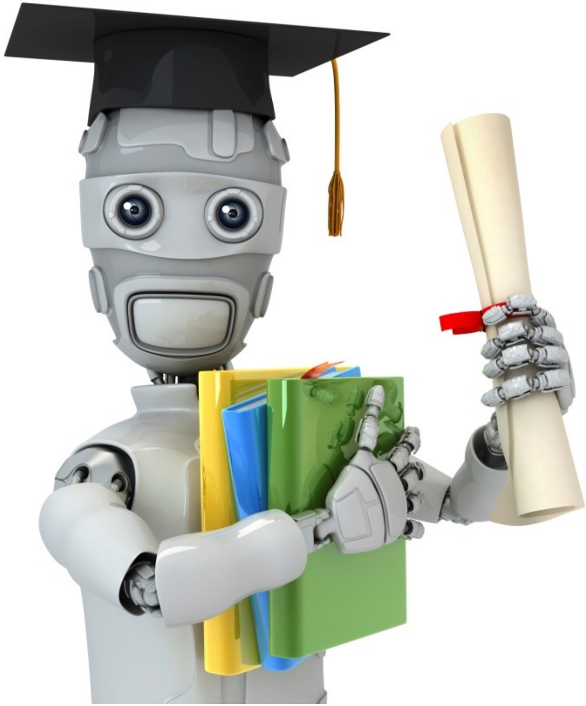


Machine Learning

Linear Regression with multiple variables

Multiple features

Multiple features (variables).

Size (feet^2)	Price (\$1000)
x	y
2104	460
1416	232
1534	315
852	178
…	…

$$h_\theta(x) = \theta_0 + \theta_1 x$$

Multiple features (variables).

Size ($feet^2$)	Number of bedrooms	Number of floors	Age of home (years)	Price ($1000)
2104	5	1	45	460
1416	3	2	40	232
1534	3	2	30	315
852	2	1	36	178
…	…	…	…	…

Multiple features (variables).

Size ($feet^2$) x_1	Number of bedrooms x_2	Number of floors x_3	Age of home (years) x_4	Price (\$1000) y
2104	5	1	45	460
1416	3	2	40	232
1534	3	2	30	315
852	2	1	36	178
…	…	…	…	…

$m = 47$

$$x^{(2)} = \begin{bmatrix} 1416 \\ 3 \\ 2 \\ 40 \end{bmatrix}$$

$x_3^{(2)} = 2$

Notation:

- n = number of features $n = 4$
- $x^{(i)}$ = input (features) of i^{th} training example.
- $x_j^{(i)}$ = value of feature j in i^{th} training example.

Hypothesis:

Previously: $h_\theta(x) = \cancel{\theta_0 + \theta_1 x}$

$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_4 x_4$

E.g. $\underline{h_\theta(x)} = \underline{80} + \underline{0.1} x_1 + \underline{0.01} x_2 + 3 x_3 - 2 x_4$

age

$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$$

For convenience of notation, define $\boxed{x_0 = 1.}$ $(x_0^{(i)} = 1)$

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n+1} \qquad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$\underbrace{[\theta_0 \ \theta_1 \cdots \theta_n]}_{\theta^T} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$(n+1) \times 1$ matrix

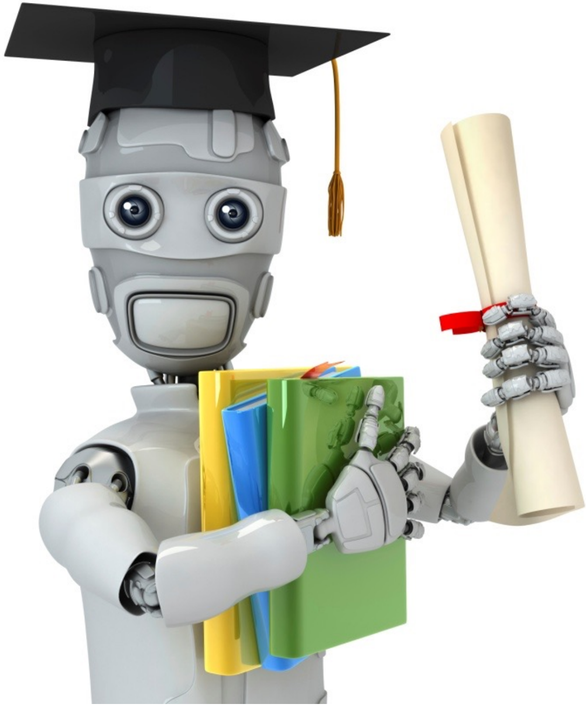
x

$\theta^T x$

$$h_\theta(x) = \theta_0 \overset{=1}{x_0} + \theta_1 x_1 + \cdots + \theta_n x_n$$

$$= \boxed{\theta^T x.}$$

Multivariate linear regression.



