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Strategic disclosure of opinions on a social network

Umberto Grandi, Emiliano Lorini and Laurent Perrussel*

IRIT and CNRS
University of Toulouse
France

Abstract

We study the strategic aspects of social influence in a society of agents linked by a trust network, introducing a new class of games called games of influence. A game of influence is an infinite repeated game with incomplete information in which, at each stage of interaction, an agent can make her opinions visible (public) or invisible (private) in order to influence other agents' opinions. The influence process is mediated by a trust network, as we assume that the opinion of a given agent is only affected by the opinions of those agents that she considers trustworthy (i.e., the agents in the trust network that are directly linked to her). Each agent is endowed with a goal, expressed in a suitable temporal language inspired from linear temporal logic (LTL). We show that games of influence provide a simple abstraction to explore the effects of the trust network structure on the agents' behaviour, by considering solution concepts from game-theory such as Nash equilibrium, weak dominance and winning strategies.

1 Introduction

At the micro-level, social influence can be conceived as a process where an agent forms her opinion on the basis of the opinions expressed by other agents in the society. Social influence depends on trust since an agent can be influenced by another agent, so that her opinions are affected by the expressed opinions of the other, only if she trusts her. At the macro-level, social influence is the basic mechanism driving the diffusion of opinions in human societies: certain agents in the society influence other agents in the society towards a given view, and these agents, in turn, influence other agents to acquire the same view, and so on. In other words, social influence can be seen as the driving force of opinion diffusion in human and human-like agent societies. This view is resonant of existing studies in social sciences and social psychology which

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emphasize the role of interpersonal processes in how people construe and form their perceptions, judgments, and impressions (see, e.g., [1, 9, 15]).

Recent work in multi-agent systems [18, 12] proposed a formal model of opinion diffusion that combined methods and techniques from social network analysis with methods and techniques from belief merging and judgment aggregation. The two models aim at studying how opinions of agents on a given set of issues evolve over time due to the influence of other agents in the population. The basic component of these models is the trust network, as it is assumed that the opinions of a certain agent are affected only by the opinions of the agents that she trusts (i.e., the agents in the trust network that are directly linked to her). Specifically, the opinions of a certain agent at a given time are the result of aggregating the opinions of the trustworthy agents at the previous time.

In this work we build on these models to look at social influence from a strategic perspective. We do so by introducing a new class of games, called games of influence. Specifically, a game of influence is an infinite repeated game with incomplete information in which, at each stage of interaction, an agent can make her opinions visible (public) or invisible (private) to the other agents. Incompleteness of information is determined by the fact that an agent has uncertainty about the private opinions of the other agents, as she cannot see them. At each stage of the game, every agent is influenced by the *public* opinions of the agents she trusts (i.e., her neighbors in the trust network) and changes her opinions on the basis of the aggregation criterion she uses.

Following the representation of agents' motivations given in [14], in a game of influence each agent is identified with the goal that she wants to achieve. This goal is represented by a formula of a variant of linear temporal logic (LTL), in which we can express properties about agents' present and future opinions. For example, an agent might have the achievement goal that at some point in the future there will be consensus about a certain proposition p (i.e., either everybody has the opinion that p is true or everybody has the opinion that p is false), or the maintenance goal that two different agents will always have the same opinion about p.

Games of influence provide a simple abstraction to explore the effects of the trust network structure on the agents' behaviour. We consider solution concepts from game-theory such as Nash equilibrium, weak dominance and winning strategies. For instance, in the context of games of influence, we can study how the relative position of an agent in the trust network determines her influencing power, that is, her capacity to influence opinions of other agents, no matter what the others decide to do (which corresponds to the concept of uniform strategy). Moreover, in games of influence one can study how the structure of the trust network determines existence of Nash equilibria, depending on the form of the agents' goals. For instance, we will show that if the trust network is fully connected and every agent wants to reach a consensus about a certain proposition p, then there always exists a least one Nash equilibrium.

Related work and paper outline

Apart from the above mentioned work on opinion diffusion via judgment aggregation [12] and belief merging [18], there is a vast interest in providing formal models of social influence. The most relevant is probably the work of Gosh and Velázquez-

Quesada [10], which does not however consider strategic aspects in their preference update model. The Facebook logic introduced by Seligman *et al.* [19] is also relevant, and motivated our effort in Section 4 to get rid of epistemic operators in the goal language. The difference between private and public information is reminiscent of the work of Christoff and Hansen [6, 7], which also does not focus on strategic aspects. A related problem to opinion diffusion is that of information cascades and knowledge diffusion, which has been given formal treatment in a logical settings [16, 2]. Finally, our work is greatly indebted to the work of [14], since an influence game can be considered as a variation of an iterated boolean game in which individuals do not have direct power on all the variables – there can be several individuals influencing another one – but concurrently participate in its change. Finally, [20] recently presented an extension of iterated boolean games with a social network structure in which agents choose actions depending on the actions of those in their neighbourhood.

The paper is organized as follows. Section 2 presents the basic definition of private and public opinions, as well as our model of opinion diffusion. In Section 3 we present our language for goals based on an epistemic version of LTL, and we show that both the model-checking problem remains in PSPACE (as for LTL), by showing a reduction of the epistemic operator. Section 4 introduces the definition of influence games, and presents the main results about the effects of the network structure on solution concepts such as Nash equilibria and winning strategy, and on the complexity of checking that a given profile of strategies is a Nash equilibrium. Section 5 concludes the paper.

2 Opinion diffusion

In this section we present the model of opinion diffusion which is the starting point of our analysis. We generalise the model of propositional opinion diffusion introduced in related work [12] by separating private and public opinions, and adapting the notion of diffusion through aggregation to this more complex setting.

2.1 Private and public opinions

Let $\mathcal{I} = \{p_1, \ldots, p_m\}$ be a finite set of propositions or *issues* and let $\mathcal{N} = \{1, \ldots, n\}$ be a finite set of individuals or *agents*. Agents have opinions about all issues in $\mathcal{I}$ in the form of a propositional evaluations, or, equivalently, a vector of 0s and 1s:

Definition 1 (private opinion)**.** *The* private opinion *of agent i is a function $B_i : \mathcal{I} \to \{1, 0\}$ where $B_i(p) = 1$ and $B_i(p) = 0$ express, respectively, the agent's opinion that p is true and the agent's opinion that p is false.*

For every $J \subseteq \mathcal{N}$, we denote with $\mathcal{B}_J = \Pi_{i \in J} B_i$ the set of all tuples of opinions of the agents in J. Elements of $\mathcal{B}_J$ are denoted by $\boldsymbol{B}_J$. For notational convenience, we write $\mathcal{B}$ instead of $\mathcal{B}_\mathcal{N}$, and $\mathcal{B}_i$ instead of $\mathcal{B}_{\{i\}}$.

Let $\boldsymbol{B} = (B_1, \ldots, B_n)$ denote the profile composed by the individual opinion of each agent. Propositional evaluations can be used to represent ballots in a multiple referendum, expressions of preference over alternatives, or value judgements over correlated issues (see, e.g., [5, 11]). Depending on the application at hand, an integrity constraint can be introduced to model the propositional correlation among issues. For the sake of simplicity in this paper we do not assume any correlation among the issues, but the setting can easily be adapted to this more general framework.

We also assume that each agent has the possibility of declaring or hiding her private opinion on each of the issues.

