Introduction to Robotics

Chapter 2. Rotation Kinematics

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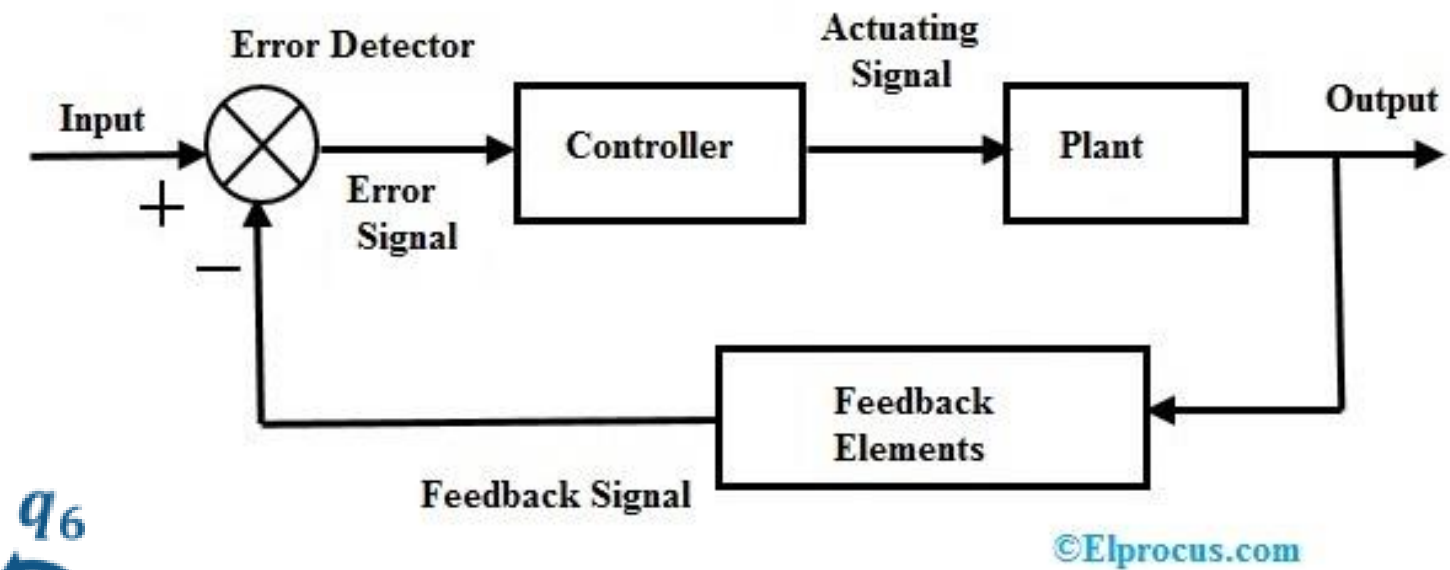
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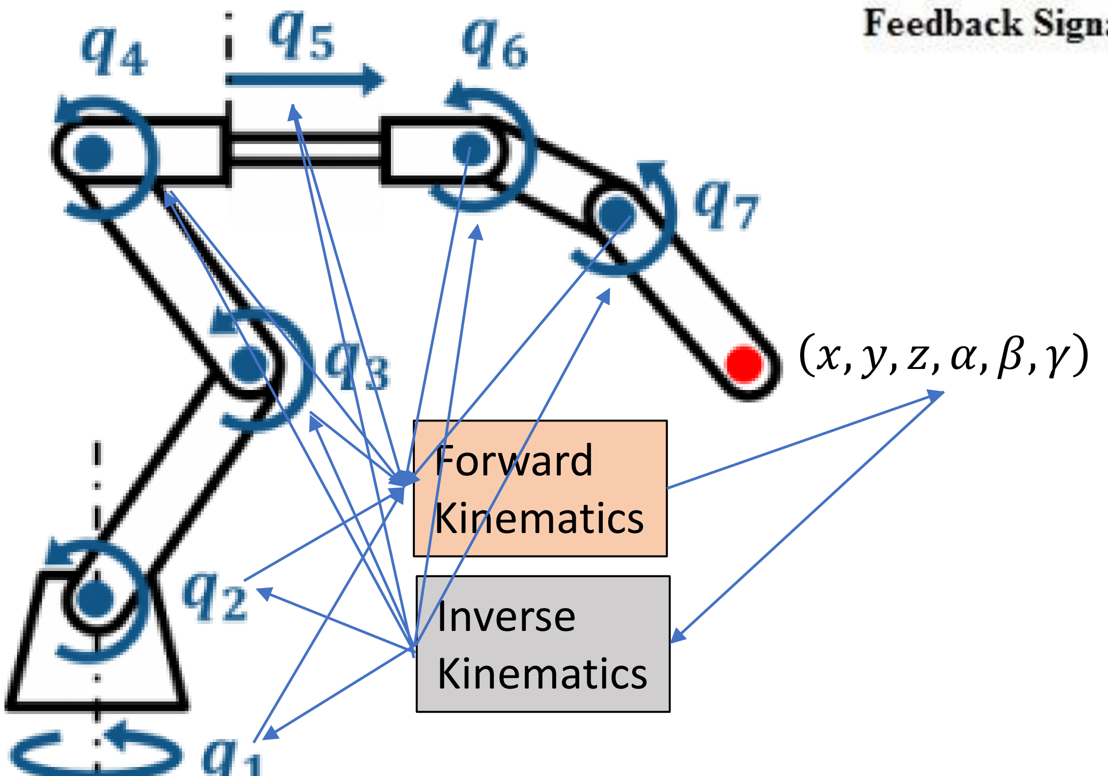
q_4
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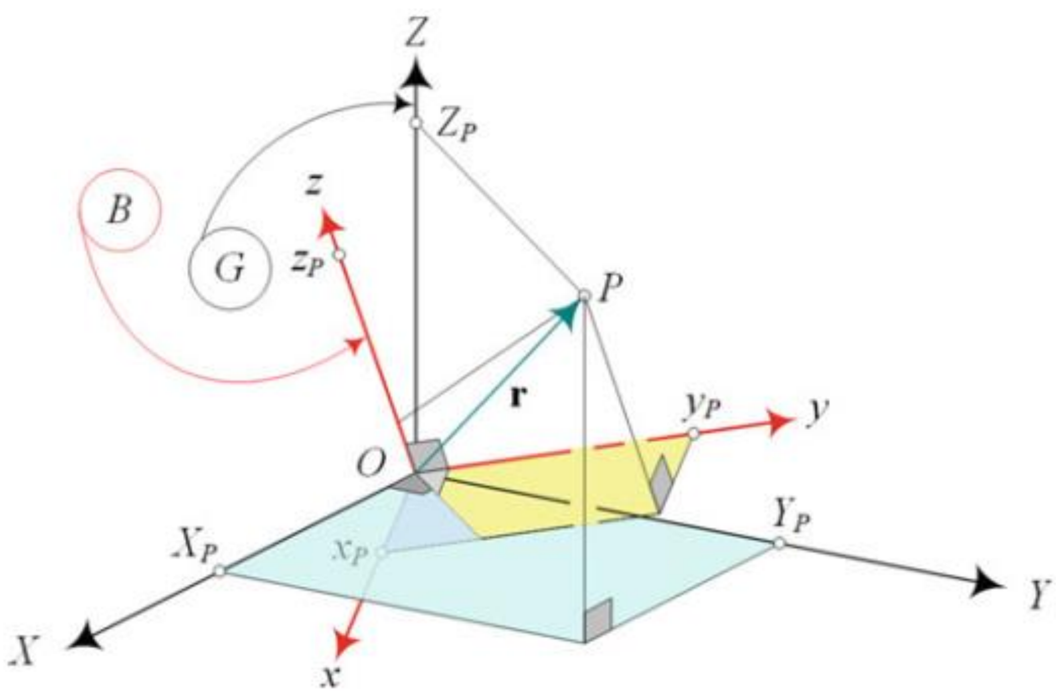
$(x, y, z, \alpha, \beta, \gamma)$
Forward
Kinematics
Inverse
Kinematics