Machine Learning

Linear Regression with multiple variables

Gradient descent for multiple variables

Hypothesis: $h_\theta(x) = \theta^T x = \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$ $\qquad x_0 = 1$

Parameters: $\cancel{\theta_0, \theta_1, \ldots, \theta_n}$ $\qquad \theta$ $\qquad$ n+1 - dimensional vector

Cost function:

$$\cancel{J(\theta_0, \theta_1, \ldots, \theta_n)} = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

$J(\theta)$

Gradient descent:

Repeat $\{$

$$\theta_j := \theta_j - \alpha \boxed{\frac{\partial}{\partial \theta_j} \cancel{J(\theta_0, \ldots, \theta_n)} \; J(\theta)}$$

$\}$ (simultaneously update for every $j = 0, \ldots, n$)

Gradient Descent

Previously (n=1):

Repeat {

$$\theta_0 := \theta_0 - \alpha \underbrace{\boxed{\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})}}_{\boxed{\frac{\partial}{\partial\theta_0}J(\theta)}}$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x^{(i)}$$

$x_1^{(i)}$

(simultaneously update θ_0, θ_1)

}

New algorithm $(n \geq 1)$:

Repeat {

$\frac{\partial}{\partial\theta_j} J(\theta)$

$$\theta_j := \theta_j - \alpha \boxed{\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}}$$

(simultaneously update θ_j for $j = 0, \ldots, n$)

}

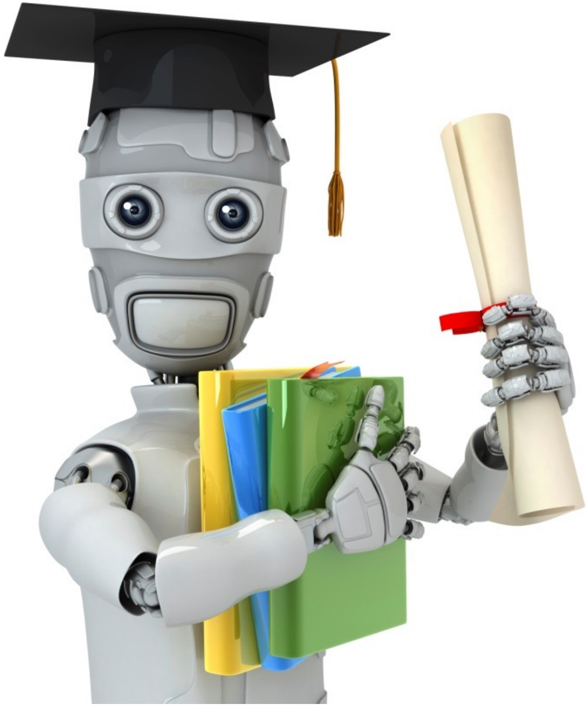
$x_0^{(i)} = 1$

$$\theta_0 := \theta_0 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_0^{(i)}$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_1^{(i)}$$

$$\theta_2 := \theta_2 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_2^{(i)}$$

...



Machine Learning

Linear Regression with multiple variables

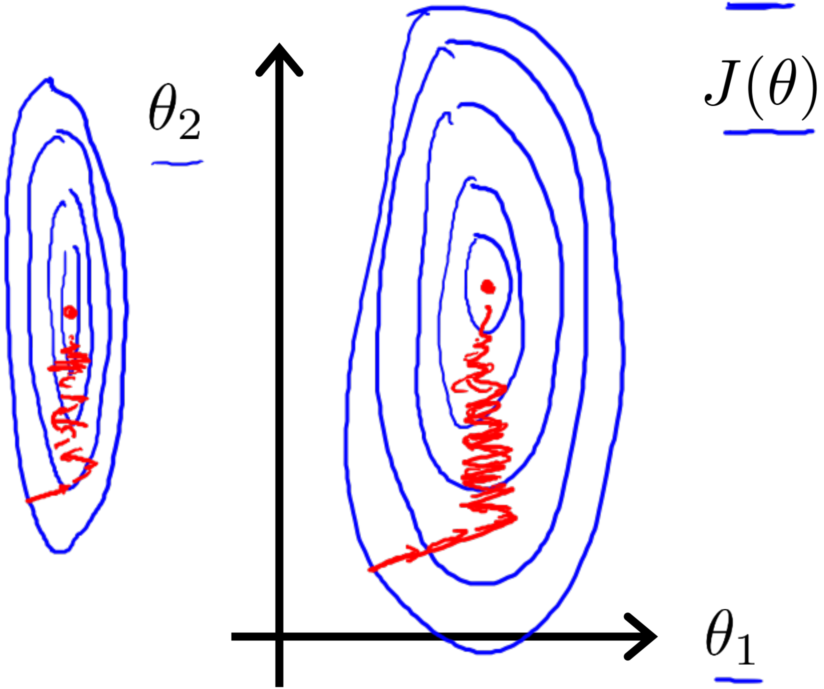
Gradient descent in practice I: Feature Scaling

Feature Scaling

Idea: Make sure features are on a similar scale.

