11

Introduction to Robotic Manipulators

CONCEPT OVERVIEW

In this chapter, the reader will gain a central understanding regarding

1. Distinctions and disadvantages of *robotic manipulators* in comparison to classical linkages
2. Matrix-based formulation of displacement equation systems for *Cartesian*, *cylindrical*, *spherical*, *articulated*, and *SCARA* robots
3. *Forward* kinematics of Cartesian, cylindrical, spherical, articulated, and SCARA robots
4. *Inverse* kinematics and workspace of Cartesian, cylindrical, spherical, articulated, and SCARA robots

11.1 Introduction

As explained in Chapter 1 and demonstrated throughout this textbook, a *linkage* (also commonly called a *mechanism*) is an assembly of links and joints where the motion of one link compels the motion of another link in a controlled manner. To enable controlled mechanism motion, they are either initially designed to have a single degree of freedom or ultimately configured (in the case of the geared five-bar mechanism) to have a single degree of freedom. Conventional planar and spatial linkages include the four-bar, slider-crank, geared five-bar, Watt, Stephenson, RRSS, RSSR, and 4R spherical linkages presented in Chapters 4 and 10.

Like the linkages presented in Chapters 4 and 10, a *robotic manipulator* (commonly called a *robot*) also includes an assembly of links and joints and is designed to produce a controlled output motion. In addition to links and joints, however, a robotic manipulator also includes *electronic circuitry*, *computer-controlled actuators* to compel link motion, and is guided by a *computer program*. Because robotic manipulators include both mechanical and electronic components, they are classified as *electro-mechanical systems*.

To achieve a controlled motion, each joint in a robotic manipulator can be controlled independently. As a result, there is no degree of freedom limit that robotic manipulators can be theoretically designed to have.*

Another common distinction between linkages and robotic manipulators is in their overall design. All of the linkages presented in Chapters 4 and 10 have *closed-loop* designs. With this design, at least two joints in a linkage are connected to ground (thus forming

* Robotic manipulators are generally limited to six DOFs because a spatial body has a maximum mobility of six.

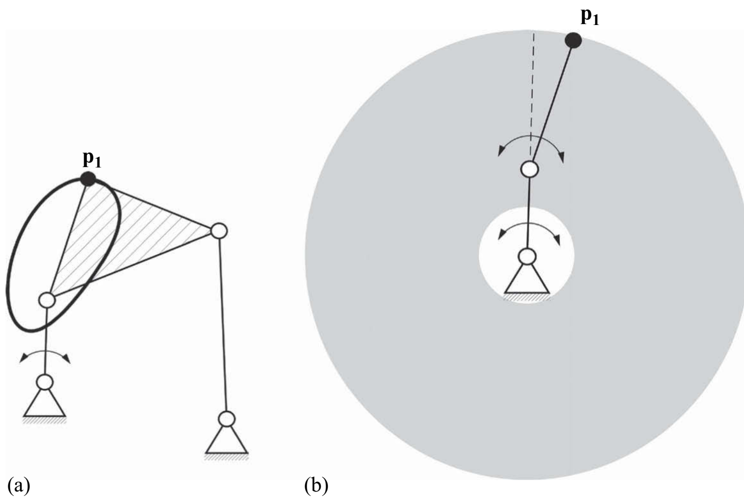

FIGURE 11.1
(a) Four-bar mechanism and coupler curve and (b) robotic manipulator and workspace.

a closed loop). While robotic manipulators can have closed-loop designs, they often have *open loops*, where only one joint is connected to ground.

Figure 11.1 includes a planar four-bar mechanism and a planar, 2-DOF, open-loop robotic manipulator. While coupler Point $\mathbf{p}_1$ on the four-bar mechanism (in the given location on the coupler link) can only trace the curve illustrated, Point $\mathbf{p}_1$ on the robotic manipulator can trace *any* path within the shaded annular area or *workspace*.* Because of the open-loop construction of the robotic manipulator and its mobility, this single manipulator can trace a greater variety of distinct paths than any number of planar four-bar mechanisms.

So, when compared to linkages, robotic manipulators offer advantages such as greater variability—specifically for motion-specific and path-specific tasks. Being computer controlled, robotic manipulators also offer advantages regarding greater precision, accuracy, and repeatability. Lastly, robotic manipulators have the capacity for *remote operation* as well as *autonomous operation* since they can be guided by computer programs (as opposed to mechanisms, which often require a degree of manual operation).

It is becoming increasingly difficult to find an industry where robotic manipulators are not employed, either directly or indirectly. Common industries where robotic manipulators are widely employed (both in product manufacturing and operation) include *automotive*, *aerospace*, *defense*, *electronics*, and *medicine*. The number of applications for robotic systems is rapidly on the increase, since new robotic manipulator capabilities and more practical robot manipulator designs are continually being developed and produced [1].

11.2 Terminology and Nomenclature

To describe the spatial position and orientation of each link in a robotic manipulator, coordinate systems are rigidly attached to each link. These coordinate systems are called

* The workspace is the area or volume (for spatial robots) of space that the robot can reach.

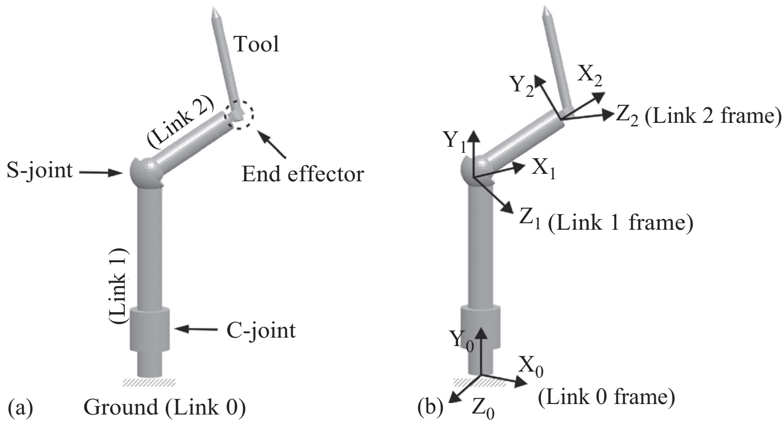

FIGURE 11.2
(a) Robotic manipulator with joint and link descriptions and (b) frames.

frames. In Figure 11.2b, frames are attached to Link 0 (called the *base frame*), Link 1, and Link 2. The positions and orientations between the frames are modeled through transformation matrices (Section 2.5). The general transformation matrix and its application will be further discussed in Sections 11.4–11.6.

A robotic manipulator often includes a component at its free end called an *end effector*. The end effector can serve to handle any tool or component. It can also be the working end of the tool itself if the end effector fully constrains the tool. One common end effector used in robotic manipulators is a *gripper*. In Figure 11.2, the end effector of the robotic manipulator is a gripper that holds the tool. The tool is assumed to be fully constrained by the gripper.*

11.3 Robotic Manipulator Mobility and Types

In Chapter 3, it was noted that Gruebler's Equation is used to determine the mobility of a linkage. These equations can also be used to determine the mobility of robotic manipulators. For planar and spatial robotic manipulators, Gruebler's Equations become

$$DOF_{PLANAR} = 3(L-1) - 2J_1 \tag{11.1}$$

$$DOF_{SPATIAL} = 6(L-1) - 5J_1 - 4J_2 - 3J_3 \tag{11.2}$$

where only 1-DOF joints are used in the planar robotic manipulator and only 1-, 2-, or 3-DOF joints are used in the spatial manipulator.†

Figure 11.3 includes the types of 1-, 2-, and 3-DOF joints used in the robotic manipulators presented in this chapter. Although a robotic manipulator can be designed to include any

* If the tool was not fully constrained by the gripper, an additional frame attached to the tool (called a *tool frame*) would have been included.

† As with mechanisms, for robotic manipulators, the ground link should also be counted among the total number of links (L) in Gruebler's Equation.

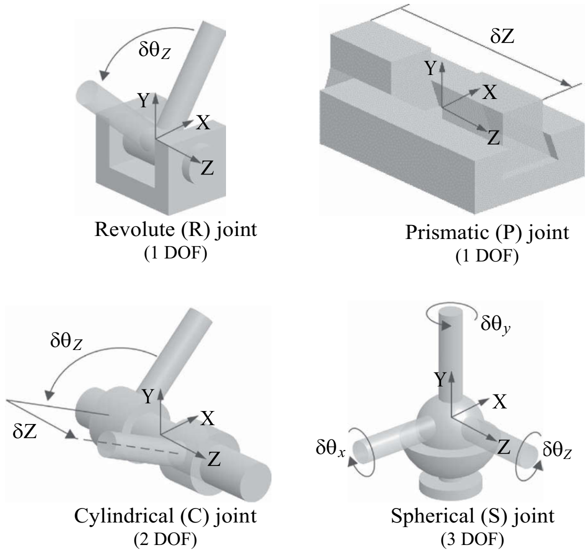

FIGURE 11.3
Example robotic manipulator joint types.

joint type, the revolute, prismatic, cylindrical, and spherical are among those joint types most commonly used in practice.

