0$, and the corresponding orthonormal system of eigenvectors by $U_1, U_2, \ldots, U_l$. The numbers $\sqrt{\lambda_i}$ are called the *singular values* of $\mathcal{X}$.
  - (ii) Let $r = \mathrm{rank}(\mathcal{X})$ (typically, $r = L$), and set $V_i = \frac{1}{\sqrt{\lambda_i}} \mathcal{X}^{\mathrm{T}} U_i$, for $i = 1, \ldots, l$.
  - (iii) Then
$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \ldots + \mathcal{X}_r, \tag{29}$$
where $\mathcal{X}_i = \sqrt{\lambda_i} U_i V_i^{\mathrm{T}}$ are rank 1 matrices, called *elementary matrices*. The triple $(\sqrt{\lambda_i}, U_i, V_i)$ is called an *eigentriple* (ET) of the SVD and the vectors $\sqrt{\lambda_i}\mathcal{X}_i = U_i V_i^{\mathrm{T}}$ are the *principal components*.
  - (iv) The numpy implementation of SVD is called `numpy.linalg.svd`.

Singular spectrum analysis

- The reconstruction stage is performed in two steps. First, we partition the set of indices $I = \{1, \ldots, r\}$ into m disjoint subsets $I = I_1 \cup \ldots \cup I_m$. For each subset I_k, form the sum

$$\mathcal{X}_{I_k} = \sum_{i \in I_k} \mathcal{X}_i. \tag{30}$$

  Clearly, this defines a decomposition of the trajectory matrix into components:

$$\mathcal{X} = \mathcal{X}_{I_1} + \ldots + \mathcal{X}_{I_m}. \tag{31}$$

- The final step is *diagonal averaging*. Each matrix $\mathcal{X}_{I_k}$ in the decomposition (31) is transformed into a new *reconstructed time series* $(\widetilde{X}_1^{(k)}, \widetilde{X}_2^{(k)}, \ldots, \widetilde{X}_T^{(k)})$ by means of the following procedure.

Singular spectrum analysis

- Let A be an $l \times k$-matrix, and let $T = l + k - 1$. We denote

$$A^*_{ij} = \begin{cases} A_{ij}, & \text{if } l < k, \\ A_{ji}, & \text{otherwise.} \end{cases} \tag{32}$$

Diagonal averaging transforms the matrix A into a time series $\widetilde{A}_1, \ldots, \widetilde{A}_T$ as follows:

$$A_j = \begin{cases} \frac{1}{j} \sum_{m=1}^{k} A^*_{m,j-m+1}, & \text{for } 1 \le j < l \wedge k, \\ \frac{1}{l \wedge k} \sum_{m=1}^{l \wedge k} A^*_{m,j-m+1}, & \text{for } l \wedge k \le j \le l \vee k, \\ \frac{1}{N-j+1} \sum_{m=k-l \vee k+1}^{T-l \vee k+1} A^*_{m,j-m+1}, & \text{for } l \vee k \le j \le T. \end{cases} \tag{33}$$

- As a result, the original time series is represented as a sun of m reconstructed series;

$$X_t = \sum_{i=1}^{m} \widetilde{X}_t^{(i)}. \tag{34}$$

Singular spectrum analysis

- The choice of the rolling window length l is an important matter. It should be suffiently large so that each lagged time series incorporates the essential features of the original series $X_1, \ldots, X_N$.
- It is a good idea to perform SSA with different choices of l.

SSA of a simulated $I(1)$ process

- The figure below shows the results of SSA of the simulated $I(1)$ process given by the following specification:

