Chapter 6. Inverse Kinematics

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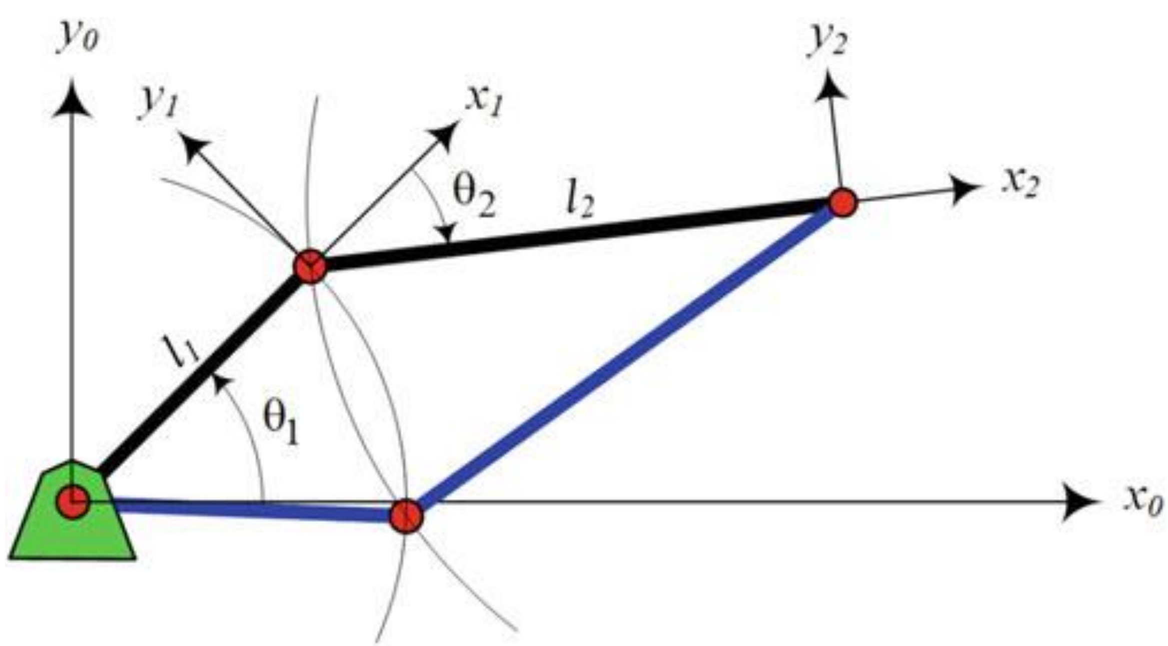
- What are **the joint variables** for a given configuration of a robot?

  ⇒ This is the problem to be answered by **inverse kinematic analysis**.
- Determination of the joint variables reduces to solving a set of nonlinear coupled.
- The main difficulty of inverse kinematic is **the multiple solutions**.



Multiple solution for inverse kinematic problem of a planar 2R manipulator

6.1. Decoupling Technique

- Computer-controlled robots are usually actuated in the joint variable space; however, objects to be manipulated are usually expressed in the **global Cartesian coordinate frame**.

- To *control the configuration of the end-effector to reach an object*, the **inverse kinematics problem must be solved**.

⇒ The required values of **joint variables** are to reach **a desired point in a desired orientation.**

6.1. Decoupling Technique

- Determination of joint variables in terms of the **end-effector position and orientation** is called **inverse kinematics**.
- Mathematically, inverse kinematics is searching for the elements of joint variable vector q,

$$\mathbf{q} = \left[\, q_1 \; q_2 \; q_3 \; \cdots \; q_n \,\right]^T$$

- A transformation ${}^{0}T_{n}$ is given as a function of the joint variables q_1, q_2, q_3, …, q_n.

$${}^{0}T_{n} = {}^{0}T_{1}(q_1)\,{}^{1}T_{2}(q_2)\,{}^{2}T_{3}(q_3)\,{}^{3}T_{4}(q_4)\,\cdots\,{}^{n-1}T_{n}(q_n)$$

6.1. Decoupling Technique

Solution of first type of trigonometric equation

The first type of trigonometric equation for the unknown angle θ is a linear combination of cos θ and sin θ.

$$a\cos\theta + b\sin\theta = c$$

This equation can be solved by introducing two new variables r and ϕ such that:

$$a = r\sin\phi \qquad b = r\cos\phi$$

$$r = \sqrt{a^2 + b^2} \qquad \phi = \operatorname{atan2}(a, b)$$

Substituting the new variables

$$\sin(\phi + \theta) = \frac{c}{r}$$

$$\cos(\phi + \theta) = \pm\sqrt{1 - \frac{c^2}{r^2}}$$

6.1. Decoupling Technique

Solution of first type of trigonometric equation

The unknown angle θ of the trigonometric equation is:

$$\begin{aligned}
\theta &= \operatorname{atan2}\left(\frac{c}{r}, \pm\sqrt{1-\frac{c^2}{r^2}}\right) - \operatorname{atan2}(a, b) \\
&= \operatorname{atan2}(c, \pm\sqrt{r^2-c^2}) - \operatorname{atan2}(a, b) \\
&\equiv \arctan\frac{c}{\pm\sqrt{r^2-c^2}} - \arctan\frac{a}{b}
\end{aligned}$$

