Chapter 5. Forward Kinematics

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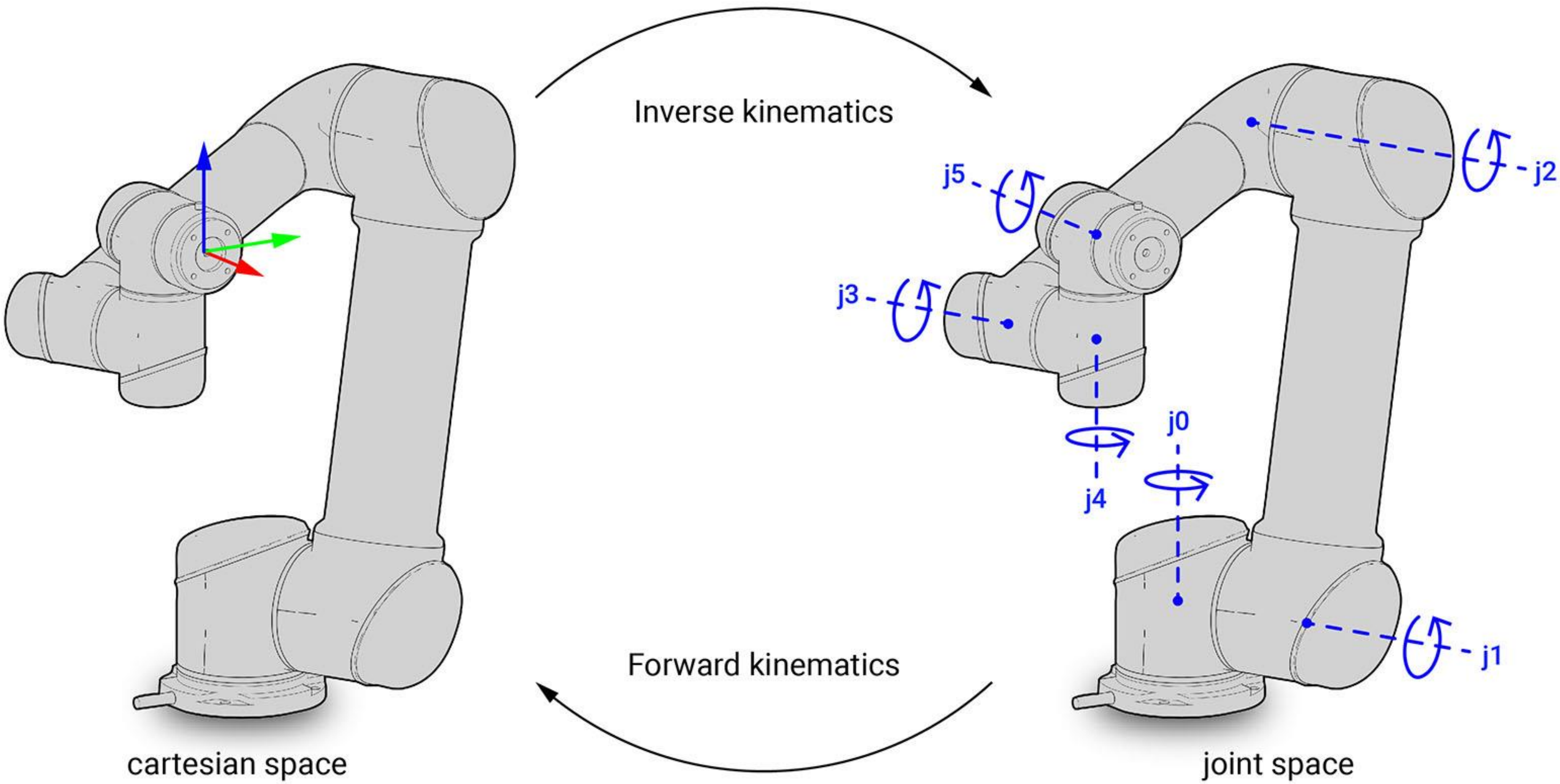
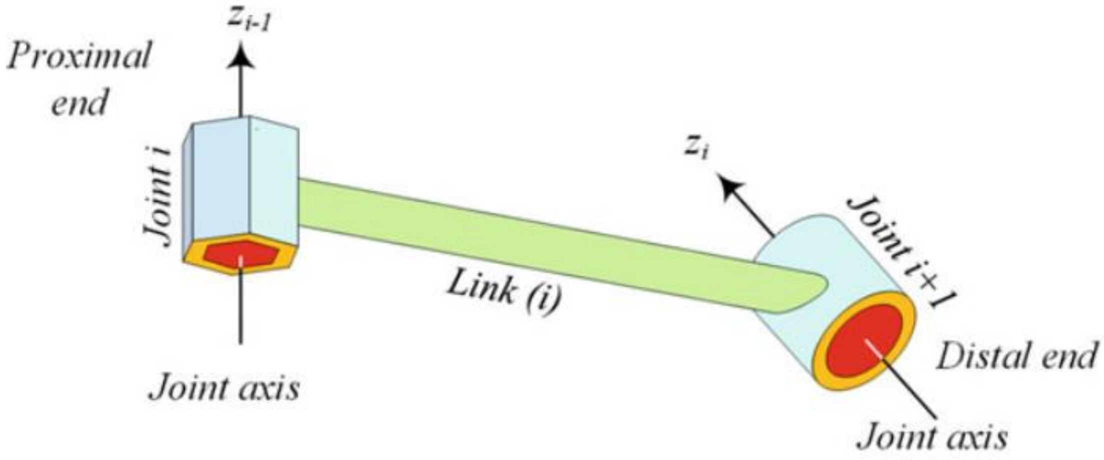


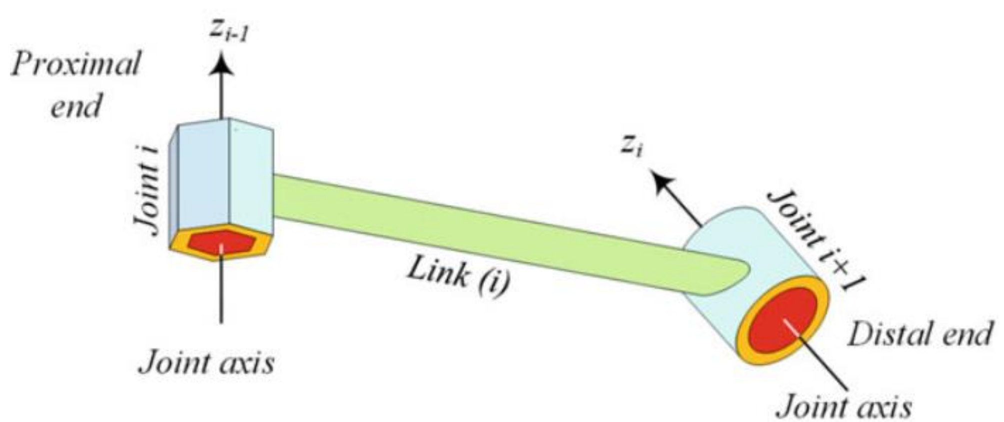
Forward and inverse kinematics

- **Forward kinematics** means having **the joint variables** of a robot, we are able to determine **the position and orientation** of every link of the robot, including **the end-effector**.
- The **analysis of determination** of **position and orientation of all links** of **a robot relative to each other** is called **forward kinematics**.



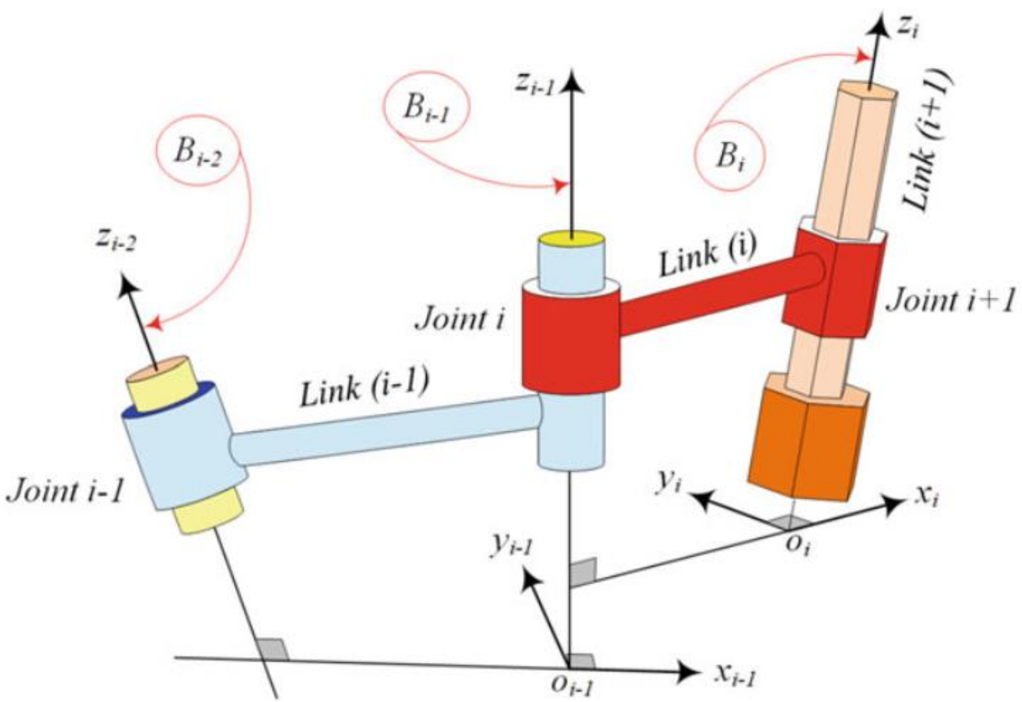
5.1. Denavit–Hartenberg Notation

- A serial robot with ***n* joints** will have ***n + 1* links**.
- Numbering of links starts from **link (0)** for the immobile **grounded base link** and increases sequentially **up to link (n) for the end-effector.**
- Numbering of **joints starts from 1**, and increases sequentially up to **joint n**.
- The link (i) is connected to its lower link ($i - 1$) at its proximal end by joint i and is connected to its upper link ($i + 1$) at its distal end by joint $i + 1$.