Definition 2 (visibility function)**.** *The* visibility function *of agent i is a map $V_i : \mathcal{I} \to \{1, 0\}$ where $V_i(p) = 1$ and $V_i(p) = 0$ express, respectively, the fact that agent i's opinion on p is visible and the fact that agent i's opinion on p is hidden.*

We denote with $\boldsymbol{V} = (V_1, \dots, V_n)$ the profile composed of the agents' visibility functions. By combining the private opinion with the visibility function of an agent we can build her public opinion as a three-valued function on the set of issues.

Definition 3 (public opinion)**.** *Let B_i be agent i's opinion and V_i her visibility function. The* public opinion *induced by B_i and V_i is a function $P_i : \mathcal{I} \to \{1, 0, ?\}$ such that*

$$P_i(p) = \begin{cases} B_i(p) & \textit{if } V_i(p) = 1 \\ ? & \textit{if } V_i(p) = 0 \end{cases}$$

For every $J \subseteq \mathcal{N}$, we denote with $\mathcal{P}_J = \Pi_{i \in J} P_i$ the set of all tuples of public opinions of the agents in J. Elements of $\mathcal{P}_J$ are denoted by P_J. For notational convenience, we write $\mathcal{P}$ instead of $\mathcal{P}_{\mathcal{N}}$, and $\mathcal{P}_i$ instead of $\mathcal{P}_{\{i\}}$.

Once more, $\boldsymbol{P} = (P_1, \dots, P_n)$ denotes the profile of public opinions of all the agents in $\mathcal{N}$. P_i is aimed at capturing the *public* expression of i's view about the issues in $\mathcal{I}$. Observe that an agent can only hide or declare her opinion about a given issue, but is not allowed to lie. Relaxing this assumption would actually represent an interesting direction for future work.

2.2 Information states

The information contained in a profile of public opinions can also be modelled using a state-based representation and an indistinguishability relation, in line with the existing work on interpreted systems (see, e.g., [8]). States will form the building blocks of our model of strategic reasoning in opinion dynamics.

Definition 4 (state)**.** *A* state *is a tuple $S = (\boldsymbol{B}, \boldsymbol{V})$ where $\boldsymbol{B}$ is a profile of private opinions and $\boldsymbol{V}$ is a profile of visibility functions. The set of all states is denoted by $\mathcal{S}$.*

The following definition formalises the uncertainty between states induced by the visibility functions. The idea is that an agent cannot distinguish between two states if and only if both the agent's individual opinion and the other agents' public opinions are the same according to the two states.

Definition 5 (Indistinguishability)**.** *Let $S, S' \in \mathcal{S}$ be two states. We say that agent i cannot distinguish between them, denoted by $S \sim_i S'$, if and only if:*

- $B_i = B'_i$,

- $\boldsymbol{V} = \boldsymbol{V}'$ *and*
- *for all* $j \in \mathcal{N} \setminus \{i\}$ *and for all* $p \in \mathcal{I}$, *if* $V_j(p) = 1$ *then* $B_j(p) = B'_j(p)$.

Let $S^{\sim_i} = \{S' \in \mathcal{S} \mid S \sim_i S'\}$ be the set of states that agent i cannot distinguish from S. Clearly $\sim_i$ is an equivalence relation. In what follows we will often use public states to represent the equivalence class of a state. Observe however that this is a too coarse representation, since each of the agents knows her own belief.

Example 1. *Let there be three agents* i, j *and* k, *and one issue* p. *Assume that agents* j *and* k *consider* p *as true as private opinion while* i *private opinion is* $p = 0$. *Suppose also that agents* i *and* j *make their opinion public while agent* k *does not. From the perspective of agent* i, *the two states* $S_0 = ((0, 1, 0), (1, 1, 0))$ *and* $S_1 = ((0, 1, 1), (1, 1, 0))$ *are indistinguishable, representing the two possible opinions of agent* k *who is hiding it. Figure 1 represents the perspective of agent* i.

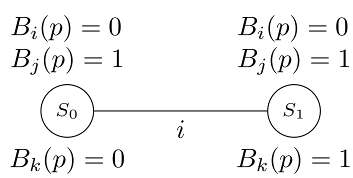

Figure 1: Indistinguishable states for i as k hides p.

2.3 Opinion diffusion through aggregation

In this section we define the influence process that is at the heart of our model. Our definition is a generalisation of the model by [12].

First, we assume that individuals are connected by an *influence network* which we model as a directed graph:

Definition 6 (influence network)**.** *An* influence network *is a directed irreflexive graph* $E \subseteq \mathcal{N} \times \mathcal{N}$. *We interpret* $(i, j) \in E$ *as "agent* j *is influenced by agent* i*".*

We also refer to E as the influence graph and to individuals in $\mathcal{N}$ as the nodes of the graph. Let $Inf(j) = \{i \in N \mid (i, j) \in E\}$ be the set of *influencers* of agent j in the network E. Given a state S, this definition can be refined by considering the set $Inf^S(i, p) = \{j \in N \mid (j, i) \in E$ and $P_j(p) \neq ?\}$ to be the subset of influencers that are actually expressing their private opinion about issue p. Clearly, $Inf^S(i, p) \subseteq Inf(i)$ for all p and S.

Example 2. *Figure 2 represents a basic influence network where some agent* i *is influenced by two agents* j *and* k. *The set* $Inf(i) = \{j, k\}$, *and, using the notation in the previous example, the set* $Inf^{S_0}(i, p) = Inf^{S_1}(i, p) = \{j\}$.

$$j \longrightarrow i \longleftarrow k$$

Figure 2: i influences by j and k.

Given a profile of public opinions and an influence network E, we model the process of opinion diffusion by means of an aggregation function, which shapes the private opinion of an agent by taking into consideration the public opinions of her influencers.

Definition 7 (Aggregation procedure)**.** *An* aggregation procedure *for agent i is a class of functions*

$$F_i : \mathcal{B} \times \mathcal{P}_J \longrightarrow \mathcal{B} \textit{ for each } J \subseteq \mathcal{N} \setminus \{i\}$$

that maps agent i's individual opinion and the public opinions of a set of agents J to agent i's individual opinion.

Aggregation functions are used to construct the new private opinion of an agent in the dynamic process of opinion diffusion. Thus, $F_i(B_i, \boldsymbol{P}_{Inf(i)})$ represents the private opinion of agent i updated with the public opinions received by its influencers.

A number of aggregation procedures have been considered in the literature on judgment aggregation and can be adapted to our setting. Notable examples are quota rules, where an agent changes her opinion if the amount of people disagreeing with her is superior of a given quota (the majority rule is such an example). These aggregation procedures give rise to the class of threshold models studied in the literature on opinion diffusion [13, 17].

For the sake of simplicity in this paper we consider that all agents use the following aggregation procedure:

Definition 8. *Let $S = (\boldsymbol{B}, \boldsymbol{V})$ be a state and $\boldsymbol{P}$ the corresponding profile of public opinions. The* unanimous issue-by-issue aggregation procedure *is defined as follows:*

$$F_i^U(B_i, \boldsymbol{P})(p) = \begin{cases} B_i(p) & \textit{if } Inf^S(i,p) = \emptyset \\ x \in \{0,1\} & \textit{if } P_j(p) = x \ \ \forall j \in Inf^S(i,p) \\ B_i(p) & \textit{otherwise} \end{cases}$$

That is, an individual will change her private opinion about issue p if and only if all her influencers that are expressing their opinion publicly are unanimous in disagreeing with her own one.

