Chapter 4. Motion Kinematics

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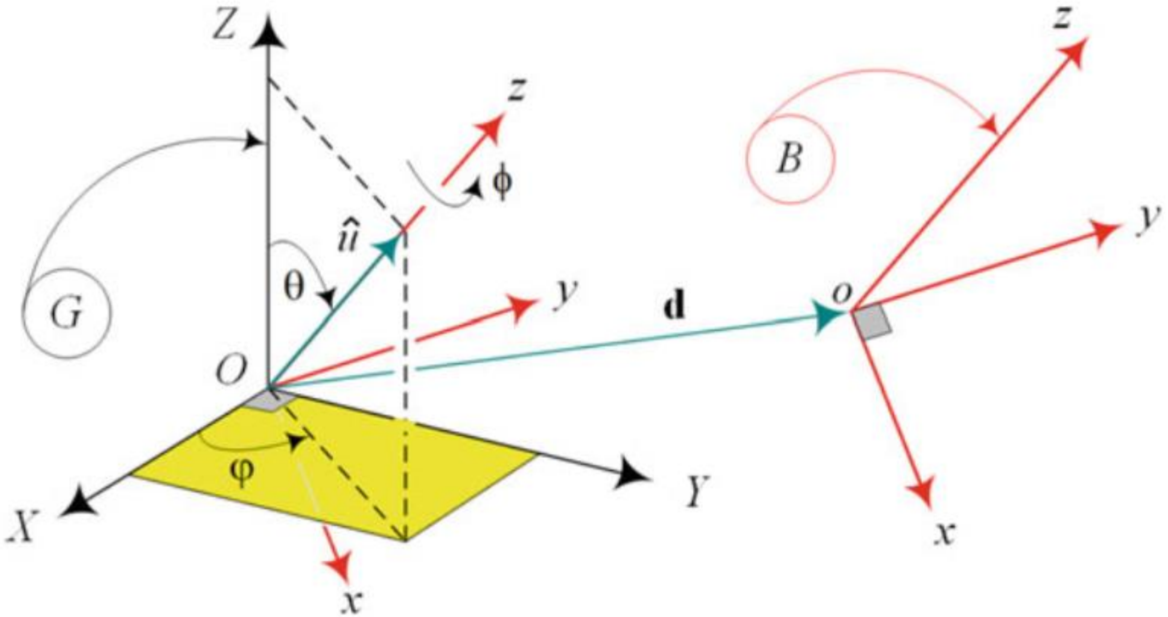
- The most general motion of **a rigid body B** in **a global frame G** is made by **a rotation ϕ** about **an axis û (vector)**, plus **a displacement d**.
- The rigid body motion may be expressed by a **3×3 rotation matrix** plus **a 3×1 displacement vector**.
- Expressing by **a 4 × 4 homogenous transformation matrix.**



Rotation and translation of a local frame with respect to a global frame

4.1. Rigid Body Motion

The displacement or translation

Consider a rigid body with an attached **body coordinate frame B** (*oxyz*) moving in a fixed **global coordinate frame G**(*OXYZ*).

- The displacement **vector** ${}^{G}\boldsymbol{d}$ indicates the position of the **moving origin o relative to the fixed origin O**, then the coordinates of a body **point P** in local and global frames are related by

$$ {}^{G}\mathbf{r}_{P} = {}^{G}\mathbf{R}_{B}\, {}^{B}\mathbf{r}_{P} + {}^{G}\mathbf{d} $$

where,
$$ {}^{G}\mathbf{r}_{P} = \begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} \qquad {}^{B}\mathbf{r}_{P} = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} \qquad {}^{G}\mathbf{d} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} $$

The vector ${}^{G}\boldsymbol{d}$ is called **the displacement or translation** of B with respect to G; ${}^{G}\boldsymbol{R}_{B}$ is the rotation matrix, when ${}^{G}\boldsymbol{d} = \mathbf{0}$.

4.1. Rigid Body Motion

Example 1: Translation and rotation of a body coordinate frame

A body coordinate frame B(*oxyz*), that is originally coincident with global coordinate frame G(*OXYZ*), rotates 45 [deg] about the *X*-axis and translates to $[3\ 5\ 7]^T$. Find, the global position of a point at $^B\mathbf{r} = [x\ y\ z]^T$.

$$^G\mathbf{r} = {}^G R_B\, {}^B\mathbf{r} + {}^G\mathbf{d}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