6.1. Decoupling Technique

Ex 1: Inverse kinematics for 2R planar manipulator

Figure illustrates a 2R planar manipulator with two R∥R links. The forward kinematics of the manipulator as:

$$
{}^{0}T_{2} = {}^{0}T_{1}\,{}^{1}T_{2}
$$

$$
= \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_1 c\theta_1 + l_2 c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & l_1 s\theta_1 + l_2 s(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The global position of the tip point of the manipulator is at:

$$
\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \end{bmatrix}
$$

6.1. Decoupling Technique

Ex 1: Inverse kinematics for 2R planar manipulator

C1: To find θ_2, we use

$$X^2 + Y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2$$

where,

$$\cos\theta_2 = \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

$$\theta_2 = \cos^{-1}\frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

6.1. Decoupling Technique

Ex 1: Inverse kinematics for 2R planar manipulator

C2: Let us employ the half angle formula,

$$\boxed{\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}}$$

Find θ_2 using an atan2 function,

$$\theta_2 = \pm 2\,\mathrm{atan2}\sqrt{\frac{(l_1+l_2)^2-\left(X^2+Y^2\right)}{\left(X^2+Y^2\right)-(l_1-l_2)^2}}$$

where,

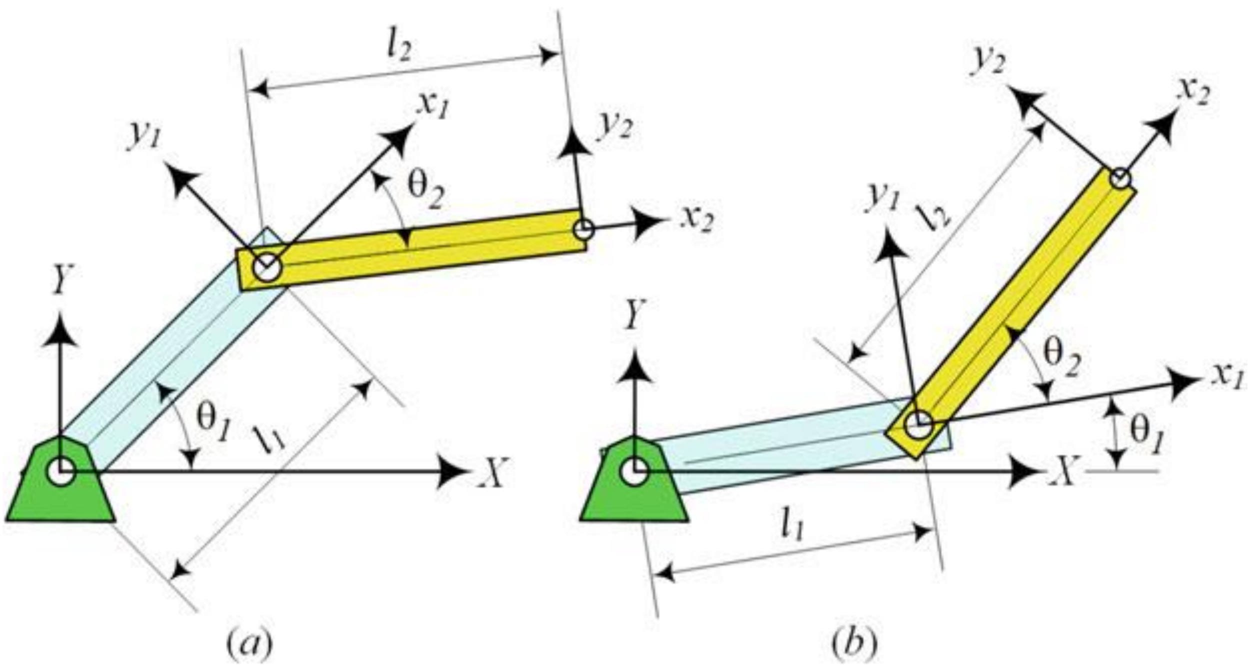
$$\mathrm{atan2}(y,x) = \begin{cases} \operatorname{sgn} y \ \arctan\left|\dfrac{y}{x}\right| & \text{if } x>0,\ y\neq 0 \\ \dfrac{\pi}{2}\operatorname{sgn} y & \text{if } x=0,\ y\neq 0 \\ \operatorname{sgn} y \ \left(\pi-\arctan\left|\dfrac{y}{x}\right|\right) & \text{if } x<0,\ y\neq 0 \\ \pi-\pi\operatorname{sgn} x & \text{if } x\neq 0,\ y=0 \end{cases}$$

6.1. Decoupling Technique

Ex 1: Inverse kinematics for 2R planar manipulator

- The first joint variable θ_1 of an elbow up/down configuration can geometrically be found from:

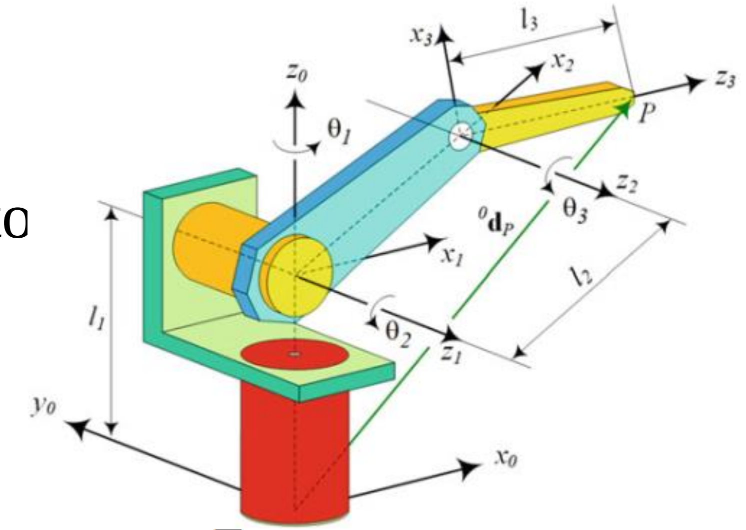
$$\theta_1 = \arctan\frac{Y}{X} + \arctan\frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2} \qquad \theta_1 = \arctan\frac{Y}{X} - \arctan\frac{l_2\sin\theta_2}{l_1 + l_2\cos\theta_2}$$



6.1. Decoupling Technique

Ex 2: An articulated manipulator.

Consider an articulated manipulator as is shown in Figure. The links of the manipulator are R⊢R(90), R∥R(0), R⊢R(90)



The forward kinematics of the manipulato is:

$$^0T_3 = {}^0T_1\,{}^1T_2\,{}^2T_3$$

$$= \begin{bmatrix} c\theta_1 c(\theta_2+\theta_3) & s\theta_1 & c\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2+\theta_3) & -c\theta_1 & s\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_2 s\theta_1 \\ s(\theta_2+\theta_3) & 0 & -c(\theta_2+\theta_3) & l_1 + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.1. Decoupling Technique

Ex 2: An articulated manipulator.