Consider a rigid body with a fixed point **O**. Rotation about the fixed point **O** is the only possible motion of the body.



A rotated body frame B in a fixed global frame G, about a fixed point at O

The rigid body is defined by **a body coordinate frame B (Oxyz)**, that rotates in another **coordinate frame G (OXYZ)**

⇒ A rotation calculus of a body point **P** in both frames is determined **a transformation matrices.**

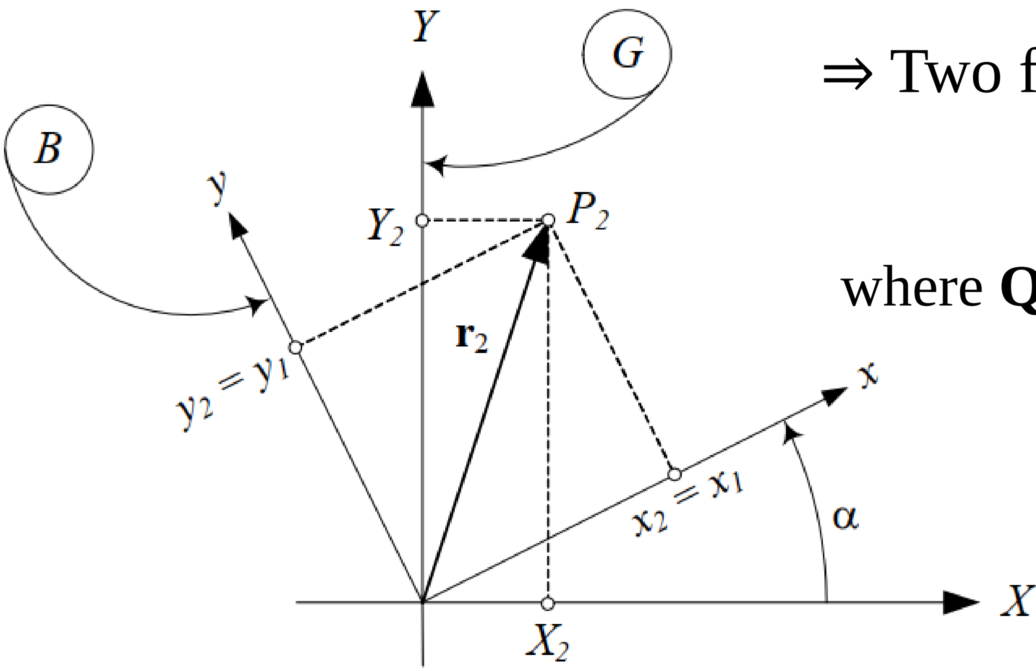
2.1. Rotation About Global Cartesian Axes

Z-axis of the global coordinate frame

- The rigid body **B** rotates **α** radians **about the Z-axis** of the global coordinate frame.
- **Point $\mathbf{P_2}$** of the rigid body:

$$^{\mathbf{B}}\mathbf{r}_2 = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T \text{ in local frame}$$

$$^{\mathbf{G}}\mathbf{r}_2 = \begin{bmatrix} X_2 & Y_2 & Z_2 \end{bmatrix}^T \text{ in global frame}$$



⇒ Two frames are related as follows:

$$\boxed{^{\mathbf{G}}\mathbf{r}_2 = \mathbf{Q}_{\mathbf{Z},\alpha}\, {}^{\mathbf{B}}\mathbf{r}_2}$$

where $\mathbf{Q}_{\mathbf{z},\alpha}$ is the **Z-rotation matrix** **?!**

$$\mathbf{Q}_{\mathbf{Z},\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.1. Rotation About Global Cartesian Axes

Y-axis of the global coordinate frame

- Similarly, rotation **β degrees about the Y–axis of the global frame** relate the local and global coordinates.
- **Consider point P:** ${}^{B}\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^{T}$

$$\boxed{{}^{G}\mathbf{r} = \mathbf{Q}_{\mathbf{Y},\boldsymbol{\beta}}\,{}^{B}\mathbf{r}}$$

where $\mathbf{Q}_{\mathbf{Y},\beta}$ is the **Y-rotation matrix**

$$\mathbf{Q}_{\mathbf{Y},\boldsymbol{\beta}} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

2.1. Rotation About Global Cartesian Axes

X-axis of the global coordinate frame

- Finally, rotation **γ degrees about the X–axis of the global frame** relate the local and global coordinates.
- **Consider point P:** ${}^{B}\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^{T}$

$${}^{G}\mathbf{r} = \mathbf{Q}_{X,\gamma}\,{}^{B}\mathbf{r}$$

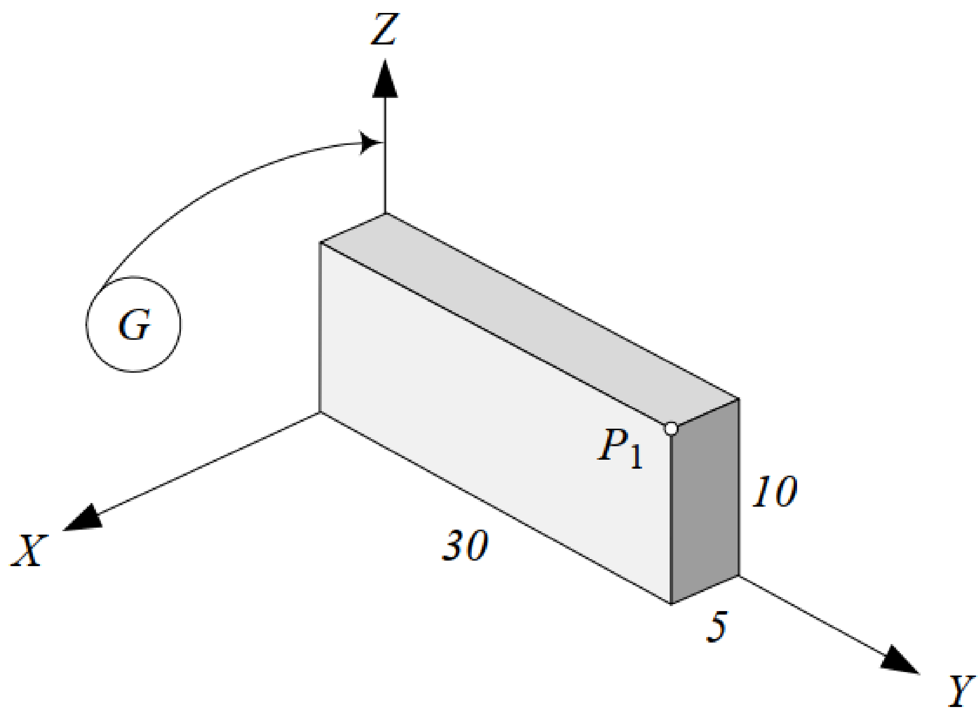
where $\mathbf{Q}_{X,\gamma}$ is the **X-rotation matrix**