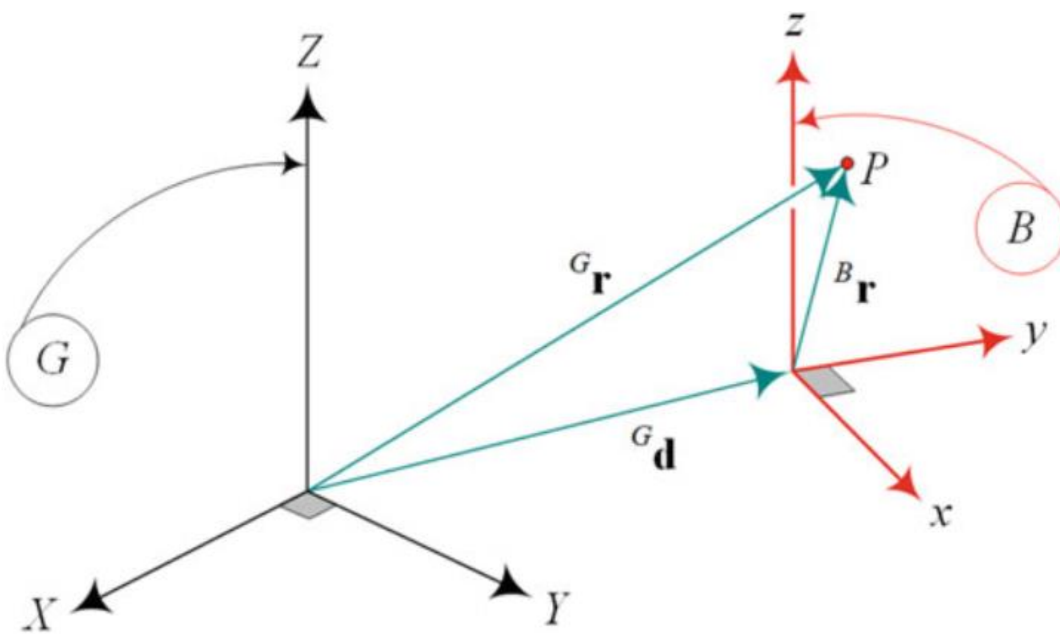
$$= \begin{bmatrix} x+3 \\ \frac{1}{2}\sqrt{2}y - \frac{1}{2}\sqrt{2}z + 5 \\ \frac{1}{2}\sqrt{2}y + \frac{1}{2}\sqrt{2}z + 7 \end{bmatrix}$$

$$= (x+3)\,\hat{I} + \left(\frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}z + 5\right)\hat{J} + \left(\frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z + 7\right)\hat{K}$$

4.1. Rigid Body Motion

Example 2: Moving body coordinate frame

A point **P** at ${}^{B}\boldsymbol{r}_{P} = 0.1\hat{\imath} + 0.3\hat{\jmath} + 0.3\hat{k}$ in **a body frame B**, which is rotated 50 [deg] about the Z-axis and translated -1 along X, 0.5 along Y, and 0.2 along the Z axes. Find the position ${}^{\mathbf{G}}\mathbf{r}_{\mathbf{P}}$ of **P in global coordinate frame.**



4.1. Rigid Body Motion

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$$
\begin{aligned}
{}^{G}\mathbf{r}_{P} &= {}^{G}R_{B}\,{}^{B}\mathbf{r}_{P} + {}^{G}\mathbf{d} \\
&= \begin{bmatrix} \cos\frac{50\pi}{180} & -\sin\frac{50\pi}{180} & 0 \\ \sin\frac{50\pi}{180} & \cos\frac{50\pi}{180} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0.5 \\ 0.2 \end{bmatrix} \\
&= \begin{bmatrix} -1.166 \\ 0.769 \\ 0.5 \end{bmatrix}
\end{aligned}
$$

4.1. Rigid Body Motion

Example 3: Rotation of a translated rigid body

A Point P of a rigid body B has an initial position vector $^{B}\boldsymbol{r}_{P} = [1\ 2\ 3]^{T}$. If the body rotates 45 [deg] about the *x*-axis and then translates to $^{G}\boldsymbol{d} = [4\ 5\ 6]^{T}$, find the final global position of **P** .

$$^{G}\mathbf{r} = {}^{B}R^{T}_{x,\pi/4}\,{}^{B}\mathbf{r}_{P} + {}^{G}\mathbf{d}$$

