

Lecture #16 - That's all folks!

- Intro to Graphs
- Graph Traversals
 - Depth-first
 - Breadth-first
- Weighted edges
- Dijkstra's Algorithm

Final Exam: Saturday, March 14th 11:30am-2:30pm
Final Exam Location: TBA

Introduction to Graphs

A **graph** is an ADT that stores a set of **entities** and also keeps track of the **relationships** between all of them.

Examples of Entities

People

Cities

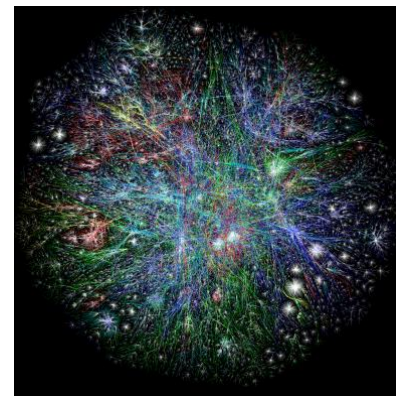
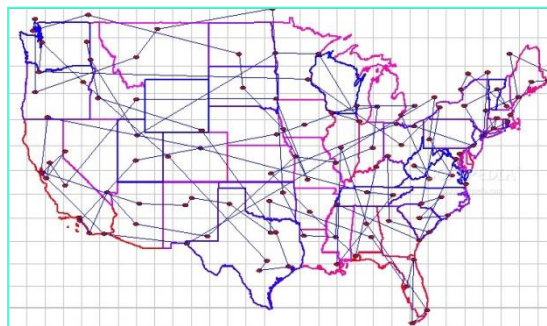
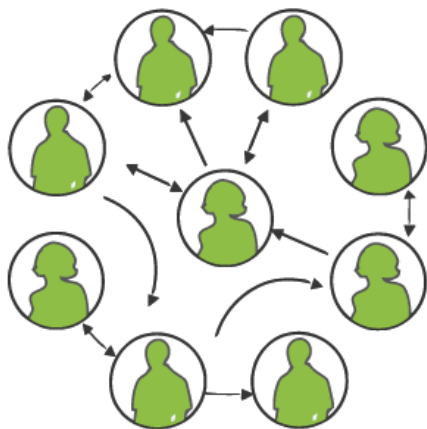
Web pages

Examples of Relationships

Joe is friends with Linda

LA is 3000 miles from NYC

ucla.edu links to awesome.com



Introduction to Graphs

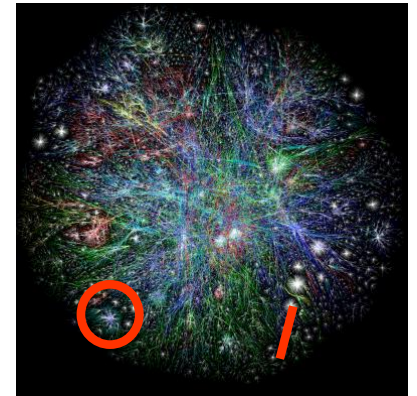
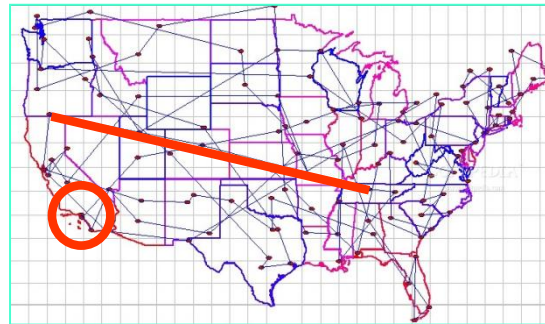
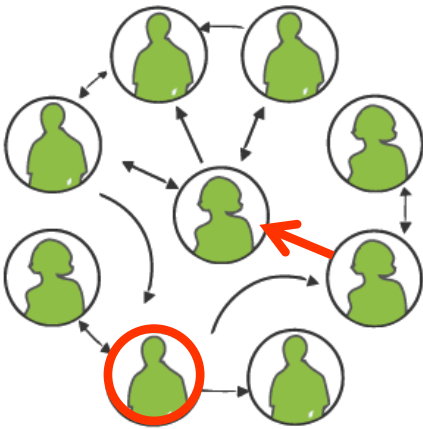
Each graph holds two types of items:

Vertices (aka **Nodes**):

A vertex might represent a **person**, a **city** or a **web page**.

Edges (aka **Arcs**):

An edge simply **connects two*** vertices to each other.



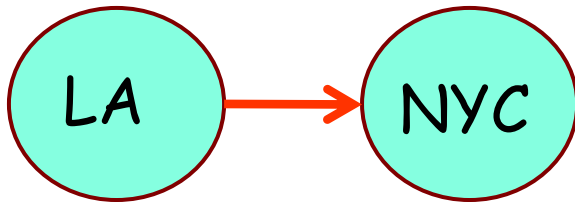
* Technically, an edge could connect a vertex to itself!

Directed vs. Undirected Graphs

There are two major types of graphs...

Directed Graph

In a **directed graph**, an edge goes from one vertex to another in a **specific direction**.

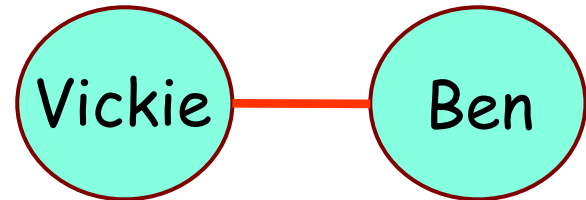


For example, above we have an edge that goes from the LA vertex to NYC vertex, but not the other way around.

(e.g., there may be a flight from LA to NYC but not the other way around)

Undirected Graph

In an **undirected graph**, all edges are **bi-directional**. You can go either way along any edge.



For example, Vickie and Ben are mutual friends on FaceBook.

(It would be kinda creepy if Vickie liked Ben, but not visa-versa)

Representing a Graph in Your Programs

The easiest way to represent a graph is with a **double-dimensional array**.

The **size** of both dimensions of the array is equal to the **number of vertices** in the graph.

```
bool graph[5][5];
```

Each element in the array indicates whether or not there is an edge between vertex **i** and vertex **j**.

Representing a Graph in Your Programs

Each element in the array indicates whether or not there is an edge between vertex *i* and vertex *j*.

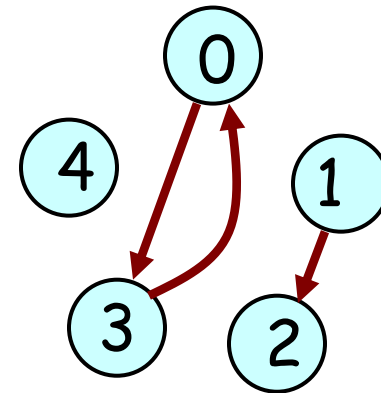
```
bool graph[5][5];
```

```
// edge from vertex 0 to vertex 3
```

```
graph[0][3] = true;
```

```
graph[1][2] = true;
```

```
graph[3][0] = true;
```



	0	1	2	3	4
0				T	
1			T		
2					
3	T				
4					

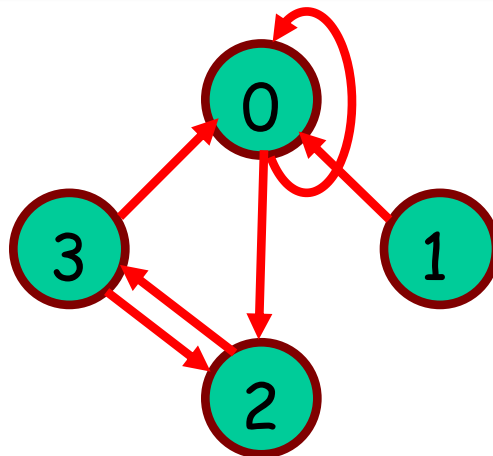
As you can see, when we set `array[i][j]` to `true`, it represents a **directed edge** from vertex *i* to vertex *j*.

This is called an **adjacency matrix**.

Representing a Graph in Your Programs

Exercise: What does the following directed graph look like?

Nodes	0	1	2	3
0	True	False	True	False
1	True	False	False	False
2	False	False	False	True
3	True	False	True	False



Representing a Graph in Your Programs

Question:

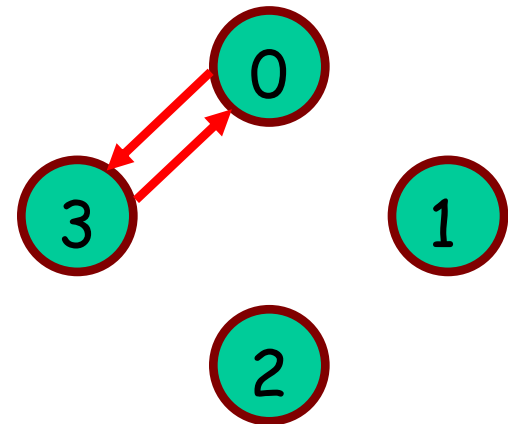
How do you represent an **undirected graph** with an adjacency matrix?

It's easy!

To bi-directionally connect vertices i and j , simply set `array[i][j]` to **true** and set `array[j][i]` to **true** as well!

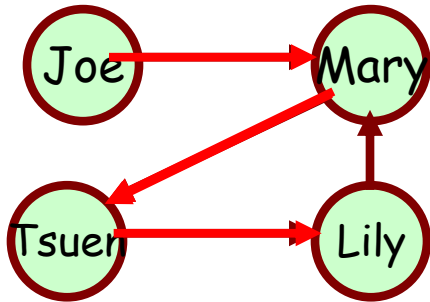
```
graph[0][3] = true;  
graph[3][0] = true;
```

Nodes	0	1	2	3
0				True
1				
2				
3	True			



⁹An Interesting Property of Adjacency Matrices

Consider the following graph:



And it's associated A.M.:

	Joe	Mary	Tsuen	Lily
Joe	0	1	0	0
Mary	0	0	1	0
Tsuen	0	0	0	1
Lily	0	1	0	0

Neato effect: If you multiply the matrix by itself something cool happens!

