Introduction to Robotics

Chapter 3. Orientation Kinematics

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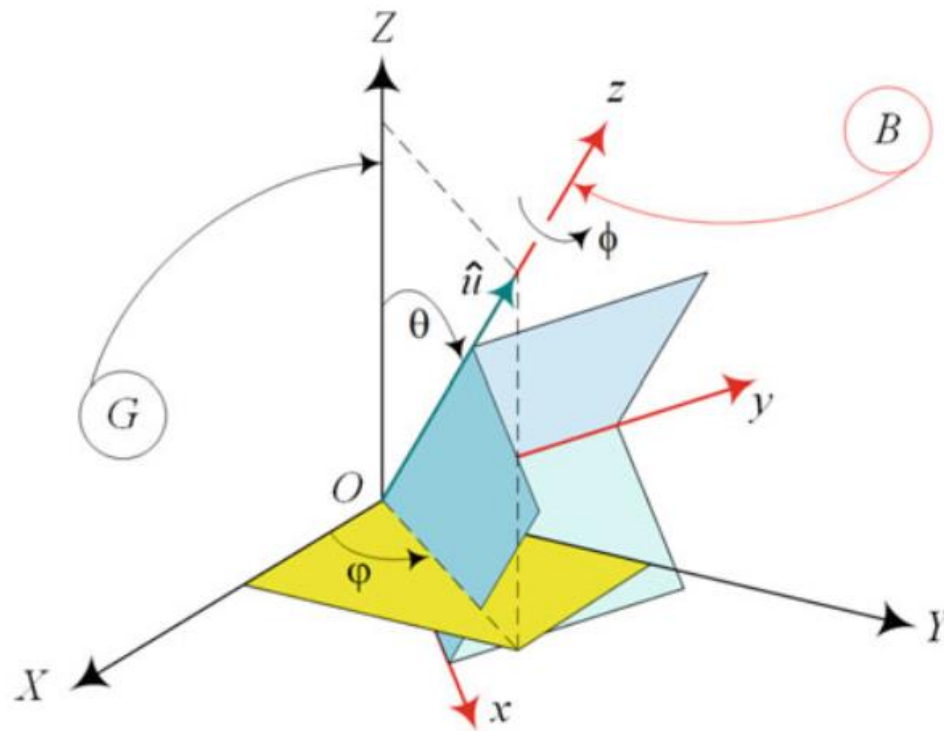
- Any rotation ϕ of a rigid body with **a fixed point O** about a fixed axis $\hat{\boldsymbol{u}}$ can be **decomposed** into three rotations about three given non-coplanar axes including the global or body principal exes.

- Determination of **the angle and axis** is ***called the*** ***orientation kinematics of rigid bodies.***



Axis of rotation $\hat{\boldsymbol{u}}$ when it is coincident with the local z-axis

3.1. Axis–Angle Rotation

- Assume **a body frame B** (*Oxyz*) rotates ϕ about a line indicated by a unit vector $\hat{\boldsymbol{u}}$ with direction cosines $\boldsymbol{u_1, u_2, u_3}$.

$$\hat{\mathbf{u}} = u_1\hat{I} + u_2\hat{J} + u_3\hat{K} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$$

$$\sqrt{u_1^2 + u_2^2 + u_3^2} = 1$$

⇒ This is called **axis–angle** representation of **a rotation**.

- Two parameters are necessary to define the unit vector $\hat{\boldsymbol{u}}$ through **O**, and one is necessary to define the amount of **rotation ϕ** of the rigid body about $\hat{\boldsymbol{u}}$.