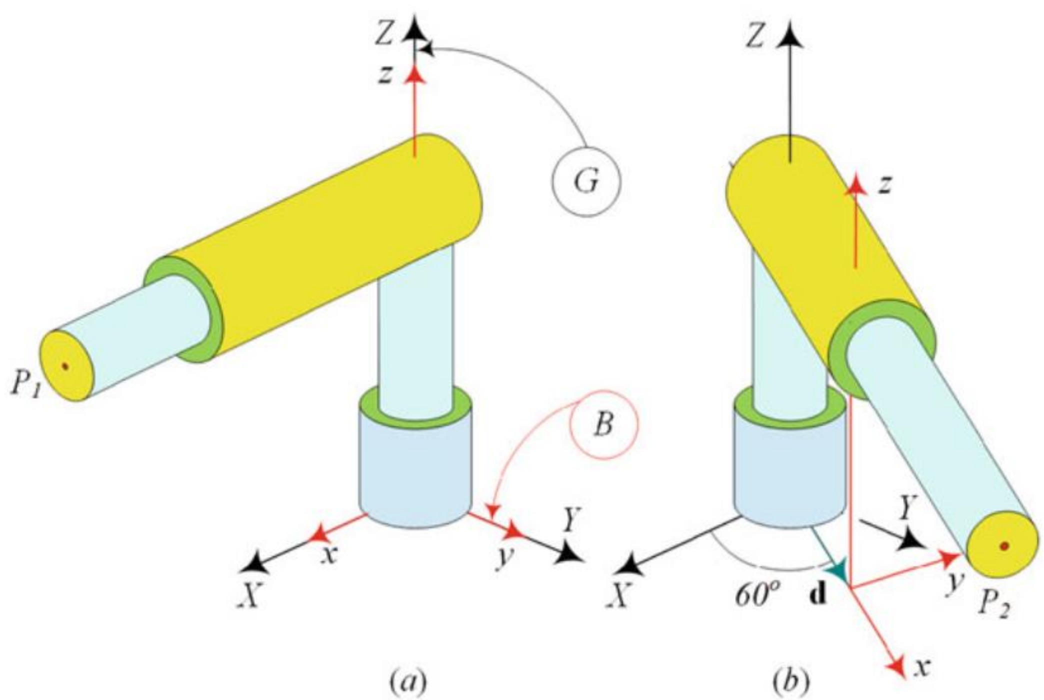
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5.0 \\ 4.29 \\ 9.53 \end{bmatrix}$$

The rotation occurs first when $^{G}\mathbf{d} = \mathbf{0}$ and then **translation happens.**

4.1. Rigid Body Motion

Example 4: Arm rotation plus elongation !?

Position vector of point P_1 at the tip of a PR arm shown in Figure is at ${}^G r_{P1} = {}^B r_{P1} = [1350\ 0\ 900]^T$ mm. The arm rotates 60 [deg] about the global Z-axis and elongates by $\mathbf{d} = 720.2\hat{\imath}$ mm. The final configuration of the arm is shown in Figure.



4.2. Homogenous Transformation

- A rigid body with coordinate frame ***B*** is moving in a globally fixed coordinate frame ***G***. The position vector of an arbitrary point P of the rigid body is denoted by $^{B}\boldsymbol{r}_{P}$ and $^{G}\boldsymbol{r}_{\mathrm{P}}$ in the frames.
- The **translation vector** $^{G}\boldsymbol{d}$ indicates the position of origin **o** of the body frame **B** in the global frame **G**.
- The general motion of a rigid body **B** (*oxyz*) in the global frame **G** (**OXYZ**) is **a combination of rotation** $^{G}\boldsymbol{R}_{B}$ **and translation** $^{G}\boldsymbol{d}$**.**

$$^{G}\mathbf{r} = {}^{G}\mathbf{R}_{B}\,{}^{B}\mathbf{r} + {}^{G}\mathbf{d}$$

4.2. Homogenous Transformation

Combining **a rotation matrix plus a vector** can be expressed better by **homogenous transformation matrix**. Introducing a **4 × 4 homogenous transformation matrix $^{G}T_{B}$**

$$^{G}\mathbf{r} = {}^{G}\mathbf{T}_{B}\, {}^{B}\mathbf{r}$$

where,

$$^{G}T_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & X_o \\ r_{21} & r_{22} & r_{23} & Y_o \\ r_{31} & r_{32} & r_{33} & Z_o \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\equiv \left[\begin{array}{c|c} {}^{G}R_{B} & {}^{G}\mathbf{d} \\ 0\ 0\ 0 & 1 \end{array}\right] \equiv \begin{bmatrix} {}^{G}R_{B} & {}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix}$$

$$^{G}\mathbf{r} = \begin{bmatrix} X_P \\ Y_P \\ Z_P \\ 1 \end{bmatrix} \qquad {}^{B}\mathbf{r} = \begin{bmatrix} x_P \\ y_P \\ z_P \\ 1 \end{bmatrix} \qquad {}^{G}\mathbf{d} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{bmatrix}$$

4.2. Homogenous Transformation

- Representation of an n-component position vector by an (n + 1)-component vector is called ***homogenous coordinate representation***.