Figure 11.4 illustrates the five spatial robotic manipulator types considered in this chapter. These particular robotic manipulator configurations are among those commonly

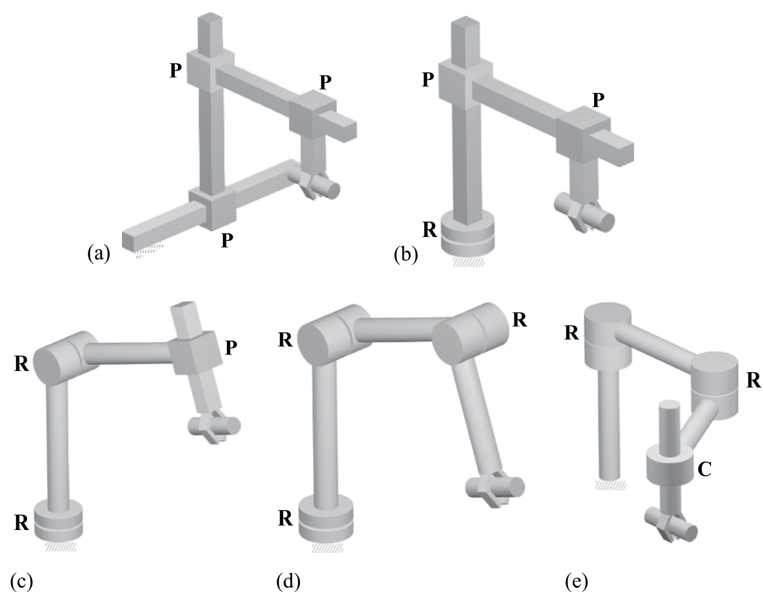

FIGURE 11.4
(a) Cartesian (P-P-P), (b) cylindrical (R-P-P), (c) spherical (R-R-P), (d) articulated (R-R-R), and (e) SCARA (R-R-C) robots.

```
>> L = 4;
>> J1 = 3;
>> DOF_RRR = 6*(L - 1) -5*J1

DOF_RRR =

     3

>> L = 4;
>> J1 = 2;
>> J2 = 1;
>> DOF_RRC = 6*(L - 1) - 5*J1 - 4*J2

DOF_RRC =

     4

>>
```

FIGURE E.11.1
Example 11.1 calculation procedure in MATLAB.

utilized in industrial applications [2]. The designations P-P-P, R-P-P, R-R-P, R-R-R, and R-R-C in Figure 11.4 denote the joint types and joint sequences used in the robotic manipulators.

The P-P-P robotic manipulator is commonly known as a *Cartesian robot* because its degrees of freedom are along the x, y, and z-axes of the Cartesian frame. The R-P-P and R-R-P robotic manipulators are commonly known as *cylindrical* and *spherical robots*, respectively, because their motion is consistent with cylindrical and spherical joints, respectively. The R-R-R and R-R-C robotic manipulators are commonly known as *articulated* and *Selective Compliance Assembly/Articulated Robot Arm* (or *SCARA*) *robots*, respectively.

Example 11.1

Problem Statement: Calculate the mobility values for the R-R-R and R-R-C robotic manipulators.

Known Information: Equation 11.2 and Figure 11.3.

Solution Approach: The R-R-R robotic manipulator is comprised of four links interconnected with three revolute joints. Since there are four links and the revolute joint has a single degree of freedom, $L = 4$ and $J_1 = 3$ in Equation 11.2 for the R-R-R robotic manipulator.

The R-R-C is comprised of four links interconnected with two revolute joints and one cylindrical joint. Since there are four links and the cylindrical joint has two degrees of freedom, $L = 4$, $J_1 = 2$, and $J_2 = 1$ in Equation 11.2 for the R-R-C robotic manipulator.

Figure E.11.1 includes the calculation procedure in the MATLAB® command window.

11.4 The General Transformation Matrix

In Section 2.5, a general spatial transformation matrix was presented. This matrix is used to calculate point coordinates originally established in one link coordinate frame (Frame j) in reference to another link coordinate frame (Frame i) in a robotic manipulator. If we recall, the general spatial transformation matrix can be expressed as

$$
{}^{i}_{j}[T] = \begin{bmatrix} R_{11} & R_{12} & R_{13} & \Delta_x \\ R_{21} & R_{22} & R_{23} & \Delta_y \\ R_{31} & R_{32} & R_{33} & \Delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11.3}
$$

where:

$R_{11} = \cos\delta_y \cos\delta_z$

$R_{12} = \sin\delta_x \sin\delta_y \cos\delta_z - \cos\delta_x \sin\delta_z$

$R_{13} = \cos\delta_x \sin\delta_y \cos\delta_z + \sin\delta_x \sin\delta_z$

$R_{21} = \cos\delta_y \sin\delta_z$

$R_{22} = \sin\delta_x \sin\delta_y \sin\delta_z + \cos\delta_x \cos\delta_z$

$R_{23} = \cos\delta_x \sin\delta_y \sin\delta_z - \sin\delta_x \cos\delta_z$

$R_{31} = -\sin\delta_y$

$R_{32} = \sin\delta_x \cos\delta_y$

$R_{33} = \cos\delta_x \cos\delta_y$

In this matrix, variables δ_x, δ_y, and δ_z are the angular rotations about a frame's *x*-, *y*-, and *z*-axes, respectively, and variables Δ_x, Δ_y, and Δ_z are the linear translations along the frame's *x*-, *y*-, and *z*-axes, respectively.*

Given ${}^{j}\{\mathbf{p}\}$, the spatial coordinates of a point $\mathbf{p}$ in Frame *j*, the coordinates of this point with respect to Frame *i* or ${}^{j}\{\mathbf{p}\}$ can be calculated as

$$
{}^{i}\{\mathbf{p}\} = {}^{i}_{j}[T]\,{}^{j}\{\mathbf{p}\} \tag{11.4}
$$

where the spatial coordinates of $\mathbf{p}$ in Frame *j* are ${}^{j}\{\mathbf{p}\} = \{p_x\ p_y\ p_z\ 1\}^T$.†

In Figure 11.5, the robotic manipulator in Figure 11.2b is again considered where Frames 0, 1, and 2 are attached to ground, Link 1 and Link 2, respectively. In this figure, Point $\mathbf{p}_1$ is attached to Link 1 and its coordinates are given with respect to Frame 1. Point $\mathbf{p}_2$ is attached to Link 2 and its coordinates are given with respect to Frame 2.

To calculate the value of Point $\mathbf{p}_1$ with respect to the base frame, Equation 11.4 becomes

$$
{}^{0}\{\mathbf{p}_1\} = {}^{0}_{1}[T]\,{}^{1}[\mathbf{p}_1] \tag{11.5}
$$

where ${}^{0}\{\mathbf{p}_1\}$ is the value of $\mathbf{p}_1$ with respect to the base frame (Frame X_0–Y_0–Z_0 in Figure 11.5).‡ The transformation matrix ${}^{0}_{1}[T]$ considers Frames 0 and 1. This matrix includes angular displacements δ_{1x}, δ_{1y}, and δ_{1z}, and linear displacements Δ_{1x}, Δ_{1y}, and Δ_{1z}.

To calculate the value of Point $\mathbf{p}_2$ with respect to the base frame, Equation 11.5 becomes

$$
{}^{0}\{\mathbf{p}_2\} = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}\{\mathbf{p}_2\} = {}^{0}_{2}[T]\,{}^{2}\{\mathbf{p}_2\} \tag{11.6}
$$

* In a general *planar* transformation matrix, variables δ_x, δ_y, and Δ_z in the general spatial transformation matrix are all zero.

† The *planar* coordinates of $\mathbf{p}$ in Frame *j* would be ${}^{j}\{\mathbf{p}\} = \{p_x\ p_y\ p_0\ 1\}^T$.

‡ A value given with respect to the base frame is also called a *global value*.

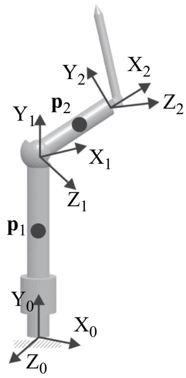

FIGURE 11.5
Robotic manipulator with frames and points.

where ${}^{0}\{\mathbf{p}_2\}$ is the value of $\mathbf{p}_2$ with respect to the base frame. Because the transformation matrix ${}^{0}_{1}[T]$ already considers Frames 0 and 1, including the transformation matrix ${}^{1}_{2}[T]$ (which considers Frames 1 and 2) and taking the product of these two matrices produces a single transformation matrix that considers Frames 0 and 2. This matrix includes all of the displacement variables in ${}^{0}_{1}[T]$ as well as the additional angular displacements δ_{2x}, δ_{2y}, and δ_{2z}, and linear displacements Δ_{2x}, Δ_{2y}, and Δ_{2z}.

Let us now assume an additional frame X_3–Y_3–Z_3 is attached to the tool and the coordinates of a point $\mathbf{p}_3$ are given with respect to this frame (Frame 3). To calculate the value of $\mathbf{p}_3$ with respect to the base frame, Equation 11.6 becomes

$$ {}^{0}\{\mathbf{p}_3\} = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T]\,{}^{3}\{\mathbf{p}_3\} = {}^{0}_{2}[T]\,{}^{2}_{3}[T]\,{}^{3}\{\mathbf{p}_3\} = {}^{0}_{3}[T]\,{}^{3}\{\mathbf{p}_3\} \tag{11.7} $$

where ${}^{0}\{\mathbf{p}_3\}$ is the value of $\mathbf{p}_3$ with respect to the base frame. Because the transformation matrix ${}^{0}_{2}[T]$ already considers Frames 0 and 2, including the transformation matrix ${}^{2}_{3}[T]$ (which considers Frames 2 and 3) and taking the product of these two matrices produces a single transformation matrix that considers Frames 0 and 3. This matrix includes all of the displacement variables in ${}^{0}_{2}[T]$ as well as the additional angular displacements δ_{3x}, δ_{3y}, and δ_{3z}, and linear displacements Δ_{3x}, Δ_{3y}, and Δ_{3z}.

From Equations 11.5 to 11.7, it can be observed that as an additional final frame is introduced, the transformation matrix to calculate the global coordinates of a point in the final frame becomes an increasing product of the transformation matrices for each frame. Knowing this, given a group of N frames, the transformation matrix to calculate the global coordinates of a point in the Nth frame can be expressed as

$$ {}^{0}_{N}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T]\ldots {}^{N-1}_{N}[T]^{*} \tag{11.8} $$

Therefore, given ${}^{N}\{\mathbf{p}_N\}$, the spatial coordinates of a point $\mathbf{p}_N$ in Frame N, the global coordinates of this point with respect to the base frame or ${}^{0}\{\mathbf{p}_N\}$ can be calculated as

$$ {}^{0}\{\mathbf{p}_N\} = {}^{0}_{N}[T]\,{}^{N}\{\mathbf{p}_N\} \tag{11.9} $$

* When calculating the product of three or more matrices in Equation 11.8, it should be done in right-to-left order.