3.1. Axis–Angle Rotation

- The axis–angle transformation matrix $^{G}\mathbf{R}_{\mathbf{B}}$ that maps the coordinates in the **local frame B** (*Oxyz*) to the corresponding coordinates in the **global frame G** (*OXYZ*)

$$^{G}\mathbf{r} = {}^{G}\mathbf{R}_{\mathbf{B}}\,{}^{B}\mathbf{r}$$

where,

$$^{G}R_{B} = \begin{bmatrix} u_1^2 \operatorname{vers}\phi + c\phi & u_1u_2 \operatorname{vers}\phi - u_3 s\phi & u_1u_3 \operatorname{vers}\phi + u_2 s\phi \\ u_1u_2 \operatorname{vers}\phi + u_3 s\phi & u_2^2 \operatorname{vers}\phi + c\phi & u_2u_3 \operatorname{vers}\phi - u_1 s\phi \\ u_1u_3 \operatorname{vers}\phi - u_2 s\phi & u_2u_3 \operatorname{vers}\phi + u_1 s\phi & u_3^2 \operatorname{vers}\phi + c\phi \end{bmatrix}$$

$$^{G}R_{B} = R_{\hat{u},\phi} = \mathbf{I}\cos\phi + \hat{u}\hat{u}^{T} \operatorname{vers}\phi + \tilde{u}\sin\phi$$

$$\operatorname{vers}\phi = versine\,\phi = 1 - \cos\phi = 2\sin^{2}\frac{\phi}{2}$$

3.1. Axis–Angle Rotation

- $\tilde{\boldsymbol{u}}$ is the skew symmetric matrix corresponding to **the vector $\hat{\boldsymbol{u}}$**.

$$\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \tilde{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

- A matrix $\tilde{\boldsymbol{u}}$ is skew symmetric if: $\tilde{u}^T = -\tilde{u}$

- Given a transformation matrix ${}^G\boldsymbol{R}_B$ we can obtain the axis $\hat{\mathbf{u}}$ and angle ϕ of the rotation by

$$\tilde{u} = \frac{1}{2\sin\phi}\left({}^G R_B - {}^G R_B^T\right)$$

$$\cos\phi = \frac{1}{2}\left(\operatorname{tr}\left({}^G R_B\right) - 1\right)$$

where, $$\operatorname{tr}\left({}^G R_B\right) = r_{11} + r_{22} + r_{33}$$

3.1. Axis–Angle Rotation

Example 1

Axis–angle rotation when $\hat{\mathbf{u}} = \hat{\boldsymbol{K}}$. Simplifying the axis–angle rotation matrix for a special known case axis of rotation.

- The local frame B (*Oxyz*) rotates about the **Z-axis**, $\hat{u} = \hat{K} = \left[0\ 0\ 1\right]^T$
- The transformation matrix reduces to

$$
{}^{G}R_B = \begin{bmatrix} 0 \operatorname{vers}\phi + \cos\phi & 0 \operatorname{vers}\phi - 1 \sin\phi & 0 \operatorname{vers}\phi + 0 \sin\phi \\ 0 \operatorname{vers}\phi + 1 \sin\phi & 0 \operatorname{vers}\phi + \cos\phi & 0 \operatorname{vers}\phi - 0 \sin\phi \\ 0 \operatorname{vers}\phi - 0 \sin\phi & 0 \operatorname{vers}\phi + 0 \sin\phi & 1 \operatorname{vers}\phi + \cos\phi \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

⇒ which is equivalent to the rotation matrix about the Z-axis of global frame.

