Chapter 8. Applied Dynamics

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- **Relation** between **kinematics and the cause of change of kinematics** is called **equations of motion**.
- **Deriving the equations of motion** and the expression of their solution is called dynamics.
- Dynamics of a robot may be considered as the motion of a rigid link with respect to a fixed global coordinate frame.

8.1. Force and Moment

Force and Moment

- In Newtonian dynamics, the acting forces on a system of connected rigid bodies can be divided into **internal and external forces**.
  - **Internal forces** are acting between connected bodies and appear as action and reaction forces.
  - **External forces** are acting from outside of the system and appear as applied driving forces.
- The **resultant F** is the vectorial sum of all the external forces acting on a body.
- The **resultant M** is the vectorial sum of all the moments of the external forces acting on the body.

$$\mathbf{F} = \sum_i \mathbf{F}_i \qquad \mathbf{M} = \sum_i \mathbf{M}_i$$

8.1. Force and Moment

Momentum

- The *momentum* of a moving rigid body is a vector quantity equal to the total mass of the body times the translational velocity of its mass center C. The momentum **p** may also be called **translational momentum or linear momentum**.

$$\mathbf{p} = m\mathbf{v}$$

- The moment of momentum **L** may also be called **angular momentum**.

$$\mathbf{L} = \mathbf{r}_C \times \mathbf{p}$$

where $\mathbf{r}_C$ is the position vector of the mass center C.

8.1. Force and Moment

Equation of Motion

- The application of a force system is emphasized by ***Newton's second and third laws of motion***. The global rate of change of linear momentum as follows

$$ {}^{G}\mathbf{F} = \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{p} = \frac{{}^{G}d}{dt}\left(m\,{}^{G}\mathbf{v}\right) $$

- The second law of motion also states that the global rate of change of angular momentum as follows

$$ {}^{G}\mathbf{M} = \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{L} $$

8.1. Force and Moment

Work and Energy

- The ***kinetic energy K*** of a moving point **P** with mass ***m*** at a position ${}^{G}r_P$ and velocity ${}^{G}v_P$ is

$$K = \frac{1}{2} m \left({}^{G}\mathbf{v} \cdot {}^{G}\mathbf{v} \right)$$

- The work done by the applied force GF on m in moving from point 1 to point 2 on a path, indicated by a vector G r, is

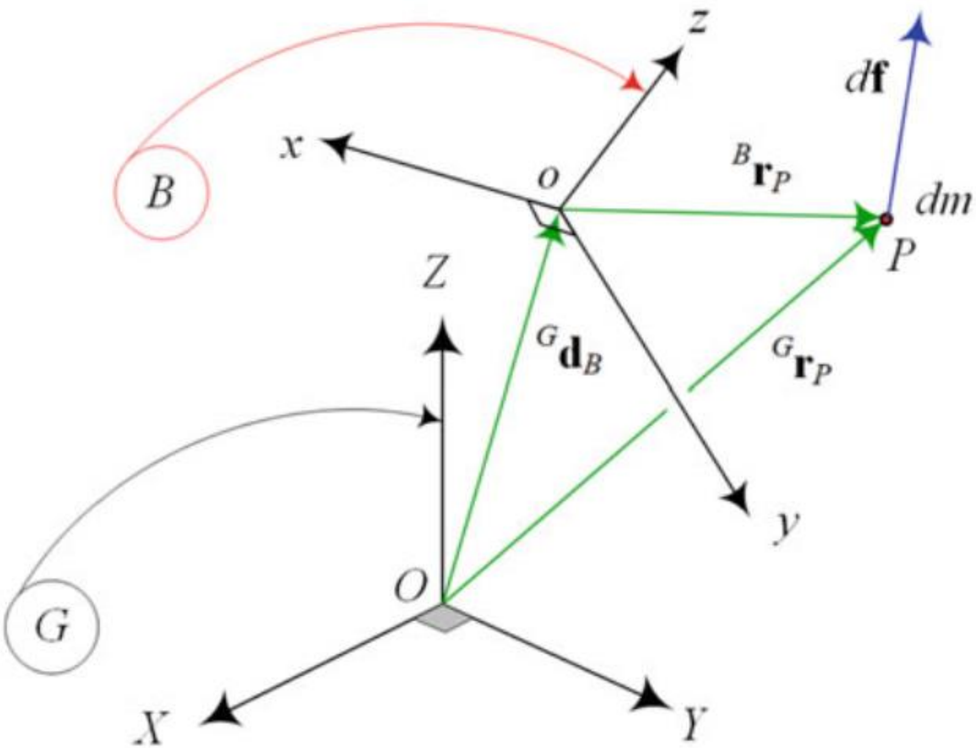
$${}_1W_2 = \int_1^2 {}^{G}\mathbf{F} \cdot d\,{}^{G}\mathbf{r}$$