The tip point P is at:

$$
{}^{0}\mathbf{d}_P = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = {}^{0}T_3 \begin{bmatrix} 0 \\ 0 \\ l_3 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} l_3 \sin(\theta_2 + \theta_3) \cos\theta_1 + l_2 \cos\theta_1 \cos\theta_2 \\ l_3 \sin(\theta_2 + \theta_3) \sin\theta_1 + l_2 \sin\theta_1 \cos\theta_2 \\ l_1 - l_3 \cos(\theta_2 + \theta_3) + l_2 \sin\theta_2 \\ 1 \end{bmatrix}
$$

The first angle can be found from:

$$
X \sin\theta_1 - Y \cos\theta_1 = 0
$$

6.1. Decoupling Technique

Ex 2: An articulated manipulator.

⇒ **That is:**

$$\theta_1 = \text{atan2}\,(Y, X)$$

We may combine **the first and second** elements of $^{0}\boldsymbol{d}_{\boldsymbol{P}}$ to find:

$$X\cos\theta_1 + Y\sin\theta_1 = l_3\sin(\theta_2+\theta_3) + l_2\cos\theta_2$$

$$X\cos\theta_1 + Y\sin\theta_1 - l_2\cos\theta_2 = l_3\sin(\theta_2+\theta_3)$$

Rewrite **the third component** as:

$$Z - l_1 - l_2\sin\theta_2 = l_3\cos(\theta_2+\theta_3)$$

A combining Equations provides:

$$(Z - l_1 - l_2\sin\theta_2)^2 + (X\cos\theta_1 + Y\sin\theta_1 - l_2\cos\theta_2)^2 = l_3^2$$

6.1. Decoupling Technique

Ex 2: An articulated manipulator.

That is a trigonometric equation of the form:

$$a\cos\theta_2 + b\sin\theta_2 = c$$

$$a = -2l_2\left(X\cos\theta_1 + Y\sin\theta_1\right)$$

$$b = 2l_2\left(l_1 - Z\right)$$

$$c = l_3^2 - \left(l_1 - Z\right)^2 - l_2^2 - Y^2 - \left(X^2 - Y^2\right)\cos^2\theta_1 - XY\sin 2\theta_1$$

⇒ We solve this equation for θ_2

$$\theta_2 = \arcsin\left(\frac{c}{\sqrt{a^2+b^2}}\right) - \arctan\frac{a}{b}$$

6.1. Decoupling Technique

Ex 2: An articulated manipulator.

The third element of $^{0}d_{P}$ determines θ_3:

$$\tan(\theta_2+\theta_3)=\frac{X\cos\theta_1+Y\sin\theta_1-l_2\cos\theta_2}{l_1+l_2\sin\theta_2-Z}$$

$$\theta_3=\operatorname{atan2}\left(\frac{X\cos\theta_1+Y\sin\theta_1-l_2\cos\theta_2}{l_1+l_2\sin\theta_2-Z}\right)-\theta_2$$

6.1. Decoupling Technique

The result of forward kinematics of such a six DOF multibody **is a 4×4 transformation matrix.**

$$
\begin{aligned}
{}^0T_6 &= {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6 \\
&= \begin{bmatrix} {}^0R_6 & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

It is possible to decouple the inverse kinematics problem into two subproblems, known as **inverse position** and **inverse orientation kinematics.**

6.1. Decoupling Technique

Following the decoupling principle, the overall transformation matrix of a robot can be decomposed to **a translation** and **a rotation.**

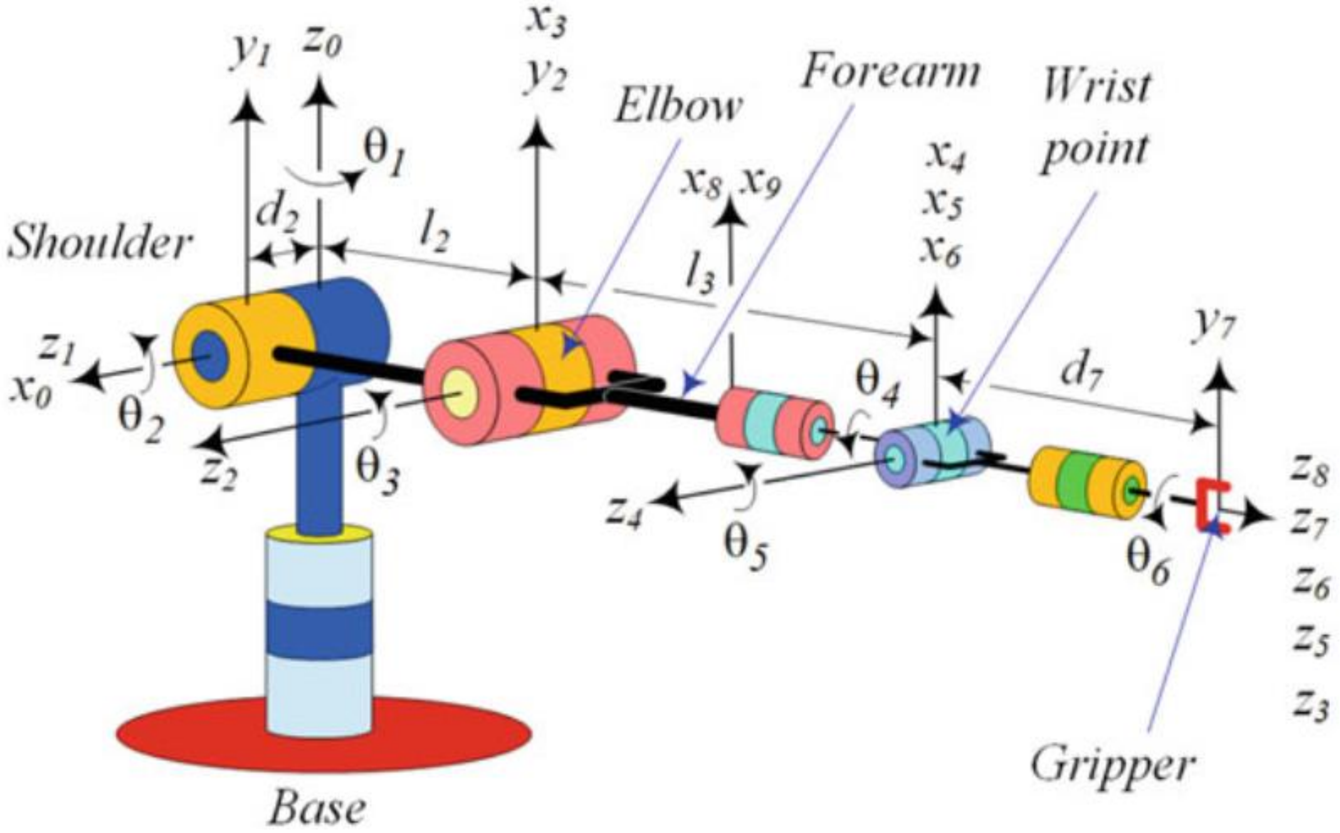
$$
\begin{aligned}
{}^0T_6 &= {}^0D_6 \, {}^0R_6 \\
&= \begin{bmatrix} {}^0R_6 & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^0R_6 & \mathbf{0} \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

