Solving American Options: A Multifaceted Approach

Columbia University

APAM 4990: Stochastic Calculus for Finance

Shuangqi Bao, Hong Cao, Ince Yang

Outline

- Introduction
- Methodology
- Methods Comparison
- Conclusion
- Future Improvement

Introduction

- An *American option* can be exercised at any time prior to its expiration date.

- Flexibility

- Due to their early exercise feature, traditional valuation methods like Black-Scholes model are not directly applicable.

Methodology

- Binomial Tree
- Monte Carlo Simulation
- PDE Method (free boundary problem)

Price of American Call Equals Euro Call (Non Dividend)

From put-call parity we have

$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

So $C_t \geq S_t - Ke^{-r(T-t)} > \mathrm{S_t} - \mathrm{K}$

This means that the price of the call C_t at any time 0 < t < T is always greater than the value of exercising the call which is S_t–K. Therefore, the optionality of exercising an American call option (with no dividends) before T has no value.

Therefore, price of American Call equals to price of European call (non-dividend-paying stock)

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Introduction to Binomial Option Pricing

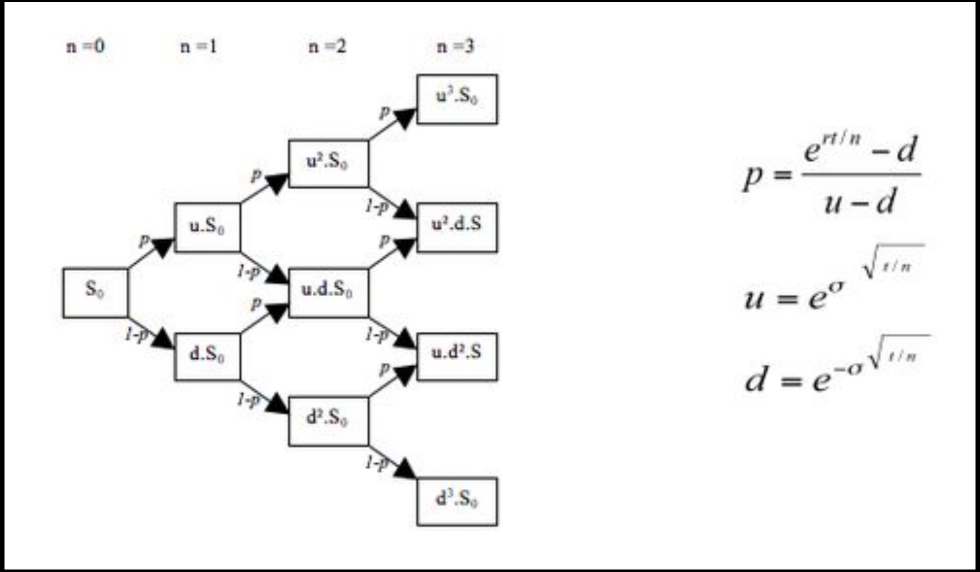
- There are two (and only two) possible prices for the underlying asset on the next date. The underlying price will either: 1. Increase by a factor of u% (an uptick) 2. Decrease by a factor of d% (a downtick)
- No dividends.
- The one-period interest rate, r, is constant over the life of the option (r% per period).

Step1: The stock Pricing Process

- S is the underlying asset. Time T is the expiration day of a option. Time T-1 is one period prior to expiration. **p** is calculated by matching the moments of Geometric Brownian Motion of the underlying stock.

$$S_{T-1} \xrightarrow{p} S_{T,u} = (1+u)S_{T-1}$$

$$S_{T-1} \xrightarrow{1-p} S_{T,d} = (1+d)S_{T-1}$$



Step2: Find Option Value at each final node

At each final node of the tree—i.e. at expiration of the option—the option value is simply its intrinsic, or exercise, value:

$$C_T = (S_T - K)^+ \text{ for a call option}$$
$$C_T = (K - S_T)^+ \text{ for a put option}$$

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Step3: Find option value at earlier nodes

$C_{t-\Delta t,i} = \left[p\, C_{t,i} + (1-p) C_{t\, i+1}\right] * e^{(-r * \Delta t)}$ where

$C_{t,i}$ is the option's value for the i node at time t.