5.1. Denavit–Hartenberg Notation

- Figure illustrates **links ($i-1$), (i), and ($i+1$)** of a serial robot, along with **joints $i-1$, i, and $i+1$**.
- Numbering of links starts from **link (0)** for the immobile **grounded base link** and increases sequentially **up to link (n) for the end-effector.** Every joint is indicated by a joint axis, which will be either translational or rotational.



- We rigidly attach a local coordinate frame $\boldsymbol{B_i}$ to every link (i) at joint $i+1$ based on the following standard method, known as **Denavit–Hartenberg (DH) method**.

5.1. Denavit–Hartenberg Notation

Denavit–Hartenberg (DH) principles

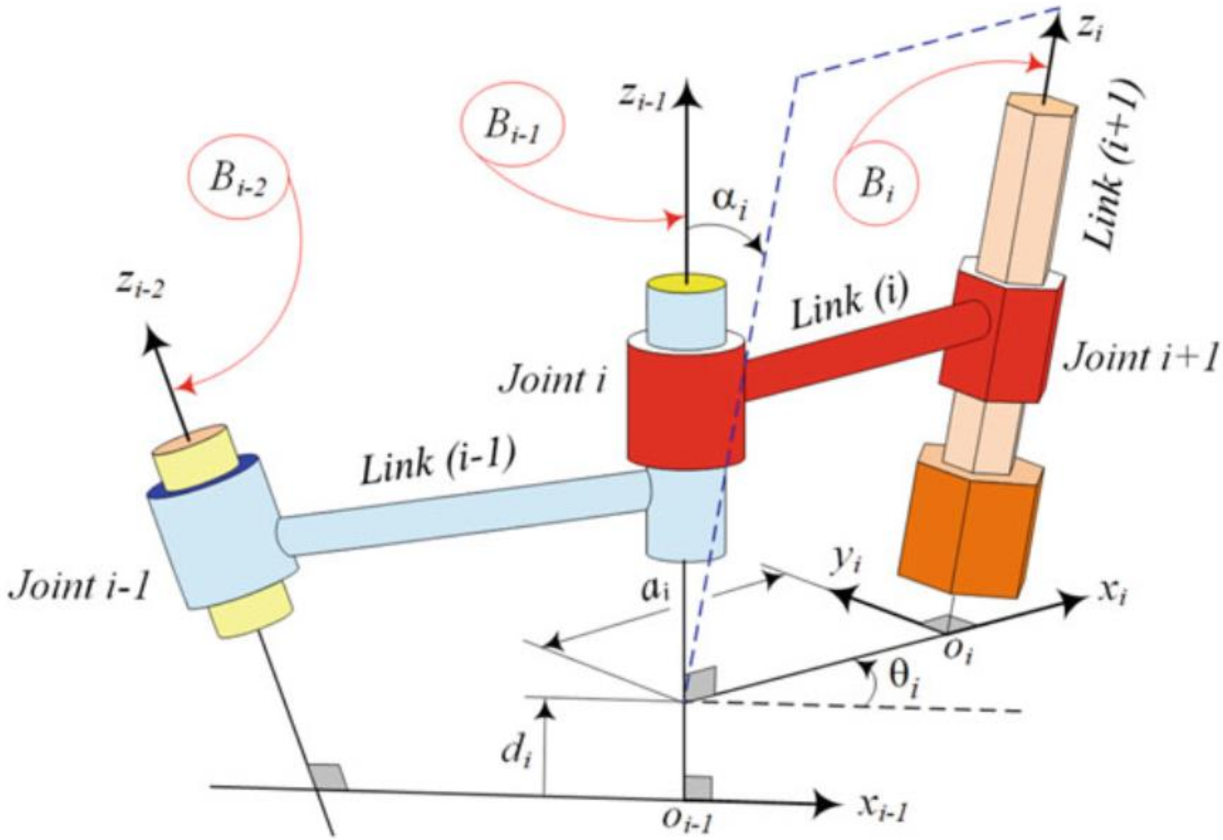
- The $\boldsymbol{z_i}$**-axis** is aligned with the $i + 1$ joint axis.
- The $\boldsymbol{x_i}$**-axis** is defined along **the common normal** between *the* $z_i{-}1$ *and* z_i *axes*, pointing from the $z_i{-}1$ to the z_i-axis.
- The $\boldsymbol{y_i}$**-axis** is determined by the right-hand rule, $y_i = z_i \times x_i$.

⇒ By applying the **DH method**, the origin o_i of the frame $\mathbf{B_i}$ (o_i, x_i, y_i, z_i), **attached to the link (*i*)**, is placed at the intersection of the joint axis $i + 1$ with the common normal between the $z_i{-}1$ and z_i axes.

5.1. Denavit–Hartenberg Notation

Denavit–Hartenberg (DH) principles

- A **DH coordinate frame** is identified by four parameters: $\boldsymbol{a_i}$, $\boldsymbol{\alpha_i}$, θ_i, and $\boldsymbol{d_i}$.



5.1. Denavit–Hartenberg Notation

Denavit–Hartenberg's (DH) rules

1. **Link length a_i** is *the distance between z_{i-1} and x_i axes along the x_i-axis*. a_i is the kinematic length of the link (i). ⇒ **a_i is along the x_i-axis, from z_{i-1} to z_i axes.**
2. **Link twist α_i (alpha)** is *the required rotation of the z_{i-1} -axis about the x_i-axis* to become *parallel to the z_i -axis*. ⇒ **α_i is about the x_i-axis, from z_{i-1} to z_i axes.**
3. **Joint distance d_i** is *the distance between x_{i-1} and x_i axes along the z_{i-1}-axis*. Joint distance is also called link offset. ⇒ **d_i is along the z_{i-1}-axis, from x_{i-1} to x_i axes.**
4. **Joint angle θ_i** is *the required rotation of x_{i-1}-axis about the z_{i-1}-axis* to become *parallel to the x_i-axis*. ⇒ **θ_i is about the z_{i-1}-axis, from x_{i-1} to x_i axes.**

5.1. Denavit–Hartenberg Notation

Denavit–Hartenberg's (DH) rules

- The parameters θ_i and d_i are called **joint parameters**, defining the relative position of two adjacent links connected at joint *i*.

⇒ For **a revolute joint (R)** at joint *i*, the θ_i is the unique joint variable, and the value of d_i is fixed.

⇒ For **a prismatic joint (P)**, the d_i is the only joint variable, while the value of θ_i is fixed.

- The joint parameters θ_i and d_i define **a screw motion** because θ_i is a rotation about the z_{i-1}-axis and d_i is a translation along the z_{i-1}-axis.

5.1. Denavit–Hartenberg Notation

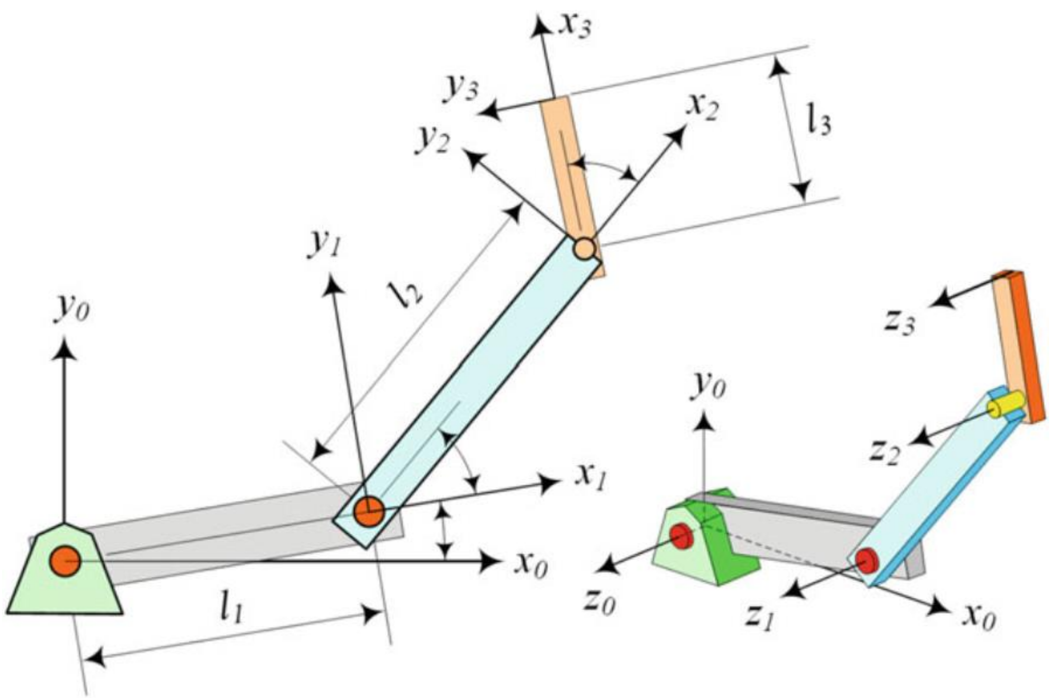
Denavit–Hartenberg's (DH) rules

- The parameters α_i and a_i are called **link parameters**, because they define relative positions of joints *i* and *i + 1* at two ends of link (*i*).
- The link twist α_i is the angle of rotation z_{i-1}-axis about x_i to become parallel with the z_i-axis.
- The other link kinematic length parameter, a_i, is the translation along the x_i-axis to bring the z_{i-1}-axis on the z_i-axis.
- The link parameters α_i and a_i define **a screw motion** because α_i is a rotation about the x_i-axis and a_i is a translation along the x_i-axis.

5.1. Denavit–Hartenberg Notation

Example 1: DH table and coordinate frames for 3R planar manipulator.

