Homework1 Solution

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Problem1

a. Prove $2n+\Theta(n^2)=\Theta(n^2)$
证明:

先证 $2n+\Theta(n^2)\subseteq\Theta(n^2)$
$\forall f(n)\in 2n+\Theta(n^2)$, $\exists c_1,c_2,n_0\in\Re^+$, $s.t.\ \forall n\geq n_0$

$$0\leq c_1(2n+n^2)\leq f(n)\leq c_2(2n+n^2) \tag{1}$$

令 $n_1=max(n_0,2)$, $\exists n_1\in\Re^+$, $s.t.\ \forall n\geq n_1$

$$0\leq c_1n^2\leq f(n)\leq c_2(2n+n^2)\leq 2c_2n^2 \tag{2}$$

则 $f(n)\in\Theta(n^2)$, 也即 $2n+\Theta(n^2)\subseteq\Theta(n^2)$

再证 $\Theta(n^2)\subseteq 2n+\Theta(n^2)$
$\forall f(n)\in\Theta(n^2),\exists c_1,c_2,n_0\in\Re^+,s.t.\forall n\geq n_0$

$$0\leq c_1n^2\leq f(n)\leq c_2n^2 \tag{3}$$

令 $n_1=max(n_0,2),\exists n_1\in\Re^+,s.t.\forall n\geq n_1$

$$0\leq\frac{c_1}{2}(2n+n^2)\leq c_1n^2\leq f(n)\leq c_2n^2\leq c_2(2n+n^2) \tag{4}$$

则 $f(n)\in 2n+\Theta(n^2)$, 也即 $\Theta(n^2)\subseteq 2n+\Theta(n^2)$

综上所述, $2n+\Theta(n^2)=\Theta(n^2)$
证毕!

b. Prove $\Theta(g(n)) \cap o(g(n)) = \emptyset$

证明:

$\forall f(n) \in \Theta(g(n)), \exists c_1, c_2, n_0 \in \Re^+, s.t. \forall n \geq n_0$

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \tag{5}$$

假设 $f(n) \in o(g(n)), \forall c_3 > 0, \exists n_1 \in \Re^+, s.t. \forall n \geq n_1$

$$0 \leq f(n) < c_3 g(n) \tag{6}$$

取 $c_3 = c_1, n_2 = max(n_0, n_1), s.t. \forall n \geq n_3$

$$c_3 g(n) = c_1 g(n) \leq f(n) \tag{7}$$

这与(6)式矛盾，故假设不成立，也即 $\forall f(n) \in \Theta(g(n)), f(n) \notin o(g(n))$
综上所述，$\Theta(g(n)) \cap o(g(n)) = \emptyset$
证毕!

c. Prove $\Theta(g(n)) \cup o(g(n)) \neq O(g(n))$

证明:

取 $g(n) = 1, f(n) = \left|sin\left(\frac{\pi n}{2}\right)\right|$
下面说明 $f(n) \in O(g(n))$ 且 $f(n) \notin \Theta(g(n)) \cup o(g(n))$
取 $c_1 = 1, n_0 = 1, \forall n \leq n_0$

$$0 \leq f(n) \leq c_1 g(n) = c_1 = 1 \tag{8}$$

则 $f(n) \in O(g(n))$
因为 $\forall n = 2k, k = 1, 2, 3..., f(n) = 0$, 所以 $\forall c_2, n_0 \in \Re^+, \exists n > n_0, n = 2k, s.t.$

$$f(n) = 0 < c_2 g(n) = c_2 \tag{9}$$

也即 $f(n) \notin \Theta(g(n))$
取 $c_3 = \frac{1}{2}, \forall n_1 \in \Re^+, \exists n > n_1, n = 2k+1, s.t.$

$$\frac{1}{2} = c_3 g(n) < f(n) = 1 \tag{10}$$

也即 $f(n) \notin o(g(n))$
综上所述，$f(n) \in O(g(n))$ 且 $f(n) \notin \Theta(g(n)) \cup o(g(n))$
则 $\Theta(g(n)) \cup o(g(n)) \neq O(g(n))$
证毕!

d. Prove $max(f(n), g(n)) = \Theta(f(n) + g(n))$

证明:

令 $h(n) = max(f(n), g(n)), \exists n_0 \in \Re^+, s.t. \forall n > n_0$

$$h(n) \geq f(n) \tag{11}$$
$$h(n) \geq g(n) \tag{12}$$
$$h(n) \leq f(n) + f(n) + 2g(n) \tag{13}$$

整理可得

$$0 \le \frac{1}{2}(f(n)+g(n)) \le h(n) = max(f(n),g(n)) \le 2(f(n)+g(n)) \tag{14}$$

则可得 $max(f(n),g(n)) = \Theta(f(n)+g(n))$
证毕!