- The appended element is *a scale factor*, *w*; hence, in general, homogenous representation of a position vector **r** = [*x y z*]^T is

$$\mathbf{r} = \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ w \end{bmatrix}$$

- Instead, it is the three ratios, r_1/w, r_2/w, and r_3/w, that are important because, provided $w \neq 0$, and $w \neq \infty$, w

$$\begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

4.2. Homogenous Transformation

Example 5: Rotation and translation of a body coordinate frame

A body coordinate frame B(*oxyz*), that is originally coincident with global coordinate frame G(*OXYZ*), rotates 45 deg about the *X*-axis and translates to $[3\ 5\ 7\ 1]^T$. Find the matrix representation of the global position of a body point at ${}^{B}r = [x\ y\ z\ 1]^T$

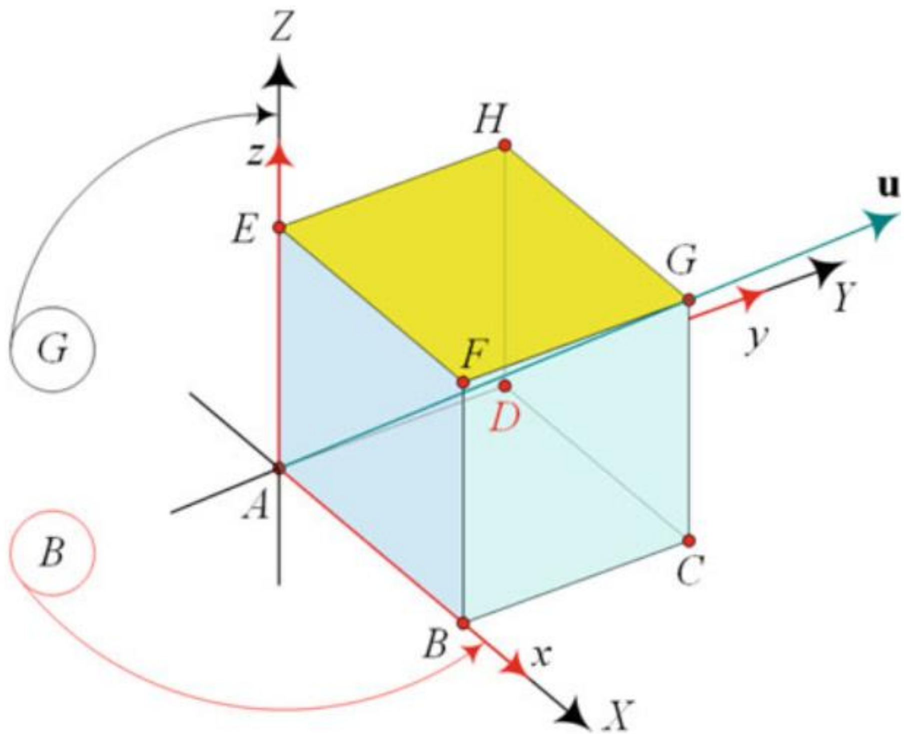
$$^{G}\mathbf{r} = {}^{G}T_{B}\,{}^{B}\mathbf{r}$$

$$= \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 5 \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+3 \\ 0.707y - 0.707z + 5 \\ 0.707y + 0.707z + 7 \\ 1 \end{bmatrix}$$

4.2. Homogenous Transformation

Example 6: An axis–angle rotation and a translation

A cubic rigid body with a unit length of edges sits at the corner of the first quadrant as is shown in Figure. If we turn the cube 45 deg about $\boldsymbol{u} = [1\ 1\ 1]^T$ and translate it by ${}^G\boldsymbol{d} = [1\ 1\ 1]^T$, determine the coordinates of the corners of the cube after the rigid body motion.



4.2. Homogenous Transformation

Decomposition of ${}^{G}T_{B}$ into translation and rotation

Homogenous transformation matrix $^{G}\mathbf{T}_{B}$ can be decomposed into a matrix multiplication of *a pure rotation matrix* $^{G}\mathbf{R}_{B}$ and *a pure translation matrix* $^{G}\mathbf{D}_{B}$.

$$
\begin{aligned}
{}^{G}T_{B} &= {}^{G}D_{B}\,{}^{G}R_{B} \\
&= \begin{bmatrix} 1 & 0 & 0 & X_o \\ 0 & 1 & 0 & Y_o \\ 0 & 0 & 1 & Z_o \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & X_o \\ r_{21} & r_{22} & r_{23} & Y_o \\ r_{31} & r_{32} & r_{33} & Z_o \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Decomposition of ***a homogenous transformation to translation*** and ***rotation*** is **not interchangeable**

$$
{}^{G}T_{B} = {}^{G}D_{B}\,{}^{G}R_{B} \neq {}^{G}R_{B}\,{}^{G}D_{B}
$$

4.2. Homogenous Transformation

Example 7: Rotation about and translation along a global and local axes

A point P is located at $^{B}\mathbf{r}$ = (0, 0, 20) in a body coordinate frame. If the rigid body rotates 30 deg about the global *X*-axis and the origin of the body frame translates to (*X*, *Y*, *Z*) = (50, 0, 60), then the coordinates of the point in the global frame are

$$\begin{bmatrix} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ 0 & \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ -10 \\ 77.3 \\ 1 \end{bmatrix}$$