An R||R||R manipulator is a planar robot with three parallel revolute joints. Figure illustrates a 3R planar manipulator robot.



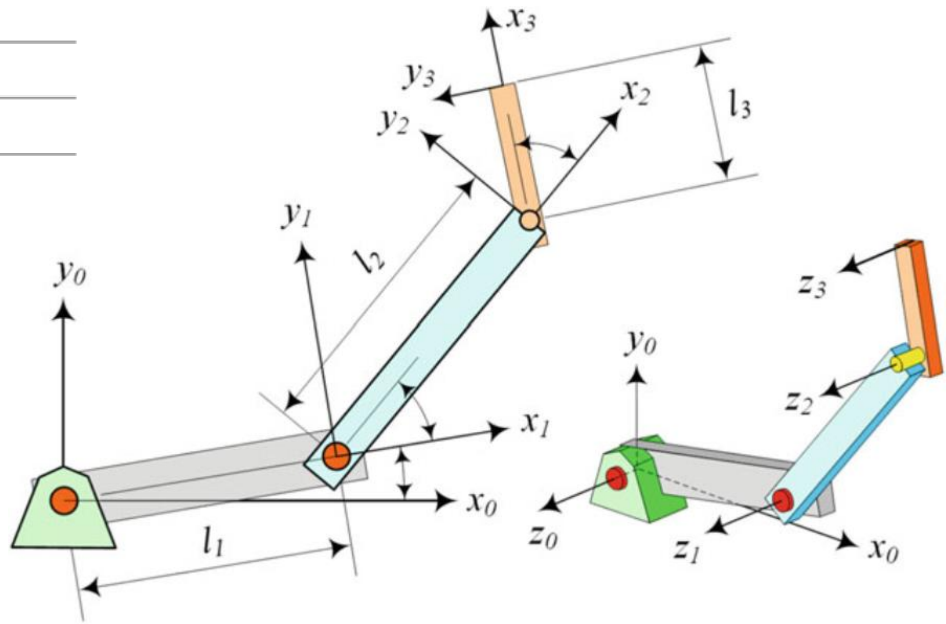
5.1. Denavit–Hartenberg Notation

Example 1: DH table and coordinate frames for 3R planar manipulator.

The DH table can be filled

Frame no.	a_i	α_i	d_i	θ_i
1	a_1	α_1	d_1	θ_1
2	a_2	α_2	d_2	θ_2
......	......	......	......	......
j	a_j	α_j	d_j	θ_j
......	......	......	......	......
n	a_n	α_n	d_n	θ_n

Frame no.	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2
3	l_3	0	0	θ_3

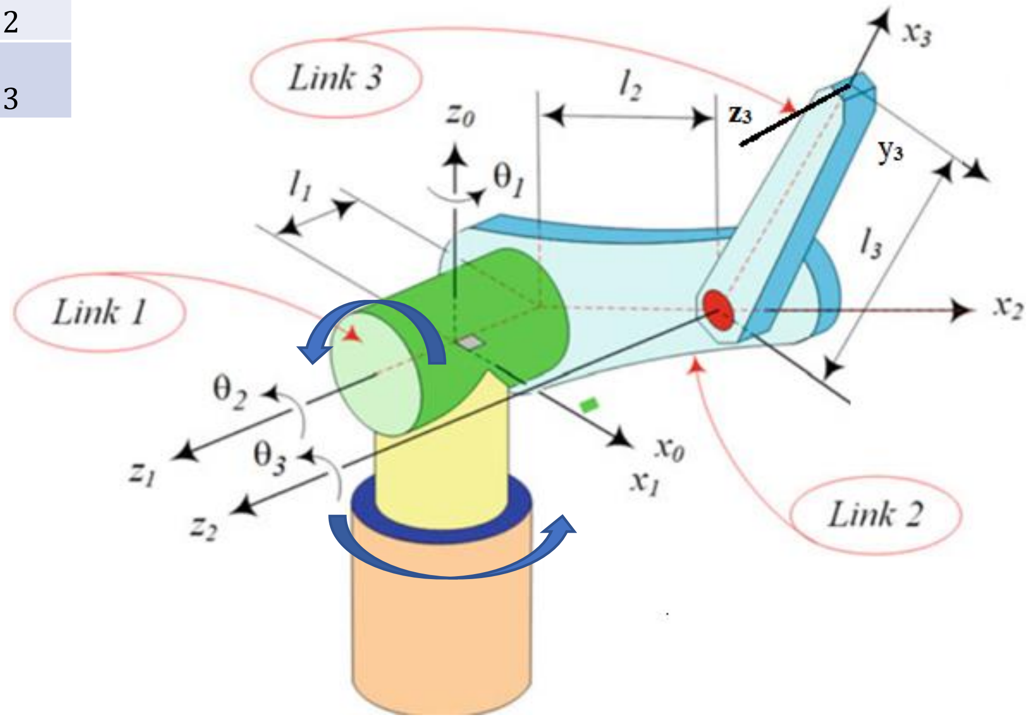


5.1. Denavit–Hartenberg Notation

Example 2: Coordinate frames for a 3R PUMA robot.

It has **R⊢R||R main structure.**

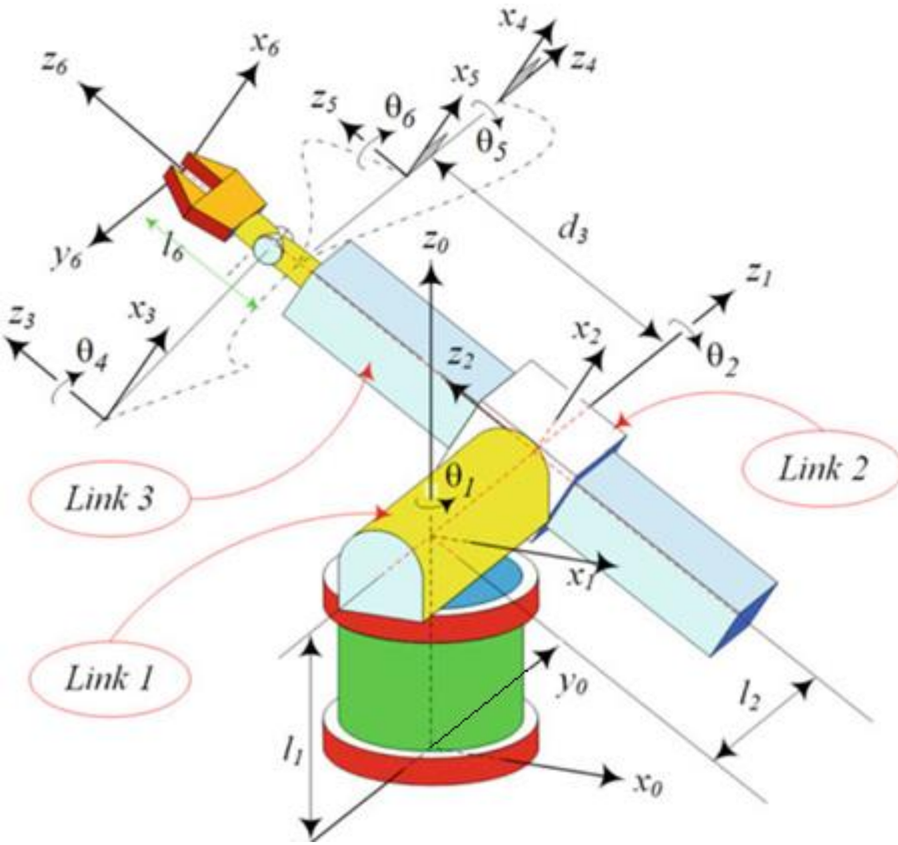
Frame no.	a_i	α_i	d_i	θ_i
1	0	90deg	0	θ_1
2	l_2	0	$-l_1$	θ_2
3	l_3	0	0	θ_3



5.1. Denavit–Hartenberg Notation

Example 3: Stanford arm.

A schematic illustration of the Stanford arm is a spherical robot $\boldsymbol{R} \vdash \boldsymbol{R} \vdash \boldsymbol{P}$ attached to a spherical wrist $\boldsymbol{R} \vdash \boldsymbol{R} \vdash \boldsymbol{R}$.



Frame no.	a_i	α_i	d_i	θ_i
1	0	-90 deg	l_1	θ_1
2	0	90 deg	l_2	θ_2
3	0	0	d_3	0

5.2. Transformation Between Adjacent Coordinate Frames

- The coordinate frame B_i is fixed to the link (i) and the coordinate frame B_{i-1} is fixed to the link ($i - 1$).
- The following prescribed set of **two rotations** and **two translations** is also a straightforward method to move the frame B_{i-1} to coincide with the frame B_i. This is a method to make **a transformation matrix $^{i}T_{i}$−1:**

    1. **Translate** frame B_{i-1} along the z_{i-1} -axis by distance d_i.
    2. **Rotate** frame B_{i-1} through θ_i around the z_{i-1} -axis.
    3. **Translate** frame B_{i-1} along the x_i-axis by distance a_i.
    4. **Rotate** frame B_{i-1} through α_i about the x_i-axis.