$$\int_1^2 {}^{G}\mathbf{F} \cdot d\,{}^{G}\mathbf{r} = m \int_1^2 \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{v} \cdot {}^{G}\mathbf{v}\,dt = \frac{1}{2} m \int_1^2 \frac{d}{dt} v^2\,dt$$

$$= \frac{1}{2} m \left(v_2^2 - v_1^2 \right) = K_2 - K_1$$

8.2. Rigid Body Rotational Kinetics

- Assume the body frame is attached at the center of mass C of the body. Point P indicates an infinitesimal sphere of the body with a very small mass *dm*.



A body point mass moving with velocity ${}^G\mathbf{v}_P$ and acted on by force ***df***

$$d\mathbf{f} = {}^G\mathbf{a}_P \, dm$$

- The equation of motion for the whole body in global coordinate frame is

$$^G\mathbf{F} = m \, {}^G\mathbf{a}_B$$

8.2. Rigid Body Rotational Kinetics

- ${}^{G}\boldsymbol{a}_{\mathbf{B}}$ is the acceleration vector of the body C in global frame, $\boldsymbol{m}$ is the total mass of the body, and $\boldsymbol{F}$ is the resultant of the external forces acted on the body at C. The motion of equation can be expressed in the body coordinate frame as

$$
{}^{B}\mathbf{F} = m\,{}_{G}^{B}\mathbf{a}_{B} + m\,{}_{G}^{B}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{v}_{B}
$$

$$
\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} ma_x + m\left(\omega_y v_z - \omega_z v_y\right) \\ ma_y - m\left(\omega_x v_z - \omega_z v_x\right) \\ ma_z + m\left(\omega_x v_y - \omega_y v_x\right) \end{bmatrix}
$$

8.3. Rigid Body Rotational Kinetics

- The rigid body rotational equation of motion is expressed by the Euler equation.

$$
\begin{aligned}
{}^{B}\mathbf{M} &= \frac{{}^{G}d}{dt}\,{}^{B}\mathbf{L} = {}^{B}\dot{\mathbf{L}} + {}^{B}_{G}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{L} \\
&= {}^{B}I\;{}^{B}_{G}\dot{\boldsymbol{\omega}}_{B} + {}^{B}_{G}\boldsymbol{\omega}_{B} \times \left({}^{B}I\;{}^{B}_{G}\boldsymbol{\omega}_{B}\right)
\end{aligned}
$$

- ***L*** is the angular momentum and [***I***] is the mass moment or moment of inertia of the rigid body

$$
{}^{B}\mathbf{L} = {}^{B}I\;{}^{B}_{G}\boldsymbol{\omega}_{B}
$$

$$
[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}
$$

8.3. Rigid Body Rotational Kinetics

- The expanded form of the Euler equation is

$$M_x = I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z$$
$$-I_{yz}\left(\omega_z^2 - \omega_y^2\right) - \omega_x\left(\omega_z I_{xy} - \omega_y I_{xz}\right)$$

$$M_y = I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - \left(I_{zz} - I_{xx}\right)\omega_z\omega_x$$
$$-I_{xz}\left(\omega_x^2 - \omega_z^2\right) - \omega_y\left(\omega_x I_{yz} - \omega_z I_{xy}\right)$$

$$M_z = I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y$$
$$-I_{xy}\left(\omega_y^2 - \omega_x^2\right) - \omega_z\left(\omega_y I_{xz} - \omega_x I_{yz}\right)$$