2.4 Strategic actions and state transitions

Showing or hiding information is a key action in the model of opinion diffusion defined above. The dynamic of opinion is rooted in two dimensions: the influence network and the visibility function. At each time step, by hiding or revealing their opinions, agents influence other agents opinions. We assume that agents can make their opinions visible or invisible by specific actions of type $\mathsf{reveal}(p)$ (i.e., action of making the opinion about p visible) and $\mathsf{hide}(p)$ (i.e., action of hiding the opinion about p). The action of doing nothing is denoted by skip. Let therefore

$$\mathcal{A} = \{\mathsf{reveal}(p) : p \in \mathcal{I}\} \cup \{\mathsf{hide}(p) : p \in \mathcal{I}\} \cup \{\mathsf{skip}\}$$

be the set of all individual actions and $\mathcal{J} = \mathcal{A}^n$ the set of all joint actions. Elements of $\mathcal{J}$ are denoted by $\boldsymbol{a} = (a_1, \ldots, a_n)$.

Each joint action $\boldsymbol{a}$ induces a transition function between states. This function is deterministic and is defined as follows:

Definition 9 (transition function)**.** *The* transition function $succ : \mathcal{S} \times \mathcal{J} \longrightarrow \mathcal{S}$ *associates to each state S and joint action $\boldsymbol{a}$ a new state $S' = (\boldsymbol{B}', \boldsymbol{V}')$ where, for all $i \in \mathcal{N}$:*

- $V_i'(p) = \begin{cases} 1 & \textit{if } a_i = \mathsf{reveal}(p) \textit{ or} \\ 0 & \textit{if } a_i = \mathsf{hide}(p) \textit{ or} \\ V_i(p) & \textit{if } a_i = \mathsf{skip} \end{cases}$
- $B_i' = F_i^U(B_i, \boldsymbol{P'}_{Inf(i)})$

Where $\boldsymbol{P'}$ is the public profile obtained from private profile $\boldsymbol{B}$ and visibility functions $\boldsymbol{V'}$.

By a slight abuse of notation we denote with $\boldsymbol{a}(S)$ the state $succ(S, \boldsymbol{a})$ obtained from S and $\boldsymbol{a}$ by applying the transition function. Observe that in our definition of transition function we are assuming that the influence process occurs after that the actions have modified the visibility of the agents' opinions. Specifically, first, actions have consequences on the visibility of the agents' opinions, then, each agent modifies her private opinions on the basis of those opinions of other agents that have become public.

We are now ready to define the concept of *history*, describing the temporal aspect of agents' opinion dynamic:

Definition 10 (history)**.** *Given a set of issues $\mathcal{I}$, a set of agents $\mathcal{N}$, and aggregation procedures F_i over a network E, an* history *is an infinite sequence of states $H = (H_0, H_1, \ldots)$. such that for all $t \in \mathbb{N}$ there exists a joint action $\boldsymbol{a}_t \in \mathcal{J}$ such that $H_{t+1} = \boldsymbol{a}_t(H_n)$.*

Let $H = (H_0, H_1, \ldots)$ be an history. For notational convenience, for any $i \in \mathcal{N}$ and for any $t \in \mathbb{N}$, we denote with $H_{i,t}^B$ agent i's private opinion in state H_t and with $H_{i,t}^V$ agent i's visibility function in state H_t.

The set of all histories is denoted by $\mathcal{H}$. Observe that our definition restricts the set of all possible histories to those that corresponds to a run of the influence dynamic described above.

Example 3. *Let us reconsider the two previous examples, with initial state $H_0 = S_0$. Consider now the following joint actions $\boldsymbol{a}_0 = (\mathsf{skip}, \mathsf{skip}, \mathsf{reveal}(p))$ and $\boldsymbol{a}_1 = (\mathsf{skip}, \mathsf{hide}(p), \mathsf{skip})$: agent k reveals her opinion, and at the next step j hides her* *opinion about p. If we assume that all individuals are using the unanimous aggregation procedure then Figure 3 shows the two states H_1 and H_2 constructed by applying the two joint actions from state S_0. In state H_1, agent i's private opinion about p has changed, i.e., $H^B_{i,1}(p) = 1$ as all her influencers are publicly unanimous about p. At the next step, instead, no opinion is updated.*

$$\underset{H_0}{((0,1,1),(1,1,0))} \xrightarrow[\boldsymbol{a}_0]{} \underset{H_1}{((1,1,1),(1,1,1))} \xrightarrow[\boldsymbol{a}_1]{} \underset{H_2}{((1,1,1),(1,0,1))}$$

Figure 3: The initial two states of a history.

3 Temporal and Epistemic Goals

As agents can hide or reveal their opinions, the strategic dimension of opinion diffusion is immediate: by revealing/hiding her opinion, an agent influences other agents. Influence is actually guided by some underlying goals. An agent reveals/hides her opinion only if she wants to influence some other agents: she aims at changing individual opinions. Temporal dimension is immediate as the dynamics of opinion has to be considered. Following [14], we define a language to express individual epistemic goals about the state of the individual opinions.

3.1 Epistemic temporal logic of influence

Let us introduce a logical language based on a combination of simple version of multi-agent epistemic logic and linear temporal logic (LTL) that can be interpreted over histories. In line with our framework, term *epistemic state* should be interpreted as *private opinion*. Goals in our perspective consists of targeting an epistemic state: typically ''agent i wants that agent j has private opinion φ in the future". The proposed language does not allow the temporal operator to be in the scope of a knowledge operator, obtaining a simpler language and a reduction that allows us to stay in the same complexity class as LTL.

We call ELTL–I this logic, from epistemic linear temporal logic of influence. Its language, denoted by $\mathcal{L}_{\mathsf{ELTL\text{–}I}}$, is defined by the following BNF:

$$\begin{array}{lcl}\alpha & ::= & \mathsf{op}(i,p) \mid \mathsf{vis}(i,p) \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \mathsf{K}_i\alpha \\ \varphi & ::= & \alpha \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{X}\varphi \mid \varphi_1 \mathsf{U} \varphi_2\end{array}$$

where i ranges over $\mathcal{N}$ and p ranges over $\mathcal{I}$. $\mathsf{op}(i,p)$ has to be read "agent i's opinion is that p is true" while $\neg\mathsf{op}(i,p)$ has to be read "agent i's opinion is that p is not true" (since we assume that agents have binary opinions). $\mathsf{vis}(i,p)$ has to be read "agent i's opinion about p is visible". Finally, $\mathsf{K}_i\alpha$ has to be read "agent i knows that α is true".

$\mathsf{X}\varphi$ and U are the standard LTL operators 'next' and 'until'. In particular, $\mathsf{X}\varphi$ has to be read "φ is going to be true in the next state" and $\varphi_1\mathsf{U}\varphi_2$ has to be read "φ_1 will be true until φ_2 is true". As usual, we can define the temporal operators 'henceforth' (G) and 'eventually' (F) by means of the 'until' operator:

$$\mathsf{G}\varphi =_{def} \neg(\top \mathsf{U} \neg\varphi)$$
$$\mathsf{F}\varphi =_{def} \neg\mathsf{G}\neg\varphi$$

The interpretation of $\mathcal{L}_{\mathsf{ELTL\text{–}I}}$-formulas relative to histories is defined as follows.