0	1	0	0
0	0	1	0
0	0	0	1
0	1	0	0

×

0	1	0	0
0	0	1	0
0	0	0	1
0	1	0	0

=

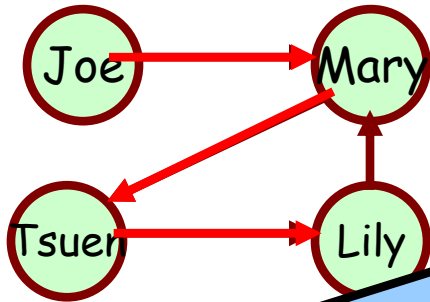
	Joe	Mary	Tsuen	Lily
Joe	0	0	1	0
Mary	0	0	0	1
Tsuen	0	1	0	0
Lily	0	0	1	0

The resulting matrix shows us which vertices are exactly **two edges** apart.

An Interesting Property of Adjacency Matrices

Consider the following graph:

And it's associated A.M.:



Neato effect:
by itself something

So now you know how
Friendster and **Facebook**
work!

Joe	Mary	Tsuen	Lily
0	1	0	0
0	0	1	0
0	0	0	1
0	1	0	0

0	1	0	0
0	0	1	0
0	0	0	1
0	1	0	0

×

0	1	0	0
0	0	1	0
0	0	0	1
0	1	0	0

=

Joe	Mary	Tsuen	Lily
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

And if we multiply our new matrix by the original matrix again,
we'll get all vertices that are exactly **3 edges apart**!

Another Way to Represent a Graph

Question:

How else can we represent a graph (without a 2D array)?

Answer:

A **directed graph** of **n** vertices can be represented by an **array of n linked lists**. This is called an **adjacency list**.

```
list<int> graph[n];
```

If we add a number **j**, to list number **i** (e.g., to `graph[i]`), this means that there is an edge from vertex **i** to vertex **j**.

The Adjacency List

If we add a number j , to list number i (e.g., to $\text{graph}[i]$), this means that there is an edge from vertex i to vertex j

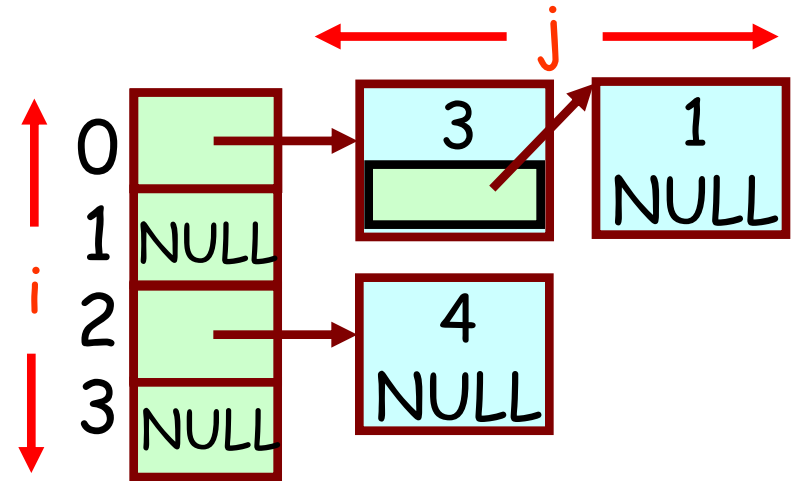
```
list<int> graph[4];
```

```
// edge from node 0 to node 3
```

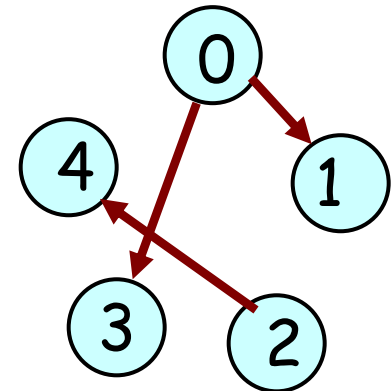
```
graph[0].push_back(3);
```

```
graph[2].push_back(4);
```

```
graph[0].push_back(1);
```



So for each entry j , in list i , this means that there is an edge from vertex i to vertex j .



Which Representation Should You Use?

When should you use an **adjacency matrix** vs. an **adjacency list**?

Scenario #1:

We've got **10,000,000 users** who have relationships with each other - typically each person is friends with just a **few hundred** other people.

What would you do?

Option A: Store the graph in a **10 million** by **10 million** array?
(That's **100 trillion** cells)



Option B: Store your graph in an array holding **10 million linked lists**, each holding roughly **500 items**?
(That's only **5 billion** pieces of data)

Which Representation Should You Use?

When should you use an **adjacency matrix** vs. an **adjacency list**?

Scenario #2:

We've got **1,000 cities**, with airlines offering flights from every city to almost every other city.

What would you do?



Option A: Store the graph in a **1000** by **1000** array?

(That's **1 million** cells)

Option B: Store your graph in an array holding **1000 linked lists**, each holding roughly **1000 items**?

(That's also **1 million** pieces of data, but it's more complex)

Which Representation Should You Use?

When should you use an **adjacency matrix** vs. an **adjacency list**?

Use an **adjacency matrix** if you have **lots of edges** between vertices but **few vertices** (< 10,000 vertices).

Use an **adjacency list** if you have **few edges** between vertices and lots of vertices (> 10,000 vertices).

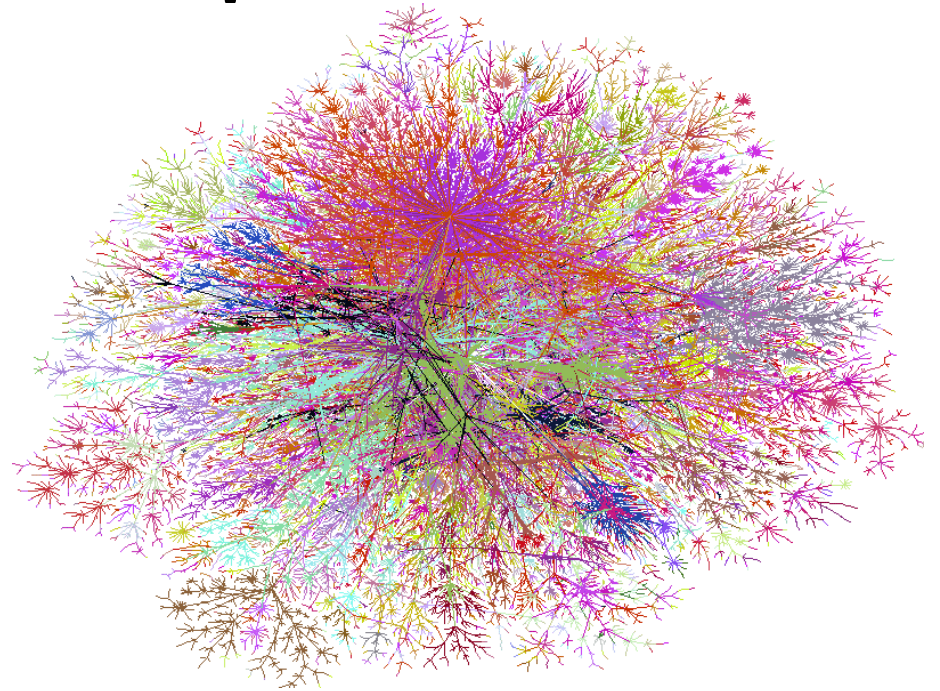
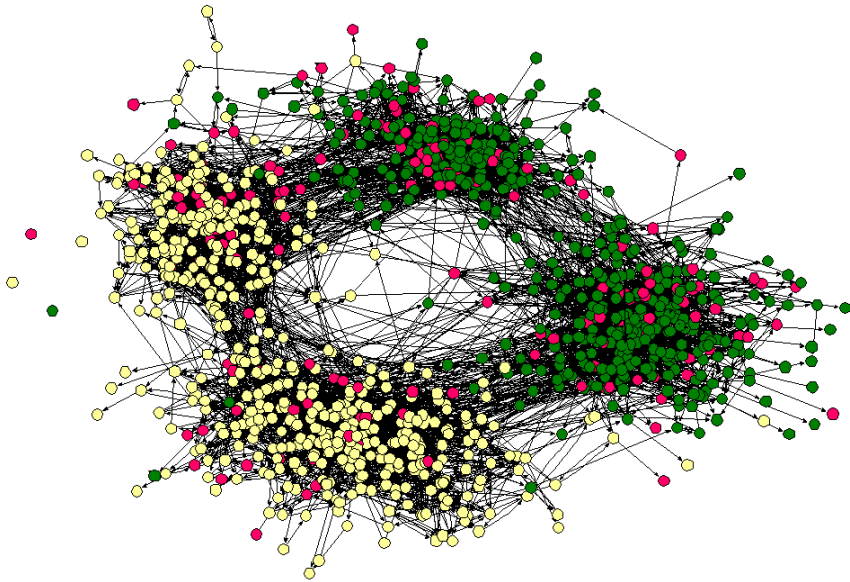
A graph that has **many edges** between the vertices is called a "**dense graph**".

A graph that has **few edges** between the vertices is called a "**sparse graph**".

Let's see examples of both...