- The Equation can be reduced to a set of simpler equations

$$M_1 = I_1\dot{\omega}_1 - \left(I_2 - I_2\right)\omega_2\omega_3$$
$$M_2 = I_2\dot{\omega}_2 - \left(I_3 - I_1\right)\omega_3\omega_1$$
$$M_3 = I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2$$

8.3. Rigid Body Rotational Kinetics

The kinetic energy of a rotating rigid body is

$$
\begin{aligned}
K &= \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) \\
&\quad - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{zx}\omega_z\omega_x \\
&= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}^T I\,\boldsymbol{\omega}
\end{aligned}
$$

The principal coordinate frame reduces to

$$
K = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)
$$

8.4. Mass Moment Matrix

- The mass moment is also called moment of inertia, centrifugal moments, or deviation moments.
- Every rigid body has a 3 × 3 moment of inertia matrix **[I]**.

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

- The diagonal elements $\mathbf{I}_{ij}$, $i = j$ are called polar mass moments,

$$I_{xx} = I_x = \int_B \left(y^2 + z^2\right) dm$$

$$I_{yy} = I_y = \int_B \left(z^2 + x^2\right) dm$$

$$I_{zz} = I_z = \int_B \left(x^2 + y^2\right) dm$$

8.4. Mass Moment Matrix

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- The diagonal elements $\mathbf{I}_{ij}$, $i = j$ are called polar mass moments,

$$I_{xx} = I_x = \int_B (y^2 + z^2)\, dm$$

$$I_{yy} = I_y = \int_B (z^2 + x^2)\, dm$$

$$I_{zz} = I_z = \int_B (x^2 + y^2)\, dm$$

8.4. Mass Moment Matrix

- The off-diagonal elements $\mathbf{I}_{ij}$, $i \neq j$ are called products mass moments

$$I_{xy} = I_{yx} = -\int_B xy\, dm$$

$$I_{yz} = I_{zy} = -\int_B yz\, dm$$

$$I_{zx} = I_{xz} = -\int_B zx\, dm$$

8.5. Lagrange's Form of Newton's Equations

- Newton's equation of motion can be transformed into

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} = F_r \qquad r = 1, 2, \cdots n$$

where

$$F_r = \sum_{i=1}^{n}\left(F_{ix}\frac{\partial f_i}{\partial q_1} + F_{iy}\frac{\partial g_i}{\partial q_2} + F_{iz}\frac{\partial h_i}{\partial q_n}\right)$$

- The Lagrange equation of motion, where K is the kinetic energy of the n DOF system, q_r, $r = 1, 2, \ldots, n$, are the generalized coordinates of the system.

- $F = [F_{ix}\ F_{iy}\ F_{iz}]^T$ is the external force acting on the i^{th} particle of the system, and Fr is the generalized force associated to q_r.

8.6. Lagrangian Mechanics

- The Lagrange equation of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_r}\right) - \frac{\partial \mathcal{L}}{\partial q_r} = Q_r \qquad r = 1, 2, \cdots n$$