E.g. x_1 = size (0-2000 feet^2)

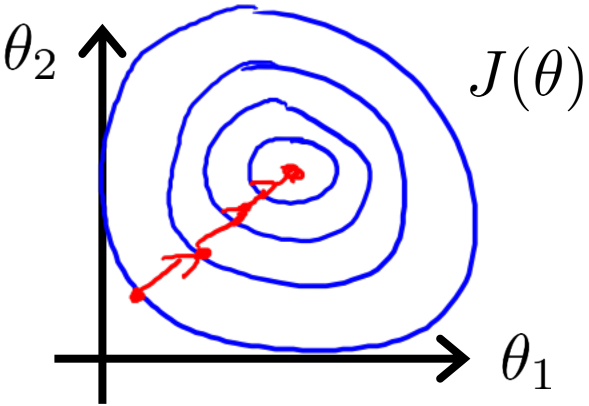
x_2 = number of bedrooms (1-5)



$$x_1 = \frac{\text{size (feet}^2)}{2000}$$

$$x_2 = \frac{\text{number of bedrooms}}{5}$$

$0 \le x_1 \le 1 \qquad 0 \le x_2 \le 1$



Feature Scaling

Get every feature into approximately a $\boxed{-1 \le x_i \le 1}$ range.

$x_0 = 1$

$0 \le x_1 \le 3$ ✓

$-2 \le x_2 \le 0.5$ ✓

$-100 \le x_3 \boxed{100}$ ✗

$-0.0001 \le x_4 \le \boxed{0.0001}$ ✗

-3 to 3 ✓

$-\frac{1}{3}$ to $\frac{1}{3}$ ✓

Mean normalization

Replace x_i with $x_i - \mu_i$ to make features have approximately zero mean (Do not apply to $x_0 = 1$).

E.g. $\boxed{x_1 = \frac{size - 1000}{2000}}$

$\boxed{x_2 = \frac{\#bedrooms - 2}{5 \quad 4}}$

Average size = 100

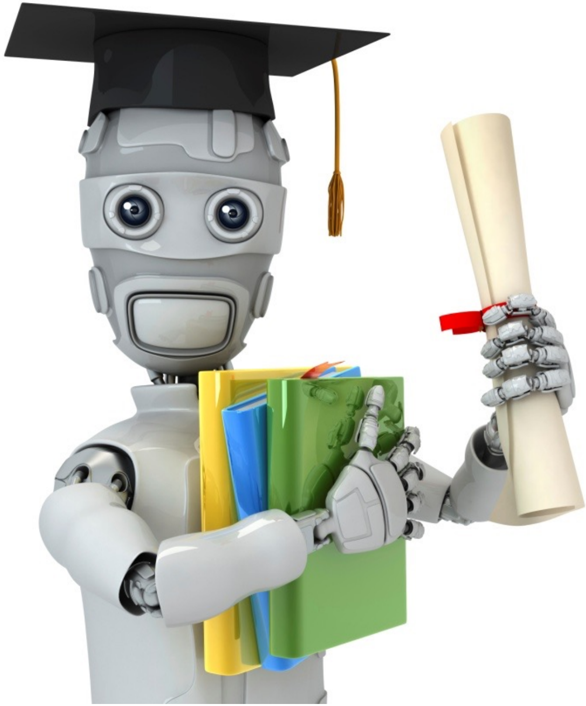
1-5 bedrooms

$\boxed{-0.5 \le x_1 \le 0.5}$, $\boxed{-0.5 \le x_2 \le 0.5}$

$x_1 \leftarrow \frac{x_1 - \mu_1}{s_1}$ ← avg value of x_1 in training set

range (max - min)
(or standard deviation)