$$X_t = 1.1 + X_{t-1} + 5.0\varepsilon_t, \tag{35}$$

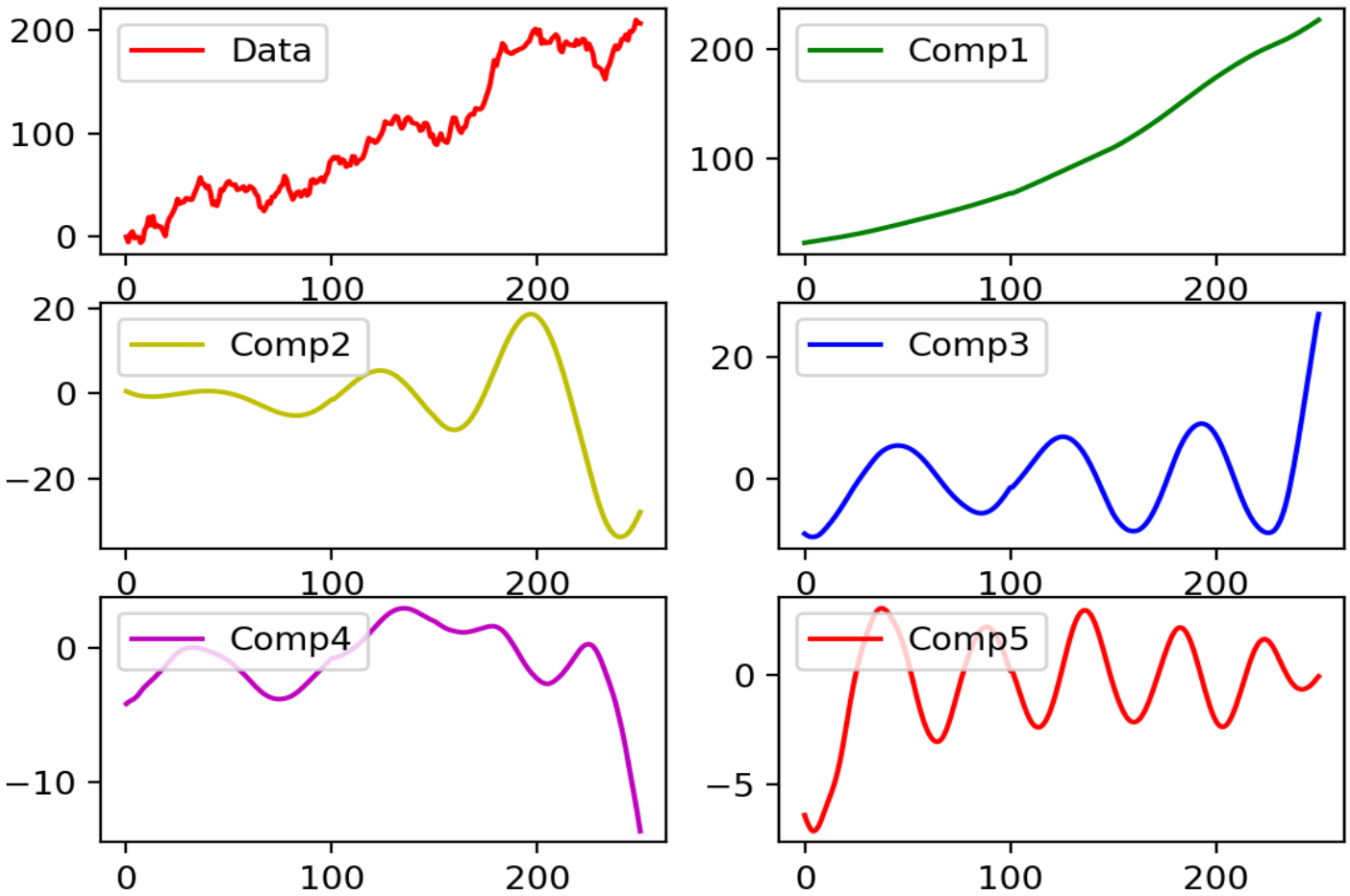
  where $\varepsilon_t \sim N(0, 1)$.
- The upper left plot shows the actual time series, while the remaining ones show the first five SSA components.
- The cumulative weights, defined as

$$CW_j = \frac{\lambda_1 + \ldots + \lambda_j}{\lambda_1 + \ldots + \lambda_l}, \tag{36}$$

  of the plotted components are:

$$\begin{aligned} CW_1 &= 0.595, \\ CW_2 &= 0.653, \\ CW_3 &= 0.698, \\ CW_4 &= 0.720, \\ CW_5 &= 0.737. \end{aligned} \tag{37}$$