where L is the Lagrangian of the system,

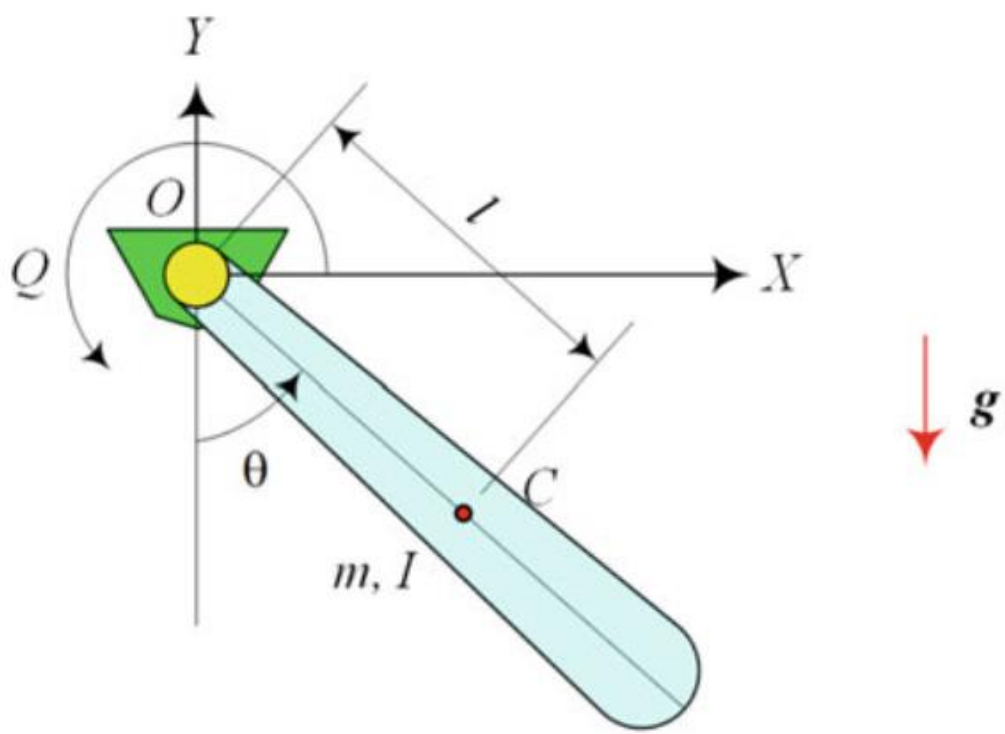
$$\mathcal{L} = K - V$$

and Q_r is the nonpotential generalized force

8.6. Lagrangian Mechanics

Ex 1. A one-link manipulator.

- A one-link manipulator is illustrated in Figure. Assume that there is viscous friction in the joint where an ideal motor applies the torque Q to move the arm. Find the equation of motion of the manipulator?



8.6. Lagrangian Mechanics

Ex 1. A one-link manipulator.

- The rotor of an ideal motor has no mass moment by assumption.
- The kinetic and potential energies of the manipulator are

$$K = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} \left(I_C + ml^2 \right) \dot{\theta}^2$$

$$V = -mg \cos \theta$$

where m is the mass and I is the mass moment of the pendulum about O.

- The Lagrangian of the manipulator will be

$$\mathcal{L} = K - V = \frac{1}{2} I \dot{\theta}^2 + mg \cos \theta$$

8.6. Lagrangian Mechanics

Ex 1. A one-link manipulator.

- The equation of motion of the manipulator can be found

$$M = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = I\,\ddot{\theta} + mgl\sin\theta$$

⇒ The generalized force M is the contribution of the motor torque Q and the viscous friction torque $-c\dot{\theta}$.