$x_2 \leftarrow \frac{x_2 - \mu_2}{s_2}$



Machine Learning

Linear Regression with multiple variables

Gradient descent in practice II: Learning rate

Gradient descent

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

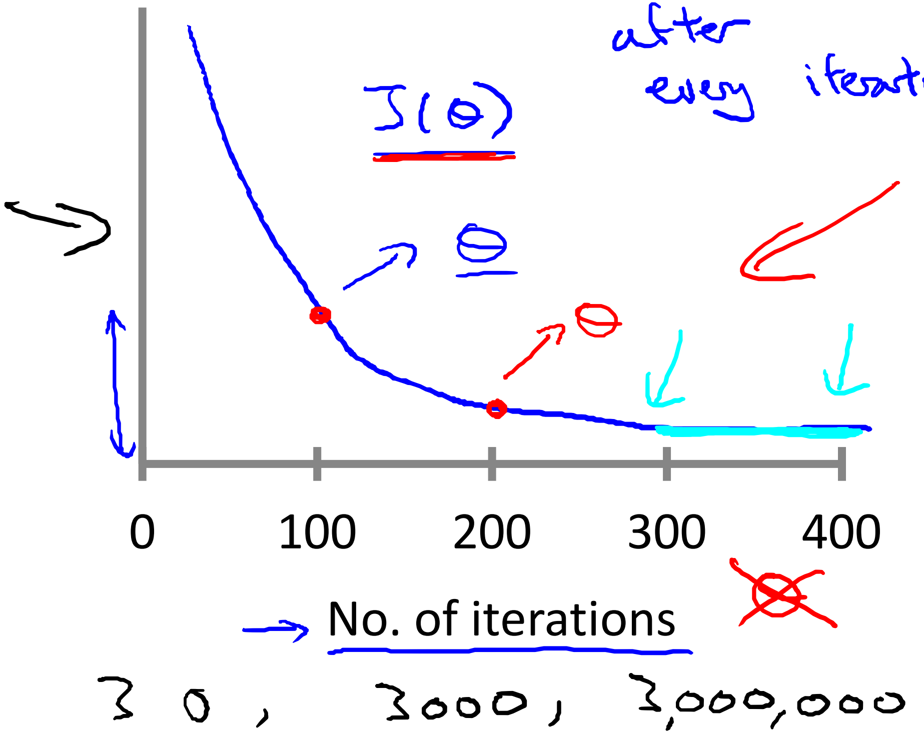
- "Debugging": How to make sure gradient descent is working correctly.

- How to choose learning rate $\boxed{\alpha}$.

Making sure gradient descent is working correctly.

$$\boxed{\min_\theta J(\theta)}$$

J(θ) should decrease after every iteration.

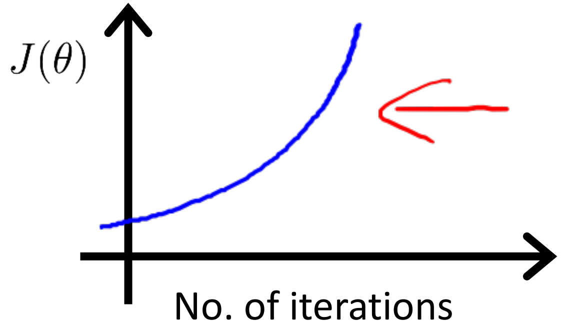


→ Example automatic convergence test:

→ Declare convergence if $J(\theta)$ decreases by less than $\boxed{10^{-3}}$ in one iteration.

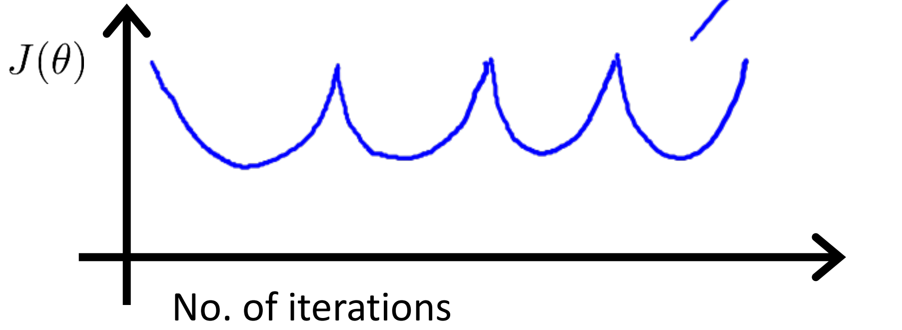
ε

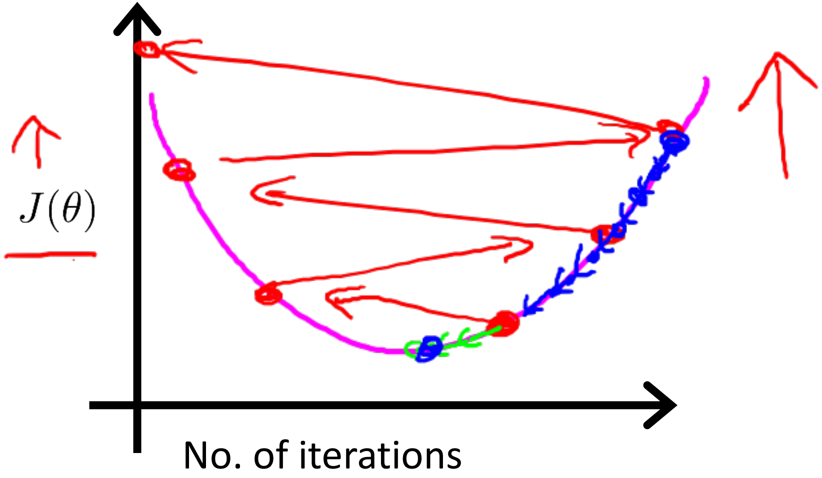
Making sure gradient descent is working correctly.