Definition 11 (Truth conditions)**.** *Let φ be a $\mathcal{L}_{\mathsf{ELTL\text{–}I}}$-formula, let H be a history and let $k \in \mathbb{N}$. Then:*

$$\begin{array}{rcl}
H,k \models \mathsf{op}(i,p) & \Leftrightarrow & H^B_{i,k}(p) = 1 \\
H,k \models \mathsf{vis}(i,p) & \Leftrightarrow & H^V_{i,k}(p) = 1 \\
H,k \models \mathsf{K}_i\alpha & \Leftrightarrow & \forall H' \in \mathcal{H} : \textit{ if } H(k) \sim_i H'(k) \textit{ then} \\
& & H',k \models \alpha \\
H,k \models \neg\varphi & \Leftrightarrow & H,k \not\models \varphi \\
H,k \models \varphi_1 \wedge \varphi_2 & \Leftrightarrow & H,k \models \varphi_1 \textit{ and } H,k \models \varphi_2 \\
H,k \models \mathsf{X}\varphi & \Leftrightarrow & H,k+1 \models \varphi \\
H,k \models \varphi_1 \mathsf{U} \varphi_2 & \Leftrightarrow & \exists k' \in \mathbb{N} : (k \leq k' \textit{ and } H,k' \models \varphi_2 \textit{ and} \\
& & \forall k'' \in \mathbb{N} : \textit{ if } k \leq k'' < k' \textit{ then } H,k'' \models \varphi_1)
\end{array}$$

The operator K_i is rather peculiar, and should not be interpreted as a classical individual epistemic operator. It mixes public and private opinions of our model. Operator K_i reading is rather "agent i is uncertain about other agents private opinion as this opinion is not visible" and $\mathsf{K}_i\alpha$ stands for agent i knows α despite this uncertainty.

The following proposition shows that K_i could also be formulated in terms of equivalence between histories rather than in terms of equivalence between states. Its proof is immediate from our definitions.

Proposition 1. *Let $H \in \mathcal{H}$ and $k \in \mathbb{N}$. Then*

$$H,k \models \mathsf{K}_i\alpha \quad \textit{iff} \quad \forall H' \in \mathcal{H} : \textit{ if } H \sim_i H' \textit{ then } H',k \models \alpha$$

where $H \sim_i H'$ iff $H(h) \sim_i H'(h)$ for all $h \in \mathbb{N}$.

Example 4. *Consider Figure 3, the following statement expresses that in state H_0, it is the case that in the future agent k knows agent i public opinion about p (as $H_1^{\sim_k} = \{H_1\}$):*

$$H,0 \models \mathsf{F}(\mathsf{K}_k(\mathsf{op}(i,p) \wedge \mathsf{vis}(i,p)))$$

This example also shows how each ELTL–I statements can be used for representing individual goals. Hence, gathering individual goals lead to the construction of a boolean game. Before detailing this aspect we conclude the section by exhibiting the key results about model checking for the ELTL–I logic.

3.2 Model checking

The aim of this section is to show that model checking for $\mathsf{ELTL{-}I}$ is as hard as model checking for LTL. Recall that epistemic temporal logic has very high complexity [8]. We do so by reducing formulas containing an epistemic modality to propositional ones.

Lemma 5. *The following formulas are valid in* $\mathsf{ELTL{-}I}$*:*

(i) $\mathsf{K}_i\mathsf{op}(i,p) \leftrightarrow \mathsf{op}(i,p)$

(ii) $\mathsf{K}_i\mathsf{op}(j,p) \leftrightarrow (\mathsf{op}(j,p) \wedge \mathsf{vis}(j,p))$ *if* $i \neq j$

(iii) $\mathsf{K}_i\mathsf{vis}(j,p) \leftrightarrow \mathsf{vis}(j,p)$ *for all* j

(iv) $\mathsf{K}_i\neg\mathsf{op}(i,p) \leftrightarrow \neg\mathsf{op}(i,p)$

(v) $\mathsf{K}_i\neg\mathsf{op}(j,p) \leftrightarrow (\neg\mathsf{op}(j,p) \wedge \mathsf{vis}(j,p))$ *if* $i \neq j$

(vi) $\mathsf{K}_i\neg\mathsf{vis}(j,p) \leftrightarrow \neg\mathsf{vis}(j,p)$ *for all* j

sketch. Straightforward from Definition 5 and the interpretation of the K_ip operator. To show the right-to-left direction of (ii) and (v), suppose that $\mathsf{op}(j,p)$ is true at every indistinguishable state for $\sim_i$, i.e., that $\mathsf{K}_i\mathsf{op}(j,p)$ is true. This implies that $\mathsf{op}(j,p)$ is true in the current state. Moreover, if $\mathsf{vis}(j,p)$ is false, then by Definition 5 there would be an indistinguishable state in which $\mathsf{op}(j,p)$ is false, contradicting the hypothesis. □

We now show two distribution laws for the operator K_i with respect to conjunction and negation:

Lemma 6. *The following formulas are valid in* $\mathsf{ELTL{-}I}$*:*

(i) $\mathsf{K}_i(\alpha_1 \wedge \alpha_2) \leftrightarrow (\mathsf{K}_i\alpha_1 \wedge \mathsf{K}_i\alpha_2)$ *for all* α_1 *and* α_2*;*

(ii) $\mathsf{K}_i(\alpha_1 \vee \alpha_2) \leftrightarrow (\mathsf{K}_i\alpha_1 \vee \mathsf{K}_i\alpha_2)$ *when* α_1 *and* α_2 *do not contain any occurrence of the modality* K_j *for any* j*.*

sketch. (i) and the right-to-left directions of (ii) are standard consequences of interpreting K_ip over equivalence relations. We now prove the left-to-right direction of (ii) by induction on the construction of a propositional formula.

Assume some history H and state S and suppose that the left part holds. This means that in all states $\in S^{\sim_i}$, either α_1 or α_2 is true. If both statements contains $\mathsf{vis}(j,p)$ or $\neg\mathsf{vis}(j,p)$ then Definition 5 entails that right-to-left direction hold, since either α_1 holds in all states $\in S^{\sim_i}$ or α_2 holds in all states $\in S^{\sim_i}$. If α_1 (respectively α_2) contains some statement $\mathsf{op}(i,p)$, then the right-to-left direction holds as α_1 either holds in all states $\in S^{\sim_i}$ or is false in all states. If it does not hold then α_2 is considered in a similar way to α_1. Now suppose that α_1 and α_2 only contain statements of the form $\mathsf{op}(j,p)$ s.t. it is always the case that $j \neq i$. Then there must exist j,p such that $\mathsf{vis}(j,p) = 1$. Otherwise, there exists a state $S' \in S^{\sim_i}$ such that neither α_1 or α_2 hold (as all possible indistinguishable states must be considered). In conclusion, the right-to-left direction holds as either α_1 (respectively α_2) either holds in all states $\in S^{\sim_i}$ or it does not hold in all states. □

Finally, we reduce the nesting of the K_ip operator:

Lemma 7. *The following formulas are valid in* $\mathsf{ELTL{-}I}$*:*

(i) $\mathsf{K}_i\mathsf{K}_i\alpha \leftrightarrow \mathsf{K}_i\alpha$

(ii) $\mathsf{K}_i\mathsf{K}_j\mathsf{K}_i\alpha \leftrightarrow \mathsf{K}_j\mathsf{K}_i\alpha$ *for all* $i \neq j$*, when* α *do not contain any occurrence of the modality* K_j *for any* j*.*

sketch. The proof of (i) is a standard consequence of using equivalence relations. To prove (ii), let α be a propositional formula. Put first α in CNF and then distribute by Lemma 6 the modalities over conjunction. We now prove that $\mathsf{K}_i\mathsf{K}_j\mathsf{K}_i\ell \leftrightarrow \mathsf{K}_j\mathsf{K}_i\ell$ where ℓ is a literal. If $\ell = (\neg)\mathsf{vis}(j,p)$, then by Lemma 5 both sides of the equivalence reduce to $(\neg)\mathsf{vis}(j,p)$. Suppose that $\ell = (\neg)\mathsf{op}(k,p)$ for some $k \in \mathcal{N}$. If $k \neq i, j$ then by Lemma 5 we can reduce $\mathsf{K}_i\ell$ to $\ell \wedge \mathsf{vis}(k,p)$, and then it is straightforward to conclude by observing that $\mathsf{K}_i\ell \wedge \mathsf{vis}(k,p) \leftrightarrow \ell \wedge \mathsf{vis}(k,p) \wedge \mathsf{vis}(k,p)$ which in turn is equivalent to $\ell \wedge \mathsf{vis}(k,p)$. The case of $k = i$ and $k = j$ is similar. □

We are now ready to present an algorithm to translate a formula of ELTL–I into an equivalent one without any occurrence of the K_i operator. Let an epistemic literal be a formula of the form $(\neg)\mathsf{K}_i\ell$ for $i \in \mathcal{N}$ and propositional literal ℓ.

Input: a formula $\varphi \in \mathcal{L}_{\mathsf{ELTL–I}}$
Output: a formula red$(\varphi) \in \mathcal{L}_{\mathsf{ELTL–I}}$ with no epistemic operators

while *there is an epistemic operator in* φ *outside an epistemic literal* **do**
  1. choose a subformula $\mathsf{K}_i\alpha$ of φ such that α is without epistemic operators and is not a propositional literal;
  2. put α in negated normal form (NNF);
  3. distribute K_i over $\wedge$ and $\vee$;
end
4. Reduce the depth modalities with Lemma 7;
5. Reduce the atoms with Lemma 5 ;

Algorithm 1: Reduction of the epistemic operator

The following proposition guarantees that the translation defined is also polynomial.

Lemma 8. *Algorithm 1 terminates,* $red(\varphi)$ *is polynomial in the size of* φ*, does not contain epistemic operators, and is equivalent to* φ*.*

Proof. Lines 3, 4 and 5 apply equivalences that are valid by Lemmas 5, 6 and 7 and the fact that the following rule of replacement of equivalents is admissible in ELTL–I:

$$\frac{\psi_1 \leftrightarrow \psi_2}{\varphi \leftrightarrow \varphi[\psi_1/\psi_2]}$$

The negated normal form of a propositional formula (treating epistemic literals as propositional literals) is constructed by propositional equivalence and is polynomial in the size of the initial formula. Finally, since the modal depth is limited by Lemma 7 to $|N|$, we add a maximum of $|N|$ extra variables of the form $\mathsf{vis}(j,p)$ for some j and some p to the translation of each epistemic literal. □

Using Lemma 8, we are able to polynomially reduce every formula of the ELTL–I to an equivalent formula of the fragment LTL–I whose language $\mathcal{L}_{\mathsf{LTL–I}}$ is defined by the following BNF:

$$\varphi \quad ::= \quad \mathsf{op}(i,p) \mid \mathsf{vis}(i,p) \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{X}\varphi \mid \varphi_1 \mathsf{U} \varphi_2$$

where i ranges over $\mathcal{N}$ and p ranges over $\mathcal{I}$. We first show the following:

Proposition 2. *The model checking problem of* LTL–I *is PSPACE-complete.*

Proof. To verify membership it is sufficient to note that LTL–I is a special instance of LTL built out of the finite set of atomic propositions $\{\mathsf{op}(i,p) : i \in \mathcal{N} \text{ and } p \in \mathcal{I}\} \cup \{\mathsf{vis}(i,p) : i \in \mathcal{N} \text{ and } p \in \mathcal{I}\}$ and interpreted over a subset of the set of all possible histories for this language. Since the model checking problem for LTL is in PSPACE [21], the model checking problem for LTL–I should also be in PSPACE.

To check that model checking of LTL–I is PSPACE-hard we are going to consider the following fragment of $\mathcal{L}_{\mathsf{LTL–I}}$:

$$\varphi \quad ::= \quad \mathsf{vis}(i,p) \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{X}\varphi \mid \varphi_1 \mathsf{U} \varphi_2$$

where i ranges over $\mathcal{N}$ and p ranges over $\mathcal{I}$.

It is straightforward to check that the set of histories $\mathcal{H}$ includes all possible interpretations for this language which is nothing but the LTL language built out of the finite set of atomic propositions $\{\mathsf{vis}(i,p) : i \in \mathcal{N} \text{ and } p \in \mathcal{I}\}$. Since the model checking for LTL is is known to be PSPACE-hard [21], it follows that the model checking for LTL–I is PSPACE-hard too. □

We can now state the following theorem about complexity of model checking for ELTL–I.

Theorem 1. *The model checking problem of* ELTL–I *is PSPACE-complete.*

Proof. By Proposition 2 we know that model checking for LTL–I is PSPACE-complete. Every formula of ELTL–I can be reduced to an equivalent formula of polynomial size in LTL–I by Lemma 8, showing membership in PSPACE of model checking for ELTL–I. Since LTL–I is a sublogic of ELTL–I we also obtain PSPACE-hardness. □

4 Games of influence

We are now ready to put together all the definitions introduced in the previous sections and give the following definition:

Definition 12 (Influence game)**.** *An influence game is a tuple* $IG = (\mathcal{N}, \mathcal{I}, E, F_i, S_0, \gamma_1, \ldots, \gamma_n)$ *where* $\mathcal{N}$, $\mathcal{I}$, E *and* S_0 *are, respectively, a set of agents, a set of issues, an influence network, and an initial state,* F_i *are aggregation procedures, one for each agent, and* $\gamma_i \in \mathcal{L}_{\mathsf{ELTL–I}}$ *is agent* i*'s goal.*

4.1 Strategies

The following definition introduces the concept of strategy. The standard definition would call for a function that assigns in each point in time an action to each player. We choose to study simpler state-based strategies:

Definition 13 (strategy)**.** *A strategy for player i is a function that associates an action to every information state, i.e., $Q_i : \mathcal{S} \to \mathcal{A}$ such that $Q_i(S) = Q_i(S')$ whenever $S \sim_i S'$. A strategy profile is a tuple $\boldsymbol{Q} = (Q_1, \ldots, Q_n)$.*

For notational convenience, we interchangeably use $\boldsymbol{Q}$ to denote a strategy profile $(Q_1, \ldots, Q_n)$ and the function $\boldsymbol{Q} : \mathcal{S} \longrightarrow \mathcal{J}$ such that $\boldsymbol{Q}(S) = \boldsymbol{a}$ if and only if $Q_i(S) = a_i$, for all $S \in \mathcal{S}$ and $i \in \mathcal{N}$. As the following definition highlights, every strategy profile $\boldsymbol{Q}$ combined with an initial state S_0 induces a history:

Definition 14 (Induced history)**.** *Let S_0 be an initial state and let $\boldsymbol{Q}$ be a strategy profile. The history $H_{S_0,\boldsymbol{Q}} \in \mathcal{H}$ induced by them is defined as follows:*

$$\begin{aligned} H_0(S_0, \boldsymbol{Q}) &= S_0 \\ H_{n+1}(S_0, \boldsymbol{Q}) &= succ(S_n, \boldsymbol{Q}(S_n)) \textit{ for all } n \in \mathbb{N} \end{aligned}$$

4.2 Solution concepts

We start with the concept of winning uniform strategy. Intuitively speaking, Q_i is a winning uniform strategy for player i if and only if i knows that, by playing this strategy, she will achieve her goal no matter what the other players will decide to do.