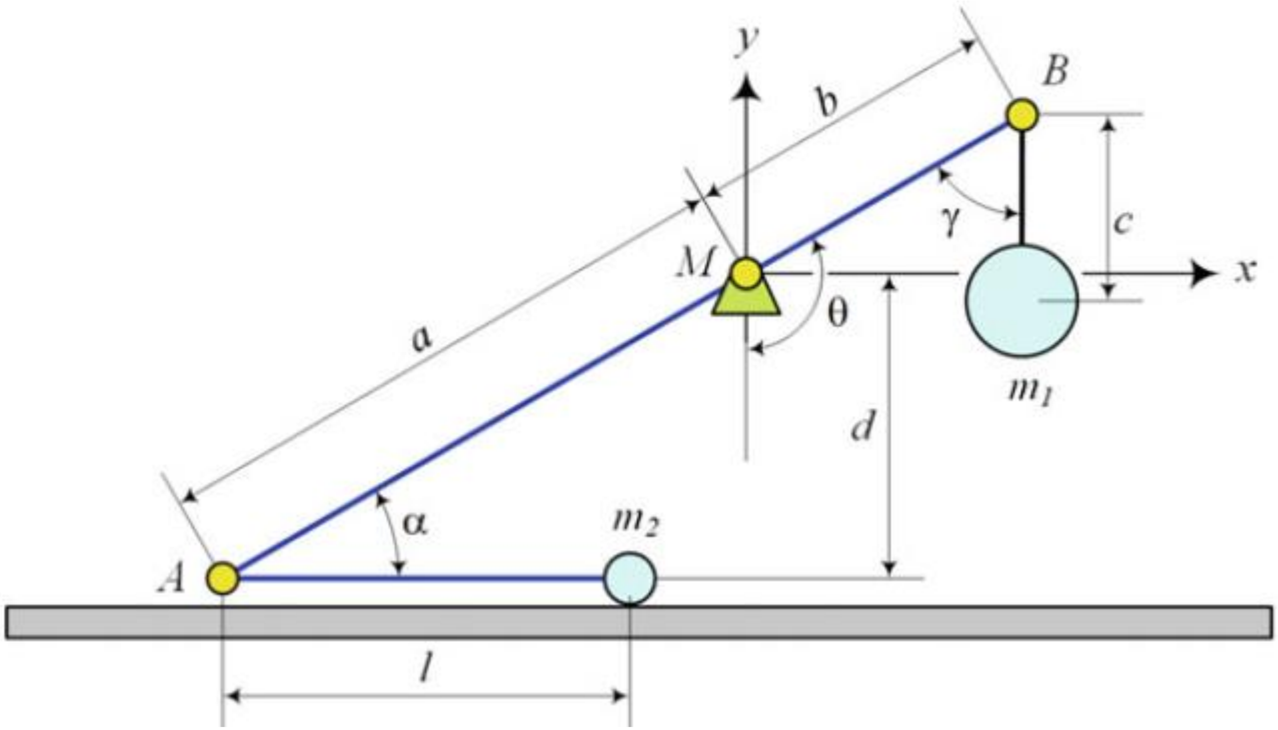
- The equation of motion of the manipulator will be