Gradient descent not working.

$\boxed{\text{Use smaller } \alpha.}$

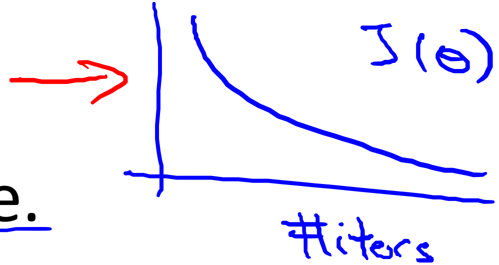




- For sufficiently small α, $J(\theta)$ should decrease on every iteration.
- But if α is too small, gradient descent can be slow to converge.

Summary:

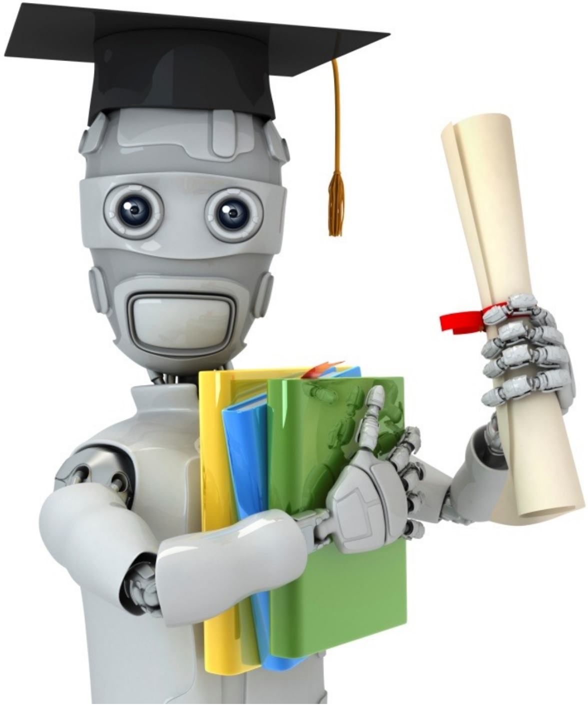
- If α is too small: slow convergence.
- If α is too large: $J(\theta)$ may not decrease on every iteration; may not converge. (Slow converge also possible.)



To choose α, try

$$\dots, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, \dots$$

3x   ≈3x   3x   ≈3x



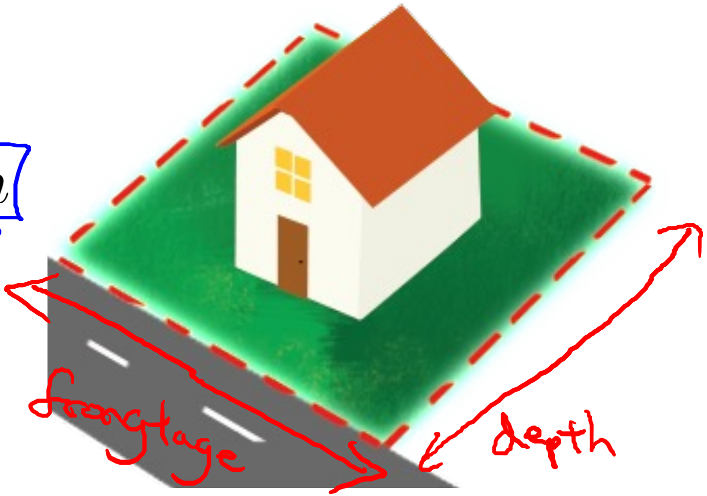
Linear Regression with multiple variables

Features and polynomial regression

Machine Learning

Housing prices prediction

$$h_\theta(x) = \theta_0 + \theta_1 \times \underbrace{\boxed{frontage}}_{\underline{x_1}} + \theta_2 \times \underbrace{\boxed{depth}}_{\underline{x_2}}$$