$$\mathbf{Q}_{X,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

2.1. Rotation About Global Cartesian Axes

Example 1: Successive rotation about global axes.

The final position of the corner **P (5, 30, 10)** of the slab after 30 [deg] rotation about the Z-axis, followed by 30 [deg] about the X-axis, and then 90 [deg] about the Y-axis. Can find the final global position of the corner **P**?



2.1. Rotation About Global Cartesian Axes

Example 1: Successive rotation about global axes.

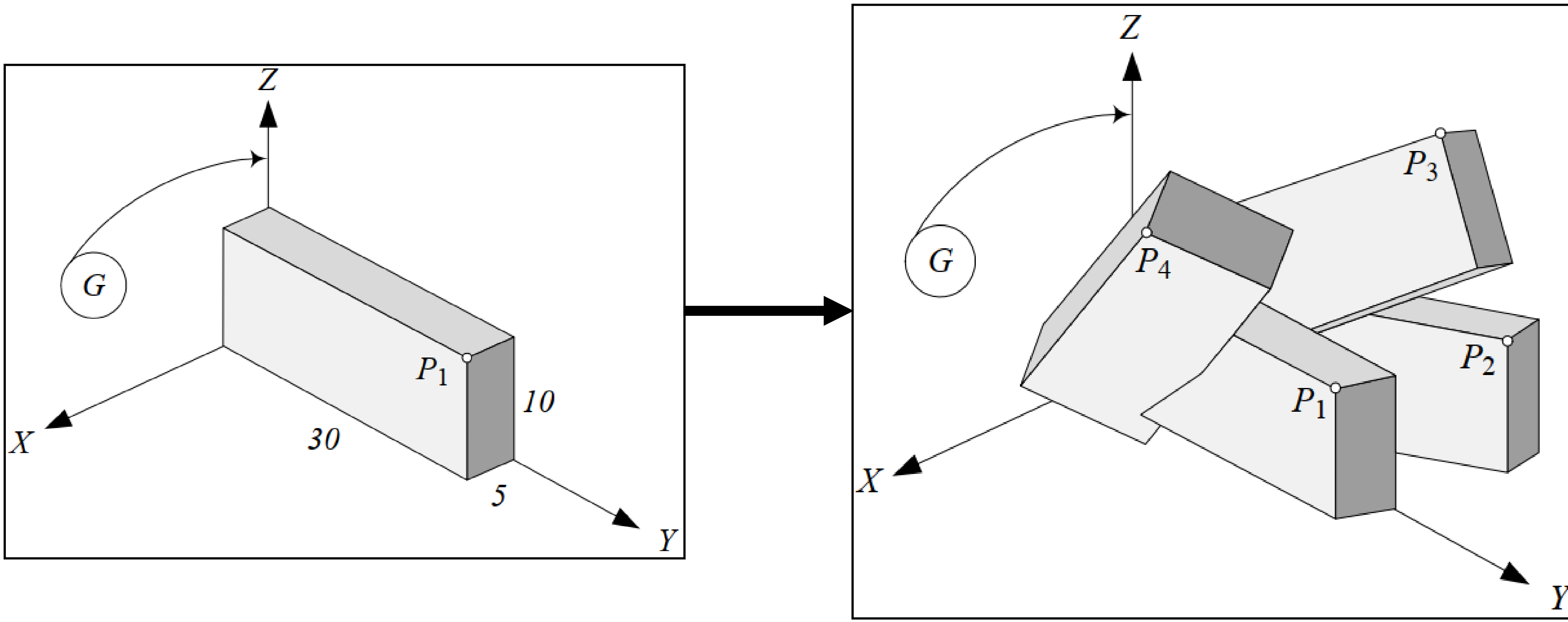
$$ {}^{G}\mathbf{r}_2 = \mathbf{Q}_{Z,\alpha} {}^{B}\mathbf{r}_2 \qquad \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix} $$

$$ {}^{G}\mathbf{r} = \mathbf{Q}_{X,\gamma} {}^{B}\mathbf{r} \qquad \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix} $$

$$ {}^{G}\mathbf{r} = \mathbf{Q}_{Y,\beta} {}^{B}\mathbf{r} \qquad \begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \cos 90 & 0 & \sin 90 \\ 0 & 1 & 0 \\ -\sin 90 & 0 & \cos 90 \end{bmatrix} \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix} = \begin{bmatrix} 22.90 \\ 19.66 \\ 10.68 \end{bmatrix} $$

2.1. Rotation About Global Cartesian Axes

Example 1: Successive rotation about global axes.