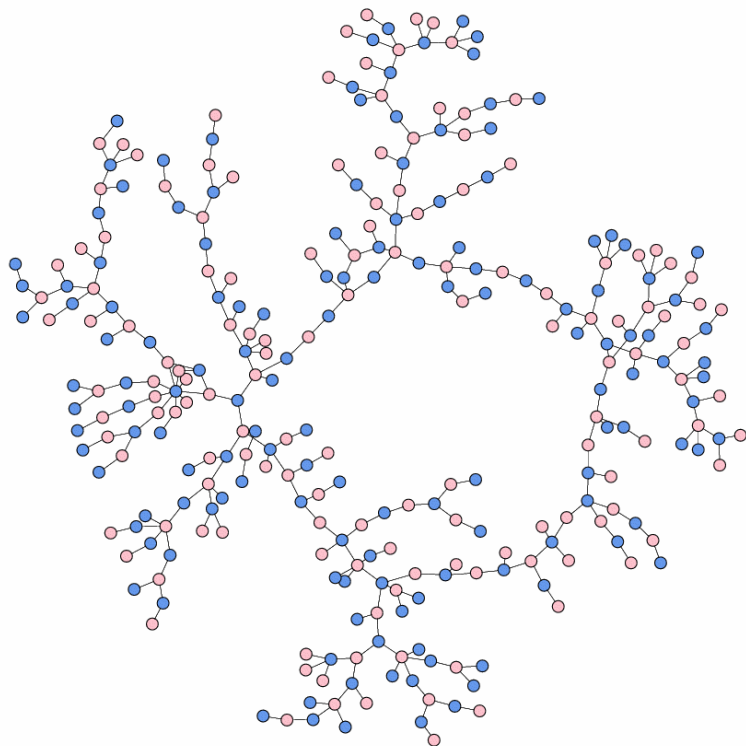
Dense Graphs

(High-school friendships)

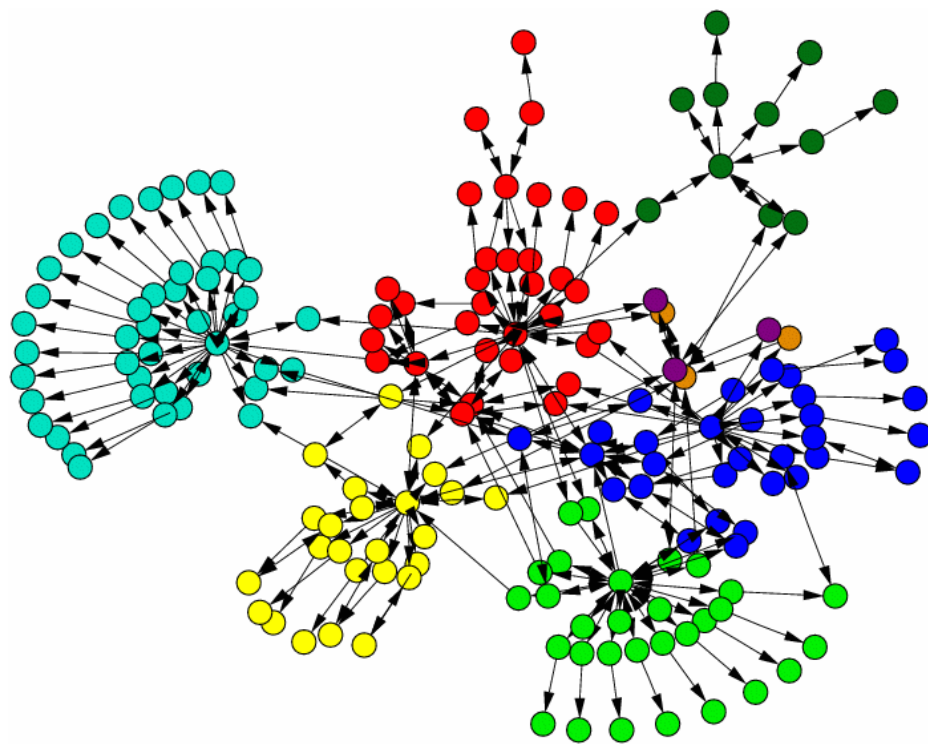


(The Internet)

Sparse Graphs



(High-school dating habits)



(Intra-website links)

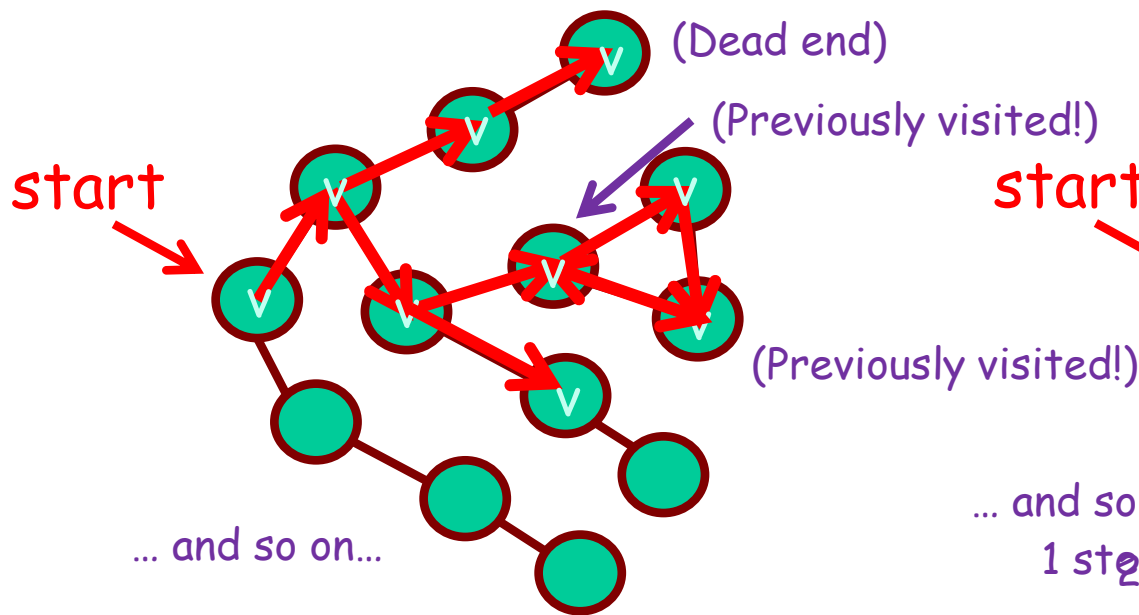
Graph Traversals

We can traverse graphs just like we traverse binary trees!

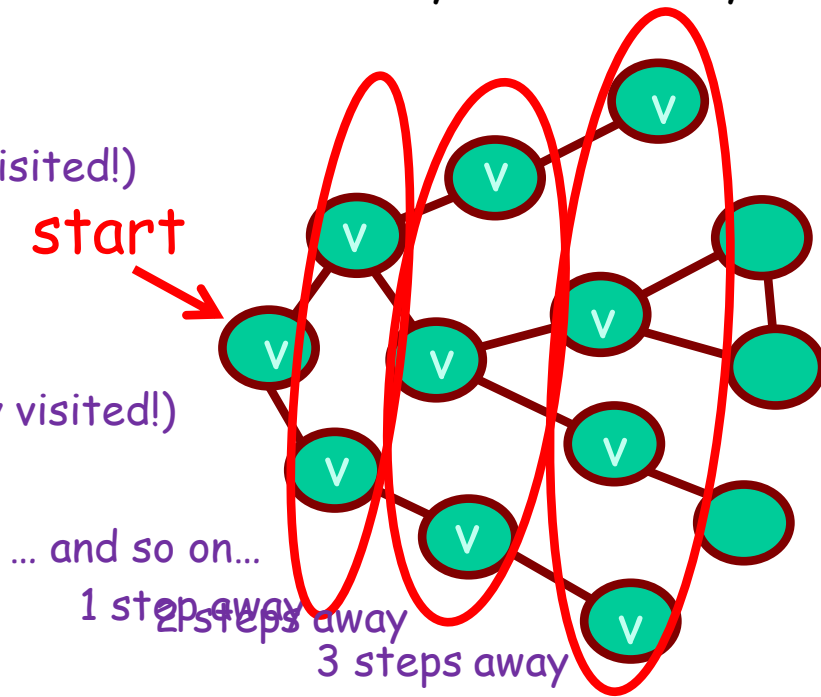
There are two types of graph traversals:

Depth-first and Breadth-first

A **Depth-first Traversal** keeps moving forward until it hits a **dead end** or a **previously-visited vertex**... then it backtracks and tries another path



A **Breadth-first Traversal** explores the graph in **growing concentric circles**, exploring all vertices 1 away from the start, then 2 away, then 3 away, etc.



Depth-first Traversals

Let's learn the Depth-first Traversal algorithm first:

```
Depth-First-Traversal(curVertex)
{
    If we've already visited the current vertex
        Return

    Otherwise
        Mark the current vertex as visited
        Process the current vertex (e.g., print it out)

        For each edge leaving the current vertex
            Determine which vertex the edge takes us to
            Call Depth-First-Traversal on that vertex
}
```

(Notice that it's recursive!)

Depth-first Traversal Demo

We haven't yet visited this

We haven't yet visit

We haven't yet visited this Vertex!

Depth-First-Traversal(*curVertex*)

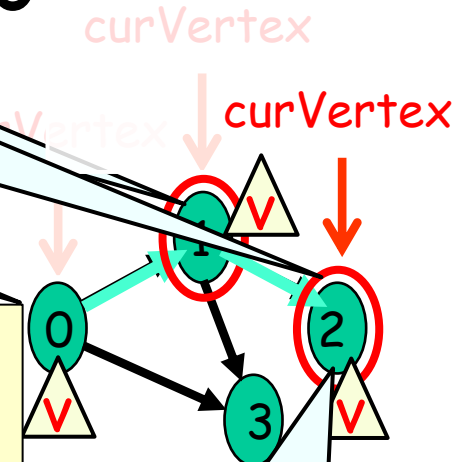
If we've already visited the *current vertex*
Return

Otherwise

Mark the *current vertex* as visited
Process the *current vertex* (e.g., print)

For each *edge* leaving the *current vertex*
Determine which *vertex* the edge takes us to
Call Depth-First-Traversal on that *vertex*

But Vertex #2 has
no outgoing edges...
So there's nothing to do!