$$\boxed{Q = I\,\ddot{\theta} + c\dot{\theta} + mgl\sin\theta}$$

8.6. Lagrangian Mechanics

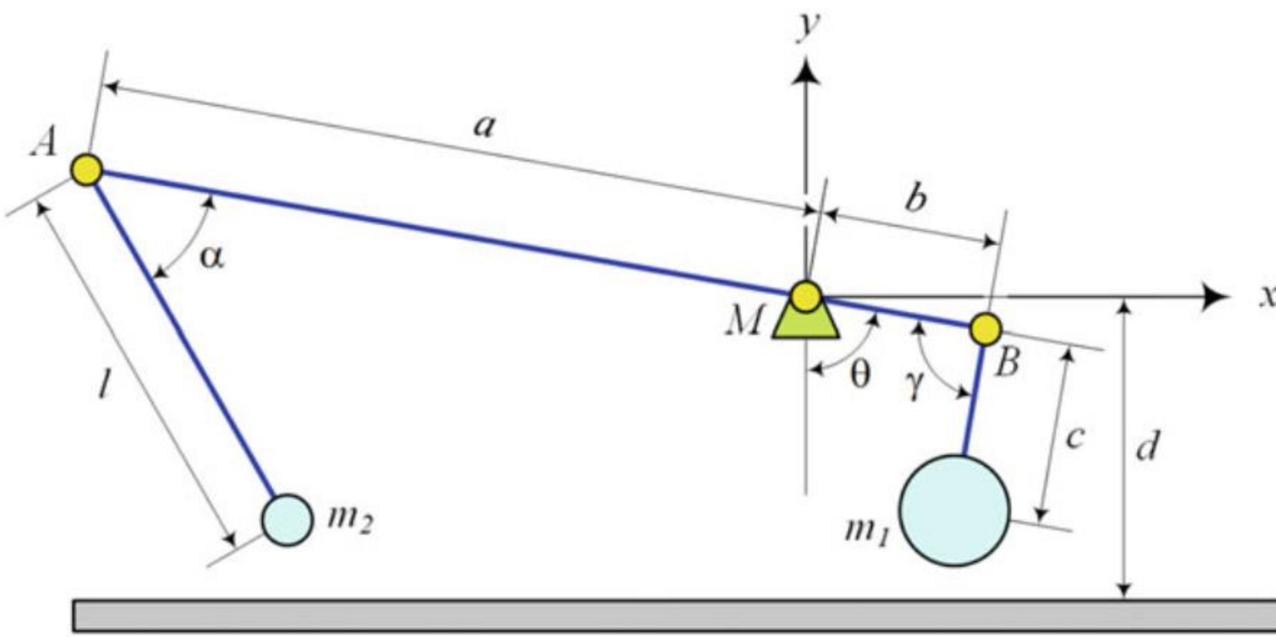
Ex 2. Trebuchet

- A trebuchet is shown schematically in Figure. Find the equations of motion of a sample trebuchet?



8.6. Lagrangian Mechanics

Ex 2. Trebuchet



A trebuchet in motion

8.6. Lagrangian Mechanics

Ex 2. Trebuchet

- The position coordinates of masses m_1 and m_2 are

$$x_1 = b\sin\theta - c\sin(\theta + \gamma)$$

$$y_1 = -b\cos\theta + c\cos(\theta + \gamma)$$

$$x_2 = -a\sin\theta - l\sin(-\theta + \alpha)$$

$$y_2 = -a\cos\theta - l\cos(-\theta + \alpha)$$

- Taking a time derivative provides the velocity components

$$\dot{x}_1 = b\dot{\theta}\cos\theta - c\left(\dot{\theta} + \dot{\gamma}\right)\cos(\theta + \gamma)$$

$$\dot{y}_1 = b\dot{\theta}\sin\theta - c\left(\dot{\theta} + \dot{\gamma}\right)\sin(\theta + \gamma)$$

$$\dot{x}_2 = l(c - \dot{\alpha})\cos(\alpha - \theta) - a\dot{\theta}\cos(\theta)$$

$$\dot{y}_2 = a\dot{\theta}\sin\theta - l\left(\dot{\theta} - \dot{\alpha}\right)\sin(\alpha - \theta)$$

8.6. Lagrangian Mechanics

Ex 2. Trebuchet

- The kinetic energy of the system is

$$
\begin{aligned}
K &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) \\
&= \frac{1}{2} m_1 \left(\left(b^2 + c^2\right) \dot{\theta}^2 + c^2 \dot{\gamma}^2 + 2c^2 \dot{\theta} \dot{\gamma}\right) \\
&\quad - m_1 b c \dot{\theta} \left(\dot{\theta} + \dot{\gamma}\right) \cos \gamma \\
&\quad + \frac{1}{2} m_2 \left(\left(a^2 + l^2\right) \dot{\theta}^2 + l^2 \dot{\alpha}^2 - 2 l^2 \dot{\theta} \dot{\alpha}\right) \\
&\quad - m_2 a l \dot{\theta} \left(\dot{\theta} - \dot{\alpha}\right) \cos \left(2\theta - \alpha\right)
\end{aligned}
$$

8.6. Lagrangian Mechanics

Ex 2. Trebuchet

- The potential energy of the system can be calculated by *y* position of the masses

$$V = m_1 g y_1 + m_2 g y_2$$
$$= m_1 g \left(-b\cos\theta + c\cos(\theta+\gamma)\right)$$
$$+ m_2 g \left(-a\cos\theta - l\cos(-\theta+\alpha)\right)$$

- We can set up the Lagrangian *L* $\boxed{\mathcal{L} = K - V}$

- Using the Lagrangian, we are able to find the three equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