The coordinate frame x, y, and z rotations and translation variables presented in this section are analogous to *Denavit–Hartenberg* (or *DH*) parameters. Like the δ and Δ variables, DH parameters are also used to attach reference frames to links. In classical DH notation, only rotations about and translations along coordinate frame x- and z-axes are used. Therefore, with the y-axis rotations and translations eliminated, the general transformation matrix becomes

$$ {}^{i}_{j}[T] = \begin{bmatrix} \cos\delta_z & -\cos\delta_x \sin\delta_z & \sin\delta_x \sin\delta_z & \Delta_x \cos\delta_z \\ \sin\delta_z & \cos\delta_x \cos\delta_z & -\sin\delta_x \cos\delta_z & \Delta_x \sin\delta_z \\ 0 & \sin\delta_x & \cos\delta_x & \Delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{*} \tag{11.10}$$

Example 11.2

Problem Statement: Calculate the elements of transformation matrix ${}^{0}_{3}[T]$ for the coordinate frame displacement values given in Table E.11.1.

Known Information: Matrix 11.3 and Table E.11.1.

Solution Approach: Because all variables in Matrix 11.3, with the exception of those given in Table E.11.1, are zero, this matrix can be simplified. Figure E.11.2 includes the calculation procedure in MATLAB's command window.

TABLE E.11.1

Frame Displacement Variables (with Unitless Link Lengths)

Frames	δ_x	δ_y	δ_z	Δ_x	Δ_y	Δ_z
1 wrt 0	0	0	55°	0	0	0
2 wrt 1	0	0	0	0	0	1.75
3 wrt 2	0	0	0	3	0	0

```
>> d1z = 55*pi/180;
>> D2z = 1.75;
>> D3x = 3;
>> T01 = [cos(d1z), -sin(d1z), 0, 0; sin(d1z), cos(d1z), 0, 0;...
0, 0, 1, 0; 0, 0, 0, 1];
>> T12 = [1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, D2z; 0, 0, 0, 1];
>> T23 = [1, 0, 0, D3x; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1];
>> T03 = T01*T12*T23

T03 =

    0.5736   -0.8192         0    1.7207
    0.8192    0.5736         0    2.4575
         0         0    1.0000    1.7500
         0         0         0    1.0000

>>
```

FIGURE E.11.2
Example 11.2 calculation procedure in MATLAB.

* In classical DH notation, the variables θ, α, d, and r are used instead of δ_z, δ_x, Δ_z, and Δ_z, respectively.

11.5 Forward Kinematics

11.5.1 Definition and Application

In *forward kinematics*, the link dimensions and joint motion of a robotic manipulator are known and the corresponding output motion of the links (usually the end effector) is calculated [3]. By formulating an equation system for a robotic manipulator (an equation system to calculate the motion of specific link points) and prescribing the link dimensions and joint motions, the resulting link motion is calculated. The most common application for forward kinematics is for determining end effector motion (e.g., tool paths and orientations).

11.5.2 P-P-P

Figure 11.6 includes the frames specified for the P-P-P robotic manipulator. The transformation matrix ${}^0_3[T]$ is required to formulate equations to calculate the global position of $\mathbf{p}_3$ on the end effector. To facilitate this procedure, Table 11.1 includes the displacement variables required to align Frame 1 to 0, Frame 2 to 1, and Frame 3 to 2. As this table indicates, a combination of three linear displacements is utilized in the P-P-P robotic manipulator.

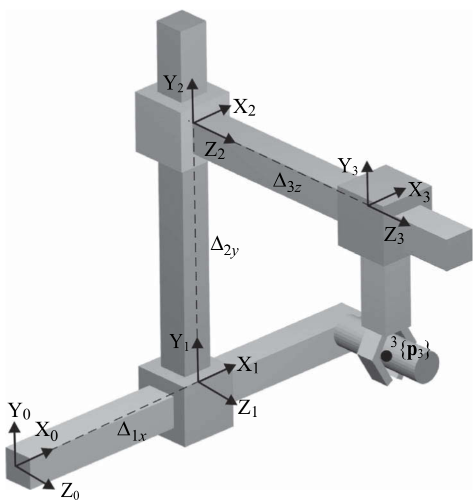

FIGURE 11.6
P-P-P robotic manipulator with frames and point.

TABLE 11.1
Frame Displacement Variables for the P-P-P Robotic Manipulator

Frames	δ_x	δ_y	δ_z	Δ_x	Δ_y	Δ_z
1 wrt 0	0	0	0	Δ_{1x}	0	0
2 wrt 1	0	0	0	0	Δ_{2y}	0
3 wrt 2	0	0	0	0	0	Δ_{3z}

From the information given in Table 11.1, the only nonzero variables are Δ_{1x}, Δ_{2y}, and Δ_{3z}. As a result, Matrix 11.8 becomes

$$
{}^{0}_{3}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T] = \begin{bmatrix} 1 & 0 & 0 & \Delta_{1x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \Delta_{2y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta_{3z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11.11}
$$

Using this transformation matrix and Equation 11.9 produces the following system of equations to calculate the global coordinates of $\mathbf{p}_3$:

$$
\begin{aligned}
{}^{0}p_{3x} &= {}^{3}p_{3x} + \Delta_{1x} \\
{}^{0}p_{3y} &= {}^{3}p_{3y} + \Delta_{2y} \\
{}^{0}p_{3z} &= {}^{3}p_{3z} + \Delta_{3z}
\end{aligned} \tag{11.12}
$$

In the P-P-P robotic manipulator, the linear displacement variables Δ_{1x}, Δ_{2y}, and Δ_{3z} are not often assigned constant values because they correspond to the translational displacements of the prismatic joints. More often, displacement ranges are assigned to these variables.

Example 11.3

Problem Statement: Using the joint displacements given in Table E.11.2, calculate the global path points achieved by the end effector of the P-P-P robotic manipulator. In this example, ${}^{3}\{\mathbf{p}_3\} = [0, -1, 0]^T$.

Known Information: Equation 11.12 and Table E.11.2.

Solution Approach: Table E.11.3 includes the global end effector path point coordinates calculated using Equation 11.12.

TABLE E.11.2

P-P-P Robotic Manipulator Joint Displacements

Point	Δ_{3x}	Δ_{2y}	Δ_{3z}
1	0.5	1.1	–0.1
2	1	1.2	–0.15
3	1.5	1.3	–0.3
4	2	1.2	–0.45
5	2.5	1.1	–0.60

TABLE E.11.3

P-P-P Robotic Manipulator Path Point Coordinates

Point	${}^{0}p_{3x}$	${}^{0}p_{3y}$	${}^{0}p_{3z}$
1	0.5	0.1	–0.1
2	1	0.2	–0.15
3	1.5	0.3	–0.3
4	2	0.2	–0.45
5	2.5	0.1	–0.60

11.5.3 R-P-P

Figure 11.7 includes the frames specified for the R-P-P robotic manipulator. The transformation matrix ${}^{0}_{3}[T]$ is required to formulate equations to calculate the global position of $\mathbf{p}_3$ on the end effector. To facilitate this procedure, Table 11.2 includes the displacement variables required to align Frame 1 to 0, Frame 2 to 1, and Frame 3 to 2. As this table indicates, a combination of a single angular displacement and two linear displacements are utilized in the R-P-P robotic manipulator.

From the information given in Table 11.2, the only nonzero variables are δ_{1z}, Δ_{2z}, and Δ_{3x}. As a result, Matrix 11.8 becomes

$$ {}^{0}_{3}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T] = \begin{bmatrix} \cos\delta_{1z} & -\sin\delta_{1z} & 0 & 0 \\ \sin\delta_{1z} & \cos\delta_{1z} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta_{2z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \Delta_{3x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11.13} $$

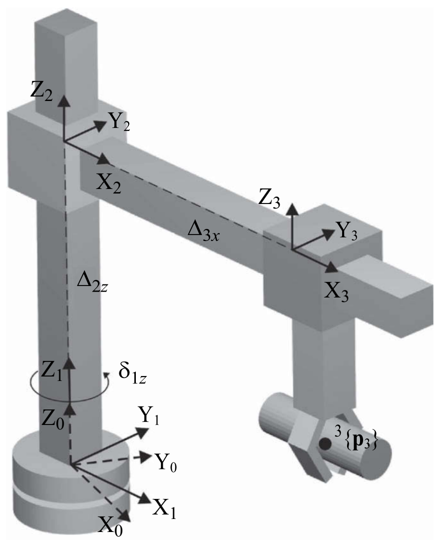

FIGURE 11.7
R-P-P robotic manipulator with frames and point.

TABLE 11.2
Frame Displacement Variables for the R-P-P Robotic Manipulator

Frames	$\boldsymbol{\delta_x}$	$\boldsymbol{\delta_y}$	$\boldsymbol{\delta_z}$	$\boldsymbol{\Delta_x}$	$\boldsymbol{\Delta_y}$	$\boldsymbol{\Delta_z}$
1 wrt 0	0	0	δ_{1z}	0	0	0
2 wrt 1	0	0	0	0	0	Δ_{2z}
3 wrt 2	0	0	0	Δ_{3x}	0	0

Using this transformation matrix and Equation 11.9 produces the following system of equations to calculate the global coordinates of $\mathbf{p}_3$:

$$
\begin{aligned}
{}^0p_{3x} &= {}^3p_{3x}\cos\delta_{1z} - {}^3p_{3y}\sin\delta_{1z} + \Delta_{3x}\cos\delta_{1z} \\
{}^0p_{3y} &= {}^3p_{3x}\sin\delta_{1z} + {}^3p_{3y}\cos\delta_{1z} + \Delta_{3x}\sin\delta_{1z} \\
{}^0p_{3z} &= {}^3p_{3z} + \Delta_{2z}
\end{aligned}
\tag{11.14}
$$

In the R-P-P robotic manipulator, the linear and angular displacement variables Δ_{2z}, Δ_{3x}, and δ_{1z} are not often assigned constant values because they correspond to the translational displacements, the prismatic joints, and the rotational displacements of the revolute joint. More often, displacement ranges are assigned to these variables.