- **The translation matrix $^0D_6$** indicates the position of the end-effector in the base frame B_0 and involves only the **three joint variables** of the manipulator. We will solve $^0\mathbf{d}_6$ for the variables that control the wrist position.
- **The rotation matrix $^0R_6$** indicates the orientation of the end-effector in B_0 and involves only the three joint variables of the wrist.

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

The decoupling method will be reviewed in this example for a 6 DOF. The forward kinematics of the articulated robot, illustrated in Figure.



6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

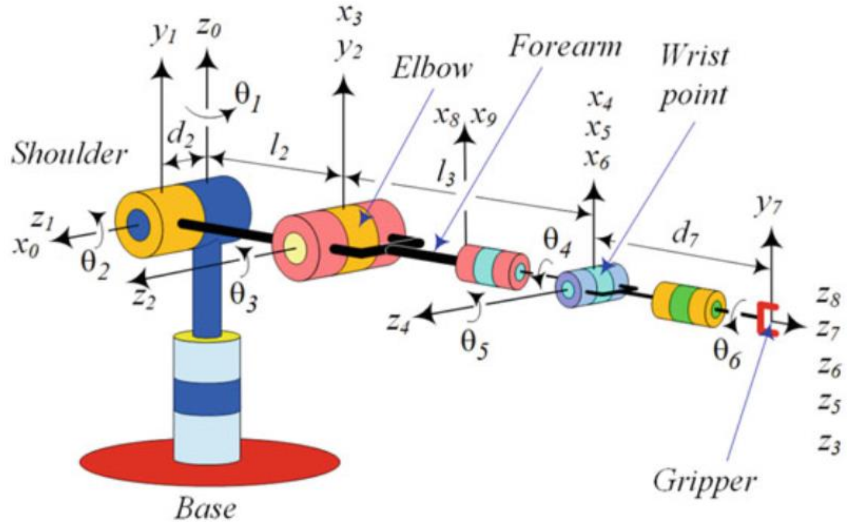
Transformation matrix of the end-effector was found

$$^{0}T_7 = T_{arm}T_{wrist} = {}^{0}T_3\,{}^{3}T_7$$

$$^{0}T_7 = {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3\,{}^{3}T_4\,{}^{4}T_5\,{}^{5}T_6\,{}^{6}T_7$$

$$= {}^{0}T_3\,{}^{3}T_6\,{}^{6}T_7$$

$$= \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



1	$R \vdash R(90)$
2	$R \parallel R(0)$
3	$R \vdash R(90)$
4	$R \vdash R(-90)$
5	$R \vdash R(90)$
6	$R \parallel R(0)$

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

where,

$$t_{11} = c\theta_1 \left(c\left(\theta_2+\theta_3\right)\left(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6\right) - c\theta_6 s\theta_5 s\left(\theta_2+\theta_3\right)\right)$$
$$+ s\theta_1 \left(c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4\right)$$

$$t_{21} = s\theta_1 \left(c\left(\theta_2+\theta_3\right)\left(-s\theta_4 s\theta_6 + c\theta_4 c\theta_5 c\theta_6\right) - c\theta_6 s\theta_5 s\left(\theta_2+\theta_3\right)\right)$$
$$- c\theta_1 \left(c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4\right)$$

$$t_{31} = s\left(\theta_2+\theta_3\right)\left(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6\right) + c\theta_6 s\theta_5 c\left(\theta_2+\theta_3\right)$$

$$t_{12} = c\theta_1 \left(s\theta_5 s\theta_6 s\left(\theta_2+\theta_3\right) - c\left(\theta_2+\theta_3\right)\left(c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6\right)\right)$$
$$+ s\theta_1 \left(c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6\right)$$

$$t_{22} = s\theta_1 \left(s\theta_5 s\theta_6 s\left(\theta_2+\theta_3\right) - c\left(\theta_2+\theta_3\right)\left(c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6\right)\right)$$
$$+ c\theta_1 \left(-c\theta_4 c\theta_6 + c\theta_5 s\theta_4 s\theta_6\right)$$

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

$$t_{32} = -s\theta_5 s\theta_6 c(\theta_2+\theta_3) - s(\theta_2+\theta_3)(c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6)$$

$$t_{13} = s\theta_1 s\theta_4 s\theta_5 + c\theta_1 (c\theta_5 s(\theta_2+\theta_3) + c\theta_4 s\theta_5 c(\theta_2+\theta_3))$$

$$t_{23} = -c\theta_1 s\theta_4 s\theta_5 + s\theta_1 (c\theta_5 s(\theta_2+\theta_3) + c\theta_4 s\theta_5 c(\theta_2+\theta_3))$$

$$t_{33} = c\theta_4 s\theta_5 s(\theta_2+\theta_3) - c\theta_5 c(\theta_2+\theta_3)$$

$$\begin{aligned} t_{14} = {} & d_6 (s\theta_1 s\theta_4 s\theta_5 + c\theta_1 (c\theta_4 s\theta_5 c(\theta_2+\theta_3) + c\theta_5 s(\theta_2+\theta_3))) \\ & + l_3 c\theta_1 s(\theta_2+\theta_3) + d_2 s\theta_1 + l_2 c\theta_1 c\theta_2 \end{aligned}$$

$$\begin{aligned} t_{24} = {} & d_6 (-c\theta_1 s\theta_4 s\theta_5 + s\theta_1 (c\theta_4 s\theta_5 c(\theta_2+\theta_3) + c\theta_5 s(\theta_2+\theta_3))) \\ & + s\theta_1 s(\theta_2+\theta_3) l_3 - d_2 c\theta_1 + l_2 c\theta_2 s\theta_1 \end{aligned}$$

$$\begin{aligned} t_{34} = {} & d_6 (c\theta_4 s\theta_5 s(\theta_2+\theta_3) - c\theta_5 c(\theta_2+\theta_3)) \\ & + l_2 s\theta_2 + l_3 c(\theta_2+\theta_3) \end{aligned}$$