Corner P and the slab at first, second, third, and final positions

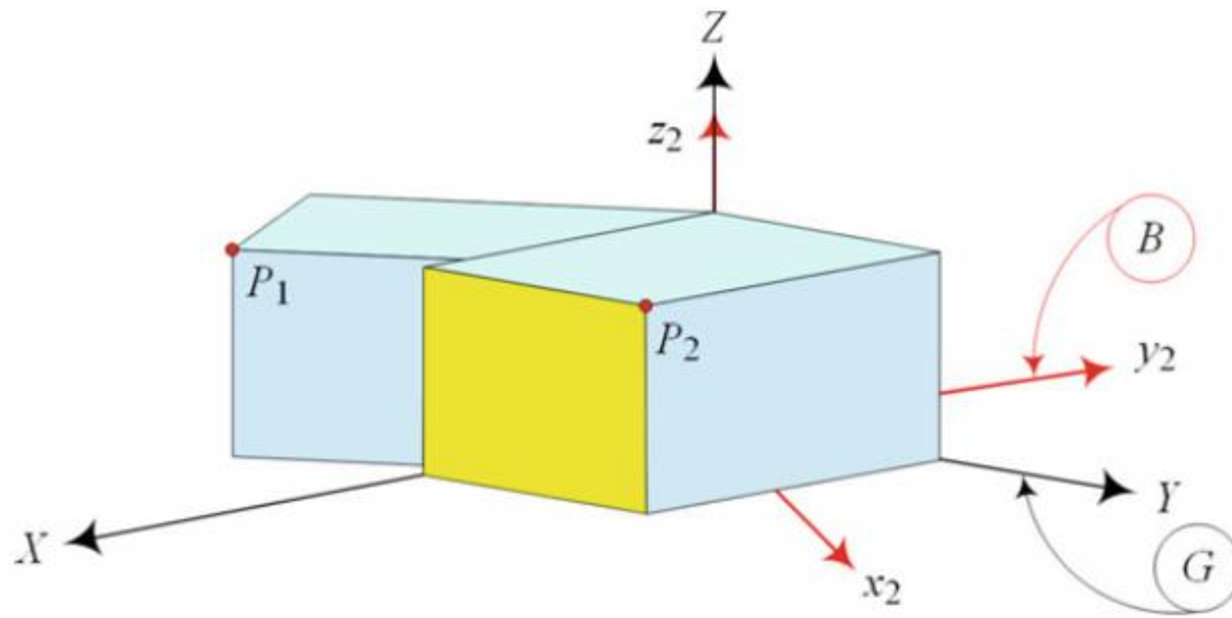
2.1. Rotation About Global Cartesian Axes

Example 2: Global rotation, local position.

A point **P** is moved to ${}^{G}\mathbf{r}_2 = [4,\ 3,\ 2]^T$ after a 60 [deg] rotation about **the Z-axis**. Find its position in the local coordinate?

$$ {}^{G}\mathbf{r}_2 = \mathbf{Q}_{Z,\alpha}\,{}^{B}\mathbf{r}_2 $$
$$ \Rightarrow {}^{B}\mathbf{r}_2 = \mathbf{Q}_{Z,\alpha}^{-1}\,{}^{G}\mathbf{r}_2 $$

$$ {}^{B}\mathbf{r}_2 = \mathbf{Q}_{Z,60}^{-1}\,{}^{G}\mathbf{r}_2 \Rightarrow $$



$$ \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.60 \\ -1.95 \\ 2.0 \end{bmatrix} $$

2.2. Successive Rotation About Global Cartesian Axes

A sequence of rotations

- The final global position of **a point P** in a rigid body B with position vector **r**, after a sequence of rotations $\mathbf{Q_1}$, $\mathbf{Q_2}$, $\mathbf{Q_3}$, ..., $\mathbf{Q_n}$ about the global axes can be found by

$$^{G}\mathbf{r} = {}^{G}\mathbf{Q}_{B}\,{}^{B}\mathbf{r}$$

where ***the global rotation matrix***,

$$^{G}\mathbf{Q}_{B} = \mathbf{Q}_{n}\ldots\mathbf{Q}_{3}\mathbf{Q}_{2}\mathbf{Q}_{1}$$

- **A rotation matrix is orthogonal**; i.e., its transpose $\mathbf{Q}^{T}$ is equal to its inverse $\mathbf{Q}^{-1}$.

$$\mathbf{Q}^{T} = \mathbf{Q}^{-1}$$

2.2. Successive Rotation About Global Cartesian Axes

Example 3: Successive global rotation matrix.

- The global rotation matrix after a rotation $Q_{Z,\alpha}$ followed by $Q_{Y,\beta}$ and then $Q_{X,\gamma}$ is:?

$$^{G}\mathbf{Q}_{\mathbf{B}} = \mathbf{Q}_{X,\gamma}\mathbf{Q}_{Y,\beta}\mathbf{Q}_{Z,\alpha}$$

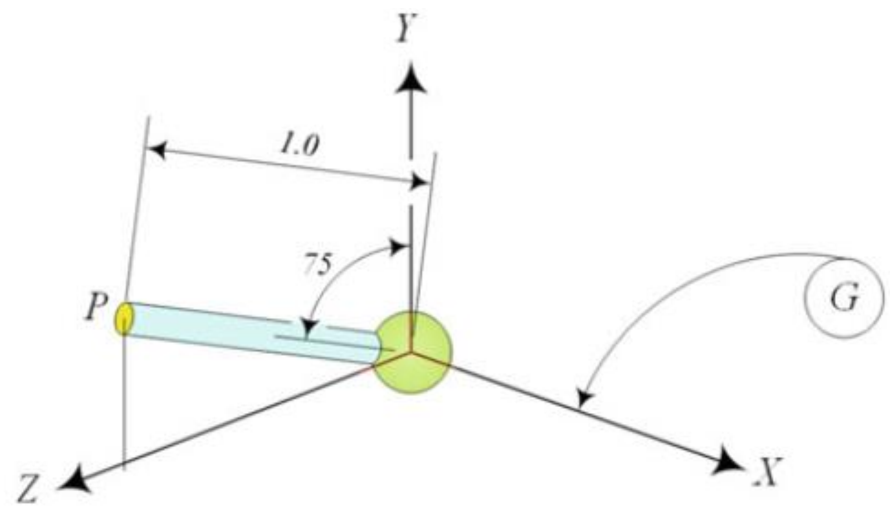
$$^{G}\mathbf{Q}_{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{G}\mathbf{Q}_{\mathbf{B}} = \begin{bmatrix} c\alpha c\beta & -c\beta s\alpha & s\beta \\ c\gamma s\alpha + c\alpha s\beta s\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -c\beta s\gamma \\ s\alpha s\gamma - c\alpha c\gamma s\beta & c\alpha s\gamma + c\gamma s\alpha s\beta & c\beta c\gamma \end{bmatrix}$$

2.2. Successive Rotation About Global Cartesian Axes

Example 4: Successive global rotations, global position.

- The end point $\mathbf{P} = [X_1 \ Y_1 \ Z_1]^T$ of the arm shown in Figure.
- The rotation matrix to find the new position of the end point after −29 [deg] rotation about the X-axis, followed by 30 [deg] about the Z-axis, and again 132 [deg] about the X-axis.



$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ l\cos\theta \\ l\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1\cos 75 \\ 1\sin 75 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.26 \\ 0.97 \end{bmatrix}$$

2.2. Successive Rotation About Global Cartesian Axes

Example 4: Successive global rotations, global position.