5.2. Transformation Between Adjacent Coordinate Frames

- The transformation matrix $^{i-1}\boldsymbol{T}_i$ to transform coordinate frames B_i into B_{i-1} is represented as a product of four basic transformations using the parameters of link (i) and joint i.

$$
\begin{aligned}
{}^{i-1}T_i &= D_{z_{i-1},d_i}\, R_{z_{i-1},\theta_i}\, D_{x_{i-1},a_i}\, R_{x_{i-1},\alpha_i} \\
&= \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\theta_i \sin\alpha_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & -\cos\theta_i \sin\alpha_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
R_{x_{i-1},\alpha_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_{x_{i-1},a_i} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

5.2. Transformation Between Adjacent Coordinate Frames

$$R_{z_{i-1},\theta_i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_{z_{i-1},d_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Therefore, the transformation equation from coordinate frame $B_i(x_i, y_i, z_i)$, to its previous coordinate frame $B_{i-1}(x_{i-1}, y_{i-1}, z_{i-1})$, is

$$\begin{bmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} \\ 1 \end{bmatrix} = {}^{i-1}T_i \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

5.2. Transformation Between Adjacent Coordinate Frames

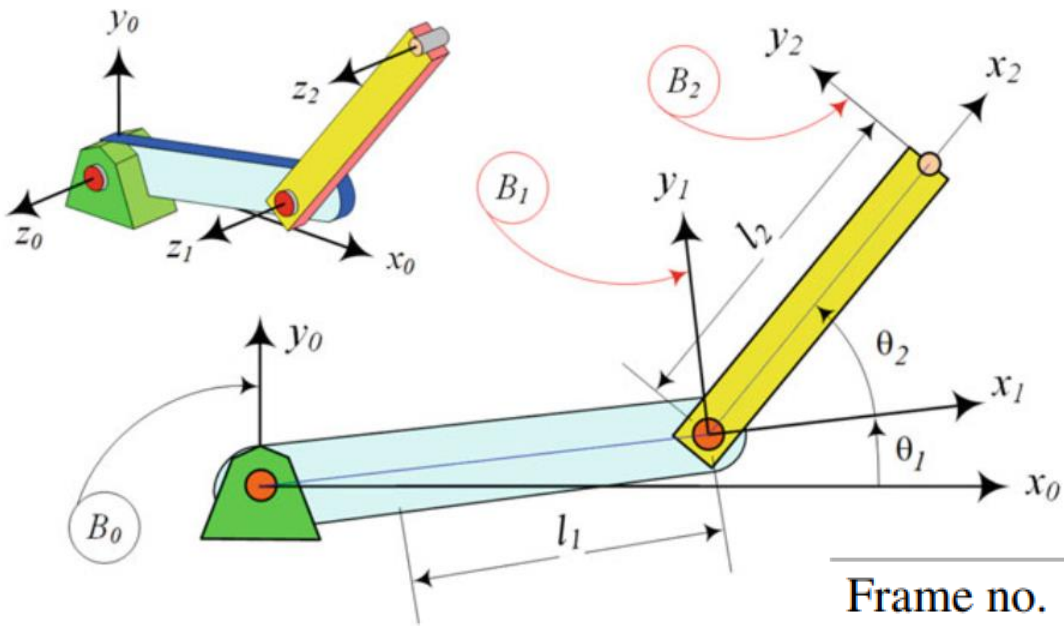
- The inverse of the homogenous transformation matrix is

$$
\begin{aligned}
{}^{i}T_{i-1} &= {}^{i-1}T_{i}^{-1} \\
&= \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\sin\theta_i\cos\alpha_i & \cos\theta_i\cos\alpha_i & \sin\alpha_i & -d_i\sin\alpha_i \\ \sin\theta_i\sin\alpha_i & -\cos\theta_i\sin\alpha_i & \cos\alpha_i & -d_i\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

5.2. Transformation Between Adjacent Coordinate Frames

Example 4: DH transformation matrices for a 2R planar manipulator.

Figure illustrates an R||R planar manipulator and its DH link coordinate frames.

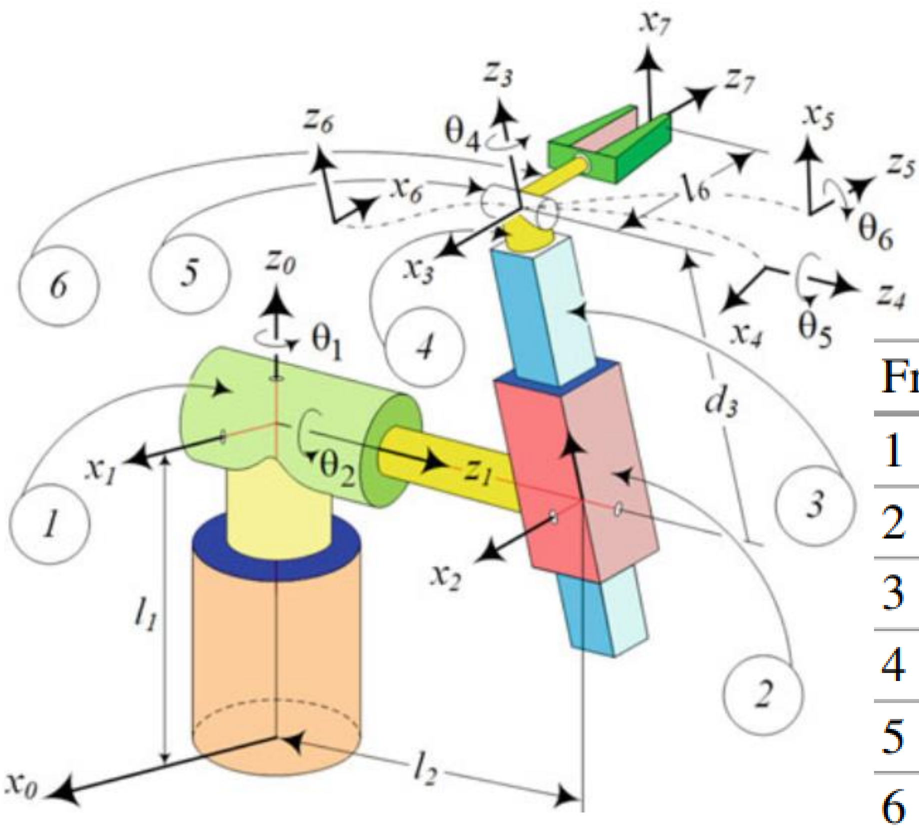


Frame no.	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2

5.2. Transformation Between Adjacent Coordinate Frames

Example 5: DH application for spherical robot. !?

Figure illustrates a spherical manipulator equipped with a spherical wrist. A spherical manipulator is an R⊢R⊢P arm.



Frame no.	a_i	α_i	d_i	θ_i
1	0	-90 deg	l_1	θ_1
2	0	90 deg	l_2	θ_2
3	0	0	d_3	0
4	0	-90 deg	0	θ_4
5	0	90 deg	0	θ_5
6	0	0	0	θ_6

5.2. Transformation Between Adjacent Coordinate Frames

Example 5: DH application for spherical robot. !?

The homogenous transformation matrices are

$$
{}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^{1}T_2 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ \sin\theta_2 & 0 & -\cos\theta_2 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^{3}T_4 = \begin{bmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & 0 \\ \sin\theta_4 & 0 & \cos\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

5.2. Transformation Between Adjacent Coordinate Frames

A closed-loop robot provides a constraint on transformation matrices,

$$[T] = {}^{1}T_2\,{}^{2}T_3\,{}^{3}T_4\,{}^{4}T_1 = \mathbf{I}_4$$

where, the transformation matrix [T] contains elements that are functions of a_2, d, a_3, θ_3, a_4, θ_4, θ_1. The parameters a_2, a_3, and a_4 are constant, while d, θ_3, θ_4, θ_1 are variables.

Assuming θ_1 is input and specified, we may solve for other unknown variables θ_3, θ_4, d by equating the corresponding elements of [T] and **I**.

5.3. Forward Position Kinematics of Robots

- **The forward or direct kinematics** is the transformation of kinematic information from **the robot joint variable space** to the **Cartesian coordinate space**.

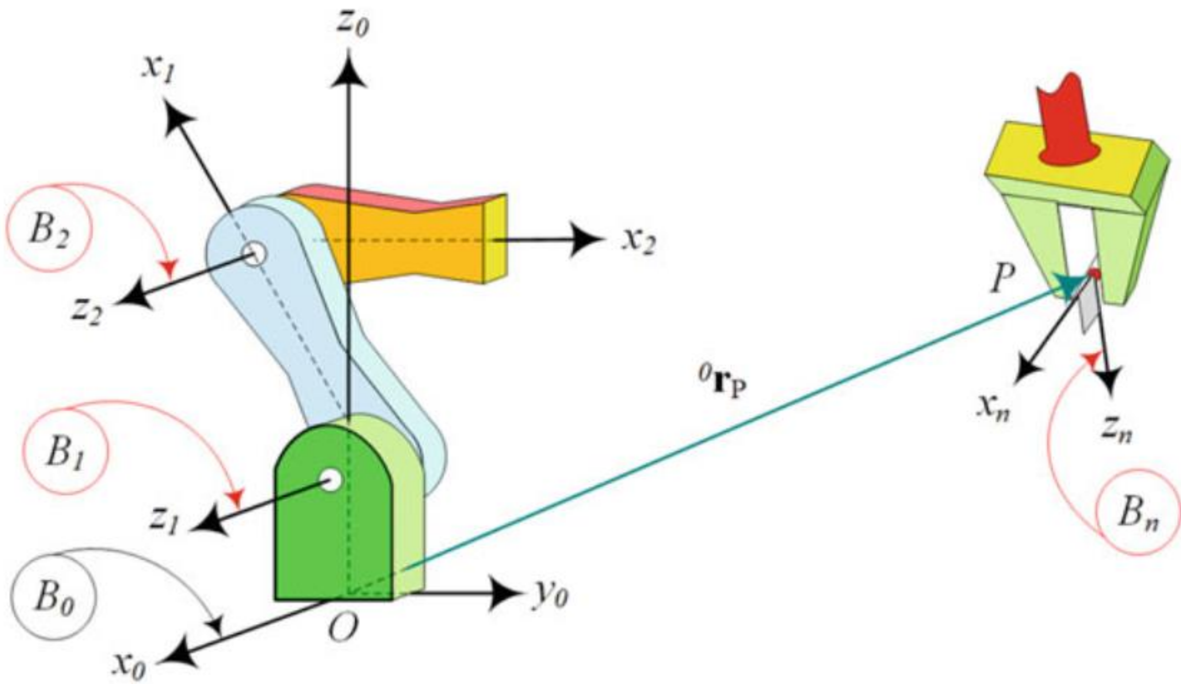
  ⇒ Determining **the end-effector position and orientatio**n for a given set of joint variables is the main problem in forward kinematics.