Appendix G.1 includes the MATLAB file user instructions for R-P-P robotic manipulator forward kinematics. In this MATLAB file (which is available for download at www.crcpress.com/product/isbn/9781498724937), Equation 11.14 is used to calculate the global coordinates of $\mathbf{p}_3$.

Example 11.4

Problem Statement: Using the Appendix G.1 MATLAB file with the joint displacements given in Table E.11.4, calculate the global path points achieved by the end effector of the R-P-P robotic manipulator. In this example, ${}^3\{\mathbf{p}_3\} = [0, 0, -1]^T$.

Known Information: Table E.11.4 and Appendix G.1 MATLAB file.

Solution Approach: The data for columns δ_{1z}, Δ_{2z}, and Δ_{3x} in Table E.11.4 are first specified in the file *RPP_Input.csv*. Figure E.11.3 includes the input specified (in bold text) in the Appendix G.1 MATLAB file and Table E.11.5 includes the global end effector path point coordinates calculated.

```
p3_3 = [0, 0, -1];
```

FIGURE E.11.3
Specified input (in bold text) in the Appendix G.1 MATLAB file for Example 11.4.

TABLE E.11.4
R-P-P Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	Δ_{2z}	Δ_{3x}
1	12	1.1	–0.1
2	24	1.2	–0.15
3	36	1.3	–0.3
4	48	1.2	–0.45
5	60	1.1	–0.60

TABLE E.11.5
R-P-P Robotic Manipulator Path Point Coordinates

Point	${}^0p_{3x}$	${}^0p_{3y}$	${}^0p_{3z}$
1	–0.0978	–0.0208	0.1000
2	–0.1370	–0.0610	0.2000
3	–0.2427	–0.1763	0.3000
4	–0.3011	–0.3344	0.2000
5	–0.3000	–0.5196	0.1000

11.5.4 R-R-P

Figure 11.8 includes the frames specified for the R-R-P robotic manipulator. The transformation matrix ${}^0_3[T]$ is required to formulate equations to calculate the global position of $\mathbf{p}_3$ on the end effector. To facilitate this procedure, Table 11.3 includes the displacement variables required to align Frame 1 to 0, Frame 2 to 1, and Frame 3 to 2. As this table indicates, a combination of two angular displacements and three linear displacements are utilized in the R-R-P robotic manipulator.

From the information given in Table 11.3, the only nonzero variables are δ_{1z}, δ_{2x}, and Δ_{3z} and the terms representing Δ_{2z} and Δ_{3y}.* As a result, Matrix 11.8 becomes

$$
{}^0_3[T] = {}^0_1[T]\,{}^1_2[T]\,{}^2_3[T]
$$

$$
= \begin{bmatrix} \cos\delta_{1z} & -\sin\delta_{1z} & 0 & 0 \\ \sin\delta_{1z} & \cos\delta_{1z} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta_{2x} & -\sin\delta_{2x} & 0 \\ 0 & \sin\delta_{2x} & \cos\delta_{2x} & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & \Delta_{3z} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\tag{11.15}
$$

Using the resulting transformation matrix and Equation 11.9 produces the following system of equations to calculate the global coordinates of $\mathbf{p}_3$:

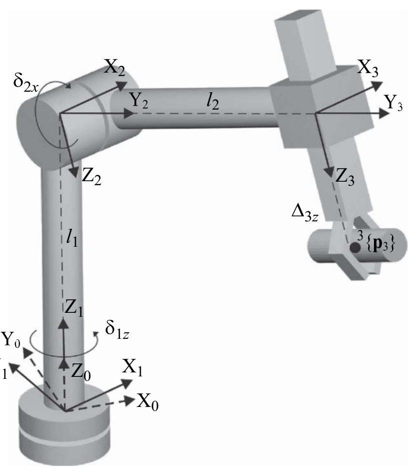

FIGURE 11.8
R-R-P robotic manipulator with frames and point.

TABLE 11.3
Frame Displacement Variables for the R-R-P Robotic Manipulator

Frames	δ_x	δ_y	δ_z	Δ_x	Δ_y	Δ_z
1 wrt 0	0	0	δ_{1z}	0	0	0
2 wrt 1	Δ_{2x}	0	0	0	0	l_1
3 wrt 2	0	0	0	0	l_2	Δ_{3z}

* In Table 11.3, $\Delta_{2z} = l_1$ and $\Delta_{3y} = l_2$.

$$
\begin{aligned}
{}^0p_{3x} &= {}^3p_{3x}\cos\delta_{1z} - {}^3p_{3y}\sin\delta_{1z}\cos\delta_{2x} + {}^3p_{3z}\sin\delta_{1z}\sin\delta_{2x} - l_2\sin\delta_{1z}\cos\delta_{2x} \\
&\quad + \Delta_{3z}\sin\delta_{1z}\sin\delta_{2x} \\
{}^0p_{3y} &= {}^3p_{3x}\sin\delta_{1z} + {}^3p_{3y}\cos\delta_{1z}\cos\delta_{2x} - {}^3p_{3z}\cos\delta_{1z}\sin\delta_{2x} + l_2\cos\delta_{1z}\cos\delta_{2x} \\
&\quad - \Delta_{3z}\cos\delta_{1z}\sin\delta_{2x} \\
{}^0p_{3z} &= {}^3p_{3y}\sin\delta_{2x} + {}^3p_{3z}\cos\delta_{2x} + l_2\sin\delta_{2x} + \Delta_{3z}\cos\delta_{2x} + l_1
\end{aligned}
\tag{11.16}
$$

In the R-R-P robotic manipulator, variables l_1 and l_2 are assigned constant values because they represent constant link lengths. The angular and linear displacement variables δ_{1z}, δ_{2x}, and Δ_{3z}, however, correspond to the rotational displacements of the revolute joints and the translational displacements of the prismatic joint. More often, displacement ranges are assigned to these variables.

Appendix G.2 includes the MATLAB file user instructions for R-R-P robotic manipulator forward kinematics. In this MATLAB file (which is available for download at www.crcpress.com/product/isbn/9781498724937), Equation 11.16 is used to calculate the global coordinates of $\mathbf{p}_3$.

Example 11.5

Problem Statement: Using the Appendix G.2 MATLAB file with the joint displacements given in Table E.11.6, calculate the global path points achieved by the end effector of the R-R-P robotic manipulator. In this example, $l_1 = l_2 = 0.5$ (unitless link lengths) and ${}^3\{\mathbf{p}_3\} = [0,0,0]^T$.

Known Information: Table E.11.6 and Appendix G.2 MATLAB file.

Solution Approach: The data for columns δ_{1z}, δ_{2x}, and Δ_{3z} in Table E.11.6 are first specified in the file *RRP_Input.csv*. Figure E.11.4 includes the input specified (in bold text) in the Appendix G.2 MATLAB file and Table E.11.7 includes the global end effector path point coordinates calculated.

```
l1 = 0.5;
l2 = 0.5;
p3_3 = [0, 0, 0];
```

FIGURE E.11.4
Specified input (in bold text) in the Appendix G.2 MATLAB file for Example 11.5.

TABLE E.11.6
R-R-P Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	δ_{2x} (°)	Δ_{3z}
1	12	10	0.5
2	24	20	1
3	36	30	1.5
4	48	40	2
5	60	50	2.5

TABLE E.11.7

R-R-P Robotic Manipulator Path Point Coordinates

Point	${}^0p_{3x}$	${}^0p_{3y}$	${}^0p_{3z}$
1	–0.0843	0.3967	1.0792
2	–0.0520	0.1168	1.6107
3	0.1863	–0.2564	2.0490
4	0.6707	–0.6039	2.3535
5	1.3802	–0.7969	2.4900

11.5.5 R-R-R

Figure 11.9 includes the frames specified for the R-R-R robotic manipulator. The transformation matrix ${}^0_3[T]$ is required to formulate equations to calculate the global position of $\mathbf{p}_3$ on the tool. To facilitate this procedure, Table 11.4 includes the displacement variables required to align Frame 1 to 0, Frame 2 to 1, and Frame 3 to 2. As this table indicates, a combination of three angular displacements and two linear displacements are utilized in the R-R-R robotic manipulator.

From the information given in Table 11.4, the only nonzero variables are δ_{1z}, δ_{2x}, and δ_{3x}, and the terms representing Δ_{2z} and Δ_{3y}.* As a result, Matrix 11.8 becomes

$$
{}^0_3[T] = {}^0_1[T]\,{}^1_2[T]\,{}^2_3[T]
$$

$$
= \begin{bmatrix} \cos\delta_{1z} & -\sin\delta_{1z} & 0 & 0 \\ \sin\delta_{1z} & \cos\delta_{1z} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta_{2x} & -\sin\delta_{2x} & 0 \\ 0 & \sin\delta_{2x} & \cos\delta_{2x} & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta_{3x} & -\sin\delta_{3x} & l_2 \\ 0 & \sin\delta_{3x} & \cos\delta_{3x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11.16}
$$

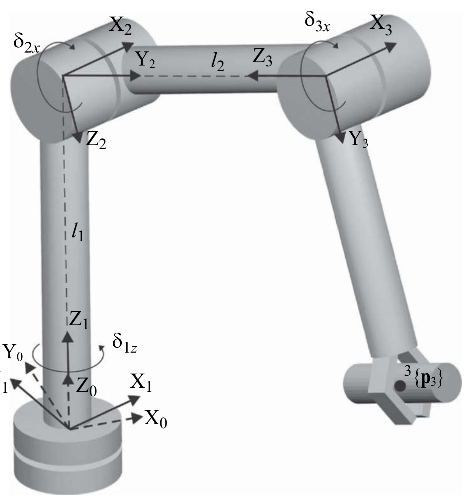

FIGURE 11.9
R-R-R robotic manipulator with frames and point.

* In Table 11.4, $\Delta_{2z} = l_1$ and $\Delta_{3y} = l_2$.