For an American option, since the option may either be held or exercised prior to expiry, the value at each node is: Max (Binomial Value, Exercise Value). Therefore:

$$C_{t-\Delta t,i} = max\,(C_{t-\Delta t,i}\,, S_t - K)$$

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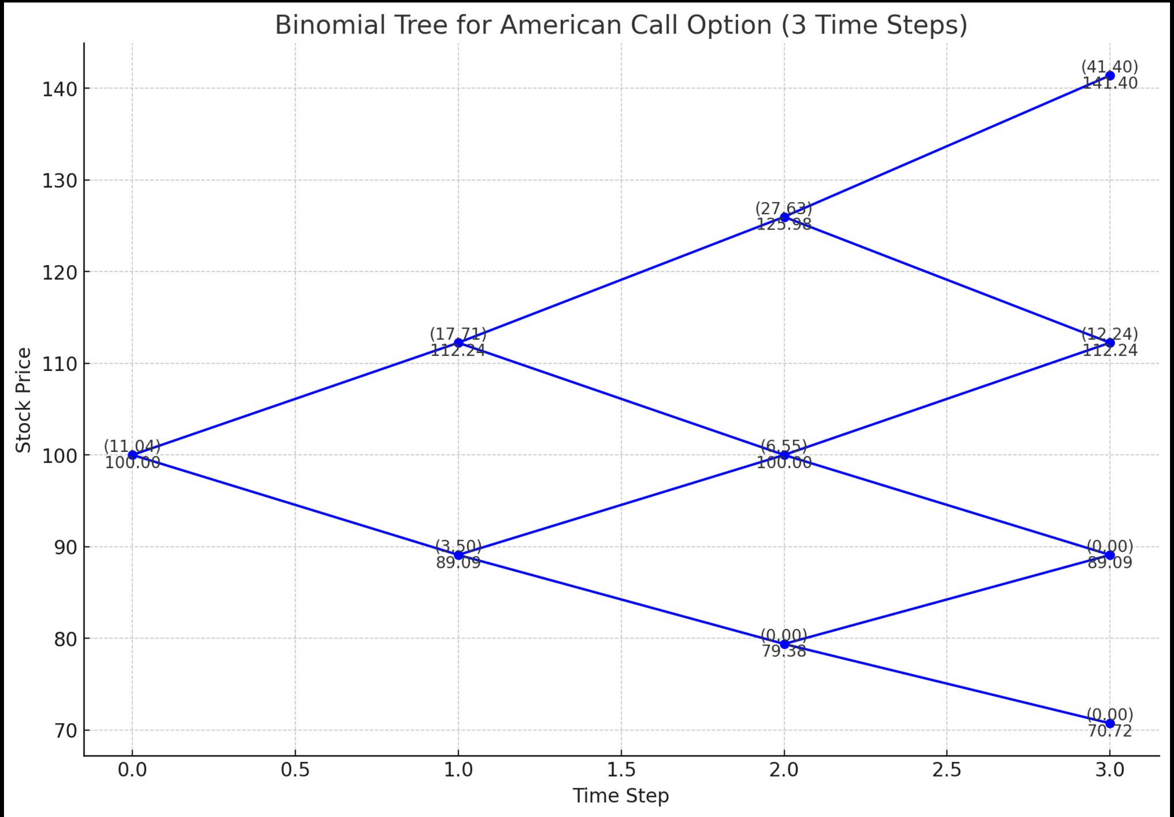
Step3: Find option value at earlier nodes

Each blue dot represents a node, indicating the stock price at that point in time.

The bottom annotation near each node is the stock price value at that node.

The top annotation (in parentheses) is the option value at that node.



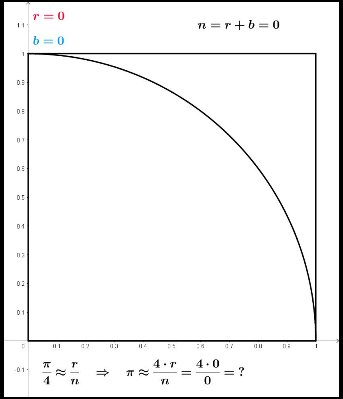
Monte-Carlo Simulation - Origin

- John von Neumann & Stanislaw Ulam
  World War II, Manhattan Project,
  Los Alamos National Lab

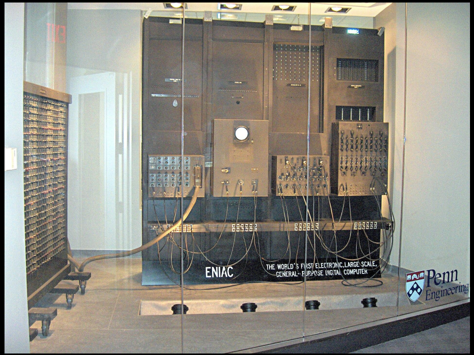
- Code Name - refers to the Monte Carlo Casino in Monaco

- Solving deterministic problems using probabilistic metaheuristics

- In principle, Monte Carlo methods can be used to solve any problem having a probabilistic interpretation.