3.1. Axis–Angle Rotation

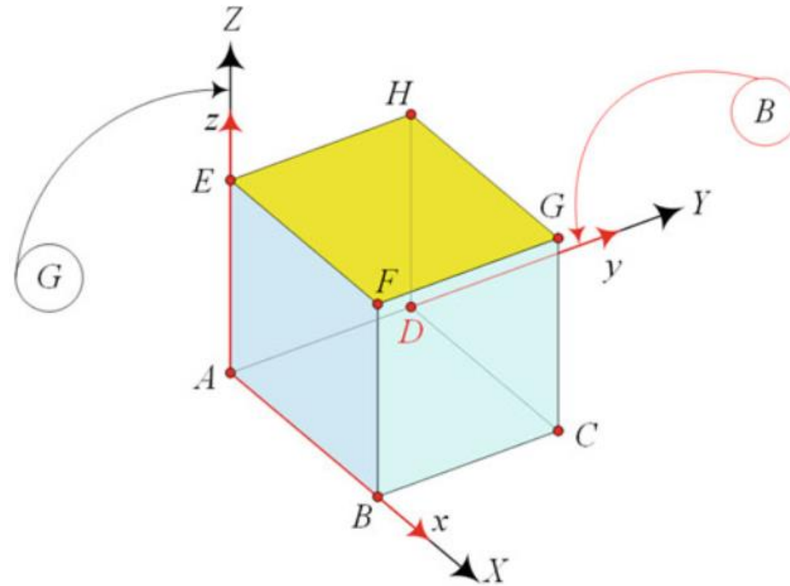
Example 2: Axis and angle of rotation

Consider a cubic rigid body with a fixed point at A and a unit length of edges as is shown in Figure. If we turn the cube 45 [deg] about **u = [1 1 1]^T**, then we can find the global coordinates of its corners using Rodriguez transformation matrix.

$$\phi = \frac{\pi}{4} \qquad \hat{u} = \frac{\mathbf{u}}{\sqrt{3}} = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}$$

$$R_{\hat{u},\phi} = \mathbf{I}\cos\phi + \hat{u}\,\hat{u}^T \operatorname{vers}\phi + \tilde{u}\sin\phi$$

$$= \begin{bmatrix} 0.80474 & -0.31062 & 0.50588 \\ 0.50588 & 0.80474 & -0.31062 \\ -0.31062 & 0.50588 & 0.80474 \end{bmatrix}$$



3.1. Axis–Angle Rotation

Example 2: Axis and angle of rotation

The local coordinates of the corners are

	${}^{B}\mathbf{r}_B$	${}^{B}\mathbf{r}_C$	${}^{B}\mathbf{r}_D$	${}^{B}\mathbf{r}_E$	${}^{B}\mathbf{r}_F$	${}^{B}\mathbf{r}_G$	${}^{B}\mathbf{r}_H$
x	1	1	0	0	1	1	0
y	0	1	1	0	0	1	1
z	0	0	0	1	1	1	1

The global coordinates of the corners after the rotation will be

$$ {}^{G}\mathbf{r} = R_{\hat{u},\phi}\, {}^{B}\mathbf{r} $$

	${}^{G}\mathbf{r}_B$	${}^{G}\mathbf{r}_C$	${}^{G}\mathbf{r}_D$	${}^{G}\mathbf{r}_E$	${}^{G}\mathbf{r}_F$	${}^{G}\mathbf{r}_G$	${}^{G}\mathbf{r}_H$
X	0.804	0.495	−0.31	0.505	1.310	1	0.196
Y	0.505	1.31	0.804	−0.31	0.196	1	0.495
Z	−0.31	0.196	0.505	0.804	0.495	1	1.31

3.1. Axis–Angle Rotation

Example 2: Axis and angle of rotation

Point G is on the axis of rotation, so its coordinates will not change. Points B, D, F, and H are in a symmetric plane indicated by $\hat{\boldsymbol{u}}$. Therefore, they will move on a circle.

Let us call the midpoint of the cube by P

$$^{B}\mathbf{r}_P = \frac{1}{2}\left(^{B}\mathbf{r}_B + {}^{B}\mathbf{r}_H\right) = \frac{1}{2}\left(^{B}\mathbf{r}_F + {}^{B}\mathbf{r}_D\right) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$^{G}\mathbf{r}_P = \frac{1}{2}\left(^{G}\mathbf{r}_B + {}^{G}\mathbf{r}_H\right) = \frac{1}{2}\left(^{G}\mathbf{r}_F + {}^{G}\mathbf{r}_D\right) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

3.1. Axis–Angle Rotation

Example 3: Axis and angle of a rotation matrix

A body coordinate frame B undergoes three Euler rotations (φ, θ, ψ) = (30, 45, 60) deg with respect to **a global frame G**.