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

The wrist position vector **d** = $[X\ Y\ Z]^T$, which is $[t_{14}\ t_{24}\ t_{34}]^T$ of ${}^0T_7$ for $d_7 = 0$, and (X, Y, Z) are coordinates of the position of the wrist point.

$$\mathbf{d} = \begin{bmatrix} (l_3 \sin(\theta_2+\theta_3) + l_2\cos\theta_2)\cos\theta_1 + d_2\sin\theta_1 \\ (l_3 \sin(\theta_2+\theta_3) + l_2\cos\theta_2)\sin\theta_1 - d_2\cos\theta_1 \\ l_3\cos(\theta_2+\theta_3) + l_2\sin\theta_2 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

It can be seen that $X\sin\theta_1 - Y\cos\theta_1 = d_2$

$$\theta_1 = 2\,\mathrm{atan2}(X \pm \sqrt{X^2+Y^2-d_2^2},\, d_2 - Y)$$

Combining the first two elements of **d** gives

$$l_3\sin(\theta_2+\theta_3) = \pm\sqrt{X^2+Y^2-d_2^2} - l_2\cos\theta_2$$

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

Then, the third element of **d** may be utilized to find

$$l_3^2 = \left(\pm\sqrt{X^2 + Y^2 - d_2^2} - l_2 \cos\theta_2 \right)^2 + (Z - l_2 \sin\theta_2)^2$$

which can be rearranged to the following form

$$a \cos\theta_2 + b \sin\theta_2 = c$$

$$a = 2l_2\sqrt{X^2 + Y^2 - d_2^2} \qquad b = 2l_2 Z$$

$$c = X^2 + Y^2 + Z^2 - d_2^2 + l_2^2 - l_3^2$$

with two solutions:

$$\theta_2 = \operatorname{atan2}\left(\frac{c}{r}, \pm\sqrt{1 - \frac{c^2}{r^2}}\right) - \operatorname{atan2}(a, b)$$

$$r^2 = a^2 + b^2$$

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

Summing the squares of the elements of **d** gives

$$X^2 + Y^2 + Z^2 = d_2^2 + l_2^2 + l_3^2 + 2l_2 l_3 \sin(2\theta_2 + \theta_3)$$

$$\theta_3 = \arcsin\left(\frac{X^2 + Y^2 + Z^2 - d_2^2 - l_2^2 - l_3^2}{2l_2 l_3}\right) - 2\theta_2$$

Find the orientation of the end-effector by solving ${}^3T_6$ or ${}^3R_6$ for θ_4, θ_5, θ_6

$$ {}^3R_6 = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 \\ c\theta_5 c\theta_6 s\theta_4 + c\theta_4 s\theta_6 & c\theta_4 cc\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 \end{bmatrix}$$

$$= \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

6.1. Decoupling Technique

Ex 3: Inverse kinematics of an articulated robot.

The angles θ_4, θ_5, θ_6 can be found by examining elements of ${}^3R_6$.

$$\theta_4 = \operatorname{atan2}(s_{23}, s_{13})$$

$$\theta_5 = \operatorname{atan2}\left(\sqrt{s_{13}^2 + s_{23}^2}, s_{33}\right)$$

$$\theta_6 = \operatorname{atan2}(s_{32}, -s_{31})$$

6.2. Inverse Transformation Technique

- Assume we have **the 4×4 transformation matrix** ${}^0T_6$ from forward kinematics expressed by numbers. The matrix ${}^0T_6$ includes the global position and the orientation of the end-effector of a 6 DOF robot in the base frame B_0.
- Assume the individual transformation matrices ${}^0T_1(q_1)$, ${}^1T_2(q_2)$, ${}^2T_3(q_3)$, ${}^3T_4(q_4)$, ${}^4T_5(q_5)$, and ${}^5T_6(q_6)$ are known as functions of joint variables analytically.

- According to forward kinematics we have:

$$
{}^0T_6 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6
$$

$$
= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

6.2. Inverse Transformation Technique

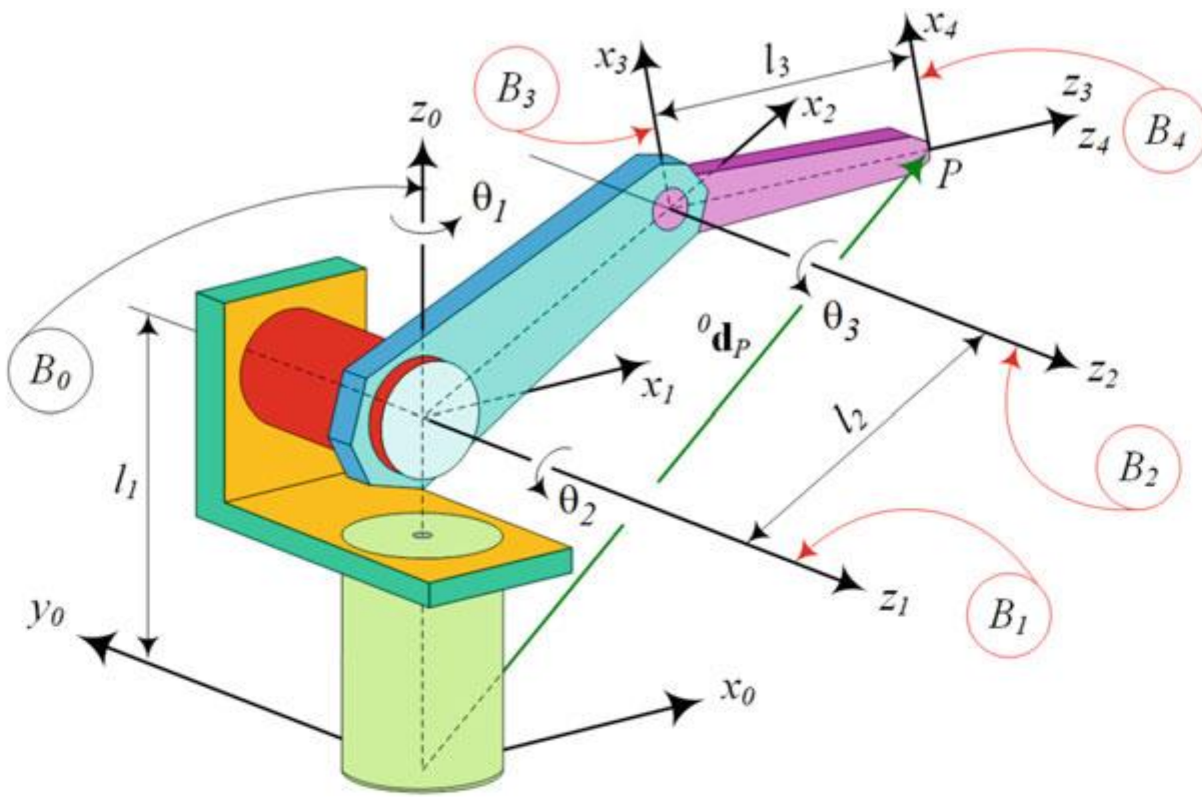
- The inverse kinematics problem as follows