Monte Carlo method applied to approximating the value of π.



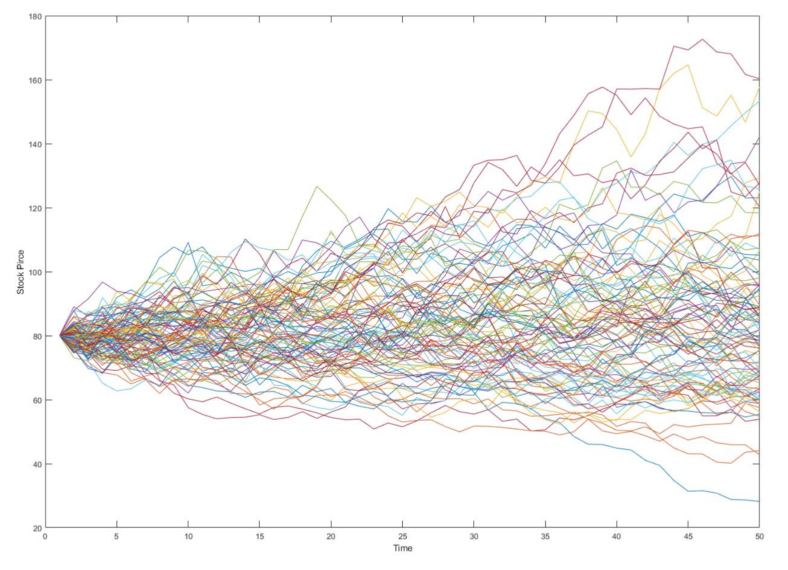
The ENIAC computer which performs the first fully automated Monte Carlo calculations in 1948

Monte-Carlo Simulation

- Path Generating (GBM)
  $dS = \mu S dt + \sigma S dW$

  Iterative step:
  $dS[t+dt] = dS[t]*\exp(\mu*dt+\sigma*\mathrm{sqrt}(dt)*e)$
  e: Standard Normal Dist

- Basic Idea:
  1.Calculate option price for each simulation path
  2.Take the average of all prices at t = 0
  3.LLN(Law of Large Number)



Simulation of stock price with S0 = 80

Monte-Carlo Simulation

- European option
  S0=80|σ=0.2|μ=0.04|T=1|K=100

- From BSM

$$c = S_0\,N(d_1) - K\,e^{-rT}N(d_2)$$

$$p = K\,e^{-rT}\,N(-d_2) - S_0\,N(-d_1)$$

where
$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

- MATLAB

```
Parameters:
Sigma=0.200000 | Initial_Stock_price=80 | Interest_rate=0.040000
Time_to_Maturity=1 | Strike_Price=100
Under 1000000 Simulation
-----------------------------------------------------------------
The price of European Call:
By Monte carlo:=1.705049
By Black-Scholes-Merton Formula=1.705573
The price of European Put:
By Monte carlo:=17.800793
By Black-Scholes-Merton Formula=17.784517
```

- What about American Option?
  Conditional Expectation is MC - “Monte Carlo on Monte Carlo”

Least Square Monte-Carlo(LSM)

- We can calculate estimated conditional expectation from the cross-sectional information in the simulation by using least squares. (Longstaff and Schwartz , 2001)

- Basis func: 1, x, x^2, x^3 (For simplicity)
- S0=80|σ=0.3|μ=0.01|T=1|K=100

- MATLAB \ : least-squares solution

```
X=SSit(valid,tt-1);
x=[ones(v_size,1),X,X.^2,X.^3];
BS=(x'*x)\(x'*A);
```

- Result

The price of American Put: 11.452917

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Free Boundary Problem

- Traditional Black-Scholes PDE
- No traditional fixed time boundary
- Early Exercise Property
- Impose new conditions

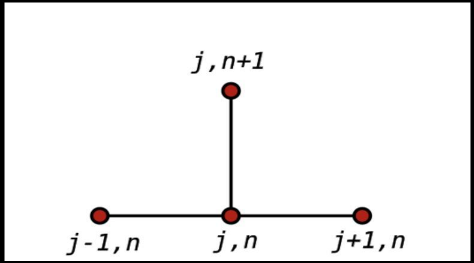
$$\frac{\partial U(t,x)}{\partial t} = \frac{D(t,x)}{2} + \frac{D(t,x)}{2}\frac{\partial^2 U(t,x)}{\partial x^2} + F(t,x)\frac{\partial U(t,x)}{\partial x} + R(t,x)$$