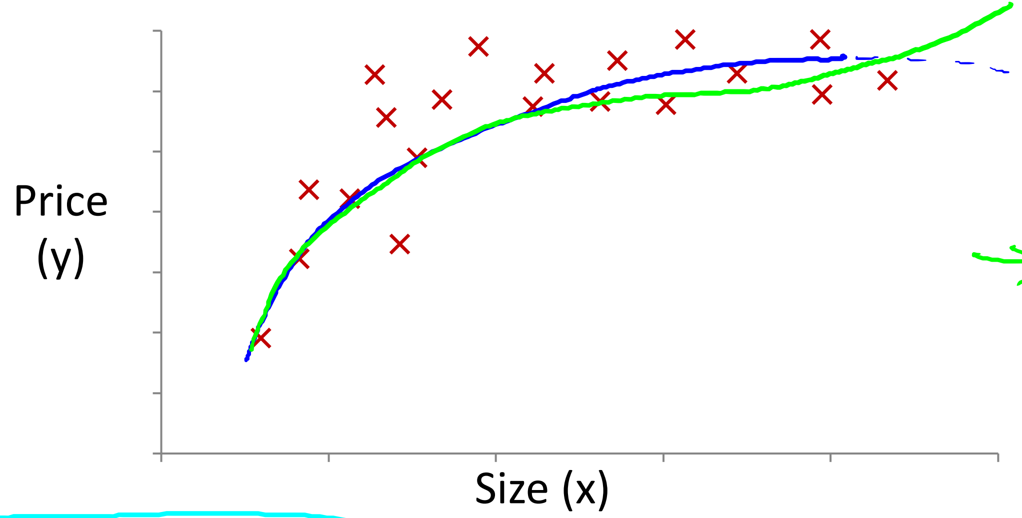
<u>Area</u>

$x = \underline{frontage * depth}$

$h_\theta(x) = \theta_0 + \theta_1 x$

↑ land area

Polynomial regression



$$\theta_0 + \theta_1 x + \theta_2 x^2$$

$$\boxed{\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3}$$

$$\boxed{h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3}$$

$$= \theta_0 + \theta_1 (size) + \theta_2 (size)^2 + \theta_3 (size)^3$$

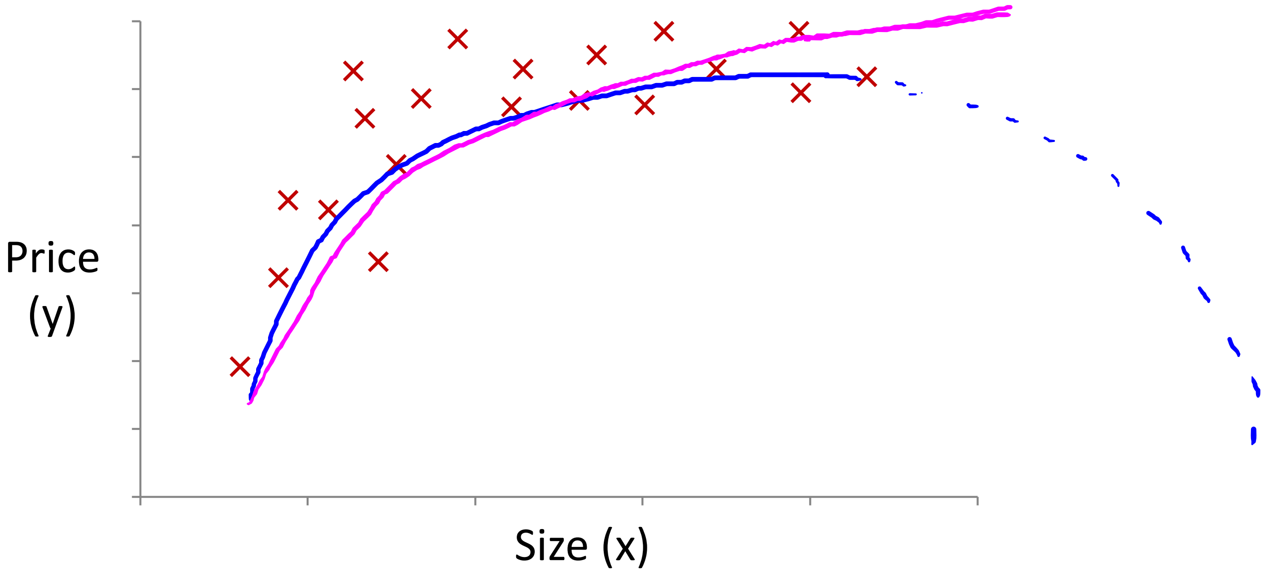
$$x_1 = (size)$$
$$x_2 = (size)^2$$
$$x_3 = (size)^3$$