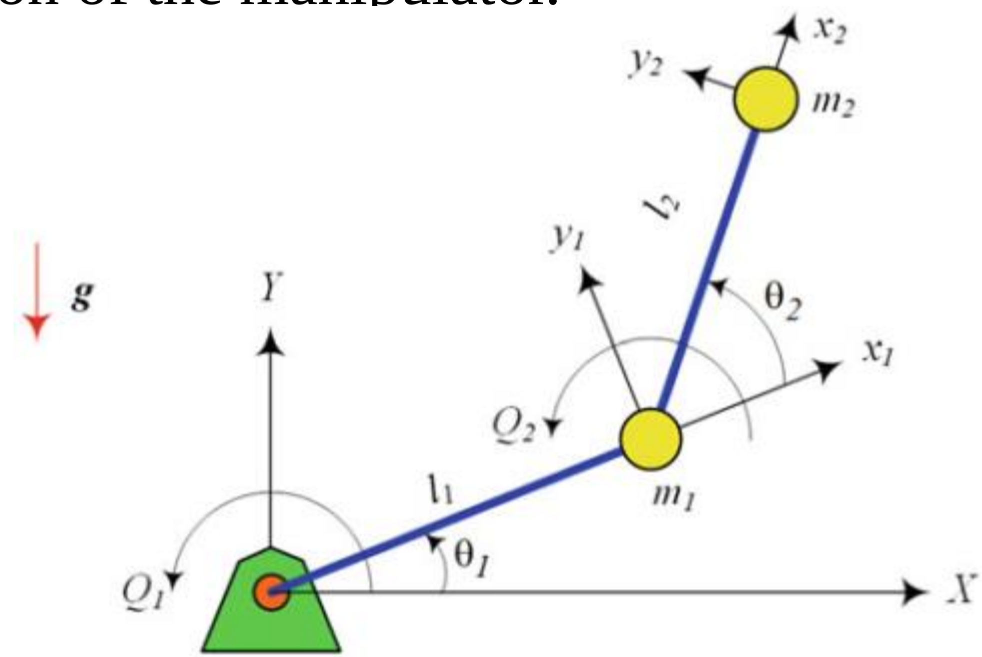
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}}\right) - \frac{\partial \mathcal{L}}{\partial \gamma} = 0$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- An ideal model of a 2R planar manipulator is illustrated in Figure. It is called ideal because we assumed the links are massless and there is no friction. The masses m_1 and m_2 are the second motor to run the second link and the load at the end point. We take the absolute angle θ_1 and the relative angle θ_2 as the generalized coordinates to express the configuration of the manipulator.



8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- The global positions of m_1 and m_2 are

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

- The global velocities of the masses are

$$\begin{bmatrix} \dot{X}_1 \\ \dot{Y}_1 \end{bmatrix} = \begin{bmatrix} -l_1 \dot{\theta}_1 \sin\theta_1 \\ l_1 \dot{\theta}_1 \cos\theta_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{X}_2 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \sin(\theta_1 + \theta_2) \\ l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \cos(\theta_1 + \theta_2) \end{bmatrix}$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- The kinetic energy of this manipulator is made of kinetic energy of the masses.

$$
\begin{aligned}
K &= K_1 + K_2 = \frac{1}{2}m_1\left(\dot{X}_1^2 + \dot{Y}_1^2\right) + \frac{1}{2}m_2\left(\dot{X}_2^2 + \dot{Y}_2^2\right) \\
&= \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 \\
&\quad + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2l_1 l_2 \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos\theta_2\right)
\end{aligned}
$$

- The potential energy of the manipulator is

$$
\begin{aligned}
V &= V_1 + V_2 = m_1 g Y_1 + m_2 g Y_2 \\
&= m_1 g l_1 \sin\theta_1 + m_2 g\left(l_1 \sin\theta_1 + l_2 \sin\left(\theta_1 + \theta_2\right)\right)
\end{aligned}
$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- The Lagrangian is then obtained

$$\mathcal{L} = K - V = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$
$$+ \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 + 2 l_1 l_2 \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \cos \theta_2 \right)$$
$$- \left(m_1 g l_1 \sin \theta_1 + m_2 g \left(l_1 \sin \theta_1 + l_2 \sin \left(\theta_1 + \theta_2 \right) \right) \right)$$

- Lagrangian provides us with the required partial derivatives.