- For each **edge** leaving the **current vertex**
 - Determine which **vertex** the edge takes us to
 - Call Depth-First-Traversal on that **vertex**

Used vertex 0!
Used vertex 1!
Used vertex 2!
Used vertex 3!

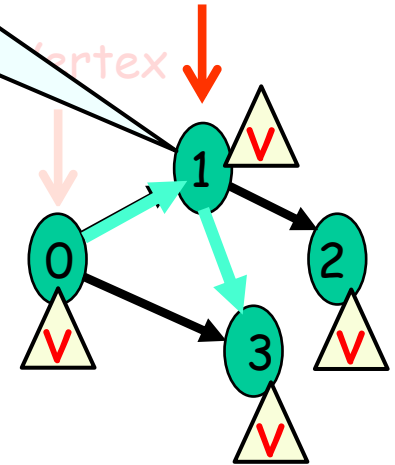
Depth-first

Vertex #1 has
no **MORE** outgoing edges...

So there's nothing to do!

mo

curVertex



Depth-First-Traversal(**curVertex**)

```
{
{
    If we've already visited the current vertex
        Return
```

Otherwise

Mark the **current vertex** as **visited**

Process the **current vertex** (e.g., print it out)

For each **edge** leaving the **current vertex**

Determine which **vertex** the edge takes us to

Call Depth-First-Traversal on that **vertex**

```
}
}
```

Processed vertex 0!

Processed vertex 1!

Processed vertex 2!

Processed vertex 3!

Depth-first Traversal Demo

Depth-First-Traversal(*curVertex*)

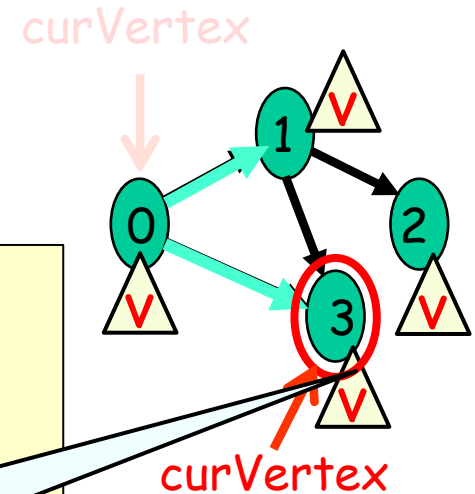
```

D {
  {
    If we've already visited the current vertex
      Return

    Otherwise
      Mark the current vertex
      Process the current vertex
      For each edge leaving the current vertex
        Determine which vertex the edge takes us to
        Call Depth-First-Traversal on that vertex
      }
    }
  }

```

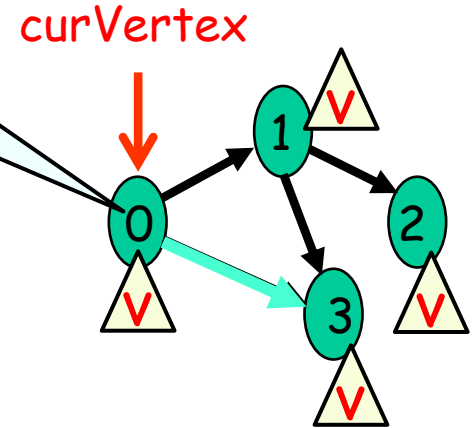
But we've already visited
Vertex #3!
So we don't want to do so
again!



Processed vertex 0!
Processed vertex 1!
Processed vertex 2!
Processed vertex 3!

Depth-first Traversal Demo

Vertex #0 has
no **MORE** outgoing edges...
So there's nothing to do!



Depth-First-Traversal(**curVertex**)

{

If we've already visited the **current vertex**
Return

Otherwise

Mark the **current vertex** as visited

Process the **current vertex** (e.g., print it out)

For each **edge** leaving the **current vertex**

Determine which **vertex** the edge takes us to

Call Depth-First-Traversal on that **vertex**

}

Processed vertex 0!

Processed vertex 1!

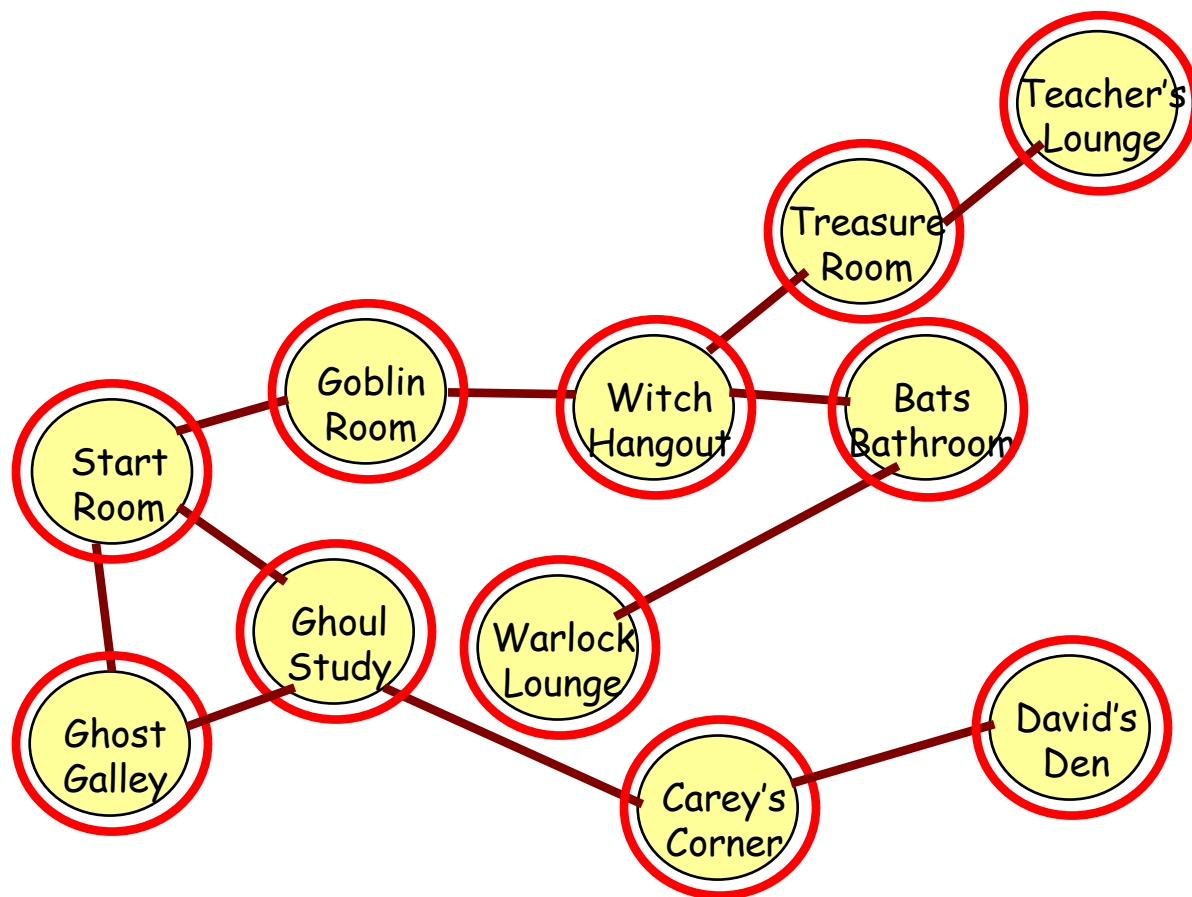
Processed vertex 2!

Processed vertex 3!

And we're done!

Depth-first Traversal Challenge

What does a **Depth-first Traversal** look like on this graph?



Implementing Depth-first Traversal w/Stack!

You can also implement your Depth-first Traversal with a stack if you like! (What's not to like???)

Depth-First-Search-With-Stack(start_room)

Push start_room on the stack

While the stack is not empty

 Pop the top item off the stack and put it in variable c

 If c hasn't been visited yet

 Drop a breadcrumb (we've visited the current room)

 For each door d leaving the room

 If the room r behind door d hasn't been visited

 Push r onto the stack.

Basically, the stack allows you to simulate recursion...

Or does the recursion allow you to simulate a stack?

Hmmmmmmmm!

Breadth-first Graph Traversal

Idea:

Process all of the vertices that are **1 edge away** from the start vertex,
then process all vertices that are **two edges away**,
then process all vertices that are **three edges away**,
etc...

Question:

What data structure could we use to implement this?

Answer:

Not a P, but a ?

Breadth-first Graph Traversal

Breadth-First-Search (startVertex)

{

Add the starting vertex to our queue

Mark the starting vertex as "discovered"

While the queue is not empty

 Dequeue the top vertex from the queue and place in c

 Process vertex c (e.g., print its contents out)

 For each vertex v directly reachable from c

 If v has not yet been "discovered"

 Mark v as "discovered"

 Insert vertex v into the queue

}

Hmmm. Does this algorithm look familiar?

It's a-maze-ingly similar to our queue-based
maze-solving algorithm!!!