Size: 1 - 1000

Size²: 1 - 1,000,000

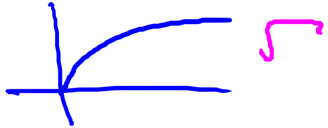
Size³: 1 - 10⁹

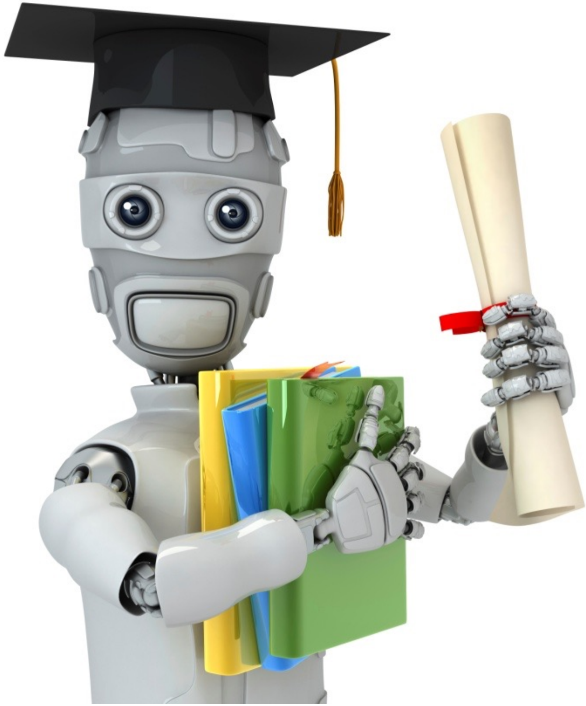
Choice of features



$$\rightarrow \quad h_\theta(x) = \theta_0 + \theta_1(size) + \theta_2(size)^2$$

$$\rightarrow \quad \boxed{h_\theta(x) = \theta_0 + \theta_1(size) + \theta_2\sqrt{(size)}}$$



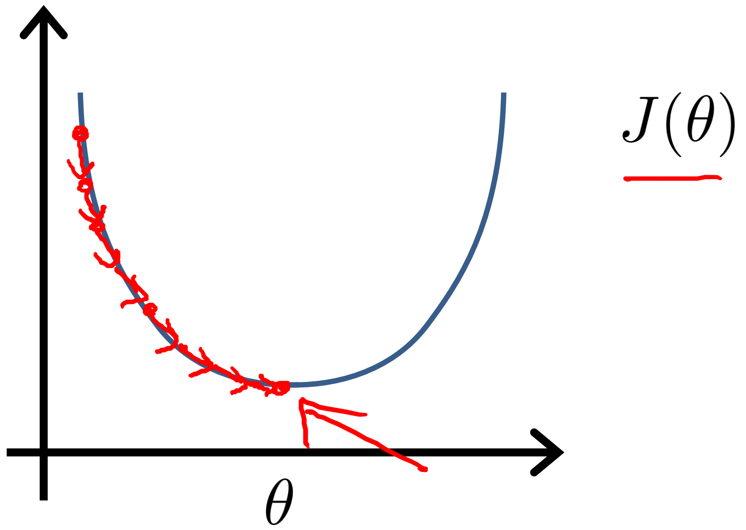


Machine Learning

Linear Regression with multiple variables

Normal equation

Gradient Descent



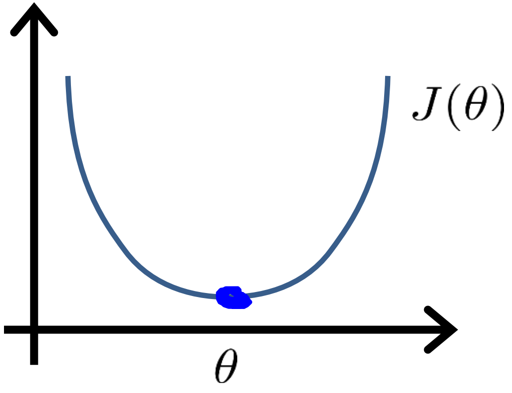
Normal equation: Method to solve for θ analytically.

Intuition: If 1D ($\theta \in \mathbb{R}$)

$$J(\theta) = a\theta^2 + b\theta + c$$

$$\frac{d}{d\theta} J(\theta) = \dots \stackrel{\text{set}}{=} 0$$

Solve for θ



$\theta \in \mathbb{R}^{n+1}$ $\qquad$ $J(\theta_0, \theta_1, \dots, \theta_m) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$

$$\frac{\partial}{\partial \theta_j} J(\theta) = \cdots \stackrel{\text{set}}{=} 0 \qquad \text{(for every } j\text{)}$$

Solve for $\theta_0, \theta_1, \dots, \theta_n$

Examples: $m = 4.$

x_0	Size (feet²) x_1	Number of bedrooms x_2	Number of floors x_3	Age of home (years) x_4	Price ($1000) y
1	2104	5	1	45	460
1	1416	3	2	40	232
1	1534	3	2	30	315
1	852	2	1	36	178

$$X = \begin{bmatrix} 1 & 2104 & 5 & 1 & 45 \\ 1 & 1416 & 3 & 2 & 40 \\ 1 & 1534 & 3 & 2 & 30 \\ 1 & 852 & 2 & 1 & 36 \end{bmatrix} \qquad y = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