4.3. Inverse and Reverse Homogenous Transformation

The advantage of simplicity to work with homogenous transformation matrices come with the penalty of losing the orthogonality property. If we show a rigid body motion by the homogenous transformation $\mathbf{^GT_B}$

$$^GT_B = \begin{bmatrix} \mathbf{I} & ^G\mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ^GR_B & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ^GR_B & ^G\mathbf{d} \\ 0 & 1 \end{bmatrix}$$

The inverse of homogenous transformation matrix $\mathbf{^GT_B}$ is

$$^BT_G = {^GT_B^{-1}} = \begin{bmatrix} ^GR_B & ^G\mathbf{d} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} ^GR_B^T & -{^GR_B^T}\,{^G\mathbf{d}} \\ 0 & 1 \end{bmatrix}$$

4.3. Inverse and Reverse Homogenous Transformation

The reverse motion of $^{\mathbf{G}}\mathbf{T}_{\mathbf{B}}$ would be $^{\mathbf{G}}\mathbf{T}_{-\mathbf{B}}$

$$ {}^{G}T_{-B} = \begin{bmatrix} {}^{G}R_B^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -{}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{G}R_B^T & -{}^{G}R_B^T\,{}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} $$

Showing that

$$ {}^{G}T_B^{-1}\,{}^{G}T_B = \mathbf{I}_4 $$

$$ {}^{G}T_{-B}\,{}^{G}T_B = \mathbf{I}_4 $$

where I_4 is the identity matrix of rank 4.

Note a shortcoming is that they lose the orthogonality property

$$ {}^{G}T_B^{-1} \neq {}^{G}T_B^{T} \qquad {}^{G}T_B^{-1} = {}^{B}T_G \qquad {}^{G}T_B^{-1}\,{}^{G}T_B = {}^{B}T_G\,{}^{G}T_B = \mathbf{I}_4 $$

4.3. Inverse and Reverse Homogenous Transformation

Example 8: Inverse of a homogenous transformation matrix

Assume that
$$ {}^{G}T_{B} = \begin{bmatrix} 0.643 & -0.766 & 0 & -1 \\ 0.766 & 0.643 & 0 & 0.5 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} $$

$$ {}^{G}R_{B} = \begin{bmatrix} 0.643 & -0.766 & 0 \\ 0.766 & 0.643 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{G}\mathbf{d} = \begin{bmatrix} -1 \\ 0.5 \\ 0.2 \end{bmatrix} $$

$$ {}^{B}T_{G} = {}^{G}T_{B}^{-1} = \begin{bmatrix} {}^{G}R_{B}^{T} & -{}^{G}R_{B}^{T}\,{}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.643 & 0.766 & 0 & 0.26 \\ -0.766 & 0.643 & 0 & -1.087 \\ 0 & 0 & 1 & -0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

4.3. Inverse and Reverse Homogenous Transformation

Quick inverse transformation

Decompose a transformation matrix into rotation **[R]** and displacement **[D]** and take advantage of the inverse of matrix multiplication.