The global rotation matrix is

$$^{G}Q_{B}=Q_{X,132}Q_{Z,30}Q_{X,-29}=\begin{bmatrix} 0.87 & -0.44 & -0.24 \\ -0.33 & -0.15 & -0.93 \\ 0.37 & 0.89 & -0.27 \end{bmatrix}$$

The new position of point **P** is

$$^{G}\mathbf{r}={}^{G}\mathbf{Q}_{B}{}^{B}\mathbf{r}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}=\begin{bmatrix} 0.87 & -0.44 & -0.24 \\ -0.33 & -0.15 & -0.93 \\ 0.37 & 0.89 & -0.27 \end{bmatrix}\begin{bmatrix} 0.0 \\ 0.26 \\ 0.97 \end{bmatrix}=\begin{bmatrix} -0.35 \\ -0.94 \\ -0.031 \end{bmatrix}$$

2.2. Successive Rotation About Global Cartesian Axes

Twelve independent triple global rotations

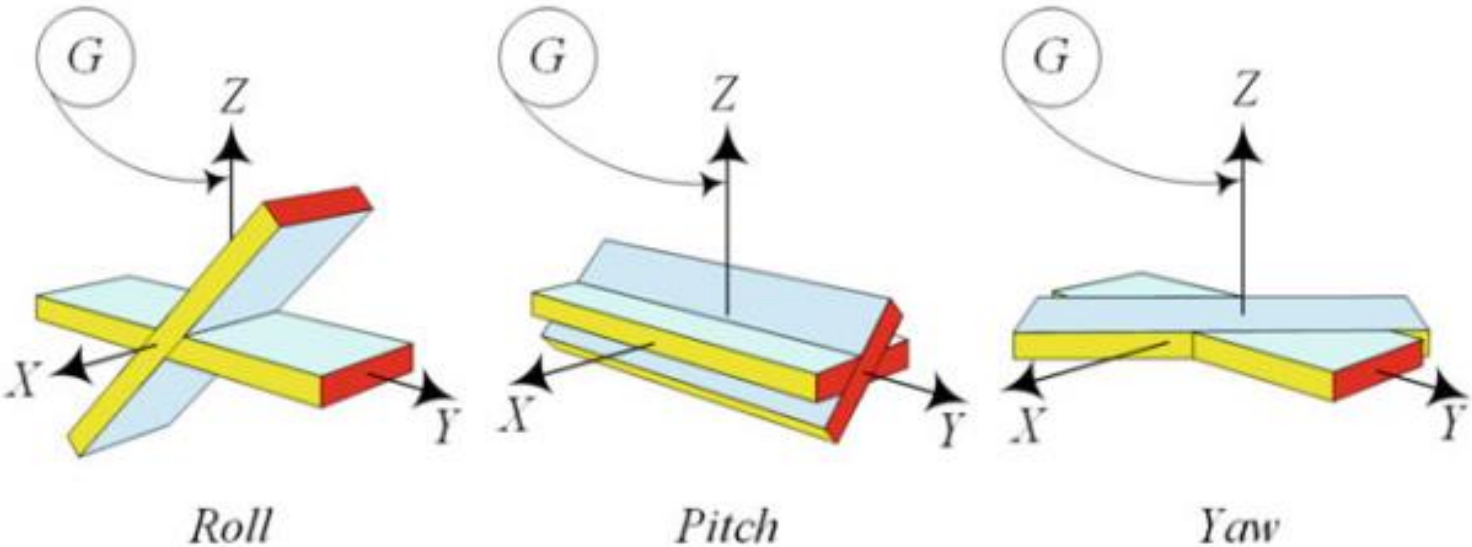
- We may transform its body coordinate frame B from the coincident position with a global frame G to any final orientation by only three rotations about the global axes provided that no two consequence rotations are about the same axis. In general, there are **12 different independent combinations of triple rotations about the global axes.**

$$
\begin{array}{ll}
1 - Q_{X,\gamma}Q_{Y,\beta}Q_{Z,\alpha} & 7 - Q_{X,\gamma}Q_{Y,\beta}Q_{X,\alpha} \\
2 - Q_{Y,\gamma}Q_{Z,\beta}Q_{X,\alpha} & 8 - Q_{Y,\gamma}Q_{Z,\beta}Q_{Y,\alpha} \\
3 - Q_{Z,\gamma}Q_{X,\beta}Q_{Y,\alpha} & 9 - Q_{Z,\gamma}Q_{X,\beta}Q_{Z,\alpha} \\
4 - Q_{Z,\gamma}Q_{Y,\beta}Q_{X,\alpha} & 10 - Q_{X,\gamma}Q_{Z,\beta}Q_{X,\alpha} \\
5 - Q_{Y,\gamma}Q_{X,\beta}Q_{Z,\alpha} & 11 - Q_{Y,\gamma}Q_{X,\beta}Q_{Y,\alpha} \\
6 - Q_{X,\gamma}Q_{Z,\beta}Q_{Y,\alpha} & 12 - Q_{Z,\gamma}Q_{Y,\beta}Q_{Z,\alpha}
\end{array}
$$

2.2. Successive Rotation About Global Cartesian Axes

Global roll, pitch, and yaw rotations

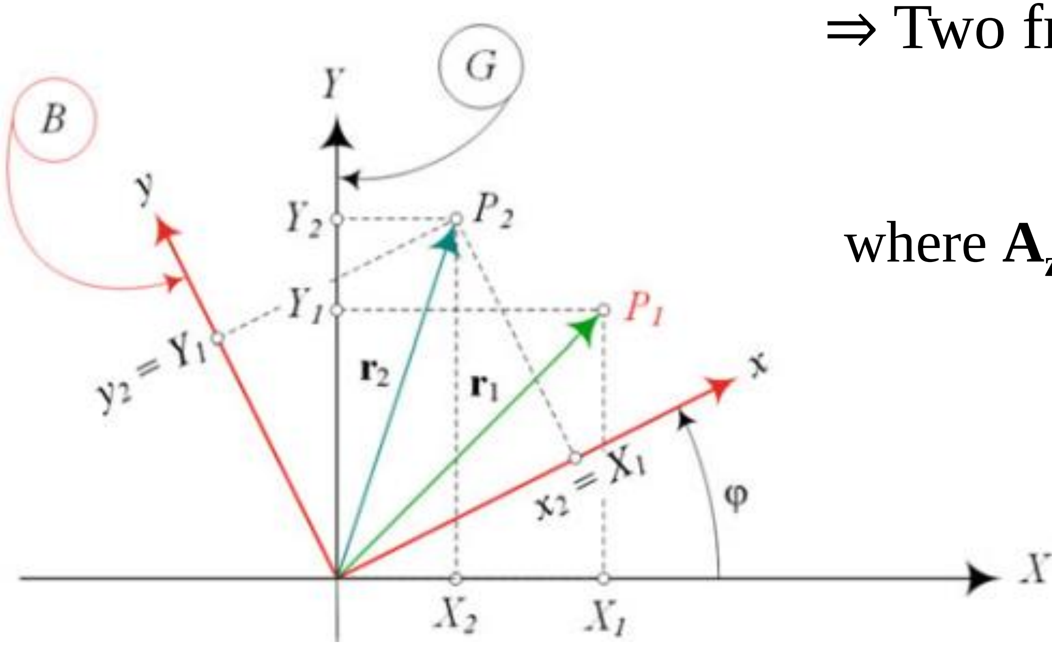
The rotation about the ***X-axis*** of the global coordinate frame is called ***a roll***, the rotation about the ***Y-axis*** is called ***a pitch***, and the rotation about the ***Z-axis*** is called ***a yaw.***



2.3. Rotation About Local Cartesian Axes

z-axis of the local coordinate frame

- Consider a rigid body B with **a local** coordinate frame **B (*Oxyz*)** that is originally coincident with **a global** coordinate frame **G (OXYZ)**.
- The body undergoes **a rotation φ** about **the z-axis** of its local coordinate frame.