SSA of a simulated $AR(1)$ process



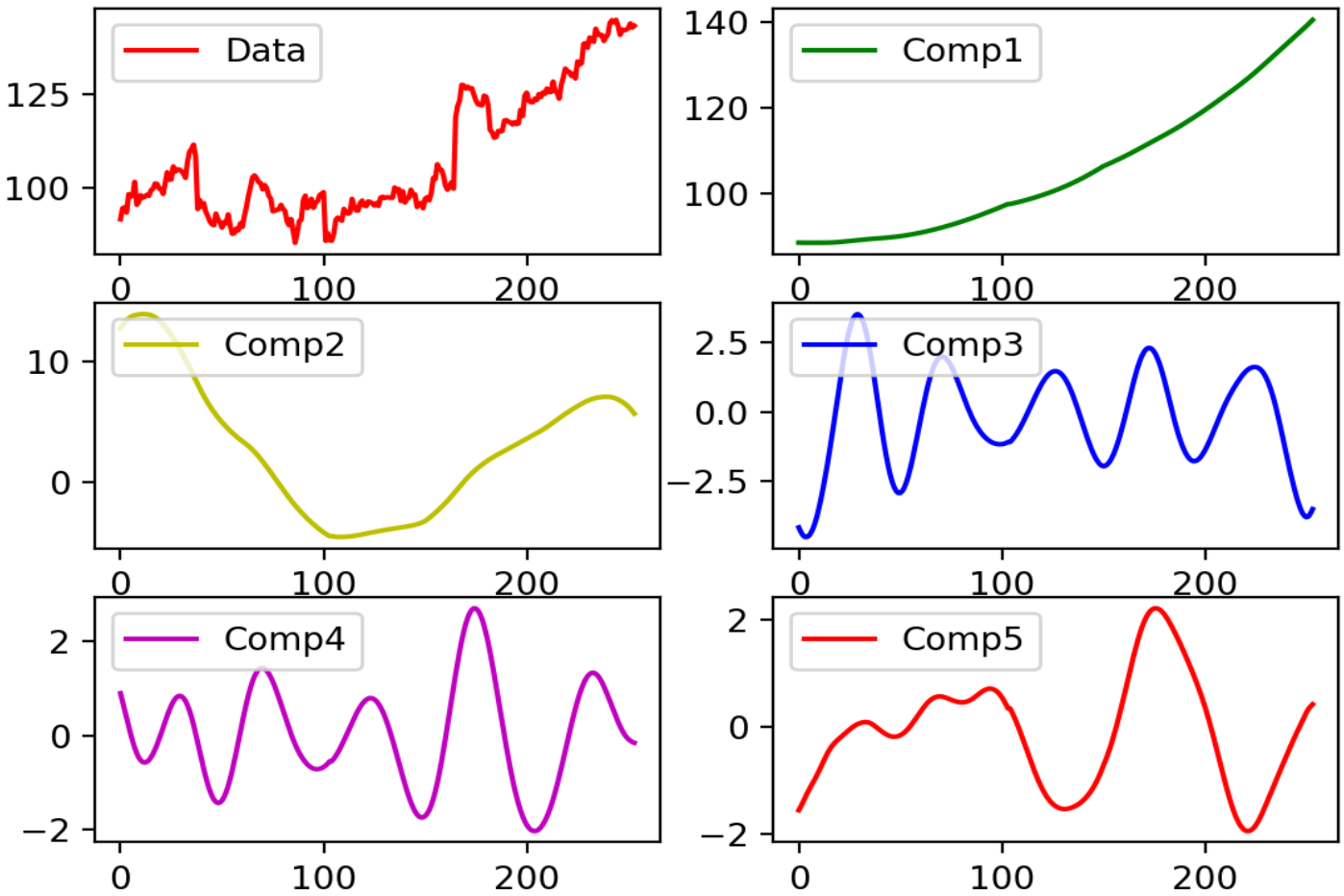
SSA of the Netflix share price

- The next figure below shows the results of SSA of the share price of Netflix (NFLX) during the one-year period 02-24-2016 through 02-24-2017.
- As before, the upper left plot shows the actual time series, while the remaining ones show the first five SSA components.
- The cumulative weights of the plotted components are:

$$\begin{aligned} CW_1 &= 0.725, \\ CW_2 &= 0.763, \\ CW_3 &= 0.778, \\ CW_4 &= 0.793, \\ CW_5 &= 0.805. \end{aligned} \tag{38}$$

- Notice that the first component (trend) is responsible for 72.5% of the dynamics.