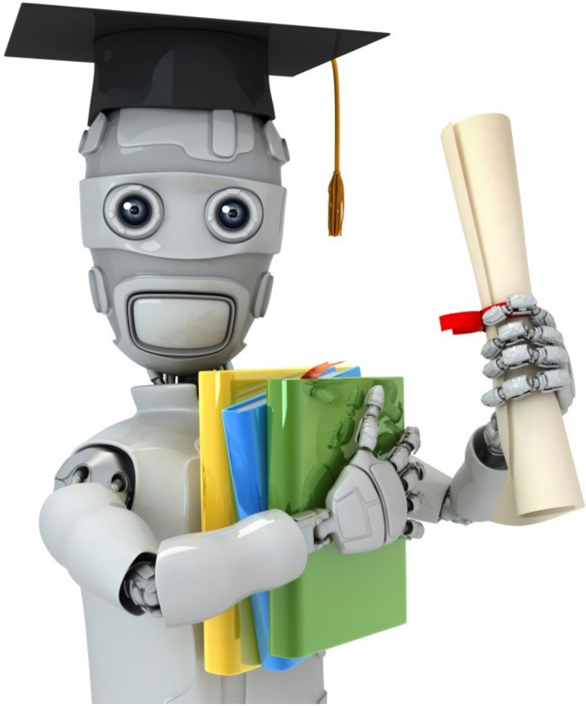
$m \times (n+1)$

m-dimensional vector

$$\boxed{\theta = (X^T X)^{-1} X^T y}$$

m training examples, n features.

Gradient Descent	Normal Equation
• Need to choose α.	• No need to choose α.
• Needs many iterations.	• Don't need to iterate.
• Works well even when n is large.	• Need to compute $\boxed{(\boxed{X^TX})^{-1}}$ $n \times n$ $O(n^3)$
$n = 10^6$	• Slow if n is very large.
	$n = 100$
	$n = 1000$
	$n = 10000$



Machine Learning

Linear Regression with multiple variables

Normal equation and non-invertibility (optional)

Normal equation

$$\theta = (X^TX)^{-1}X^Ty$$

X^TX

- What if $\boxed{X^TX}$ is non-invertible? (singular/ degenerate)
- Octave: `pinv(X'*X)*X'*y`

θ

pinv

inv

What if $\boxed{X^T X}$ is non-invertible?

- Redundant features (linearly dependent).

  E.g. x_1 = size in feet^2

  ~~x_2 = size in m^2~~

  $x_1 = (3.28)^2 x_2$

  1m = 3.28 feet

  $m = 10$

  $n = 100$

  $\theta \in \mathbb{R}^{101}$

- Too many features (e.g. $m \leq n$).
  - Delete some features, or use regularization.

    later

Hồi quy đa biến

Trong tập dữ liệu tôi gửi bao gồm 17000 hàng và 9 cột, trong đó:

- 17000 hàng là số lượng dữ liệu (m)
- 9 cột: 8 cột đầu (n) là các đặc trưng (x), cột cuối cùng là giá nhà (y)

Các bạn làm các công việc sau:

1. Chuẩn hóa dữ liệu: scale dữ liệu, format kích thước dữ liệu
2. Viết chương trình cho phép học các tham số của mô hình hồi quy tuyến tính đa biến
3. Tính J ở mỗi vòng lặp, và vẽ biểu đồ J ở các giá trị learning rate khác nhau sau khi chạy hết các vòng lặp.
4. Kiểm chứng các theta mà các bạn tìm được bằng phương pháp gradient descent với phương pháp normal equation.

Hồi quy đa bậc

Trong tập dữ liệu tôi gửi bao gồm 84 hàng và 2 cột, trong đó:

- 84 hàng là số lượng dữ liệu (m)
- 2 cột: cột đầu là đặc trưng (x), cột cuối là nhãn (y)

Các bạn làm các công việc sau:

1. Biểu diễn dữ liệu
2. Chọn mô hình "h" phù hợp (bậc bao nhiêu?)
3. Chuyển bài toán đa bậc thành bài toán đa biến và chuẩn hóa dữ liệu: scale dữ liệu, format kích thước dữ liệu
4. Viết chương trình cho phép học các tham số của mô hình hồi quy tuyến tính đa biến
5. Tính J ở mỗi vòng lặp, và vẽ biểu đồ J ở các giá trị learning rate khác nhau sau khi chạy hết các vòng lặp.
6. Biển diễn đường cong học được và dữ liệu trên cùng 1 hình ảnh.