$$T^{-1} = [DR]^{-1} = R^{-1}D^{-1} = R^T D^{-1}$$

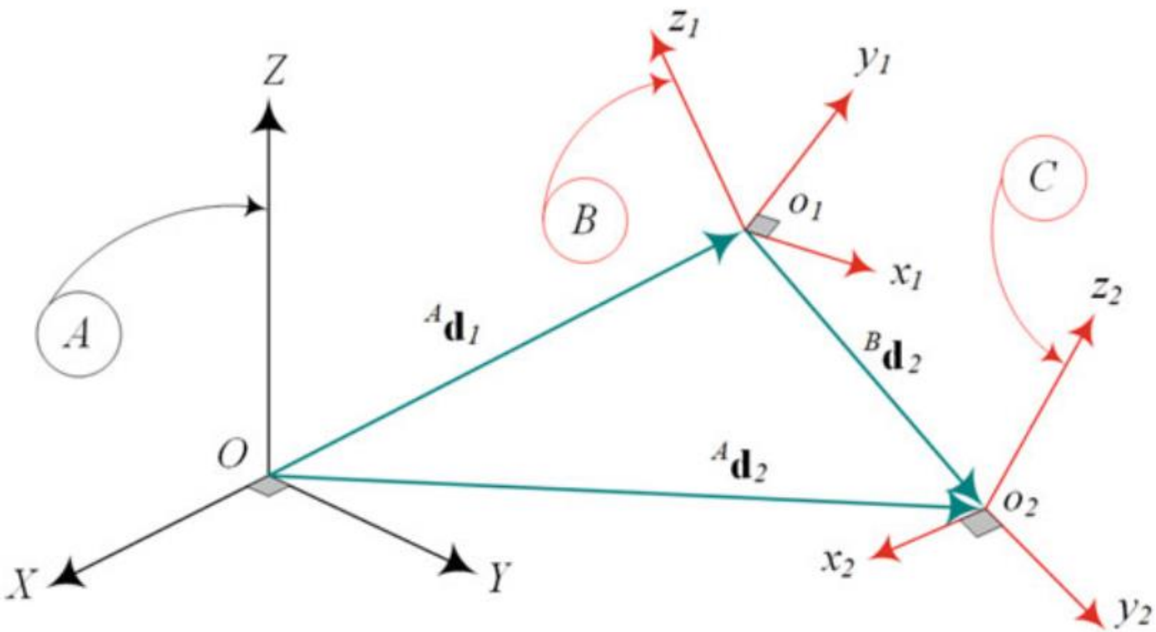
Consider a homogenous matrix **[T]**

$$[T] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & r_{14} \\ 0 & 1 & 0 & r_{24} \\ 0 & 0 & 1 & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = [DR]^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -r_{14} \\ 0 & 1 & 0 & -r_{24} \\ 0 & 0 & 1 & -r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.4. Combined Homogenous Transformation

The transformation matrices to transform coordinates from frame B to A and from frame C to B are



$$^AT_B = \begin{bmatrix} ^AR_B & ^A\mathbf{d}_1 \\ 0 & 1 \end{bmatrix}$$

$$^BT_C = \begin{bmatrix} ^BR_C & ^B\mathbf{d}_2 \\ 0 & 1 \end{bmatrix}$$

4.4. Combined Homogenous Transformation

The transformation matrix from C to A is

$$ {}^{A}T_C = {}^{A}T_B\,{}^{B}T_C = \begin{bmatrix} {}^{A}R_B & {}^{A}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}R_C & {}^{B}\mathbf{d}_2 \\ 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} {}^{A}R_B\,{}^{B}R_C & {}^{A}R_B\,{}^{B}\mathbf{d}_2 + {}^{A}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_C & {}^{A}\mathbf{d}_2 \\ 0 & 1 \end{bmatrix} $$

The inverse transformation is

$$ {}^{C}T_A = \begin{bmatrix} {}^{B}R_C^T\,{}^{A}R_B^T & -\,{}^{B}R_C^T\,{}^{A}R_B^T\,\left[{}^{A}R_B\,{}^{B}\mathbf{d}_2 + {}^{A}\mathbf{d}_1\right] \\ 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} {}^{B}R_C^T\,{}^{A}R_B^T & -\,{}^{B}R_C^T\,{}^{B}\mathbf{d}_2 - {}^{A}R_C^T\,{}^{A}\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} {}^{A}R_C^T & -\,{}^{A}R_C^T\,{}^{A}\mathbf{d}_2 \\ 0 & 1 \end{bmatrix} $$

4.4. Combined Homogenous Transformation

The value of homogenous coordinates are better appreciated when several displacements occur in succession, which, for instance, can be written as

$$ {}^{G}T_4 = {}^{G}T_1 \, {}^{1}T_2 \, {}^{2}T_3 \, {}^{3}T_4 $$

Rather than

$$ \begin{aligned} & {}^{G}R_4 \, {}^{4}\mathbf{r}_P + {}^{G}\mathbf{d}_4 \\ & \quad = {}^{G}R_1 \left({}^{1}R_2 \left({}^{2}R_3 \left({}^{3}R_4 \, {}^{4}\mathbf{r}_P + {}^{3}\mathbf{d}_4 \right) + {}^{2}\mathbf{d}_3 \right) + {}^{1}\mathbf{d}_2 \right) + {}^{G}\mathbf{d}_1 \end{aligned} $$

4.4. Combined Homogenous Transformation

Example 9: A rotating cylinder

Imagine a cylinder with radius R = 2 that is going to turn about the axis $\hat{\mathbf{u}} = [0\ 0\ 1]^T$ at $\mathbf{d} = [2\ 0\ 0]^T$. If the cylinder turns 90 deg about its axis, then every point on the periphery of the cylinder will move 90 deg on circular paths parallel to (x, y)-plane.

$$
{}^G T_B = D_{\hat{d},d}\, R_{\hat{u},\phi}\, D_{\hat{d},-d} = \begin{bmatrix} \mathbf{I} & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{\hat{K},\frac{\pi}{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{d} \\ 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\frac{\pi}{2} & -s\frac{\pi}{2} & 0 & 0 \\ s\frac{\pi}{2} & c\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