$$
\begin{aligned}
{}^{1}T_6 &= {}^{0}T_1^{-1}\ {}^{0}T_6 \\
{}^{2}T_6 &= {}^{1}T_2^{-1}\ {}^{0}T_1^{-1}\ {}^{0}T_6 \\
{}^{3}T_6 &= {}^{2}T_3^{-1}\ {}^{1}T_2^{-1}\ {}^{0}T_1^{-1}\ {}^{0}T_6 \\
{}^{4}T_6 &= {}^{3}T_4^{-1}\ {}^{2}T_3^{-1}\ {}^{1}T_2^{-1}\ {}^{0}T_1^{-1}\ {}^{0}T_6 \\
{}^{5}T_6 &= {}^{4}T_5^{-1}\ {}^{3}T_4^{-1}\ {}^{2}T_3^{-1}\ {}^{1}T_2^{-1}\ {}^{0}T_1^{-1}\ {}^{0}T_6 \\
\mathbf{I} &= {}^{5}T_6^{-1}\ {}^{4}T_5^{-1}\ {}^{3}T_4^{-1}\ {}^{2}T_3^{-1}\ {}^{1}T_2^{-1}\ {}^{0}T_1^{-1}\ {}^{0}T_6
\end{aligned}
$$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

Here is the use of inverse transformation technique to solve its inverse kinematics. Consider the articulated manipulator shown in Figure.



6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

The forward kinematics of the manipulator is:

$$^{0}T_3 = {}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3$$

$$= \begin{bmatrix} c\theta_1 c(\theta_2+\theta_3) & s\theta_1 & c\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2+\theta_3) & -c\theta_1 & s\theta_1 s(\theta_2+\theta_3) & l_2 c\theta_2 s\theta_1 \\ s(\theta_2+\theta_3) & 0 & -c(\theta_2+\theta_3) & l_1 + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, we attach a coordinate frame B_4 at P that is at a constant distance l_3 from B_3

$$^{3}T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

The overall forward kinematics of the manipulator is:

$$^{0}T_4 = {}^{0}T_3\,{}^{3}T_4 = \begin{bmatrix} c(\theta_2+\theta_3)c\theta_1 & s\theta_1 & s(\theta_2+\theta_3)c\theta_1 & l_3 s(\theta_2+\theta_3)c\theta_1 + l_2 c\theta_1 c\theta_2 \\ c(\theta_2+\theta_3)s\theta_1 & -c\theta_1 & s(\theta_2+\theta_3)s\theta_1 & l_3 s(\theta_2+\theta_3)s\theta_1 + l_2 c\theta_2 s\theta_1 \\ s(\theta_2+\theta_3) & 0 & -c(\theta_2+\theta_3) & l_1 - l_3 c(\theta_2+\theta_3) + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the following dimensions: l_1= 1m, l_2= 1.05m, l_3= 0.89m

$$^{0}\mathbf{d}_P = \left[1\ 1.1\ 1.2\right]^T$$

$$^{0}T_4 = \begin{bmatrix} \cos(\theta_2+\theta_3)\cos\theta_1 & \sin\theta_1 & \sin(\theta_2+\theta_3)\cos\theta_1 & 1 \\ \cos(\theta_2+\theta_3)\sin\theta_1 & -\cos\theta_1 & \sin(\theta_2+\theta_3)\sin\theta_1 & 1.1 \\ \sin(\theta_2+\theta_3) & 0 & -\cos(\theta_2+\theta_3) & 1.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

Let us multiply both sides of ${}^{0}T_{4}$ by ${}^{0}T_{1}^{-1}$ to have:

$$ {}^{0}T_{1}^{-1}\,{}^{0}T_{4} = {}^{0}T_{1}^{-1}\left({}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4}\right) = {}^{1}T_{4} $$

$$ {}^{0}T_{1}^{-1}\,{}^{0}T_{4} = {}^{1}T_{4} $$

$$ = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{0}T_{4} $$

$$ = \begin{bmatrix} \cos(\theta_2+\theta_3) & 0 & \sin(\theta_2+\theta_3) & \cos\theta_1 + 1.1\sin\theta_1 \\ \sin(\theta_2+\theta_3) & 0 & -\cos(\theta_2+\theta_3) & 0.2 \\ 0 & 1 & 0 & \sin\theta_1 - 1.1\cos\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