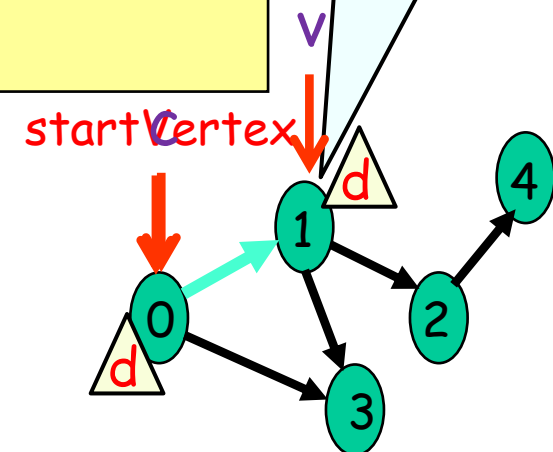
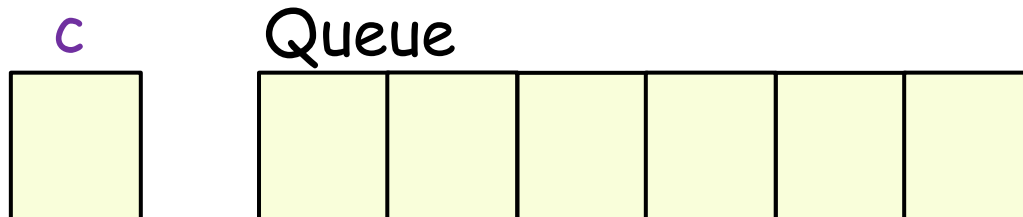
Breadth-first Traversal Demo

Breadth-First-Search (startVertex)

```

{
  Add the starting vertex to our queue
  Mark the starting vertex as "discovered"
  While the queue is not empty
    Dequeue the top vertex from the queue and place in c
    Process vertex c (e.g., print its contents out)
    For each vertex v directly reachable from c
      If v has not yet been "discovered"
        Mark v as "discovered"
        Insert vertex v into the queue
}
  
```

We haven't discovered this Vertex yet!



Breadth-first Traversal Demo

Breadth-First-Search (startVertex)

```
{
```

Add the starting vertex to our **queue**

Mark the starting vertex as "discovered"

While the queue is not empty

Dequeue the top vertex from the queue and place in **c**

Process vertex **c** (e.g., print its contents out)

For each vertex **v** directly reachable from **c**

If **v** has not yet been "discovered"

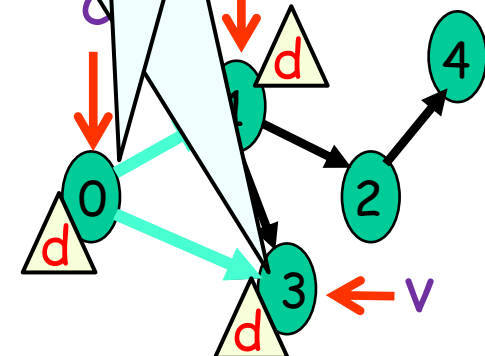
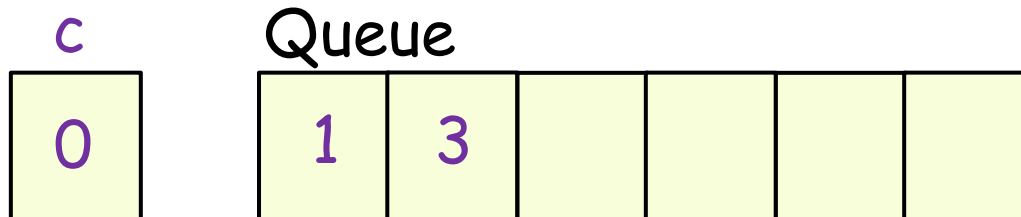
Mark **v** as "discovered"

Insert vertex **v** into the **queue**

```
}
```

Vertex **c** has no other edges, so we're done with it.

We have discovered this vertex



Breadth-first Traversal Demo

Processed vertex 0!

Processed vertex 1!

Breadth-First-Search (startVertex)

```
{
  Add the starting vertex to our queue
  Mark the starting vertex as "discovered"
```

While the queue is not empty

Dequeue the top vertex from the queue and place in c

Process vertex **c** (e.g., print its con

For each vertex v directly reachable

If v has not yet been "discovered"

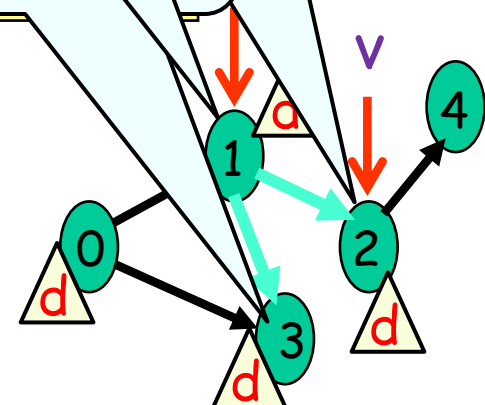
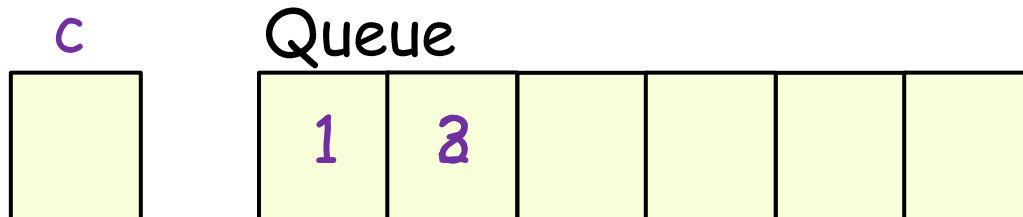
Mark v as "discovered"

Insert vertex v into the queue

}

Vertex #1 has no
more outgoing
edges...

~~Ah ha! We've already discovered this this Verte~~



Breadth-first Traversal Demo

Processed vertex 0!
 Processed vertex 1!
 Processed vertex 3!

Breadth-First-Search (startVertex)

{

Add the starting vertex to our **queue**

Mark the starting vertex as "discovered"

While the queue is not empty

Dequeue the top vertex from the queue and place in **c**

Process vertex **c** (e.g., print its contents out)

For each vertex **v** directly reachable from **c**

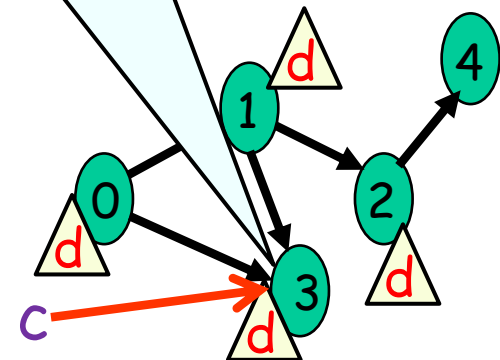
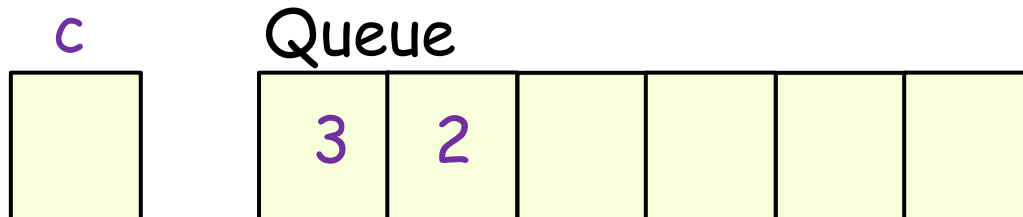
If **v** has not yet been "discovered"

Mark **v** as "discovered"

Insert vertex **v** into the **queue**

}

Vertex #3 has NO
 outgoing edges at
 all! So we're done.



Breadth-First-Search Demo

And finally we're done!

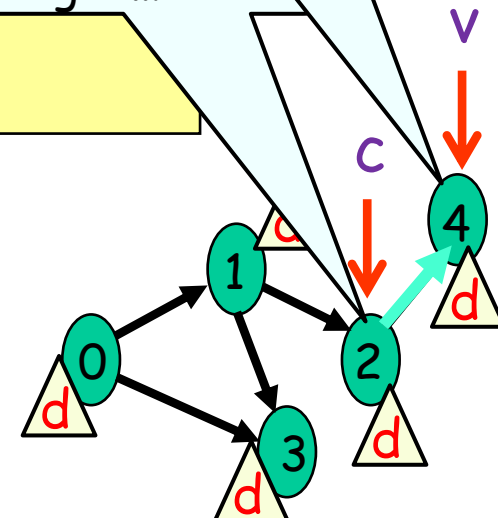
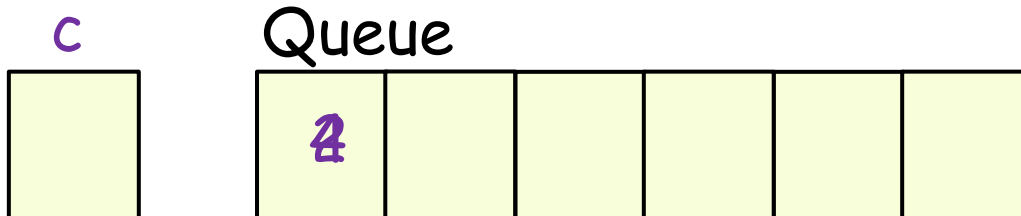
Processed vertex 0!
 Processed vertex 1!
 Processed vertex 3!
 Processed vertex 2!
 Processed vertex 4!