⇒ Two frames are related as follows:

$$ {}^{B}\mathbf{r} = \mathbf{A}_{z,\varphi}\, {}^{G}\mathbf{r} $$

where $\mathbf{A}_{z,\varphi}$ is the **z-rotation matrix ?!**

$$ \mathbf{A}_{z,\varphi} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

2.3. Rotation About Local Cartesian Axes

x,y-axes of the local coordinate frame

Similarly, rotation **θ about the y-axis** and **rotation ψ about the x-axis** of the local frame relate the local and global coordinates of **point P** by the following equations

$$^{B}\mathbf{r} = \mathbf{A}_{y,\theta}\,^{G}\mathbf{r}$$

$$^{B}\mathbf{r} = \mathbf{A}_{x,\psi}\,^{G}\mathbf{r}$$

where $\mathbf{A}_{y,\theta}$ is the ***y*-rotation matrix** and $\mathbf{A}_{x,\psi}$ is the ***x*-rotation matrix**

$$\mathbf{A}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \mathbf{A}_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

2.3. Rotation About Local Cartesian Axes

Example 5: Local rotation, local position. Local rotation, global position.

If a local coordinate frame ***Oxyz*** has been rotated 60 [deg] about the z-axis and **a point P** in the local coordinate frame ***Oxyz*** is at (4, 3, 2)

⇒ its position **in the global coordinate frame *OXYZ*** is at

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} & 0 \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.60 \\ 4.96 \\ 2.0 \end{bmatrix}$$

2.3. Rotation About Local Cartesian Axes

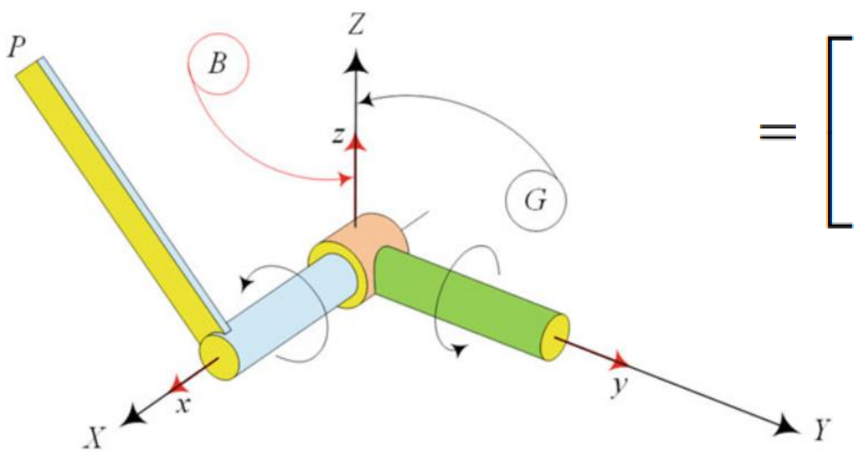
Example 6: Successive local rotation, global position.

The first actuator rotates the arm −90 [deg] about y-axis, and then the second actuator rotates the arm 90 [deg] about x-axis. If, before the rotations, **the end point P** is at $^{B}\mathbf{r}_{P} = [9.5\ -10.1\ 10.1]^{T}$

⇒ its position **in the global coordinate frame** is

$$^{G}\mathbf{r}_{P} = \left[A_{x,\pi/2}\, A_{y,-\pi/2}\right]^{-1}\, {}^{B}\mathbf{r}_{P} = A_{y,-\pi/2}^{-1}\, A_{x,\pi/2}^{-1}\, {}^{B}\mathbf{r}_{P}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 9.5 \\ -10.1 \\ 10.1 \end{bmatrix} = \begin{bmatrix} 10.1 \\ -10.1 \\ 9.5 \end{bmatrix}$$



2.4. Successive Rotation About Local Cartesian Axes

A sequence of rotations

- Consider a point P in a rigid body B(*Oxyz*) at position vector **r**. Having the final global position vector ${}^{G}\mathbf{r}$ of P , we can determine its local position vector ${}^{B}\mathbf{r}$ after a series of sequential rotations $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, . . ., $\mathbf{A}_n$ about the local axes, by

$$ {}^{B}\mathbf{r} = {}^{B}\mathbf{A}_{G}\,{}^{G}\mathbf{r} $$

where ***the local rotation matrix***,

$$ {}^{B}\mathbf{A}_{G} = \mathbf{A}_n \ldots \mathbf{A}_3 \mathbf{A}_2 \mathbf{A}_1 $$

- The multiplication of rotation transformation matrices is associative

$$ \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1 = \mathbf{A}_3\left[\mathbf{A}_2\mathbf{A}_1\right] = \left[\mathbf{A}_3\mathbf{A}_2\right]\mathbf{A}_1 $$

2.4. Successive Rotation About Local Cartesian Axes

Example 7: Successive local rotation, local position.

A local coordinate frame B(*Oxyz*) that initially is coincident with a global coordinate frame G(*OXYZ*) undergoes a rotation φ = 30 [deg] about the z-axis, then θ = 30 [deg] about the *x*-axis, and then ψ = 30 [deg] about the *y*-axis. The local coordinates of **a point P** located at X = 5, Y = 30, and Z = 10 can be found?

The **local** rotation matrix is $$ {}^{B}A_{G} = A_{y,30}\, A_{x,30}\, A_{z,30} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} $$

The coordinates of P **in the local frame** are

$$ \begin{bmatrix} x & y & z \end{bmatrix}^{T} = A_{y,\psi}\, A_{x,\theta}\, A_{z,\varphi} \begin{bmatrix} 5 & 30 & 10 \end{bmatrix}^{T} $$

$$ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 18.28 \\ 25.33 \\ 7.0 \end{bmatrix} $$

2.4. Successive Rotation About Local Cartesian Axes

Twelve independent rotation about local axes

- We may transform a body coordinate frame B from the coincident position with a global frame G to any final orientation by minimum three rotations about the local axes provided that no two consequence rotations are about the same axis. In general, there are 12 different independent combinations of triple rotation about local axes.