TABLE 11.4

Frame Displacement Variables for the R-R-R Robotic Manipulator

Frames	δ_x	δ_y	δ_z	Δ_x	Δ_y	Δ_z
1 wrt 0	0	0	δ_{1z}	0	0	0
2 wrt 1	δ_{2x}	0	0	0	0	l_1
3 wrt 2	δ_{3x}	0	0	0	l_2	0

Using the resulting transformation matrix and Equation 11.9 produces the following system of equations to calculate the global coordinates of $\mathbf{p}_3$:

$$\begin{aligned}
{}^0p_{3x} &= {}^3p_{3x}\cos\delta_{1z} + {}^3p_{3y}\left(\sin\delta_{1z}\sin\delta_{2x}\sin\delta_{3x} - \sin\delta_{1z}\cos\delta_{2x}\cos\delta_{3x}\right) \\
&\quad + {}^3p_{3z}\left(\sin\delta_{1z}\cos\delta_{2x}\sin\delta_{3x} + \sin\delta_{1z}\sin\delta_{2x}\cos\delta_{3x}\right) - l_2\sin\delta_{1z}\cos\delta_{2x} \\
{}^0p_{3y} &= {}^3p_{3x}\sin\delta_{1z} + {}^3p_{3y}\left(\cos\delta_{1z}\cos\delta_{2x}\cos\delta_{3x} - \cos\delta_{1z}\sin\delta_{2x}\sin\delta_{3x}\right) \\
&\quad + {}^3p_{3z}\left(-\cos\delta_{1z}\cos\delta_{2x}\sin\delta_{3x} - \cos\delta_{1z}\sin\delta_{2x}\cos\delta_{3x}\right) + l_2\cos\delta_{1z}\cos\delta_{2x} \\
{}^0p_{3z} &= {}^3p_{3y}\left(\sin\delta_{2x}\cos\delta_{3x} + \cos\delta_{2x}\sin\delta_{3x}\right) + {}^3p_{3z}\left(\cos\delta_{2x}\cos\delta_{3x} - \sin\delta_{2x}\sin\delta_{3x}\right) \\
&\quad + l_2\sin\delta_{2x} + l_1
\end{aligned} \tag{11.18}$$

In the R-R-R robotic manipulator, variables l_1 and l_2 are assigned constant values because they represent constant link lengths. The angular and linear displacement variables δ_{1z}, δ_{2x}, and δ_{3x}, however, correspond to the rotational displacements of the revolute joints. More often, displacement ranges are assigned to these variables.

Appendix G.3 includes the MATLAB file user instructions for R-R-R robotic manipulator forward kinematics. In this MATLAB file (which is available for download at www.crcpress.com/ product/isbn/9781498724937), Equation 11.18 is used to calculate the global coordinates of $\mathbf{p}_3$.

Example 11.6

Problem Statement: Using the Appendix G.3 MATLAB file with the joint displacements given in Table E.11.8, calculate the global path points achieved by the end effector of the R-R-R robotic manipulator. In this example, $l_1 = l_2 = 0.5$ (unitless link lengths) and ${}^3\{\mathbf{p}_3\} = [0, 1, 0]^T$.

Known Information: Table E.11.8 and Appendix G.3 MATLAB file.

Solution Approach: The data for columns δ_{1z}, δ_{2x}, and δ_{3x} in Table E.11.8 are first specified in the file *RRR_Input.csv*. Figure E.11.5 includes the input specified (in bold text) in

TABLE E.11.8

R-R-R Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	δ_{2x} (°)	Δ_{3x} (°)
1	12	10	–5
2	24	20	–10
3	36	30	–15
4	48	40	–20
5	60	50	–25

```
l1 = 0.5;
l2 = 0.5;
p3_3 = [0, 1, 0];
```

FIGURE E.11.5
Specified input (in bold text) in the Appendix G.3 MATLAB file for Example 11.3.

TABLE E.11.9
R-R-R Robotic Manipulator Path Point Coordinates

Point	${}^0p_{3x}$	${}^0p_{3y}$	${}^0p_{3z}$
1	–0.3095	1.4561	0.6740
2	–0.5917	1.3289	0.8447
3	–0.8223	1.1318	1.0088
4	–0.9830	0.8851	1.1634
5	–1.0632	0.6139	1.3056

the Appendix G.3 MATLAB file and Table E.11.9 includes the global end effector path point coordinates calculated.

11.5.6 R-R-C

Figure 11.10 includes the frames specified for the R-R-C robotic manipulator. The transformation matrix ${}^0_3[T]$ is required to formulate equations to calculate the global position of $\mathbf{p}_3$ on the tool. To facilitate this procedure, Table 11.5 includes the displacement variables required to align Frame 1 to 0, Frame 2 to 1, and Frame 3 to 2. As this table indicates, a

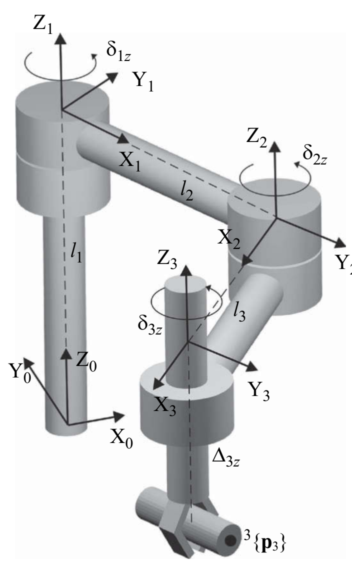

FIGURE 11.10
R-R-C robotic manipulator with frames and point.

TABLE 11.5

Frame Displacement Variables for the R-R-C Robotic Manipulator

Frames	$\boldsymbol{\delta_x}$	$\boldsymbol{\delta_y}$	$\boldsymbol{\delta_z}$	$\boldsymbol{\Delta_x}$	$\boldsymbol{\Delta_y}$	$\boldsymbol{\Delta_z}$
1 wrt 0	0	0	δ_{1z}	0	0	l_1
2 wrt 1	0	0	δ_{2z}	l_2	0	0
3 wrt 2	0	0	δ_{3z}	l_3	0	Δ_{3z}

combination of three angular displacements and four linear displacements are utilized in the R-R-C robotic manipulator.

Using the information given in Table 11.5, the nonzero variables are δ_{1z}, δ_{2z}, δ_{3z}, Δ_{3z}, and the terms representing Δ_{1z}, Δ_{2x}, and Δ_{3x}.* As a result, Matrix 11.8 becomes

$$
{}^0_3[T] = {}^0_1[T]\;{}^1_2[T]\;{}^2_3[T]
$$

$$
= \begin{bmatrix} \cos\delta_{1z} & -\sin\delta_{1z} & 0 & 0 \\ \sin\delta_{1z} & \cos\delta_{1z} & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta_{2z} & -\sin\delta_{2z} & 0 & l_2 \\ \sin\delta_{2z} & \cos\delta_{2z} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta_{3z} & -\sin\delta_{3z} & 0 & l_3 \\ \sin\delta_{3z} & \cos\delta_{3z} & 0 & 0 \\ 0 & 0 & 1 & \Delta_{3z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11.19}
$$

Using the resulting transformation matrix and Equation 11.9 produces the following system of equations to calculate the global coordinates of $\mathbf{p}_3$:

$$
\begin{aligned}
{}^0p_{3x} = {}^3p_{3x}\big[&(\cos\delta_{1z}\cos\delta_{2z} - \sin\delta_{1z}\sin\delta_{2z})\cos\delta_{3z} + (-\cos\delta_{1z}\sin\delta_{2z} - \sin\delta_{1z}\cos\delta_{2z})\sin\delta_{3z}\big] \\
&+ {}^3p_{3y}\big[-(\cos\delta_{1z}\cos\delta_{2z} - \sin\delta_{1z}\sin\delta_{2z})\sin\delta_{3z} + (-\cos\delta_{1z}\sin\delta_{2z} - \sin\delta_{1z}\cos\delta_{2z})\cos\delta_{3z} \\
&+ l_3(\cos\delta_{1z}\cos\delta_{2z} - \sin\delta_{1z}\sin\delta_{2z}) + l_2\cos\delta_{1z} \\
{}^0p_{3y} = {}^3p_{3x}\big[&(\sin\delta_{1z}\cos\delta_{2z} + \cos\delta_{1z}\sin\delta_{2z})\cos\delta_{3z} + (\cos\delta_{1z}\cos\delta_{2z} - \sin\delta_{1z}\sin\delta_{2z})\sin\delta_{3z}\big] \\
&+ {}^3p_{3y}\big[-(\sin\delta_{1z}\cos\delta_{2z} + \cos\delta_{1z}\sin\delta_{2z})\sin\delta_{3z} + (\cos\delta_{1z}\cos\delta_{2z} - \sin\delta_{1z}\sin\delta_{2z})\cos\delta_{3z}\big] \\
&+ l_3(\sin\delta_{1z}\cos\delta_{2z} + \cos\delta_{1z}\sin\delta_{2z}) + l_2\sin\delta_{1z} \\
{}^0p_{3z} = {}^3p_{3z} &+ l_1 + \Delta_{3z}.
\end{aligned} \tag{11.20}
$$

In the R-R-C robotic manipulator, variables l_1, l_2, and l_3 are assigned constant values because they represent constant link lengths. The angular and linear displacement variables δ_{1z}, δ_{2z}, δ_{3z}, and Δ_{3y}, however, correspond to the rotational and translational

* In Table 11.5, $\Delta_{1z} = l_1$, $\Delta_{2x} = l_2$, and $\Delta_{3x} = l_3$.

displacements of the revolute and prismatic joints, respectively. More often, displacement ranges are assigned to these variables.

Appendix G.4 includes the MATLAB file user instructions for R-R-C robotic manipulator forward kinematics. In this MATLAB file (which is available for download at www.crcpress.com/ product/isbn/9781498724937), Equation 11.20 is used to calculate the global coordinates of $\mathbf{p}_3$.

Example 11.7

Problem Statement: Using the Appendix G.4 MATLAB file with the joint displacements given in Table E.11.10, calculate the global path points achieved by the end effector of the R-R-C robotic manipulator. In this example, $l_1 = 1$, $l_2 = l_3 = 0.5$ (unitless link lengths), and ${}^3\{\mathbf{p}_3\} = [1, 0, 0]^T$.

Known Information: Table E.11.10 and Appendix G.4 MATLAB file.