  ⇒ Determining **transformation matrices** $^{0}\mathbf{T}_{i}$ to express the kinematic information of link (i) in the base link coordinate frame.

- The traditional way of producing forward kinematic equations for robotic manipulators is to proceed link by link using the Denavit–Hartenberg transformation matrices.

5.3. Forward Position Kinematics of Robots

- For an ***n*-DOF robot**, at least ***n* transformation matrices**, one for every link, are required to determine the global coordinate of any point in any frame.
- The configuration of the multibody when all the joint variables are zero is called the *rest position*.



5.3. Forward Position Kinematics of Robots

- If the links of a robot are arranged such that every link (i) has only one coordinate frame B_i and the frames are arranged sequentially, then:

$$ {}^0T_i = {}^0T_1\, {}^1T_2\, {}^2T_3\, {}^3T_4 \cdots {}^{i-1}T_i \qquad i = 1, 2, 3, \cdots, n $$

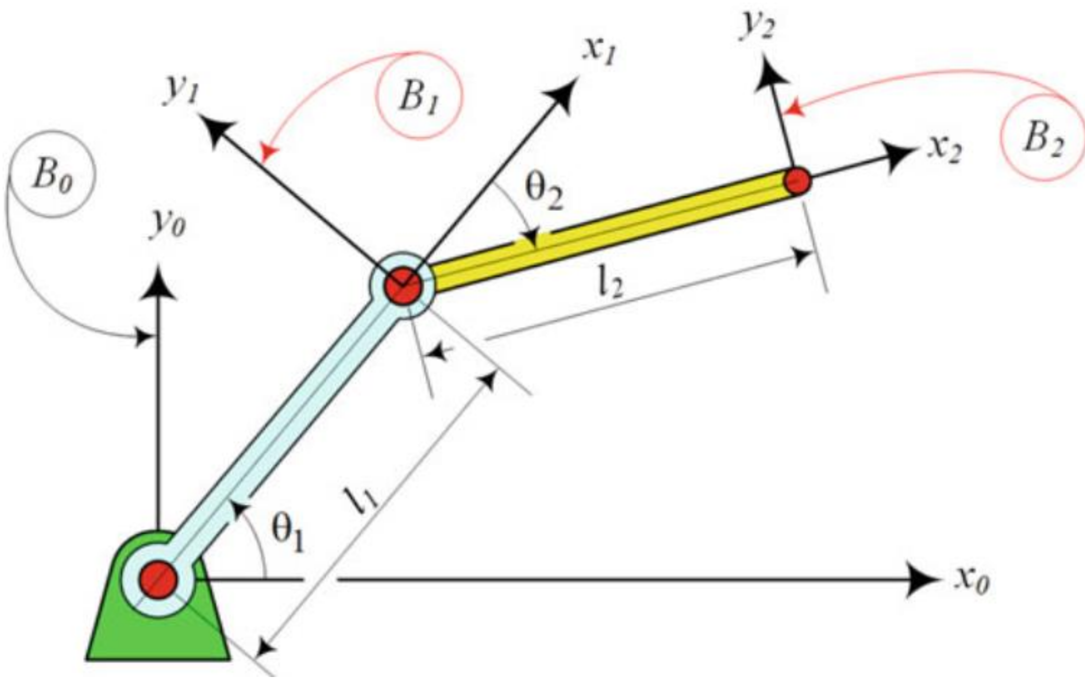
- Determine the coordinates of any point P of link (i) in the base frame

$$ {}^0\mathbf{r}_P = {}^0T_i\, {}^i\mathbf{r}_P \qquad i = 1, 2, 3, \cdots, n $$

5.3. Forward Position Kinematics of Robots

Example 7: A 2R planar manipulator. !?

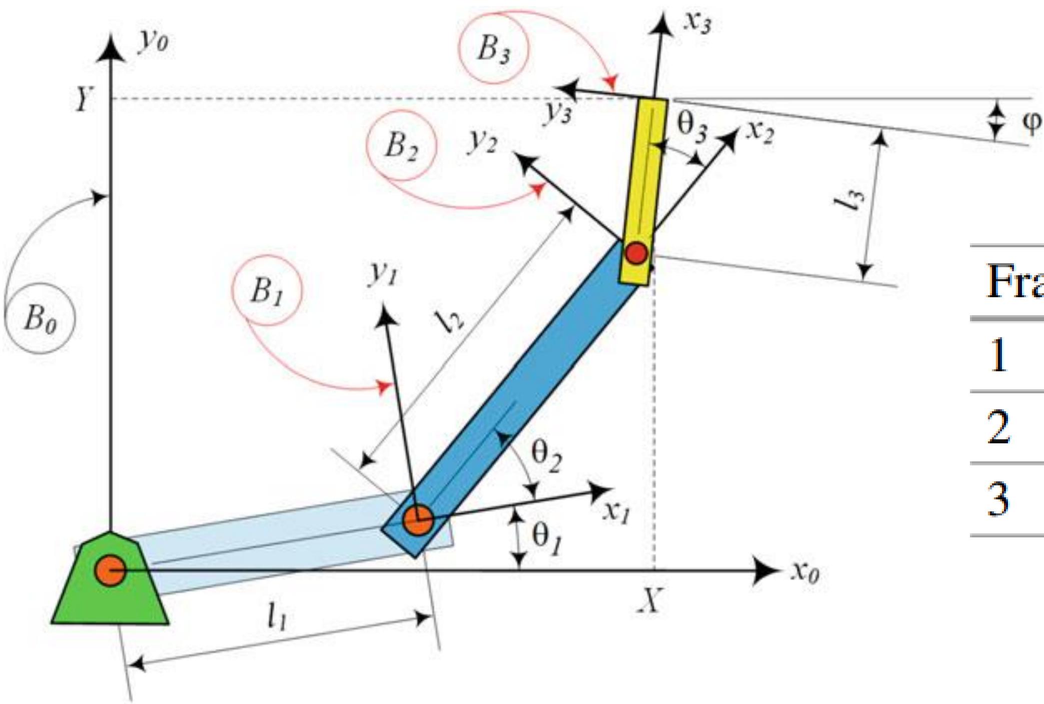
Figure illustrates a 2R or R||R planar manipulator with two parallel revolute joints. Find, the transformation matrices ${}^{0}T_{1}$, ${}^{1}T_{2}$, ${}^{0}T_{2}$



5.3. Forward Position Kinematics of Robots

Example 9: R||R||R, planar manipulator forward kinematics.

Application of DH matrices in forward kinematic analysis of a planar 3 DOF robot.



Frame no.	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2
3	l_3	0	0	θ_3

5.3. Forward Position Kinematics of Robots

Example 9: R||R||R, planar manipulator forward kinematics.

The transformation matrices ${}^{i-1}T_i$ for i = 3, 2, 1 can be found

$${}^{2}T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & l_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & l_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.3. Forward Position Kinematics of Robots

Example 9: R||R||R, planar manipulator forward kinematics.

The transformation matrices ${}^{i-1}T_i$ for i = 3, 2, 1 can be found

$$
{}^{0}T_3 = {}^{0}T_1 \ {}^{1}T_2 \ {}^{2}T_3
$$

$$
= \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & r_{14} \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & r_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
r_{14} = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3)
$$

$$
r_{24} = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_3 \sin(\theta_1+\theta_2+\theta_3)
$$

5.3. Forward Position Kinematics of Robots

Example 9: R||R||R, planar manipulator forward kinematics.

The origin of the frame B_3 is the tip point of the robot. Its position is at

$$
{}^{0}T_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 c\theta_1 + l_2 c\left(\theta_1 + \theta_2\right) + l_3 c\left(\theta_1 + \theta_2 + \theta_3\right) \\ l_1 s\theta_1 + l_2 s\left(\theta_1 + \theta_2\right) + l_3 s\left(\theta_1 + \theta_2 + \theta_3\right) \\ 0 \\ 1 \end{bmatrix}
$$

It means we can find the coordinate of the tip point in the base Cartesian coordinate frame

$$
X = l_1 \cos\theta_1 + l_2 \cos\left(\theta_1 + \theta_2\right) + l_3 \cos\left(\theta_1 + \theta_2 + \theta_3\right)
$$

$$
Y = l_1 \sin\theta_1 + l_2 \sin\left(\theta_1 + \theta_2\right) + l_3 \sin\left(\theta_1 + \theta_2 + \theta_3\right)
$$

5.3. Forward Position Kinematics of Robots

Example 9: R||R||R, planar manipulator forward kinematics.