$$1 - A_{x,\psi}\, A_{y,\theta}\, A_{z,\varphi} \qquad 7 - A_{x,\psi}\, A_{y,\theta}\, A_{x,\varphi}$$
$$2 - A_{y,\psi}\, A_{z,\theta}\, A_{x,\varphi} \qquad 8 - A_{y,\psi}\, A_{z,\theta}\, A_{y,\varphi}$$
$$3 - A_{z,\psi}\, A_{x,\theta}\, A_{y,\varphi} \qquad 9 - A_{z,\psi}\, A_{x,\theta}\, A_{z,\varphi}$$
$$4 - A_{z,\psi}\, A_{y,\theta}\, A_{x,\varphi} \qquad 10 - A_{x,\psi}\, A_{z,\theta}\, A_{x,\varphi}$$
$$5 - A_{y,\psi}\, A_{x,\theta}\, A_{z,\varphi} \qquad 11 - A_{y,\psi}\, A_{x,\theta}\, A_{y,\varphi}$$
$$6 - A_{x,\psi}\, A_{z,\theta}\, A_{y,\varphi} \qquad 12 - A_{z,\psi}\, A_{y,\theta}\, A_{z,\varphi}$$

2.5. Euler Angles

Definition

- The rotation about the **Z-axis (z-axis) of the global coordinate** is called ***precession***, the rotation about **the x-axis of the local coordinate** is called ***nutation***, and the rotation about **the z-axis of the local coordinate** is called ***spin***.
- The ***precession*** **(*φ*) –** ***nutation*** **(*θ*) –** ***spin*** **(*ψ*)** rotation angles are also called **Euler angles**.
- The kinematics and dynamics of axisymmetric **rigid bodies** have simpler and more understandable expression based on Euler angles.

2.5. Euler Angles

Definition

- The Euler angle rotation matrix ${}^{B}\mathbf{A}_{G}$ to transform a position vector from G (*OXYZ*) to B (*Oxyz*)

$${}^{B}\mathbf{r} = {}^{B}\mathbf{A}_{G}\,{}^{G}\mathbf{r}$$

where ${}^{B}\mathbf{A}_{G}$ is the **rotation matrix**

$$\begin{aligned}
{}^{B}A_{G} &= A_{z,\psi}\,A_{x,\theta}\,A_{z,\varphi} \\
&= \begin{bmatrix} c\varphi\,c\psi - c\theta\,s\varphi\,s\psi & c\psi\,s\varphi + c\theta\,c\varphi\,s\psi & s\theta\,s\psi \\ -c\varphi\,s\psi - c\theta\,c\psi\,s\varphi & -s\varphi\,s\psi + c\theta\,c\varphi\,c\psi & s\theta\,c\psi \\ s\theta\,s\varphi & -c\varphi\,s\theta & c\theta \end{bmatrix}
\end{aligned}$$

2.5. Euler Angles

Example 8: Euler angle rotation matrix

The Euler or ***precession–nutation–spin*** rotation matrix for φ = 79.15 [deg], θ = 41.41 [deg], and ψ = −40.7 [deg] or φ = 1.38 [rad], θ = 0.72 [rad], and ψ = −0.71 [rad]. The **rotation matrix** is?

$$
\begin{aligned}
{}^{B}A_{G} &= A_{z,-0.71}\, A_{x,0.72}\, A_{z,1.38} \\
&= \begin{bmatrix} 0.623 & 0.652 & -0.431 \\ -0.436 & 0.747 & 0.501 \\ 0.649 & -0.124 & 0.75 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}Q_{B} &= {}^{B}A_{G}^{-1} \\
&= \begin{bmatrix} 0.623 & -0.436 & 0.65 \\ 0.653 & 0.748 & -0.124 \\ -0.431 & 0.501 & 0.751 \end{bmatrix}
\end{aligned}
$$

2.5. Euler Angles

Example 9: Euler angles of a local transformation matrix.

The local rotation matrix after rotation of 30 [deg] about the z-axis, then 30 [deg] about the x-axis, and then 30 [deg] about the y-axis.

The local rotation matrix:

$$
{}^{B}A_{G} = A_{y,\frac{\pi}{6}} A_{x,\frac{\pi}{6}} A_{z,\frac{\pi}{6}}
$$

$$
= \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix}
$$

The Euler angles of the corresponding ***precession–nutation–spin*** rotation matrix are **?!**

$$
\theta = \arccos\left(r_{33}\right) = \arccos 0.75 = 0.72\,\mathrm{rad} = 41.4\,\mathrm{deg}
$$

$$
\varphi = -\arctan\frac{r_{31}}{r_{32}} = -\arctan\frac{0.65}{-0.125} = 1.38\,\mathrm{rad} = 79.1\,\mathrm{deg}
$$

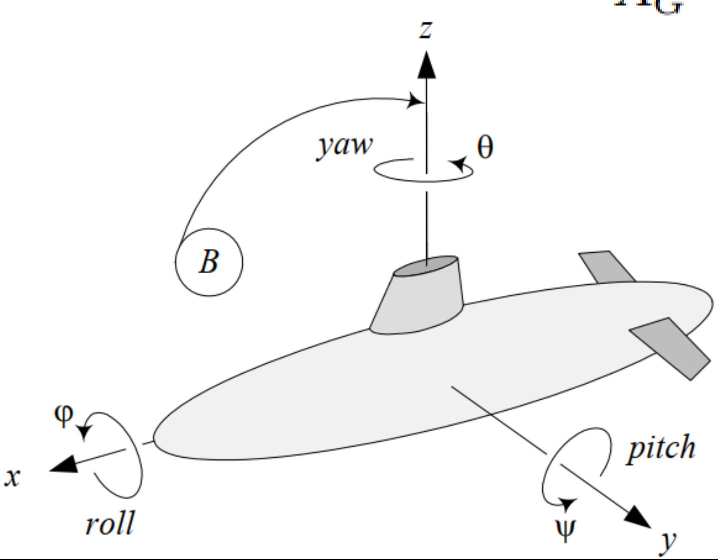
$$
\psi = \arctan\frac{r_{13}}{r_{23}} = \arctan\frac{-0.43}{0.50} = -0.71\,\mathrm{rad} = -40.7\,\mathrm{deg}
$$

2.6. Local Roll-Pitch-Yaw Angles

Definition

- Rotation about the ***x*-axis** of **the local frame** is called ***roll*** or *bank*, rotation about ***y*-axis** of **the local frame** is called ***pitch*** or *attitude*, and rotation about the ***z*-axis** of **the local frame** is called ***yaw***, *spin*, or *heading*.
- The local ***roll (φ)-pitch (θ)-yaw (ψ)*** rotation matrix is

$$
\begin{aligned}
{}^{B}A_{G} &= A_{z,\psi}A_{y,\theta}A_{x,\varphi} \\
&= \begin{bmatrix} c\theta c\psi & c\varphi s\psi + s\theta c\psi s\varphi & s\varphi s\psi - c\varphi s\theta c\psi \\ -c\theta s\psi & c\varphi c\psi - s\theta s\varphi s\psi & c\psi s\varphi + c\varphi s\theta s\psi \\ s\theta & -c\theta s\varphi & c\theta c\varphi \end{bmatrix}
\end{aligned}
$$



Note: The difference between ***roll-pitch-yaw*** and Euler angles, although we show both utilizing ***roll (φ)-pitch (θ)-yaw (ψ).***