- The rotation matrix to transform coordinates of **B** to **G** is

$$
{}^{G}R_{B} = {}^{B}R_{G}^{T} = \left[R_{z,\psi}\, R_{x,\theta}\, R_{z,\varphi}\right]^{T} = R_{z,\varphi}^{T}\, R_{x,\theta}^{T}\, R_{z,\psi}^{T}
$$

$$
= \begin{bmatrix} 0.12683 & -0.92678 & 0.35355 \\ 0.78033 & -0.12683 & -0.61237 \\ 0.61237 & 0.35355 & 0.70711 \end{bmatrix}
$$

The unique angle-axis of rotation for this rotation matrix can then be found

$$
\phi = \arccos\left(\frac{1}{2}\left(\mathrm{tr}\left({}^{G}R_{B}\right) - 1\right)\right)
$$

$$
= \arccos(-0.14645) = 1.7178\,\mathrm{rad} = 98\,\mathrm{deg}
$$

3.1. Axis–Angle Rotation

Example 3: Axis and angle of a rotation matrix

$$\tilde{u} = \frac{1}{2\sin\phi}\left({}^G R_B - {}^G R_B^T\right)$$

$$= \begin{bmatrix} 0.0 & -0.86285 & -0.13082 \\ 0.86285 & 0.0 & -0.48822 \\ 0.13082 & 0.48822 & 0.0 \end{bmatrix}$$

$$\hat{u} = \begin{bmatrix} 0.48822 \\ -0.13082 \\ 0.86285 \end{bmatrix}$$

- We may verify the angle-axis rotation formula and derive the same rotation matrix

$${}^G R_B = R_{\hat{u},\phi} = \mathbf{I}\cos\phi + \hat{u}\,\hat{u}^T \operatorname{vers}\phi + \tilde{u}\sin\phi$$

3.2. Euler Parameters

- Assume ϕ to be the angle of rotation of a body coordinate frame B(*Oxyz*) about $\hat{\boldsymbol{u}} = \boldsymbol{u_1}\hat{\boldsymbol{I}} + \boldsymbol{u_2}\hat{\boldsymbol{J}} + \boldsymbol{u_3}\hat{\boldsymbol{K}}$ relative to a global frame G(*OXYZ*).
- **Euler Rigid Body Rotation Theorem**. The most general displacement of a rigid body with one fixed point is a rotation about an axis.
- To find the axis and angle of rotation we introduce the Euler parameters e_0, e_1, e_2, e_3 such that e_0 is a scalar and e_1, e_2, e_3 are components of a vector **e,**

$$e_0 = \cos\frac{\phi}{2}$$

$$\mathbf{e} = e_1\hat{I} + e_2\hat{J} + e_3\hat{K} = \hat{u}\sin\frac{\phi}{2}$$

$$e_1^2 + e_2^2 + e_3^2 + e_0^2 = e_0^2 + \mathbf{e}^T\ \mathbf{e} = 1$$

3.2. Euler Parameters

- The transformation matrix $^{G}\boldsymbol{R}_{B}$ to satisfy the equation $^{G}\boldsymbol{r} = {}^{G}\boldsymbol{R}_{B}{}^{B}\boldsymbol{r}$ can be derived based on Euler parameters.

$$
\begin{aligned}
{}^{G}R_{B} = R_{\hat{u},\phi} &= \left(e_0^2 - \mathbf{e}^2\right)\mathbf{I} + 2\mathbf{e}\,\mathbf{e}^T + 2e_0\tilde{e} \\
&= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2\left(e_1e_2 - e_0e_3\right) & 2\left(e_0e_2 + e_1e_3\right) \\ 2\left(e_0e_3 + e_1e_2\right) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2\left(e_2e_3 - e_0e_1\right) \\ 2\left(e_1e_3 - e_0e_2\right) & 2\left(e_0e_1 + e_2e_3\right) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix}
\end{aligned}
$$

where, $\tilde{\mathbf{e}}$ is *the skew symmetric matrix* corresponding to vector **e**.