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = - \left(m_1 + m_2 \right) g l_1 \cos \theta_1 - m_2 g l_2 \cos \left(\theta_1 + \theta_2 \right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \left(m_1 + m_2 \right) l_1^2 \dot{\theta}_1 + m_2 l_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)$$
$$+ m_2 l_1 l_2 \left(2 \dot{\theta}_1 + \dot{\theta}_2 \right) \cos \theta_2$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- Lagrangian provides us with the required partial derivatives.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = (m_1 + m_2)\, l_1^2 \ddot{\theta}_1 + m_2 l_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)$$

$$+ m_2 l_1 l_2 \left(2\,\ddot{\theta}_1 + \ddot{\theta}_2\right) \cos\theta_2$$

$$- m_2 l_1 l_2 \dot{\theta}_2 \left(2\dot{\theta}_1 + \dot{\theta}_2\right) \sin\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = -m_2 l_1 l_2 \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \sin\theta_2 - m_2 g l_2 \cos\left(\theta_1 + \theta_2\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2 l_1 l_2 \dot{\theta}_1 \cos\theta_2$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- Lagrangian provides us with the required partial derivatives.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 l_1 l_2 \ddot{\theta}_1 \cos\theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

- The equations of motion for the 2R manipulator are

$$\begin{aligned} Q_1 &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} \\ &= \left(m_1 + m_2\right) l_1^2 \ddot{\theta}_1 + m_2 l_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ &\quad + m_2 l_1 l_2\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - m_2 l_1 l_2 \dot{\theta}_2\left(2\dot{\theta}_1 + \dot{\theta}_2\right)\sin\theta_2 \\ &\quad + \left(m_1 + m_2\right) g l_1 \cos\theta_1 + m_2 g l_2 \cos\left(\theta_1 + \theta_2\right) \end{aligned}$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- The equations of motion for the 2R manipulator are

$$
\begin{aligned}
Q_2 &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} \\
&= m_2 l_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 l_1 l_2 \ddot{\theta}_1 \cos\theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 \\
&\quad + m_2 l_1 l_2 \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \sin\theta_2 + m_2 g l_2 \cos\left(\theta_1 + \theta_2\right)
\end{aligned}
$$

8.6. Lagrangian Mechanics

Ex 3. The ideal 2R planar manipulator dynamics.

- The generalized forces $\mathbf{Q_1}$ and $\mathbf{Q_2}$ are the required forces to vary the generalized coordinates. In this case, **$\mathbf{Q_1}$ is the torque at the base motor** and **$\mathbf{Q_2}$ is the torque of the motor at $\mathbf{m_1}$.**
- The equations of motion can be rearranged to have a more systematic form.

$$\boxed{\begin{aligned} Q_1 = {} & \left((m_1+m_2)\, l_1^2 + m_2 l_2 \left(l_2 + 2 l_1 \cos\theta_2\right)\right)\ddot{\theta}_1 \\ & + m_2 l_2 \left(l_2 + l_1 \cos\theta_2\right)\ddot{\theta}_2 \\ & - 2 m_2 l_1 l_2 \sin\theta_2\, \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 \sin\theta_2\, \dot{\theta}_2^2 \\ & + (m_1+m_2)\, g l_1 \cos\theta_1 + m_2 g l_2 \cos(\theta_1+\theta_2) \\ Q_2 = {} & m_2 l_2 \left(l_2 + l_1 \cos\theta_2\right)\ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 \\ & + m_2 l_1 l_2 \sin\theta_2\, \dot{\theta}_1^2 + m_2 g l_2 \cos(\theta_1+\theta_2) \end{aligned}}$$

C8. End!