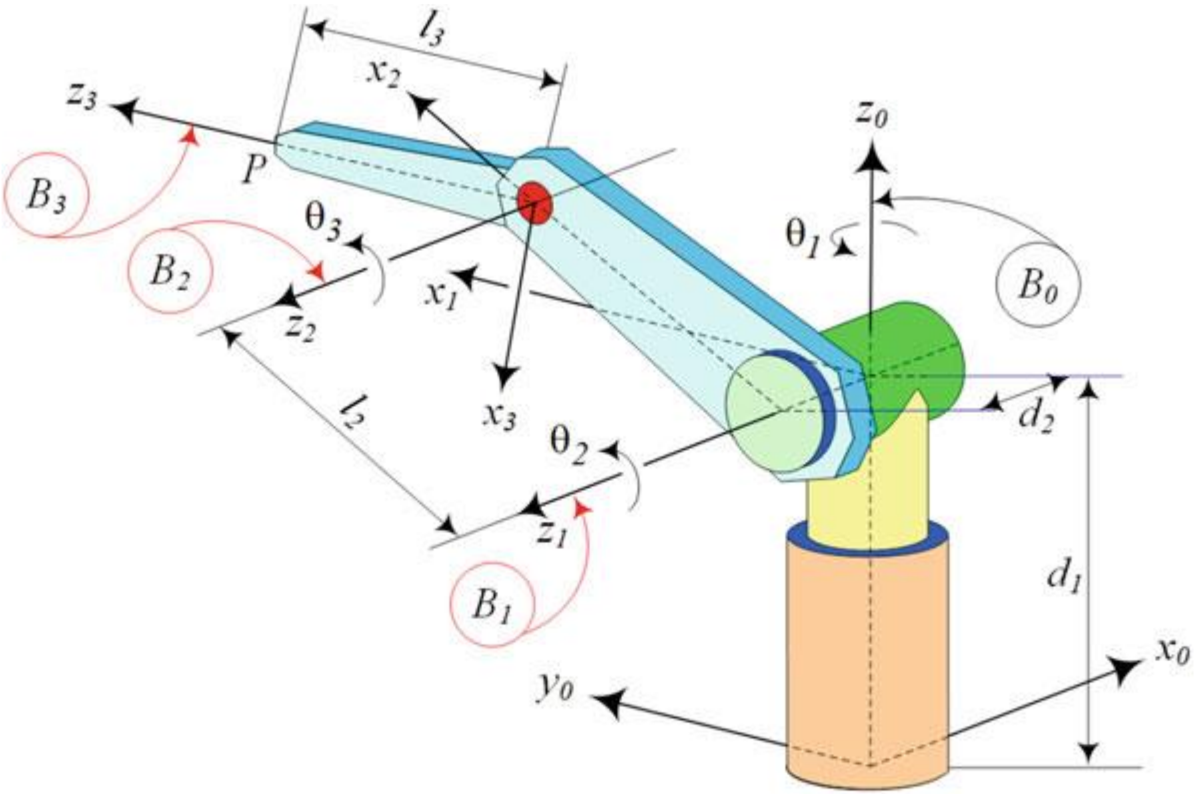
The rest position of the manipulator is lying on the x_0-axis where $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 0$ because ${}^0T_3$ becomes

$$
{}^0T_3 = \begin{bmatrix} 1 & 0 & 0 & l_1 + l_2 + l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

5.3. Forward Position Kinematics of Robots

Example 10: 3R, R⊢R||R, articulated arm forward kinematics.

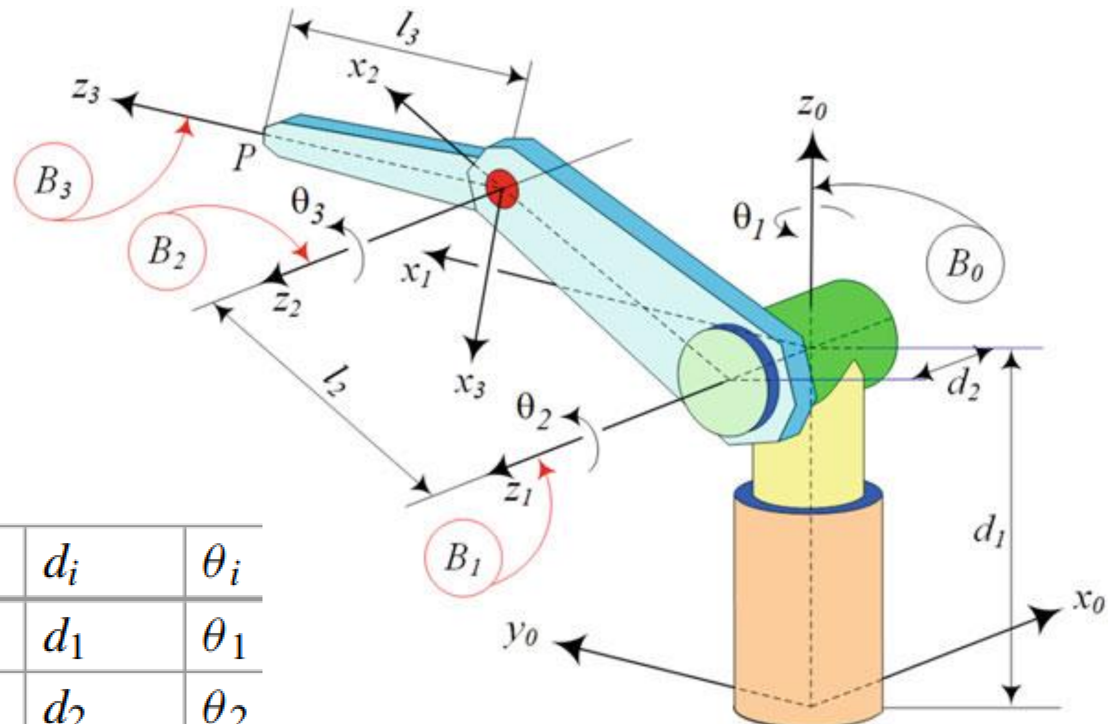
How to determine forward kinematics of the robot?



5.3. Forward Position Kinematics of Robots

Example 10: 3R, R⊢R||R.

Link no.	Type
1	R⊢R(−90)
2	R‖R(0)
3	R⊢R(90)



Frame no.	a_i	α_i	d_i	θ_i
1	0	−90 deg	d_1	θ_1
2	l_2	0	d_2	θ_2
3	0	90 deg	l_3	θ_3

5.3. Forward Position Kinematics of Robots

Example 10: 3R, R⊢R||R.

The successive transformation matrices have the following expressions:

$$ {}^0T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^1T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2T_3 = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

5.3. Forward Position Kinematics of Robots

Example 10: 3R, R⊢R||R.

To express the complete forward kinematics transformation:

$$ {}^{0}T_{3} = {}^{0}T_{1}\ {}^{1}T_{2}\ {}^{2}T_{3} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ r_{11} = \cos\theta_1 \cos(\theta_2 + \theta_3) $$

$$ r_{21} = \sin\theta_1 \cos(\theta_2 + \theta_3) \qquad r_{12} = -\sin\theta_1 \qquad r_{22} = \cos\theta_1 \qquad r_{32} = 0 $$

$$ r_{31} = -\sin(\theta_2 + \theta_3) $$

$$ r_{14} = l_2 \cos\theta_1 \cos\theta_2 - d_2 \sin\theta_1 $$

$$ r_{13} = \cos\theta_1 \sin(\theta_2 + \theta_3) $$

$$ r_{24} = l_2 \cos\theta_2 \sin\theta_1 + d_2 \cos\theta_1 $$

$$ r_{23} = \sin\theta_1 \sin(\theta_2 + \theta_3) $$

$$ r_{34} = d_1 - l_2 \sin\theta_2 $$

$$ r_{33} = \cos(\theta_2 + \theta_3) $$

5.3. Forward Position Kinematics of Robots

Example 10: 3R, R⊢R||R.

The tip point P of the third arm is at ${}^{3}r_{P} = [0\ 0\ l_3]^T$ in B_3

$$
\begin{aligned}
{}^{0}\mathbf{r}_P &= {}^{0}T_3\, {}^{3}\mathbf{r}_P \\
&= {}^{0}T_3 \begin{bmatrix} 0 \\ 0 \\ l_3 \\ 1 \end{bmatrix} = \begin{bmatrix} -d_2 s\theta_1 + l_2 c\theta_1 c\theta_2 + l_3 c\theta_1 s(\theta_2+\theta_3) \\ d_2 c\theta_1 + l_2 c\theta_2 s\theta_1 + l_3 s\theta_1 s(\theta_2+\theta_3) \\ d_1 - l_2 s\theta_2 + l_3 c(\theta_2+\theta_3) \\ 1 \end{bmatrix}
\end{aligned}
$$

The transformation matrix at rest position, where $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 0$, is

$$
{}^{0}T_3 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

5.3. Forward Position Kinematics of Robots

Example 10: 3R, R⊢R||R.

The tip point P of the third arm is at in B_3

$$l_2 = 0.75\,\mathrm{m} \qquad l_3 = 0.65\,\mathrm{m}$$

$$d_1 = 0.48\,\mathrm{m} \qquad d_2 = 0.174\,\mathrm{m}$$

$$ {}^{0}T_3 = {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3 $$

$$ = \begin{bmatrix} c\theta_1 c\left(\theta_2+\theta_3\right) & -s\theta_1 & c\theta_1 s\left(\theta_2+\theta_3\right) & r_{11} \\ s\theta_1 c\left(\theta_2+\theta_3\right) & c\theta_1 & s\theta_1 s\left(\theta_2+\theta_3\right) & r_{12} \\ -s\left(\theta_2+\theta_3\right) & 0 & c\left(\theta_2+\theta_3\right) & r_{13} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$r_{11} = 0.75\cos\theta_1\cos\theta_2 - 0.174\,\sin\theta_1$$