Let us multiply both sides of ${}^{0}T_{4}$ by ${}^{0}T_{1}{}^{-1}$ to have:

$${}^{0}T_{1}^{-1}\,{}^{0}T_{4} = {}^{0}T_{1}^{-1}\left({}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4}\right) = {}^{1}T_{4}$$

$${}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4} =$$

$$\begin{bmatrix} \cos(\theta_2+\theta_3) & 0 & \sin(\theta_2+\theta_3) & 0.89\sin(\theta_2+\theta_3)+1.05\cos\theta_2 \\ \sin(\theta_2+\theta_3) & 0 & -\cos(\theta_2+\theta_3) & 1.05\sin\theta_2-0.89\cos(\theta_2+\theta_3) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

Equating the element r_{24} of both sides of provides an equation to determine $\boldsymbol{\theta_1}$.

$$\sin\theta_1 - 1.1\cos\theta_1 = 0$$

$$\theta_1 = \operatorname{atan2}(1.1, 1) = \arctan\frac{1.1}{1}$$

$$= 0.8329812667\,\text{rad} \approx 47.72631098\,\text{deg}$$

Substituting $\theta_1 = 0.83298$ rad in provides a matrix ${}^1T_4$

$${}^1T_4 = \begin{bmatrix} \cos(\theta_2+\theta_3) & 0 & \sin(\theta_2+\theta_3) & 1.4866 \\ \sin(\theta_2+\theta_3) & 0 & -\cos(\theta_2+\theta_3) & 0.2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

We multiply both sides of ${}^{1}T_{2}^{-1}$ to have:

$$ {}^{1}T_{2}^{-1}\,{}^{1}T_{4} = {}^{1}T_{2}^{-1}\left({}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4}\right) = {}^{2}T_{4} $$

where,

$$ {}^{1}T_{2}^{-1}\,{}^{1}T_{4} = {}^{2}T_{4} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & -1.05 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{1}T_{4} $$

$$ = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 1.4866\cos\theta_2 + 0.2\sin\theta_2 - 1.05 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0.2\cos\theta_2 - 1.4866\sin\theta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

$$^2T_3\,^3T_4 = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0.89\sin\theta_3 \\ \sin\theta_3 & 0 & -\cos\theta_3 & -0.89\cos\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Squaring the elements r_{14} and r_{24} of the left-hand sides, to determine θ_2.

$$(1.4866\cos\theta_2 + 0.2\sin\theta_2 - 1.05)^2 + (0.2\cos\theta_2 - 1.4866\sin\theta_2)^2$$
$$= (0.89\sin\theta_3)^2 + (-0.89\cos\theta_3)^2$$

$$3.1219\cos\theta_2 + 0.42\sin\theta_2 = 2.5604$$

6.2. Inverse Transformation Technique

Ex 4: Articulated manipulator and numerical case.

$$\theta_2 = 0.7555 \text{ rad} \approx 43.29 \text{ deg}$$

$$\theta_2 = -0.4881 \text{ rad} \approx -27.96 \text{ deg}$$

Having θ_2, we can calculate θ_3 from the last column

$$\theta_3 = \arctan\left(\frac{1.4866\cos\theta_2 + 0.2\sin\theta_2 - 1.05}{0.2\cos\theta_2 - 1.4866\sin\theta_2}\right) + \pi$$

If θ_2= 0.7555 rad, $\theta_3 = 2.95 \text{ rad} \approx 169 \text{ deg}$

If θ_2= –0.0.488 rad, $\theta_3 = 0.19198 \text{ rad} \approx 11 \text{ deg}$

6.3. Iterative Technique

The inverse kinematics problem of robots can be interpreted as searching for the unknowns q_k of a set of nonlinear algebraic equations

$$
\begin{aligned}
{}^{0}T_n &= \mathbf{T}(\mathbf{q}) \\
&= {}^{0}T_1(q_1)\,{}^{1}T_2(q_2)\,{}^{2}T_3(q_3)\,{}^{3}T_4(q_4)\,\cdots\,{}^{n-1}T_n(q_n) \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad r_{ij} = r_{ij}(q_k) \qquad k = 1, 2, \cdots n
\end{aligned}
$$

where n is the number of degree of freedom (DOF) of the robot.

6.3. Iterative Technique

The most common method is known as the Newton-Raphson method. The iteration technique can be set in an algorithm.

Inverse kinematics iteration technique.

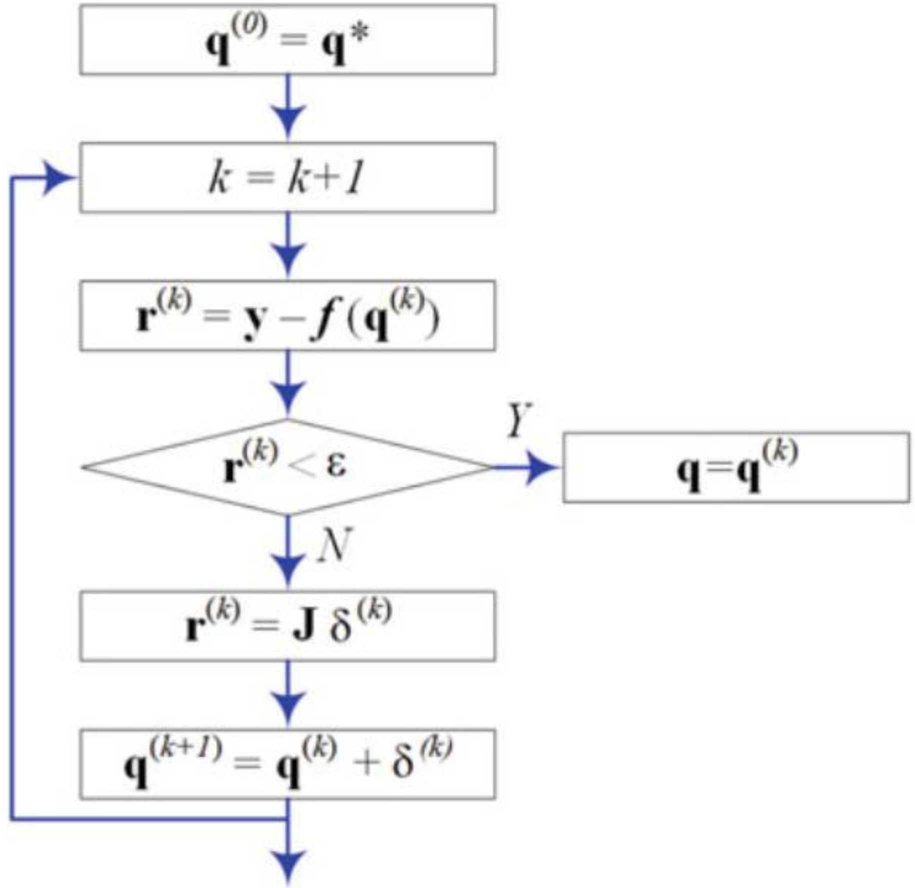
1. Set the initial counter i = 0.
2. Find or guess an initial estimate **q**(0).
3. Calculate the residue δ**T**(**q**(i)) = **J**(**q**(i)) δ**q**(i).