$$
\tilde{e} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}
$$

3.2. Euler Parameters

- Given a transformation matrix $^{G}\boldsymbol{R}_{\boldsymbol{B}}$ we may obtain Euler parameters

$$e_0^2 = \frac{1}{4}\left(\mathrm{tr}\left({}^{G}R_B\right) + 1\right)$$

$$\tilde{e} = \frac{1}{4e_0}\left({}^{G}R_B - {}^{G}R_B^T\right)$$

- Determine the angle ϕ and axis of rotation $\hat{\boldsymbol{u}}$.

$$\cos\phi = \frac{1}{2}\left(\mathrm{tr}\left({}^{G}R_B\right) - 1\right)$$

$$\tilde{u} = \frac{1}{2\sin\phi}\left({}^{G}R_B - {}^{G}R_B^T\right)$$

- Euler parameters provide a well-suited, redundant, and non-singular rotation description for arbitrary and large rotations.

3.2. Euler Parameters

Example 4: Axis–angle rotation ${}^{G}R_{B}$ for a φ and û

Euler parameters for rotation φ = 30 deg about û = $(\hat{I} + \hat{J} + \hat{K})/\sqrt{3}$

$$e_0 = \cos\frac{\pi}{12} = 0.966$$

$$\mathbf{e} = \hat{u}\sin\frac{\phi}{2} = e_1\hat{I} + e_2\hat{J} + e_3\hat{K} = 0.149\left(\hat{I} + \hat{J} + \hat{K}\right)$$

- The corresponding transformation matrix ${}^{G}R_{B}$ is

$$\begin{aligned} {}^{G}R_{B} &= \left(e_0^2 - \mathbf{e}^2\right)\mathbf{I} + 2\mathbf{e}\,\mathbf{e}^T + 2e_0\tilde{e} \\ &= \begin{bmatrix} 0.91 & -0.244 & 0.333 \\ 0.333 & 0.91 & -0.244 \\ -0.244 & 0.333 & 0.91 \end{bmatrix} \end{aligned}$$

3.2. Euler Parameters

Euler parameters and Euler angles relationship

- The following relationships between Euler angles and Euler parameters

$$e_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}$$

$$e_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}$$

$$e_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}$$

$$e_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}$$

Or

$$\varphi = \cos^{-1}\frac{2\left(e_2e_3+e_0e_1\right)}{\sin\theta}$$

$$\theta = \cos^{-1}\left[2\left(e_0^2+e_3^2\right)-1\right]$$

$$\psi = \cos^{-1}\frac{-2\left(e_2e_3-e_0e_1\right)}{\sin\theta}$$

3.2. Euler Parameters

Calculate the Euler parameters

Case 1

$$e_0 = \pm\frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$e_1 = \frac{1}{4}\frac{r_{32} - r_{23}}{e_0} \quad e_2 = \frac{1}{4}\frac{r_{13} - r_{31}}{e_0} \quad e_3 = \frac{1}{4}\frac{r_{21} - r_{12}}{e_0}$$

Case 2

$$e_1 = \pm\frac{1}{2}\sqrt{1 + r_{11} - r_{22} - r_{33}}$$

$$e_2 = \frac{1}{4}\frac{r_{21} + r_{12}}{e_1} \quad e_3 = \frac{1}{4}\frac{r_{31} + r_{13}}{e_1} \quad e_0 = \frac{1}{4}\frac{r_{32} + r_{23}}{e_1}$$

Case 3

$$e_2 = \pm\frac{1}{2}\sqrt{1 - r_{11} + r_{22} - r_{33}}$$

$$e_3 = \frac{1}{4}\frac{r_{32} + r_{23}}{e_2} \quad e_0 = \frac{1}{4}\frac{r_{13} - r_{31}}{e_2} \quad e_1 = \frac{1}{4}\frac{r_{21} + r_{12}}{e_2}$$

Case 4

$$e_3 = \pm\frac{1}{2}\sqrt{1 - r_{11} - r_{22} + r_{33}}$$

$$e_0 = \frac{1}{4}\frac{r_{21} - r_{12}}{e_3} \quad e_1 = \frac{1}{4}\frac{r_{31} + r_{13}}{e_3} \quad e_2 = \frac{1}{4}\frac{r_{32} + r_{23}}{e_3}$$

C3. End!