Problem2

a. $T(n) = 2T(\sqrt{n}) + 1$
解:

令 $m = lgn$, 则有 $n = 2^m$, 带入原式得

$$T(2^m) = 2T(2^{\frac{m}{2}}) + 1 \tag{15}$$

令 $G(n) = T(2^n)$, 可得

$$G(m) = 2G(\frac{m}{2}) + 1 \tag{16}$$

根据主定理, $a = 2, b = 2, f(n) = 1$, 则 $n^{log_b a} = n$, 取 $\epsilon = 1$, 则有

$$1 = f(n) = \Theta(n^{log_b a - \epsilon}) = \Theta(1) \tag{17}$$

则 $G(n) = \Theta(n)$, 又因为 $G(n) = T(2^n)$, 可得

$$T(n) = \Theta(lgn) \tag{18}$$

b. $nT(n) = (n-2)T(n-1) + 2$
解:

原式可化为

$$n(T(n)-1) = (n-2)(T(n-1)-1) \tag{19}$$

令 $G(n) = T(n) - 1$, 则上式可化为

$$\frac{G(n)}{G(n-1)} = \frac{n-2}{n} \tag{20}$$

累乘可得

$$\frac{G(n)}{G(2)} = \frac{G(n)}{G(n-1)}\frac{G(n-1)}{G(n-2)}\cdots\frac{G(3)}{G(2)} = \frac{n-2}{n}\frac{n-3}{n-1}\cdots\frac{1}{3} = \frac{2(n-2)!}{n!} = \frac{2}{n(n-1)} \tag{21}$$

则可得

$$T(n) = \frac{2G(2)}{n(n-1)} + 1 \tag{22}$$

也即

$$T(n) = \Theta\left(\frac{1}{n^2}\right) \tag{23}$$

Problem3

a. 解:

下表给出排序结果，从上到下渐进增长率增大，同一行中从左向右渐进增长率增大

等价类	函数
$\Theta(1)$	$1, n^{\frac{1}{lgn}}$
$\Theta(lg(lg^*n))$	$lg(lg^*n)$
$\Theta(lg^*(lgn))$	$lg^*(lgn)$
$\Theta(lg^*n)$	lg^*n
$\Theta(2^{lg^*n})$	2^{lg^*n}
$\Theta(lg(lgn))$	$ln(lnn)$
$\Theta(\sqrt{lgn})$	$\sqrt{lgn}$
$\Theta(lgn)$	lnn
$\Theta(lg^2n)$	lg^2n
$\Theta(2^{\sqrt{2lgn}})$	$2^{\sqrt{2lgn}}$
$\Theta(n^{\frac{1}{2}})$	$\sqrt{2}^{lgn}$
$\Theta(n)$	$2^{lgn}, n$
$\Theta(nlgn)$	$nlgn, lg(n!)$
$\Theta(n^2)$	$n^2, 4^{lgn}$
$\Theta(n^3)$	n^3
$\Theta((lgn)!)$	$(lgn)!$
$\Theta(lgn^{lgn})$	$lgn^{lgn}, n^{lg(lgn)}$
$\Theta((\frac{3}{2})^n)$	$(\frac{3}{2})^n$
$\Theta(2^n)$	2^n
$\Theta(n2^n)$	$n2^n$
$\Theta(e^n)$	e^n
$\Theta(n!)$	$n!$
$\Theta((n+1)!)$	$(n+1)!$
$\Theta(2^{2^n})$	2^{2^n}
$\Theta(2^{2^{n+1}})$	$2^{2^{n+1}}$

b. 解:

令 $f(n) = 2^{2^{n+2}} + (-1)^n 2^{2^{n+2}}$, 当$n$为偶数时, $f(n) = \Theta(2^{2^{n+2}})$, 当$n$为奇数时, $f(n) = 0$
因此, $\forall c_1, c_2, n_0 \in \Re^+, \exists n > n_0, s.t.$

$$f(n) > c_1 g_i(n) \quad (n = 2k) \tag{24}$$
$$f(n) < c_2 g_i(n) \quad (n = 2k+1) \tag{25}$$

则对于所有$g_i(n)$, 上述$f(n)$既不是$O(g_i(n))$也不是$\Omega(g_i(n))$