   If every element of **T**(**q**(i)) or its norm , ,**T**(**q**(i)), , is less than a tolerance, , ,**T**(**q**(i)), , < then terminate the iteration. The **q**(i) is the desired solution.
4. Calculate **q**(i+1) = **q**(i) + **J**-1(**q**(i)) δ**T**(**q**(i)).
5. **Set i = i + 1 and return to step 3.**

6.3. Iterative Technique

The most common method is known as the Newton-Raphson method. The iteration technique can be set in an algorithm.

Inverse kinematics iteration technique.



6.3. Iterative Technique

The tolerance ϵ can equivalently be set up on variables

$$\mathbf{q}^{(i+1)} - \mathbf{q}^{(i)} < \epsilon$$

Or, the condition Jacobian ***J***:

$$\mathbf{J} - \mathbf{I} < \epsilon$$

$$\mathbf{J}(\mathbf{q}) = \left[\frac{\partial T_i}{\partial q_j} \right]$$

6.3. Iterative Technique

Ex 5: Inverse kinematics for a 2R planar manipulator.

The position of tip point of a 2R planar manipulator is calculated

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 s \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Define $\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} X \\ Y \end{bmatrix}$

The Jacobian of the equations

$$\mathbf{J}(\mathbf{q}) = \left[\frac{\partial T_i}{\partial q_j}\right] = \begin{bmatrix} \dfrac{\partial X}{\partial \theta_1} & \dfrac{\partial X}{\partial \theta_2} \\ \dfrac{\partial Y}{\partial \theta_1} & \dfrac{\partial Y}{\partial \theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

6.3. Iterative Technique

Ex 4: Inverse kinematics for a 2R planar manipulator.

The inverse of the Jacobian is

$$\mathbf{J}^{-1} = \frac{-1}{l_1 l_2 s\theta_2}\begin{bmatrix} -l_2 c(\theta_1+\theta_2) & -l_2 s(\theta_1+\theta_2) \\ l_1 c\theta_1 + l_2 c(\theta_1+\theta_2) & l_1 s\theta_1 + l_2 s(\theta_1+\theta_2) \end{bmatrix}$$

The iterative formula is set up as:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(i+1)} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(i)} + \mathbf{J}^{-1}\left(\begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix}^{(i)}\right)$$

$$l_1 = l_2 = 1$$

Assume,

$$\mathbf{T} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

6.3. Iterative Technique

Ex 4: Inverse kinematics for a 2R planar manipulator.

Start from a guess value $\mathbf{q}^{(0)}$

$$\mathbf{q}^{(0)} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(0)} = \begin{bmatrix} \pi/3 \\ -\pi/3 \end{bmatrix}$$

$$\delta\mathbf{T} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \cos\pi/3 + \cos(\pi/3 + -\pi/3) \\ \sin\pi/3 + \sin(\pi/3 + -\pi/3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} + 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \qquad \mathbf{J}^{-1} = \begin{bmatrix} -\frac{2}{3}\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{bmatrix}$$

6.3. Iterative Technique

Ex 4: Inverse kinematics for a 2R planar manipulator.

Therefore,

$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}^{(1)} = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}^{(0)} + \mathbf{J}^{-1}\,\delta\mathbf{T}$$

$$= \begin{bmatrix}\pi/3\\ -\pi/3\end{bmatrix} + \begin{bmatrix}-\frac{2}{3}\sqrt{3} & 0\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ -\frac{1}{2}\sqrt{3}+1\end{bmatrix}$$

$$= \begin{bmatrix}1.6245\\ -1.7792\end{bmatrix}$$

6.3. Iterative Technique

Ex 4: Inverse kinematics for a 2R planar manipulator.

Iteration 1.

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \qquad \delta\mathbf{T} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} + 1 \end{bmatrix} \qquad \mathbf{q}^{(1)} = \begin{bmatrix} 1.6245 \\ -1.7792 \end{bmatrix}$$

Iteration 2.

$$\mathbf{J} = \begin{bmatrix} -0.844 & 0.154 \\ 0.934 & 0.988 \end{bmatrix} \qquad \delta\mathbf{T} = \begin{bmatrix} 6.516 \times 10^{-2} \\ 0.15553 \end{bmatrix} \qquad \mathbf{q}^{(2)} = \begin{bmatrix} 1.583 \\ -1.582 \end{bmatrix}$$

6.2. Inverse Transformation Technique

Ex 4: Inverse kinematics for a 2R planar manipulator.

Iteration 3.

$$\mathbf{J} = \begin{bmatrix} -1.00 & -.433 \times 10^{-3} \\ .988 & .999 \end{bmatrix} \qquad \delta\mathbf{T} = \begin{bmatrix} .119 \times 10^{-1} \\ -.362 \times 10^{-3} \end{bmatrix}$$

$$\mathbf{q}^{(3)} = \begin{bmatrix} 1.570795886 \\ -1.570867014 \end{bmatrix}$$

Iteration 4.

$$\mathbf{J} = \begin{bmatrix} -1.000 & 0.0 \\ 0.998\,50 & 1.0 \end{bmatrix} \quad \delta\mathbf{T} = \begin{bmatrix} -.438 \times 10^{-6} \\ .711 \times 10^{-4} \end{bmatrix} \quad \mathbf{q}^{(4)} = \begin{bmatrix} 1.570796329 \\ -1.570796329 \end{bmatrix}$$

The result of the fourth iteration q(4) is close enough to the exact value $\boldsymbol{q} = [\ \pi/2\ -\pi/2]^T$.

C6. End!