MTH 9893 Time Series HW 4

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1.

Let $Y_t = \psi(L) X_t = \alpha + \varphi(L) \varepsilon_t$,

Take a look at $Y_t = \psi(L) X_t = (1 - \beta_1 L - \cdots - \beta_p L^p) X_t = X_t - \beta_1 X_{t-1} - \cdots - \beta_p X_{t-p}$

$$
\begin{aligned}
cov(Y_t, Y_{t+k}) &= cov(\psi(L) X_t, \psi(L) X_{t+k}) \\
&= \sum_{0 \le r \le p} \sum_{0 \le s \le p} \beta_r \beta_s cov(X_{t-r}, X_{t+k-s}) \\
&= \sum_{0 \le r,s \le p} \beta_r \beta_s \int_{-\pi}^{\pi} s_X(\omega) e^{i\omega(k+r-s)} d\omega \\
&= \int_{-\pi}^{\pi} s_X(\omega) \psi(e^{i\omega}) \psi(e^{-i\omega}) e^{i\omega k} d\omega \\
\int_{-\pi}^{\pi} s_Y(\omega) e^{i\omega k} d\omega &= \int_{-\pi}^{\pi} s_X(\omega) |\psi(e^{i\omega})|^2 e^{i\omega k} d\omega
\end{aligned}
$$

Thus we have

$$s_Y(\omega) = |\psi(e^{i\omega})|^2 s_X(\omega) \qquad (1)$$

It is an application of *the filter theorem*. Similarly from $Y_t = \alpha + \varphi(L)\varepsilon_t$we have,

$$
\begin{aligned}
s_Y(\omega) &= |\varphi(e^{i\omega})|^2 s_\varepsilon(\omega) \\
&= \frac{\sigma^2}{2\pi} |\varphi(e^{i\omega})|^2 \qquad (2)
\end{aligned}
$$

Combine (1) and (2), we have

$$
\begin{aligned}
|\psi(e^{i\omega})|^2 s_X(\omega) &= \frac{\sigma^2}{2\pi} |\varphi(e^{i\omega})|^2 \\
s_X(\omega) &= \frac{\sigma^2}{2\pi} \left|\frac{\varphi(e^{i\omega})}{\psi(e^{i\omega})}\right|^2
\end{aligned}
$$

which is formula (8) in Lecture Notes 4.

If we factorize the polynomials $\varphi(e^{i\omega})$ and $\psi(e^{i\omega})$,

$$
\begin{aligned}
\varphi(e^{i\omega}) &= (1 - \mu_1 e^{i\omega}) \cdots (1 - \mu_q e^{i\omega}) \\
|\varphi(e^{i\omega})|^2 &= \varphi(e^{i\omega}) \varphi(e^{-i\omega}) \\
&= (1 - \mu_1 e^{i\omega}) \cdots (1 - \mu_q e^{i\omega}) (1 - \mu_1 e^{-i\omega}) \cdots (1 - \mu_q e^{-i\omega})
\end{aligned}
$$

$$
\begin{aligned}
(1 - \mu_1 e^{i\omega})(1 - \mu_1 e^{-i\omega}) &= 1 - \mu_1 e^{i\omega} - \mu_1 e^{-i\omega} + \mu_1^2 \\
&= 1 - \mu_1 (cos\omega + isin\omega + cos\omega - isin\omega) + \mu_1^2 \\
&= 1 - 2\mu_1 cos\omega + \mu_1^2
\end{aligned}
$$

Similarly for others, finally we have

$$
\begin{aligned}
s_X(\omega) &= \frac{\sigma^2}{2\pi} \left|\frac{\varphi(e^{i\omega})}{\psi(e^{i\omega})}\right|^2 \\
&= \frac{\sigma^2}{2\pi} \frac{(1 - \mu_1 e^{i\omega}) \cdots (1 - \mu_q e^{i\omega})(1 - \mu_1 e^{-i\omega}) \cdots (1 - \mu_q e^{-i\omega})}{(1 - \lambda_1 e^{i\omega}) \cdots (1 - \lambda_p e^{i\omega})(1 - \lambda_1 e^{-i\omega}) \cdots (1 - \lambda_p e^{-i\omega})} \\
&= \frac{\sigma^2}{2\pi} \frac{(1 - 2\mu_1 cos\omega + \mu_1^2) \cdots (1 - 2\mu_q cos\omega + \mu_q^2)}{(1 - 2\lambda_1 cos\omega + \lambda_1^2) \cdots (1 - 2\lambda_p cos\omega + \lambda_p^2)}
\end{aligned}
$$

So formula (9) in Lecture Notes 4 is proved.