4.4. Combined Homogenous Transformation

Example 9: A rotating cylinder

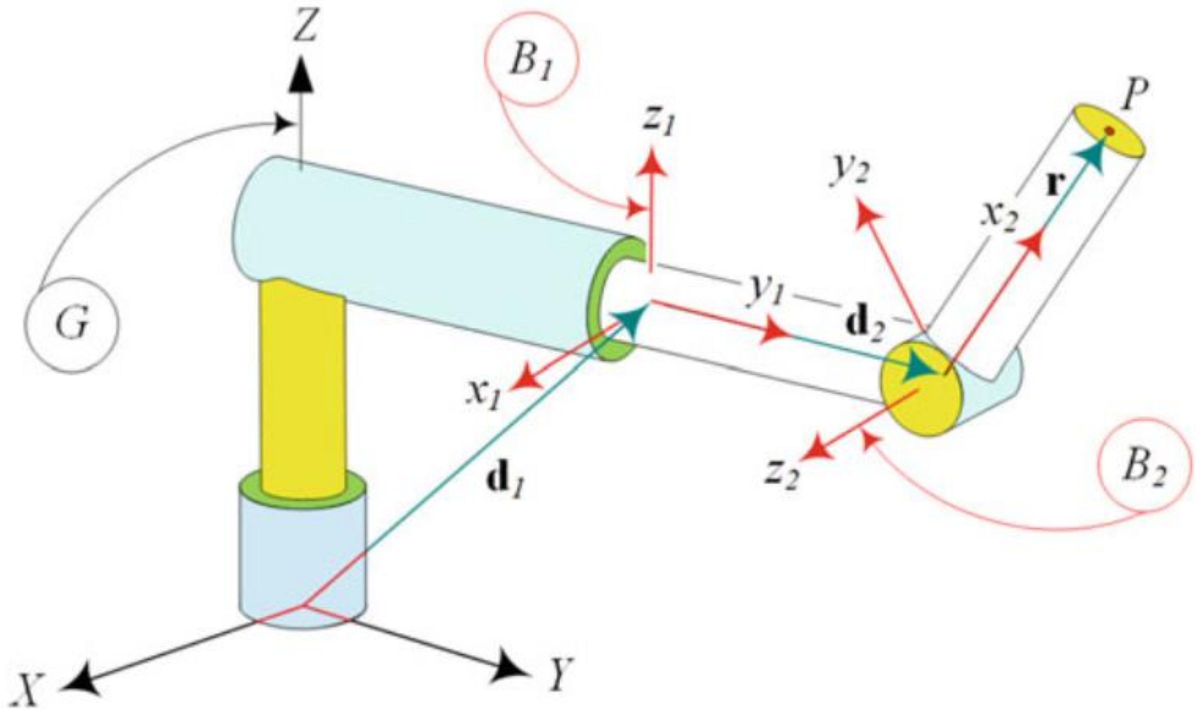
Consider a point of cylinder that was on the origin. After the rotation, the point would be seen at:

$$
\begin{aligned}
{}^{G}\mathbf{r} &= {}^{G}T_{B}\,{}^{G}\mathbf{r} \\
&= \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

4.4. Combined Homogenous Transformation

Example 10: End-effector of an RPR robot in a global frame

Position vector of ***P*** in frame B_2 $(x_2y_2z_2)$ is ${}^2\mathbf{r}_P$. Frame B_2 $(x_2y_2z_2)$ at location ${}^G\mathbf{d}_1$ can rotate about z_2 and slide along y_1. Frame B_1 $(x_1y_1z_1)$ can rotate about the Z-axis of the global frame G($OXYZ$).



4.4. Combined Homogenous Transformation

Example 10: End-effector of an RP R robot in a global frame

The position of the origin of B_1 is shown by ${}^1\mathbf{d}_2$ in B_2. To determine the position of P in G($OXYZ$), we add ${}^G\mathbf{d}_1$ and ${}^G\mathbf{d}_2$ and ${}^G\mathbf{r}_P$.

$$
\begin{aligned}
{}^G\mathbf{r} &= {}^GR_1\,{}^1R_2\,{}^2\mathbf{r}_P + {}^GR_1\,{}^1\mathbf{d}_2 + {}^G\mathbf{d}_1 = {}^GT_1\,{}^1T_2\,{}^2\mathbf{r}_P \\
&= {}^GT_2\,{}^2\mathbf{r}_P
\end{aligned}
$$