2.7. Local Axes Versus Global Axes Rotation

The global rotation matrix ${}^{G}\boldsymbol{Q}_{B}$ is equal to the inverse of the local rotation matrix ${}^{B}\boldsymbol{A}_{G}$ and vice versa

$$ {}^{G}\mathbf{Q}_{B} = {}^{B}\mathbf{A}_{G}^{-1} \qquad {}^{B}\mathbf{A}_{G} = {}^{G}\mathbf{Q}_{B}^{-1} $$

where,

$$ {}^{G}\mathbf{Q}_{B} = \mathbf{A}_{1}^{-1}\mathbf{A}_{2}^{-1}\mathbf{A}_{3}^{-1}\ldots\mathbf{A}_{n}^{-1} $$

$$ {}^{B}\mathbf{A}_{G} = \mathbf{Q}_{1}^{-1}\mathbf{Q}_{2}^{-1}\mathbf{Q}_{3}^{-1}\ldots\mathbf{Q}_{n}^{-1} $$

Also, pre-multiplication of the global rotation matrix is equal to post-multiplication of the local rotation matrix.

2.8. General Transformation

- Consider a general situation in which two coordinate frames, **G (*OXYZ*)** and **B (*Oxyz*)** with a common origin **O**, are employed to express the components of **a given vector *r***. There is always **a transformation matrix** $^{G}R_{B}$ to map the components of ***r*** from the reference frame **B** (***Oxyz***) to the other reference frame **G** (***OXYZ***).

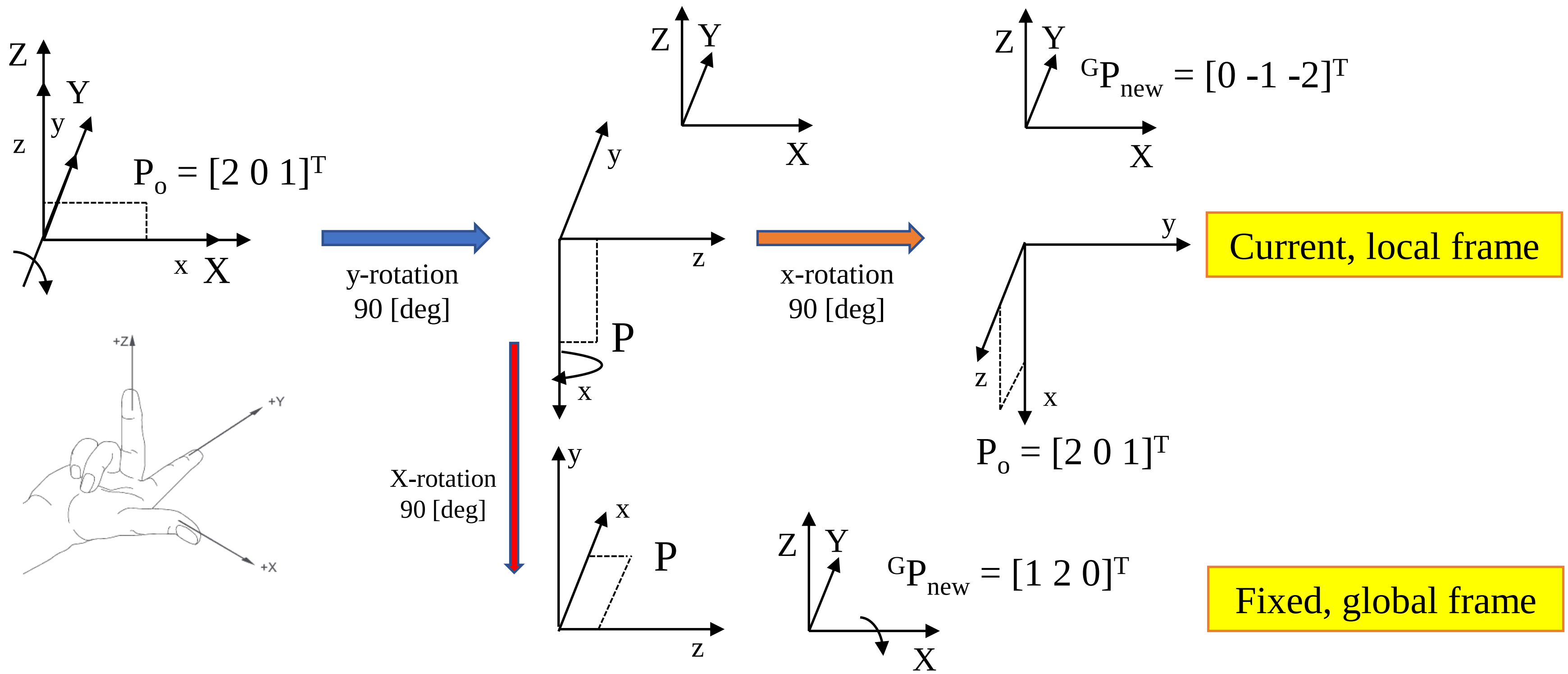
$$\mathbf{{}^{G}r = {}^{G}R_{B}\,{}^{B}r \Rightarrow {}^{B}r = {}^{G}R_{B}^{-1}\,{}^{G}r}$$

- Map the components of ***r*** from the reference frame ***G (OXYZ)*** to the other reference frame B ***(Oxyz)*** by introducing $^{B}R_{G}$

$$\mathbf{{}^{B}r = {}^{B}R_{G}\,{}^{G}r}$$

where, $$\mathbf{{}^{B}R_{G} = {}^{G}R_{B}^{-1} = {}^{G}R_{B}^{T} \qquad \left|{}^{G}R_{B}\right| = \left|{}^{B}R_{G}\right| = 1}$$

2.8. General Transformation



2.8. General Transformation

- **Red arrow (fixed, global frame)**

$$ {}^{G}P_{new} = {}^{G}RP_o = \left(\underbrace{{}^{G}R_{X,90}}_{Q_{X,90}} \; \underset{Q_{Y,90}}{{}^{G}R_{Y,90}} \right) P_o = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} $$

- **Orange arrow (current, local frame)**

$$ {}^{B}P_o = \left({}^{B}R \right) {}^{G}P_{new} = \left({}^{B}R_{x,90} \, {}^{B}R_{y,90} \right) {}^{G}P_{new} $$

$$ \Rightarrow {}^{G}P_{new} = \left({}^{B}R_{x,90} \, {}^{B}R_{y,90} \right)^{-1} {}^{B}P_o $$

$$ \Rightarrow {}^{G}P_{new} = \left({}^{B}R_{y,90} \right)^{-1} \left({}^{B}R_{x,90} \right)^{-1} {}^{B}P_o $$

$$ \Rightarrow {}^{G}P_{new} = \underset{Q_Y}{{}^{G}R_{y,90}} \; \underset{Q_X}{{}^{G}R_{x,90}} \; {}^{B}P_o = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} $$

C2. End!