Definition 15 (Winning strategy)**.** *Let IG be an influence game and let Q_i be a strategy for player i. We say that Q_i is a winning strategy for player i if and only if*

$$H_{S_0,(Q_i,\boldsymbol{Q}_{-i})} \models \gamma_i \tag{1}$$

for all profiles $\boldsymbol{Q}_{-i}$ of strategies of players other than i. A winning strategy is called uniform *if (1) is true for all states $S \in S_0^{\sim_i}$.*

Observe that a winning strategy is not necessarily winning uniform, as the private state of an agent is not necessarily accessible to the other players.

Example 9. *Let Ann, Bob and Jesse be three agents. Let p be an issue, and suppose that $B_{Ann}(p) = 1$, $B_{Bob}(p) = 0$, $B_{Jesse}(p) = 0$. Ann influences Bob and Jesse, while Bob influences Jesse as shown in the following picture:*

$$\begin{tikzcd} Jesse & \\ & Bob \arrow[ul] \\ Ann \arrow[uu] \arrow[ur] & \end{tikzcd}$$

Suppose that the goal of Ann is $\mathsf{FGop}(Jesse, p)$. Her winning (uniform) strategy is $\mathsf{reveal}(p)$ in all states: Bob will be influenced to believe p in the second stage, and subsequently Jesse will also do so, since her influencers are unanimous (even if Bob plays $\mathsf{hide}(p)$).

As we will show in the following section, the concept of winning strategy is too strong for our setting. Let us then define the less demanding notion of weak dominance:

Definition 16. *Let IG be an influence game and let Q_i be a strategy for player i. We say that Q_i is a* weakly dominant strategy *for player i and initial state S_0 if and only if for all profiles $\boldsymbol{Q}_{-i}$ of strategies of players other than i and for all strategies Q_i' we have:*

$$H_{S_0,(Q_i',\boldsymbol{Q}_{-i})} \models \gamma_i \Rightarrow H_{S_0,(Q_i,\boldsymbol{Q}_{-i})} \models \gamma_i \tag{2}$$

A weakly dominant strategy is called uniform *if (2) is true for all initial states $S \in S_0^{\sim i}$.*

Example 10. *Let us go back to the previous example and suppose now that Ann still believes p, but does not influence Jesse any longer. In this case, Ann does not have a winning strategy: if neither Bob nor Jesse do not believe p, it is sufficient for Bob to play* $\mathsf{reveal}(p)$ *to make sure that she will never satisfy her goal. However, the strategy* $\mathsf{reveal}(p)$ *is a weakly dominant strategy for Ann.*

Now let us consider the following concept of best response. Intuitively speaking, Q_i is a best response to $\boldsymbol{Q}_{-i}$ if and only if player i knows that the worst she could possibly get by playing Q_i, when the others play $\boldsymbol{Q}_{-i}$, is better or equal to the worst she could possibly get by playing a strategy different from Q_i.

Definition 17 (Best response)**.** *Let IG be an influence game, let $Q_i \in \mathcal{Q}_i$ and let $\boldsymbol{Q}_{-i} \in \mathcal{Q}_{-i}$. We say that Q_i is a best response to $\boldsymbol{Q}_{-i}$ wrt. initial state S_0 if and only if for all $Q_i' \in \mathcal{Q}_i$:*

$$(\forall S \in S_0^{\sim i}\ (H_{S,(Q_i,\boldsymbol{Q}_{-i})} \models \gamma_i) \text{ or } (\exists S, S' \in S_0^{\sim i}\ H_{S,(Q_i,\boldsymbol{Q}_{-i})} \not\models \gamma_i \text{ and } H_{S',(Q_i',\boldsymbol{Q}_{-i})} \not\models \gamma_i))$$

If we rephrase this definition through some utility notion, then we can consider a fictitious utility $U_i(S) = 1$ for states S that satisfy the goal of agent i, and $U_i(S) = 0$ for states where the goal is not satisfied. In that case, our definition of best response corresponds to $\min_{S \in S_0^{\sim i}} U_i(H_{S,(Q_i,\boldsymbol{Q}_{-i})}) \geq \min_{S \in S_0^{\sim i}} U_i(H_{S,(Q_i',\boldsymbol{Q}_{-i})})$. This definition is justified on the basis of the prudential criterion according to which, if an agent does not have a probability distribution over the set of possible states, she should focus on the worst possible outcome and choose the action whose worst possible outcome is at least as good as the worst possible outcome of any other actions (see, e.g., [22, 3, 4]).

Definition 17 allow us to define the concept of Nash equilibrium for games with incomplete information such as influence games:

Definition 18 (Nash equilibrium)**.** *Let IG be an influence game and let $\boldsymbol{Q}$ be a strategy profile. $\boldsymbol{Q}$ is a Nash equilibrium if and only if, for all $i \in \mathcal{N}$, Q_i is a best response to $\boldsymbol{Q}_{-i}$.*

4.3 Influence network and solution concepts

In this section we show some preliminary results about the interplay between the network structure and the existence of solutions concepts. In what follows we only consider influence games where the aggregation function is the unanimous one (see Definition 8). In the interest of space, most proofs will only be sketched. Let us first give the following:

Definition 19. *A goal γ is coherent with an initial state S_0 in game IG if and only if there exists a strategy profile Q inducing history H such that $H_{S_0,Q} \models \gamma$.*

Clearly, if γ_i is not coherent with initial state S_0, then all strategies for player i are equivalent. The following lemma shows that visibility goals cannot be enforced by means of a winning strategy:

Proposition 3. *If γ_i entails one formula of the form $\mathsf{vis}(j,p)$, $\mathsf{K}_j\mathsf{op}(i,p)$, or $\mathsf{Xvis}(j,p)$ for $j \neq i$, with belief and visibility atoms eventually negated, then i does not have a winning strategy.*

To see this, consider that if an individual goal γ_i concerns the visibility of another agent about a given issue p, then this second agent can always respond $\mathsf{hide}(p)$ and make sure that γ_i is false.