Black Scholes PDE

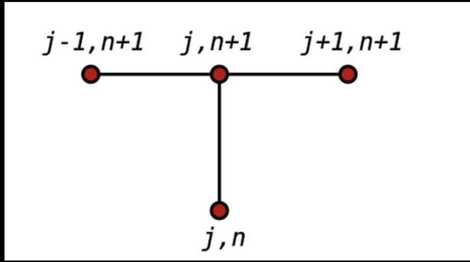
Finite Difference Method

- Discretization Scheme
- Numerically estimate derivatives
- Unconditional stability
- Precision

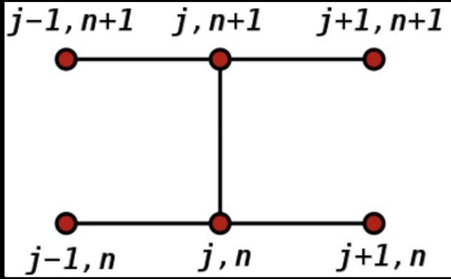
Explicit Method



Implicit Method



Crank Nicolson Method



Slope and Value Equality

- Linear Complementarity Problem
- Optimal Exercise Boundary
- Impose new boundary
- Condition ("slope and value")
- Smooth & computational fast

$$\tilde{u}_j - \tilde{u}_{j-1} = E_j - E_{j-1}$$

$$\tilde{u}_{j-1} = E_{j-1}$$

Optimal Exercise Boundary

$$\begin{aligned}
&\frac{f_{ij} - f_{i,j-1}}{\delta t} + \frac{ri\,\delta S}{2}\left(\frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\,\delta S}\right) + \frac{ri\,\delta S}{2}\left(\frac{f_{i+1,j} - f_{i-1,j}}{2\,\delta S}\right) \\
&\quad + \frac{\sigma^2 i^2 (\delta S)^2}{4}\left(\frac{f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1}}{(\delta S)^2}\right) \\
&\quad + \frac{\sigma^2 i^2 (\delta S)^2}{4}\left(\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\delta S)^2}\right) \\
&\quad = \frac{r}{2} f_{i,j-1} + \frac{r}{2} f_{ij}.
\end{aligned}$$

PDE after Crank-Nicolson Discretization

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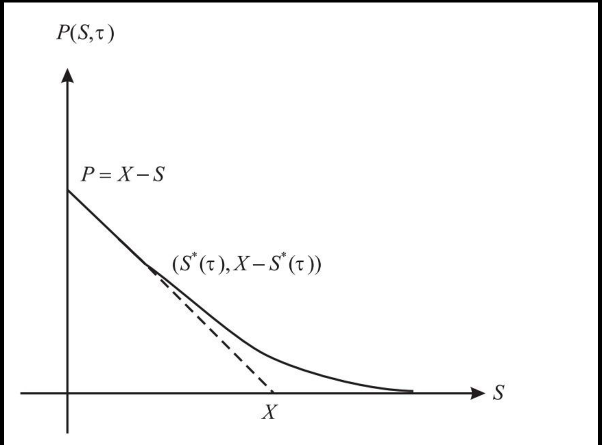
Thomas Algorithm

- Tridiagonal System
- Solving Forward and Backward
- Estimates of optimal exercise Boundary
- Interpolate and found the desired option price

$$\begin{bmatrix} 1+\beta_1 & \gamma_1 & & & & \\ \alpha_2 & 1+\beta_2 & \gamma_2 & & & \\ & \alpha_3 & 1+\beta_3 & \gamma_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{M-2} & 1+\beta_{M-2} & \gamma_{M-2} \\ & & & & \alpha_{M-1} & 1+\beta_{M-1} \end{bmatrix}$$

Tridiagonal Matrix



Optimal Exercise Boundary

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Methods Comparison

Pricing results:

- Binomial: 11.4527
- Monte Carlo: 11.452917
- PDE with Free Boundary: 11.4465

*Relative Error is less than 0.05%

Conclusions

- All three methods returned accture price with small error/difference
- Binomial Tree Method is the easiest to implement
- Monte Carlo Method has biggest potential to adjust more variations
- Finite difference method is the fastest

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Future Improvements

- Improve the model efficiency, reduce computing complexity by choosing more optimal time and space grids
- Implement the models in C++

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Thank you!