Solution Approach: The data for columns δ_{1z}, δ_{2x}, δ_{3z}, and Δ_{3z} in Table E.11.10 are first specified in the file *RRC_Input.csv*. Figure E.11.6 includes the input specified (in bold text) in the Appendix G.4 MATLAB file and Table E.11.11 includes the global end effector path point coordinates calculated.

TABLE E.11.10

R-R-C Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	δ_{2z} (°)	δ_{3z} (°)	Δ_{3z}
1	12	−5	15	−0.1000
2	24	−10	30	−0.2000
3	36	−15	45	−0.3000
4	48	−20	60	−0.2000
5	60	−25	75	−0.1000

```
l1 = 1;
l2 = 0.5;
l3 = 0.5;
p3_3 = [1, 0, 0];
```

FIGURE E.11.6
Specified input (in bold text) in the Appendix G.4 MATLAB file for Example 11.4.

TABLE E.11.11

R-R-C Robotic Manipulator Path Point Coordinates

Point	${}^0p_{3x}$	${}^0p_{3y}$	${}^0p_{3z}$
1	1.9125	0.5395	0.9000
2	1.6613	1.0190	0.8000
3	1.2780	1.3866	0.7000
4	0.8109	1.6057	0.8000
5	0.3176	1.6595	0.9000

11.6 Inverse Kinematics

11.6.1 Definition and Application

While the robotic manipulator dimensions are known and the end effector motion is calculated in forward kinematics, in *inverse kinematics*, the end effector motion and link dimensions are known and the joint motion required to achieve the end effector motion is calculated [4–7]. Inverse kinematics is often described as the *reverse of forward kinematics*. It is accomplished by calculating joint motion solutions (using the same forward-kinematics-based equation system for a robotic manipulator) given the end effector motion and link lengths. The most common application for inverse kinematics is to determine the joint motion required to achieve the required end effector motion (e.g., the joint motion required to achieve tool paths and orientations).*

11.6.2 P-P-P

Because Equation 11.12 includes three equations each containing a single unknown variable (Δ_{1x}, Δ_{2y}, and Δ_{3z}, respectively), these unknowns can be calculated algebraically for the inverse kinematics of the P-P-P robotic manipulator.

Solving for Δ_{1x}, Δ_{2y}, and Δ_{3z} in Equation 11.12 produces

$$\begin{aligned}
\Delta_{1x} &= {}^{0}p_{3x} - {}^{3}p_{3x} \\
\Delta_{2y} &= {}^{0}p_{3y} - {}^{3}p_{3y} \\
\Delta_{3z} &= {}^{0}p_{3z} - {}^{3}p_{3z}
\end{aligned} \tag{11.21}$$

From Equation 11.21, the user can specify the global end effector coordinates ${}^{0}\{\mathbf{p}_3\}$, the end effector coordinates in Frame 3 ${}^{0}\{\mathbf{p}_3\}$, and calculate the required displacements of each prismatic joint in the P-P-P robotic manipulator (Δ_{1x}, Δ_{2y}, and Δ_{3z}).

The global path point coordinates ${}^{0}\{\mathbf{p}_3\}$ should be prescribed from within the workspace of the P-P-P robotic manipulator. This ensures that the prescribed points will be achieved by the robotic manipulator. As shown in Figure 11.11, the P-P-P robotic manipulator has a cubic (or a rectangular cuboid) workspace with outer x, y, and z dimensions of $\Delta_{1x_{\max}}$, $\Delta_{2y_{\max}}$ and $\Delta_{3z_{\max}}$, respectively (the maximum prismatic joint translations).

Example 11.8

Problem Statement: Calculate the P-P-P joint displacements required to achieve the global path points given in Table E.11.12. In this example, ${}^{3}\{\mathbf{p}_3\} = [0, -1, 0]^T$.

Known Information: Equation (11.21) and Table E.11.12.

Solution Approach: Table E.11.13 includes the joint displacements calculated using Equation (11.21). Figure E.11.7 includes the initial and final positions of the P-P-P robotic manipulator over the range of prescribed global end effector points.

* Inverse kinematics is similar to motion generation and path generation (see Chapter 5). The key distinction between dimensional synthesis and inverse kinematics is that, with the latter, *joint displacements* are calculated (rather than link dimensions).

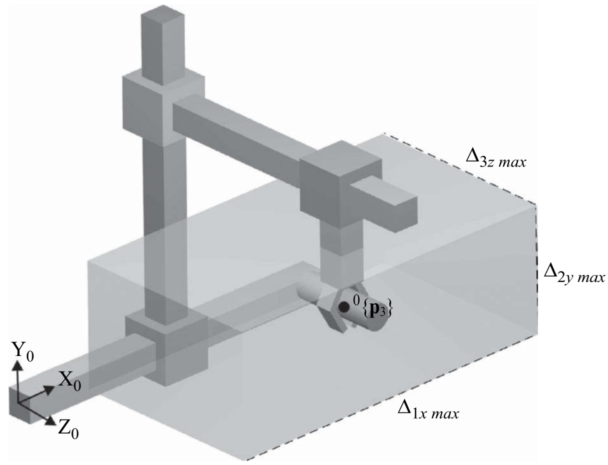

FIGURE 11.11
P-P-P robotic manipulator and workspace.

TABLE E.11.12
P-P-P Robotic Manipulator End Effector Path Point Coordinates

Point	$^0p_{3x}$	$^0p_{3y}$	$^0p_{3z}$
1	0.683	0.375	0.2165
2	0.6764	0.4521	0.2566
3	0.6569	0.5306	0.289
4	0.625	0.6083	0.3125
5	0.5817	0.6826	0.3266
6	0.5283	0.7514	0.3307
7	0.4665	0.8125	0.3248
8	0.3981	0.8641	0.309
9	0.3252	0.9047	0.2838
10	0.25	0.933	0.25

TABLE E.11.13
P-P-P Robotic Manipulator Joint Displacements

Point	Δ_{1x}	Δ_{2y}	Δ_{3z}
1	0.683	1.375	0.2165
2	0.6764	1.4521	0.2566
3	0.6569	1.5306	0.289
4	0.625	1.6083	0.3125
5	0.5817	1.6826	0.3266
6	0.5283	1.7514	0.3307
7	0.4665	1.8125	0.3248
8	0.3981	1.8641	0.309
9	0.3252	1.9047	0.2838
10	0.25	1.933	0.25

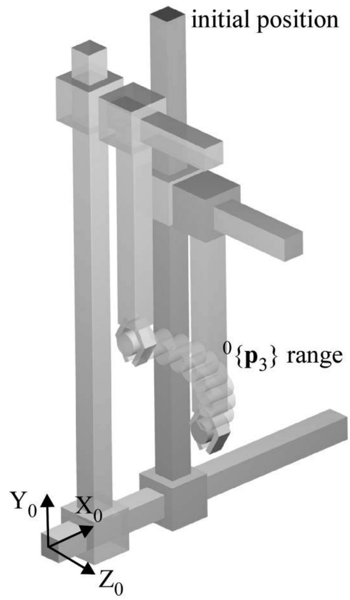

FIGURE E.11.7
P-P-P robotic manipulator in (semitransparent) initial and final positions.

11.6.3 R-P-P

Unlike Equation 11.12, the joint displacement variables δ_{1z}, Δ_{2x}, and Δ_{3x} in Equation 11.14 cannot be calculated algebraically. With unknowns δ_{1z}, Δ_{2x}, and Δ_{3x}, Equation 11.14 becomes a set of three nonlinear simultaneous equations. A root-finding method (see Section 4.2) is required for the inverse kinematics of the R-P-P robotic manipulator. Appendix G.5 includes the user instructions for the MATLAB file to calculate joint displacement solutions for Equation 11.14 given $^3\{\mathbf{p}_3\}$ and a prescribed range of values for $^0\{\mathbf{p}_3\}$.

As shown in Figure 11.12, the R-P-P robotic manipulator has a cylindrical workspace (having a center axis that is collinear with Δ_{2z}) with outer cylinder height and radius dimensions of $\Delta_{2z_{\text{max}}}$ and $\Delta_{3x_{\text{max}}}$, respectively (the maximum prismatic joint translations).

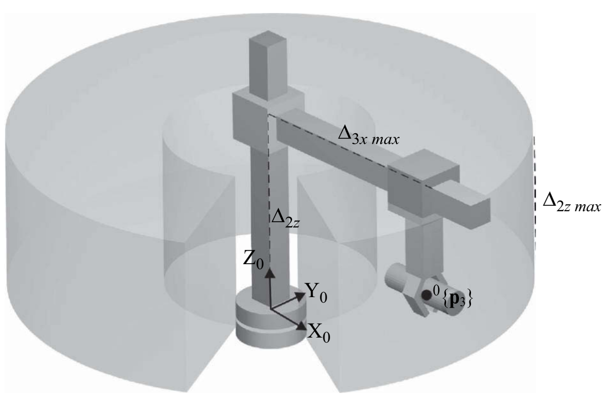

FIGURE 11.12
R-P-P robotic manipulator and workspace.

Example 11.9

Problem Statement: Using the Appendix G.5 MATLAB file, calculate the R-P-P joint displacements required to achieve the global path points given in Table E.11.12. In this example, ${}^{3}\{\mathbf{p}_3\} = [0, 0, -1]^T$.

Known Information: Table E.11.12 and Appendix G.5 MATLAB file.

Solution Approach: The data for columns ${}^{0}p_{3x}$, ${}^{0}p_{3y}$, and ${}^{0}p_{3z}$ in Table E.11.12 are first specified in the file *RPP_Input.csv*. Figure E.11.8 includes the input specified (in bold text) in the Appendix G.5 MATLAB file and Table E.11.14 includes the joint displacements calculated. Figure E.11.9 includes the initial and final positions of the R-P-P robotic manipulator over the range of prescribed global end effector points.

```
p3_3 = [0, 0, -1];
```

FIGURE E.11.8
Specified input (in bold text) in the Appendix G.5 MATLAB file for Example 11.9.