Breadth-First-Search (startVertex)

```
{
  Add the starting vertex to our queue
  Mark the starting vertex as "discovered"
  While the queue is not empty
    Dequeue the top vertex from the queue and
    Process vertex c (e.g., print its contents out)
    For each vertex v directly reachable from c
      If v has not yet been "discovered"
        Mark v as "discovered"
        Insert vertex v into the queue
}
```

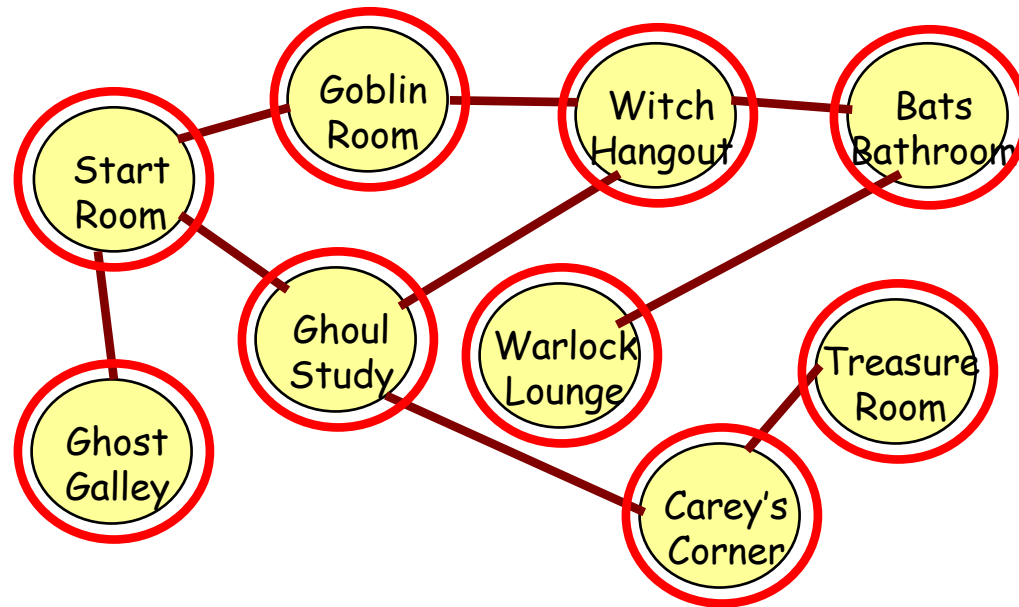
Vertex #4 has NO outgoing edges...

Vertex #4 has NO more outgoing edges...



Breadth-first Traversal Challenge

What does a **Breadth-first Traversal** look like on this graph?

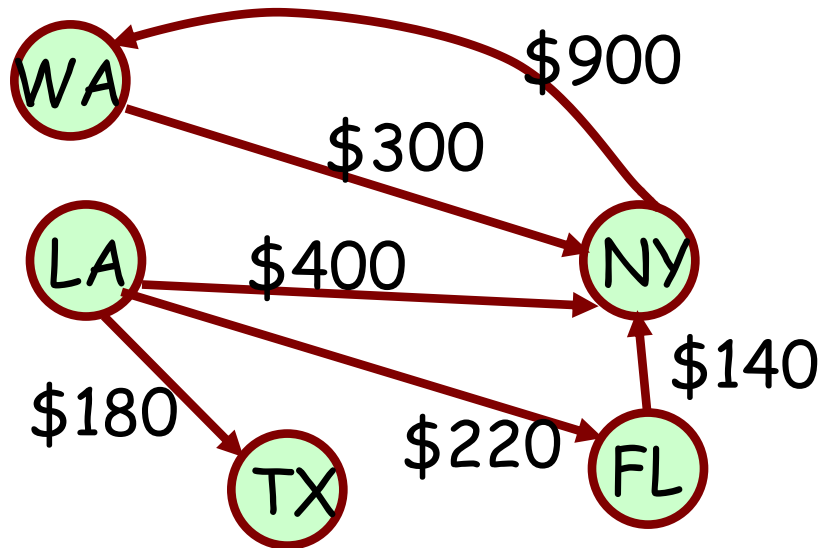


Graphs With Weighted Edges

What does it mean for a graph to have **weighted edges**?

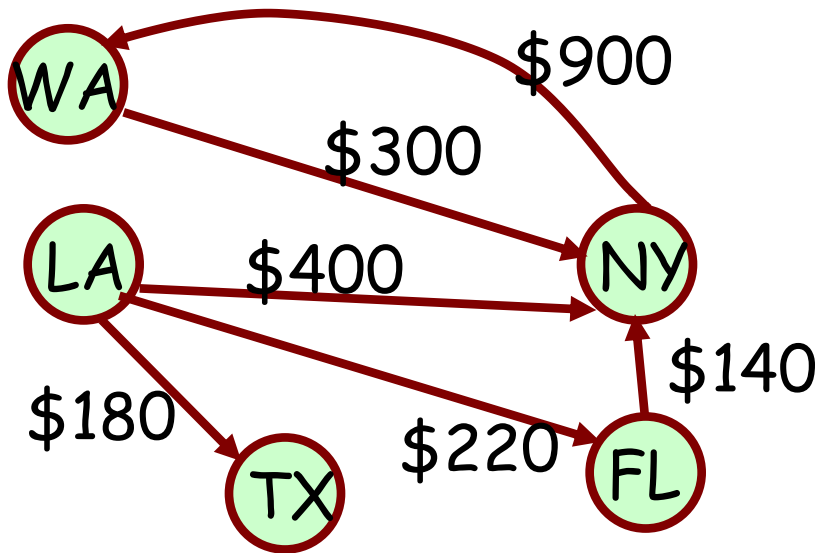
Definition: Each **edge** connecting **vertex** u with **vertex** v has a **weight** or **cost** associated with it.

Question: Why would we want to have weighted edges?



Graphs With Weighted Edges

Definition: The **weight of a path** from **vertex u** to **vertex v** is the **sum of the weights of the edges** between the two vertices.



Question: What's the cost of traveling from LA to NY to WA?

Question: What's the **shortest path** from LA to WA?

Definition: The **shortest path** between two vertices is the path with the lowest total cost of edges between the two vertices. (The shortest path is a set of vertices)

Finding the Shortest Path

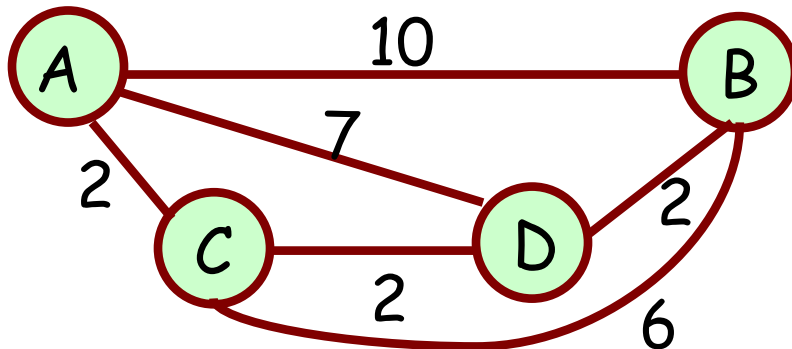
Question: How can we find the shortest path between any two nodes in a graph?

Answer: Dijkstra's Algorithm (the dorm guy?)

Dijkstra's Algorithm:

the length of

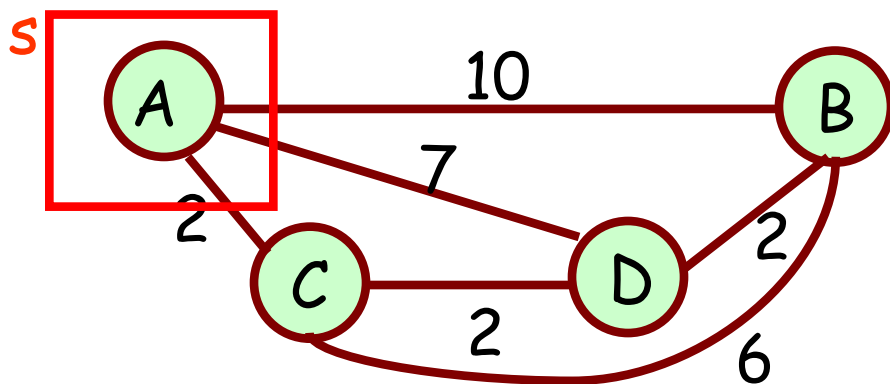
This algorithm determines the **shortest path** (i.e. set of vertices) from a start vertex **s** to all **other vertices** in the graph.



So **Dijkstra(A)** would give us a value of **6** for **A** to **B**, a value of **2** for **A** to **C**, and **4** for **A** to **D**.

Dijkstra's Algorithm

Input: A graph G , and a starting vertex s



G may not have any negative edge values.

Output: An array called **Dist** of *optimal distances* from s to every other node in the graph.

Dist	A	B	C	D
	0	6	2	4

Dijkstra's Algorithm: Basic Idea

Dijkstra's algorithm splits vertices in two distinct sets: the set of *unsettled* vertices and the set of *settled* vertices.

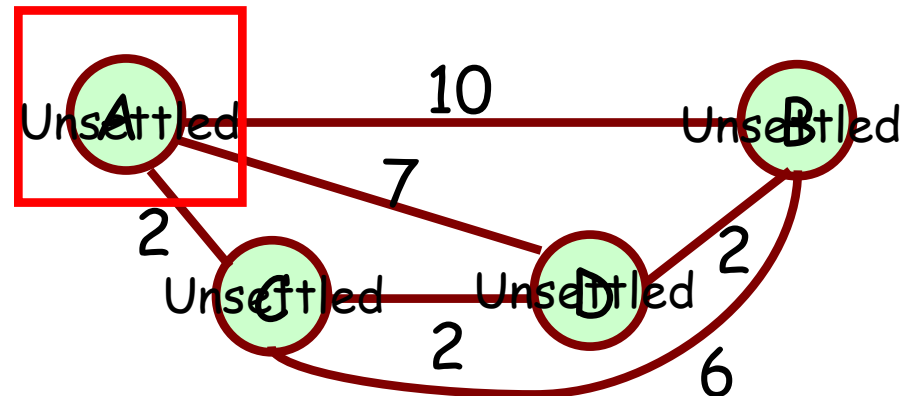
Unsettled vertex: A vertex v is *unsettled* if we don't know the optimal distance to it from the starting vertex s .