SSA of the Netflix share price



Transfer entropy

- The concept of Granger causality defined earlier in these lectures can be reformulated in a model free manner, using the concept of *transfer entropy*.
- The price for the model freeness is a bit of formalism required. In order to lighten up on the math, we will assume that X_t can take on only one of finitely many state values in $A = \{x^1, \ldots, x^K\}$.
- The *Shannon entropy* of the probability distribution $p_j = \mathrm{P}(X_t = x^j)$ is given by:

Shannon entropy from information theory

Physics: Gibbs entropy

$$\mathrm{H}(X_t) = -\sum_{j=1}^{K} p_j \log p_j. \tag{39}$$

- The Shannon entropy is always nonnegaitive. Its value is 0, if one of the p_j's is 1. It reaches is maximum value $\log K$, if the distribution is uniform, $p_j = 1/K$, for all $j = 1, \ldots, K$.
- It is interpreted as a measure of information in the probability distribution: the lower the entropy, the higher its information content.

Transfer entropy

- Suppose q_j, $j = 1, \ldots, K$ is another probability distribution (possibly an *a priori* guess of p). A useful measure of distance between p and q is given by the *Kullback-Leibler divergence*, a.k.a. *relative entropy*:

Practice: p observed from real data
q: model parameters

$$D_{p\|q}(X_t) = \sum_{j=1}^{K} p_j \log \frac{p_j}{q_j} \,. \tag{40}$$

This D is not symmetry by construction

- One can show that $D_{p\|q}(X_t) \geq 0$, and $D_{p\|q}(X_t) = 0$, iff $p = q$.
- Consider now a second process Y_t with state values in $B = \{y^1, \ldots, y^K\}$. We define the *joint entropy*:

$$\mathrm{H}(X_t, Y_s) = -\sum_{i,j=1}^{K} p_{i,j} \log p_{i,j}, \tag{41}$$

where $p_{i,j} = \mathrm{P}(X_t = x^i, Y_s = y^j)$ is the joint probability distribution.

Transfer entropy

- The mutual entropy of X_t and Y_t is defined by

$$\begin{aligned} \mathrm{M}(X_t, Y_s) &= \mathrm{H}(X_t) + \mathrm{H}(Y_s) - \mathrm{H}(X_t, Y_s) \\ &= \sum_{i,j=1}^{K} \mathrm{P}(X_t = x^i, Y_s = y^j) \log \frac{\mathrm{P}(X_t = x^i, Y_s = y^j)}{\mathrm{P}(X_t = x^i)\mathrm{P}(Y_s = y^j)} . \end{aligned} \tag{42}$$

- The mutual entropy is thus identital with the Kullback-Leibler distance between the joint distribution of X_t, Y_s and the product of the marginal distributions, $q_{i,j} = \mathrm{P}(X_t = x^i)\mathrm{P}(Y_s = y^j)$.

Entropy rate of a process

- The dynamic character of a time series is captured by the transistion probabilities $\mathrm{P}(X_{t+1} = x^j | x_{t-l+1:t})$. The associated Shannon entropy is given by

$$\mathrm{H}(X_{t+1}|x_{t-l+1:t}) = -\sum_{j=1}^{K} \mathrm{P}(X_{t+1} = x^j | x_{t-l+1:t}) \log \mathrm{P}(X_{t+1} = x^j | x_{t-l+1:t}). \quad (43)$$