$$r_{12} = 0.174\,\cos\theta_1 + 0.75\cos\theta_2\sin\theta_1$$

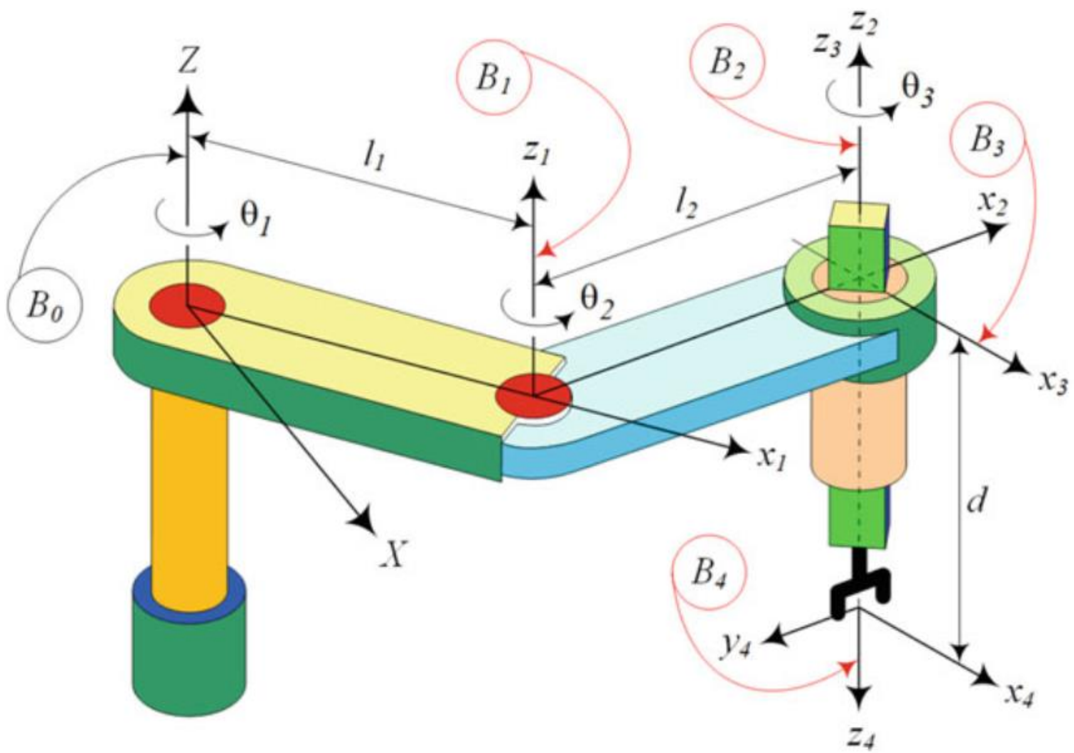
$$r_{13} = 0.48 - 0.75\sin\theta_2$$

$$ {}^{0}\mathbf{r}_P = {}^{0}T_3\,{}^{3}\mathbf{r}_P = {}^{0}T_3 \begin{bmatrix} 0 \\ 0 \\ 0.65 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.174 \\ 1.13 \\ 1 \end{bmatrix} $$

5.3. Forward Position Kinematics of Robots

Example 11: SCARA robot (R||R||R||P).

Consider the R||R||R||P robot shown in Figure.



5.3. Forward Position Kinematics of Robots

Example 11: SCARA robot (R||R||R||P).

The first link is an R||R(0), which has the following transformation matrix:

$$ {}^{0}T_{1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The second link is also an R||R(0)

$$ {}^{1}T_{2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

5.3. Forward Position Kinematics of Robots

Example 11: SCARA robot (R||R||R||P).

The third link is an R||R(0) with zero length,

$$^2T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The fourth link is an R||P(180)

$$^3T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi & 0 \\ 0 & \sin\pi & \cos\pi & -d \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.3. Forward Position Kinematics of Robots

Example 11: SCARA robot (R||R||R||P).

The configuration of the end-effector in the base coordinate frame is

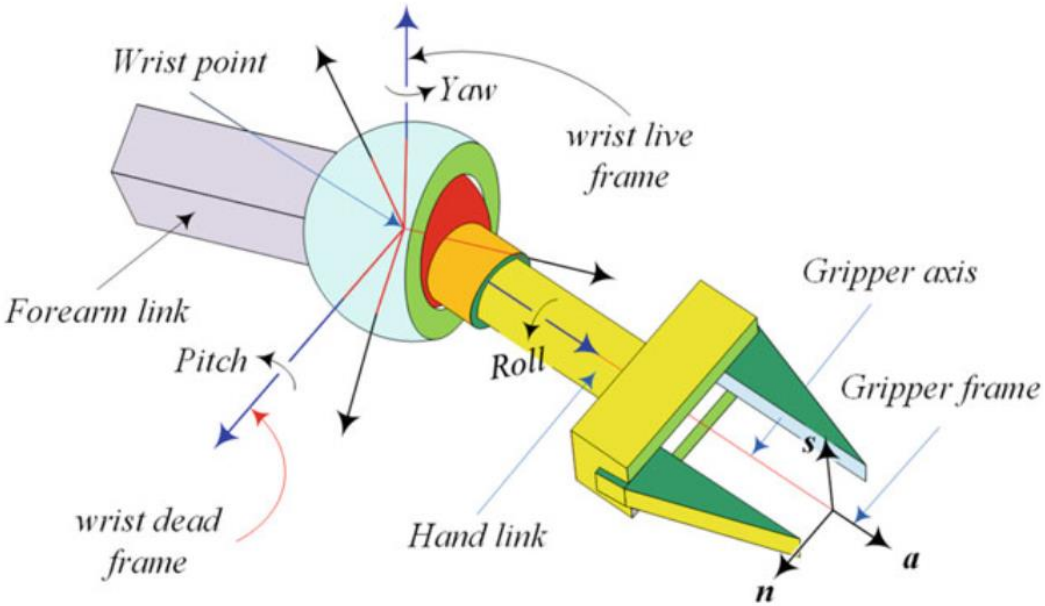
$$
\begin{aligned}
{}^0T_4 &= {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4 \\
&= \begin{bmatrix} c(\theta_1+\theta_2+\theta_3) & s(\theta_1+\theta_2+\theta_3) & 0 & l_1c\theta_1+l_2c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2+\theta_3) & -c(\theta_1+\theta_2+\theta_3) & 0 & l_1s\theta_1+l_2s(\theta_1+\theta_2) \\ 0 & 0 & -1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

It shows the rest position of the robot $\theta_1 = \theta_2 = \theta_3 = d = 0$ is at

$$
{}^0T_4 = \begin{bmatrix} 1 & 0 & 0 & l_1+l_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

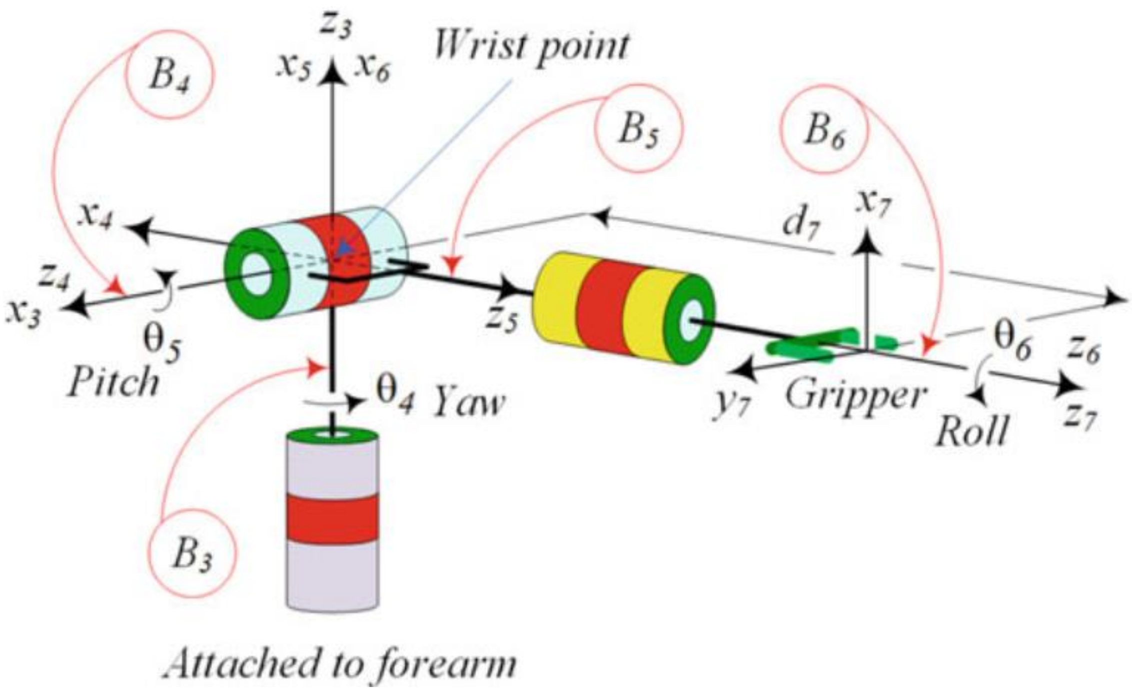
5.4. Spherical Wrist

- The **spherical joint** connects two links: **the forearm link** and **hand link**.
- The axis of **the forearm and hand** are colinear at the rest position of the hand.
- An **industrial spherical wrist** is to simulate a spherical joint and provide **3 rotational DOF** for the gripper link.



5.4. Spherical Wrist

- To classify spherical wrists, let us decompose the rotations of the spherical wrist into three rotations about three orthogonal axes, calling the rotations, **Roll, Pitch**, and **Yaw.**



Type	Rotation order
1	Roll–Pitch–Roll
2	Roll–Pitch–Yaw
3	Pitch–Yaw–Roll

5.4. Spherical Wrist

- A Roll–Pitch–Roll spherical wrist with the following transformation matrix are illustrated

$$
{}^{3}T_{6} = {}^{3}T_{4}\,{}^{4}T_{5}\,{}^{5}T_{6}
$$

$$
= \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 & 0 \\ c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4 & c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 & 0 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- The following transformation matrix provides the configuration of the tool frame B_7 in the forearm coordinate frame B_3

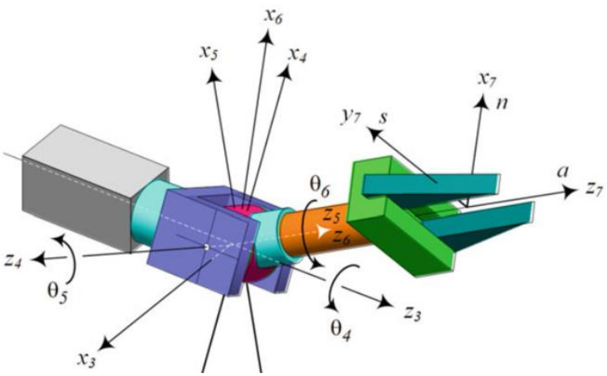
$$
{}^{3}T_{7} = {}^{3}T_{4}\,{}^{4}T_{5}\,{}^{5}T_{6}\,{}^{6}T_{7}
$$