Settled vertex: A vertex v is *settled* if we have learned the optimal distance to it from the starting vertex s .

Initially all vertices are *unsettled*.

The algorithm ends once all vertices are in the *settled* set.

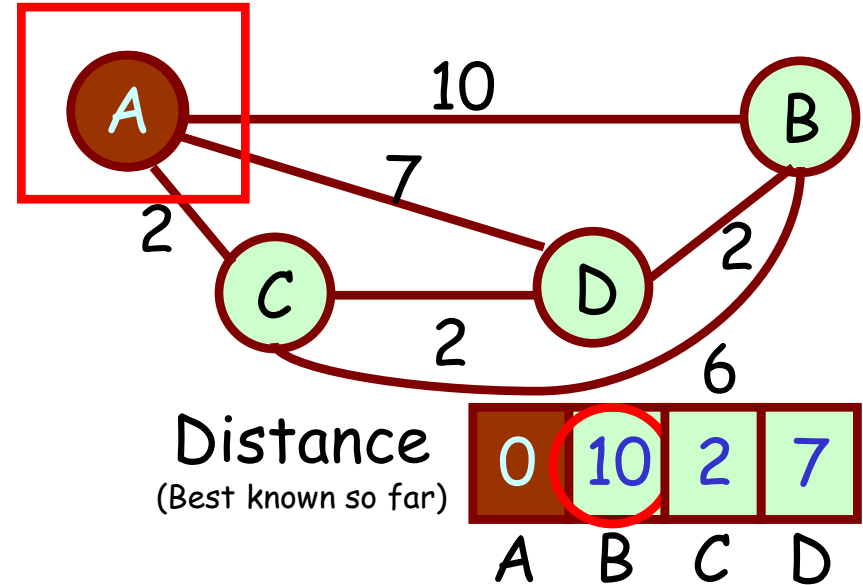
Start vertex



Dijkstra on a Graph

Assume that **all vertices** are **infinitely far away** to start...

Since we start at vertex **A**, we know it's the closest vertex to us! How far is it? **Zero** steps away! We can **settle** it immediately!



Now let's see which unsettled vertices we can reach *directly* from **A**.

- **B** is **10** units away.
- **C** is **2** units away.
- **D** is **7** units away.

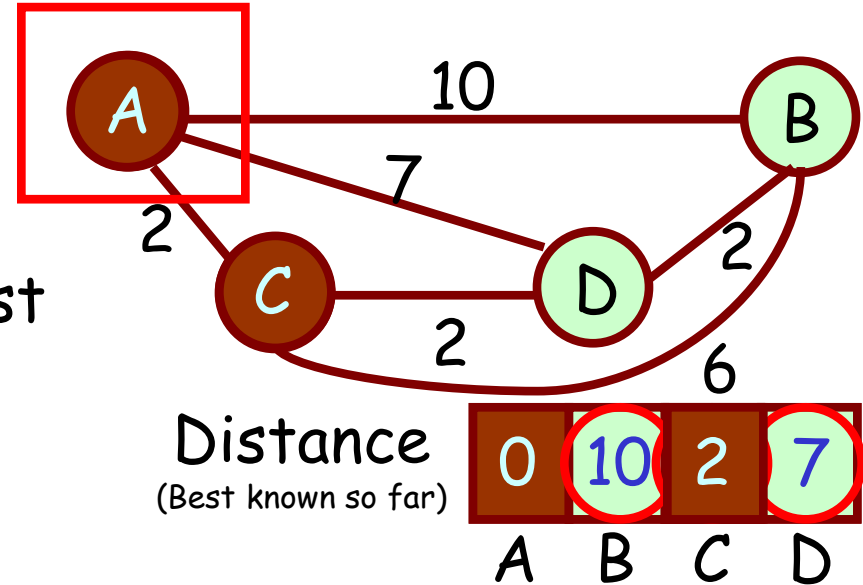
And going directly from **A** to **D** is only **7** units away, which is less than infinity, so I'll update this entry too...

Dijkstra on a Graph

Ok, so now we know the costs to travel to all vertices directly reachable from **A**.

Which unsettled vertex is closest to **A**?

Right! **C** is closest to **A**.



If we go directly to **C** ($A \rightarrow C$), it costs us 2 units. Is there any possible way I can travel to **C** cheaper by going through **B** or through **D** first?

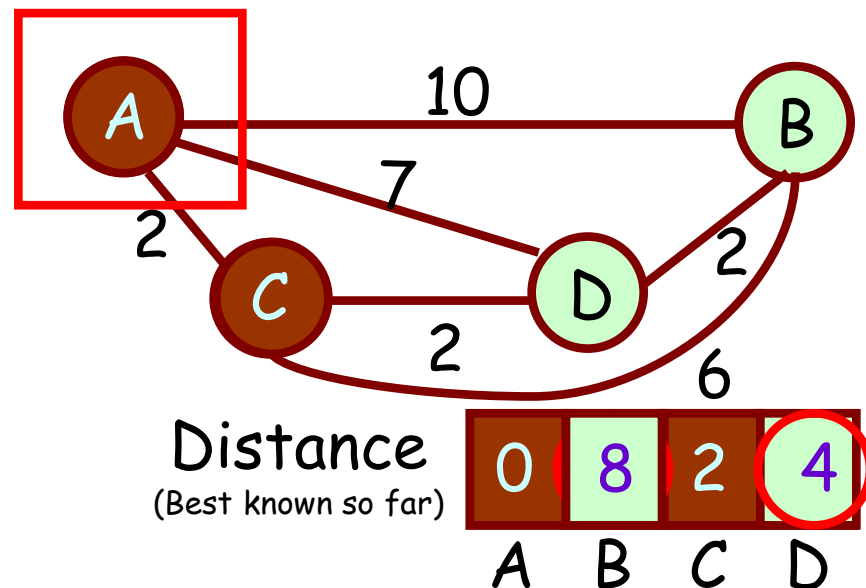
So I know that if I travel directly from **A** to **C**, at a cost of 2 units, that's the fastest possible route. Therefore we can settle **C** at 2 units.

Dijkstra on a Graph

At this point, we know the shortest path from **A** to **C**. Now let's see if we can travel through **C** to reach one of our other unsettled vertices faster.

Ok, which unsettled vertices can be reached directly from **C**?

- **B** is 6 units away.
- **D** is 2 units away.



Let's do **D** next. We know we can get from **A** to **C** in 2 units, and we can directly go from **C** to **D** in 2 units, so we can reach **D** in just 4 units!

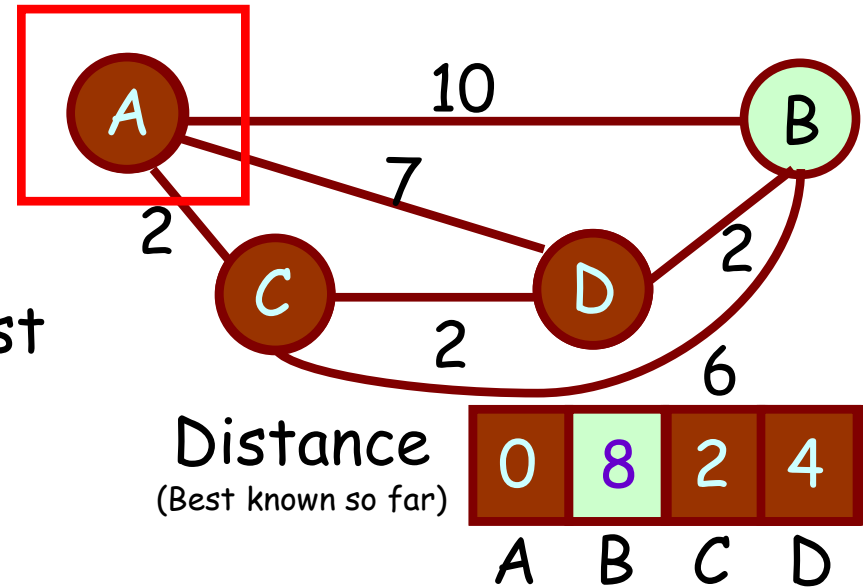
Is our new distance to **D** better than our old one?
Yup!! Let's update our table again!

Dijkstra on a Graph

Ok, so now we know the best cost to get to all unsettled vertices, assuming we travel through **C**.

Which unsettled vertex is closest to **A** now?

Right! **D** is closest.



If we take the path **A** \rightarrow **C** \rightarrow **D**, it costs us 4 units. Is there any possible way I can travel to **D** cheaper by going through **B** first?

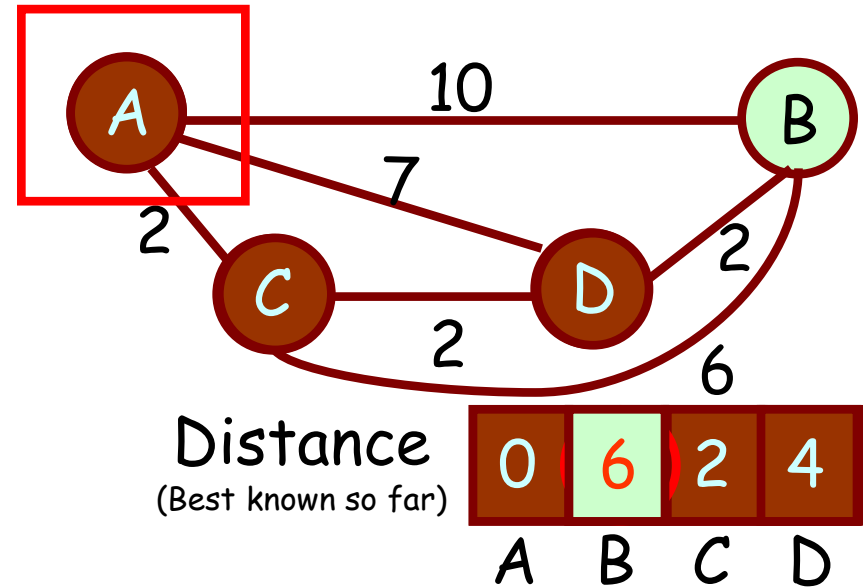
So I know that if I travel from **A** \rightarrow **C** \rightarrow **D**, at a cost of 4 units, that's the *fastest* possible route. Therefore we can settle **D** at 4 units.