- The entropy concept below (which is related to Shannon's *entropy rate*) measures the information content in these transition probabilities.
- Namely, the entropy rate of a time series is defined as the expected value of $\mathrm{H}(X_{t+1}|x_{t-l+1:t})$ with respect to all histories $x_{t-l+1:t}$:

$$\begin{aligned} H(X_{t+1}|X_{t-l+1:t}) &= \sum_{x_{t-l+1:t}} p(x_{t-l+1:t}) \mathrm{H}(X_{t+1}|x_{t-l+1:t}) \\ &= -\sum_{x_{t-l+1:t+1}} p(x_{t-l+1:t+1}) \log p(x_{t+1}|x_{t-l+1:t}). \end{aligned} \quad (44)$$

Entropy rate of a process

- If q is another probability distribution, we can define the following Kullback-Leibler divergence:

$$D_{p||q}(X_{t+1}|X_{t-l+1:t}) = \sum_{x_{t-l+1:t+1}} p(x_{t-l+1:t+1}) \log \frac{p(x_{t+1}|x_{t-l+1:t})}{q(x_{t+1}|x_{t-l+1:t})} . \quad (45)$$

- The same concepts can be extended to multivariate time series, with the corresponding increase in the notational complexity.
- In particular, if X_t, Y_t is a bivariate time series, the Kullback-Leibler divergence with respect to

$$q(x_{t+1}, y_{t+1}|x_{t-l+1:t}, y_{t-l+1:t}) = p(x_{t+1}|x_{t-l+1:t})p(y_{t+1}|y_{t-l+1:t})$$

is the corresponding mutual entropy.

Entropy rate of a process

- Explicitly,

$$\mathrm{M}(X_{t+1}, Y_{t+1}|X_{t-l+1:t}, Y_{t-l+1:t}|) = \sum_{x_{t-l+1:t+1}} \sum_{y_{t-l+1:t+1}} p(x_{t-l+1:t+1}, y_{t-l+1:t+1}) \log \frac{p(x_{t+1}, y_{t+1}|x_{t-l+1:t}, y_{t-l+1:t})}{p(x_{t+1}|x_{t-l+1:t})p(x_{t+1}|x_{t-l+1:t})} .$$

- While this concept is an approprite information measure for the bivariate time series X_t, Y_t, it has the property that it is symmetric in X_t and Y_t. As a consequence, it cannot be used for quantifing causal dependence of one series on the other.
- The causal dependence measure related the concepts introduced above is *transfer entropy*.

Transfer entropy

- If two processes X_t and Y_t are independent, then, for any number of lags j,

$$p(x_{t+1}|x_{t-j+1:t}) = p(x_{t+1}|x_{t-j+1:t}, y_{t-j+1:t}) \tag{46}$$

- This leads to the following concept of *transfer entropy*:

$$\begin{aligned} &\mathrm{T}(X_{t+1}|X_{t-l+1:t}, Y_{t-l+1:t}|) \\ &= \sum_{x_{t-l+1:t+1}} \sum_{y_{t-l+1:t}} p(x_{t-j+1:t+1}, y_{t-j+1:t}) \log \frac{p(x_{t+1}|x_{t-j+1:t}, y_{t-j+1:t})}{p(x_{t+1}|x_{t-j+1:t})} . \end{aligned} \tag{47}$$

- This also can be rewritten as

$$\begin{aligned} &\mathrm{T}(X_{t+1}|X_{t-l+1:t}, Y_{t-l+1:t}|) \\ &\quad = \mathrm{H}(X_{t+1}|X_{t-j+1:t}) - \mathrm{H}(X_{t+1}|X_{t-j+1:t}, Y_{t-j+1:t}). \end{aligned} \tag{48}$$

Transfer entropy

- Transfer entropy is a very elegant and economic concept of causal dependence among time series.
- It applies to time series models that are not necessarily linear, or whose disturnaces are necessarily normally distributed.
- In case of autoregressive models with normally distributed disturbances, transfer entropy is essentially identical with the statistics used to test Granger causality.
- Estimation of transfer entropy from observed data is a bit of a challenge, as reliable estimates require large sample sets.

References


Golyandina, N., and Zhiglyavsky, A.: *Singular Spectrum Analysis for Time Series*, Springer (2013).


Hamilton, J. D.: *Time Series Analysis*, Princeton University Press (1994).


Schreiber, T.: Measuring Information transfer, *Phys. Rev. Lett.*, **85** ,461 - 464 (2000).