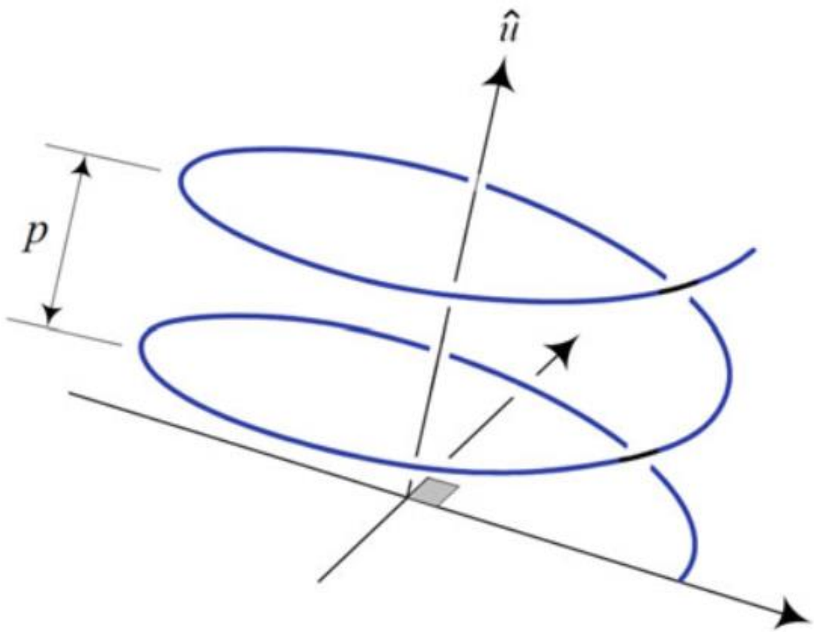
where,

$$
{}^GT_1 = \begin{bmatrix} {}^GR_1 & {}^G\mathbf{d}_1 \\ 0 & 1 \end{bmatrix} \qquad {}^1T_2 = \begin{bmatrix} {}^1R_2 & {}^1\mathbf{d}_2 \\ 0 & 1 \end{bmatrix}
$$

$$
{}^GT_2 = \begin{bmatrix} {}^GR_1\,{}^1R_2 & {}^GR_1\,{}^1\mathbf{d}_2 + {}^G\mathbf{d}_1 \\ 0 & 1 \end{bmatrix}
$$

4.5. Screw Coordinates

Any rigid body motion can be replaced by **a single translation** along an axis combined with **a unique rotation** about that axis. Such a motion is called **screw motion**.



A point **P** that **rotates about the screw** axis **û** and **simultaneously translates** along the same axis **û**. Hence, any point on the **screw axis** moves along the axis, while any point off the axis moves along ***a helix***.

4.5. Screw Coordinates

The **angular rotation** of the rigid body about the screw is called **twist**. A screw motion is indicated by its **pitch**, p, that is **the ratio of translation**, h, to rotation, ϕ.

$$p = \frac{h}{\phi}$$

The rectilinear distance h through which the rigid body translates parallel to the axis of screw $\hat{\mathbf{u}}$ for a unit rotation ϕ is the pitch p. If $\boldsymbol{p} > \mathbf{0}$, then the screw is **right-handed**, and if $\boldsymbol{p} < \mathbf{0}$, it is **left-handed**.

A screw motion $\check{\mathbf{s}}$ is shown by $\check{s}(h, \phi, \hat{u}, \mathbf{s})$ and is indicated by a twist axis unit vector $\hat{u}$, a location vector $\mathbf{s}$, a twist angle ϕ, and a translation h (or a pitch p).

The location vector $\mathbf{s}$ indicates the global position of a point on the screw axis. The twist angle ϕ, the twist axis $\hat{u}$, and the pitch p (or translation h) are called **screw parameters**.

4.5. Screw Coordinates

For a central screw motion, we have

$$^G\check{s}_B(h, \phi, \hat{u}) = D_{\hat{u},h}\ R_{\hat{u},\phi}$$

where,

$$D_{\hat{u},h} = \begin{bmatrix} 1 & 0 & 0 & hu_1 \\ 0 & 1 & 0 & hu_2 \\ 0 & 0 & 1 & hu_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{\hat{u},\phi} = \begin{bmatrix} u_1^2 \operatorname{vers}\phi + c\phi & u_1u_2 \operatorname{vers}\phi - u_3 s\phi & u_1u_3 \operatorname{vers}\phi + u_2 s\phi & 0 \\ u_1u_2 \operatorname{vers}\phi + u_3 s\phi & u_2^2 \operatorname{vers}\phi + c\phi & u_2u_3 \operatorname{vers}\phi - u_1 s\phi & 0 \\ u_1u_3 \operatorname{vers}\phi - u_2 s\phi & u_2u_3 \operatorname{vers}\phi + u_1 s\phi & u_3^2 \operatorname{vers}\phi + c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.5. Screw Coordinates

Hence,

$$
{}^{G}\check{s}_{B}(h,\phi,\hat{u}) =
\begin{bmatrix}
u_1^2 \operatorname{vers}\phi + c\phi & u_1u_2 \operatorname{vers}\phi - u_3 s\phi & u_1u_3 \operatorname{vers}\phi + u_2 s\phi & hu_1 \\
u_1u_2 \operatorname{vers}\phi + u_3 s\phi & u_2^2 \operatorname{vers}\phi + c\phi & u_2u_3 \operatorname{vers}\phi - u_1 s\phi & hu_2 \\
u_1u_3 \operatorname{vers}\phi - u_2 s\phi & u_2u_3 \operatorname{vers}\phi + u_1 s\phi & u_3^2 \operatorname{vers}\phi + c\phi & hu_3 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

As a result, a central screw transformation matrix indicates ***a pure translation*** corresponds to $\phi = 0$, and a pure rotation corresponds to h = 0 (or p = ∞).

A reverse central screw is defined as $\check{s}(-h, -\phi, \hat{u})$.

C4. End!