Let us now introduce a simpler language for goals, in order to study the limitations of considering winning strategies in this setting. Let $\mathsf{LTL\text{–}I}_{\mathcal{J}}$ be the language of future goals about a subset of agent $\mathcal{J} \subseteq \mathcal{N}$, which focuses on the future opinions of agents in $\mathcal{J}$ without considering the visibility. This language is defined by the following BNF:

$$\begin{array}{rcl} \alpha & ::= & \mathsf{op}(j,p) \mid \neg\mathsf{op}(j,p) \mid \alpha \wedge \alpha \\ \varphi & ::= & \mathsf{X}\alpha \mid \mathsf{X}\varphi \mid \mathsf{G}\varphi \mid \mathsf{F}\varphi \end{array}$$

where $j \in \mathcal{J}$ and p ranges over $\mathcal{I}$.

Let us introduce some further notation. If $i, j \in \mathcal{N}$ we say that i *controls* j if either $\mathit{Inf}(j) = \{j\}$, or for all paths $l_1, \ldots, l_n$ such that $(l_i, l_{i+1}) \in E$, $l_1 = i$ and $l_n = j$, we have that i controls each l_k. We can now prove the following:

Proposition 4. *If $\gamma_i \in \mathsf{LTL\text{–}I}_{\mathcal{J}}$ then i has a winning strategy for all initial states S_0 if and only if i controls j for all $j \in \mathcal{J}$.*

Proof. One direction is easier: if i controls agent j, then her winning strategy is to always play $\mathsf{reveal}(p)$ in case her goal is consistent with her opinion, e.g. if her goal is $\mathsf{op}(j,p)$ and $H^B_{i,0}(p) = 1$. Otherwise always playing $\mathsf{hide}(p)$ guarantees that her goal will be satisfied. Note that the consistency of γ_i is crucial here.

For the other direction, consider a network in which j has more than one influencer, say k, which is however not controlled by i. A simple case study shows that there always exists an initial state in which i does not have a winning strategy. For instance, if $\gamma_i = \mathsf{Xop}(j,p)$, then in an initial state in which $B_j(p) = 0$ it is sufficient for agent k to play $\mathsf{reveal}(p)$ to make sure that agent j never updates her belief, and hence that γ_i will not be satisfied. $\square$

Proposition 4 shows that the concept of winning strategy is too strong in influence games, as it can only be applied in situations in which an agent has exclusive control over the opinion of another.

Let us now focus on a particular influence game, in which the agents' goal is to reach a consensus about p. That is, each agent i adopts the following goal:

$$\gamma_i^+ =_{def} \mathsf{FX}\left(\bigwedge_{j\neq i} \mathsf{op}(j,p)\right) \qquad \gamma_i^- =_{def} \mathsf{FX}\left(\bigwedge_{j\neq i} \neg\mathsf{op}(j,p)\right)$$

Consensus means that the conjunction of each individual goal leads to a state where all agents have p as opinion.

Theorem 2. *If all agent i has the goal of consensus represented by γ_i^+ (respectively γ_i^-), and the network E is fully connected, then for any initial state S_0 there always exists a Nash equilibrium $\widehat{\boldsymbol{Q}}$ such that $H_{S_0,\widehat{\boldsymbol{Q}}} \models \bigwedge_{i\in\mathcal{N}} \gamma_i^+$ (respectively, γ_i^-). Moreover, each strategy is weakly dominant.*

Proof. Take an initial state $S_0 = (\boldsymbol{B}, \boldsymbol{V})$, and assume that all individuals have goal γ_i^+. Consider the four possible states about p for agent i: either p is true or false, and p is visible or not. Assume $B_i(p) = 1$, then, regardless of visibility, we show that strategy $\mathsf{reveal}(p)$ is weakly dominant. For all $S \in S_0^{\sim_i}$, either $H_{S,(\mathsf{reveal}(p),Q_{-i})} \models \bigwedge_{j\neq i} \mathsf{op}(j,p)$ or $H_{S,(\mathsf{reveal}(p),Q_{-i})} \not\models \bigwedge_{j\neq i} \mathsf{op}(j,p)$; for that former case, definition of unanimity entails that $H_{S,(\mathsf{skip},Q_{-i})} \not\models \bigwedge_{j\neq i} \mathsf{op}(j,p)$ and $H_{S,(\mathsf{hide}(p),Q_{-i})} \not\models \bigwedge_{j\neq i} \mathsf{op}(j,p)$. In a similar way, we can show that strategy $\mathsf{hide}(p)$ is a best response to any strategy $\boldsymbol{Q_{-i}}$ if $B_i(p) = 0$. We then built up a strategy profile $\widehat{\boldsymbol{Q}}$ w.r.t. B_i: $Q_i = \mathsf{reveal}(p)$ if $B_i(p) = 1$ otherwise $Q_i = \mathsf{hide}(p)$, this strategy profile is a Nash equilibrium in weakly dominant strategies. ☐

Observe that if the network E is not fully connected this result does not hold since, for instance, the opinion of an isolated agent cannot be changed. Let us now focus on specific shapes of the influence graph where weakly dominant strategy exists. Those strategies concern the sources the agents who have no influencers.

Theorem 3. *If E is acyclic, I^* is the set of sources, and S_0 is coherent with $\bigwedge_{i^*\in I^*} \gamma_i$, and each source* influences only one individual, *then for all i^* there exists a weakly dominant strategy.*

sketch. If $\bigwedge_{i^*\in I^*} \gamma_{i^*}$ is coherent with S_0 then there exists some induced history H by some strategy $\boldsymbol{Q}^c$ such that $H_{S_0,\boldsymbol{Q}^c} \models \bigwedge_{i^*\in I^*} \gamma_{i^*}$. Consider a source i^* and its goal γ_{i^*}. Subformulas of its goal refer to agents that it directly influences or not. In the latter case all the strategies of the source will be equivalent (hence weak-dominant). If they talk about the (only) individuals that is influenced by the source, then by the monotonicity of the aggregation procedure it is weakly dominant to play $\mathsf{reveal}(p)$ if the source goal is coherent with the source's belief. And $\mathsf{hide}(p)$ otherwise. (A case study is required to obtain the full proof). ☐

This result, once more, shows the difficulty of playing an influence game. It is actually possible to exhibit examples of acyclic influence graphs with sources influencing multiple agents where no weakly dominant strategies exist for these sources.

4.4 Computational complexity

In this section we exemplify the use of $\mathsf{ELTL\text{–}I}$ and the complexity results presented in Section 3 for the computation of strategic aspects of influence games. We do so by providing a PSPACE algorithm to decide whether a strategy profile is a Nash equilibrium.

Let MEMBERSHIP(F) be the following problem: given as input a set of individuals $\mathcal{N}$, issues $\mathcal{I}$, goals γ_i for $i \in \mathcal{N}$ – which together with F form an influence game IG – and a strategy profile $\boldsymbol{Q}$, we want to know whether $\boldsymbol{Q}$ is a Nash equilibrium of IG.

The algorithm presented in [14] in the setting of iterated boolean games cannot be directly applied to our setting for two reasons. First, our histories are generated by means of an aggregation function F that models the diffusion of opinions – i.e., agents have a concurrent control on a set of propositional variables. Second, not all conceivable strategies are available to players, as we focus on state-based strategies.