TABLE E.11.14
R-P-P Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	Δ_{2z}	Δ_{3x}
1	28.7689	1.2165	0.7792
2	33.7584	1.2566	0.8136
3	38.929	1.289	0.8444
4	44.2242	1.3125	0.8722
5	49.5629	1.3266	0.8968
6	54.8894	1.3307	0.9185
7	60.1375	1.3248	0.9369
8	65.264	1.309	0.9514
9	70.2287	1.2838	0.9614
10	74.9998	1.25	0.9659

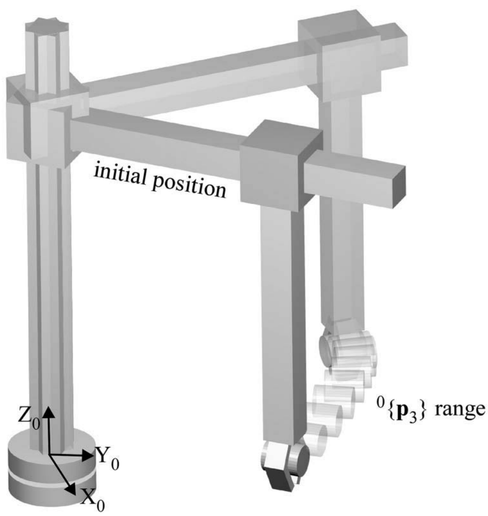

FIGURE E.11.9
R-P-P robotic manipulator in initial and final positions.

11.6.4 R-R-P

The joint displacement variables δ_{1z}, δ_{2x}, δ_{3z} in Equation 11.16 cannot also be calculated algebraically. With unknowns δ_{1z}, δ_{2x}, δ_{3z}, Equation 11.16 becomes a set of three nonlinear simultaneous equations. Like the R-P-P robotic manipulator, a root-finding method is also required for the inverse kinematics of the R-R-P robotic manipulator. Appendix G.6 includes the user instructions for the MATLAB file to calculate joint displacement solutions for Equation 11.16 given ${}^{3}\{\mathbf{p}_3\}$ and a prescribed range of values for ${}^{0}\{\mathbf{p}_3\}$.

As shown in Figure 11.13, the R-R-P robotic manipulator has a spherical workspace (having center coordinates center = (0, 0, l_1)) with an outer radius dimension of

$$r_{\text{outer}} = \sqrt{(l_2)^2 + (\Delta_{3z\,\max})^2}$$

Example 11.10

Problem Statement: Using the Appendix G.6 MATLAB file, calculate the R-R-P joint displacements required to achieve the global path points given in Table E.11.12. In this example, $l_1 = l_2 = 0.5$ (unitless link lengths) and ${}^{3}\{\mathbf{p}_3\} = [0,0,0]^T$.

Known Information: Table E.11.12 and Appendix G.6 MATLAB file.

Solution Approach: The data for columns ${}^{0}p_{3x}$, ${}^{0}p_{3y}$, and ${}^{0}p_{3z}$ in Table E.11.12 are first specified in the file *RRP_Input.csv*. Figure E.11.10 includes the input specified (in bold text) in the Appendix G.6 MATLAB file and Table E.11.15 includes the joint displacements calculated. Figure E.11.11 includes the initial and final positions of the R-R-P robotic manipulator over the range of prescribed global end effector points.

```
l1 = 0.5;
l2 = 0.5;
p3_3 = [0, 0, 0];
```

FIGURE E.11.10
Specified input (in bold text) in the Appendix G.6 MATLAB file for Example 11.10.

TABLE E.11.15
R-R-P Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	δ_{2x} (°)	Δ_{3z}
1	–61.2311	32.9191	0.6614
2	–56.2416	37.2735	–0.6864
3	–51.0710	40.9089	–0.7124
4	–45.7758	43.7775	–0.7388
5	–40.4371	45.8699	–0.7644
6	–35.1105	47.1905	–0.7889
7	–29.8625	47.7679	–0.8115
8	–24.7360	47.6333	–0.8316
9	–19.7713	46.8339	–0.8491
10	–15.0002	45.4141	–0.8634

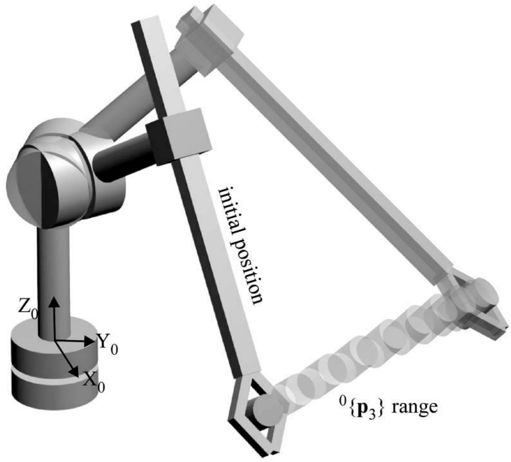

FIGURE E.11.11
R-R-P robotic manipulator in initial and final positions.

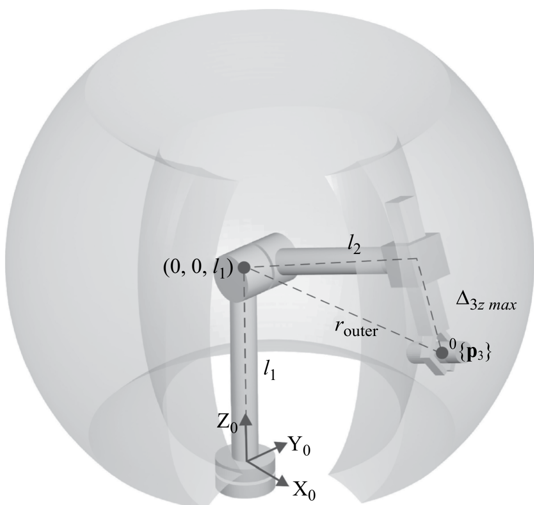

FIGURE 11.13
R-R-P robotic manipulator and workspace.

11.6.5 R-R-R

The joint displacement variables δ_{1z}, δ_{2x}, and δ_{3x} in Equation 11.18 cannot also be calculated algebraically. With unknowns δ_{1z}, δ_{2x}, and δ_{3x}, Equation 11.18 becomes a set of three nonlinear simultaneous equations. Like the R-P-P and R-R-P robotic manipulators, a root-finding method is also required for the inverse kinematics of the R-R-R robotic manipulator. Appendix G.7 includes the user instructions for the MATLAB file to calculate joint displacement solutions for Equation 11.18 given ${}^{3}\{\mathbf{p}_3\}$ and a prescribed range of values for ${}^{0}\{\mathbf{p}_3\}$.

As shown in Figure 11.14, the R-R-R robotic manipulator has a spherical workspace (having center coordinates center = $(0, 0, l_1)$) with an outer radius dimension of $r_{outer} = l_2 + \|{}^{3}\{\mathbf{p}_3\}\|$ where ${}^{3}\{\mathbf{p}_3\} = [0, {}^{0}p_{3y}, {}^{0}p_{3z}]^T$.

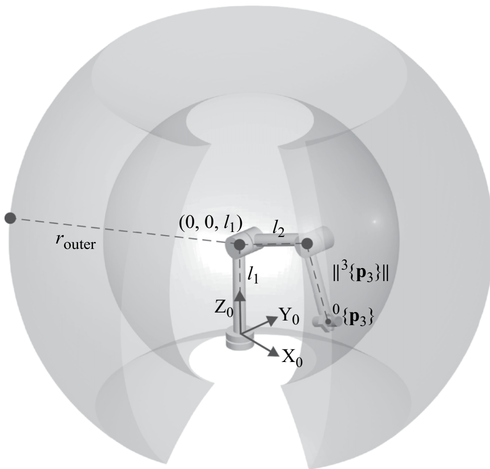

FIGURE 11.14
R-R-R robotic manipulator and workspace.

Example 11.11

Problem Statement: Using the Appendix G.7 MATLAB file, calculate the R-R-R joint displacements required to achieve the global path points given in Table E.11.12. In this example, $l_1 = l_2 = 0.5$ (unitless link lengths) and ${}^3\{\mathbf{p}_3\} = [0, 1, 0]^T$.

Known Information: Table E.11.12 and Appendix G.7 MATLAB file.

Solution Approach: The data for columns ${}^0p_{3x}$, ${}^0p_{3y}$, and ${}^0p_{3z}$ in Table E.11.12 are first specified in the file *RRR_Input.csv*. Figure E.11.12 includes the input specified (in bold text) in the Appendix G.7 MATLAB file and Table E.11.16 includes the joint displacements calculated. Figure E.11.13 includes the initial and final positions of the R-R-R robotic manipulator.

```
l1 = 0.5;
l2 = 0.5;
p3_3 = [0, 1, 0];
```

FIGURE E.11.12
Specified input (in bold text) in the Appendix G.7 MATLAB file for Example 11.11.

TABLE E.11.16
R-R-R Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	δ_{2x} (°)	δ_{3x} (°)
1	–61.2311	74.3302	–124.2298
2	–56.2416	75.2908	–121.9275
3	–51.0710	75.4719	–119.5001
4	–45.7758	74.9234	–117.0128
5	–40.4371	73.7565	–114.5580
6	–35.1106	72.0287	–112.1872
7	–29.8625	69.8373	–109.9699
8	–24.7360	67.2585	–107.9608
9	–19.7713	64.3666	–106.2017
10	–15.0002	61.2454	–104.7446

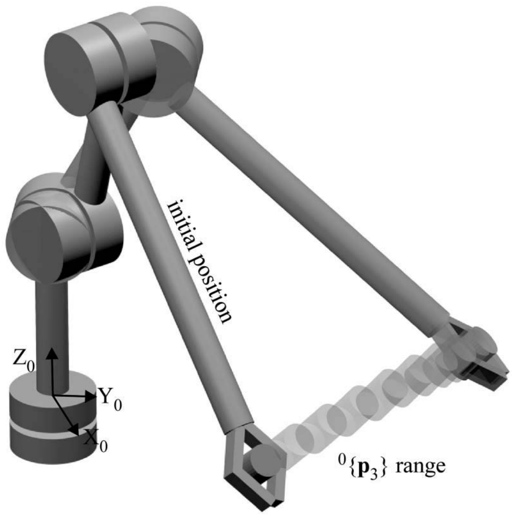

FIGURE E.11.13
R-R-R robotic manipulator in initial and final positions.