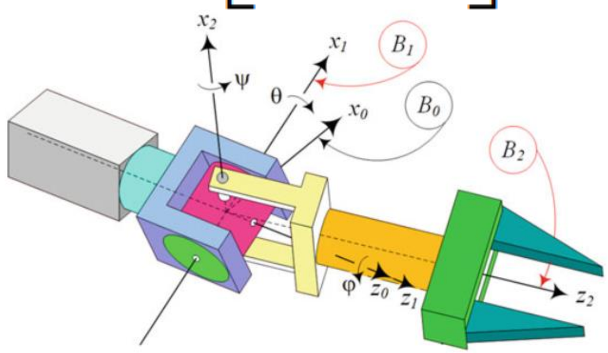
$$
= \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 & d_7 c\theta_4 s\theta_5 \\ c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4 & c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 & d_7 s\theta_4 s\theta_5 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 & d_7 c\theta_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

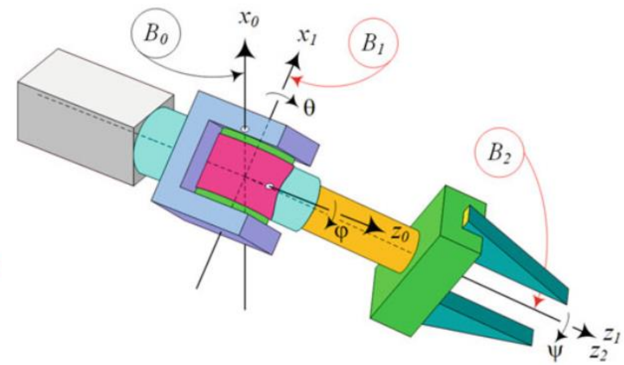
5.4. Spherical Wrist

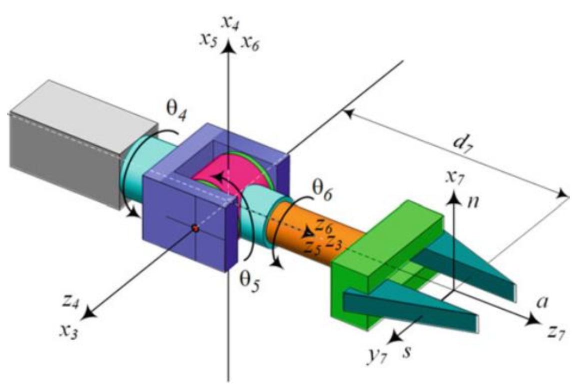
- The transformation matrix at rest position, where $\theta_4 = 0$, $\theta_5 = 0$, $\theta_6 = 0$, is

$$
{}^3T_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



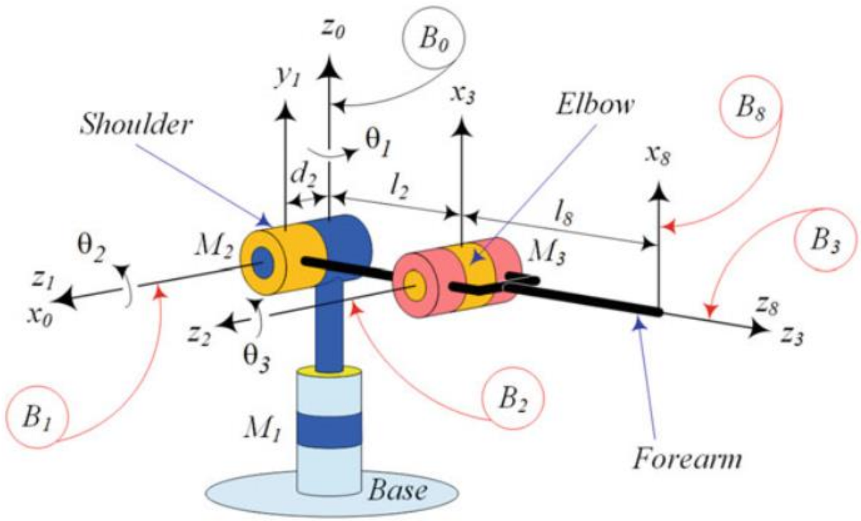




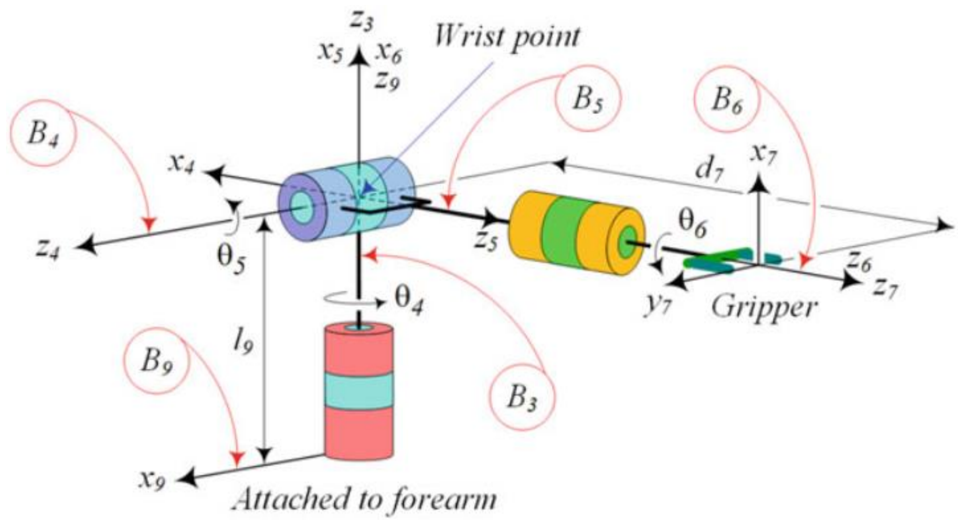


5.5. Assembling Kinematics

- Most modern industrial robots have a main manipulator and a series of changeable *wrists*. The manipulator is multibody so that holds the main power units and provides a powerful motion for the wrist point.



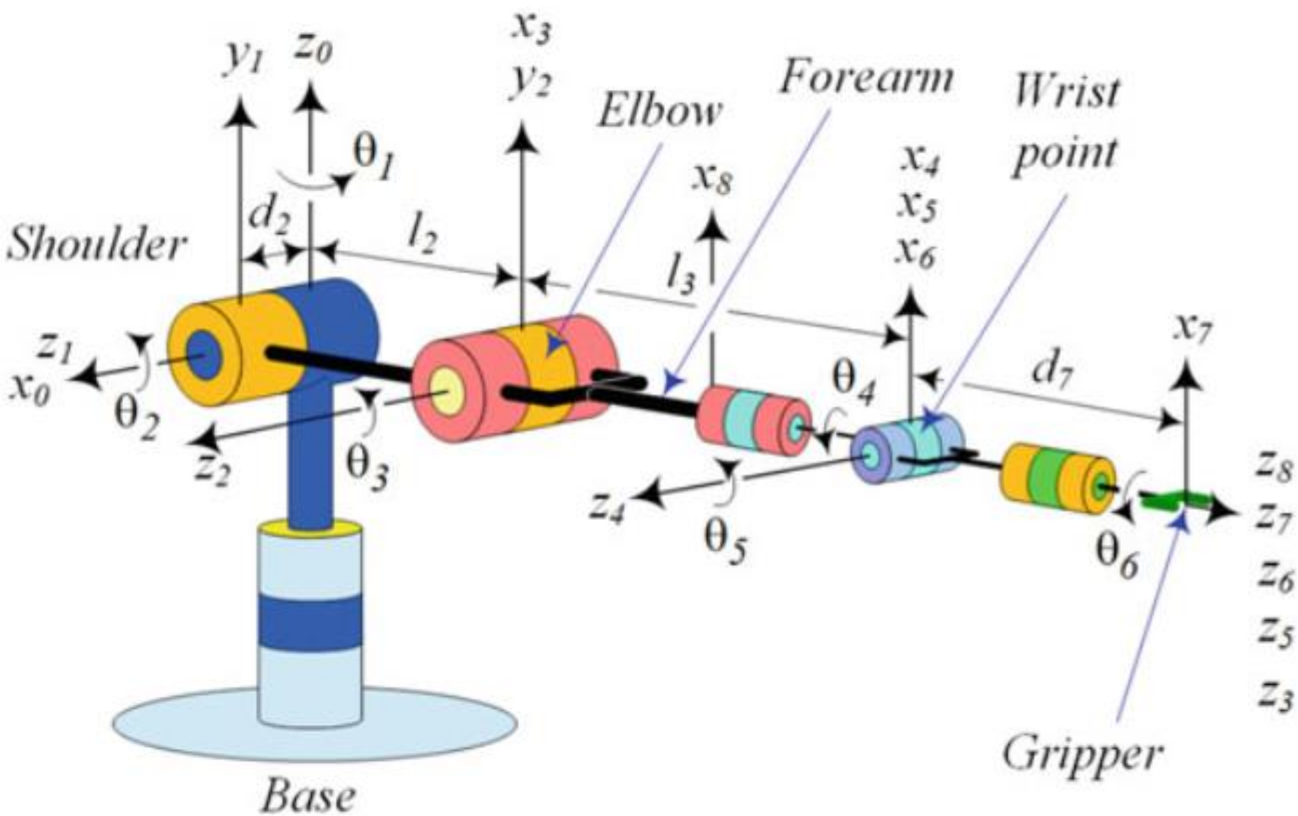
An articulator manipulator with 3 DOFs



A spherical wrist and its kinematics

5.5. Assembling Kinematics

- The articulated robot that is made by assembling the spherical wrist and articulated manipulator.



An articulated robot that is made by assembling a spherical wrist to an articulated manipulator

C5. End!