We therefore begin by translating a state-based strategy in $\mathsf{ELTL\text{–}I}$. Clearly, a conjunction of literals $\alpha(S)$ can be defined to uniquely identify a state S: $\alpha(S)$ will specify the private opinion of all individuals and their visibility function. Given action a, let

$$\beta_i(a) = \begin{cases} \mathsf{X}\mathsf{vis}(i,p) & \text{if } a = \mathsf{reveal}(p) \\ \mathsf{X}\neg\mathsf{vis}(i,p) & \text{if } a = \mathsf{hide}(p) \\ \top & \text{if } a = \mathsf{skip} \end{cases}$$

We can now associate a $\mathsf{ELTL\text{–}I}$ formula to each strategy Q_i:

$$\tau_i(Q_i) =_{def} \bigwedge_{S \in \mathcal{S}} \alpha(S) \to \beta_i(Q_i(S))$$

If $\boldsymbol{Q}$ is a strategy profile, let $\tau(\boldsymbol{Q}) = \bigwedge_{i \in \mathcal{N}} \tau_i(Q_i)$. We now need to encode the aggregation function into a formula as well. Recall the unanimous issue-by-issue aggregation function of Definition 8 and consider the following formulas *unan*(i,p):

$$\begin{aligned}
&\mathsf{X}\ \mathsf{op}(i,p) \leftrightarrow \\
&\Big(\big[\bigwedge_{j \in \mathit{Inf}(i)} \mathsf{X}\neg\mathsf{vis}(j,p) \wedge \mathsf{op}(i,p)\big] \vee \\
&\big[\bigvee_{j \in \mathit{Inf}(i)} \mathsf{X}\ \mathsf{vis}(j,p) \wedge \bigwedge_{j \in \mathit{Inf}(i)} (\mathsf{X}\ \mathsf{vis}(j,p) \to \mathsf{op}(j,p))\big] \vee \\
&\big[\bigvee_{j,z \in \mathit{Inf}(i):} (\mathsf{X}\ \mathsf{vis}(j,p) \wedge \mathsf{X}\ \mathsf{vis}(z,p) \wedge \\
&\mathsf{op}(j,p) \wedge \neg\mathsf{op}(j,p)) \wedge \mathsf{op}(i,p)\big]\Big)
\end{aligned}$$

as well as the following formula *unan*$(i, \neg p)$:

$$
\begin{aligned}
&\mathsf{X}\,\neg\mathsf{op}(i,p) \leftrightarrow \\
&\Big(\big[\bigwedge_{j\in Inf(i)} \mathsf{X}\neg\mathsf{vis}(j,p) \wedge \neg\mathsf{op}(i,p)\big]\vee \\
&\big[\bigvee_{j\in Inf(i)} \mathsf{X}\,\mathsf{vis}(j,p) \wedge \bigwedge_{j\in Inf(i)} (\mathsf{X}\,\mathsf{vis}(j,p) \rightarrow \neg\mathsf{op}(j,p))\big]\vee \\
&\big[\bigvee_{j,z\in Inf(i):} (\mathsf{X}\,\mathsf{vis}(j,p) \wedge \mathsf{X}\,\mathsf{vis}(z,p)\wedge \\
&\neg\mathsf{op}(j,p) \wedge \mathsf{op}(j,p)) \wedge \neg\mathsf{op}(i,p)\big]\Big)
\end{aligned}
$$

This formula ensures that if the influencers of agent i are unanimous, then agent i's opinion should be defined according to the three cases described in Definition 8. Recall that, while actions take one time unit to be effectuated (hence the X operator in front of $\mathsf{vis}(j,p)$), the diffusion of opinions is simultaneous. Let now:

$$\tau(F_i^U) =_{def} \bigwedge_{\{i\in\mathcal{N} \mid Inf(i)\neq\emptyset\}} \bigwedge_{\{p\in\mathcal{I}\}} (\textit{unan}(i,p) \wedge \textit{unan}(i,\neg p))$$

This formula encodes the transition process defined by the opinion diffusion. $\tau(F_i^U)$ is polynomial in both the number of individuals and the number of issues (in the worst case it is quadratic in n and linear in m). We are now ready to prove the following result:

Theorem 4. MEMBERSHIP(F_i^U) *is in PSPACE.*

Proof. Let $\boldsymbol{Q}$ be a strategy profile for game IG. The following algorithm can be used to check whether $\boldsymbol{Q}$ is a Nash equilibrium. For all individuals $i \in \mathcal{N}$, we first check the following entailment:

$$\tau(\boldsymbol{Q}) \wedge \tau(F_i) \models_{\mathsf{LTL}} \mathrm{red}(\gamma_i)$$

in the language of LTL built out the set of atomic propositions $\{\mathsf{op}(i,p) : i \in \mathcal{N} \text{ and } p \in \mathcal{I}\} \cup \{\mathsf{vis}(i,p) : i \in \mathcal{N} \text{ and } p \in \mathcal{I}\}$, where $\mathrm{red}(\gamma_i)$ is defined as in Algorithm 1 in Section 3.2.

If this is not the case, we consider all the possible strategies $Q_i' \neq Q_i$ for agent i – there are exponentially many, but each one can be specified in space polynomial in the size of the input – and check the following entailment:

$$\tau(\boldsymbol{Q}_{-i}, Q_i) \wedge \tau(F_i) \models_{\mathsf{LTL}} \mathrm{red}(\gamma_i)$$

If the answer is positive we output NO, otherwise we proceed until all strategies and all individuals have been considered. The entailment for LTL can be reduced to the problem of checking validity in LTL. Indeed, the following equivalence holds:

$$\psi \models_{\mathsf{LTL}} \varphi \text{ iff } \models_{\mathsf{LTL}} \mathsf{G}\psi \rightarrow \varphi$$

Since the problem of checking validity in LTL can be solved in PSPACE [21], we obtain the desired upper bound. □

We conjecture that the problem is also PSPACE-complete, as a reduction in line with the one by [14] is likely to be obtained.

Observe that Theorem 4 can easily be generalised to all aggregation procedures that can be axiomatised by means of polynomially many ELTL–I formulas – with the eventual use of Lemma 8 to translate ELTL–I-formulas in LTL. This is not the case for all aggregation procedures: the majority rule – i.e., the rule that updates the opinion of an individual to copy that of the majority of its influencers – would for instance require an exponential number of formulas, one for each subset of influencers that forms a relative majority. The study of the axiomatisation of aggregation procedures for opinion diffusion constitutes a promising direction for future work.

5 Conclusions and future work

In this paper we proposed a model, inspired from related work on iterated boolean games [14], that allows us to explore some basic aspects of strategic reasoning in social influence. We grounded our model on related work [18, 12], which modelled the process of social influence by means of aggregation procedures from either judgment aggregation or belief merging, and we augmented it with the introduction of a simple logical language for the expression of temporal and epistemic goals. This allowed us to inquire into the multiple aspects of the relation between the structure of the influence network, and the existence of well-known game-theoretic solution concepts. Moreover, we were able to show that model checking for our language, as well as the problem of checking whether a given profile is a Nash equilibrium, is in PSPACE, hence no harder than the linear temporal logic on which our language is based.

There are multiple directions in which this work can be expanded. First, the introduction of extra actions to add or sever trust links may add an important dynamic aspect to the network structure. Second, we may allow agents to lie about their private preferences, hence providing them with more strategies to attain their goals. Third, to develop our framework to its full generality we could introduce integrity constraints among the issues at hand. In all these cases, a deeper study of the interconnection between the network structure and the strategies played by the agents of extreme interest, and has the potential to unveil general insights about the problem of social influence.

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