11.6.6 R-R-C

The joint displacement variables δ_{1z}, δ_{2x}, δ_{3z}, and Δ_{3y} in Equation 11.20 cannot also be calculated algebraically. With unknowns δ_{1z}, δ_{2x}, δ_{3z}, and Δ_{3y}, Equation 11.20 becomes a set of three nonlinear simultaneous equations. Like the R-P-P, R-R-P, and R-R-R robotic manipulators, a root-finding method is also required for the inverse kinematics of the R-R-C robotic manipulator. Appendix G.8 includes the user instructions for the MATLAB file to calculate joint displacement solutions for Equation 11.20 given $^3\{\mathbf{p}^3\}$ and a prescribed range of values for $^0\{\mathbf{p}_3\}$.

As shown in Figure 11.15, the R-R-C robotic manipulator has a cylindrical workspace (having a center axis that is collinear with l_1) with outer height and radius dimensions of $\Delta_{3z_{\max}}$ and $r_{\text{outer}} = l_1 + l_2$, respectively.

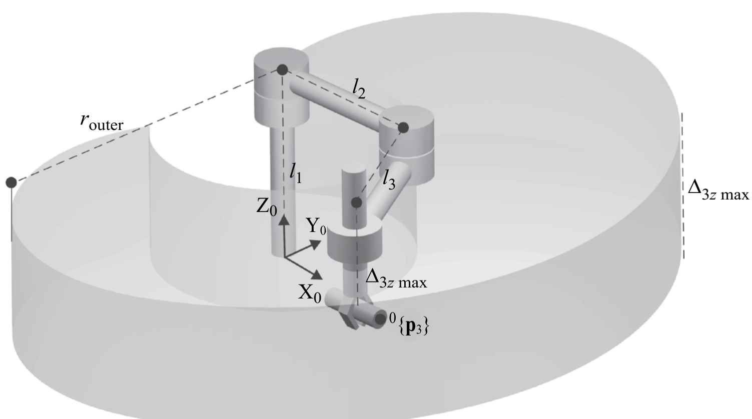

FIGURE 11.15
R-R-C robotic manipulator and workspace.

Example 11.12

Problem Statement: Using the Appendix G.8 MATLAB file, calculate the R-R-C joint displacements required to achieve the global path points given in Table E.11.12. In this example, $l_1 = 1, l_2 = l_3 = 0.5$ (unitless link lengths), and ${}^{3}\{\mathbf{p}_3\} = [0, 1, -1]^T$.

Known Information: Table E.11.14 and Appendix G.8 MATLAB file.

Solution Approach: The data for columns ${}^{0}p_{3x}$, ${}^{0}p_{3y}$, and ${}^{0}p_{3z}$ in Table E.11.12 are first specified in the file *RRC_Input.csv*. Figure E.11.14 includes the input specified (in bold text) in the Appendix G.8 MATLAB file and Table E.11.17 includes the joint displacements calculated. Figure E.11.15 includes the initial and final positions of the R-R-C robotic manipulator over the range of prescribed global end effector points.

```
l1 = 1;
l2 = 0.5;
l3 = 0.5;
p3_3 = [0, 1, -1];
```

FIGURE E.11.14
Specified input (in bold text) in the Appendix G.8 MATLAB file for Example 11.12.

TABLE E.11.17
R-R-C Robotic Manipulator Joint Displacements

Point	δ_{1z} (°)	δ_{2z} (°)	δ_{3z} (°)	Δ_{3z}
1	−35.4860	−5.7728	46.9980	0.2165
2	−30.0170	−4.5237	44.2350	0.2566
3	−24.5330	−3.1631	41.6230	0.2890
4	−19.0690	−1.7162	39.1470	0.3125
5	−13.6890	−0.21255	36.8220	0.3266
6	−8.4370	1.3269	34.6550	0.3307
7	−3.3822	2.8823	32.6820	0.3248
8	1.4188	4.4435	30.9450	0.3090
9	5.9070	6.0036	29.4950	0.2838
10	10.0190	7.5616	28.3930	0.2500

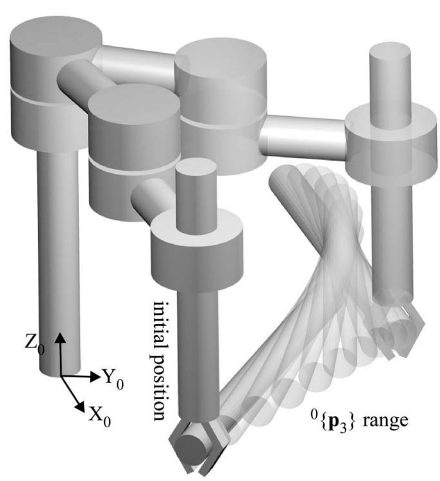

FIGURE E.11.15
R-R-C robotic manipulator in initial and final positions.

11.7 Robotic Manipulator Kinematic Analysis and Modeling in Simmechanics®

As has been noted throughout this chapter, Appendices G.1–G.8 include MATLAB file user instructions for R-P-P, R-R-P, R-R-R, and R-R-C forward and inverse kinematics. In these files, the global end effector coordinates are calculated for prescribed joint displacements (in Appendices G.1–G.4) and the joint displacements are calculated for prescribed global end effector coordinates (in Appendices G.5–G.8).

This textbook also utilizes SimMechanics as an alternate approach for simulation-based kinematic analysis. A library of SimMechanics files is also available for download at www.crcpress. com/product/isbn/9781498724937 for R-P-P, R-R-P, R-R-R, and R-R-C forward kinematics. In addition to calculating the global end effector coordinates for prescribed joint displacements, the motion of the robotic manipulator is also simulated over the joint displacements. The SimMechanics file user instructions for the forward kinematic analysis of the R-P-P, R-R-P, R-R-R, and R-R-C robotic manipulators are given in Appendices L.1–L.4, respectively.

Example 11.13

Problem Statement: Using the Appendix L.4 SimMechanics files, calculate the R-R-C end effector coordinates in Example 11.12.

Known Information: Example 11.12 and Appendix L.4 SimMechanics files.

Solution Approach: The data for columns δ_{1z}, δ_{2x}, δ_{3z}, and Δ_{3z} in Table E.11.17 are first specified in the file *RRC_Input.csv*. Figure E.11.16 includes the input specified (in bold text) in the Appendix L.4 SimMechanics file and Table E.11.18 includes the global R-R-C end effector coordinates calculated. These coordinates are identical to the prescribed end effector coordinates given in Table E.11.12.

```
l1 = 1;
l2 = 0.5;
l3 = 0.5;
p3_3 = [0, 1, -1];
```

FIGURE E.11.16
Specified input (in bold text) in the Appendix L.4 SimMechanics file for Example 11.13.

TABLE E.11.18
Calculated R-R-C Robotic Manipulator End Effector Coordinates

Point	${}^0p_{3x}$	${}^0p_{3y}$	${}^0p_{3z}$
1	0.683	0.375	0.2165
2	0.6764	0.4521	0.2566
3	0.6569	0.5306	0.289
4	0.625	0.6083	0.3125
5	0.5817	0.6826	0.3266
6	0.5283	0.7514	0.3307
7	0.4665	0.8125	0.3248
8	0.3981	0.8641	0.309
9	0.3252	0.9047	0.2838
10	0.25	0.933	0.25

Identical end effector coordinates to those in Table E.11.18 can be replicated for The R-P-P, R-R-P, and R-R-R robotic manipulators by using the joint displacements and dimension input data in Examples 11.9, 11.10, and 11.11, respectively (in the Appendix L.1, L.2 and L.3 SimMechanics files, respectively).

11.8 Summary

Like a linkage, a robotic manipulator (commonly called a robot) includes an assembly of links and joints and is designed to produce a controlled output motion. In addition to links and joints, however, a robotic manipulator also includes electronic circuitry, computer-controlled actuators to compel link motion, and is guided by a computer program. Because robotic manipulators include both mechanical and electronic components, they are classified as electro-mechanical systems.

Another common distinction between linkages and robotic manipulators is in their overall design. Linkages commonly have closed-loop designs. With this design, at least two joints in a linkage are connected to ground (thus forming a closed loop). While robotic manipulators can have closed-loop designs, they often have open loops, where only one joint is connected to ground.

The five spatial robotic manipulator types considered in this chapter are the Cartesian, cylindrical, spherical, articulated, and SCARA robots. They are commonly known as the P-P-P, R-P-P, R-R-P, R-R-R, and R-R-C robotic manipulators, respectively. By prescribing coordinate frames for each link in a robotic manipulator and establishing displacement variables between each frame, equation systems are formulated (using the general spatial transformation matrix) to calculate the motion of any link in the robotic manipulator.

This textbook includes a library of MATLAB files for the forward kinematics (Appendices G.1–G.4) and the inverse kinematics (Appendices G.5–G.8) of the R-P-P, R-R-P, R-R-R, and R-R-C robotic manipulators. In forward kinematics, the link dimensions and joint motion of a robotic manipulator are known and the corresponding output motion of the links (usually the end effector) is calculated. In inverse kinematics, the end effector motion and link dimensions are known and the joint motion required to achieve the end effector motion is calculated. Inverse kinematics is often described as the reverse of forward kinematics.

This textbook also includes a library of MATLAB and SimMechanics files for the forward kinematics (Appendices L.1–L.4) of the R-P-P, R-R-P, R-R-R, and R-R-C robotic manipulators. In addition to calculating the global values for the end effector point $\mathbf{p}_3$, the motion of the robotic manipulator is also simulated in the Appendix L files.

References

1. International Federation of Robotics. Industrial robots: Statistics. http://www.ifr.org. Accessed May 18, 2015.
2. Lewis, F. L., D. M. Dawson, and C. T. Abdallah. 2004. *Robot Manipulator Control: Theory and Practice*. 2nd edn. Section 1.2. New York: Marcel Dekker.