Dijkstra on a Graph

At this point, we know the shortest path from **A** to **D**. Now let's see if we can travel through **D** to reach one of our other unsettled vertices faster.

Ok, which unsettled vertices can be reached directly from **D**?

- **B** is 2 units away.



Let's check **B**. We know we can get from **A** to **D** in 4 units, and we can directly go from **D** to **B** in 2 units, so we can reach **B** in just 6 units!

Is our new distance to **B** better than our old one?

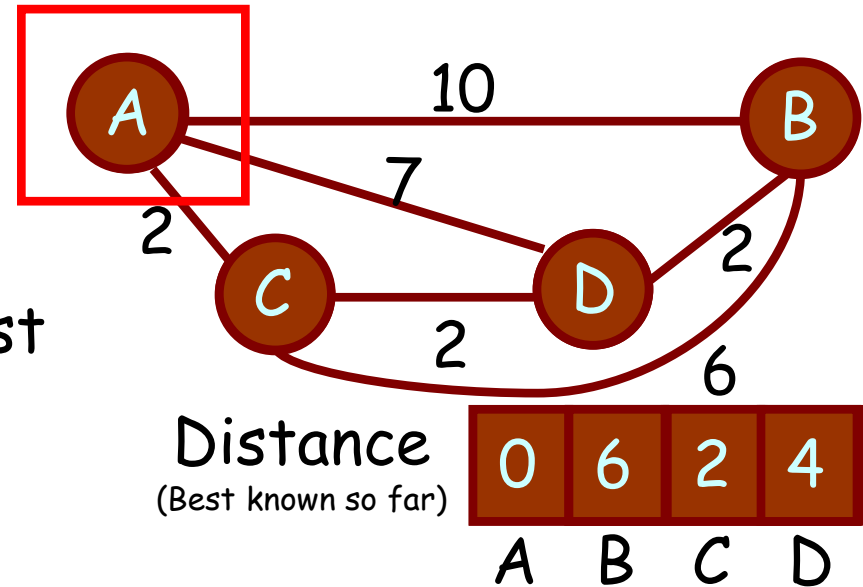
You bet!! Let's update our table!

Dijkstra on a Graph

Ok, so now we know the best cost to get to all unsettled vertices, assuming we travel through **D**.

Which unsettled vertex is closest to **A** now?

Right! **B** is closest.



And now that all of our vertices are settled, we are guaranteed to have found the **minimum** travel distances to each of our vertices!

Dijkstra

And now I'll give you the more formal algorithm...



Born: 11 May 1930, Rotterdam, Netherlands
Died: 6 August 2002, Nuenen, Netherlands

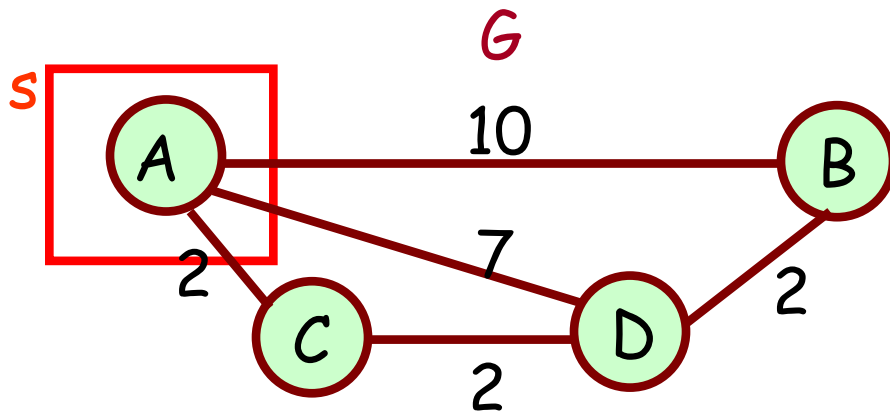
Dijkstra's Algorithm

Dijkstra's Algorithm uses 2 data structures:

1. An array called **Dist** that holds the **the current best known cost** to get from **s** to every other vertex in the graph.

For each vertex **i**, **Dist[i]** starts out with a value of:

- **0** for vertex **s**
- **Infinity** for all other vertices



Dist from
vertex **s**
to...

A	B	C	D
0	∞	∞	∞

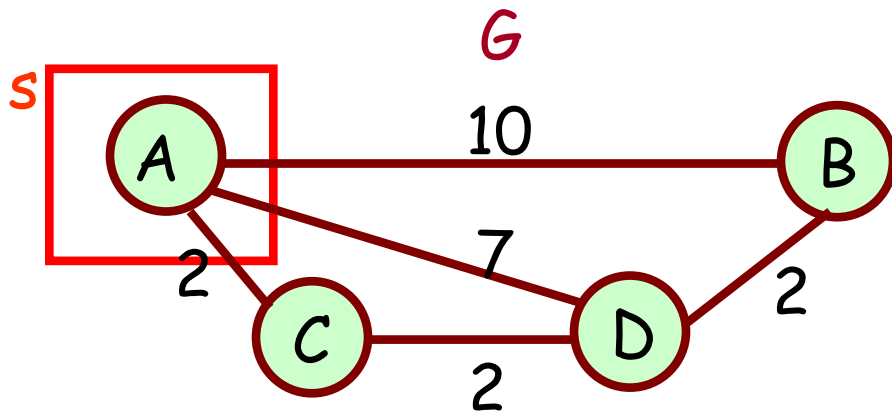
Idea: We start at node **A** so we're 0 steps away from node **A**. We assume the other vertices are infinitely far away from **A**.

Dijkstra's Algorithm

Dijkstra's Algorithm uses 2 data structures:

2. An array called **Done** that holds **true** for each vertex that has been fully processed, and **false** otherwise.

For each vertex i , **Done** $[i]$ starts out with a value of **false**.



Dist from
vertex s
to...

A	B	C	D
0	∞	∞	∞

Done

A	B	C	D
false	false	false	false

Dijkstra's Algorithm

While there are still unprocessed vertices:

Set u = the closest unprocessed vertex to the start vertex s

Mark vertex u as processed: $\text{Done}[u] = \text{true}$.

We now know how to reach u optimally from s

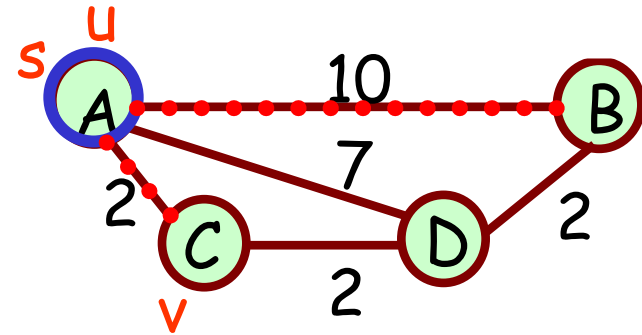
Loop through all unprocessed vertices:

Set v = the next unprocessed vertex

If there's an edge from u to v then compare:

- the previously computed path from s to v (i.e. $\text{Dist}[v]$) OR
- the path from s to u , and then from u to v (I.e. $\text{Dist}[u] + \text{weight}(u,v)$)

If the new cost is less than old cost then
Set $\text{Dist}[v] = \text{Dist}[u] + \text{weight}(u,v)$



Dist from vertex s to...	A	B	C	D
	0	10	2	∞
Done	A	B	C	D
	true	false	false	false
	u		v	

Dijkstra's Algorithm

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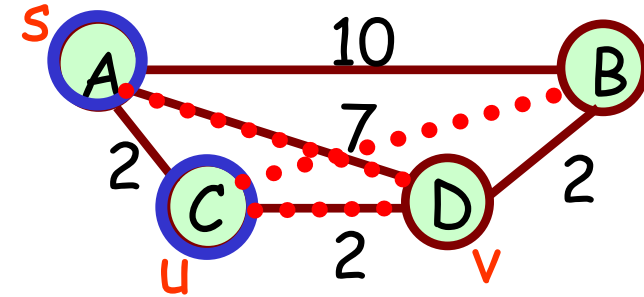
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Previous cost: 7

New cost: $2 + 2 = 4$

Dist from vertex s to...	A	B	C	D
	0	10	2	4
Done	A	B	C	D
	true	false	true	false
			u	v

Dijkstra's Algorithm

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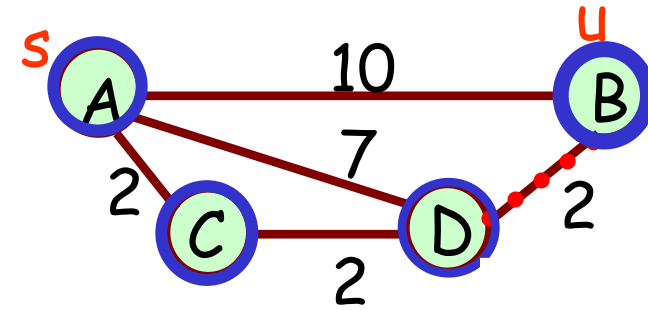
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If the new cost is less than old cost then
Set $\text{Dist}[v] = \text{Dist}[u] + \text{weight}(u,v)$



Previous cost: 10

New cost: $4 + 2 = 6$

Dist from vertex s to...	A	B	C	D
	0	6	2	4
Done	A	B	C	D
	true	true	true	true

u

And we're done! The **Dist** array contains the results.